

# Moving layers and damage coupling with elasto-plasticity

Claude Stolz-UMR9219 CNRS, Palaiseau, France

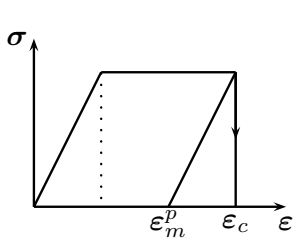
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- Objective : what is coupling between plasticity and damage.
- Simple model : perfect plasticity or linear kinematic hardening, and isotropic hardening
- Influence of the choice of rupture criterion.

$$\mathbf{E} = \mathbf{E}_E + \mathbf{E}_{ir}, \quad \mathbf{E}_{ir} = \mathbf{E}_p + \mathbf{E}_f$$

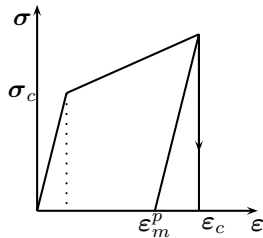
$$D_m = \frac{1}{2}(\Sigma : \dot{\mathbf{E}} - \mathbf{E} : \dot{\Sigma}) + \Sigma : \dot{\mathbf{E}}_{ir} - \frac{d}{dt} W_b = \int_{\Omega} D_p \, d\Omega + \int_{\Gamma} \mathcal{G}(s)\phi \, ds$$

# Rupture criterion



Perfect plasticity

$\epsilon_c, \alpha_m$

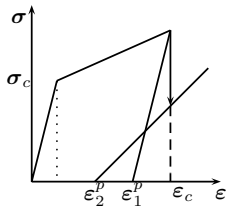


Hardening

$\sigma^m, \alpha_m, \epsilon_c, \Phi_c$

$$\Phi(\epsilon, \alpha) = \frac{1}{2} C(\epsilon - \alpha)^2 + \frac{1}{2} H\alpha^2$$

# Quasi-brittle material



At critical point, a jump of energy occurs

$$\Phi_1(\epsilon, \alpha) = \frac{1}{2}(\epsilon - \alpha) : \mathbb{C}_1 : (\epsilon - \alpha_1) + \frac{1}{2}\alpha_1 : \mathbb{H}_1 : \alpha_1$$

$$\Phi_2(\epsilon, \alpha_2) = \frac{1}{2}(\epsilon - \alpha_2) : \mathbb{C}_2 : (\epsilon - \alpha_2) + \frac{1}{2}\alpha_2 : \mathbb{H}_2 : \alpha_2$$

At critical point the plastic yielding function  $f(\sigma - \mathbb{H} : \alpha) \leq 0$

Material 2 has initial strain  $\alpha_2$ , which must be compatible with

$$n \cdot \llbracket \sigma \rrbracket_r = 0$$

- Material 1 changes to material 2 homogeneously on the bar
- Another answer is possible : let a damage phase appears at  $x = 0$  ,

$$\Phi_1(\varepsilon, \alpha_1) = \frac{1}{2} C_1 (\varepsilon - \alpha_1)^2 + \frac{1}{2} H_1 \alpha_1^2$$

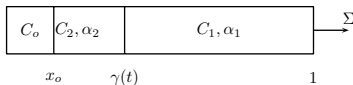
Dissipation is now

$$D_m = A_1 \dot{\alpha}_1 (1 - \gamma) + \mathcal{G} \dot{\gamma}$$

$$\mathcal{G} = [\Phi]_r - \Sigma[\varepsilon]_r = \frac{1}{2} \Sigma[\varepsilon - \alpha]_r - \Sigma[\alpha]_r + \frac{1}{2} [H\alpha^2]_r$$

# Uniaxial investigation

Consider a bar under uni-axial tension



$$x < x_0 : \Phi = \frac{1}{2} C_0 \varepsilon_0^2$$

$$x_0 < x < \gamma(t) : \Phi = \frac{1}{2} C_0 (\varepsilon_2 - \alpha_2)^2 + \frac{1}{2} H_2 \alpha_2^2$$

$$\gamma(t) < x < 1 : \Phi = \frac{1}{2} (\varepsilon_1 - \alpha_1(t))^2 C_1 + \frac{1}{2} H_1 \alpha_1^2$$

$$\Sigma = C_0 \varepsilon_0 = C_0 (\varepsilon_2 - \alpha_2) = C_1 (\varepsilon_1 - \alpha_1)$$

