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POLYNOMIAL STABILITY OF THERMOELASTIC TIMOSHENKO SYSTEM WITH NON-GLOBAL TIME-DELAYED CATTANEO'S LAW

HAIDAR BADAWI^{1,2} AND HAWRAA ALSAYED^{1,2}

ABSTRACT. In this paper, we consider a one dimensional thermoelastic Timoshenko system in which the heat flux is given by Cattaneo's law and acts locally on the bending moment with a time delay. We prove its well-posedness, strong stability, and polynomial stability.

1. INTRODUCTION

In this paper, we investigate the stability of a thermoelastic Timoshenko system in which the heat flux given by Cattaneo's law acts locally on the bending moment with a time delay. More precisely, we consider the following system:

$$(1.1) \quad \begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + d(\theta) = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_3 \theta_t + q_x + \delta \psi_{tx} = 0 & \text{in } (0, \ell_0) \times (0, \infty), \ell_0 \in (0, \ell), \\ \gamma q_t + \mu_1 q + \mu_2 q(x, t - \tau) + \theta_x = 0 & \text{in } (0, \ell_0) \times (0, \infty), \end{cases}$$

with boundary and initial conditions

$$(1.2) \quad \begin{cases} \varphi(0, t) = \varphi(\ell, t) = \psi(0, t) = \psi(\ell, t) = q(0, t) = \theta(\ell_0, t) = 0 & \text{in } (0, \infty), \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \psi(x, 0) = \psi_0(x) & \text{in } (0, \ell), \\ \psi_t(x, 0) = \psi_1(x), \theta(x, 0) = \theta_0(x), q(x, 0) = q_0(x) & \text{in } (0, \ell_0), \\ q(x, t) = f_0(x, t) & \text{in } (0, \ell_0) \times (-\tau, 0), \end{cases}$$

where φ is the transverse displacement of the beam, ψ is the rotational angle of a filament, θ is the temperature deviations, q is the heat flux. The coefficients $\rho_1, \rho_2, \rho_3, k_1, k_2, \gamma, \mu_1, \delta$ are positive real numbers, μ_2 is a non-zero real number. We define

$$(1.3) \quad d(\theta) = \begin{cases} \delta \theta_x & \text{in } (0, \ell_0) \times (0, \infty), \\ 0 & \text{in } (\ell_0, \ell) \times (0, \infty). \end{cases}$$

In [17], the classical Timoshenko-beam system is expressed by

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) = 0 & \text{in } (0, \ell) \times (0, \infty). \end{cases}$$

There are several publications concerning the stabilization of Timoshenko system with different kinds of damping (see [3], [7], [8] and [18]). Alves and al. [5] studied the thermoelastic Timoshenko system by coupling the

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thermal laws on both the shear force and the bending moment under Fourier's law, by considering:

$$(1.4) \quad \begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x + \delta_1 \theta_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) - \delta_1 \theta + \delta_2 \vartheta_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_3 \theta_t - \mu_1 \theta_{xx} + \delta_1(\varphi_x + \psi)_t = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_4 \vartheta_t - \mu_2 \vartheta_{xx} + \delta_2 \psi_{tx} = 0 & \text{in } (0, \ell) \times (0, \infty), \end{cases}$$

they proved an exponential stability result without any relation between the coefficients of the system. Moreover,

- Fernandez Sare and Racke [12] studied system (1.4) by taking $\delta_1 = 0$ and they showed that the system is exponentially stable if and only if

$$(1.5) \quad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}.$$

- Almeida Junior and al. [4] studied system (1.4) by taking $\delta_2 = 0$ and they also proved exponential stability result provided that (1.5) holds.

In [11], the authors studied the thermoelastic Timoshenko system where the heat flux given by Cattaneo's law is acting on both shear force and the bending moment, by considering:

$$(1.6) \quad \begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x + \delta_1 \theta_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) - \delta_1 \theta + \delta_2 \vartheta_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_3 \theta_t + q_x + \delta_1(\varphi_x + \psi)_t = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \gamma q_t + q + \mu_1 \theta_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_4 \vartheta_t + p_x + \delta_2 \psi_{tx} = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \gamma p_t + p + \mu_2 \vartheta_x = 0 & \text{in } (0, \ell) \times (0, \infty), \end{cases}$$

with boundary conditions

$$\varphi_x(0, t) = \varphi_x(\ell, t) = \psi(0, t) = \psi(\ell, t) = \theta(0, t) = \theta(\ell, t) = p(0, t) = p(\ell, t) \text{ in } (0, \infty),$$

they established an exponential stability result irrespective of the values of the coefficients of the system. However, Fernandez Sare and Racke [12] considered system (1.6) with $\delta_1 = 0$ and they showed that the system is not exponentially stable even if (1.5) holds.

The originality of this work lies in the study of the thermoelastic Timoshenko system (1.1)-(1.2), in which the heat flux given by Cattaneo's law acts only on the bending moment and is distributed locally with a time delay.

In fact, from a physical point of view, models with partial differential equations that take into account a certain time delay are generally more realistic than those that do not take into account the effect of the delay, we refer the reader to [1, 2] for systems with time delay.

This paper is organized as follows. In Section 2, we prove the well-posedness of our system by using semigroup approach. In Section 3, following a general criteria of Arendt and Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Finally, in Section 4, by combining the frequency domain approach with a specific multiplier multiplier method, we prove that the energy of our system decays polynomially with the rate t^{-1} .

2. WELL-POSEDNESS OF THE SYSTEM

In this section, we will establish the well-posedness of system (1.1)-(1.2) by using semigroup approach. For this aim, as in [15], we introduce the following auxiliary change of variable

$$(2.1) \quad z(x, s, t) := q(x, t - s\tau) \quad \text{in } (0, \ell_0) \times (0, 1) \times (0, \infty).$$

Then, system (1.1)-(1.2) becomes

$$(2.2) \quad \begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + d(\theta) = 0 & \text{in } (0, \ell) \times (0, \infty), \\ \rho_3 \theta_t + q_x + \delta \psi_{tx} = 0 & \text{in } (0, \ell_0) \times (0, \infty), \\ \gamma q_t + \mu_1 q + \mu_2 z(x, 1, t) + \theta_x = 0 & \text{in } (0, \ell_0) \times (0, \infty), \\ \tau z_t(x, s, t) + z_s(x, s, t) = 0 & \text{in } (0, \ell_0) \times (0, 1) \times (0, \infty), \end{cases}$$

with the following boundary conditions

$$(2.3) \quad \varphi(0, t) = \varphi(\ell, t) = \psi(0, t) = \psi(\ell, t) = q(0, t) = \theta(\ell_0, t) = 0 \quad \text{in } (0, \infty)$$

and the following initial conditions

$$(2.4) \quad \begin{cases} \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x) & \text{in } (0, \ell), \\ \psi_t(x, 0) = \psi_1(x), \quad \theta(x, 0) = \theta_0(x), \quad q(x, 0) = q_0(x) & \text{in } (0, \ell_0), \\ z(x, s, 0) = f_0(x, -s\tau) & \text{in } (0, \ell_0) \times (0, 1). \end{cases}$$

We set

$$\Phi = (\varphi, \varphi_t, \psi, \psi_t, \theta, q, z)^\top,$$

then system (2.2)-(2.4) can be written as the following first order evolution equation

$$(2.5) \quad \Phi_t = \mathcal{A}\Phi, \quad \Phi(0) = \Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0(\cdot, -\cdot\tau))^\top \in \mathcal{H},$$

where the Hilbert space \mathcal{H} defined by

$$\mathcal{H} = [H_0^1(0, \ell) \times L^2(0, \ell)]^2 \times [L^2(0, \ell_0)]^2 \times L^2((0, \ell_0) \times (0, 1)),$$

is endowed with the inner product

$$\begin{aligned} (\Phi, \Phi^1)_{\mathcal{H}} &= \int_0^\ell \left(k_1(\varphi_x + \psi)(\overline{\varphi_x^1 + \psi^1}) + \rho_1 u \overline{u^1} + k_2 \psi_x \overline{\psi_x^1} + \rho_2 v \overline{v^1} \right) dx + \int_0^{\ell_0} (\rho_3 \theta \overline{\theta^1} + \gamma q \overline{q^1}) dx \\ &\quad + \tau |\mu_2| \int_0^{\ell_0} \int_0^1 z \overline{z^1} ds dx, \end{aligned}$$

where $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top$, $\Phi^1 = (\varphi^1, u^1, \psi^1, v^1, \theta^1, q^1, z^1)^\top \in \mathcal{H}$. Besides, the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \mapsto \mathcal{H}$ is defined by

$$(2.6) \quad \mathcal{A} \begin{pmatrix} \varphi \\ u \\ \psi \\ v \\ \theta \\ q \\ z \end{pmatrix} = \begin{pmatrix} u \\ k_1 \rho_1^{-1}(\varphi_x + \psi)_x \\ v \\ \rho_2^{-1}[k_2 \psi_{xx} - k_1(\varphi_x + \psi) - d(\theta)] \\ -\rho_3^{-1}(q_x + \delta v_x) \\ -\gamma^{-1}[\mu_1 q + \mu_2 z(\cdot, 1) + \theta_x] \\ -\tau^{-1} z_s \end{pmatrix},$$

with domain

$$(2.7) \quad D(\mathcal{A}) = \left\{ \Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in \mathcal{H} \text{ such that } \begin{aligned} &\varphi, \psi \in H^2(0, \ell), \quad u, v \in H_0^1(0, \ell), \\ &\theta \in H_R^1(0, \ell_0) := \{\theta \in H^1(0, \ell_0) \mid \theta(\ell_0) = 0\}, \\ &q \in H_L^1(0, \ell_0) := \{q \in H^1(0, \ell_0) \mid q(0) = 0\}, \\ &z \in L^2((0, \ell_0), H^1(0, 1)), \quad z(\cdot, 0) = q \text{ in } (0, \ell_0) \end{aligned} \right\}$$

The energy of system (2.2)-(2.4) is given by

$$E(t) = \frac{1}{2} \left\| (\varphi, \varphi_t, \psi, \psi_t, \theta, q, z)^\top \right\|_{\mathcal{H}}^2.$$

Here and throughout the paper, the assumptions on μ_1 and μ_2 satisfy

$$|\mu_2| < \mu_1.$$

Proposition 2.1. The operator \mathcal{A} defined above is m-dissipative.

