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## Diffeologies and generalized Kadomtsev-Petviashvili hierarchies

Anahita Eslami-Rad, Jean-Pierre Magnot, Enrique G. Reyes,  
and Vladimir Rubtsov

ABSTRACT. We present a simplified, but extended proof of the well-posedness of the Kadomtsev-Petviashvili hierarchy with coefficients in an abstract diffeological differential algebra, and with initial values of any order.

### 1. Introduction

The Kadomtsev-Petviashvili hierarchy is an integrable system with an infinite number of independent variables studied since a very long time, inherited from the so-called Kadomtsev-Petviashvili equation which can be deduced from the hierarchy. From the Kadomtsev-Petviashvili hierarchy, one can specialize and deduce other equations of interest, including Korteweg-de Vries and Boussinesq equations, see e.g. [3].

Surprisingly, the full study of this system have created unexpected links with infinite dimensional geometry (more specifically with Sato grassmanians and loop groups), groups of diffeomorphisms and Hodge theory, which raised many years ago. Therefore, it is intuitively normal that generalizations of the KP hierarchy carry newly discovered correspondences with unexpected partial differential equations which are still under investigations. For such examples, see e.g. [13]. These investigations have been motivated, for the authors, by the statement of the well-posedness of the generalized KP hierarchy [6, 12] by the adjunction of diffeologies to the algebraic setting.

In this note, we review how diffeologies invite themselves in this non-linear, infinite dimensional integrable system along the lines of our previous works [6, 12] in a general framework and in a clarified way that the intrinsic progression of our discoveries did not allow us to state and prove in this way or under this formulation. We take profit of this short presentation to expose two new results, a grading in the Mulase group and an extension of the statement of integrability and well-posedness for a wider class of initial values. The reader is assumed to be familiar with diffeologies.

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## 2. Preliminaries on formal pseudo-differential operators in a diffeological setting

We set the following framework. Let  $\mathbb{K}$  be a field and let  $(A, +, *, \partial)$  be a differential algebra over  $\mathbb{K}$ .

DEFINITION 2.1. The set of pseudo-differential symbols is

$$\Psi DO(A) = \left\{ \sum_{-\infty < k \leq N} a_k \xi^k : a_k \in A \text{ and } N \in \mathbb{Z} \right\},$$

where  $\xi$  is a formal variable.

We always assume that  $A$  is a unital algebra, that is, there exists  $1 \in A$  such that  $\forall a \in A, 1 * a = a * 1 = a$ . The set  $\Psi DO(A)$  can be equipped with an algebra structure.

DEFINITION 2.2. We define addition and multiplication on  $\Psi DO(A)$  as follows.

$$\begin{aligned} + : \Psi DO(A) \times \Psi DO(A) &\longrightarrow \Psi DO(A) \\ (\sum a_n \xi^n, \sum b_m \xi^m) &\longmapsto \sum (a_p + b_p) \xi^p \end{aligned}$$

and

$$\begin{aligned} * : \Psi DO(A) \times \Psi DO(A) &\longrightarrow \Psi DO(A) \\ (a(\xi) = \sum a_n \xi^n, b(\xi) = \sum b_m \xi^m) &\longmapsto \sum_{m,n} \sum_{\alpha \in \mathbb{N}} \frac{1}{\alpha!} (a_n \partial^\alpha b_m) (D_\xi^\alpha \xi^n) \xi^m \end{aligned}$$

THEOREM 2.3.  $(\Psi DO(A), +, *)$  is a unital (associative)  $\mathbb{K}$ -algebra.

This theorem is well-known, see for example [3], or the more recent papers [5, 6]. The following observation is at the basis of the results presented herein.

PROPOSITION 2.1. *The decomposition*

$$\Psi DO(A) = DO(A) \oplus IO(A),$$

in which  $DO(A)$  is the subalgebra of symbols of differential operators, and  $IO(A)$  is the algebra of symbols of integral operators, is a vector space splitting. We note that  $IO(A)$  is a non unital algebra.

