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An optimal transport approach for the multiple quantile hedging problem

Cyril Bénézet*, Jean-François Chassagneux[†], Mohan Yang[‡]

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Abstract

We consider the multiple quantile hedging problem, which is a class of partial hedging problems containing as special examples the quantile hedging problem (Föllmer & Leukert 1999) and the PnL matching problem (introduced in Bouchard & Vu 2012). In complete non-linear markets, we show that the problem can be reformulated as a kind of Monge optimal transport problem. Using this observation, we introduce a Kantorovitch version of the problem and prove that the value of both problems coincide. In the linear case, we thus obtain that the multiple quantile hedging problem can be seen as a semi-discrete optimal transport problem, for which we further introduce the dual problem. We then prove that there is no duality gap, allowing us to design a numerical method based on SGA algorithms to compute the multiple quantile hedging price.

*Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), ENSIIE, UMR CNRS 8071, IBGBI 23 Boulevard de France, 91037 Évry Cedex, France. Email: cyril.benezet@ensiie.fr.

[†]Université Paris Cité, Laboratoire de Probabilités, Statistiques et Modélisation, F-75013 Paris, France. Email: chassagneux@lpsm.paris.

[‡]ADIA, Email: mohan.yang@adia.ae.

1 Introduction

In this work, we introduce and study a novel class of stochastic target problems with controlled loss. In financial terms, these are partial hedging problems and encapsulate as particular examples the *quantile hedging problem* [1] and *PnL matching problem* [2]. In the quantile hedging problem, a financial agent looks for the minimal initial wealth which allows to replicate a given contingent claim with a 95% (say) probability of success. A natural extension is the so-called *PnL matching problem* introduced in [2]. In their article, the authors consider a finite number of quantile constraints targeting a given discrete distribution on the PnL. These partial hedging problems are in sharp contrast with the classical *super-hedging problem*, which imposes an almost sure constraint at the terminal date. Indeed, in complete linear markets, the later can be computed using only the risk neutral probability distribution \mathbb{Q} , while the former introduces an interplay between \mathbb{Q} and the physical probability distribution \mathbb{P} , as the quantile constraints are naturally formulated under \mathbb{P} . One motivation behind partial hedging is, when the option seller is willing to take on some risk, to be able to sell the option at a lower price than the super-replication price. In incomplete markets, this allows then to define an alternative pricing notion, complementary to utility indifference pricing. The target distribution can be seen as a pricing-time risk profile and could be easier to define than an utility function for risk management purposes.

Various mathematical methods have been considered to solve these partial hedging problems since the pioneering work [1]. An important contribution was made in [3] by interpreting this problem as a stochastic target problem on an extended state space in order to recover time-consistency, and obtaining a dynamic programming principle and a Partial Differential Equation (PDE) characterization. They applied this approach to the quantile hedging problem in particular and, in complete linear markets, were able to obtain a dual formulation for the quantile hedging price. The non-Markovian case is investigated in [4] and leads to the concept of Backward Stochastic Differential Equation (BSDE) with weak terminal condition. Other theoretical works have focused on extending the framework by considering jump processes [5], american type constraints [6, 7] [8], optimal book liquidation [9], other type of risk constraints see e.g. [10, 11]. We should also mention that it has triggered a new literature on the concept of BSDEs constrained in law [12, 13]. Few numerical methods have been proposed, whether relying on the PDE approach, see e.g. [2, 14] or on a dual characterization available in some cases [6].

We focus here on the case of multiple quantile hedging constraints, close to the *PnL matching problem* introduced in [2]. We work in a complete market but consider a non-linear dynamics for the wealth process of the trader. We define the *multiple quantile hedging price* as the minimal initial wealth allowing to reach at terminal time the multiple constraints in law, modeled by a probability distribution μ specified at pricing time. Quantile hedging and PnL matching, that have been already studied, are particular instances of this problem. Our first goal is to characterize this price. However, we do not follow the methods previously introduced in the literature. We first recast the *multiple quantile hedging problem* as a classical control problem involving the minimisation of a

non-linear expectation over a tailored class of random variables. We then make the new and crucial observation that this problem resembles an optimal transportation problem ‘à la Monge’ up to considering a non-linear expectation instead of a classical one and a target set of dominating probabilities instead of a given target probability. We then naturally introduce a ‘Kantorovitch version’ for this ‘Monge problem’. Our first main result is to show that the values of these two problems coincide. We then prove that, up to considering non-linear expectation, the multiple quantile hedging price is the minimal value associated to the ‘transport’ of the underlying probability \mathbb{P} to the target distribution μ . We prove these results in a Brownian setting with a target probability distribution μ having finite and discrete support. We then consider a more classical linear framework. In this case, the multiple quantile hedging price is the value of a semi-discrete optimal transport problem [15, Chapter 5]. We then naturally profit from this observation by considering the dual formulation of the problem and proving that, in our setting, duality does hold. The dual characterization that we obtain is new, in particular for the PnL matching problem. This characterization eventually allows us to derive a numerical method for the multiple quantile hedging problem based on stochastic gradient methods.

The rest of the paper is organized as follows. In Section 2, we first rigorously introduce the *multiple quantile hedging problem*. We present our main examples of *multiple quantile hedging problem*, in particular the PnL distribution hedging problem, that we solve in a linear setting. We then prove some key reformulation of the price in a non-linear setting including Monge and Kantorovitch like formulation. Section 3 focuses then on a linear framework in which we obtain our main duality result. We conclude by presenting a numerical algorithm based on the dual representation and demonstrate its efficiency in practice.

Notations that will be used throughout the paper:

- If $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ is a filtered probability space and $(E, |\cdot|)$ a normed space, we define $\mathcal{H}^2(\mathbb{F}, \mathbb{P}; E)$ as the set of progressively measurable processes $U : \Omega \times [0, T] \rightarrow E$ with $T > 0$ fixed satisfying

$$\mathbb{E} \left[\int_0^T |U_t|^2 dt \right] < +\infty,$$

and $\mathcal{S}^2(\mathbb{F}, \mathbb{P}; E)$ as the set of processes $U : \Omega \times [0, T] \rightarrow E$ continuous and adapted such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |U_t|^2 \right] < +\infty.$$

- Given a measurable space (E, \mathcal{E}) , we denote by $\mathcal{P}(E)$ the set of probability distributions on E .

- For two probability measures μ, ν on \mathbb{R} , $\nu \geq \mu$ denotes the *first order stochastic dominance*, i.e

$$\nu([\cdot, \infty)) =: F_\nu \geq F_\mu := \mu([\cdot, \infty)) \text{ on } \mathbb{R}.$$

- For all $N \geq 1$, we set $\Delta^N := \{x \in \mathbb{R}^N \mid x_1 \leq \dots \leq x_N\}$, $\Delta_+^N := \{x \in \mathbb{R}^N \mid 0 \leq x_1 \leq \dots \leq x_N\}$, and $\mathfrak{Q}^N = \{q \in \mathbb{R}^{N+1} \mid q^1 = 1 \geq \dots \geq q^\ell \geq \dots \geq q^{N+1} = 0\}$.
- For two metric spaces E, E' , $\mathbf{Lip}(E, E')$ is the set of Lipschitz continuous function from E to E' .

For $S = \{s_1 < \dots < s_N\}$ a finite subset of \mathbb{R} , a σ -alegra \mathcal{A} and a measure $\mu \in \mathcal{P}(S)$, we consider:

- $\mathcal{P}_\mu(\mathcal{A})$: the set of \mathcal{A} -measurable random vector $(P^n)_{n=1}^N$ such that $\sum_{n=1}^N P^n \delta_{s_n} \in \mathcal{P}(S)$ and $\sum_{n=1}^N \mathbb{E}[P^n] \delta_{s_n} = \mu$.
- $\mathcal{P}_\mu^+(\mathcal{A}) = \cup_{\nu \in \mathcal{P}(S), \nu \geq \mu} \mathcal{P}_\nu(\mathcal{A})$.
- $\mathcal{Q}_\mu(\mathcal{A})$: the set of \mathcal{A} -measurable random vector $(Q^n)_{n=1}^{N+1}$ such that Q is valued in \mathfrak{Q}^N and such that $\mathbb{E}[Q^n] = F_\mu(s_n)$, $1 \leq n \leq N$.
- $\mathcal{Q}_\mu^+(\mathcal{A}) = \cup_{\nu \in \mathcal{P}(S), \nu \geq \mu} \mathcal{Q}_\nu(\mathcal{A})$.

2 Multiple quantile hedging problem

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space supporting a m -dimensional Brownian Motion W , where m is a positive integer. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the natural \mathbb{P} -augmented filtration of W . In the sequel, we work with a finite time horizon $T > 0$.

2.1 Problem formulation

We consider controlled processes of the following form: For $y \in \mathbb{R}$ and $Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m)$,

$$Y_t^{y,Z} = y - \int_0^t \mathfrak{f}(s, Y_s, Z_s) ds + \int_0^t Z_s dW_s, \quad t \in [0, T], \quad (2.1)$$

where $(\mathfrak{f}(t, \cdot))_{t \in [0, T]}$ is a progressively measurable process taking values in $\mathbf{Lip}(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ and such that $\mathfrak{f}(t, 0, 0) \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R})$. Note that the assumptions on \mathfrak{f} guarantee that the stochastic differential equation (SDE) (2.1) admits a unique solution $Y \in \mathcal{S}^2(\mathbb{F}, \mathbb{P}; \mathbb{R})$.

We financially interpret the above controlled process as the wealth process of an investor with initial wealth y and an auto-financing admissible strategy based on risky financial assets. The Brownian motion then represents the underlying financial risk. In the whole paper, we shall follow this intuition but we will stay at this abstract level of description. We refer to e.g. [16] for classical examples of non-linear specification of the wealth process.

A key result in this context is the completeness of the market, i.e. the existence and uniqueness of solutions to BSDEs with a prescribed terminal value $\xi \in \mathcal{L}^2(\mathcal{F}_T)$, see e.g. [16]. More specifically, there exists a unique $(\mathcal{Y}_0, \mathcal{Z}) \in \mathbb{R} \times \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m)$ such that $Y_T^{\mathcal{Y}_0, \mathcal{Z}} = \xi$. In this case, we define $\mathcal{Y}_t := Y_t^{\mathcal{Y}_0, \mathcal{Z}}$ for all $t \in [0, T]$, so that $(\mathcal{Y}, \mathcal{Z})$ is the solution to the BSDE with driver \mathfrak{f} and terminal condition ξ , namely

$$\mathcal{Y}_t = \xi + \int_t^T \mathfrak{f}(s, \mathcal{Y}_s, \mathcal{Z}_s) ds - \int_t^T \mathcal{Z}_s dW_s, \quad t \in [0, T]. \quad (2.2)$$

To insist on the dependence upon the terminal condition ξ , we shall sometimes denote the solution of the above BSDE by $(\mathcal{Y}_t[\xi], \mathcal{Z}_t[\xi])_{0 \leq t \leq T}$.

In the next section, we will consider the following linear setup for \mathfrak{f} .

Assumption 2.1. *There exists two bounded progressively measurable stochastic processes (\mathbf{a}, \mathbf{b}) valued in $\mathbb{R} \times \mathbb{R}^m$ such that*

$$\mathfrak{f}(t, y, z) := \mathbf{a}_t y + \mathbf{b}_t^\top z, \quad \text{for } (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m. \quad (2.3)$$

Remark 2.1. *Under Assumption 2.1, it is well known that $\mathcal{Y}_0[\xi]$ for $\xi \in \mathcal{L}^2(\mathcal{F}_T)$ rewrites as an expectation, see e.g. Proposition 2.2 in [16]. More precisely, let us introduce the process Γ solution to*

$$\Gamma_t = 1 + \int_0^t \Gamma_s \mathbf{a}_s ds + \int_0^t \Gamma_s \mathbf{b}_s^\top dW_s, \quad t \in [0, T], \quad (2.4)$$

which satisfies, for any $p \geq 1$, to

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\Gamma_t|^p \right] \leq C_p. \quad (2.5)$$

Then, one has, for any $\xi \in \mathcal{L}^2(\mathcal{F}_T)$,

$$\mathcal{Y}_0[\xi] = \mathbb{E}[\Gamma_T \xi]. \quad (2.6)$$

The financial interpretation here is that Γ_T represents the pricing kernel and $\mathcal{Y}_0[\xi]$ the replication price of the European contingent claim ξ .