Proof. For all $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$, we have

$$\begin{aligned} \Re(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} &= \Re \left[k_1 \int_0^\ell (u_x + v)(\overline{\varphi_x} + \overline{\psi}) dx + k_1 \int_0^\ell (\varphi_x + \psi)_x \overline{u} dx + k_2 \int_0^\ell v_x \overline{\psi}_x dx + k_2 \int_0^\ell \psi_{xx} \overline{v} dx \right. \\ &\quad - k_1 \int_0^\ell (\varphi_x + \psi) \overline{v} dx - \delta \int_0^{\ell_0} \theta_x \overline{v} dx - \int_0^{\ell_0} (q_x + \delta v_x) \overline{\theta} dx - \mu_1 \int_0^{\ell_0} |q|^2 dx - \mu_2 \int_0^{\ell_0} z(\cdot, 1) \overline{q} dx \\ &\quad \left. - \int_0^{\ell_0} \theta_x \overline{q} dx - |\mu_2| \int_0^{\ell_0} |z(\cdot, 1)|^2 dx + |\mu_2| \int_0^{\ell_0} |q|^2 dx \right], \end{aligned}$$

using integration by parts and the fact that $\Phi \in D(\mathcal{A})$, we get

$$\Re(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} = -\mu_1 \int_0^{\ell_0} |q|^2 dx - \Re \left[\mu_2 \int_0^{\ell_0} z(\cdot, 1) \overline{q} dx \right] - \frac{|\mu_2|}{2} \int_0^{\ell_0} |z(\cdot, 1)|^2 dx + \frac{|\mu_2|}{2} \int_0^{\ell_0} |q|^2 dx.$$

Consequently, by Young's inequality, we claim

$$(2.8) \quad \Re(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \leq -(\mu_1 - |\mu_2|) \int_0^{\ell_0} |q|^2 dx \leq 0,$$

which implies that \mathcal{A} is dissipative. Let us prove that \mathcal{A} is maximal. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7)^\top \in \mathcal{H}$, we look for $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$ a unique solution of

$$(2.9) \quad -\mathcal{A}\Phi = F \in \mathcal{H}.$$

Detailing (2.9), we obtain

$$(2.10) \quad u = -f^1 \in H_0^1(0, \ell),$$

$$(2.11) \quad -k_1(\varphi_x + \psi)_x = \rho_1 f^2 \in L^2(0, \ell)$$

$$(2.12) \quad v = -f^3 \in H_0^1(0, \ell),$$

$$(2.13) \quad -k_2 \psi_{xx} + k_1(\varphi_x + \psi) + d(\theta) = \rho_2 f^4 \in L^2(0, \ell),$$

$$(2.14) \quad q_x + \delta v_x = \rho_3 f^5 \in L^2(0, \ell_0),$$

$$(2.15) \quad \mu_1 q + \mu_2 z(\cdot, 1) + \theta_x = \gamma f^6 \in L^2(0, \ell_0),$$

$$(2.16) \quad z_s = \tau f^7 \in L^2((0, \ell_0) \times (0, 1)),$$

with the following boundary conditions

$$(2.17) \quad \varphi(0) = \varphi(\ell) = \psi(0) = \psi(\ell) = q(0) = \theta(\ell_0) = 0 \quad \text{and} \quad z(\cdot, 0) = q \text{ in } (0, \ell_0).$$

From (2.16), we obtain $z \in L^2((0, \ell_0); H^1(0, 1))$. Moreover, from (2.17), we deduce that

$$(2.18) \quad z = \tau \int_0^s f^7(\cdot, \xi_1) d\xi_1 + q.$$

Inserting (2.12) in (2.14), we get

$$q_x = \delta f_x^3 + \rho_3 f^5,$$

consequently, from (2.17) and the fact that $F \in \mathcal{H}$, we deduce that $q \in H_L^1(0, \ell_0)$ and

$$(2.19) \quad q = \delta f^3 + \rho_3 \int_0^x f^5(\xi_2) d\xi_2.$$

Now, from (2.15), we deduce that $\theta \in H_R^1(0, \ell_0)$ and

$$(2.20) \quad \theta_x = -(\mu_1 + \mu_2) \left(\delta f^3 + \rho_3 \int_0^x f^5(\xi_2) d\xi_2 \right) - \mu_2 \tau \int_0^1 f^7(\cdot, \xi_1) d\xi_1 + \gamma f^6,$$

as a consequence, we get

$$\theta = \int_x^{\ell_0} \left[(\mu_1 + \mu_2) \left(\delta f^3(\xi_3) + \rho_3 \int_0^{\xi_3} f^5(\xi_2) d\xi_2 \right) + \mu_2 \tau \int_0^1 f^7(\xi_3, \xi_1) d\xi_1 - \gamma f^6(\xi_3) \right] d\xi_3.$$

Let $(\phi^1, \phi^2) \in [H_0^1(0, \ell)]^2$, multiplying (2.11) and (2.13) by $\overline{\phi^1}$ and $\overline{\phi^2}$ respectively, integrating over $(0, \ell)$, then using formal integrations by parts, we obtain

$$(2.21) \quad \mathcal{B}((\varphi, \psi), (\phi^1, \phi^2)) = \mathcal{L}(\phi^1, \phi^2), \quad \forall (\phi^1, \phi^2) \in [H_0^1(0, \ell)]^2,$$

where

$$\mathcal{B}((\varphi, \psi), (\phi^1, \phi^2)) = \int_0^\ell \left[k_1(\varphi_x + \psi)(\overline{\phi_x^1} + \overline{\phi_x^2}) + k_2 \psi_x \overline{\phi_x^2} \right] dx$$

and

$$\begin{aligned} \mathcal{L}(\phi^1, \phi^2) &= \int_0^\ell \left(\rho_1 f^2 \overline{\phi^1} + \rho_2 f^4 \overline{\phi^2} \right) dx \\ &\quad + \delta \int_0^{\ell_0} \left[(\mu_1 + \mu_2) \left(\delta f^3 + \rho_3 \int_0^x f^5(\xi_2) d\xi_2 \right) + \mu_2 \tau \int_0^1 f^7(\cdot, \xi_1) d\xi_1 - \gamma f^6 \right] dx. \end{aligned}$$

It is easy to see that, \mathcal{B} is a sesquilinear, continuous and coercive form on $[H_0^1(0, \ell)]^2 \times [H_0^1(0, \ell)]^2$ and \mathcal{L} is an antilinear and continuous form on $[H_0^1(0, \ell)]^2$. Then, it follows by Lax-Milgram theorem that (2.21) admits a unique solution $(\varphi, \psi) \in [H_0^1(0, \ell)]^2$. By taking test-functions $(\phi^1, \phi^2) \in [\mathcal{D}(0, \ell)]^2$, we see that ((2.11), (2.13)) hold in the distributional sense, from which we deduce that $(\varphi, \psi) \in [H^2(0, \ell) \cap H_0^1(0, \ell)]^2$ and hence, $\Phi \in D(\mathcal{A})$ is a unique solution of (2.9). Accordingly, \mathcal{A} is an isomorphism, and since $\rho(\mathcal{A})$ is open set of \mathbb{C} (see Theorem 6.7 (Chapter III) in [13]), we easily get $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This, together with the dissipativeness of \mathcal{A} , imply that $D(\mathcal{A})$ is dense in \mathcal{H} and that \mathcal{A} is m-dissipative in \mathcal{H} (see Theorems 4.5, 4.6 in [16]). \square

According to Lumer-Phillips theorem (see [16]), Proposition 2.1 implies that the operator \mathcal{A} generates a C_0 -semigroup of contractions $e^{t\mathcal{A}}$ in \mathcal{H} which gives the well-posedness of (2.5). Then, we have the following result:

Theorem 2.1. For all $\Phi_0 \in \mathcal{H}$, system (2.5) admits a unique weak solution $\Phi(t) = e^{t\mathcal{A}}\Phi_0 \in C^0(\mathbb{R}_+, \mathcal{H})$. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then the system (2.5) admits a unique strong solution $\Phi(t) = e^{t\mathcal{A}}\Phi_0 \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})$.

3. STRONG STABILITY

In this section, we will prove the strong stability of system (2.2)-(2.4). The main result of this section is the following theorem.

Theorem 3.1. The C_0 -semigroup of contraction $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable in \mathcal{H} ; i.e., for all $\Phi_0 \in \mathcal{H}$, the solution of (2.5) satisfies

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}}\Phi_0\|_{\mathcal{H}} = 0.$$

According to Theorem A.2, to prove Theorem 3.1, we need to prove that the operator \mathcal{A} has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. The proof of these results is not reduced to the analysis of the point spectrum of \mathcal{A} on the imaginary axis since its resolvent is not compact. Hence the proof of Theorem 3.1 is divided into the following two Lemmas.