PROOF. The proof relies on standard arguments:

- If  $(n, m) \in \mathbb{N}^2$ , then for all  $\alpha \in \mathbb{N}$ , we have that

$$\deg_\xi ((D_\xi^\alpha \xi^n) \xi^m) \in \mathbb{N},$$

which shows that  $DO(A)$  is stable under multiplication,

- On the other hand, if  $(n, m) \in (\mathbb{N}^*)^2$ , then for all  $\alpha \in \mathbb{N}$ , we have that

$$\deg_\xi ((D_\xi^\alpha \xi^{-n}) \xi^{-m}) < 0,$$

which shows that  $DO(A)$  is stable under multiplication. □

This proposition implies that a symbol  $a \in \Psi DO(A)$  decomposes uniquely under this direct sum as

$$a = a_+ + a_- , \quad a_+ \in DO(A) \text{ and } a_- \in IO(A) .$$

In the most classical framework,  $A$  is an algebra of  $\mathbb{R}$ - (or  $\mathbb{C}$ -)valued functions (most usually with domain  $S^1$ , see [7, 8, 9]) equipped with a derivation  $\partial$ ; taking

this fact into account, we usually replace the formal variable  $\xi$  for  $\partial$  in  $\Psi DO(A)$ , and speak about “formal pseudo-differential operators” or, rather informally, of “pseudo-differential operators”<sup>1</sup> Of great importance for us is the fact that the algebras of functions

$$A = C^\infty(S^1, \mathbb{K}),$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$ , (or  $\mathbb{H}$ ), are Fréchet algebras, and that they are equipped with a natural notion of differentiability making addition, multiplication and differentiation smooth. Considering addition and multiplication in  $\Psi DO(A)$ , we can say that addition and multiplication on  $\Psi DO(A)$  are *smooth* in the following sense: if

$$a = \sum_{n \in \mathbb{Z}} a_n \xi^n, \quad b = \sum_{n \in \mathbb{Z}} b_n \xi^n,$$

then setting

$$a + b = c = \sum_{n \in \mathbb{Z}} c_n \xi^n$$

and

$$ab = d = \sum_{n \in \mathbb{Z}} d_n \xi^n,$$

the map

$$((a_n)_{n \in \mathbb{Z}}, (b_n)_{n \in \mathbb{Z}}) \mapsto ((c_n)_{n \in \mathbb{Z}}, (d_n)_{n \in \mathbb{Z}})$$

is smooth in the relevant infinite product.

These observations motivate us to consider very general settings, and to understand smoothness on infinite products by means of the infinite product diffeology. Hereafter, smoothness will be understood in this way on objects that arise as infinite products of infinite-dimensional diffeological spaces.

We assume henceforth that the algebra  $A$  is a **diffeological algebra**. This means that addition, scalar multiplication, and multiplication are smooth operations, and also that inversion is a smooth operation on  $A^*$ , the group of the units of the algebra  $A$ , in which  $A^*$  is equipped with the subset diffeological structure of

$$A^* \subset Diff(A).$$

In this picture, we understand  $A^*$  as acting on the left on  $A$  by multiplication. For completeness, we recall that the group of diffeomorphisms  $Diff(X)$  of a diffeological space  $X$  is not endowed by the subset diffeology of functional diffeology  $\mathcal{P}_f$  of  $C^\infty(X, X)$ , but by the pull-back diffeology

$$\Phi^*(\mathcal{P}_f \times \mathcal{P}_f)$$

where

$$\Phi : g \in Diff(X) \mapsto (g, g^{-1}) \in C^\infty(X, X)^2.$$

Recall that the inversion may not be smooth for the subset diffeology of the functional diffeology, which justifies the use of this refined construction. We also assume that  $\partial$  is smooth. It follows that, identifying a formal pseudo-differential operator  $a \in \Psi(A)$  with its sequence of partial symbols, the space  $\Psi(A)$  is a linear subspace of  $A^{\mathbb{Z}}$  and it carries a natural diffeological structure. We obtain, see [6]:

**PROPOSITION 2.2.**  *$\Psi(A)$  is a diffeological algebra.*

<sup>1</sup>There are other possible settings, for instance (see [10, 16]), we can consider functions with values in  $\mathbb{H}$ , and  $\partial = \frac{d}{dx}$ .

### 3. An algebraic resolution of the KP hierarchy

**3.1. The Mulase bicross product and its subgroups.** Let  $t$  be a countable family of (time) independent variables  $t_1, t_2, \dots$ . We note by  $A_t$  the diffeological algebra of formal series

$$A_t = A[[t_1, t_2, \dots]] ,$$

and we fix a valuation on  $A_t$  by setting

$$val_t(t_n) = n.$$

The interested reader may find an explicit construction of  $A_t$  in [1]. After [5], other choices of valuations can be performed but we choose to concentrate our review on the standard one given in Mulase's original works [14, 15].