2.1.1 Definition

Let N be a positive integer and $\mathcal{S} := \{1, \dots, N\}$.

We consider a $(\mathcal{F}_T, \mathcal{B}(\mathbb{R}^N))$ -measurable map

$$\Omega \ni \omega \mapsto (G^n(\omega), 1 \leq n \leq N) \in \mathbb{R}^N.$$

We assume that $\mathcal{S} \ni n \mapsto G^n \in \mathbb{R}$ is \mathbb{P} -almost surely non-decreasing and that there exists $\mathfrak{C} \in \mathcal{L}^2(\mathcal{F}_T)$ such that

$$\max_{1 \leq n \leq N} |G^n| = \max(|G^1|, |G^N|) \leq \mathfrak{C}, \quad \mathbb{P}\text{-almost surely.} \quad (2.7)$$

We refer to Section 2.1.2 below for motivating examples and discussion of possible extensions of our setting.

For $\omega \in \Omega$, we denote by $\Psi(\omega, \cdot)$ the generalized inverse of $G(\omega)$, defined by

$$\Omega \times \mathbb{R} \ni (\omega, y) \mapsto \Psi(\omega, y) := \max \{1 \leq n \leq N \mid y \geq G^n(\omega)\} \in \bar{\mathcal{S}}, \quad (2.8)$$

where $\bar{\mathcal{S}} := \mathcal{S} \cup \{-\infty\}$, and using the convention $\max \emptyset = -\infty$. We naturally extend G on $\bar{\mathcal{S}}$ by setting $G^{-\infty} := -\infty$, \mathbb{P} -almost surely. Note that Ψ is $(\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}))$ -measurable. We observe that Ψ is càdlàg, upper bounded by N and satisfies

$$\Psi(\omega, y) = -\infty \iff y < G^1(\omega), \text{ and} \quad (2.9)$$

$$G^{\Psi(\omega, \cdot)}(\omega) \leq \text{Id}(\cdot) \text{ on } \mathbb{R}, \quad \text{Id}(\cdot) \leq \Psi \circ G(\omega) \text{ on } \bar{\mathcal{S}}. \quad (2.10)$$

We consider a probability measure $\mu \in \mathcal{P}(\mathcal{S})$, such that $\mu(\{n\}) > 0$ for all $1 \leq n \leq N$, i.e. the support of μ is exactly \mathcal{S} .

We now introduce the *multiple quantile hedging problem*. The set of multiple quantile superhedging prices is denoted by

$$\mathfrak{H}(\mu) := \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \mathbb{P}(Y_T^{y,Z} \geq G^n) \geq F_\mu(n), 1 \leq n \leq N \right\}. \quad (2.11)$$

We then classically define the *multiple quantile hedging price* as

$$\mathcal{V}_{\text{WH}}(\mu) := \inf \mathfrak{H}(\mu). \quad (2.12)$$

As mentioned in the introduction, the multiple quantile hedging problem is general enough to encapsulate in particular the quantile hedging problem and the PnL matching problem, that we review in the next section.

2.1.2 Examples

Two cases of the above multiple constraints problem are of particular interest.

- For $N = 1$, one gets the super-replication problem (see e.g. [17], [18]): $G^1 = \xi$ for some $\xi \in \mathcal{L}^2(\mathcal{F}_T)$, $\mu = \delta_1$, and then

$$\mathcal{V}_{\text{WH}}(\mu) = \inf \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \mathbb{P}(Y_T^{y,Z} \geq \xi) \geq 1 \right\}.$$

- For $N = 2$, one recovers the classical quantile hedging problem (see e.g. [1], [3], [14]) and we denote \mathcal{V}_{WH} by \mathcal{V}_{QH} in this case: $G^1 = 0$, $G^2 = \xi$ for some $0 \leq \xi \in \mathcal{L}^2(\mathcal{F}_T)$, $\mu = (1-p)\delta_1 + p\delta_2$ for $p \in (0, 1)$, and then

$$\mathcal{V}_{\text{QH}}(\mu) = \inf \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), Y_T^{y,Z} \geq 0, \mathbb{P}\text{-a.s.}, \text{ and } \mathbb{P}(Y_T^{y,Z} \geq \xi) \geq p \right\}. \quad (2.13)$$

Let us now consider another example with more structure and which is inspired by the article [2].

Example 2.1 (PnL distribution hedging). *Let $\gamma : \mathcal{S} \ni n \mapsto \gamma(n) \in \mathbb{R}$ an increasing map. For latter use, we denote by $\gamma^{-1} : \gamma(\mathcal{S}) \rightarrow \mathcal{S}$ the inverse mapping of γ . We are given a contingent claim $\xi \in \mathcal{L}^2(\mathcal{F}_T)$ and a probability distribution $\mu \in \mathcal{P}(\mathcal{S})$. The map G is now defined by*

$$G^n := \xi + \gamma(n), \quad \text{for } n \in \mathcal{S}. \quad (2.14)$$

According to (2.12), we have

$$\begin{aligned} \mathcal{V}_{\text{WH}}(\mu) &= \inf \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \mathbb{P}(Y_T^{y,Z} \geq G^n) \geq F_\mu(n), \forall n \in \mathcal{S} \right\} \\ &= \inf \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \mathbb{P}(Y_T^{y,Z} - \xi \geq \gamma(n)) \geq F_\mu(n), \forall n \in \mathcal{S} \right\} \\ &= \inf \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \mathcal{L}(Y_T^{y,Z} - \xi) \geq \gamma_\# \mu \right\}. \end{aligned} \quad (2.15)$$

In (2.15), we observe that a constraint on the terminal PnL distribution, given by the net position $Y_T^{y,Z} - \xi$, is imposed using the probability distribution $\gamma_\# \mu$.

To conclude this section and to motivate the use of optimal transport tools, we are going to solve the above PnL distribution hedging problem in the following setting:

Assumption 2.2. 1. *Assumption 2.1 is in force.*

2. *The \mathbb{P} -distribution of Γ_T , denoted by $\mathcal{L}(\Gamma_T)$, is absolutely continuous with respect to the Lebesgue measure.*

The starting point to solve the problem anticipates slightly the results to come. Indeed, in the next section, see Proposition 2.2, we are going to prove

$$\mathcal{V}_{\text{WH}}(\mu) = \inf \left\{ y \in \mathbb{R} \mid \exists \chi \in \mathcal{T}_+(\mu), \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), Y_T^{y,Z} - \xi \geq \gamma(\chi) \right\} \quad (2.16)$$

$$= \inf_{\chi \in \mathcal{T}_+(\mu)} \mathbb{E}[\Gamma_T(\xi + \gamma(\chi))]. \quad (2.17)$$

where $\mathcal{T}_+(\mu) = \{\chi \in \mathcal{L}^2(\mathcal{F}_T) \mid \mathcal{P}(\mathcal{S}) \ni \chi_\# \mathbb{P} \geq \mu\}$. Intuitively, equation (2.16) is a lift at the level of random variable of the formulation given in terms of law in (2.15). Once this formulation is obtained, (2.17) comes from classical pricing result in complete market. We observe also that (2.17) resembles a Monge problem except that the target is not exactly the distribution μ but possibly any dominating measure in the sense of \geq , the first order stochastic dominance. To solve this problem, we first solve the more restrictive problem when the constraint is saturated, namely when $\chi_\# \mathbb{P} = \mu$. It is defined as follows. For $\mu \in \mathcal{P}(\mathcal{S})$, we set

$$\mathcal{V}_{\text{OT}}(\mu) := \inf_{\chi \in \mathcal{T}(\mu)} \mathbb{E}[\Gamma_T(\xi + \gamma(\chi))], \quad (2.18)$$

where $\mathcal{T}(\mu) = \{\chi \in \mathcal{L}^2(\mathcal{F}_T) \mid \chi_\# \mathbb{P} = \mu\}$.

Lemma 2.1. *Under Assumption 2.2, we have*

$$\mathcal{V}_{\text{OT}}(\mu) = \mathbb{E}[\Gamma_T \xi] - \frac{1}{2} \mathbb{E}[(\Gamma_T)^2] - \frac{1}{2} \int x^2 \gamma_{\#} \mu(dx) + \frac{1}{2} \mathcal{W}_2^2(\mathcal{L}(-\Gamma_T), \gamma_{\#} \mu).$$

In addition, there exists $\chi^ \in \mathcal{T}(\mu)$ such that*

$$\mathcal{V}_{\text{OT}}(\mu) = \mathbb{E}[\Gamma_T (\xi + \gamma(\chi^*))],$$

which writes explicitly $\chi^ = \gamma^{-1}(N_{\gamma_{\#} \mu}^{-1} \circ N_{\mathcal{L}(-\Gamma_T)}(-\Gamma_T))$. Here, $N_{\gamma_{\#} \mu}$ stands for the c.d.f. of the law $\gamma_{\#} \mu$ and $N_{\gamma_{\#} \mu}^{-1}$ its generalized inverse.*

Proof. We compute, starting from (2.18), as each $\chi \in \mathcal{T}(\mu)$ has law μ ,

$$\begin{aligned} \mathcal{V}_{\text{OT}}(\mu) &= \mathbb{E}[\Gamma_T \xi] + \inf_{\chi \in \mathcal{T}(\mu)} \mathbb{E}[\Gamma_T \gamma(\chi)] \\ &= \mathbb{E}[\Gamma_T \xi] + \inf_{\chi \in \mathcal{T}(\mu)} -\mathbb{E}[(-\Gamma_T) \gamma(\chi)] \\ &= \mathbb{E}[\Gamma_T \xi] + \frac{1}{2} \inf_{\chi \in \mathcal{T}(\mu)} (\mathbb{E}[(-\Gamma_T - \gamma(\chi))^2] - \mathbb{E}[(-\Gamma_T)^2] - \mathbb{E}[\gamma(\chi)^2]) \\ &= \mathbb{E}[\Gamma_T \xi] - \frac{1}{2} \mathbb{E}[(\Gamma_T)^2] - \frac{1}{2} \int x^2 \gamma_{\#} \mu(dx) + \frac{1}{2} \inf_{\chi \in \mathcal{T}(\mu)} \mathbb{E}[(-\Gamma_T - \gamma(\chi))^2]. \end{aligned}$$

Denote $\nu := \mathcal{L}(-\Gamma_T)$, we straightforwardly observe that

$$\inf_{\chi \in \mathcal{T}(\mu)} \mathbb{E}[(-\Gamma_T - \gamma(\chi))^2] \geq \inf_{X \sim \nu, Y \sim \gamma_{\#} \mu} \mathbb{E}[|X - Y|^2] = \mathcal{W}_2^2(\nu, \gamma_{\#} \mu). \quad (2.19)$$

Since $\mathcal{L}(-\Gamma_T)$ is absolutely continuous with respect to the Lebesgue measure, using Brenier's theorem (see for example [19, Theorem 5.20]), there exists an optimal transport map T from ν to $\gamma_{\#} \mu$, i.e. such that $T_{\#} \nu = \gamma_{\#} \mu$ and

$$\mathcal{W}_2^2(\nu, \gamma_{\#} \mu) = \int (x - T(x))^2 \nu(dx).$$

It is well-known, in our one-dimensional context, see for example [19, Remark 5.15], that such an optimal transport map is given by $x \mapsto T(x) = N_{\gamma_{\#} \mu}^{-1} \circ N_{\nu}(x)$. Defining $\chi^* := \gamma^{-1}(T(-\Gamma_T)) \in \mathcal{T}(\mu)$, we have

$$\mathcal{W}_2^2(\nu, \gamma_{\#} \mu) = \int |-x - T(-x)|^2 d\Gamma_{T\#} \mathbb{P}(x) = \mathbb{E}[(-\Gamma_T - \gamma(\chi^*))^2] = \inf_{\chi \in \mathcal{T}(\mu)} \mathbb{E}[(-\Gamma_T - \gamma(\chi))^2],$$

which concludes the proof. \square

We now make the following simple observation.