Lemma 3.1. For all $\lambda \in \mathbb{R}$, we have $i\lambda I - \mathcal{A}$ is injective i.e.,

$$\ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof. From Proposition 2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, suppose that $\lambda \neq 0$ and let $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$ such that

$$(3.1) \quad \mathcal{A}\Phi = i\lambda\Phi.$$

Equivalently, we have the following system

$$\begin{aligned}
(3.2) \quad & u = i\lambda\varphi, \\
(3.3) \quad & k_1(\varphi_x + \psi)_x = i\lambda\rho_1 u, \\
(3.4) \quad & v = i\lambda\psi, \\
(3.5) \quad & k_2\psi_{xx} - k_1(\varphi_x + \psi) - d(\theta) = i\lambda\rho_2\lambda v, \\
(3.6) \quad & q_x + \delta v_x = -i\lambda\rho_3\theta, \\
(3.7) \quad & \mu_1 q + \mu_2 z(\cdot, 1) + \theta_x = -i\lambda\gamma q, \\
(3.8) \quad & z_s = -i\lambda\tau z.
\end{aligned}$$

From (2.8) and (3.1), we obtain

$$(3.9) \quad 0 = \Re(i\lambda\Phi, \Phi)_{\mathcal{H}} = \Re(\mathcal{A}\Phi, \Phi)_{\mathcal{H}} \leq -(\mu_1 - |\mu_2|) \int_0^{\ell_0} |q|^2 dx \leq 0.$$

Thus, we have

$$(3.10) \quad q = 0 \text{ in } (0, \ell_0),$$

consequently, from (3.8) and the fact that $z(\cdot, 0) = q$ in $(0, \ell_0)$, we deduce that

$$(3.11) \quad z = qe^{-i\lambda\tau s} = 0 \text{ in } (0, \ell_0) \times (0, 1).$$

Inserting (3.10) and (3.11) in (3.6), we get

$$\theta_x = 0 \text{ in } (0, \ell_0) \text{ and } d(\theta) = 0 \text{ in } (0, \ell),$$

since $\theta(\ell_0) = 0$, we infer

$$(3.12) \quad \theta = 0 \text{ in } (0, \ell_0).$$

Now, (3.4) and (3.6) imply that

$$(3.13) \quad v_x = \psi_x = 0 \text{ in } (0, \ell_0).$$

Since $\psi(0) = v(0) = 0$, then

$$(3.14) \quad v = \psi = 0 \text{ in } (0, \ell_0),$$

From (3.5), (3.12), (3.14), we obtain

$$\varphi_x = 0 \text{ in } (0, \ell_0),$$

hence, using $\varphi(0) = 0$ and (3.2), we get

$$u = \varphi = 0 \text{ in } (0, \ell_0).$$

Inserting (3.2) and (3.4) in (3.3) and (3.5), we obtain

$$(3.15) \quad \begin{cases} \varphi_{xx} = -\psi_x - \frac{\lambda^2\rho_1}{k_1}\varphi & \text{in } (\ell_0, \ell), \\ \psi_{xx} = \frac{k_1}{k_2}\varphi_x + \frac{k_1 - \lambda^2\rho_2}{k_2}\psi & \text{in } (\ell_0, \ell). \end{cases}$$

Let $\Psi = (\varphi, \varphi_x, \psi, \psi_x)^\top$. From the above system and the regularity of φ and ψ , we deduce that

$$(3.16) \quad \begin{cases} \Psi_x = A_\lambda \Psi & \text{in } (\ell_0, \ell), \\ \Psi(\ell_0) = 0, \end{cases}$$

where

$$A_\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\lambda^2\rho_1}{k_1} & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{k_1}{k_2} & \frac{k_1 - \lambda^2\rho_2}{k_2} & 0 \end{pmatrix}.$$

The solution of the above differential equation is given by

$$\Psi(x) = \exp(A_\lambda(x - \ell_0))\Psi(\ell_0) = 0, \quad \forall x \in (\ell_0, \ell).$$

From the above estimation, (3.2) and (3.4), we deduce that

$$u = \varphi = v = \psi = 0 \quad \text{in } (\ell_0, \ell).$$

Finally, we conclude that

$$\Phi = 0.$$

□

Lemma 3.2. For all $\lambda \in \mathbb{R}$, we have

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Proof. From Proposition 2.1, we have $0 \in \rho(\mathcal{A})$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5, f^6, f^7)^\top \in \mathcal{H}$, we look for $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$ a solution of

$$(3.17) \quad (i\lambda I - \mathcal{A})\Phi = F \in \mathcal{H}.$$

Detailing (3.17), we obtain

$$(3.18) \quad i\lambda\varphi - u = f^1 \in H_0^1(0, \ell),$$

$$(3.19) \quad i\rho_1\lambda u - k_1(\varphi_x + \psi)_x = \rho_1 f^2 \in L^2(0, \ell),$$

$$(3.20) \quad i\lambda\psi - v = f^3 \in H_0^1(0, \ell),$$

$$(3.21) \quad i\rho_2\lambda v - k_2\psi_{xx} + k_1(\varphi_x + \psi) + d(\theta) = \rho_2 f^4 \in L^2(0, \ell),$$

$$(3.22) \quad i\rho_3\lambda\theta + q_x + \delta v_x = \rho_3 f^5 \in L^2(0, \ell_0),$$

$$(3.23) \quad i\gamma\lambda q + \mu_1 q + \mu_2 z(\cdot, 1) + \theta_x = \gamma f^6 \in L^2(0, \ell_0),$$

$$(3.24) \quad i\tau\lambda z + z_s = \tau f^7 \in L^2((0, \ell_0) \times (0, 1)),$$

with the following boundary condition

$$\varphi(0) = \varphi(\ell) = \psi(0) = \psi(\ell) = q(0) = \theta(\ell_0) = 0 \text{ and } z(\cdot, 0) = q \text{ in } (0, \ell_0).$$

From (3.24) and because $z(\cdot, 0) = q$ in $(0, \ell_0)$, we deduce that

$$(3.25) \quad z = qe^{-i\tau\lambda s} + \tau \int_0^s e^{i\lambda\tau(\xi-s)} f^7(\cdot, \xi) d\xi.$$

Consequently, from (3.23), we deduce that

$$(3.26) \quad \theta_x = -(i\gamma\lambda + \mu_1 + \mu_2 e^{-i\tau\lambda})q - \mu_2\tau \int_0^1 e^{i\lambda\tau(\xi-1)} f^7(\cdot, \xi) d\xi + \gamma f^6.$$

Moreover, since $\theta(\ell_0) = 0$, then we get

$$(3.27) \quad \theta = (i\gamma\lambda + \mu_1 + \mu_2 e^{-i\tau\lambda}) \int_x^{\ell_0} q dy + \mu_2\tau \int_x^{\ell_0} \int_0^1 e^{i\lambda\tau(\xi-1)} f^7(y, \xi) d\xi dy - \gamma \int_x^{\ell_0} f^6 dy.$$

Inserting (3.20) and the above equation in (3.22), we get

$$(3.28) \quad i\rho_3\lambda(i\gamma\lambda + \mu_1 + \mu_2 e^{-i\tau\lambda}) \int_x^{\ell_0} q dy + q_x + i\delta\lambda\psi_x = G,$$

where

$$G = -i\rho_3\mu_2\tau\lambda \int_x^{\ell_0} \int_0^1 e^{i\lambda\tau(\xi-1)} f^7(y, \xi) d\xi dy + i\rho_3\gamma\lambda \int_x^{\ell_0} f^6 dy + \rho_3 f^5 + \delta f_x^3.$$

Inserting (3.18) in (3.19), (3.20) and (3.26) in (3.21), then using the fact that

$$-q = \frac{d}{dx} \int_x^{\ell_0} q dy,$$

we obtain

$$(3.29) \quad -\rho_1\lambda^2\varphi - k_1(\varphi_x + \psi)_x = \rho_1 f^2 + i\rho_1\lambda f^1,$$

$$(3.30) \quad -\rho_2\lambda^2\psi - k_2\psi_{xx} + k_1(\varphi_x + \psi) + d_1 \left(\frac{d}{dx} \int_x^{\ell_0} q dy \right) = \rho_2 f^4 + i\rho_2\lambda f^3 + d_2(f^6, f^7),$$

where

$$d_1(X) = \begin{cases} (i\gamma\lambda + \mu_1 + \mu_2 e^{-i\tau\lambda})X & \text{in } (0, \ell_0), \\ 0 & \text{in } (\ell_0, \ell), \end{cases}$$

with $X = \frac{d}{dx} \int_x^{\ell_0} q dy$ or $\int_x^{\ell_0} q dy$, and

$$d_2(f^6, f^7) = \begin{cases} \mu_2 \tau \int_0^1 e^{i\lambda\tau(\xi-1)} f^7(\cdot, \xi) d\xi - \gamma f^6 & \text{in } (0, \ell_0), \\ 0 & \text{in } (\ell_0, \ell). \end{cases}$$

We set

$$\mathbb{H} := [H_0^1(0, \ell)]^2 \times H_R^1(0, \ell_0), \quad V := (\varphi, \psi, \int_x^{\ell_0} q dy), \quad \text{and} \quad W := (\Phi, \Psi, \int_x^{\ell_0} Q dy) \in \mathbb{H}.$$

Multiplying (3.29), (3.30), and (3.28) by $\overline{\Phi}$, $\overline{\Psi}$, and $-\int_x^{\ell_0} \overline{Q} dy$, respectively, then integrating over $(0, \ell)$ and using formal integration by parts:

$$(3.31) \quad (\mathcal{B}_1 + \mathcal{B}_2)(V, W) = \mathcal{L}(W), \quad \forall W \in \mathbb{H},$$

where

$$(3.32) \quad \begin{aligned} \mathcal{B}_1(V, W) &= k_1 \int_0^\ell (\varphi_x + \psi)(\overline{\Phi}_x + \overline{\Psi}) dx + k_2 \int_0^\ell \psi_x \overline{\Psi}_x dx + \rho_3 \gamma \lambda^2 \int_0^{\ell_0} \left(\int_x^{\ell_0} q dy \right) \left(\int_x^{\ell_0} \overline{Q} dy \right) dx \\ &\quad - i \rho_3 \mu_1 \lambda \int_0^{\ell_0} \left(\int_x^{\ell_0} q dy \right) \left(\int_x^{\ell_0} \overline{Q} dy \right) dx + \int_0^{\ell_0} q \overline{Q} dx, \end{aligned}$$

$$(3.33) \quad \begin{aligned} \mathcal{B}_2(V, W) &= -\rho_1 \lambda^2 \int_0^\ell \varphi \overline{\Phi} dx - \rho_2 \lambda^2 \int_0^\ell \psi \overline{\Psi} dx - \int_0^\ell d_1 \left(\int_x^{\ell_0} q dy \right) \overline{\Psi}_x dx \\ &\quad - i \rho_3 \mu_2 \lambda e^{-i\tau\lambda} \int_0^{\ell_0} \left(\int_x^{\ell_0} q dy \right) \left(\int_x^{\ell_0} \overline{Q} dy \right) dx + i \delta \lambda \int_0^{\ell_0} \psi \overline{Q} dx \end{aligned}$$

and

$$(3.34) \quad \mathcal{L}(W) = \int_0^\ell (\rho_1 f^2 + i \rho_1 \lambda f^1) \overline{\Phi} dx + \int_0^\ell [\rho_2 f^4 + i \rho_2 \lambda f^3 + d_2(f^6, f^7)] \overline{\Psi} dx - \int_0^{\ell_0} G \left(\int_x^{\ell_0} \overline{Q} dy \right) dx.$$