DEFINITION 3.1 (Mulase, [15]). The Mulase spaces of formal pseudo-differential and differential operators of infinite order,  $\widehat{\Psi}(A)$  and  $\widehat{\mathcal{D}}_A$  respectively, are:

$$(3.1) \quad \widehat{\Psi}(A_t) = \left\{ \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : a_\alpha \in A_t \text{ and } \exists (C, N) \in \mathbb{R} \times \mathbb{N} \text{ so that } val_t(a_\alpha) > C\alpha - N \forall \alpha \gg 0 \right\}$$

and

$$(3.2) \quad \widehat{\mathcal{D}}_{A_t} = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \partial^\alpha : P \in \widehat{\Psi}(A_t) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\} .$$

REMARK 3.2. As a set of formal series, the invertibility of an element  $A \in \widehat{\Psi}(A_t)$  is dependent on its monomial of  $T$ -valuation 0. This monomial is of non-positive  $\partial$ -order. Therefore, one can prove that  $\widehat{\Psi}(A_t)^*$  is a diffeological group if and only if  $A^*$  is a diffeological group, by analyzing classical formulas of inversion of series.

In addition, we define

$$\mathcal{I}_{A_t} = \left\{ P \in \widehat{\Psi}(A_t) : \forall \alpha \geq 0, a_\alpha = 0 \right\}$$

and

$$G_{A_t} = 1 + \mathcal{I}(A_t) .$$

REMARK 3.3. In order to prove that  $G_{A_t}$  is a diffeological group, it is not necessary to assume that  $A^*$  is a diffeological group.

We notice that

$$A_t \subset \widehat{\Psi}(A_t)$$

as the 0-order term,

$$G_{A_t} \subset \Psi(A_t) \subset \widehat{\Psi}(A_t) ,$$

(in fact,  $\Psi(A_t)$  is composed of elements of  $\widehat{\Psi}(A_t)$  with uniformly bounded degree in  $\partial$ ) and that

$$\widehat{\mathcal{D}}_{A_t} \not\subset \Psi(A_t).$$

Analysing the operations on  $\widehat{\Psi}(A_t)$ , see [15] and our later paper [5], we obtain the following result, as a consequence of our work in [6]:

LEMMA 3.4. *The spaces  $\widehat{\Psi}(A_t)$  and  $\widehat{\mathcal{D}}_{A_t}$  are diffeological algebras.*

The constraints appearing in Mulase's definitions of  $\widehat{\Psi}(A_t)$  allow us to construct *diffeological groups* of pseudo-differential operators. We proceed thus.

DEFINITION 3.5. Let  $\mathcal{K}$  be the ideal of  $A_t$  generated by  $t_1, t_2, \dots$ . If  $P \in \widehat{\Psi}(A_t)$ , we denote by  $P|_{t=0}$  the equivalence class  $P \bmod \mathcal{K}$  (i.e. the projection on the constant coefficient of the  $t$ -series), and we identify it with an element of  $\Psi(A)$ . We also set  $G_A = 1 + IO(A)$ , and we define the spaces

$$(3.3) \quad G(\widehat{\Psi}(A_t)) = \{P \in \widehat{\Psi}(A_t) : P|_{t=0} \in G_A\}$$

and

$$(3.4) \quad \widehat{\mathcal{D}}_{A_t}^\times = \{P \in \widehat{\mathcal{D}}_A : P|_{t=0} = 1\}.$$

We now give a result established in [12] and, for algebras  $A$  equipped with an abstract valuation, in [6]. The proof we give here is inspired by [12, 15]; it is different to the one given in [6].

THEOREM 3.6.

(1) For any  $U \in G(\widehat{\Psi}(A_t))$  there exist unique  $S \in G_{A_t}$  and  $Y \in \widehat{\mathcal{D}}_A^\times$  such that

$$U = S^{-1} Y.$$

*In other words, there exists a unique global factorization of the diffeological group  $G(\widehat{\Psi}(A_t))$  as a group defined by bicross product of diffeological groups,*

$$G(\widehat{\Psi}(A_t)) = G_{A_t} \bowtie \widehat{\mathcal{D}}_{A_t}^\times.$$

(2) The factorization  $U \mapsto (S, Y)$  is (diffeologicaly) smooth.