Lemma 2.2. *Under Assumption 2.2, for any $\nu, \mu \in \mathcal{P}(\mathcal{S})$ such that $\nu \geq \mu$, we have $\mathcal{V}_{\text{OT}}(\nu) \geq \mathcal{V}_{\text{OT}}(\mu)$.*

Proof. By Lemma 2.1, we have that

$$\begin{aligned}\mathcal{V}_{\text{OT}}(\nu) - \mathcal{V}_{\text{OT}}(\mu) &= \mathbb{E}\left[\Gamma_T \left(\gamma^{-1}(N_{\gamma\sharp\nu}^{-1}(N_{\mathcal{L}(-\Gamma_T)}(-\Gamma_T))) - \gamma^{-1}(N_{\gamma\sharp\mu}^{-1}(N_{\mathcal{L}(-\Gamma_T)}(-\Gamma_T))) \right)\right] \\ &= \mathbb{E}\left[\Gamma_T \left(N_{\gamma\sharp\nu}^{-1}(U) - N_{\gamma\sharp\mu}^{-1}(U) \right)\right],\end{aligned}$$

with $U := N_{\mathcal{L}(-\Gamma_T)}(-\Gamma_T)$ is uniformly distributed on $[0, 1]$ as Γ_T has no atom. Since $\nu \geq \mu$ and γ is increasing, we have $\gamma\sharp\nu \geq \gamma\sharp\mu$ and $N_{\gamma\sharp\nu}^{-1} \geq N_{\gamma\sharp\mu}^{-1}$, hence the result, as $\Gamma_T > 0$, \mathbb{P} -almost surely. \square

Proposition 2.1. *Under Assumption 2.2, for all $\mu \in \mathcal{P}(\mathcal{S})$, we have*

$$\mathcal{V}_{\text{WH}}(\mu) = \mathcal{V}_{\text{OT}}(\mu).$$

Proof. Since $\mathcal{T}(\mu) \subset \mathcal{T}_+(\mu)$, we have $\mathcal{V}_{\text{WH}}(\mu) \leq \mathcal{V}_{\text{OT}}(\mu)$.

Conversely, let $(\chi_n, n \geq 1)$, a sequence of elements of $\mathcal{T}_+(\mu)$ such that $\mathcal{V}_{\text{WH}}(\mu) + \frac{1}{n} \geq \mathbb{E}[\Gamma_T(\xi + \gamma(\chi_n))]$. We have $\mathbb{E}[\Gamma_T(\xi + \gamma(\chi_n))] \geq \mathcal{V}_{\text{OT}}((\chi_n)\sharp\mathbb{P}) \geq \mathcal{V}_{\text{OT}}(\mu)$, thanks to Lemma 2.2. We then have $\mathcal{V}_{\text{WH}}(\mu) + \frac{1}{n} \geq \mathcal{V}_{\text{OT}}(\mu)$ and sending n to infinity gives the reverse inequality. \square

Remark 2.2. *i) In this case, the multiple quantile hedging price is thus the replicating price of the payoff $\xi + \gamma(\chi^*)$ given in Lemma 2.1.*

ii) For the PnL distribution hedging problem, one could follow the above steps to obtain the result for more general probability density $\mu \in \mathcal{P}(\mathbb{R})$, general $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ such that $\gamma\sharp\mu \in \mathcal{P}_2(\mathbb{R})$ and $G^x(\omega) = \xi(\omega) + \gamma(x)$. The study in our setting of the general weak hedging problem with an arbitrary, possibly uncountable, number of quantile constraints, is however left for further research.

2.2 The Monge representation

We now turn to the systematic study of the multiple quantile hedging problem defined in (2.12). In particular, we first prove that the multiple quantile hedging price can be obtained as an optimisation of non-linear expectation.

Proposition 2.2. *Under our standing assumptions, the following equivalent formulations hold for the multiple quantile hedging price*

$$\mathcal{V}_{\text{WH}}(\mu) = \mathcal{V}_{\text{RM}}(\mu) := \inf_{\chi \in \mathcal{T}_+(\mu)} \mathcal{Y}_0[G^\chi]. \quad (2.20)$$

and

$$\mathcal{V}_{\text{WH}}(\mu) = \inf \tilde{\mathfrak{H}}(\mu), \quad \text{with } \tilde{\mathfrak{H}}(\mu) := \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \Psi(Y_T^{y,Z})\sharp\mathbb{P} \geq \mu \right\}, \quad (2.21)$$

recall (2.8).

Proof. 1. We first prove (2.20). Set $\hat{\mathfrak{H}}(\mu) := \{\mathcal{Y}_0[G^\chi] \mid \chi \in \mathcal{T}_+(\mu)\}$.

1.a Let $\chi \in \mathcal{T}_+(\mu)$ and set $\xi := G^\chi$. Observe that, since G is non-decreasing, for each $n \in \mathcal{S}$, we have $\{\chi \geq n\} \subset \{\xi \geq G^n\}$. Thus $\mathbb{P}(\xi \geq G^n) \geq \mathbb{P}(\chi \geq n) \geq F_\mu(n)$ as $\chi \# \mathbb{P} \geq \mu$. Using (2.7), we have that $\xi \in \mathcal{L}^2(\mathcal{F}_T)$ and then the BSDE $(\mathcal{Y}[\xi], \mathcal{Z}[\xi])$, recall (2.2), is well posed. We deduce $\mathcal{Y}_0[\xi] \in \mathfrak{H}(\mu)$. Thus $\hat{\mathfrak{H}}(\mu) \subset \mathfrak{H}(\mu)$ and hence $\mathcal{V}_{\text{WH}}(\mu) \leq \mathcal{V}_{\text{RM}}(\mu)$.

1.b Conversely, let $y \in \mathfrak{H}(\mu)$, so that there exists $Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m)$ s.t. $Y_T^{y,Z}$ satisfies to $\mathbb{P}(Y_T^{y,Z} \geq G^n) \geq F_\mu(n)$ for all $n \in \mathcal{S}$. We set $\chi := \Psi(Y_T^{y,Z})$ which is valued in \mathcal{S} , as $F_\mu(1) = 1$ and Ψ satisfies (2.9). Recalling (2.10), we have that $Y_T^{y,Z} \geq G^n$ implies $\chi = \Psi(Y_T^{y,Z}) \geq \Psi \circ G^n \geq n$. Thus, $\mathbb{P}(\chi \geq n) \geq \mathbb{P}(Y_T^{y,Z} \geq G^n) \geq F_\mu(n)$ yielding $\chi \# \mathbb{P} \geq \mu$. Again from (2.10), we also observe that $G^\chi = G^{\Psi(Y_T^{y,Z})} \leq Y_T^{y,Z}$ leading to $\mathcal{Y}_0[G^\chi] \leq \mathcal{Y}_0[Y_T^{y,Z}] = y$ by the comparison theorem for Lipschitz BSDEs. We then obtain $\mathcal{V}_{\text{RM}}(\mu) \leq y$, hence $\mathcal{V}_{\text{RM}}(\mu) \leq \mathcal{V}_{\text{WH}}(\mu)$.

2. We now turn to the equivalent formulation (2.21). Set $\tilde{v}(\mu) := \inf \tilde{\mathfrak{H}}(\mu)$. From the step 1.b above, we obtain that $\mathcal{V}_{\text{WH}}(\mu) \geq \tilde{v}(\mu)$. Now, let $y \in \tilde{\mathfrak{H}}(\mu)$, then there exists Z , s.t. $\chi := \Psi(Y_T^{y,Z})$ satisfies $\chi \# \mathbb{P} \geq \mu$. Using (2.10), we observe that $Y_T^{y,Z} \geq G^\chi$. In particular on $\{\chi \geq n\}$, we have, since G is non-decreasing, that $Y_T^{y,Z} \geq G^n$. Thus $\mathbb{P}(Y_T^{y,Z} \geq G^n) \geq F_\mu(n)$ proving that $y \in \mathfrak{H}(\mu)$. \square

Remark 2.3. 1. The formulation (2.21) given in the previous proposition shows that the multiple quantile hedging problem is a particular instance of more generic weak stochastic target problem: Consider a set $\mathcal{M} \subset \mathcal{P}(\mathbb{R})$ and a right continuous non decreasing random function \mathfrak{D} and solve

$$\inf \left\{ y \in \mathbb{R} \mid \exists Z \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}; \mathbb{R}^m), \mathfrak{D}(Y_T^{y,Z}) \# \mathbb{P} \in \mathcal{M} \right\}. \quad (2.22)$$

The multiple quantile hedging problem corresponds to the case $\mathfrak{D} = \Psi$ and $\mathcal{M} := \{\nu \in \mathcal{P}(\mathcal{S}) \mid \nu \geq \mu\}$ for a given μ . In [4], the authors introduce a similar problem to (2.22) where $\mathcal{M} := \{\nu \in \mathcal{P}(\mathbb{R}) \mid \int \gamma \nu(d\gamma) \geq p\}$ for a given level $p \in \mathbb{R}$.

2. The formulation (2.21) opens the way to a dynamic approach to characterize the solution of the quantile hedging problems as a BSDE, as done in [4] in a special case or to a PDE characterization in a Markovian setting, as done in [3]. The extension of these results to our framework is left for further research.

Remark 2.4. 1. In the problem (2.20), “RM” stands for “Relaxed Monge”. Indeed, let Assumption 2.1 hold. Then \mathcal{Y}_0 is a linear operator, since $\mathcal{Y}_0[\xi] = \mathbb{E}[\Gamma_T \xi]$ where Γ_T is defined by (2.4), according to Remark 2.1. Then, one has

$$\mathcal{V}_{\text{RM}}(\mu) = \inf_{\chi \in \mathcal{T}_+(\mu)} \mathbb{E}[\Gamma_T G^\chi] = \inf_{\chi \in \mathcal{T}_+(\mu)} \int \Gamma_T(\omega) G^{\chi(\omega)}(\omega) \mathbb{P}(d\omega), \quad (2.23)$$

which can be interpreted as an “à la Monge” optimal transport problem with cost function

$$(\omega, n) \mapsto \Gamma_T(\omega) G^n(\omega).$$

We use the term ‘relaxed’ as the target distribution is not exactly μ , but can be any probability distribution (with support included in \mathcal{S}) stochastically dominating μ . The linear framework will be extensively studied in Section 3.

2. In connection with the previous point, we introduce the Monge Problem (MP). In our non-linear context, it reads:

$$\mathcal{V}_{\text{MP}}(\mu) := \inf_{\chi \in \mathcal{T}(\mu)} \mathcal{Y}_0[G^\chi]. \quad (2.24)$$

In the next section, we shall show that the values \mathcal{V}_{MP} and \mathcal{V}_{RM} are equal. This has already been observed in the specific context of Example 2.1, recall Proposition 2.1.