Let \mathbb{H}' be the dual space of \mathbb{H} , define the following operators

$$(3.35) \quad \begin{aligned} \mathbb{B}_j : \mathbb{H} &\longmapsto \mathbb{H}' \\ V &\longmapsto \mathbb{B}_j V, \quad j \in \{1, 2\}, \end{aligned}$$

such that

$$(3.36) \quad (\mathbb{B}_j V)(W) = \mathcal{B}_j(V, W), \quad \forall W \in \mathbb{H}.$$

We need to prove that the operator $\mathbb{B}_1 + \mathbb{B}_2$ is an isomorphism. For this purpose, we divide the proof into two steps:

Step 1. In this step, we prove that the operator \mathbb{B}_2 is compact. Indeed, from (3.33) and (3.36) we have

$$(3.37) \quad |\mathcal{B}_2(V, W)| \leq C \|V\|_{[L^2(0, \ell)]^2 \times L^2(0, \ell_0)} \|W\|_{\mathbb{H}},$$

for some positive constant C , and consequently, using the compact embedding from \mathbb{H} into $[L^2(0, \ell)]^2 \times L^2(0, \ell_0)$, we deduce that \mathbb{B}_2 is a compact operator.

The compactness property and the fact that \mathbb{B}_1 is an isomorphism imply that the operator $\mathbb{B}_1 + \mathbb{B}_2$ is a Fredholm operator of index

$$0 = \dim \ker(\mathbb{B}_1 + \mathbb{B}_2) - \dim R(\mathbb{B}_1 + \mathbb{B}_2)^\perp.$$

Now, following Fredholm alternative, we simply need to prove that the operator $\mathbb{B}_1 + \mathbb{B}_2$ is injective to obtain that it is an isomorphism.

Step 2. In this step, we want to prove that the operator $\mathbb{B}_1 + \mathbb{B}_2$ is injective (i.e. $\ker(\mathbb{B}_1 + \mathbb{B}_2) = \{0\}$). So, let $(\widehat{\varphi}, \widehat{\psi}, \int_x^{\ell_0} \widehat{q} dy) \in \ker(\mathbb{B}_1 + \mathbb{B}_2)$ which gives

$$(\mathcal{B}_1 + \mathcal{B}_2) \left((\widehat{\varphi}, \widehat{\psi}, \int_x^{\ell_0} \widehat{q} dy), W \right) = 0, \quad \forall W \in \mathbb{H}.$$

Thus, we find that

$$(3.38) \quad \begin{cases} -\rho_1 \lambda^2 \widehat{\varphi} - k_1 (\widehat{\varphi}_x + \widehat{\psi})_x = 0 \\ -\rho_2 \lambda^2 \widehat{\psi} - k_2 \widehat{\psi}_{xx} + k_1 (\widehat{\varphi}_x + \widehat{\psi}) + d_1 \left(\frac{d}{dx} \int_x^{\ell_0} \widehat{q} dy \right) = 0, \\ i\rho_3 \lambda (i\gamma\lambda + \mu_1 + \mu_2 e^{-i\tau\lambda}) \int_x^{\ell_0} \widehat{q} dy + \widehat{q}_x + i\delta\lambda \widehat{\psi}_x = 0, \\ \widehat{\varphi}(0) = \widehat{\varphi}(\ell) = \widehat{\psi}(0) = \widehat{\psi}(\ell) = \widehat{q}(0) = 0. \end{cases}$$

Therefore, the vector $\widehat{\Phi}$ defined by

$$\widehat{\Phi} = \left(\widehat{\varphi}, i\lambda\widehat{\varphi}, \widehat{\psi}, i\lambda\widehat{\psi}, (i\gamma\lambda + \mu_1 + \mu_2 e^{-i\tau\lambda}) \int_x^{\ell_0} \widehat{q} dy, \widehat{q}, \widehat{q} e^{-i\tau\lambda s} \right)^\top$$

belongs to $D(\mathcal{A})$ and satisfies

$$i\lambda\widehat{\Phi} - \mathcal{A}\widehat{\Phi} = 0,$$

and consequently $\widehat{\Phi} \in \ker(i\lambda I - \mathcal{A})$. Hence, Lemma 3.1 yields $\widehat{\Phi} = 0$ as a result $\widehat{\varphi} = \widehat{\psi} = 0$ and $\ker(\mathbb{B}_1 + \mathbb{B}_2) = \{0\}$.

Steps 1 and 2 guarantee that the operator $\mathbb{B}_1 + \mathbb{B}_2$ is an isomorphism. Furthermore, it is easy to see that the operator \mathcal{L} is an antilinear and continuous form on \mathbb{H} . Therefore, (3.31) admits a unique solution $V \in \mathbb{H}$. In (3.31), by taking the test functions $W \in [\mathcal{D}(0, \ell)]^2 \times \mathcal{D}(0, \ell_0)$, we see that ((3.28), (3.29), (3.30)) holds in distributional sense, hence

$$q \in H_L^1(0, \ell_0) \quad \text{and} \quad \varphi, \psi \in H^2(0, \ell) \cap H_0^1(0, \ell).$$

Moreover, from (3.25) and (3.27), we deduce that

$$z \in L^2((0, \ell_0); H^1(0, 1)) \quad \text{and} \quad \theta \in H_R^1(0, \ell_0).$$

Finally, we claim that

$$\Phi = (\varphi, i\lambda\varphi - f^1, \psi, i\lambda\psi - f^3, \theta, q, z)^\top \in D(\mathcal{A}),$$

is the unique solution of (3.2). \square

Proof of Theorem 3.1. From Lemma 3.1, the operator \mathcal{A} has no pure imaginary eigenvalues (i.e. $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$). Moreover, from Lemma 3.1 and Lemma 3.2, $i\lambda I - \mathcal{A}$ is bijective for all $\lambda \in \mathbb{R}$ and since \mathcal{A} is closed, we conclude, with the help of the closed graph theorem, that $i\lambda I - \mathcal{A}$ is an isomorphism for all $\lambda \in \mathbb{R}$, hence $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. \square

4. POLYNOMIAL STABILITY

In this section, we will prove the polynomial stability of system (2.2)-(2.4). The main result of this section is the following theorem.

Theorem 4.1. For all $\Phi_0 \in D(\mathcal{A})$, there exists a constant $C > 0$ independent of Φ_0 such that the energy of system (2.2)-(2.4) satisfies the following estimation

$$E(t) \leq \frac{C}{t} \|\Phi_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0.$$

According to Theorem A.3, to prove Theorem 4.1, we need to prove the following two conditions

$$(4.1) \quad i\mathbb{R} \subset \rho(\mathcal{A})$$

and

$$(4.2) \quad \limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{\lambda^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

As condition (4.1) was checked in Section 3, we only need to prove the second condition. Condition (4.2) is proved by a contradiction argument. For this purpose, suppose that (4.2) is false, then there exists $\{(\lambda_n, \Phi_n := (\varphi_n, u_n, \psi_n, v_n, \theta_n, q_n, z_n)^\top)\}_{n \geq 1} \subset \mathbb{R}_+^* \times D(\mathcal{A})$ with

$$(4.3) \quad \lambda_n \rightarrow \infty \text{ as } n \rightarrow \infty \quad \text{and} \quad \|\Phi_n\|_{\mathcal{H}} = 1, \forall n \geq 1,$$

such that

$$(4.4) \quad (\lambda_n)^2 (i\lambda_n I - \mathcal{A})\Phi_n = F_n := (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6, f_n^7)^\top \rightarrow 0 \quad \text{in } \mathcal{H} \quad \text{as } n \rightarrow \infty.$$

For simplicity, we drop the index n . Equivalently, from (4.4), we have

$$(4.5) \quad i\lambda\varphi - u = \lambda^{-2}f^1,$$

$$(4.6) \quad i\rho_1\lambda u - k_1(\varphi_x + \psi)_x = \rho_1\lambda^{-2}f^2,$$

$$(4.7) \quad i\lambda\psi - v = \lambda^{-2}f^3,$$

$$(4.8) \quad i\rho_2\lambda v - k_2\psi_{xx} + k_1(\varphi_x + \psi) + d(\theta) = \rho_2\lambda^{-2}f^4,$$

$$(4.9) \quad i\rho_3\lambda\theta + q_x + \delta v_x = \rho_3\lambda^{-2}f^5,$$

$$(4.10) \quad i\gamma\lambda q + \mu_1 q + \mu_2 z(\cdot, 1) + \theta_x = \gamma\lambda^{-2}f^6,$$

$$(4.11) \quad i\tau\lambda z + z_s = \tau\lambda^{-2}f^7.$$

Inserting (4.5) in (4.6), (4.7) in (4.8) and (4.9), we get

$$(4.12) \quad \rho_1\lambda^2\varphi + k_1(\varphi_x + \psi)_x = -\rho_1\lambda^{-2}f^2 - i\rho_1\lambda^{-1}f^1,$$

$$(4.13) \quad \rho_2\lambda^2\psi + k_2\psi_{xx} - k_1(\varphi_x + \psi) - d(\theta) = -\rho_2\lambda^{-2}f^4 - i\rho_2\lambda^{-1}f^3,$$

$$(4.14) \quad i\rho_3\lambda\theta + q_x + i\delta\lambda\psi_x = \rho_3\lambda^{-2}f^5 + \delta\lambda^{-2}f^3.$$

