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# Regression models with repeated functional data

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## Abstract

Linear regression and classification models with repeated functional data are considered. For each statistical unit in the sample, a real-valued parameter is observed over time under different conditions. Two regression models based on fusion penalties are presented. The first one is a generalization of the variable fusion model based on the 1-nearest neighbor. The second one, called group fusion lasso, assumes some grouping structure of conditions and allows for homogeneity among the regression coefficient functions within groups. A finite sample numerical simulation and an application on EEG data are presented.

**Keywords.** linear models, regression, classification, variable fusion, fused lasso, multivariate functional data, repeated functional data, group lasso

## 1 Introduction

Let  $X$  be a functional random variable valued in some space of real-valued functions defined on the time interval  $[0, T]$ ,  $T > 0$ . Without loss of generality, we assume that this is the space of squared integrable functions  $L_2([0, T])$  (Ramsey and Silverman, 2005). The setting we consider in this paper assumes that  $X$  is observed under  $p$  different conditions  $\{\mathcal{C}_1, \dots, \mathcal{C}_p\}$ ,  $p \geq 1$ . For instance, these conditions can represent times or/and locations (regions) in some metrical space  $(\mathcal{S}, d)$ , typically  $(\mathbb{R}^s, \|\cdot\|_2)$ , for some natural integer  $s \geq 1$ . Thus, proximity between conditions can be considered through the distance  $d$ . As an example, EEG data (Ruiz et al., 2021) measure the brain activity through the electric field intensity over a time interval of  $T = 500ms$  and at different regions of the brain using  $p = 28$  electrodes/sensors evenly distributed (Figure 1).

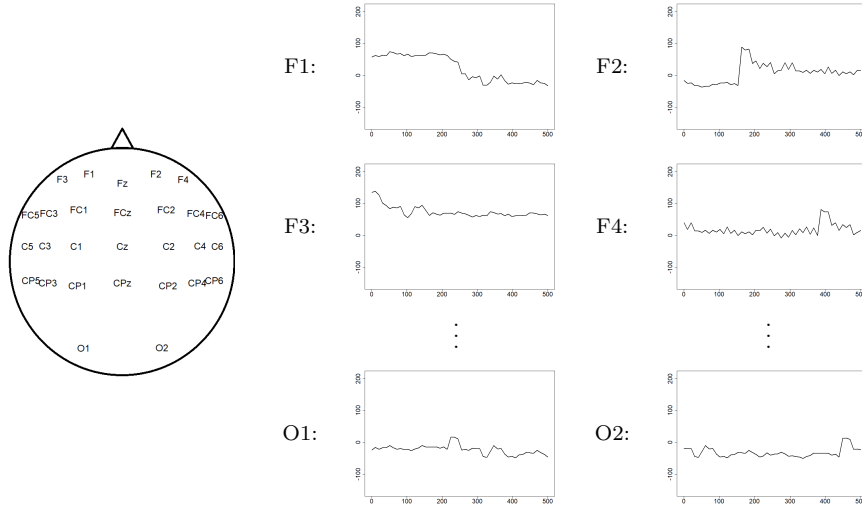


Figure 1: *FingerMovements* data. Each subject is represented by  $p = 28$  EEG recordings (right) corresponding to 28 sensors located on the scalp (left).

Let denote with  $X^{(j)}$  the observation of  $X$  under the condition  $\mathcal{C}_j$ ,  $j = 1, \dots, p$ , and with  $\mathbf{X}$ , the random vector

$$\mathbf{X} = (X^{(1)}, \dots, X^{(p)})^\top.$$

Known as repeated functional data, principal component analysis (PCA) methods are developed to deal with the dependency between the components  $X^{(j)}$ 's. In Chen and Müller (2012) the authors use a double PCA exploiting the metric structure of the space of conditions  $\{\mathcal{C}_1, \dots, \mathcal{C}_p\}$  belong. In Jacques and Preda (2014) that structure is ignored and  $\mathbf{X}$  is viewed as a  $p$ -dimensional functional random vector of which principal components are used for clustering and visualization.

In this paper we are interested in the estimation of the linear regression model with scalar or binary response  $Y$  and predictor  $\mathbf{X}$  taking into account the topology of the measurement conditions through the distance  $d$ . Thus, as a difference of the classical framework of linear regression with multivariate functional data, in our approach, we use the (*spatially*) distance between the conditions' measurement of components  $X^{(j)}$  for model estimation. To the best of our knowledge, there are no proposed methods in the multivariate functional data framework that explicitly take into account the information carried by the spatial feature of data. The existing contributions mainly consider  $\mathbf{X}$  as a  $p$ -dimensional functional vector and methodologies were designed to take into account the dependence between the dimensions of  $\mathbf{X}$  (see e.g Yi et al. (2022), Górecki et al. (2015), Beyaztas and Lin Shang (2022)).

We tackle the problem from the interpretation point of view in the sense that components  $X^{(j)}$ 's spatially close should provide similar information in the regression model. Our motivation application is on electroencephalography recordings (EEG) classification. Each subject is writing a text and the electric field intensity  $X$  is measured simultaneously at  $p = 28$  spatial positions (sensors) of the scalp during  $T = 500ms$ . Two groups of subjects are considered: the right-handed ( $Y = 0$ ) and left-handed ( $Y = 1$ ) writers. The question is to know and interpret in what measure the ability of a person to be left or right-handed is associated with some different activity of the brain. In statistical terms, to understand the

relationship between  $Y$  and  $\mathbf{X}$ .

The standard functional linear regression model assumes that there exists  $\beta^{(0)} \in \mathbb{R}$  and the regression coefficient function  $\beta = (\beta^{(1)}, \dots, \beta^{(p)})^\top \in \mathcal{H} = \{L_2([0, T])\}^p$  such that

$$\mathbb{E}(Y|\mathbf{X}) \approx \beta^{(0)} + \sum_{j=1}^p \int_0^T X^{(j)}(t)\beta^{(j)}(t)dt \quad (1)$$

If  $\{(X_i, Y_i)\}_{i=1:n}$  is an i.i.d. sample of size  $n$ ,  $n \geq 1$ , drawn from the same distribution as  $(\mathbf{X}, Y)$  and  $\{(x_i, y_i)\}_{i=1:n}$  is an observation of that sample, the estimation of the model (1) is based on the minimization of the sum squared errors (MSE), that is,

$$(\hat{\beta}_0, \hat{\beta}) = \arg \min_{(\psi_0, \psi) \in \mathbb{R} \times \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \left( y_i - \left( \psi_0 + \sum_{j=1}^p \langle x_i^{(j)}, \psi^{(j)} \rangle_{L_2} \right) \right)^2, \quad (2)$$

Because of the non-invertibility of the covariance operator, the direct estimation of the coefficient  $\beta$  under the minimization of the MSE criterion is an ill-posed problem (Cardot et al., 1999). The principal component regression (PCR) and the partial least squares (PLS) have been successful alternatives in this case (see e.g Escabias et al. (2005), Aguilera et al. (2006), Preda and Saporta (2002), Moindjié et al. (2022)). However, in these approaches, the estimate coefficient functions are difficult to interpret: why two components  $X^{(j)}$  and  $X^{(j')}$  which are close in the space of measurements, i.e. the distance  $d(\mathcal{C}_j, \mathcal{C}_{j'})$  is small, have associated very different regression coefficient functions  $\beta^{(j)}$  and  $\beta^{(j')}$ ? This situation occurs especially when  $p$  is large. In Godwin (2013), the authors propose to add the constraint  $\mathcal{P} = \sum_{j=1}^p \|\psi^{(j)}\|_{L_2}$  in the regression model. In this case,  $\mathcal{P}$  is a generalization to functional multivariate variables of the penalty group lasso (GL), originally introduced in the multivariate data case (Meier et al. (2008), Yuan and Lin (2006)).

This penalty leads to achieving a trade-off between a minimum number of contributing components  $X^{(j)}$  and model fit. Our hypothesis is that closeness between components  $X^{(j)}$ , in the sense of the distance  $d$  between the corresponding conditions  $\mathcal{C}_j$ , can help for a better interpretation of  $\beta$ . For this purpose, the *fusion penalty* was introduced in the finite multivariate setting in Land and Friedman (1997).

Let  $v$  be a surjective function,  $v : \{\mathcal{C}_1, \dots, \mathcal{C}_p\} \mapsto \{1, \dots, K\}$ ,  $K \leq p$ , and define the fusion penalty (in the functional framework) as

$$\mathcal{P}(\beta) = \sum_{k=1}^K \sqrt{\sum_{j \in \mathcal{I}_k} \|\beta^{(j)} - \bar{\beta}_{\mathcal{I}_k}\|_{L_2}^2},$$

where for each  $k = 1 : K$ ,  $\mathcal{I}_k = \{j : v(\mathcal{C}_j) = k\}$  and  $\bar{\beta}_{\mathcal{I}_k}(t) = \frac{1}{|\mathcal{I}_k|} \sum_{j \in \mathcal{I}_k} \beta^{(j)}(t)$  for  $t \in [0, T]$  and  $|\mathcal{I}_k|$  denotes the cardinal of  $\mathcal{I}_k$  for  $k = 1, \dots, p$ . Then, the proximity between conditions  $\mathcal{C}_1, \dots, \mathcal{C}_p$  can be integrated through the function  $v$  and the distance  $d : v^{-1}(k)$  represents all the conditions closest to  $\mathcal{C}_k$ . Obviously, this penalty favors close dimensions of  $\mathbf{X}$  to have similar corresponding dimensions of the regression function (the  $\beta^{(j)}$ 's functions).