PROOF. Part (1) is already proved in [14, 15] (see also the exposition in [5]), but we have to examine carefully the original proof in order to check that the map  $U \mapsto S$  is smooth. Once we have this first result, we finish the proof of the theorem by remarking that  $Y = SU$ .

Let us follow the proof of [14, Lemma 3]. We solve the equation

$$(SU)_- = 0$$

for given

$$U = \sum_{\nu \in \mathbb{Z}} u_\nu \partial^\nu$$

and unknown

$$S = \sum_{\mu \in \mathbb{Z} - \mathbb{N}} s_\mu \partial^\mu.$$

This equation turns into the infinite system

$$\sum_{\nu \in \mathbb{Z} - \mathbb{N}} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{\mu+i-\nu} \right) s_\nu = -u_\mu \quad \text{for } \mu \in \mathbb{Z} \setminus \mathbb{N}$$

with the convention

$$\binom{\nu}{i} = (-1)^{|\nu|} \binom{|\nu|}{i} \quad \text{if } \nu < 0.$$

We define the notation  $[\cdot]_k$  as the sum of the monomials in  $T$  with valuation equal to  $k$ , and by  $[\cdot]_{m(T)}$  the coefficient related the  $T$ -monomial  $m(T)$ .

We then solve this infinite system by induction on  $\mu$ . For  $\mu = -1$ , we study the  $t$ -monomials.

$$\left[ \sum_{\nu \in \mathbb{Z}-\mathbb{N}} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-1+i-\nu} \right) s_\nu \right]_0 = \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) s_{-1} \right]_0 = -[s_{-1}]_0$$

thus  $[s_{-1}]_0 = [u_{-1}]_0$ . Then, at the monomial of valuation 1, which is only the  $t_1$ -monomial,

$$\begin{aligned} \left[ \sum_{\nu \in \mathbb{Z}-\mathbb{N}} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-1+i-\nu} \right) s_\nu \right]_{t_1} &= \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) \right]_{t_1} [s_{-1}]_0 \\ &\quad + \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) \right]_0 [s_{-1}]_{t_1} \\ &= \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) \right]_{t_1} [u_{-1}]_0 \\ &\quad - [s_{-1}]_{t_1} \end{aligned}$$

we get

$$[s_{-1}]_{t_1} = [u_{-1}]_{t_1} + \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) \right]_{t_1} [u_{-1}]_0$$

which shows that the maps

$$u \mapsto [s_{-1}]_0$$

and

$$u \mapsto [s_{-1}]_{t_1}$$

are smooth. The same calculations can be used for any  $t$ -monomial for  $s_{-1}$ , and they show that

$$u \mapsto s_{-1}$$

is smooth. Let us now assume that  $s_{-1}, \dots, s_{-k}$  are determined, and are smooth with respect to  $U$ , and let us determine  $s_{-k-1}$ . Then we write, for  $\mu = -k-1$ ,

$$\begin{aligned} &\sum_{\nu \leq -k-1} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-k-1+i-\nu} \right) s_\nu \\ &= -u_{-k-1} - \sum_{\nu=-k}^{-1} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-k-1+i-\nu} \right) s_\nu \end{aligned}$$

noticing that the right side is fully determined. Then we carry out induction with respect to the valuation on  $t$ , for which we sketch the first two steps as we did before:

- for the  $t = 0$  term:

$$[s_{-k-1}]_0 = (-1)^{-k-1} \left[ u_{-k-1} + \sum_{\nu=-k}^{-1} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-k-1+i-\nu} \right) s_\nu \right]_0,$$

- for the  $t_1$ -monomial:

$$[s_{-k-1}]_{t_1} = (-1)^{-k-1} \left( \left[ u_{-k-1} + \sum_{\nu=-k}^{-1} \left( \sum_{i \in \mathbb{N}} \binom{\nu}{i} \partial^i u_{-k-1+i-\nu} \right) s_\nu \right]_{t_1} + \left[ \left( \sum_{i \in \mathbb{N}} \binom{-1}{i} \partial^i u_i \right) \right]_{t_1} [s_{-k-1}]_0 \right)$$