Here we will check condition (4.2) by finding a contradiction with (4.3) i.e. by showing $\|\Phi\|_{\mathcal{H}} = o(1)$. For clarity, we divide the proof into three Lemmas. From the above system and the fact that $\|\Phi\|_{\mathcal{H}} = 1$ and $\|F\|_{\mathcal{H}} = o(1)$, we infer that

$$(4.15) \quad \begin{cases} \|\lambda\varphi\|_{L^2(0,\ell)}, \|\lambda\psi\|_{L^2(0,\ell)}, \|\varphi_x\|_{L^2(0,\ell)}, \|\psi_x\|_{L^2(0,\ell)}, \|\theta\|_{L^2(0,\ell_0)} = O(1), \\ \|\varphi_{xx}\|_{L^2(0,\ell)}, \|\psi_{xx}\|_{L^2(0,\ell)} = O(\lambda). \end{cases}$$

Also, for all $\Omega \subseteq (0, \ell)$, we have

$$(4.16) \quad \|\psi_{xx}\|_{L^2(\Omega)} \leq O(\lambda)\|\lambda\psi\|_{L^2(\Omega)} + O(1).$$

Lemma 4.1. The solution $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$ of (4.5)-(4.11) satisfies the following estimations

$$\begin{aligned} \int_0^{\ell_0} |q|^2 dx &= o(\lambda^{-2}), \\ \int_0^{\ell_0} \int_0^1 |z|^2 ds dx &= o(\lambda^{-2}), \\ \int_0^{\ell_0} |z(\cdot, 1)|^2 dx &= o(\lambda^{-2}), \\ \int_0^{\ell_0} |\theta_x|^2 dx &= o(1), \\ \int_0^{\ell_0} |\theta|^2 dx &= o(\lambda^{-1}). \end{aligned}$$

Proof. First, taking the inner product of (4.4) with Φ in \mathcal{H} and using (2.8), we get

$$\int_0^{\ell_0} |q|^2 dx \leq \lambda^{-2}(\mu_1 - |\mu_2|)^{-1} \Re(F, \Phi)_{\mathcal{H}} \leq \lambda^{-2}(\mu_1 - |\mu_2|)^{-1} \|F\|_{\mathcal{H}} \|\Phi\|_{\mathcal{H}} = o(\lambda^{-2}).$$

From (4.11) and since $z(\cdot, 0) = q$ in $(0, \ell_0)$, we have

$$z = qe^{-i\tau\lambda s} + \tau\lambda^{-2} \int_0^s e^{i\lambda\tau(\xi-s)} f^7(\cdot, y) dy \quad \text{in } (0, \ell_0) \times (0, 1),$$

consequently, by using Cauchy-Schwarz's inequalities, we deduce that

$$\begin{aligned}
\int_0^{\ell_0} \int_0^1 |z|^2 ds dx &\leq 2 \int_0^{\ell_0} |q|^2 dx + 2\tau^2 \lambda^{-4} \int_0^{\ell_0} \int_0^1 \left(\int_0^s |f^7(\cdot, y)| dy \right)^2 ds dx \\
&\leq 2 \int_0^{\ell_0} |q|^2 dx + 2\tau^2 \lambda^{-4} \int_0^{\ell_0} \int_0^1 s \int_0^s |f^7(\cdot, y)|^2 dy ds dx \\
&\leq 2 \int_0^{\ell_0} |q|^2 dx + 2\tau^2 \lambda^{-4} \left(\int_0^1 s ds \right) \int_0^{\ell_0} \int_0^1 |f^7(\cdot, y)|^2 dy dx \\
&= 2 \int_0^{\ell_0} |q|^2 dx + \tau^2 \lambda^{-4} \int_0^{\ell_0} \int_0^1 |f^7(\cdot, y)|^2 dy dx \\
&= o(\lambda^{-2}),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\ell_0} |z(\cdot, 1)|^2 dx &= \int_0^{\ell_0} \left| qe^{-i\tau\lambda} + \tau\lambda^{-2} \int_0^1 e^{i\lambda\tau(\xi-1)} f^7(\cdot, y) dy \right|^2 dx \\
&\leq 2 \int_0^{\ell_0} |q|^2 dx + 2\tau^2 \lambda^{-4} \int_0^{\ell_0} \left(\int_0^1 |f^7(\cdot, y)| dy \right)^2 dx \\
&\leq 2 \int_0^{\ell_0} |q|^2 dx + 2\tau^2 \lambda^{-4} \int_0^{\ell_0} \int_0^1 |f^7(\cdot, y)|^2 dy dx \\
&= o(\lambda^{-2}).
\end{aligned}$$

Now, from (4.9), we obtain

$$\begin{aligned}
\int_0^{\ell_0} |\theta_x|^2 dx &= \int_0^{\ell_0} |-i\gamma\lambda q - \mu_1 q - \mu_2 z(\cdot, 1) + \gamma\lambda^{-2} f^6|^2 dx \\
&\leq (\gamma^2 \lambda^2 + \mu_1^2) \int_0^{\ell_0} |q|^2 dx + \mu_2^2 \int_0^{\ell_0} |z(\cdot, 1)|^2 dx + \gamma^2 \lambda^{-4} \int_0^{\ell_0} |f^6|^2 dx \\
&= o(1).
\end{aligned}$$

Finally, multiplying (4.13) by $(\rho_3 \lambda)^{-1} \bar{\theta}$, integrating over $(0, \ell_0)$, using integration by parts:

$$\begin{aligned}
\int_0^{\ell_0} |\theta|^2 dx &= \Im \left[(\rho_3 \lambda)^{-1} \int_0^{\ell_0} q \bar{\theta}_x dx - i\delta \int_0^{\ell_0} \psi \bar{\theta}_x dx + \lambda^{-2} \int_0^{\ell_0} (\rho_3 f^5 + \delta f_x^3) \bar{\theta} dx \right] \\
&\leq (\rho_3 \lambda)^{-1} \|q\|_{L^2(0, \ell_0)} \|\theta_x\|_{L^2(0, \ell_0)} + \delta \underbrace{\|\psi\|_{L^2(0, \ell_0)}}_{=O(\lambda^{-1})} \|\theta_x\|_{L^2(0, \ell_0)} + \lambda^{-2} (\rho_3 + \delta) \|F\|_{\mathcal{H}} \underbrace{\|\theta\|_{L^2(0, \ell_0)}}_{=O(1)} \\
&= o(\lambda^{-1}).
\end{aligned}$$

□

Lemma 4.2. The solution $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$ of (4.5)-(4.11) satisfies the following estimations

$$\|\psi_x\|_{L^2(\omega_4)}^2 = o(\lambda^{-2}) \quad \text{and} \quad \|\lambda\psi\|_{L^2(\omega_5)}^2 = o(\lambda^{-2}).$$

Proof. To prove this Lemma, for some $j, k \in \mathbb{N}$, $1 \leq j \leq 7$, and $0 < \varepsilon < \frac{\ell_0}{14}$, we need to define the following intervals:

$$\omega_k = (k\varepsilon, \ell_0 - k\varepsilon), \quad \forall j-1 \leq k \leq j$$

and as well we need to fix functions $\mathbf{h}_j \in C^1(\overline{\omega_0})$ such that $0 \leq \mathbf{h}_j(x) \leq 1$, for all $x \in \overline{\omega_0}$ and

$$\mathbf{h}_j(x) = \begin{cases} 1 & \text{if } x \in \overline{\omega_j}, \\ 0 & \text{if } x \in \overline{\omega_0} \setminus \omega_{j-1}. \end{cases}$$

For simplicity, let us divide the proof into four steps:

Step1. In this step, we prove:

$$(4.17) \quad \begin{aligned} \|\psi_x\|_{L^2(\omega_j)}^2 &\leq \max\left(\frac{\rho_3}{\delta} \max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)|, \frac{\rho_3}{\delta}, \frac{1}{\delta} \max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)|, \frac{1}{\delta}, 1\right) [(\|\theta\|_{L^2(\omega_{j-1})} + \|\theta_x\|_{L^2(\omega_{j-1})}) \|\psi\|_{L^2(\omega_{j-1})} \\ &\quad + \lambda^{-1} (\|\psi_x\|_{L^2(\omega_{j-1})} + \|\psi_{xx}\|_{L^2(\omega_{j-1})}) \|q\|_{L^2(\omega_{j-1})} + \lambda^{-3} \|F\|_{\mathcal{H}} \|\psi_x\|_{L^2(\omega_{j-1})}]. \end{aligned}$$

For this aim, Multiplying (4.14) by $\delta^{-1}\lambda^{-1}\mathbf{h}_j\bar{\psi}_x$ and integrating over ω_{j-1} , we get

$$(4.18) \quad \int_{\omega_{j-1}} \mathbf{h}_j |\psi_x|^2 dx = \Im \left[-i \frac{\rho_3}{\delta} \int_{\omega_{j-1}} \mathbf{h}_j \theta \bar{\psi}_x dx - \frac{1}{\delta \lambda} \int_{\omega_{j-1}} \mathbf{h}_j q_x \bar{\psi}_x dx + \lambda^{-3} \int_{\omega_{j-1}} \mathbf{h}_j (\rho_3 \delta^{-1} f^5 + f_x^3) \bar{\psi}_x dx \right].$$