2.3 The Kantorovitch representation

Let us denote $\bar{\Omega} := \Omega \times \mathcal{S}$ and $\bar{\mathcal{F}}_T := \mathcal{F}_T \otimes \mathcal{F}(\mathcal{S})$, with $\mathcal{F}(\mathcal{S})$ the discrete sigma-algebra of \mathcal{S} . We define the following projections

$$\text{pr}_1 : \bar{\Omega} \ni (\omega, n) \mapsto \omega \in \Omega, \text{ and } \text{pr}_2 : \bar{\Omega} \ni (\omega, n) \mapsto n \in \mathcal{S}.$$

We then define the following set of couplings

$$\mathcal{C}_+(\mathbb{P}, \mu) := \{ \Pi \in \mathcal{P}(\bar{\Omega}) \mid (\text{pr}_1)_\# \Pi = \mathbb{P} \text{ and } (\text{pr}_2)_\# \Pi \geq \mu \}. \quad (2.25)$$

Using the previous notations, we introduce the Kantorovitch representation of the multiple quantile hedging price. We define

$$\mathcal{V}_{\text{KP}}(\mu) := \inf_{\Pi \in \mathcal{C}_+(\mathbb{P}, \mu)} \mathcal{Y}_0 \left[\sum_{n=1}^N G^n \rho^\Pi(\{n\}) \right], \quad (2.26)$$

where ρ^Π is obtained from the disintegration of the measure Π : for all $\Pi \in \mathcal{P}(\bar{\Omega})$, there exists a \mathcal{F}_T -measurable map $\rho^\Pi : \Omega \rightarrow \mathcal{P}(\mathcal{S})$ such that (see e.g. Villani [20, p.209]) $\Pi(d\omega, di) = \rho^\Pi(\omega, di) \mathbb{P}(d\omega) = \sum_{n=1}^N \rho^\Pi(\omega, \{n\}) \delta_n(di) \mathbb{P}(d\omega)$, i.e. the $\rho^\Pi(\{n\})$ are \mathcal{F}_T -measurable random variables in $[0, 1]$ such that $\sum_{n=1}^N \rho^\Pi(\{n\}) = 1$, \mathbb{P} -almost surely.

If additionally $\Pi \in \mathcal{C}_+(\mathbb{P}, \mu)$, one easily observes that $(\rho^\Pi(\{n\}), 1 \leq n \leq N) \in \mathcal{P}_\mu^+(\mathcal{F}_T)$, and conversely, any element $P \in \mathcal{P}_\mu^+(\mathcal{F}_T)$ induces a probability distribution $\Pi \in \mathcal{C}_+(\mathbb{P}, \mu)$, hence

$$\mathcal{V}_{\text{KP}}(\mu) = \inf_{P \in \mathcal{P}_\mu^+(\mathcal{F}_T)} \mathcal{Y}_0 \left[\sum_{n=1}^N G^n P^n \right]. \quad (2.27)$$

Remark 2.5. In the linear setting of Remark 2.4, (2.20) writes

$$\mathcal{V}_{\text{RM}}(\mu) = \inf_{\chi \in \mathcal{T}_+(\mu)} \int \Gamma_T(\omega) G^{\chi(\omega)}(\omega) d\mathbb{P}(\omega),$$

and the associated Kantorovitch Problem (KP) (2.26)

$$\mathcal{V}_{\text{KP}}(\mu) = \inf_{\Pi \in \mathcal{C}_+(\mathbb{P}, \mu)} \int \Gamma(\omega) G^n(\omega) d\Pi(\omega, n).$$

Moreover, using disintegration $\Pi(d\omega, di) = \rho^\Pi(\omega, di)\mathbb{P}(d\omega) = \sum_{n=1}^N \rho^\Pi(\omega, \{n\})\delta_n(di)\mathbb{P}(d\omega)$ as above, we obtain

$$\begin{aligned} \mathcal{V}_{\text{KP}}(\mu) &= \inf_{\Pi \in \mathcal{C}_+(\mathbb{P}, \mu)} \int \Gamma_T(\omega) \sum_{n=1}^N G^n(\omega) \rho^\Pi(\omega, \{n\}) \mathbb{P}(d\omega) \\ &= \inf_{\Pi \in \mathcal{C}_+(\mathbb{P}, \mu)} \mathbb{E} \left[\Gamma_T \sum_{n=1}^N G^n \rho^\Pi(\{n\}) \right], \end{aligned}$$

which motivates our definition (2.26). Proposition 2.5 shows that it is indeed the natural non-linear counterpart of the previous relation.

The following proposition, shows, that in our setting the target constraint is saturated in the Kantorovitch problem, namely, one can restrict to couplings satisfying to $(\text{pr}_2)_\# \Pi = \mu$.

Proposition 2.3. *The Kantorovitch problem (2.27) writes*

$$\mathcal{V}_{\text{KP}}(\mu) = \inf_{P \in \mathcal{P}_\mu(\mathcal{F}_T)} \mathcal{Y}_0 \left[\sum_{n=1}^N G^n P^n \right], \quad (2.28)$$

$$= \inf_{Q \in \mathcal{Q}_\mu(\mathcal{F}_T)} \mathcal{Y}_0 \left[G^1 + \sum_{n=2}^N Q^n (G^n - G^{n-1}) \right]. \quad (2.29)$$

Proof. 1. From (2.27) and the definition of $\mathcal{Q}_\mu^+(\mathcal{F}_T)$, we straightforwardly deduce

$$\mathcal{V}_{\text{KP}}(\mu) = \inf_{Q \in \mathcal{Q}_\mu^+(\mathcal{F}_T)} \mathcal{Y}_0 \left[G^1 + \sum_{n=2}^N Q^n (G^n - G^{n-1}) \right]. \quad (2.30)$$

For each $Q \in \mathcal{Q}_\mu^+(\mathcal{F}_T)$, we are going to build a $\check{Q} \in \mathcal{Q}_\mu(\mathcal{F}_T)$ such that $\check{Q}^n \leq Q^n$ for $n \in \{1, \dots, N+1\}$. Since $n \mapsto G^n$ is non-decreasing by assumption, we get

$$G^1 + \sum_{n=2}^N Q^n (G^n - G^{n-1}) \geq G^1 + \sum_{n=2}^N \check{Q}^n (G^n - G^{n-1}) \quad (2.31)$$

The proof is then concluded by invoking the comparison theorem for BSDE.

2. In this step, we build by induction on the component $1 \leq n \leq N+1$, for $Q \in \mathcal{Q}_\mu^+(\mathcal{F}_T)$, the \check{Q} used above. In preparation, we denote $\nu \in \mathcal{P}(\mathcal{S})$ defined by $F_\nu(n) := \mathbb{E}[Q^n] \geq F_\mu(n)$, $n \in \{1, \dots, N\}$. With a slight abuse of notations, let $(Q_t)_{t \in [0, T]}$ be the martingale defined by $Q_t = \mathbb{E}_t[Q]$. We observe that it is valued in \mathcal{Q}^N (by convexity) and satisfies

$Q_0^n = F_\nu(n)$, $n \in \{1, \dots, N\}$. We now build a martingale $(\hat{Q}_t)_{t \in [0, T]}$ s.t. $Q_T^n \geq \hat{Q}_T^n$ and such that $\mathbb{E}[\hat{Q}_T^n] = \hat{Q}_0^n = F_\mu(n)$, for all $n \in \{1, \dots, N\}$. The proof is done by induction on the components and we introduce the following induction hypothesis for $n \in \{1, \dots, N+1\}$:

- H_n : for $j \in \{n, \dots, N+1\}$, there are martingales $(\hat{Q}_t^j)_{t \in [0, T]}$, such that $\hat{Q}^j \leq Q^j$,

$$\hat{Q}_t^n \geq \dots \geq \hat{Q}_t^j \geq \dots \geq \hat{Q}_t^{N+1} = 0, \quad t \in [0, T],$$

and for $j \in \{n+1, \dots, N\}$, $\mathbb{E}[\hat{Q}_t^j] = F_\mu(j)$.

- For component $n = N+1$, H_{N+1} holds trivially by setting $\hat{Q}_t^{N+1} := 0$, $t \in [0, T]$;
- Assume now that, for $n \in \{1, \dots, N\}$, H_{n+1} holds. Then, set

$$\tilde{Q}_t^n := Q_t^n - Q_0^n + F_\mu(n), \quad t \in [0, T], \quad \text{and} \quad \tau_n := \inf \left\{ t \geq 0 \mid \tilde{Q}_t^n = \hat{Q}_t^{n+1} \right\} \wedge T.$$

Observe that $\tau_n > 0$ since $\tilde{Q}_0^n = F_\mu(n) > F_\mu(n+1) = \hat{Q}_0^{n+1}$, by induction hypothesis. We finally define

$$\hat{Q}_t^n := \tilde{Q}_{t \wedge \tau_n}^n + \left(\hat{Q}_t^{n+1} - \hat{Q}^{n+1}(\tau_n) \right) \mathbf{1}_{\{t > \tau_n\}}, \quad t \in [0, T].$$

It is a martingale and we thus have that $\mathbb{E}[\hat{Q}_t^n] = F_\mu(\gamma_n)$. Moreover, by construction, we observe that

$$\begin{aligned} \text{on } \{t \leq \tau_n\} : \hat{Q}_t^{n+1} &\leq \hat{Q}_t^n = \tilde{Q}_t^n \leq Q_t^n, \\ \{t > \tau_n\} : \hat{Q}_t^n &= \hat{Q}_t^{n+1} \leq Q_t^{n+1}, \end{aligned}$$

where we use the induction hypothesis for the last inequality. This allows us to obtain H_n .

We then simply set $\check{Q} := \hat{Q}_T$ to conclude the proof of this step. \square

We conclude this section by observing that there exists an optimal coupling in the KP problem above, under a convexity assumption.

Proposition 2.4. *Assume $\xi \mapsto \mathcal{V}_0[\xi]$ is convex, then there exists a $\bar{P} \in \mathcal{P}_\mu(\mathcal{F}_T)$, s.t.*

$$\mathcal{V}_{\text{KP}}(\mu) = \mathcal{V}_0 \left[\sum_{n=1}^N G^n \bar{P}^n \right].$$

Proof. The set $\mathcal{P}_\mu(\mathcal{F}_T)$ is convex and strongly closed in the separable Hilbert space $\mathcal{L}^2(\mathcal{F}_T, \mathbb{R}^N)$, hence it is also weakly closed. We consider a minimizing sequence (P^k) for the problem \mathcal{V}_{KP} . The variables P^k are uniformly bounded and thus there exists subsequence (still denoted (P^k)) which converges weakly to some $\bar{P} \in \mathcal{P}_\mu(\mathcal{F}_T)$. Invoking Mazur's Lemma, we know that there exists a strongly convergent sequence using convex

combination of (P^k) , namely: $\tilde{P}^k = \sum_{j=k}^{K_k} \lambda_j^k P^j$ with $\sum_{j=k}^{K_k} \lambda_j^k = 1$ and $\tilde{P}^k \rightarrow \bar{P}$. By convexity, we also have

$$\mathcal{Y}_0\left[\sum_{n=1}^N G^n(\tilde{P}^k)^n\right] \leq \sum_{j=k}^{K_k} \lambda_j^k \mathcal{Y}_0\left[\sum_{n=1}^N G^n(P^j)^n\right] \quad (2.32)$$

By strong convergence $\mathcal{Y}_0\left[\sum_{n=1}^N G^n(\tilde{P}^k)^n\right] \rightarrow \mathcal{Y}_0\left[\sum_{n=1}^N G^n(\bar{P})^n\right]$. Moreover, we know that $\mathcal{Y}_0\left[\sum_{n=1}^N G^n(P^j)^n\right] \rightarrow \mathcal{V}_{\text{KP}}(\mu)$ and thus $\sum_{j=k}^{K_k} \lambda_j^k \mathcal{Y}_0\left[\sum_{n=1}^N G^n(P^j)^n\right] \rightarrow \mathcal{V}_{\text{KP}}(\mu)$. We then obtain $\mathcal{Y}_0\left[\sum_{n=1}^N G^n(\bar{P})^n\right] \leq \mathcal{V}_{\text{KP}}(\mu)$ from (2.32), which concludes the proof. \square

2.4 Equality between KP and MP

In this section, we will show that the Monge Problem and the Kantorovitch Problem, respectively introduced in (2.24) and (2.26), coincide. This will allow us to obtain the representation of the multiple quantile hedging price.

In preparation of the proofs, we introduce an approximation of $\mathcal{Y}[\cdot]$ which will also be useful: For $s \in [0, T]$ a terminal time and $\zeta \in \mathcal{L}^2(\mathcal{F}_s)$, we denote $(\mathcal{Y}_t^s[\zeta], \mathcal{Z}_t^s[\zeta])_{t \in [0, s]}$ the solution to

$$Y_t = \zeta + \int_t^s f(u, Y_u, Z_u) du - \int_t^s Z_u dW_u, \quad t \in [0, s]. \quad (2.33)$$

With this notation, we have in particular that $\mathcal{Y}_t[\xi] = \mathcal{Y}_t^T[\xi] = \mathcal{Y}_t^s[\mathcal{Y}_s^T[\xi]] = \mathcal{Y}_t^s[\mathcal{Y}_s[\xi]]$ for $\xi \in \mathcal{L}^2(\mathcal{F}_T)$ and $0 \leq t \leq s \leq T$.