Thus we have here a construction of  $S$  made of smooth operations, so that the map

$$U \mapsto S$$

is smooth. Hence the map

$$U \mapsto Y = SU$$

is also smooth. This completes the proof.  $\square$

In order to solve the KP hierarchy, we need the following groups (see [6, 12] for the  $n = 1$  case):

DEFINITION 3.7.

$$\bar{\Psi}_n = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \xi^\alpha : a_\alpha \in A[[T]] \text{ and } \text{val}_T(a_\alpha) \geq n\alpha \text{ and } P|_{t=0} \in 1 + IO(A) \right\}$$

and

$$\bar{\mathcal{D}}_n = \left\{ P = \sum_{\alpha \in \mathbb{Z}} a_\alpha \xi^\alpha : P \in \bar{\Psi}_n(A_t) \text{ and } a_\alpha = 0 \text{ for } \alpha < 0 \right\}.$$

PROPOSITION 3.1. *Let us assume that the algebra  $A$ , understood as a diffeological group with respect to its addition, has its internal tangent space at 1 that is diffeologically isomorphic to  $A$ .*

- (1) *The groups  $\bar{\Psi}_n$ ,  $G_{A_t}$  and  $\bar{\mathcal{D}}_n$  are diffeological Lie groups, that is, they have an internal tangent space at their neutral element which carries a smooth Lie bracket.*
- (2) *Moreover, we have a diffeological matched pair (or bicross product)*

$$\bar{\Psi}_n = G_{A_t} \bowtie \bar{\mathcal{D}}_n.$$

REMARK 3.8. The condition that a diffeological vector space is isomorphic with its tangent space at a given point seems natural in a Fréchet setting but becomes not so obvious for a general diffeology. A recent work develops a counter-example in [2].

PROOF. If  $A$  has its internal tangent space at  $A$  with is diffeologically isomorphic to  $A$ , for the addition law, then it is trivial that we have an inclusion of diffeological vector spaces

$$T_1 \bar{\mathcal{D}}_n \subset T_1 \bar{\Psi}_n \subset \widehat{\Psi}(A_t)$$

and therefore that the Ad-action of  $\bar{\mathcal{D}}_n$  (resp.  $\bar{\Psi}_n$ ) on  $T_1 \bar{\mathcal{D}}_n$  (resp.  $T_1 \bar{\Psi}_n$ ) differentiates as in the usual case of a Fréchet algebra. Therefore, the Lie bracket is defined in a classical way by commutators  $[A, B] = AB - BA$ . The same holds for  $G_{A_t}$ , for which it is trivial that

$$T_1 G_{A_t} = IO(A_t).$$

The claim (2) follows from the inclusion of  $\bar{\Psi}_n$  into  $G(\widehat{\Psi}(A_t))$  and Theorem 3.4.  $\square$

We conclude that we have two diffeological inclusion sequences,

$$\overline{\Psi}_1 \subset \cdots \subset \overline{\Psi}_n \subset \cdots \subset G(\widehat{\Psi}(A_t))$$

and

$$\overline{\mathcal{D}}_1 \subset \cdots \subset \overline{\mathcal{D}}_n \subset \cdots \subset \widehat{\mathcal{D}}_{A_t}^\times .$$

REMARK 3.9. In all the present constructions, we do not need to define a tangent space on any of our groups. This is a good point since actually the notion of the tangent space of a manifold has multiple generalizations in the setting of diffeologies, and there often happens that, in one or another definition of the tangent space at one point of a diffeological vector space, the tangent space is not isomorphic to the diffeological vector space. Even more, for close reasons, considering a diffeological group, the tangent space at the neutral element may not carry a smooth Lie bracket. This is the difference between diffeological groups and diffeological Lie groups: in the second framework, one considers only diffeological groups which have a diffeological Lie algebra structure on their tangent space at the identity [11]. This is very difficult to prove that a diffeological group is not a diffeological Lie group and actually there is no such rigorously proven example. Assuming that,  $\forall a \in A$ , the kinematic or internal tangent space at  $a$  is isomorphic to  $A$ , we think that we can propose such a counter-example. Indeed, the group  $G(\widehat{\Psi}(A_t))$  may well be an example of diffeological group which is not a diffeological Lie group (that is, for which the Lie bracket is ill-defined) but this has not been proven yet, while it is proven in [12] that  $\overline{\Psi}_1$  is a diffeological Lie group if  $A$  is a diffeological algebra with tangent space isomorphic to  $A$  as stated before.