Since $\mathbf{h}_j(x) = 0$, $\forall x \in \partial\omega_{j-1}$, then integration by parts and Cauchy-Schwarz's inequality yield,

$$\left| \int_{\omega_{j-1}} \mathbf{h}_j \theta \bar{\psi}_x dx \right| = \left| \int_{\omega_{j-1}} (\mathbf{h}'_j \theta + \mathbf{h}_j \theta_x) \bar{\psi} dx \right| \leq \left(\max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)| \|\theta\|_{L^2(\omega_{j-1})} + \|\theta_x\|_{L^2(\omega_{j-1})} \right) \|\psi\|_{L^2(\omega_{j-1})}$$

and

$$\left| \int_{\omega_{j-1}} \mathbf{h}_j q_x \bar{\psi}_x dx \right| = \left| \int_{\omega_{j-1}} (\mathbf{h}'_j \bar{\psi}_x + \mathbf{h}_j \bar{\psi}_{xx}) q dx \right| \leq \left(\max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)| \|\psi_x\|_{L^2(\omega_{j-1})} + \|\psi_{xx}\|_{L^2(\omega_{j-1})} \right) \|q\|_{L^2(\omega_{j-1})}.$$

Moreover, by Cauchy-Schwarz's inequality, we get

$$\left| \int_{\omega_{j-1}} \mathbf{h}_j (\rho_3 \delta^{-1} f^5 + f_x^3) \bar{\psi}_x dx \right| \leq (\rho_3 \delta^{-1} + 1) \|F\|_{\mathcal{H}} \|\psi_x\|_{L^2(\omega_{j-1})}.$$

Using (4.18), the above inequalities, and the fact that $\|\psi_x\|_{L^2(\omega_j)}^2 \leq \|\sqrt{\mathbf{h}_j} \psi_x\|_{L^2(\omega_{j-1})}^2$, we deduce (4.17).

Step 2. In this step, we prove:

$$(4.19) \quad \begin{aligned} \|\lambda \psi\|_{L^2(\omega_j)}^2 &\leq \max\left(k_2 \rho_2^{-1}, k_2 \rho_2^{-1} \max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)|, k_1 \rho_2^{-1} \max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)|, k_1 \rho_2^{-1}, \delta \rho_2^{-1}, \rho_2\right) \left[\|\psi_x\|_{L^2(\omega_{j-1})}^2 \right. \\ &\quad + (\|\psi_x\|_{L^2(\omega_{j-1})} + \|\theta_x\|_{L^2(\omega_{j-1})}) \|\psi\|_{L^2(\omega_{j-1})} \\ &\quad + (\|\psi\|_{L^2(\omega_{j-1})} + \|\psi_x\|_{L^2(\omega_{j-1})}) \|\varphi\|_{L^2(\omega_{j-1})} \\ &\quad \left. + \|\psi\|_{L^2(\omega_{j-1})}^2 + (\lambda^{-2} + \lambda^{-1}) \|F\|_{\mathcal{H}} \|\psi\|_{L^2(\omega_{j-1})} \right]. \end{aligned}$$

So, multiplying (4.13) by $\rho_2^{-1}\mathbf{h}_j\bar{\psi}$ and integrating over ω_{j-1} , we infer

$$(4.20) \quad \begin{aligned} \int_{\omega_{j-1}} \mathbf{h}_j |\lambda \psi|^2 dx &= \Re \left[-k_2 \rho_2^{-1} \int_{\omega_{j-1}} \mathbf{h}_j \psi_{xx} \bar{\psi} dx + k_1 \rho_2^{-1} \int_{\omega_{j-1}} \mathbf{h}_j (\varphi_x + \psi) \bar{\psi} dx + \delta \rho_2^{-1} \int_{\omega_{j-1}} \mathbf{h}_j \theta_x \bar{\psi} dx \right. \\ &\quad \left. - \rho_2 \int_{\omega_{j-1}} \mathbf{h}_j (\lambda^{-2} f^4 + i \lambda^{-1} f^3) \bar{\psi} dx \right]. \end{aligned}$$

Using integrations by parts and Cauchy-Schwarz's inequality, we get

$$\left| \int_{\omega_{j-1}} \mathbf{h}_j \psi_{xx} \bar{\psi} dx \right| = \left| - \int_{\omega_{j-1}} \mathbf{h}_j |\psi_x|^2 dx - \int_{\omega_{j-1}} \mathbf{h}'_j \psi_x \bar{\psi} dx \right| \leq \|\psi_x\|_{L^2(\omega_{j-1})}^2 + \max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)| \|\psi_x\|_{L^2(\omega_{j-1})} \|\psi\|_{L^2(\omega_{j-1})}$$

and

$$\begin{aligned} \left| \int_{\omega_{j-1}} \mathbf{h}_j (\varphi_x + \psi) \bar{\psi} dx \right| &= \left| - \int_{\omega_{j-1}} (\mathbf{h}'_j \bar{\psi} + \mathbf{h}_j \bar{\psi}_x) \varphi dx + \int_{\omega_{j-1}} \mathbf{h}_j |\psi|^2 dx \right| \\ &\leq \left(\max_{x \in \bar{\omega}_j} |\mathbf{h}'_j(x)| \|\psi\|_{L^2(\omega_{j-1})} + \|\psi_x\|_{L^2(\omega_{j-1})} \right) \|\varphi\|_{L^2(\omega_{j-1})} + \|\psi\|_{L^2(\omega_{j-1})}^2. \end{aligned}$$

Moreover, owing to Cauchy-Schwarz's inequality, we obtain

$$\left| \int_{\omega_{j-1}} \mathbf{h}_j \theta_x \bar{\psi} dx \right| \leq \|\theta_x\|_{L^2(\omega_{j-1})} \|\psi\|_{L^2(\omega_{j-1})}$$

and

$$\left| \int_{\omega_{j-1}} \mathbf{h}_j (\lambda^{-2} f^4 + i \lambda^{-1} f^3) \bar{\psi} dx \right| \leq (\lambda^{-2} + \lambda^{-1}) \|F\|_{\mathcal{H}} \|\psi\|_{L^2(\omega_{j-1})}.$$

Using (4.20) and the above inequalities, we deduce (4.19).

Step 3. In this step, we prove that

$$(4.21) \quad \|\lambda \psi\|_{L^2(\omega_3)}^2 = O(\lambda^{-2}) \quad \text{and} \quad \|\psi_{xx}\|_{L^2(\omega_3)}^2 = O(1).$$

First, taking $j = 1$ in (4.17), using Lemma 4.1, and (4.15), we get

$$(4.22) \quad \|\psi_x\|_{L^2(\omega_1)}^2 = o(\lambda^{-1}).$$

Now, take $j = 2$ in (4.19) and use the above estimation with Lemma 4.1 and (4.15), we deduce that

$$(4.23) \quad \|\lambda \psi\|_{L^2(\omega_2)}^2 = o(\lambda^{-1}).$$

Multiplying (4.13) and (4.14) by $-ik_2(\delta\lambda)^{-1}\mathbf{h}_3\bar{\psi}_x$ and $\mathbf{h}_3\bar{\psi}$, respectively, and integrating over ω_2 , we have

$$(4.24) \quad k_2\rho_3\delta^{-1}\mathbf{I}_1 - ik_2(\delta\lambda)^{-1}\mathbf{I}_2 + k_2 \int_{\omega_2} \mathbf{h}_3 |\psi_x|^2 dx = -ik_2 \int_{\omega_2} \mathbf{h}_3 (\delta^{-1}\rho_3\lambda^{-3}f^5 + i\lambda^{-3}f_x^3) \bar{\psi}_x dx$$

and

$$(4.25) \quad \rho_2 \int_{\omega_2} \mathbf{h}_3 |\lambda \psi|^2 dx + k_2 \mathbf{I}_3 - k_1 \int_{\omega_2} \mathbf{h}_3 (\varphi_x + \psi) \bar{\psi} dx - \delta \int_{\omega_2} \mathbf{h}_3 \theta_x \bar{\psi} dx = -\rho_2 \int_{\omega_2} \mathbf{h}_3 (\lambda^{-2} f^4 + i \lambda^{-1} f^3) \bar{\psi} dx,$$

where

$$\mathbf{I}_1 = \int_{\omega_2} \mathbf{h}_3 \theta \bar{\psi}_x dx, \quad \mathbf{I}_2 = \int_{\omega_2} \mathbf{h}_3 q_x \bar{\psi}_x dx \quad \text{and} \quad \mathbf{I}_3 = \int_{\omega_2} \mathbf{h}_3 \psi_{xx} \bar{\psi} dx.$$

Using integration by parts, \mathbf{I}_1 , \mathbf{I}_2 , and \mathbf{I}_3 become

$$\mathbf{I}_1 = - \int_{\omega_2} \mathbf{h}_3 \theta_x \bar{\psi} dx - \int_{\omega_2} \mathbf{h}_3' \theta \bar{\psi} dx,$$

$$\mathbf{I}_2 = - \int_{\omega_2} \mathbf{h}_3 q \bar{\psi}_{xx} dx - \int_{\omega_2} \mathbf{h}_3' q \bar{\psi}_x dx$$

and

$$(4.26) \quad \mathbf{I}_3 = - \int_{\omega_2} \mathbf{h}_3 |\psi_x|^2 dx - \int_{\omega_2} \mathbf{h}_3' \psi_x \bar{\psi} dx$$

Now, Cauchy-Schwarz's inequality yields

$$|\mathbf{I}_1| \leq \underbrace{\|\sqrt{\mathbf{h}_3} \theta_x\|_{L^2(\omega_2)} \|\sqrt{\mathbf{h}_3} \psi\|_{L^2(\omega_2)}}_{=: A_1} + \max_{x \in \omega_3} |\mathbf{h}_3'(x)| \|\theta\|_{L^2(\omega_2)} \|\psi\|_{L^2(\omega_2)}.$$