To our knowledge, this penalty has not been explored in the case of regression with repeated functional variables (nor multivariate functional variables). In the classical multivariate setting, the models that have this penalty are known as the variable fusion model

(FU) (Land and Friedman, 1997) and, when a lasso penalty is added, as the fused lasso model (FL) (Tibshirani et al., 2005). More recently, the group fusion model (GFL) introduced in Bleakley and Vert (2011) extended this penalty from unique conditions to groups of conditions. However, in these cases, the function  $v$  has been oriented as a way to integrate consecutive conditions (or dimensions), i.e.  $v$  is defined as  $v(\mathcal{C}_j) = j + 1$ , for  $1 \leq j \leq p - 1$  and  $v(\mathcal{C}_p) = p$ . Even if this case can be well-suited for the setting  $\mathcal{S} \subset \mathbb{R}$  ( $s = 1$ ), it limits the number of applications as for multivariate locations, i.e.  $s \geq 2$ . In this case, the function  $v$  can be defined through the distance  $d$  and can be used naturally to extend the variable fusion model (Land and Friedman, 1997) to such a spatial structure of components of  $\mathbf{X}$ .

In this paper we introduce two new fusion-like penalties for the functional linear regression estimation under mean squares error criterion. The first one is an extension of the variable fusion model (FU) where the 1-nearest neighbor graph (1-NN) is used to define  $v$ . We show that the estimation of FU in this setting is equivalent to a group lasso model such as those studied in Godwin (2013). The second model we introduce takes into account a more general grouping structure of conditions based on neighborhood relationships. We call it the group fusion lasso (GFUL) model. It allows also to test the equality among the dimensions of the coefficient regression function belonging to the same cluster of conditions.

The paper is organized as follows. Section 2 presents the proposed models and their estimation strategies using basis function expansion techniques. A comparison study of the two models and the group lasso model is performed using simulated data in Section 3. A real data application from the EEG classification task is presented in Section 4. The paper ends with a discussion in Section 5.

## 2 Two new fusion models for linear regression with multivariate functional data

Without loss of generality let assume that  $\mathbf{X}$  and  $Y$  are zero mean random variables and  $\{(x_i, y_i)\}_{i=1, \dots, n}$  is the realization of a random sample of size  $n$ ,  $n \geq 1$ , drawn from the joint distribution of  $(\mathbf{X}, Y)$ . The intercept  $\beta^{(0)}$  in (1) is zero and the mean square criterion (2) becomes :

$$\hat{\beta} = \arg \min_{\psi \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \left( y_i - \sum_{j=1}^p \langle x_i^{(j)}, \psi^{(j)} \rangle_{L_2} \right)^2.$$

Remind that for each  $i = 1, \dots, n$ ,  $y_i \in \mathbb{R}$  and  $x_i$  is a multivariate function defined on  $[0, T]$ ,

$$x_i(t) = \left( x_i^{(1)}(t), \dots, x_i^{(j)}(t), \dots, x_i^{(p)}(t) \right)^\top, \quad t \in [0, T],$$

where each dimension  $x_i^{(j)}$  is observed under the condition  $\mathcal{C}_j$ ,  $j = 1, \dots, p$ .

Let us now introduce our first model of penalty based on distance among conditions.

## 2.1 Fusion model based on the neighbor relationship among conditions

The basic idea is that if two conditions  $\mathcal{C}_j$  and  $\mathcal{C}_{j'}$  are close in the space  $\mathcal{S}$  (with respect to distance  $d$ ), then the contributions brought by the components  $X^{(j)}$  and  $X^{(j')}$  in the linear model (1), i.e.,  $\beta^{(j)}$  and  $\beta^{(j')}$ , might be comparable. Allowing for identical coefficients  $\beta^{(j)}$  associated with close conditions, the variable fusion methodology is a candidate to obtain a parsimonious model and to compete with existing linear model approaches (Land and Friedman (1997), Tibshirani et al. (2005), Bleakley and Vert (2011)).

When  $\mathcal{C}_j$  are conditions in  $\mathbb{R}^s$  and  $s \geq 2$ , the distance  $d$  defines a neighbor relationship between conditions and thus it can be used to estimate the regression coefficient functions accordingly.

Following the ideas in Land and Friedman (1997), the *1-nearest neighbor (1-NN) variable fusion model* can be formulated as the following optimization problem:

$$\hat{\beta}_\lambda = \arg \min_{\beta \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n \left( y_i - \sum_{j=1}^p \langle x_i^{(j)}, \beta^{(j)} \rangle_{L_2} \right)^2 + \lambda \sum_{j=1}^p \|\beta^{(j)} - \beta^{(v(\mathcal{C}_j))}\|_{L_2} \quad (3)$$

where  $v : \{\mathcal{C}_1, \dots, \mathcal{C}_p\} \rightarrow \{1, \dots, p\}$  denotes the neighbor function

$$v(\mathcal{C}_j) = \arg \min_{i \in \{1, \dots, p\} / \{j\}} d(\mathcal{C}_i, \mathcal{C}_j), \quad j = 1, \dots, p. \quad (4)$$

The function  $v$  helps to integrate into the estimation process of  $\beta$  the information brought by the conditions (locations, spatial distributions). Notice that if the set of  $\arg \min$  in (4) is not unique, then we choose randomly or experimentally an element of this set.

For ease of notation, let denote with  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  the inner product between  $f, g \in \mathcal{H}$  on  $\mathcal{H}$  given by :

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{i=1}^p \langle f^{(i)}, g^{(i)} \rangle_{L_2}.$$

Then, the penalty function in (3) can then be rewritten as

$$\sum_{j=1}^p \|\beta^{(j)} - \beta^{(v(\mathcal{C}_j))}\|_{L_2} = \|\mathbf{L}\beta\|_{L_2,1},$$

where  $\mathbf{L} = \mathbf{W} - \mathbb{I}_{p \times p}$  and  $\mathbf{W} = (w_{i,j})_{1 \leq i \leq p, 1 \leq j \leq p}$  the adjacency matrix with elements

$$w_{i,j} = \begin{cases} 1 & \text{if } v(\mathcal{C}_i) = j \\ 0 & \text{otherwise,} \end{cases}$$

$\mathbb{I}_{p \times p}$  is the  $p \times p$  unitary matrix and  $\|\cdot\|_{L_2,1}$  is defined as :

$$\|f\|_{L_2,1} = \sum_{i=1}^p \|f^{(i)}\|_{L_2}, \quad f \in \mathcal{H}.$$

For illustrative purposes, consider the toy example shown in Figure 2, with  $\mathcal{S} \subset \mathbb{R}^2$  and  $p = 8$ . It represents  $p = 8$  points corresponding to the conditions  $\mathcal{C}_j, j = 1, \dots, p$ . The neighborhood

relationship among the conditions is given by the following  $v$  function:  $v(1) = 8$ ,  $v(2) = 5$ ,  $v(3) = 4$ ,  $v(4) =$ ,

Remark that the rank of the matrix  $\mathbf{L}$  is generally lower than  $p$ , since symmetric relationships are possible (contrary to consecutive conditions case, see e.g Land and Friedman (1997)). For example, Figure 2 shows that 1 is the neighbor of 8 and 8 is the neighbor of 1, the same for the couple (5, 4).

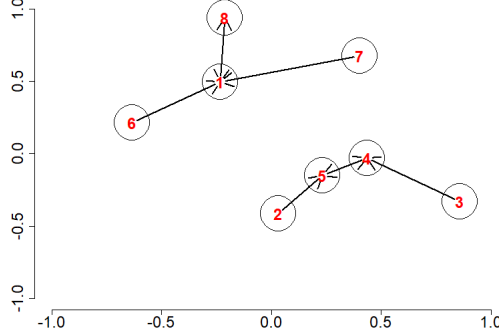


Figure 2: One nearest neighbor graph:  $a \rightarrow b$  means  $b$  is the neighbor of  $a$ .

**Lemma 1.** *If  $r$  is the rank of the matrix  $\mathbf{L}$ , there exists a  $r \times p$  full rank matrix  $\mathbf{L}_0$  such as*

$$\|\mathbf{L}f\|_{L_{2,1}} = \|\mathbf{L}_0f\|_{L_{2,1}}, \quad f \in \mathcal{H}. \quad (5)$$

Thus,  $\mathbf{L}_0$  avoids redundancy. In our case, its construction consists of finding the two rows where symmetric relations occur and replacing one of the rows with the other times 2. The rank of the matrix  $\mathbf{L}$  coincides with the number of vertices of the undirected version of the 1-NN graph.

To fix ideas, in our example (Figure 2), the following matrix has full rank and holds the equation (5)

$$\begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Hence, lemma 1 implies that there's an alternative reformulation of (3). The next proposition shows that the fusion model, similar to the classical fusion model(Land and Friedman, 1997), can be put in relation to a lasso alike problem.

**Proposition 1.** *The solution of (3) holds  $\hat{\beta}_\lambda = \mathbf{D}^{-1}\hat{\psi}_\lambda$  where*

$$\hat{\psi}_\lambda = \arg \min_{f \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i - \langle (\mathbf{D}^{-1})^\top x_i, f \rangle_{\mathcal{H}})^2 + \lambda \sum_{j=1}^r \|f^{(j)}\|_{L_2}, \quad (6)$$

$\mathbf{D} = \begin{pmatrix} \mathbf{L}_0 \\ \mathbf{T} \end{pmatrix}$ ,  $\mathbf{L}_0$  is the  $r \times p$  reduced matrix of  $\mathbf{L}$  and  $\mathbf{T}$  is the  $(p - r) \times p$  matrix such as its rows are orthogonal to  $\mathbf{L}_0$ , i.e.  $\mathbf{L}_0 \mathbf{T}^\top = \mathbf{0}_p$  and  $\mathbf{0}_p$  is the  $p$  column vector of zeros.