The proof of the next lemma, which is based on classical arguments and estimates of BSDEs solution is postponed to the end of the section.

Lemma 2.3. *For $\xi \in \mathcal{L}^2(\mathcal{F}_T)$ and $0 < \epsilon < T$, we have that*

$$|\mathcal{Y}_0^{T-\epsilon}[\mathbb{E}[\xi | \mathcal{F}_{T-\epsilon}]] - \mathcal{Y}_0^T[\xi]| \leq C_{\|\xi\|_{\mathcal{L}^2}} \epsilon^{\frac{1}{4}}. \quad (2.34)$$

Proposition 2.5. *The following holds true*

$$\mathcal{V}_{\text{MP}}(\mu) = \mathcal{V}_{\text{KP}}(\mu). \quad (2.35)$$

Proof. 1. We first observe that, for all $\epsilon > 0$, there exists a \mathcal{F}_T -measurable random variable \mathfrak{U}^ϵ with uniform distribution and independent of $\mathcal{F}_{T-\epsilon}$, for example $\mathfrak{U}^\epsilon = N\left(\frac{W_T - W_{T-\epsilon}}{\sqrt{\epsilon}}\right)$. Here, N denotes the c.d.f of the standard centered gaussian law.

2. Let $\chi \in \mathcal{T}(\mu)$. We have

$$G^\chi = \sum_{i=1}^N G^i \mathbf{1}_{\{\chi=i\}},$$

Defining the \mathcal{F}_T -measurable random variables $P^n := \mathbf{1}_{\{\chi=n\}}$ for all $1 \leq n \leq N$. Since $P \in \mathcal{P}_\mu(\mathcal{F}_T)$, we have

$$\mathcal{Y}_0[G^\chi] \geq \mathcal{V}_{\text{KP}}(\mu),$$

where we used Proposition 2.3. Taking the infimum over $\chi \in \mathcal{T}(\mu)$, we obtain $\mathcal{V}_{\text{MP}}(\mu) \geq \mathcal{V}_{\text{KP}}(\mu)$.

3. We now prove the converse inequality.

3.a Let $\eta \in (0, 1)$ and $P^\eta \in \mathcal{P}_\mu(\mathcal{F}_T)$, such that

$$\mathcal{V}_{\text{KP}}(\mu) \geq \mathcal{Y}_0 \left[\sum_{n=1}^N G^n (P^\eta)^n \right] - \eta. \quad (2.36)$$

We set $Q^\eta \in \mathcal{Q}_\mu(\mathcal{F}_T)$ such that $(P^\eta)^n = (Q^\eta)^{n+1} - (Q^\eta)^n$, $1 \leq n \leq N$ and define

$$P_{T-\epsilon}^\eta := \mathbb{E}[P^\eta | \mathcal{F}_{T-\epsilon}] \text{ and } Q_{T-\epsilon}^\eta := \mathbb{E}[Q^\eta | \mathcal{F}_{T-\epsilon}], \text{ for } \epsilon \in (0, T). \quad (2.37)$$

We observe that $P_{T-\epsilon}^\eta \in \mathcal{P}_\mu(\mathcal{F}_{T-\epsilon})$ and $Q_{T-\epsilon}^\eta \in \mathcal{Q}_\mu(\mathcal{F}_{T-\epsilon})$. We now introduce the \mathcal{F}_T -measurable random variable

$$\chi^{\eta, \epsilon} := \sum_{n=1}^N n \mathbf{1}_{\{(Q_{T-\epsilon}^\eta)^n \geq \mathfrak{U}^\epsilon > (Q_{T-\epsilon}^\eta)^{n+1}\}}, \quad (2.38)$$

where \mathfrak{U}^ϵ is constructed in step 1. We compute that, for all $1 \leq n \leq N$,

$$\mathbb{E}[\mathbf{1}_{\{\chi^{\eta, \epsilon}=n\}} | \mathcal{F}_{T-\epsilon}] = (Q_{T-\epsilon}^\eta)^{n+1} - (Q_{T-\epsilon}^\eta)^n = (P_{T-\epsilon}^\eta)^n. \quad (2.39)$$

Since $P_{T-\epsilon}^\eta \in \mathcal{P}_\mu(\mathcal{F}_{T-\epsilon})$, we deduce, for all $1 \leq n \leq N$,

$$\mathbb{P}(\chi^{\eta, \epsilon} = n) = \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\chi^{\eta, \epsilon}=n\}} | \mathcal{F}_{T-\epsilon}]] = \mathbb{E}[(P_{T-\epsilon}^\eta)^n] = p_n.$$

which implies that $\chi^{\eta, \epsilon} \in \mathcal{T}(\mu)$. Assume that

$$\mathcal{Y}_0 \left[\sum_{n=1}^N G^n (P^\eta)^n \right] \geq \mathcal{Y}_0[G^{\chi^{\eta, \epsilon}}] + o_{\eta, \epsilon}(1) \quad (2.40)$$

where $o_{\eta, \epsilon}(1) \xrightarrow{\epsilon \rightarrow 0} 0$, for all $\eta \in (0, 1)$.

From the definition of $\mathcal{V}_{\text{MP}}(\mu)$, we straightforwardly obtain

$$\mathcal{Y}_0 \left[\sum_{n=1}^N G^n (P^\eta)^n \right] \geq \mathcal{V}_{\text{MP}}(\mu) + o_{\eta, \epsilon}(1), \quad (2.41)$$

which, combined with (2.36), leads to

$$\mathcal{V}_{\text{KP}}(\mu) \geq \mathcal{V}_{\text{MP}}(\mu) - o_{\eta, \epsilon}(1) - \eta.$$

Sending first ϵ to 0 and then η to 0 yields the inequality for this step and thus (2.35).
 3.b To conclude, it remains to prove (2.40).

We define, for all $1 \leq n \leq N$, $G_{T-\epsilon}^n := \mathbb{E}[G^n | \mathcal{F}_{T-\epsilon}]$. According to (2.38), we observe that

$$\mathbb{E}\left[G_{T-\epsilon}^{\chi^{\eta,\epsilon}} \middle| \mathcal{F}_{T-\epsilon}\right] = \sum_{n=1}^N G_{T-\epsilon}^n (P_{T-\epsilon}^\eta)^n = \mathbb{E}\left[\sum_{n=1}^N G^n (P_{T-\epsilon}^\eta)^n \middle| \mathcal{F}_{T-\epsilon}\right]. \quad (2.42)$$

We have, using (2.42) and the fact that $\mathcal{Y}_0^T[\cdot] = \mathcal{Y}_0^{T-\epsilon}[\mathcal{Y}_{T-\epsilon}^T[\cdot]]$,

$$\begin{aligned} \mathcal{Y}_0\left[\sum_{n=1}^N G^n (P^\eta)^n\right] - \mathcal{Y}_0[G^{\chi^{\eta,\epsilon}}] &= A^{\eta,\epsilon} + B^{\eta,\epsilon} + C^{\eta,\epsilon} + D^{\eta,\epsilon}, \text{ with} \\ A^{\eta,\epsilon} &:= \mathcal{Y}_0\left[\sum_{n=1}^N G^n (P^\eta)^n\right] - \mathcal{Y}_0\left[\sum_{n=1}^N G_{T-\epsilon}^n (P^\eta)^n\right], \\ B^{\eta,\epsilon} &:= \mathcal{Y}_0^{T-\epsilon}\mathcal{Y}_{T-\epsilon}^T\left[\sum_{n=1}^N G_{T-\epsilon}^n (P^\eta)^n\right] - \mathcal{Y}_0^{T-\epsilon}\left[\mathbb{E}_{T-\epsilon}\left[\sum_{n=1}^N G_{T-\epsilon}^n (P^\eta)^n\right]\right], \\ C^{\eta,\epsilon} &:= \mathcal{Y}_0^{T-\epsilon}\left[\mathbb{E}_{T-\epsilon}\left[G_{T-\epsilon}^{\chi^{\eta,\epsilon}}\right]\right] - \mathcal{Y}_0^{T-\epsilon}\mathcal{Y}_{T-\epsilon}^T\left[G_{T-\epsilon}^{\chi^{\eta,\epsilon}}\right], \text{ and} \\ D^{\eta,\epsilon} &:= \mathcal{Y}_0\left[G_{T-\epsilon}^{\chi^{\eta,\epsilon}}\right] - \mathcal{Y}_0\left[G^{\chi^{\eta,\epsilon}}\right]. \end{aligned}$$

Using stability property of BSDEs, see e.g. [16, Remark (b) p.20], we have that

$$|A^{\eta,\epsilon}| \leq C \left\| \sum_{n=1}^N (G^n - G_{T-\epsilon}^n) (P^\eta)^n \right\|_{\mathcal{L}^2}.$$

Using Cauchy-Schwarz inequality and dominated convergence theorem, we obtain easily that $\left\| \sum_{n=1}^N (G^n - G_{T-\epsilon}^n) (P^\eta)^n \right\|_{\mathcal{L}^2} = o_{\eta,\epsilon}(1)$, which, combined with the previous inequality leads to,

$$A^{\eta,\epsilon} = o_{\eta,\epsilon}(1). \quad (2.43)$$

For the B term, we first observe that $\left\| \sum_{n=1}^N G_{T-\epsilon}^n (P^\eta)^n \right\|_{\mathcal{L}^2} \leq \|\mathfrak{C}\|_{\mathcal{L}^2}$, recall (2.7). Then invoking Lemma 2.3, we obtain

$$B^{\eta,\epsilon} \leq C\epsilon^{\frac{1}{4}}. \quad (2.44)$$

For the C -term (resp. D -term), we use similar arguments as for the B -term (resp. A -term), to obtain

$$C^{\eta,\epsilon} + D^{\eta,\epsilon} = o_{\eta,\epsilon}(1). \quad (2.45)$$

The proof for this step is then concluded by combining (2.43), (2.44) and (2.45). \square

Theorem 2.1. *The following representation holds true*

$$\mathcal{V}_{\text{WH}}(\mu) = \inf_{P \in \mathcal{P}_\mu(\mathcal{F}_T)} \mathcal{Y}_0 \left[\sum_{n=1}^N G^n P^n \right]. \quad (2.46)$$

Proof. According to the definitions of $\mathcal{V}_{\text{MP}}(\mu)$ in , $\mathcal{V}_{\text{RM}}(\mu)$ in and $\mathcal{V}_{\text{KP}}(\mu)$ in , we have

$$\mathcal{V}_{\text{MP}}(\mu) \geq \mathcal{V}_{\text{RM}}(\mu) \geq \mathcal{V}_{\text{KP}}(\mu).$$

Proposition 2.5 allows then to get

$$\mathcal{V}_{\text{MP}}(\mu) = \mathcal{V}_{\text{RM}}(\mu) = \mathcal{V}_{\text{KP}}(\mu).$$

The proof is concluded by combining (2.28) and (2.20) with the previous equalities. \square

We conclude the section with the proof of the auxiliary lemma.