In view of (4.13), we have

$$k_2 \bar{\psi}_{xx} = -\rho_2 \lambda^2 \bar{\psi} + k_1 (\bar{\varphi}_x + \bar{\psi}) + \delta \bar{\theta}_x - \rho_2 \lambda^{-2} \bar{f}^4 + i \rho_2 \lambda^{-1} \bar{f}^3 \quad \text{in } \omega_2.$$

Hence,

$$\begin{aligned} k_2 \mathbf{I}_2 &= \rho_2 \lambda^2 \int_{\omega_2} \mathbf{h}_3 q \bar{\psi} dx - k_1 \int_{\omega_2} \mathbf{h}_3 q (\bar{\varphi}_x + \bar{\psi}) dx - \delta \int_{\omega_2} \mathbf{h}_3 q \bar{\theta}_x dx + \rho_2 \int_{\omega_2} \mathbf{h}_3 q (\lambda^{-2} \bar{f}^4 - i \lambda^{-1} \bar{f}^3) dx \\ &\quad - k_2 \int_{\omega_2} \mathbf{h}_3' q \bar{\psi}_x dx. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
k_2 \lambda^{-1} |I_2| &\leq \underbrace{\rho_2 \|\sqrt{h_3} q\|_{L^2(\omega_2)} \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}}_{=: A_2} + k_1 \lambda^{-1} \|q\|_{L^2(\omega_2)} (\|\varphi_x\|_{L^2(\omega_2)} + \|\psi\|_{L^2(\omega_2)}) \\
&\quad + \delta \lambda^{-1} \|q\|_{L^2(\omega_2)} \|\theta_x\|_{L^2(\omega_2)} + \rho_2 (\lambda^{-3} + \lambda^{-2}) \|q\|_{L^2(\omega_2)} \|F\|_{\mathcal{H}} \\
&\quad + k_2 \lambda^{-1} \max_{x \in \bar{\omega}_3} |h'_3(x)| \|q\|_{L^2(\omega_2)} \|\psi_x\|_{L^2(\omega_2)}.
\end{aligned}$$

Adding (4.24) and (4.25), then using (4.26), we infer

$$\begin{aligned}
\rho_2 \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2 &= \left| -\frac{k_2 \rho_3}{\delta} I_1 + \frac{i k_2}{\delta \lambda} I_2 + k_2 \int_{\omega_2} h'_3 \psi_x \bar{\psi} dx + k_1 \int_{\omega_2} h_3 (\varphi_x + \psi) \bar{\psi} dx + \delta \int_{\omega_2} h_3 \theta_x \bar{\psi} dx \right| \\
&\leq k_2 \rho_3 \delta^{-1} |I_1| + \delta^{-1} k_2 \lambda^{-1} |I_2| + k_2 \max_{x \in \bar{\omega}_3} |h'_3(x)| \|\psi_x\|_{L^2(\omega_2)} \|\psi\|_{L^2(\omega_2)} \\
&\quad + \underbrace{k_1 \|\sqrt{h_3} \varphi_x\|_{L^2(\omega_2)} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}}_{=: A_3} + k_1 \|\psi\|_{L^2(\omega_2)}^2 + \underbrace{\delta \|\sqrt{h_3} \theta_x\|_{L^2(\omega_2)} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}}_{=: A_4}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\rho_2 \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2 &\leq k_2 \rho_3 \delta^{-1} A_1 + k_2 \rho_3 \delta^{-1} \max_{x \in \bar{\omega}_3} |h'_3(x)| \|\theta\|_{L^2(\omega_2)} \|\psi\|_{L^2(\omega_2)} + \delta^{-1} A_2 \\
&\quad + \delta^{-1} k_1 \lambda^{-1} \|q\|_{L^2(\omega_2)} (\|\varphi_x\|_{L^2(\omega_2)} + \|\psi\|_{L^2(\omega_2)}) + \lambda^{-1} \|q\|_{L^2(\omega_2)} \|\theta_x\|_{L^2(\omega_2)} \\
&\quad + \delta^{-1} \rho_2 (\lambda^{-3} + \lambda^{-2}) \|q\|_{L^2(\omega_2)} \|F\|_{\mathcal{H}} + \delta^{-1} k_2 \lambda^{-1} \max_{x \in \bar{\omega}_3} |h'_3(x)| \|q\|_{L^2(\omega_2)} \|\psi_x\|_{L^2(\omega_2)} \\
&\quad + k_2 \max_{x \in \bar{\omega}_3} |h'_3(x)| \|\psi_x\|_{L^2(\omega_2)} \|\psi\|_{L^2(\omega_2)} + A_3 + k_1 \|\psi\|_{L^2(\omega_2)}^2 + A_4,
\end{aligned}$$

as a consequence of (4.15), Lemma 4.1, (4.22), (4.23), we deduce that

$$\rho_2 \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2 \leq k_2 \rho_3 \delta^{-1} A_1 + \delta^{-1} A_2 + A_3 + A_4 + o(\lambda^{-2}).$$

Using Young's inequalities, we obtain

$$\begin{cases} A_1 \leq \frac{1}{2c_1(\lambda)} \|\sqrt{h_3} \theta_x\|_{L^2(\omega_2)}^2 + \frac{c_1(\lambda)}{2} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}^2, \\ A_2 \leq \frac{\rho_2}{2c_2} \|\sqrt{h_3} q\|_{L^2(\omega_2)}^2 + \frac{c_2 \rho_2}{2} \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2, \\ A_3 \leq \frac{k_1^2}{2c_3(\lambda) \rho_2} \|\sqrt{h_3} \varphi_x\|_{L^2(\omega_2)}^2 + \frac{\rho_2 c_3(\lambda)}{2} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}^2, \\ A_4 \leq \frac{\delta^2}{2c_4(\lambda) \rho_2} \|\sqrt{h_3} \theta_x\|_{L^2(\omega_2)}^2 + \frac{\rho_2 c_4(\lambda)}{2} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}^2, \end{cases}$$

where $c_1(\lambda), c_2, c_3(\lambda), c_4(\lambda)$ are positive constants to be determined. Hence we can infer that

$$\begin{aligned}
\rho_2 \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2 &\leq \frac{k_2 \rho_3 \delta^{-1}}{2c_1(\lambda)} \|\sqrt{h_3} \theta_x\|_{L^2(\omega_2)}^2 + \frac{k_2 \rho_3 \delta^{-1} c_1(\lambda)}{2} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}^2 + \frac{\delta^{-1} \rho_2}{2c_2} \|\sqrt{h_3} q\|_{L^2(\omega_2)}^2 \\
&\quad + \frac{c_2 \rho_2 \delta^{-1}}{2} \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2 + \frac{k_1^2}{2c_3(\lambda) \rho_2} \|\sqrt{h_3} \varphi_x\|_{L^2(\omega_2)}^2 + \frac{\rho_2 c_3(\lambda)}{2} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}^2 \\
&\quad + \frac{\delta^2}{2c_4(\lambda) \rho_2} \|\sqrt{h_3} \theta_x\|_{L^2(\omega_2)}^2 + \frac{\rho_2 c_4(\lambda)}{2} \|\sqrt{h_3} \psi\|_{L^2(\omega_2)}^2 + o(\lambda^{-2}).
\end{aligned}$$

Now, taking

$$c_1(\lambda) = \frac{\lambda^2 \delta \rho_2}{4k_2 \rho_3}, \quad c_2 = \frac{\delta}{4}, \quad c_3(\lambda) = c_4(\lambda) = \frac{\lambda^2}{4},$$

we get

$$\frac{\rho_2}{2} \|\sqrt{h_3} \lambda \psi\|_{L^2(\omega_2)}^2 \leq \frac{2}{\rho_2 \lambda^2} \left[\frac{(k_2 \rho_3)^2}{\delta^2} + \delta^2 \right] \|\theta_x\|_{L^2(\omega_2)}^2 + \frac{2\rho_2}{\delta^2} \|q\|_{L^2(\omega_2)}^2 + \frac{2k_1^2}{\lambda^2 \rho_2} \|\varphi_x\|_{L^2(\omega_2)}^2 + o(\lambda^{-2}).$$

From the above estimation, Lemma 4.1, (4.15), and (4.16), we deduce that

$$\|\lambda\psi\|_{L^2(\omega_3)}^2 \leq \|\sqrt{h_3}\lambda\psi\|_{L^2(\omega_2)}^2 = O(\lambda^{-2}) \text{ and } \|\psi_{xx}\|_{L^2(\omega_3)}^2 = O(1).$$

Step 4: In this step, we finish the proof, with this intention, take $j = 4$ in (4.17), then using (4.21), (4.15) and Lemma 4.1, we obtain

$$\|\psi_x\|_{L^2(\omega_4)}^2 = o(\lambda^{-2}).$$

Finally, by taking $j = 5$ in (4.19) and by using the above estimation with (4.15) and Lemma 4.1, we get

$$\|\lambda\psi\|_{L^2(\omega_5)}^2 = o(\lambda^{-2}).$$

□

Lemma 4.3. The solution $\Phi = (\varphi, u, \psi, v, \theta, q, z)^\top \in D(\mathcal{A})$ of (4.5)-(4.11) satisfies the following estimations

$$\|\varphi_x\|_{L^2(\omega_6)}^2 = o(1) \text{ and } \|\lambda\varphi\|_{L^2(\omega_7)}^2 = o(1).$$