**Remark.** The non-penalized part from the orthogonal matrix might have a real influence in equation (6). Indeed, it might lead to putting the maximum weight on the non-constrained part of  $\beta$  and estimating overfitted coefficients. To fix this issue, one can add in (3) the orthogonal part in the optimization problem :

$$\|\mathbf{L}\beta\|_{L_{2,1}} + \sqrt{p-r}\|\mathbf{T}\beta\|_{L_{2,2}},$$

where  $\|\cdot\|_{L_{2,2}}$  is the Frobenius norm of  $\mathcal{H}$  :

$$\|f\|_{L_{2,2}} = \sqrt{\sum_{i=1}^p \|f^{(i)}\|_{L_2}^2}$$

for  $f \in \mathcal{H}$ .

This last penalty can be easily integrated into the optimization problem (6).

## 2.2 The group fusion lasso

The previous model has some limitations, as illustrated in Figure 3. In this case, the conditions are around centers  $c_1, c_2, c_3$ . For this configuration, more than one neighbor must be considered. Indeed, the following sets (1, 6, 8) (3, 4, 7), (2, 5) have symmetric neighborhood relations (i.e. 8 has (1,6) as neighbours, 6 has (1,8) as neighbours, etc.). Rather than examining the interactions of the conditions individually, we propose in this section to test the resulting group relations.

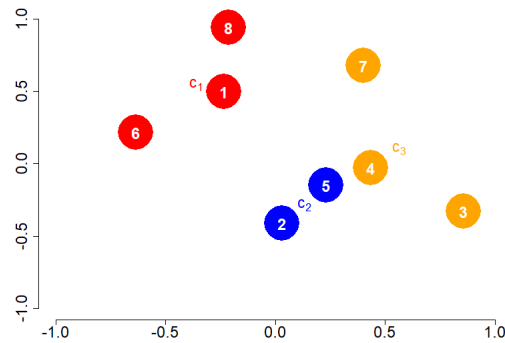


Figure 3: Conditions around centers

For doing that, an alternative definition of the function  $v$  is presented. We assumed that conditions are around  $K < p$  known centers  $c_k \in \mathbb{R}^s$ ,  $k = 1, \dots, K$ , and

$$v(\mathcal{C}_j) = \arg \min_{k=1, \dots, K} d(\mathcal{C}_j, c_k)$$



for  $j = 1, \dots, K$ . We recall the definition of the sets of index

$$\mathcal{I}_k = \{j \in \{1, \dots, p\}, v(\mathcal{C}_j) = k\}, \quad k = 1, \dots, K.$$

The group fusion model (GFU) is presented as

$$\begin{aligned} \hat{\beta}_\lambda &= \arg \min_{\beta \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i - \langle x_i, \beta \rangle_{\mathcal{H}})^2 + \lambda \sum_{k=1}^K \mathcal{P}_k(\beta) \\ &\text{with } \mathcal{P}_k(\beta) = \sqrt{|\mathcal{I}_k|} \sqrt{\sum_{i \in \mathcal{I}_k} \|\beta^{(i)} - \bar{\beta}_{\mathcal{I}_k}\|_{L_2}^2} \end{aligned} \quad (7)$$

where  $\bar{\beta}_{\mathcal{I}_k}(t) = \frac{1}{|\mathcal{I}_k|} \sum_{j \in \mathcal{I}_k} \beta^{(j)}(t)$ ,  $t \in [0, T]$ .

As already noticed in the last remark, the non-constrained part of  $\beta$  with such penalty can yield to overfitting. To overcome this difficulty in the model (7) we can add a penalty that favors sparsity. The resulting model, called group fusion lasso (GFUL), is then defined as

$$\begin{aligned} \hat{\beta}_{\lambda, \alpha} &= \arg \min_{\beta \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i - \langle x_i, \beta \rangle_{\mathcal{H}})^2 + \lambda \sum_{k=1}^K \mathcal{P}_{\alpha, k}(\beta) \\ &\text{with } \mathcal{P}_{\alpha, k}(\beta) = (1 - \alpha) \mathcal{P}_k(\beta) + \alpha \|\bar{\beta}_{\mathcal{I}_k}\|_{L_2}. \end{aligned} \quad (8)$$

where  $0 \leq \alpha \leq 1$ .

The purpose of GFUL is related to the group lasso model. Indeed, in the group lasso (Meier et al. (2008), Yuan and Lin (2006)), the objective is to discover groups with zero coefficients. GFUL aims to discover groups with the same coefficients function, which is a more general statement.

In practice, it is worth noting that the groups  $\mathcal{I}_k$ ,  $k = 1, \dots, K$  can be constructed on another type of consideration than spatial ones. They just have to be partition sets of  $\{1, \dots, p\}$ , similar to the group lasso case.

**Remark.**

- *The penalty function is composed of two parts, the first one:  $\mathcal{P}_k(\beta)$  is of fusion type. Indeed, the norm is zero only if  $\beta^{(j)} = \beta^{(k)}$ ,  $\forall j, k \in \mathcal{I}_k$ . The second one:  $\|\bar{\beta}_{\mathcal{I}_j}\|_{L_2}$  is a group-lasso-like penalty.*
- *In the case  $0 < \alpha < 1$ , GFUL similarly to the group lasso can cancel the contribution of a group. Indeed, if for  $k \in \{1, \dots, K\}$   $\mathcal{P}_{\alpha, k} = 0 \iff$  all components of  $\beta_{\mathcal{I}_k}$  are null.*
- *In the case  $\alpha = 0$ , and  $|\mathcal{I}_k| = 2$  for  $k = 1, \dots, K$ , the GFUL method is equivalent to the fused lasso for unconnected pairs of points.*

In GFUL, more than neighborhood, the central notion is membership. This notion can be encoded in the matrix  $\mathbf{M} = (m_{k,j})_{1 \leq k \leq K, 1 \leq j \leq p}$  which recapitulates  $\{\mathcal{I}_k\}_{k=1}^K$  where

$$m_{k,j} = \begin{cases} 1 & \text{if } j \in \mathcal{I}_k \\ 0 & \text{otherwise.} \end{cases}$$

To fix ideas, in the toy example (Figure 3), the membership matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Nonetheless, in a general case, up to a permutation,  $\mathbf{M}$  can be written as

$$\mathbf{M} = \begin{pmatrix} \mathbf{I}_{|\mathcal{I}_1|}^\top & \mathbf{0}_{|\mathcal{I}_2|}^\top & \cdots & 0 \\ \mathbf{0}_{|\mathcal{I}_1|}^\top & \mathbf{I}_{|\mathcal{I}_2|}^\top & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{|\mathcal{I}_1|}^\top & \mathbf{0}_{|\mathcal{I}_2|}^\top & \cdots & \mathbf{I}_{|\mathcal{I}_K|}^\top \end{pmatrix},$$

where  $\mathbf{I}_{|\mathcal{I}_K|}$  is the  $|\mathcal{I}_K|$  column vector of ones.

Let denote by  $\bar{\mathbf{M}}$  the standardized version of  $\mathbf{M}$ , i.e.  $\bar{\mathbf{M}} = \text{diag}(1/|\mathcal{I}_1|, 1/|\mathcal{I}_2|, \dots, 1/|\mathcal{I}_K|)\mathbf{M}$ . Similarly to FU, the idea of the following lemma is to show that the penalty  $\mathcal{P}$  can be written more conveniently.

**Lemma 2.** *Let  $f \in \mathcal{H}$ ,  $0 < \alpha < 1$ , and  $\mathbf{R}_k$  is the  $(|\mathcal{I}_k| - 1) \times |\mathcal{I}_k|$  matrix from the reduced rank QR decomposition of  $\mathbf{P}_k = \mathbb{I}_{|\mathcal{I}_k| \times |\mathcal{I}_k|} - \frac{1}{|\mathcal{I}_k|} \mathbf{I}_{|\mathcal{I}_k| \times |\mathcal{I}_k|}$ , where  $\mathbf{I}_{|\mathcal{I}_k| \times |\mathcal{I}_k|}$  is the  $|\mathcal{I}_k| \times |\mathcal{I}_k|$  matrix of ones, for  $k = 1, \dots, K$ . The penalty function of GFUL can be written as*

$$\sum_{k=1}^K \mathcal{P}_{\alpha,k}(f) = \sum_{k=1}^{2K} \sqrt{\sum_{i \in \tilde{\mathcal{I}}_k} \|(\mathbf{G}_\alpha f)^{(i)}\|_{L_2}^2},$$

where  $\mathbf{G}_\alpha^\top = (\mathbf{R}^\top \quad \bar{\mathbf{M}}^\top) ((1 - \alpha)\mathbf{I}_K, \alpha\mathbf{I}_K)$  is a non-singular matrix,  $\mathbf{R}$  is the block diagonal matrix composed of the following elements  $\sqrt{|\mathcal{I}_1|}\mathbf{R}_1, \dots, \sqrt{|\mathcal{I}_K|}\mathbf{R}_K$ , and  $\{\tilde{\mathcal{I}}_k\}_{k=1}^{2K}$  are  $2K$  synthetic groups. These new groups are sorted and have the following cardinals:  $p_k - 1$  for  $k = 1, \dots, K$ , and 1 for  $k = K + 1, \dots, 2K$ .

Using the non-singularity of  $\mathbf{G}$ , the following proposition provides a way to estimate GFUL using a simpler model.

**Proposition 2.** *If  $\alpha \notin \{0, 1\}$ , the solution of (8), holds  $\beta_{\alpha,\lambda} = \mathbf{G}_\alpha^{-1} \hat{\psi}_\lambda$ , where*

$$\hat{\psi}_\lambda = \arg \min_{f \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i - \langle (\mathbf{G}_\alpha^{-1})^\top x_i, f \rangle_{\mathcal{H}})^2 + \lambda \sum_{k=1}^{2K} \sqrt{\sum_{i \in \tilde{\mathcal{I}}_k} \|f^{(j)}\|_{L_2}^2}, \quad (9)$$

**Remark.** *The case of  $\alpha = 1$  or  $\alpha = 0$  can be resolved using the same technique as in Proposition 1. Indeed, the non null part of  $\mathbf{G}_\alpha$  is full rank for  $\alpha \in \{0, 1\}$*

The direct estimation of  $\beta$ , under least squares regression, is generally an ill-posed inverse problem (Cardot et al. (1999), Aguilera et al. (2006)). The basis expansion technique, a well-known dimension reduction technique as an alternative to solve this problem, is presented in the next section.