Proof of Lemma 2.3 From stability estimates for BSDEs, see e.g. [16, Remark (b) p.20], and the fact that $\mathcal{Y}_0^T[\cdot] = \mathcal{Y}_0^{T-\epsilon}[\mathcal{Y}_{T-\epsilon}^T[\cdot]]$, we have

$$|\mathcal{Y}_0^{T-\epsilon}[\mathbb{E}[\xi|\mathcal{F}_{T-\epsilon}]] - \mathcal{Y}_0^T[\xi]| \leq C \|\mathbb{E}[\xi|\mathcal{F}_{T-\epsilon}] - \mathcal{Y}_{T-\epsilon}^T[\xi]\|_{\mathcal{L}^2}. \quad (2.47)$$

Let us denote (U, V) the solution of the BSDE with terminal value ξ and null generator i.e. for all $t \leq T$,

$$U_t = \xi - \int_t^T V_s dW_s = \mathbb{E}[\xi|\mathcal{F}_t],$$

and to alleviate the notation $(\mathcal{Y}_t, \mathcal{Z}_t) = (\mathcal{Y}_t^T[\xi], \mathcal{Z}_t^T[\xi])$, $t \leq T$. Define finally $\delta Y = U - \mathcal{Y}$ and $\delta Z = V - \mathcal{Z}$. Classical computations for comparison of two BSDEs lead to

$$|\delta Y_t|^2 \leq C \mathbb{E} \left[\int_t^T |f(s, U_s, V_s) \delta Y_s| ds | \mathcal{F}_t \right]. \quad (2.48)$$

We also know, see e.g. [16, Remark (b) p.20], that

$$\mathbb{E}_t \left[\sup_{s \in [t, T]} |\mathcal{Y}_s|^2 + \int_t^T |\mathcal{Z}_s|^2 ds \right] \leq C \mathbb{E}_t \left[|\xi|^2 + \int_t^T |f(s, 0, 0)|^2 ds \right], \quad (2.49)$$

$$\mathbb{E}_t \left[\sup_{s \in [t, T]} |U_s|^2 + \int_t^T |V_s|^2 ds \right] \leq C \mathbb{E}_t[|\xi|^2]. \quad (2.50)$$

From (2.48), we compute

$$|\delta Y_t|^2 \leq C \mathbb{E} \left[\sup_{s \in [t, T]} |\delta Y_s| \int_t^T |f(s, U_s, V_s)| ds | \mathcal{F}_t \right], \quad (2.51)$$

and then using Cauchy-Schwarz inequality,

$$|\delta Y_t|^2 \leq C\sqrt{T-t}\mathbb{E}_t\left[\sup_{s\in[t,T]}|\delta Y_s|^2\right]^{\frac{1}{2}}\mathbb{E}_t\left[\int_t^T|f(s,U_s,V_s)|^2ds\right]^{\frac{1}{2}}. \quad (2.52)$$

Using the Lipschitz property of f , we get

$$\mathbb{E}_t\left[\int_t^T|f(s,U_s,V_s)|^2ds\right] \leq C\left(\mathbb{E}_t\left[\int_t^T|f(s,0,0)|^2ds\right] + \mathbb{E}_t\left[\int_t^T|U_s|^2ds\right] + \mathbb{E}_t\left[\int_t^T|V_s|^2ds\right]\right) \quad (2.53)$$

which using (2.50), leads to

$$\mathbb{E}_t\left[\int_t^T|f(s,U_s,V_s)|^2ds\right] \leq C\mathbb{E}_t\left[|\xi|^2 + \int_t^T|f(s,0,0)|^2ds\right]. \quad (2.54)$$

Combining (2.49) and (2.50), we obtain also

$$\mathbb{E}_t\left[\sup_{s\in[t,T]}|\delta Y_s|^2\right] \leq C\mathbb{E}_t\left[|\xi|^2 + \int_t^T|f(s,0,0)|^2ds\right]. \quad (2.55)$$

Inserting back (2.54) and (2.55) into (2.48), we get

$$|\delta Y_t|^2 \leq C\sqrt{T-t}\mathbb{E}_t\left[|\xi|^2 + \int_t^T|f(s,0,0)|^2ds\right]. \quad (2.56)$$

Combining the previous inequality at $t = T - \epsilon$ and (2.47), we get

$$|\mathcal{Y}_0^{T-\epsilon}[\mathbb{E}[\xi|\mathcal{F}_{T-\epsilon}]] - \mathcal{Y}_0^T[\xi]| \leq C\epsilon^{\frac{1}{4}}\mathbb{E}\left[|\xi|^2 + \int_t^T|f(s,0,0)|^2ds\right],$$

which concludes the proof. \square

3 Linear setting: duality and numerical illustration

We now focus on a special framework, useful for application, where tractable and implementable formula can be derived, namely the linear setting. We thus assume that Assumption 2.1 is in force throughout the section. It turns out that the multiple quantile hedging problem corresponds then to a semi-discrete optimal transport problem see e.g. [21, Chapter 5] and the references therein. We shall thus rely here further on the approach of optimal transport to solve our problem and in particular we use duality methods.

3.1 Dual representation

In the setting of Remark 2.1, we define the random vector

$$H(\omega) = (H^n(\omega) := \Gamma_T(\omega)G^n(\omega))_{n=1}^N. \quad (3.1)$$

We now assume

$$H \in \mathcal{L}^2(\mathcal{F}_T, \mathbb{R}^N). \quad (3.2)$$

In this framework, Theorem 2.1 reads as follows.

Corollary 3.1. *Under Assumptions 2.1, the following holds*

$$\mathcal{V}_{\text{WH}}(\mu) = \inf_{P \in \mathcal{P}_\mu(\mathcal{F}_T)} \mathbb{E} \left[\sum_{n=1}^N H^n P^n \right]. \quad (3.3)$$

We introduce a dual formulation for the multiple quantile hedging problem, which is the classical optimal transport formulation of the dual Kantorovitch problem. We set

$$\mathcal{V}_{\text{DP}}(\mu) := \sup_{(X, \Phi) \in \mathfrak{P}} \left(\mathbb{E}[X] + \sum_{n=1}^N \Phi^n p^n \right), \quad (3.4)$$

where

$$\mathfrak{P} := \{(X, \Phi) \in \mathcal{L}^2(\mathcal{F}_T) \times \mathbb{R}^N \mid H^n \geq X + \Phi^n, 1 \leq n \leq N, \mathbb{P} - \text{a.s.}\}. \quad (3.5)$$

We first make the classical observation without proof,

Lemma 3.1. *Under Assumption 2.1, for $\mu \in \mathcal{P}(\mathcal{S})$,*

$$\mathcal{V}_{\text{WH}}(\mu) = \mathcal{V}_{\text{KP}}(\mu) \geq \mathcal{V}_{\text{DP}}(\mu).$$

The goal of this section is to prove that there is no duality gap, i.e. that $\mathcal{V}_{\text{KP}}(\mu) = \mathcal{V}_{\text{DP}}(\mu)$. To this effect, we introduce, for $\mu = \sum_{n=1}^N p^n \delta_n \in \mathcal{P}(\mathcal{S})$ with $\sum_{n=1}^N p^n = 1$ and $p^n \in (0, 1)$ for all $1 \leq n \leq N$, the map

$$D : L^2(\mathcal{F}_T; \mathbb{R}^N) \rightarrow \mathbb{R} \quad (3.6)$$

$$\mathfrak{h} \mapsto - \sup_{(X, \Phi) \in \mathfrak{P}(\mathfrak{h})} \left(\mathbb{E}[X] + \sum_{i=1}^N \Phi^i p^i \right), \quad (3.7)$$

where

$$\mathfrak{P}(\mathfrak{h}) := \{(X, \Phi) \in \mathcal{L}^2(\mathcal{F}_T) \times \mathbb{R}^N \mid H^n - \mathfrak{h}^n \geq X + \Phi^n, 1 \leq n \leq N, \mathbb{P} - \text{a.s.}\}.$$

Observe that $-V_{\text{DP}}(\mu) = D(0)$ and $\mathfrak{P} = \mathfrak{P}_{H, \mu}(0)$. We first collect some properties of the map D .

Proposition 3.1. *The map D is continuous and convex. Moreover, for $\mathfrak{h} \in L^2(\mathcal{F}_T; \mathbb{R}^N)$, we have*

$$D(\mathfrak{h}) = - \sup_{\Phi \in (\mathbb{R}_+)^N} \left(\mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n - \Phi^n) \right] + \sum_{n=1}^N \Phi^n p^n \right), \quad (3.8)$$

$$= - \left(\mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n - \Phi_{\mathfrak{h}}^n) \right] + \sum_{n=1}^N \Phi_{\mathfrak{h}}^n p^n \right), \quad (3.9)$$

for some $\Phi_{\mathfrak{h}} \in (\mathbb{R}_+)^N$.

Proof. 1. We first observe that in Definition (3.6), we can replace $(X, \Phi) \in \mathfrak{P}(\mathfrak{h})$ by $(\min_n (H^n - \mathfrak{h}^n - \Phi^n), \Phi) \in \mathfrak{P}(\mathfrak{h})$ as the criterion is improved. Thus, we get

$$D(\mathfrak{h}) = - \sup_{\Phi \in \mathbb{R}^N} \left(\mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n - \Phi^n) \right] + \sum_{n=1}^N \Phi^n p^n \right)$$

Let us now consider the set

$$E = \{x \in (\mathbb{R}_+)^N \mid \exists j \in \{1, \dots, N\}, \text{ s.t. } x^j = 0\}. \quad (3.10)$$

Observe to $\Phi \in \mathbb{R}^N$, we can associate $\tilde{\Phi} \in E$ by setting

$$\tilde{\Phi}^n = \Phi^n - \min_{1 \leq j \leq N} \Phi^j,$$

and we have, as $\sum_n p^n = 1$,

$$\mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n - \Phi^n) \right] + \sum_{n=1}^N \Phi^n p^n = \mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n - \tilde{\Phi}^n) \right] + \sum_{n=1}^N \tilde{\Phi}^n p^n.$$

We then have proved

$$D(\mathfrak{h}) = - \sup_{\Phi \in E} w_{\mathfrak{h}}(\Phi) \quad \text{with} \quad w_{\mathfrak{h}}(\Phi) := \mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n - \Phi^n) \right] + \sum_{n=1}^N \Phi^n p^n, \quad (3.11)$$

which *a fortiori* implies (3.8).

2. We now prove the existence of optimal potential, namely (3.9). First, for $\Phi \in E$, there is $k \in \{1, \dots, N\}$ such that $\Phi^k = 0$ and thus

$$\sum_{n=1}^N \Phi^n p^n = \sum_{n \neq k} \Phi^n p^n \leq |\Phi|_{\infty} \sum_{n \neq k} p^n \leq |\Phi|_{\infty} (1 - q) \quad \text{with} \quad q := \min_n p^n > 0.$$

Then, we observe that

$$\begin{aligned} w_{\mathfrak{h}}(\Phi) &= \sum_{n=1}^N \Phi^n p^n - \mathbb{E} \left[\max_{1 \leq n \leq N} (\Phi^n - H^n + \mathfrak{h}^n) \right] \\ &\leq |\Phi|_{\infty} \left((1 - q) - \frac{1}{|\Phi|_{\infty}} \mathbb{E} \left[\max_{1 \leq n \leq N} (\Phi^n + \mathfrak{h}^n - H^n) \right] \right). \end{aligned} \quad (3.12)$$

Since

$$\frac{1}{|\Phi|_\infty} \mathbb{E} \left[\max_{1 \leq n \leq N} (\Phi^n + \mathfrak{h}^n - H^n) \right] \geq 1 - \frac{1}{|\Phi|_\infty} \mathbb{E} \left[\min_{1 \leq n \leq N} |\mathfrak{h}^n - H^n| \right] \quad (3.13)$$

Combining the previous inequality with (3.12), we obtain, for $\Phi \in E$,

$$w_{\mathfrak{h}}(\Phi) \leq |\Phi|_\infty \left(-q + \frac{1}{|\Phi|_\infty} \mathbb{E} \left[\min_{1 \leq n \leq N} |\mathfrak{h}^n - H^n| \right] \right). \quad (3.14)$$

We also observe that $w_{\mathfrak{h}}(0) = \mathbb{E}[\min_{1 \leq n \leq N} (H^n - \mathfrak{h}^n)]$ and thus for Φ s.t.

$$|\Phi|_\infty > M := \frac{2}{q} \mathbb{E} \left[\min_{1 \leq n \leq N} |\mathfrak{h}^n - H^n| \right] \quad (3.15)$$

we obtain that $w_{\mathfrak{h}}(\Phi) < -|w_{\mathfrak{h}}(0)|$. Thus, the supremum in (3.11) is achieved for Φ s.t. $|\Phi| \leq M$. Combining this with the fact that $w_{\mathfrak{h}}(\cdot)$ is continuous on the closed set E , we obtain the existence of $\Phi^{\mathfrak{h}}$ s.t. the supremum is achieved, concluding the proof of this step.