**3.2. The algebraic initial value problem.** First of all we state the solution to the IVP for the KP hierarchy after [15] and the more recent re-workings of Mulase’s result appearing in our papers [5] and [6].

THEOREM 3.10. *Consider the KP system of equations*

$$(3.5) \quad \frac{dL}{dt_k} = [(L^k)_+, L]$$

with initial condition  $L(0) = L_0 \in \Psi(A)$ , and let  $Y \in G_+ = G_+(\mathcal{D}_A)$  and  $S \in G_- = G_-(\mathcal{I}_A)$  be the unique solution to the factorization problem  $\exp(t_k L_0^k) = S^{-1}(t_k) Y(t_k)$ . The unique solution to Equation (3.5) with  $L(0) = L_0$  is

$$L(t_k) = Y L_0 Y^{-1} = S L_0 S^{-1} .$$

What is important for us at this point is that at this level of generality *the initial condition is an arbitrarily given formal pseudo-differential operator*. We postpone the proof of Theorem 3.5 until the next section dedicated to well-posedness. Before that, we have to stress that a tentative for a statement for a more general setting is given in [6], which may include the case of Laurent series in  $T$  under mild assumptions on the control of the order, and also sets of coefficients with arbitrary valuations. However, the proof given in [6] is drastically more complicated than the one that we present here, and “concrete” examples in such general settings still have to be constructed.

#### 4. Proof of well-posedness in a diffeological setting

The following result, stated in [6] for more specific initial values  $L_0$ , gives a synthesized statement of our main results on the KP hierarchy (3.5), and it states smooth dependence on the initial conditions.

THEOREM 4.1.

Consider the KP hierarchy 3.5 with initial condition  $L(0) = L_0$  of order  $n \in \mathbb{N}$ . Then,

- (1) There exists a pair  $(S, Y) \in \bar{G} \times \bar{\mathcal{D}}_n$  such that the unique solution to Equation (3.5) with  $L|_{t=0} = L_0$  is

$$L(t_1, t_2, \dots) = Y L_0 Y^{-1} = S L_0 S^{-1} .$$

and  $L \in \Psi^n(A_t)$ .

- (2) The pair  $(S, Y)$  is uniquely determined by the smooth decomposition problem

$$\exp\left(\sum_{k \in \mathbb{N}} t_k L_0^k\right) = S^{-1} Y$$

and the solution  $L$  depends smoothly on the initial condition  $L_0$ .

- (3) The solution operator  $L$  is smoothly dependent on the initial value  $L_0$ .

PROOF: THEOREM 3.10 AND THEOREM 4.1. We model our proof after Reyman and Semenov-Tian-Shansky's [18], see also [6, 12, 17], in a proof based on r-matrix theory.

We set

$$U = \exp\left(\sum_{k \in \mathbb{N}} t_k L_0^k\right) .$$

Then,

$$U \in \bar{\Psi}_n .$$

We consider the smooth decomposition

$$U \mapsto (S, Y) \in G_{A_t} \times \bar{\mathcal{D}}_n$$

following Proposition 3.1, and we set

$$L = Y L_0 Y^{-1} .$$

We make two obvious observations:

- (1)  $L^k = Y L_0^k Y^{-1}$   
 (2)  $U L_0^k U^{-1} = L_0^k$  since  $L_0$  commutes with  $U = \exp(\sum_k t_k L_0^k)$ .

It follows that

$$L^k = Y L_0^k Y^{-1} = S S^{-1} Y L_0^k Y^{-1} S S^{-1} = S L_0^k S^{-1} .$$

We take  $t_k$ -derivative of  $U$  for each  $k \geq 1$ . We get the equation

$$L_0^k U = -S^{-1} S_{t_k} S^{-1} Y + S^{-1} Y_{t_k}$$

and so, using  $U = S^{-1} Y$ , we obtain the decomposition

$$S L_0^k S^{-1} = -S_{t_k} S^{-1} + Y_{t_k} Y^{-1} .$$

Since  $S_{t_k} S^{-1} \in \mathcal{I}_A$  and  $Y_{t_k} Y^{-1} \in \mathcal{D}_A$ , we conclude that