## 2.3 Computational aspect: Basis expansion

The basis expansion technique assumes that there is a set of independent functions  $\{\phi_k\}_{k=1}^M$ , such as each  $x_i$  can be written as

$$\forall t \in \mathcal{T} \quad x_i(t) = \Phi(t)a_i, \quad i = 1, \dots, n$$

with  $a_i \in \mathbb{R}^{pM}$  and

$$\Phi = \begin{pmatrix} \phi_1 & \dots & \phi_1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \phi_1 & \dots & \phi_M & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \phi_1 & \dots & \phi_M \end{pmatrix}.$$

Note that the set of functions is not always the same for all dimensions of  $X$  as presented here. However, this seems realistic in the context that  $X^{(1)}, \dots, X^{(p)}$  are in the same unit. Under this hypothesis, the coefficient function  $\beta$  can also be written as

$$\beta(t) = \Phi(t)b, \quad \text{where } b \in \mathbb{R}^{pM} \quad t \in \mathcal{T}.$$

Hence, an equivalent matrix notation is

$$\beta(t) = \mathbf{B}\phi(t) \quad \text{and} \quad x_i(t) = \mathbf{A}_i\phi(t) \quad (10)$$

where  $\mathbf{B}$ ,  $\mathbf{A}_i$  are  $p \times M$  matrix,

- $\phi = (\phi_1 \quad \phi_2 \quad \dots \quad \phi_M)^\top$  is the vector of functions,
- $a_i = \text{vec}(\mathbf{A}_i^\top)$ ,  $b = \text{vec}(\mathbf{B}^\top)$  with  $\text{vec}(\cdot)$  denotes the vectorization operator. More precisely,  $\mathbf{A}_i = \left( a_i^{(1)} \quad a_i^{(2)} \quad \dots \quad a_i^{(p)} \right)^\top$  with  $a_i^{(j)}$  verifies  $x_i^{(j)} = (a_i^{(j)})^\top \phi$ ,  $j = 1, \dots, p$ .

**Proposition 3.** *The following equations hold as consequences of the vectorization operator*

1.  $\|\beta\|_{L_{2,1}} = \|\mathbf{B}\mathbf{F}^{1/2}\|_{2,1}$ , where  $\|\cdot\|_{2,1}$  is the 2,1 matrix norm.

$$\mathbf{F} = \{\langle \phi_i, \phi_j \rangle\}_{i,j} \quad \text{and} \quad \mathbf{F} = (\mathbf{F}^{1/2})^\top \mathbf{F}^{1/2}.$$

2. Let  $\mathbf{Z}$  a  $(p-k) \times p$  matrix, such as  $\beta_0(t) = \mathbf{Z}\beta(t)$  then  $\beta_0(t) = b_0\Phi(t)$ , with  $b_0 = (\mathbf{Z} \otimes \mathbb{I}_{M \times M})b$ ,  $k = 0, \dots, p-1$ .

$\otimes$  denotes the Kronecker product.

The first point shows that the norm of  $\beta$  depends on the vector  $b$  and on the chosen basis via the matrix  $\mathbf{F}$ . In the penalty function, we choose to ignore the  $\mathbf{F}$  matrix to ease the model estimation, similarly to Godwin (2013) for the group lasso case.

Hence, FU and GFUL are related to the following latent group lasso (Propositions 1 and 2):

$$\hat{b}_\lambda^{\text{Proj}} = \arg \min_{h \in \mathbb{R}^{pM}} \frac{1}{2} \sum_{i=1}^n (y_i - a_i^\top \text{Proj}^{-1} \mathbf{F}h)^2 + \lambda \sum_{k=1}^{\tilde{K}} \|h_{\tilde{\mathcal{I}}_k}\|_2 \quad (11)$$

with  $\{\tilde{\mathcal{I}}_k\}_{k=1}^K$  is the set of  $\tilde{K}$  considered groups. The solution coefficient function is given by  $\hat{\beta}_\lambda = \text{Proj} \hat{b}_\lambda^{\text{Proj}} \Phi$ .

Using the notations of sections 2.1 and 2.2, the corresponding  $\tilde{\mathcal{I}}$  and Proj matrix are given below for each method.

- **FU**: the matrix  $\text{Proj} = \mathbf{D} \otimes \mathbb{I}_{M \times M}$  and the considered groups of variables are given by  $\tilde{\mathcal{I}}_l = \{(l-1)M + 1, \dots, lM\}$ ,  $l = 1, \dots, r$ .
- **GFUL**:  $0 < \alpha < 1$ , the matrix  $\text{Proj}$  is given by  $\mathbf{G}_\alpha \mathbf{S} \otimes \mathbb{I}_{M \times M}$ , and the considered groups are

$$\begin{aligned} \tilde{\mathcal{I}}_l &= \{1 + M \sum_{k=0}^{l-1} p_k, \dots, p_l M\} & l = 1, \dots, K \\ \tilde{\mathcal{I}}_l &= \{p' + (l-1)M + 1, \dots, p' + lM\} & l = K + 1, \dots, 2K, \end{aligned}$$

where  $p_0 = 0$ ,  $p' = (p - K)M$ , and  $\mathbf{S}$  is the  $p \times p$  matrix such it permits sorting  $\mathbf{X}$  as

$$\mathbf{S}\mathbf{X}(t) = \begin{pmatrix} X_{\mathcal{I}_1} \\ X_{\mathcal{I}_2} \\ \dots \\ X_{\mathcal{I}_K} \end{pmatrix},$$

with  $X_{\mathcal{I}_k}$  is the  $|\mathcal{I}_k| \geq 2$  set of index which belong to group  $k = 1, \dots, K$ .

The problem (11) is studied in Godwin (2013). The authors show that principal component analysis helps to avoid multicollinearity and high-dimension issues (Aguilera et al. (2006), Escabias et al. (2005)). We use the same procedure to estimate the vector in (11).

## 2.4 Logistic regression model

The case of binary response can be naturally taken into account in our proposed methodologies. The previous relationships still hold in the logistic regression case.

In the following, all models are estimated by the maximum likelihood estimation (thanks to Lukas and Meier (2020) R package).

## 3 Simulations

We propose a small simulation study to compare the performance of the proposed methods with competitor lasso methods. In this simulation among the  $p$  conditions, four groups of conditions of  $\kappa$  elements ( $p = 4\kappa$ ) are considered.

More precisely,  $\mathcal{C}_j$   $j = 1, \dots, p$  are randomly distributed around the centers  $c_1, c_2, c_3, c_4$ .

- $\mathcal{C}_j \sim \mathcal{N}(c_1, \eta)$ ,  $1 \leq j \leq \kappa$
- $\mathcal{C}_j \sim \mathcal{N}(c_2, \eta)$ ,  $\kappa + 1 \leq j \leq 2\kappa$

- $\mathcal{C}_j \sim \mathcal{N}(c_3, \eta)$ ,  $2\kappa + 1 \leq j \leq 3\kappa$
- $\mathcal{C}_j \sim \mathcal{N}(c_4, \eta)$ ,  $3\kappa + 1 \leq j \leq p$

where  $\eta^2 = 0.05\mathbb{I}_{2 \times 2}$  and  $c_1 = (-2, 2)^\top$ ,  $c_2 = (0, 2)^\top$ ,  $c_3 = (0, -2)^\top$ ,  $c_4 = (2, 2)^\top$ . And the membership matrix  $\mathbf{M} = (m_{kj})_{k,j}$  is defined by

$$m_{k,j} = \begin{cases} 1 & \text{if } k = \arg \min_{s=1,\dots,4} d(\mathcal{C}_j, c_s) \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the groups are sorted as the first group is composed of the conditions from 1 to  $\kappa$ , the second group the ones from  $\kappa + 1$  to  $2\kappa$ , and so on.

The coefficient function  $\beta$  is generated by

$$\beta^{(j)}(t) = \sum_{k=1}^4 m_{k,j} h_k(t), \quad j = 1, \dots, p$$

where  $t \in [0, 1]$ ,  $h_1 = 0$ ,  $h_2 = 10(\Delta_1 + \Delta_2 + \Delta_3)$ ,  $h_3 \stackrel{\mathcal{L}}{=} v_1(\sum_{i=1}^9 \Delta_i)$ ,  $h_4 \stackrel{\mathcal{L}}{=} v_2(\sum_{i=1}^9 \Delta_i)$  and  $\stackrel{\mathcal{L}}{=}$  means equal in distribution.

- The functions  $\Delta_1, \dots, \Delta_9$  denote the triangle functions defined by:  $\Delta_s(t) = (1 - |10t - s|)_+$ , where  $(\cdot)_+$  is the positive function.
- The variables  $v_1, v_2$  are drawn using the discrete uniform distribution:  $v_1 \sim \mathcal{U}(-1, 4)$ , and  $v_2 \sim \mathcal{U}(-1, 4)$ .

Hence, only the two first groups share the same coefficient functions, respectively the (null) function  $h_1$  and the function  $h_2$ .

Each  $X^{(j)}$  is generated using a normal distribution and the triangle functions

$$X^{(j)}(t) = \sum_{s=1}^9 a_s \Delta_s(t),$$

where  $t \in [0, 1]$ ,  $j = 1, \dots, p$ , and  $a_s \sim \mathcal{N}(0, 1)$ ,  $s = 1, \dots, 9$ .

Finally, the response variable  $Y$  is given by

$$Y = \langle X, \beta \rangle_{\mathcal{H}} + \epsilon,$$

where  $\epsilon \sim \mathcal{N}(0, 1)$ .