3.a Since $\mathfrak{h} \mapsto w_{\mathfrak{h}}(\Phi)$ is continuous, we straightforwardly obtain that D is upper semi-continuous. Now let $(\mathfrak{h}_k)_{k \geq 0}$ converging in $L^2(\mathcal{F}_T; \mathbb{R}^N)$ to $\hat{\mathfrak{h}}$ and denote Φ_k an optimal potential associated to the optimisation $D(\mathfrak{h}_k)$. We compute, by suboptimality of Φ_k for the optimisation $D(\hat{\mathfrak{h}})$, according to (3.8),

$$D(\hat{\mathfrak{h}}) \leq -\mathbb{E} \left[\sum_n \Phi_k^n p^n + \min_n (H^n - \hat{\mathfrak{h}}^n - \Phi_k^n) \right] \quad (3.16)$$

$$\leq -\mathbb{E} \left[\sum_n \Phi_k^n p^n + \min_n (H^n - \mathfrak{h}_k^n - \Phi_k^n) \right] + \mathbb{E} \left[\min_n |\mathfrak{h}_k^n - \hat{\mathfrak{h}}^n| \right] \quad (3.17)$$

$$\leq D(\mathfrak{h}^k) + \mathbb{E} [|\mathfrak{h} - \mathfrak{h}^k|]. \quad (3.18)$$

Taking the liminf in this last inequality,* we obtain that D is lower semi-continuous, hence continuous.

3.b Let $\lambda \in [0, 1]$, $\mathfrak{h}_1, \mathfrak{h}_2$ in $L^2(\mathcal{F}_T; \mathbb{R}^N)$ and denote Φ_1, Φ_2 the associated optimal potentials. By suboptimality, we obtain

$$D(\lambda \mathfrak{h}_1 + (1 - \lambda) \mathfrak{h}_2) \leq -\sum_n (\lambda \Phi_1^n + (1 - \lambda) \Phi_2^n) p^n \quad (3.19)$$

$$- \mathbb{E} \left[\min_n (\lambda (H^n - \mathfrak{h}_1^n - \Phi_1^n) + (1 - \lambda) (H^n - \mathfrak{h}_2^n - \Phi_2^n)) \right]. \quad (3.20)$$

Since

$$\begin{aligned} \min_n (\lambda (H^n - \mathfrak{h}_1^n - \Phi_1^n) + (1 - \lambda) (H^n - \mathfrak{h}_2^n - \Phi_2^n)) &\geq \\ &\lambda \min_n (H^n - \mathfrak{h}_1^n - \Phi_1^n) + (1 - \lambda) \min_n (H^n - \mathfrak{h}_2^n - \Phi_2^n), \end{aligned}$$

we obtain

$$D(\lambda \mathfrak{h}_1 + (1 - \lambda) \mathfrak{h}_2) \leq \lambda D(\mathfrak{h}_1) + (1 - \lambda) D(\mathfrak{h}_2),$$

which conclude the proof for the convexity of D . \square

Proposition 3.2. *Let D^\star be the Fenchel transform of D , namely*

$$D^\star : L^2(\mathcal{F}_T; \mathbb{R}^N) \rightarrow \mathbb{R},$$

$$P \mapsto \sup_{\mathfrak{h} \in L^2(\mathcal{F}_T; \mathbb{R}^N)} \mathbb{E} \left[\sum_{n=1}^N \mathfrak{h}^n P^n \right] - D(\mathfrak{h}).$$

We then have

$$D^\star(P) = \mathbb{E} \left[\sum_{n=1}^N H^n P^n \right] \mathbf{1}_{\{P \in \mathcal{P}_\mu(\mathcal{F}_T)\}} + (+\infty) \mathbf{1}_{\{P \notin \mathcal{P}_\mu(\mathcal{F}_T)\}}.$$

Proof. 1 From the definition of D , we observe that

$$D^\star(P) = \sup_{(\mathfrak{h}, X, \Phi) \in \mathfrak{E}} \mathbb{E} \left[\sum_{n=1}^N \mathfrak{h}^n P^n + X \right] + \sum_{n=1}^N \Phi^n p^n \quad (3.21)$$

with

$$\mathfrak{E} = \{(\mathfrak{h}, X, \Phi) \in L^2(\mathcal{F}_T; \mathbb{R}^N) \times \mathcal{L}^2(\mathcal{F}_T) \times \mathbb{R}^N \mid H^n - \mathfrak{h}^n \geq X + \Phi^n, 1 \leq n \leq N \text{ and } \mathbb{P}\text{-a.s.}\}. \quad (3.22)$$

For a given $P \in L^2(\mathcal{F}_T; \mathbb{R}^N)$ and $\lambda \geq 0$, we introduce $\mathfrak{h}_\lambda \in L^2(\mathcal{F}_T; \mathbb{R}^N)$ defined by

$$\mathfrak{h}_\lambda^n = H^n - \lambda \mathbf{1}_{\{P^n < 0\}},$$

and we observe that $(\mathfrak{h}_\lambda, 0, 0) \in \mathfrak{E}$. We then obtain

$$D^\star(P) \geq \mathbb{E} \left[\sum_{n=1}^N \mathfrak{h}_\lambda^n P^n \right] = \lambda \mathbb{E} \left[\sum_{n=1}^N -P^n \mathbf{1}_{\{P^n < 0\}} \right]. \quad (3.23)$$

Thus, as soon as $\mathbb{P}(P \in (\mathbb{R}_+)^N) < 1$, we get $D^\star(P) = +\infty$ by letting $\lambda \rightarrow \infty$ in the inequality above.

2. We now consider $P \in L^2(\mathcal{F}_T; (\mathbb{R}_+)^N)$. To $(\mathfrak{h}, X, \Phi) \in \mathfrak{E}$, we associate $(\tilde{\mathfrak{h}}, X, \Phi) \in \mathfrak{E}$ where $\tilde{\mathfrak{h}}^n = H^n - (X + \Phi^n)$, $1 \leq n \leq N$. We observe, since P has non-negative components, that

$$\mathbb{E} \left[\sum_{n=1}^N \mathfrak{h}^n P^n \right] \leq \mathbb{E} \left[\sum_{n=1}^N \tilde{\mathfrak{h}}^n P^n \right] = \mathbb{E} \left[\sum_{n=1}^N H^n P^n - X \sum_{n=1}^N P^n - \sum_{n=1}^N \Phi^n P^n \right].$$

We therefore get that

$$D^\star(P) = \mathbb{E} \left[\sum_{n=1}^N H^n P^n \right] + \sup_{X \in \mathcal{L}^2(\mathcal{F}_T), \Phi \in \mathbb{R}^N} \mathbb{E} \left[X \left(1 - \sum_{n=1}^N P^n \right) \right] + \sum_{n=1}^N \Phi^n (p^n - \mathbb{E}[P^n]).$$

From this expression, we deduce that, as soon as $P \notin \mathcal{P}_\mu(\mathcal{F}_T)$, $D^\star(P) = +\infty$. If $P \in \mathcal{P}_\mu(\mathcal{F}_T)$, we then get $D^\star(P) = \mathbb{E} \left[\sum_{n=1}^N H^n P^n \right]$ which concludes the proof. \square

Theorem 3.1. *Duality holds and then*

$$\mathcal{V}_{\text{WH}}(\mu) = \sup_{\Phi \in (\mathbb{R}_+)^N} \left(\mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \Phi^n) \right] + \sum_{n=1}^N \Phi^n p^n \right). \quad (3.24)$$

Proof. By Proposition 3.1, D is convex and continuous. It implies that $D = D^{**}$ by the Fenchel-Moreau theorem, with

$$D^{**}(\mathfrak{h}) := \sup_{P \in L^2(\mathcal{F}_T; \mathbb{R}^N)} \mathbb{E} \left[\sum_{n=1}^N \mathfrak{h}^n P^n \right] - D^*(P), \quad \mathfrak{h} \in L^2(\mathcal{F}_T; \mathbb{R}^N).$$

By Proposition 3.2, we then have

$$D^{**}(\mathfrak{h}) = \sup_{P \in \mathcal{P}_\mu(\mathcal{F}_T)} \mathbb{E} \left[\sum_{n=1}^N (\mathfrak{h}^n - H^n) P^n \right].$$

Taking $\mathfrak{h} = 0$, we obtain

$$-V_{\text{DP}}(\mu) = D(0) = D^{**}(0) = \sup_{P \in \mathcal{P}_\mu(\mathcal{F}_T)} -\mathbb{E} \left[\sum_{n=1}^N H^n P^n \right] = - \inf_{P \in \mathcal{P}_\mu(\mathcal{F}_T)} \mathbb{E} \left[\sum_{n=1}^N H^n P^n \right] = -V_{\text{KP}}(\mu).$$

The proof is concluded by invoking Lemma 3.1 and using the representation of $D(0)$ given in (3.8). \square

Corollary 3.2. *The multiple quantile hedging problem solution is also given by*

$$\mathcal{V}_{\text{WH}}(\mu) = \mathbb{E}[H^1] + \sup_{\zeta \in \Delta_+^{N-1}} \left(\mathbb{E} \left[\min_{1 \leq n \leq N-1} \left(\tilde{H}^{n+1} - \zeta^n \right)_- \right] + \sum_{n=1}^{N-1} \zeta^n p^{n+1} \right).$$

where $x_- := \min(x, 0)$ for $x \in \mathbb{R}$ and $\tilde{H}^n = H^n - H^1$, $1 \leq n \leq N$.

Proof. We first go back to the definition of $\mathcal{V}_{\text{DP}}(\mu)$ in (3.4)-(3.5). In particular, we observe that to $(X, \Phi) \in \mathfrak{P}$, we can associate $(X, \tilde{\Phi}) \in \mathfrak{P}$ s.t. $\tilde{\Phi}^n \geq \Phi^n$ and $\tilde{\Phi} \in \Delta^N$. Indeed, one simply sets $\tilde{\Phi}^n = \text{ess inf}_{\omega \in \Omega} H^n(\omega) - X(\omega)$. The fact that $\tilde{\Phi}^n \geq \tilde{\Phi}^{n-1}$ comes from the fact that G is non-decreasing. Then, we get the equivalent formulation

$$\mathcal{V}_{\text{WH}}(\mu) = \sup_{\Phi \in \Delta^N} \left(\sum_{n=1}^N \Phi^n p^n + \mathbb{E} \left[\min_{1 \leq n \leq N} (H^n - \Phi^n) \right] \right). \quad (3.25)$$

Setting $\zeta_n = \Phi^{n+1} - \Phi^1$, for $1 \leq n \leq N-1$, concludes the proof. \square

3.2 Numerical illustration

We now turn to numerical considerations associated to the above partial hedging problem. To this end, we first introduce a Markovian setting where our previous results can be straightforwardly restated.

We consider an underlying process X solution of the following SDE

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \quad (3.26)$$

where $b : \mathbb{R}^m \rightarrow \mathbb{R}^m$, $\sigma : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ are Lipschitz continuous functions. The starting point $X_0 \in \mathbb{R}^m$ at time 0 is arbitrary and we omit it in the notation. In this setting, the controlled process $Y^{y,Z}$ satisfies, according to (2.1),

$$Y_s^{y,Z} = y - \int_t^s f(r, X_r, Y_r^{y,Z}, Z_r) dr + \int_t^s Z_r dW_r, \quad (3.27)$$

where $f : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a Lipschitz-continuous function.