Proof. First, multiplying (4.13) by $h_6\overline{\varphi_x}$ and integrating over ω_5 , then using integration by parts, Cauchy-Schwarz's inequality, (4.15) and Lemmas 4.1-4.2, we deduce that

$$\begin{aligned} k_1 \int_{\omega_5} h_6 |\varphi_x|^2 dx &= \left| \int_{\omega_5} h_6 (\rho_2 \lambda^2 \psi + k_2 \psi_{xx} - k_1 \psi - \delta \theta_x + \rho_2 \lambda^{-2} f^4 + i \rho_2 \lambda^{-1} f^3) \overline{\varphi_x} dx \right| \\ &= \left| \int_{\omega_5} h_6 (\rho_2 \lambda^2 \psi - k_1 \psi - \delta \theta_x + \rho_2 \lambda^{-2} f^4 + i \rho_2 \lambda^{-1} f^3) \overline{\varphi_x} dx - k_2 \int_{\omega_5} (h'_6 \overline{\varphi_x} + h_6 \overline{\varphi_{xx}}) \psi_x dx \right| \\ &\leq \max(\rho_2, k_1, \delta) [\lambda^2 \|\psi\|_{L^2(\omega_5)} + \|\psi\|_{L^2(\omega_5)} + \|\theta_x\|_{L^2(\omega_5)} + (\lambda^{-2} + \lambda^{-1}) \|F\|_{\mathcal{H}}] \|\varphi_x\|_{L^2(\omega_5)} \\ &\quad + k_2 \max \left(\max_{x \in \overline{\omega_6}} |h'_6(x)|, 1 \right) (\|\varphi_x\|_{L^2(\omega_5)} + \|\varphi_{xx}\|_{L^2(\omega_5)}) \|\psi_x\|_{L^2(\omega_5)} \\ &= o(1). \end{aligned}$$

Thus, the first estimation in Lemma 4.3 is proved. Now, multiplying (4.12) by $h_7 \overline{\varphi}$ and integrating over ω_6 , then using integration by parts, Cauchy-Schwarz's inequality, (4.15), Lemmas 4.1-4.2, the first estimation of Lemma 4.3, we obtain

$$\begin{aligned} \rho_1 \int_{\omega_6} h_7 |\lambda\varphi|^2 dx &= \left| \int_{\omega_6} h_7 (-k_1 (\varphi_x + \psi)_x - \rho_1 \lambda^{-2} f^2 - i \rho_1 \lambda^{-1} f^1) \overline{\varphi} dx \right| \\ &= \left| \int_{\omega_6} h_7 (-\rho_1 \lambda^{-2} f^2 - i \rho_1 \lambda^{-1} f^1) \overline{\varphi} dx + k_1 \int_{\omega_6} (h'_7 \overline{\varphi} + h_7 \overline{\varphi_x}) (\varphi_x + \psi) dx \right| \\ &\leq \rho_1 (\lambda^{-2} + \lambda^{-1}) \|F\|_{\mathcal{H}} \|\varphi\|_{L^2(\omega_6)} \\ &\quad + k_1 \max \left(\max_{x \in \overline{\omega_7}} |h'_7(x)|, 1 \right) (\|\varphi\|_{L^2(\omega_6)} + \|\varphi_x\|_{L^2(\omega_6)}) (\|\varphi_x\|_{L^2(\omega_6)} + \|\psi\|_{L^2(\omega_6)}) \\ &= o(1). \end{aligned}$$

Finally, the above estimation finished the proof. □

Proof of Theorem 4.1: First, from Lemmas 4.2-4.3, we deduce that

$$(4.27) \quad \int_{\omega_7} (\rho_1 |\lambda\varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda\psi|^2 + k_2 |\psi_x|^2) dx = o(1).$$

Now, let us fix a function $\chi \in C^1([0, \ell])$ such that $\chi(0) = \chi(\ell) = 0$. Multiplying (4.12) and (4.13) by $-2\chi\overline{\varphi_x}$ and $-2\chi\overline{\psi_x}$, respectively and integrating over $(0, \ell)$, then taking the real part, yield

$$-\int_0^\ell \chi \left(\rho_1 |\lambda\varphi|^2 + k_1 |\varphi_x|^2 \right)_x dx - \Re \left[2k_1 \int_0^\ell \chi \psi_x \overline{\varphi_x} dx \right] = \Re \left[2 \int_0^\ell \chi (\rho_1 \lambda^{-2} f^2 + i \rho_1 \lambda^{-1} f^1) \overline{\varphi_x} dx \right]$$

and

$$\begin{aligned} & - \int_0^\ell \chi \left(\rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx + \Re \left[2k_1 \int_0^\ell \chi \varphi_x \overline{\psi_x} dx + 2k_1 \int_0^\ell \chi \psi \overline{\varphi_x} dx \right] \\ & + \Re \left[2\delta \int_0^{\ell_0} \chi \theta_x \overline{\psi_x} dx \right] = \Re \left[2 \int_0^\ell \chi \left(\rho_2 \lambda^{-2} f^4 + i \rho_2 \lambda^{-1} f^3 \right) \overline{\psi_x} dx \right]. \end{aligned}$$

Adding the above equations, then using integration by parts, we get

$$\begin{aligned} & \int_0^\ell \chi' \left(\rho_1 |\lambda \varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx = \Re \left[-2k_1 \int_0^\ell \chi \psi \overline{\varphi_x} dx - 2\delta \int_0^{\ell_0} \chi \theta_x \overline{\psi_x} dx \right. \\ & \left. + 2 \int_0^\ell \chi \left(\rho_1 \lambda^{-2} f^2 + i \rho_1 \lambda^{-1} f^1 \right) \overline{\varphi_x} dx + 2 \int_0^\ell \chi \left(\rho_2 \lambda^{-2} f^4 + i \rho_2 \lambda^{-1} f^3 \right) \overline{\psi_x} dx \right]. \end{aligned}$$

Using Cauchy-Schwarz's inequality in the above equation, we deduce that

$$\begin{aligned} \int_0^\ell \chi' \left(\rho_1 |\lambda \varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx & \leq \max_{x \in [0, \ell]} \chi(x) \left[2k_1 \|\psi\|_{L^2(0, \ell)} + 2\delta \|\theta_x\|_{L^2(0, \ell_0)} \right. \\ & \left. + 2(\rho_1 + \rho_2)(\lambda^{-2} + \lambda^{-1}) \|F\|_{\mathcal{H}} \right] \|U\|_{\mathcal{H}}. \end{aligned}$$

Now, by taking $\chi = x\chi_0 + (x - \ell)\chi_1$ with $\chi_0, \chi_1 \in C^1([0, \ell], [0, 1])$ such that

$$\begin{cases} \chi_0 = 1 & \text{in } [0, 7\varepsilon], \\ \chi_0 = 0 & \text{in } [\ell_0 - 7\varepsilon, \ell], \\ \chi_1 = 0 & \text{in } [0, 7\varepsilon], \\ \chi_1 = 1 & \text{in } [\ell_0 - 7\varepsilon, \ell], \end{cases}$$

we deduce that $\chi' = \chi_0' + \chi_1' + x\chi_0' + (x - \ell)\chi_1'$ and

$$\begin{aligned} & \int_{(0, \ell) \setminus \omega_7} \left(\rho_1 |\lambda \varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx \\ & \leq \left[(\ell_0 - 7\varepsilon) \max_{x \in \overline{\omega_7}} |\chi_0'(x)| + (\ell - 7\varepsilon) \max_{x \in \overline{\omega_7}} |\chi_1'(x)| \right] \int_{\omega_7} \left(\rho_1 |\lambda \varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx \\ & \quad + \max_{x \in [0, \ell]} \chi(x) \left[2k_1 \|\psi\|_{L^2(0, \ell)} + 2\delta \|\theta_x\|_{L^2(0, \ell_0)} + 2(\rho_1 + \rho_2)(\lambda^{-2} + \lambda^{-1}) \|F\|_{\mathcal{H}} \right] \|\Phi\|_{\mathcal{H}}, \end{aligned}$$

consequently, from Lemmas 4.1, (4.15), and (4.27), we get

$$(4.28) \quad \int_{(0, \ell) \setminus \omega_7} \left(\rho_1 |\lambda \varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx = o(1).$$

Now, from (4.27) and (4.28), we obtain

$$(4.29) \quad \int_0^\ell \left(\rho_1 |\lambda \varphi|^2 + k_1 |\varphi_x|^2 + \rho_2 |\lambda \psi|^2 + k_2 |\psi_x|^2 \right) dx = o(1).$$

Finally, from the above estimation and Lemma 4.1, we deduce that

$$\|\Phi\|_{\mathcal{H}} = o(1).$$

□

APPENDIX A. SOME NOTIONS AND STABILITY THEOREMS

In order to make this paper more self-contained, we recall in this short appendix some notions and stability results used in this work.

Definition A.1. Assume that A is the generator of C_0 -semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space H . The C_0 -semigroup $(e^{tA})_{t \geq 0}$ is said to be

(1) Strongly stable if

$$\lim_{t \rightarrow +\infty} \|e^{tA} x_0\|_H = 0, \quad \forall x_0 \in H.$$

(2) Exponentially (or uniformly) stable if there exists two positive constants M and ε such that

$$\|e^{tA}x_0\|_H \leq Me^{-\varepsilon t}\|x_0\|_H, \quad \forall t > 0, \forall x_0 \in H.$$

(3) Polynomially stable if there exists two positive constants C and α such that

$$\|e^{tA}x_0\|_H \leq Ct^{-\alpha}\|Ax_0\|_H, \quad \forall t > 0, \forall x_0 \in D(A).$$

□

To show the strong stability of a C_0 -semigroup we rely on the following result due to Arendt-Batty [6].

Theorem A.2. Assume that A is the generator of a C_0 -semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space H . If A has no pure imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ is countable, where $\sigma(A)$ denotes the spectrum of A , then the C_0 -semigroup $(e^{tA})_{t \geq 0}$ is strongly stable. □

Finally for the polynomial stability of a C_0 -semigroup of contractions we use the following result due to Borichev and Tomilov [10] (see also [9] and [14]).

Theorem A.3. Assume that A is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t \geq 0}$ on H . If $i\mathbb{R} \subset \rho(A)$, then for a fixed $\ell > 0$ the following conditions are equivalent

$$(A.1) \quad \limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^\ell} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty,$$

$$(A.2) \quad \|e^{tA}\Phi_0\|_H^2 \leq \frac{C}{t^\ell} \|\Phi_0\|_{D(A)}^2, \quad \forall t > 0, \Phi_0 \in D(A), \text{ for some } C > 0.$$

□

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