Two scenarios are studied, according to  $\kappa$  (the size of groups): **(S1)**:  $\kappa = 2$ , **(S2)**:  $\kappa = 20$ . In each, 100 simulations are done, and  $n = 300$  copies of  $X$  (with 100 sampling points) and  $Y$  are generated. For all repeated variables, 20 B-splines are used as basis functions to reconstruct the functional version of the predictors. To assess the models' performances, the data are split into training and test sets with the following rate 80% and 20%.

The variable fusion model(FU) is computed thanks to 1-NN based on the conditions, and the group fusion lasso (GFUL) is computed using the membership matrix. For comparison,

two group lasso methods are also computed, using a similar methodology as in Godwin (2013)). The first one denoted by LA uses each variable as a group, as in the classical lasso it tests the variable contributions. The second one denoted by GL uses the same group definitions as in GFUL.

In addition, to these two models, we compute the regression model using only the group’s mean covariates (G0). This model always gives the same coefficient by groups, it can be seen as a naive competitor to GFUL.

Except for the model G0 which doesn’t have a penalty term, the hyperparameters  $(\alpha, \lambda)$  are tuned by 10-fold cross-validation:  $\lambda$  is chosen in the grid composed  $\{\lambda_{\max}, (0.96)\lambda_{\max}, (0.96)^2\lambda_{\max}, \dots, (0.96)^{148}\lambda_{\max}, 0\}$ ,  $\alpha$  in  $\{0, 0.1, \dots, 1\}$ , where  $\lambda_{\max}$  is the minimum value such all constraints verify.

The models are assessed by the mean square error (MSE) on the test set and their ability to recover true equality among coefficients by sensibility (Sens) and specificity (Spec) metrics.

### 3.1 Results

The summary of the obtained metrics is presented in Table 1. Figure 4 depicts the norms  $\|\hat{\beta}^{(j)}\|_{L_2}$   $j = 1, \dots, p$  of the estimated coefficients ( $\hat{\beta}$ ) obtained by each model (except for the benchmark model G0).

In the first scenario ( $\kappa = 2$ ), all models give close results. However, the variable fusion model (FU) and the group fusion lasso model (GFUL) reach the highest scores of sensibility and specificity. This demonstrates their ability to find true equalities among coefficients in this setting. In the group lasso models (LA and GL), the scores obtained are almost the same. Their sensibility metrics are around 0.5: they found half of the true equalities. Figure 4 shows that they well-estimated the  $\kappa = 2$  null coefficients (first group).

In the second scenario, Table 1 clearly shows GFUL as the best model for all metrics performance. High specificity and low sensitivity indicate that most estimated coefficients by GFUL competitors are all distinct, which is confirmed in Figure 4.

		<b>Equality</b>		
		<b>Test-error(MSE)</b>	<b>Sens</b>	<b>Spec</b>
<b>S1</b>	LA	2.06(0.43)	0.48(0.11)	0.92(0.08)
	GL	2.06(0.41)	0.50(0.05)	0.96(0.08)
	FU	2.05(0.44)	1.00(0.00)	0.97(0.02)
	G0	1.71(0.60)	1.00(0.00)	0.80(0.00)
	GFUL	1.68(0.49)	1.00(0.00)	0.94(0.02)
<b>S2</b>	LA	09.56(2.91)	0.08(0.05)	0.98(0.01)
	GL	09.60(2.73)	0.02(0.07)	1.00(0.00)
	FU	10.40(3.52)	0.05(0.02)	1.00(0.00)
	G0	11.92(2.75)	1.00(0.00)	0.72(0.00)
	GFUL	02.20(0.66)	0.84(0.30)	1.00(0.00)

Table 1: Results of the simulation study  
Standards error are given in parentheses

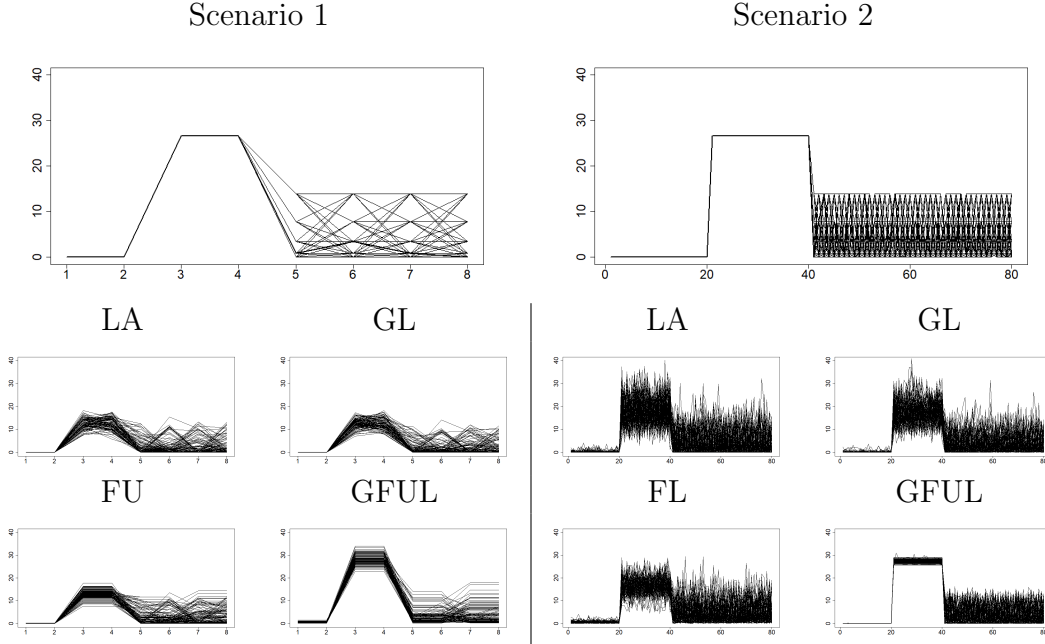


Figure 4: The 2-norms dimensions of the estimated coefficients

## 4 Application: FingerMovements

In this part, we apply our methods to FingerMovements<sup>1</sup>. These datasets are from the brain-computer interface area and are used for binary classification. More precisely, a subject has been asked to type characters using only the index and the pinky fingers. The challenge is to determine, based on their electroencephalography (EEG) recording, the hand which has been used. Each EEG subject is characterized by 50 times points (500 ms of recording) and  $p = 28$  sensors (electrodes). The data are split into 316 and 100 training and validation instances.

These datasets have been used in Ruiz et al. (2021), where the authors found the Inception Time model (Ismail Fawaz et al., 2020) to give the best accuracy metric on the validation set. Its accuracy is compared to the performance of our proposed models.

Our models have been computed thanks to the locations of the 28 electrodes. The 1-NN graph is used on their location to compute the variable fusion model(FU). Since no predefined groups of variables are given, for the group fusion lasso we used a Gaussian mixture model to cluster sensors based on their locations. The Bayesian Information Criteria gives  $K = 11$  as the optimal number of clusters. However, in this partition, one cluster has a unique element. As for GFUL, we want the groups to have at least two elements, we merge the one-sensor cluster with its nearest one (based on membership probabilities). At the end,  $K = 10$  clusters are considered (see Figure 5).

Two group lasso models are also computed: LA and GL. Similarly to the previous part, LA uses each dimension as a group and GL uses the same groups as GFUL.

For all dimensions, 30 B-splines are used as basis functions to reconstruct the functional version of the predictors. The hyperparameters  $\lambda$  and  $\alpha$  are tuned by a 10-cross-validation

<sup>1</sup><https://www.timeseriesclassification.com/description.php?Dataset=FingerMovements>

procedure, on the following grids  $\{\lambda_{\max}, (0.96)\lambda_{\max}, \dots, (0.96)^{148}\lambda_{\max}, 0\}, \{0, 0.1, \dots, 1\}$ , where  $\lambda_{\max}$  is the minimum value such all constraints verify.

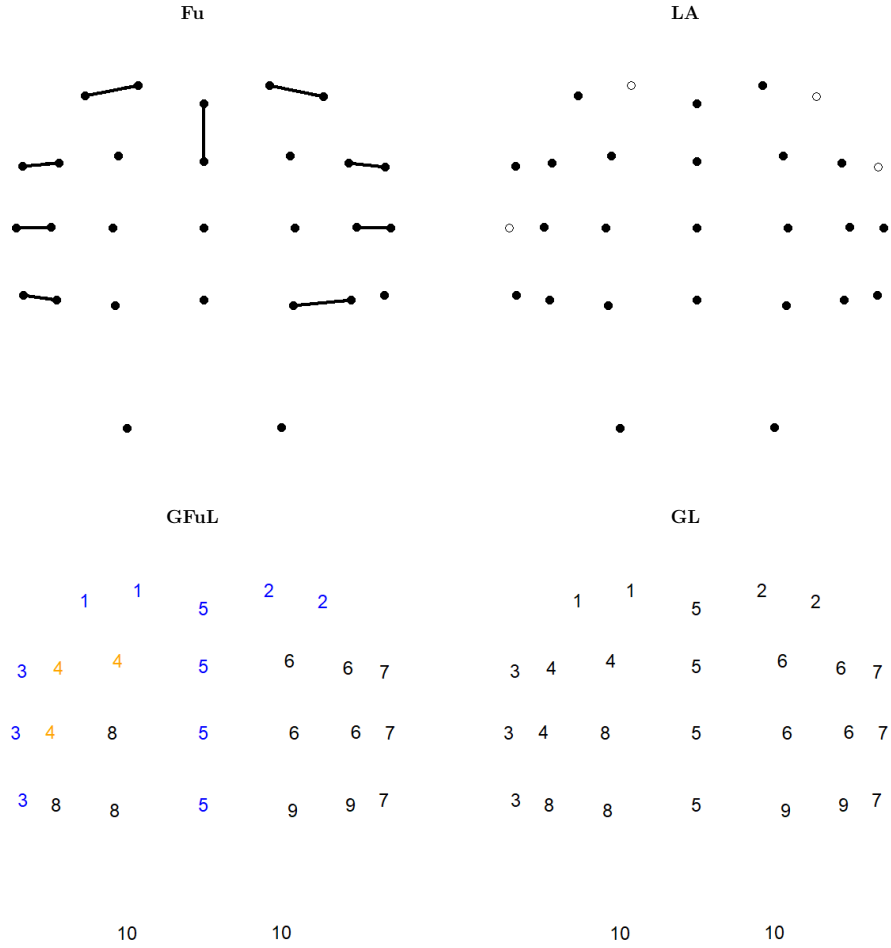


Figure 5: Estimated structures

**FU**: connected points share the same coefficients, **LA**: white points have zero as coefficient function, **GFUL**: groups in black share the same coefficients, groups in orange have zero as mean coefficients function, **GL**: all groups have not null coefficient functions.