In the linear framework, we are considering, Assumption 2.1 on f reads as follows,

Assumption 3.1. *There exists a bounded continuous map $(\alpha, \beta) : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R} \times \mathbb{R}^m$ such that*

$$f(t, x, y, z) = \alpha(t, x)y + \beta(t, x)^\top z, \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m.$$

Under Assumption 3.1, the process Γ , following Remark 2.1, is solution to

$$\Gamma_t = 1 + \int_0^t \Gamma_s \alpha(s, X_s) ds + \int_0^t \Gamma_s \beta(s, X_s)^\top dW_s, \quad 0 \leq t \leq T. \quad (3.28)$$

Regarding the terminal constraint, we consider a measurable function:

$$\mathbb{R}^m \ni x \mapsto (g^n(x))_{n=1}^N \in \mathbb{R}^N, \quad (3.29)$$

such that $n \mapsto g^n(\cdot)$ is non-decreasing. And we set, for all $n \in \mathcal{S}$,

$$G^n = g^n(X_T) \quad \text{and} \quad H^n = \Gamma_T G^n. \quad (3.30)$$

The link made in the previous section to semi-discrete optimal transport allows us to use some numerical methods already developed in this field, see e.g. [21, Chapter 5]. In particular, we use the dual representation formula obtained in Corollary 3.2. Since the value we want to compute is obtained via the maximisation of the expectation of a concave function, we will naturally rely here on Stochastic Gradient Ascent (SGA), methods. In this regard, we shall use the ADAM optimiser [22], described below in Definition 3.1. Let us mention that stochastic gradient algorithms have already been considered to compute quantile in a financial context, see e.g. [23].

We introduce the function $\mathcal{W} : \Delta_+^{N-1} \times \mathcal{L}^2(\mathcal{F}_T) \times \mathcal{L}^2(\mathcal{F}_T) \rightarrow \mathbb{R}$, given by,

$$(\zeta, \Gamma, X) \mapsto \mathcal{W}(\zeta, \Gamma, X) = \sum_{n=1}^{N-1} \zeta^n p^{n+1} + \min_{1 \leq n \leq N-1} (\Gamma[g^{n+1}(X) - g^1(X)] - \zeta^n)_-. \quad (3.31)$$

and $w : \Delta_+^{N-1} \rightarrow \mathbb{R}$, given by ,

$$\zeta \mapsto w(\zeta) = \mathbb{E}[\mathcal{W}(\zeta, \Gamma_T, X_T)]. \quad (3.32)$$

With this notation, we get that

$$\mathcal{V}_{\text{WH}}(\mu) = \mathbb{E}[g^1(X_T)] + \sup_{\zeta \in \Delta_+^{N-1}} w(\zeta). \quad (3.33)$$

We introduce the following partition, for $1 \leq n \leq N-1$,

$$L^n(\zeta, \Gamma, X) := \left\{ \tilde{H}^{n+1} - \zeta^n < 0, \tilde{H}^{n+1} - \zeta^n < \min_{j < n} \tilde{H}^{j+1} - \zeta^j, \tilde{H}^{n+1} - \zeta^n = \min_{j \geq n} \tilde{H}^{j+1} - \zeta^j \right\} \quad (3.34)$$

recalling that $\tilde{H}^{n+1} = \Gamma[g^{n+1}(X) - g^1(X)]$. We have that

$$\mathcal{W}(\zeta, \Gamma, X) = \sum_{n=1}^{N-1} \zeta^n p^{n+1} + (\tilde{H}^{n+1} - \zeta^n) \mathbf{1}_{L^n(\zeta, \Gamma, X)}.$$

We then observe that the function

$$(\zeta, \Gamma, X) \mapsto \mathfrak{d}(\zeta, \Gamma, X) = p^{n+1} - \mathbf{1}_{L^n(\zeta, \Gamma, X)} \quad (3.35)$$

is a subgradient of $\zeta \mapsto \mathcal{W}(\zeta, \Gamma, X)$. We use this function to perform the gradient ascent, in the ADAM optimizer, see Definition 3.1 below. The algorithm requires in particular the introduction of a stepsize $(\eta_m)_{m \geq 1}$ satisfying generally

$$\sum_{m \geq 1} \eta_m = +\infty \text{ and } \sum_{n \geq 1} \eta_m^2 < +\infty. \quad (3.36)$$

We are also given a maximal number of iteration $M_{\text{iter}} \geq 1$ and we allow the use of batch with batchsize $B \geq 1$.

Definition 3.1 (ADAM optimizer [22] for SGA). *1. At initial step $m = 0$, set randomly ζ_0 and initialize first moment $\mathbf{m}_0 = 0$, and second moment $\mathbf{v}_0 = 0$.*

2. For $m = 1, \dots, M_{\text{iter}}$:

(a) generate $(\bar{\Gamma}_{b+(m-1)B}, \bar{X}_{b+(m-1)B})_{1 \leq b \leq B}$ independent copies of (Γ_T, X_T) independently from the past values $(\bar{\Gamma}_k, \bar{X}_k)_{k \leq (m-1)B}$ (if $m > 1$) and ζ_0 .

(b) Compute then

$$\mathbf{g}_m = \frac{1}{B} \sum_{b=1}^B \mathfrak{d}(\zeta_{m-1}, \bar{\Gamma}_{b+(m-1)B}, \bar{X}_{b+(m-1)B}), \quad (3.37)$$

$$\mathbf{m}_m = \beta_1 \mathbf{m}_{m-1} + (1 - \beta_1) \mathbf{g}_m, \quad (3.38)$$

$$\mathbf{v}_m = \beta_2 \mathbf{v}_{m-1} + (1 - \beta_2) |\mathbf{g}_m|^2, \quad (3.39)$$

$$\hat{\mathbf{m}}_m = \frac{\mathbf{m}_m}{1 - \beta_1^m}, \hat{\mathbf{v}}_m = \frac{\mathbf{v}_m}{1 - \beta_2^m}, \quad (3.40)$$

$$\zeta_m = \zeta_{m-1} - \eta_m \frac{\hat{\mathbf{m}}_m}{\sqrt{\hat{\mathbf{v}}_m + \epsilon}}, \quad (3.41)$$

for some small $\epsilon > 0$ and $(\beta_1, \beta_2) \in (0, 1)^2$.

(c) If $|\zeta_m - \zeta_{m-1}| < \iota$, then stop the computation, for a given small ι .

In all our numerical experiments, we set $\epsilon = 10^{-8}$, $\beta_1 = 0.9$, $\beta_2 = 0.999$ and the early stopping criterion $\iota = 10^{-6}$. The stepsize is given by $\eta_m = \eta_0/m$, $m \geq 1$ with η_0 possibly varying in each numerical test.

Then, to obtain the approximation of $\mathcal{V}_{\text{WH}}(\mu)$, we perform the two following steps:

1. Compute, by using the above SGA algorithm, ζ^* realizing approximately the maximum of w above;
2. Compute, by Monte-Carlo simulation, the value $\mathbb{E}[g^1(X_T)] + w(\zeta^*)$ which then approximates $\mathcal{V}_{\text{WH}}(\mu)$ according to (3.33).

The numerical experiments are done using the Black & Scholes model, which has already been used extensively in the context of quantile hedging problem for numerical purposes in [14, 1].

Example 3.1 (Black & Scholes Model). *The process X satisfies*

$$X_t = X_0 + \int_0^t \hat{b} X_s ds + \int_0^t \sigma X_s dW_s,$$

with $\hat{b} \in \mathbb{R}$, $\sigma > 0$ and $X_0 > 0$. The function f is given by:

$$f(t, x, y, z) = -ry - \frac{\hat{b} - r}{\sigma} z =: -ry - \lambda z, \quad (3.42)$$

with $r \geq 0$ the interest rate, and $\lambda := \frac{\hat{b} - r}{\sigma}$ the risk premium. The Radon-Nikodym derivative defined in (3.28) is thus:

$$\Gamma_T = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_T = \exp \left(\frac{r - \hat{b}}{\sigma} W_T - \frac{1}{2} \frac{(r - \hat{b})^2}{\sigma^2} T \right).$$

Quantiles p_2, p_3	γ_2, γ_3	SGA (std)	OT-SOLVER
(0.3,0.5)	(10,20)	17.38 (0.001)	17.48
(0.05,0.05)		8.48 (0.002)	8.41
(0.05,0.9)		24.53 (0.002)	24.44
(0.3,0.5)	(10,100)	42.07 (0.002)	42.19
(0.05,0.05)		9.77 (0.002)	9.62
(0.05,0.9)		87.15 (0.01)	87.57

Table 1: Numerics of measure $\mu = (1 - p_2 - p_3)\delta_1 + p_2\delta_2 + p_3\delta_3$ with different probabilities and quantiles with SGD algorithm and OT-SOLVER. Algorithm's parameters as follows: $M_{iter} = 10^5$, $B = 256$ and $\gamma_0 = 0.01$.

3.2.1 Numerical solution for the PnL distribution hedging problem

We work in the setting of Example 2.1 with $N = 3$, $\gamma_1 = 0 < \gamma_2 < \gamma_3$, and

$$g^i(x) := (x - K)_+ + \gamma_i, \quad (3.43)$$

for some $K \in \mathbb{R}$.

We consider in this part the discrete measure $\mu = (1 - p_2 - p_3)\delta_1 + p_2\delta_2 + p_3\delta_3$. To validate empirically the results obtained via the SGA algorithm, we rely on the formula obtained for Example 2.1 in Lemma 2.1. We call this approach below OT-SOLVER. We do not use the PDE method proposed in [2] which is more delicate to implement. The numerical results are reported in the Table 1. We observe that the numerical solutions by SGD perform well in comparison to the OT-SOLVER.

3.2.2 SGA for quantile hedging problem

We conclude this section by revisiting the numerical approximation of the classical quantile hedging problem using SG algorithms. We are thus in the setting of (2.13).

The two terminal constraints imposed on the portfolio value $Y_T^{y,Z}$ are: $\mathbb{P}(Y_T^{y,Z} \geq 0) = 1$ and $\mathbb{P}(Y_T^{y,Z} \geq \tilde{g}(X_T)) = p$, with \tilde{g} a Lipschitz continuous function. In our numerical test, we consider for \tilde{g} the payoff of European put and call options. We also compute the quantile hedging price on set of probabilities $p := \{\frac{i}{20}, 0 \leq i \leq 20\}$, reaching thus some extreme quantile value. The numerical results are compared to the value given by the theoretical formula in [1]. We observe that our quantile hedging price approximation is able to reproduce perfectly the true solution of call and put options claim, even for extreme values of p , as reported in Figure 1. In term of SGA algorithm to obtain the approximated quantile, we observe that the call option example is less demanding computationally than the put option example. Both are much less computationally demanding than the example of the previous section.

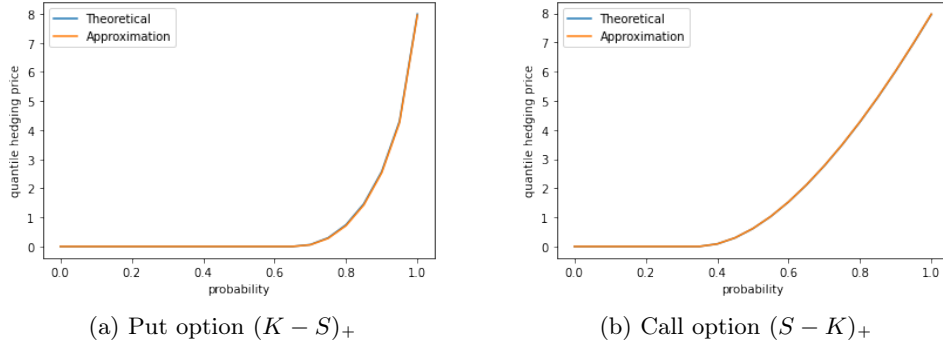


Figure 1: Comparison of two methods: Approximated & Exact solution [14, 1] for put and call option, with parameters $X_0 = 100$, $r = 0$, $\sigma = 0.2$ and $\hat{b} = 0.1$, strike $K = 100$, terminal time $T = 1$. For the put, the algorithm's parameters are as follows: $M_{iter} = 5000$, $B = 256$ and $\gamma_0 = 0.01$. For the call, the algorithm's parameters are as follows: $M_{iter} = 2500$, $B = 64$ and $\gamma_0 = 0.01$.

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