$$(L^k)_+ = Y_{t_k} Y^{-1} \quad \text{and} \quad (L^k)_- = -S_{t_k} S^{-1} .$$

Now we take  $t_k$ -derivative of  $L$ :

$$\begin{aligned} \frac{dL}{dt_k} &= Y_{t_k} L_0 Y^{-1} - Y L_0 Y^{-1} Y_{t_k} Y^{-1} \\ &= Y_{t_k} Y^{-1} Y L_0 Y^{-1} - Y L_0 Y^{-1} Y_{t_k} Y^{-1} \\ &= (L^k)_+ L - L (L^k)_+ \\ &= [(L^k)_+, L]. \end{aligned}$$

It remains to check the initial condition: We have

$$L(0) = Y(0) L_0 Y(0)^{-1},$$

but  $Y(0) = 1$  since  $Y \in \overline{\mathcal{D}}_n$ . Therefore

$$L(0) = L_0.$$

Smoothness with respect to  $t$  is already proved by construction, and we have established smoothness of the map  $L_0 \mapsto Y$  at the beginning of the proof. Thus, the map

$$L_0 \mapsto L(t) = Y(t) L_0 Y^{-1}(t)$$

is smooth in  $\overline{\Psi}(A_t)$ .

In order to finish the proof, we need to prove that the solution  $L$  belongs to  $\Psi^n(A_t)$ . For this, we consider above mentioned relation

$$L^k = S L_0^k S^{-1},$$

which for  $k = 1$ , implies that

$$L = L_0 + S[L_0, S^{-1}].$$

$[L_0, S^{-1}] \in \Psi^{n-1}(A_t)$ , which ends the proof since uniqueness is obtained algebraically in [5].  $\square$

## 5. Outlook

There are several examples we can give. Our approach can be applied to  $A = C^\infty(S^1, \mathbb{R})$ , and therefore we obtain a solution to the Cauchy problem for the KP hierarchy in this classical case treated by algebraic and geometric means in, for instance, [9, 7, 8]. There is no reason why  $A$  should be a commutative algebra. We can therefore prove the well-posedness of the KP hierarchy posed on the algebra of quaternions that appears in [16]. Very general non-commutative algebras have been considered in [4] and [13]. Our main theorems imply well-posedness also in these cases, as long as we start with diffeological algebras  $A$ .

Let us finish with two questions: First, What about the Kadomtsev-Petviashvili equations? Our theory, in fact, implies smooth existence theorems for them. We present our results on this issue in [12] and in more depth, in [13]. Second, What about other hierarchies such as the Gelfand-Dickey flows? We are investigating this issue and we will present our results elsewhere. Finally, one can wonder whether the algebraic deformation described in [4] has a pending diffeological realization or a generalization in other settings.

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A. E. R.: DEPARTMENT OF MATHEMATICS, FAMAF, UNIVERSIDAD NACIONAL DE CORDOBA, MEDINA ALLENDE, CIUDAD UNIVERSITARIA , 5000-CORDOBA, ARGENTINA  
*Email address:* [aeslami@famaf.unc.edu.ar](mailto:aeslami@famaf.unc.edu.ar)

UNIV. ANGERS, CNRS, LAREMA, SFR MATHSTIC, F-49000 ANGERS, FRANCE, AND, LYCÉE JEANNE D'ARC, AVENUE DE GRANDE BRETAGNE,, 63000 CLERMONT-FERRAND, FRANCE  
*Email address:* [magnot@math.cnrs.fr](mailto:magnot@math.cnrs.fr)

DEPARTAMENTO DE MATEMÁTICA Y CIENCIA DE LA COMPUTACIÓN, UNIVERSIDAD DE SANTIAGO DE CHILE (USACH), CASILLA 307 CORREO 2, SANTIAGO, CHILE  
*Email address:* [enrique.reyes@usach.cl](mailto:enrique.reyes@usach.cl) ; [e\\_g.reyes@yahoo.ca](mailto:e_g.reyes@yahoo.ca)

UNIV. ANGERS, CNRS, LAREMA, SFR MATHSTIC, F-49000 ANGERS, FRANCE  
*Email address:* [volodya@univ-angers.fr](mailto:volodya@univ-angers.fr)