## 4.1 Results

Table 2 shows that our proposed models perform better than competitors. Figure 5 shows the estimated coefficients structure of linear models. Hence, our models can find information on the distribution of coefficients and be competitive with a black-box model such as Inception Time.

For the sake of readability, the associated coefficient functions are given in appendix A.



<b>Methods</b>	<b>Accuracy</b>
<b>Fu</b>	63%
<b>GFuL</b>	60%
<b>Inception Time</b>	56.7%
<b>LA</b>	55%
<b>GL</b>	54 %

Table 2: Accuracy obtained on the test dataset

## 5 Discussion

In this paper, we have presented methods that can integrate location variables of repeated functional data to ease the interpretation of linear models. The simulation study shows a case where the group fusion lasso (GFUL) outperforms the variable fusion model(FU) and competitors. The application to real datasets demonstrates the competitiveness of FU over GFUL. Both proposed models outperform the lasso method competitors.

The group fusion lasso can be seen as a generalization of the fusion method in more than one neighbor. However, it tests group relations at once instead of testing one-on-one interactions. This is an assertive hypothesis, as it assumes that relations in a group can be either all true or all false. The use of smaller overlap groups could be an alternative model. In this setting, the resolution is related to the group lasso with overlap, which is more challenging (Yuan et al., 2011). An extensive study of adapted optimization problems should be done. One can also explore the model group lasso proposed in Jacob et al. (2009). Yet, it seems that using this approach leads to losing the diffusion between overlapped groups, the penalty is no longer defined on the 2, 1 norm (See Jacob et al. (2009) for details).

Future works should focus on the further study of the group fusion lasso. The integration of sparsity conditions and the study of other types of neighborhood structures in the fusion method should also be explored.

# Appendices

## A Additional figures: FingerMovements

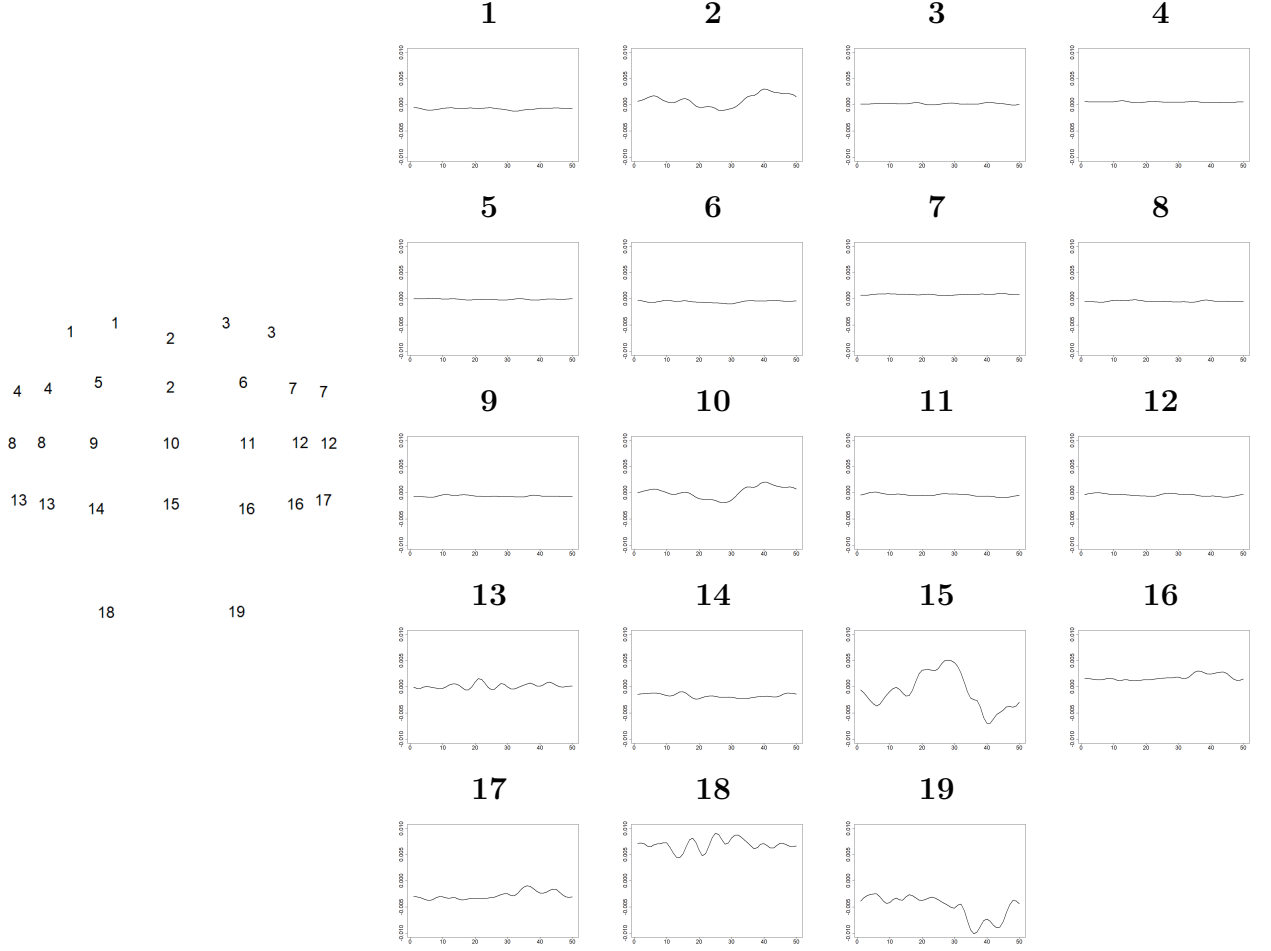


Figure 6: Fusion variable model

## B Technical arguments

*Proof of Lemma 1.* Let the neighbor function  $v: \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_p\} \rightarrow \{1, 2, \dots, p\}$ , as defined in Section 2.1 and  $\mathcal{V}_0, \mathcal{V}_1$  define as:  $\mathcal{V}_0 = \{i \in \{1, 2, \dots, p\}, v^2(\mathcal{C}_i) = i\}$  and  $\mathcal{V}_1 = \{i \in \{1, 2, \dots, p\}, i > v(\mathcal{C}_i), v^2(\mathcal{C}_i) = i\}$ , with  $v^2(\mathcal{C}_i) = v(\mathcal{C}_{v(\mathcal{C}_i)})$ , for  $i = 1, \dots, p$ .

If  $i \in \mathcal{V}_0 \iff v(\mathcal{C}_i) \in \mathcal{V}_0$ , if  $i \in \mathcal{V}_1 \implies v(\mathcal{C}_i) \notin \mathcal{V}_1$ . In other words,  $\mathcal{V}_0$  denotes the set of 2-cycle and  $\mathcal{V}_1$  is a partition of  $\mathcal{V}_0$  with  $\text{card}\mathcal{V}_1 = \frac{1}{2}\text{card}\mathcal{V}_0$ .

Then  $\forall f \in \mathcal{H}$

$$\begin{aligned}
\sum_{j=1}^p \|(\mathbf{L}f)^{(j)}\|_2 &= \sum_{j \in \mathcal{V}_0} \|f^{(j)} - f^{(v(\mathcal{C}_j))}\|_2 + \sum_{j \notin \mathcal{V}_0} \|f^{(j)} - f^{(v(\mathcal{C}_j))}\|_2 \\
&= 2 \sum_{j \in \mathcal{V}_1} \|f^{(j)} - f^{(v(\mathcal{C}_j))}\|_2 + \sum_{j \notin \mathcal{V}_0} \|f^{(j)} - f^{(v(\mathcal{C}_j))}\|_2 \\
\sum_{j=1}^p \|(\mathbf{L}f)^{(j)}\|_2 &= \sum_{j \in \mathcal{V}_1} \|2(f^{(j)} - f^{(v(\mathcal{C}_j))})\|_2 + \sum_{j \notin \mathcal{V}_0} \|f^{(j)} - f^{(v(\mathcal{C}_j))}\|_2
\end{aligned}$$

Then, there exists a matrix  $\mathbf{L}_0 \in \mathbb{R}^{r \times p}$ , with  $r = p - \frac{1}{2}\text{card}(\mathcal{V}_0)$ , as

$$\sum_{j=1}^p \|(\mathbf{L}f)^{(j)}\|_2 = \sum_{j=1}^r \|(\mathbf{L}_0 f)^{(j)}\|_2.$$

Moreover,  $v$  is constructed by the one-nearest neighbor graph. In this case, only the 2-cycle can occur (Eppstein et al., 1997). Hence,  $\mathbf{L}_0$  is full rank.  $\square$

*Proof of Proposition 1.* Here, we use the same reasoning as Tibshirani and Taylor (2011) for the case of the full-rank matrix in generalized lasso models. Indeed, if  $\mathbf{L}_0$  is full rank,  $\begin{pmatrix} \mathbf{L}_0 \\ \mathbf{T} \end{pmatrix}$  is non-singular, where  $\mathbf{T}$  is not null and verifies  $\mathbf{L}_0 \mathbf{T}^\top = 0_{\mathbb{R}^p}$ . Notice that  $\|\mathbf{D}f\|_{L_{2,1}} = \sum_{j=1}^r \|(\mathbf{L}_0 f)^{(j)}\| + \sum_{j=r+1}^p \|(\mathbf{T}f)^{(j)}\|$ ,  $f \in \mathcal{H}$ . Using the definition of  $\mathbf{L}_0$ ,  $\|\mathbf{D}f\|_{L_{2,1}} = \|\mathbf{L}f\|_{L_{2,1}} + \sum_{j=r+1}^p \|(\mathbf{T}f)^{(j)}\|$ . Hence, the problem (3) can be written as

$$\hat{\beta}_\lambda = \arg \min_{f \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i - \langle x_i, f \rangle_{\mathcal{H}})^2 + \sum_{j=1}^p \lambda \mathbb{I}(j \leq r) \|(\mathbf{D}f)^{(j)}\|_{L_2} \quad (12)$$

with  $\mathbb{I}(\cdot)$  is the indicator function. The non-singularity of  $\mathbf{D}$  implies

$$\hat{\psi}_\lambda = \arg \min_{f \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i - \langle x_i, \mathbf{D}^{-1}f \rangle_{\mathcal{H}})^2 + \sum_{j=1}^p \lambda \mathbb{I}(j \leq r) \|f^{(j)}\|_{L_2}, \quad (13)$$

with  $\hat{\psi}_\lambda = \mathbf{D}\hat{\beta}_\lambda$ . Moreover, since  $\langle (\mathbf{D}^{-1})^\top x_i, f \rangle_{\mathcal{H}} = \langle x_i, \mathbf{D}^{-1}f \rangle_{\mathcal{H}}$ , this concludes the proof.  $\square$

*Proof of Lemma 2.*

- First, let prove that for  $f \in \mathcal{H}$ ,  $k = 1, \dots, K$ ,

$$\|f_{\mathcal{I}_k} - \bar{f}_{\mathcal{I}_k} \mathbf{I}_{|\mathcal{I}_k|}\|_{L_{2,2}} = \|\mathbf{R}_k f_{\mathcal{I}_k}\|_{L_{2,2}}. \quad (14)$$

Direct calculation shows that

$$f_{\mathcal{I}_k} - \bar{f}_{\mathcal{I}_k} \mathbf{I}_{|\mathcal{I}_k|} = \underbrace{\left[ \mathbf{I}_{|\mathcal{I}_k| \times |\mathcal{I}_k|} - \frac{1}{|\mathcal{I}_k|} \mathbb{I}_{|\mathcal{I}_k| \times |\mathcal{I}_k|} \right]}_{\mathbf{P}_k} f_{\mathcal{I}_k} \quad (15)$$

The rank of  $\mathbf{P}_k$  is  $|\mathcal{I}_k| - 1$  (due to the freedom degree of the mean). The norm of  $f_{\mathcal{I}_k} - \bar{f}_{\mathcal{I}_k} \mathbf{I}_{|\mathcal{I}_k|}$ , can be written as  $\|f_{\mathcal{I}_k} - \bar{f}_{\mathcal{I}_k} \mathbf{I}_{|\mathcal{I}_k|}\|_{L_{2,2}}^2 = \|\mathbf{P}_k f_{\mathcal{I}_k}\|_{L_{2,2}}^2$ . Let  $\mathbf{R}_k \in \mathbb{R}^{(|\mathcal{I}_k|-1) \times |\mathcal{I}_k|}$  the  $\mathbf{R}$  reduced rank matrix obtained by  $\mathbf{QR}$  decomposition of  $\mathbf{P}_k$ , since  $\|\cdot\|_{L_{2,2}}$  is the Frobenius function norm,  $\|\mathbf{P}_k f_{\mathcal{I}_k}\|_{L_{2,2}}^2 = \|\mathbf{R}_k f_{\mathcal{I}_k}\|_{L_{2,2}}^2$ .

- The second point is to show that  $\mathbf{R}_k u_k^\top = \mathbf{0}_{|\mathcal{I}_k|-1}$  where  $u_k = \frac{1}{|\mathcal{I}_k|} \mathbf{I}_{|\mathcal{I}_k|}$ , for  $k = 1, \dots, K$ . Since all element of  $u_k$  are the same  $\mathbf{P}_k u_k = \mathbf{0}_{|\mathcal{I}_k|}$ .

Recall the definition of  $\mathbf{R}_k$ ,  $\mathbf{P}_k = \mathbf{Q}_k \begin{pmatrix} \mathbf{R}_k \\ \mathbf{0}_{|\mathcal{I}_k|}^\top \end{pmatrix}$ , where  $\mathbf{Q}_k$  is an orthogonal matrix. This means that the null space of  $\mathbf{Q}_k$  is composed of only the null vector. Then  $\mathbf{P}_k u_k = \mathbf{0}_{|\mathcal{I}_k|} \iff \begin{pmatrix} \mathbf{R}_k \\ \mathbf{0}_{|\mathcal{I}_k|}^\top \end{pmatrix} u_k = \mathbf{0}_{|\mathcal{I}_k|}$ , then  $\mathbf{R}_k u_k = \mathbf{0}_{|\mathcal{I}_k|-1}$ .

This concludes the proof. □

*Proof of Proposition 2.* The matrix  $\mathbf{G}$  helps to reformulate the problem 8 as

$$\hat{\beta}_\lambda = \arg \min_{f \in \mathcal{H}} \frac{1}{2} \sum_{i=1}^n (y_i, \langle x_i, f \rangle_{\mathcal{H}})^2 + \lambda \sum_{j=1}^{2g} \|(\mathbf{G}f)^{(j)}\|_{L_{2,2}}.$$

Since  $\mathbf{G}_\alpha$  is non-singular, the result is straightforward. □

*Proof of Proposition 3.*

1. The equation (10) implies that each dimension  $j = 1, \dots, p$

$$\beta^{(j)}(t) = (b^{(j)})^\top \phi(t) \text{ where } b^{(j)} \in \mathbb{R}^M \text{ } t \in \mathcal{T}.$$

Since  $\beta^{(j)}(t) \in \mathbb{R}$ ,  $t \in \mathcal{T} \implies (\beta^{(j)}(t))^\top = \beta^{(j)}(t)$ , then  $\|\beta^{(j)}\|_{L_2} = \sqrt{\langle (b^{(j)})^\top \phi, \phi^\top b^{(j)} \rangle}$ . Define  $\mathbf{F} = \{\langle \phi_i, \phi_j \rangle\}_{i,j}$  and  $\mathbf{F} = (\mathbf{F}^{1/2})^\top \mathbf{F}^{1/2}$ , thus  $\|\beta^{(j)}\|_{L_2} = \|(\mathbf{F}^{1/2})^\top b^{(j)}\|_2 = \|(b^{(j)})^\top \mathbf{F}^{1/2}\|_2$ . Moreover,

$$\mathbf{B}\mathbf{F}^{1/2} = \begin{pmatrix} (b^{(1)})^\top \\ (b^{(2)})^\top \\ \dots \\ (b^{(p)})^\top \end{pmatrix} \mathbf{F}^{1/2} = \begin{pmatrix} (b^{(1)})^\top \mathbf{F}^{1/2} \\ (b^{(2)})^\top \mathbf{F}^{1/2} \\ \dots \\ (b^{(p)})^\top \mathbf{F}^{1/2} \end{pmatrix},$$

then  $\|\beta\|_{L_{2,1}} = \|\mathbf{B}\mathbf{F}^{1/2}\|_{2,1}$ .

2. Indeed,

$$\begin{aligned} b_0 &= \text{vec}((\mathbf{Z}\mathbf{B})^\top) = \text{vec}(\mathbf{B}^\top \mathbf{Z}^\top) \\ b_0 &= (\mathbf{Z} \otimes \mathbf{I}_{M \times M}) \text{vec}(\mathbf{B}^\top) = (\mathbf{Z} \otimes \mathbf{I}_{M \times M}) b, \end{aligned}$$

with  $\otimes$  denotes the Kronecker Product. □

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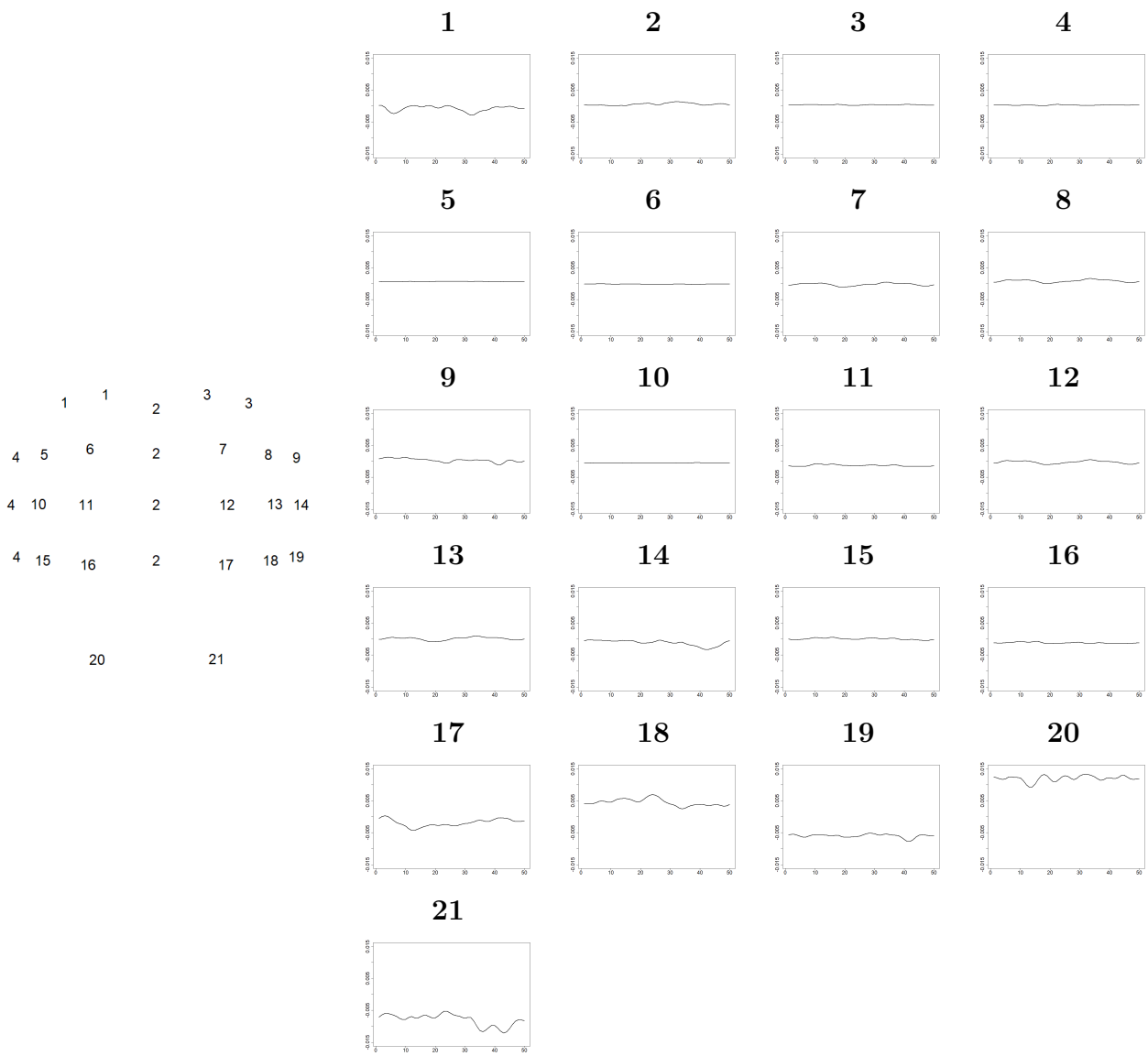


Figure 7: Group Fusion variable model

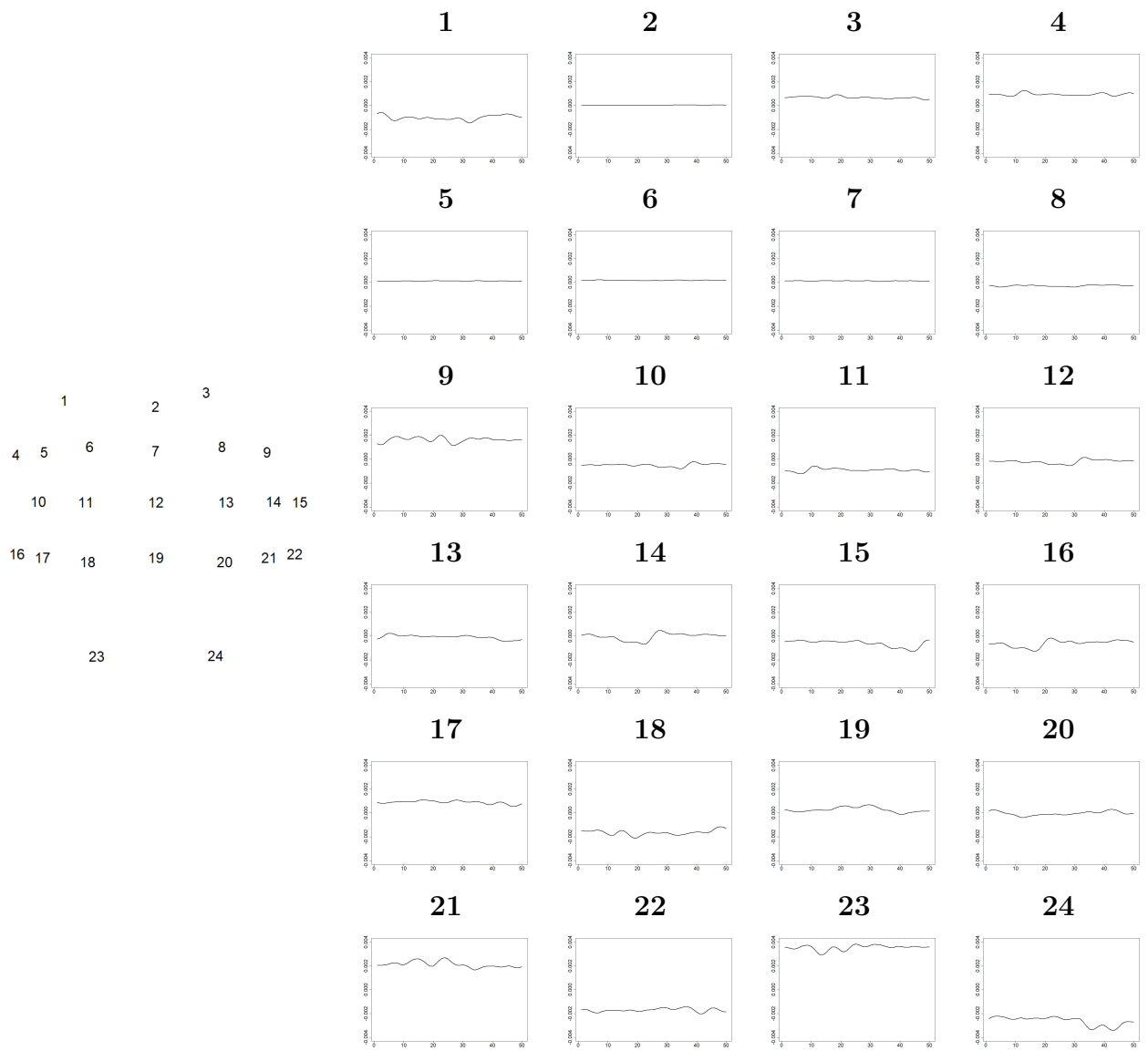


Figure 8: LA model



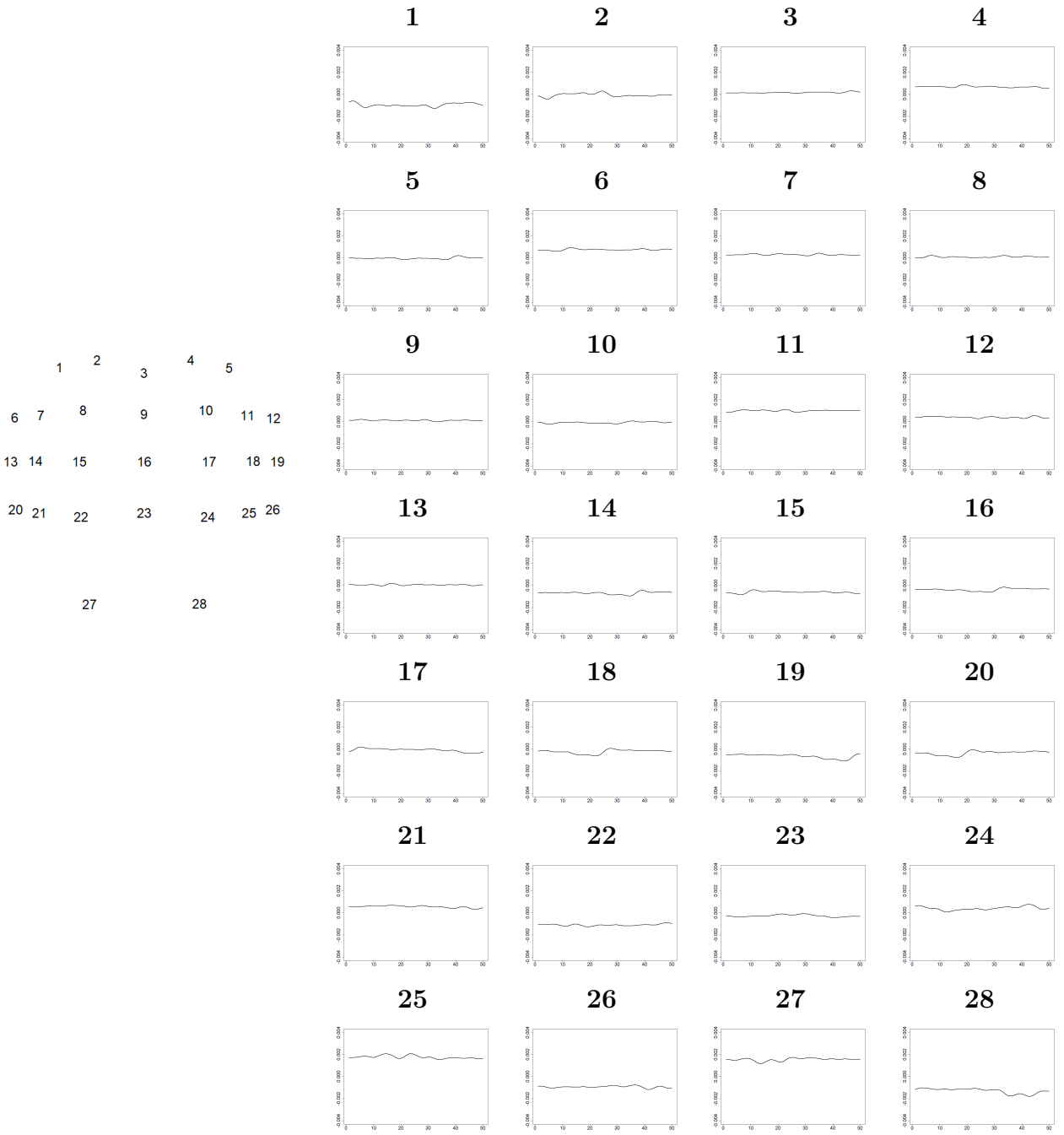


Figure 9: GL model