Global strain

$$E = x_0 \varepsilon_0 + (\gamma - x_0) \varepsilon_2 + (1 - \gamma) \varepsilon_1$$

Global irreversible strain

$$E_{ir} = (\gamma - x_0) \alpha_2 + (1 - \gamma) \alpha_1$$

Rigidity

$$E = \frac{1}{K}\Sigma, \quad \frac{1}{K} = \frac{\gamma}{C_o} + \frac{1-\gamma}{C_1}$$

Rigidity changes with the motion of  $\gamma$ .

$$E_{ir} = (x_o - \gamma)\alpha_2 + (1 - \gamma)\alpha_1, \quad \dot{E}_{ir} = \dot{\gamma}(\alpha_2 - \alpha_1) + (1 - \gamma)\dot{\alpha}_1$$

Two terms = rupture + plasticity

Plasticity is governed by normality law

$$f(\Sigma - H_1\alpha_1) = \|\Sigma - H_1\alpha_1\| - \sigma_o \leq 0, \quad \dot{\alpha}_1 \geq 0, \quad f\dot{\alpha}_1 = 0$$

The solution is given as

$$\alpha_1 = \frac{C_1}{C_1 + H_1} \left( E - \frac{\sigma_o}{C_1} \right), \quad \Sigma = \sigma_o + H_1\alpha_1$$

When  $\alpha_1 = \alpha_c$ , rupture occurs



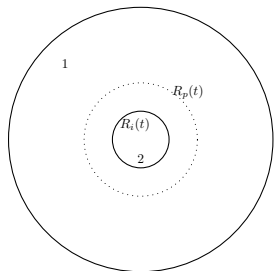
$$\begin{aligned} \mathcal{G} &= \frac{1}{2} \Sigma [\varepsilon - \alpha]_r - \Sigma [\varepsilon]_r + \frac{1}{2} [H\alpha^2]_r \\ &= \frac{1}{2} \Sigma^2 \left( \frac{1}{K_2} - \frac{1}{K_1} \right) - \Sigma [\alpha]_r + \frac{1}{2} [H\alpha^2]_r \end{aligned}$$

- For  $\alpha$  continuous and  $H_2 \leq H_1$ , then  $\mathcal{G} \geq 0$   
 If  $\mathcal{G} = G_c$  then  $\Sigma, \alpha$  are determined, The transformation occurs, and the state is determined by  $E$ .

$$E = x\varepsilon_1 + (1-x)\varepsilon_o, \quad \dot{E} = \dot{x}(\varepsilon_1 - \varepsilon_o) = \dot{x} \Sigma \left( \frac{1}{C_1} - \frac{1}{C_o} \right)$$

- For  $[\alpha]_r < 0, \alpha_2 - \alpha_1 > 0, H_2 = H_1, \mathcal{G} > 0$   
 A criterion  $\mathcal{G} = G_c$  gives  $\Sigma = \Sigma_c$   
 $\Sigma - H\alpha_1 < \Sigma - H\alpha_o = \sigma_o$  Plasticity remains unchanged.

Consider a sphere under radial loading



During monotone process : three phases occur

- : Elasticity
- : Plasticity
- : Propagation of interface

# Case of a composite sphere

$$c = \frac{R_i^3}{R_e^3}$$

Elastic domain  $R_p, R_e$

$$\frac{u}{r} = a + b \frac{R_p^3}{r^3}; \quad E = a + bp, \quad p = \frac{R_p^3}{R_e^3}$$

$$\sigma_{rr} = 3K_o a - 4\mu b \frac{R_p^3}{r^3}, \quad \sigma_{\theta\theta} = 3K_o a + 2\mu_o b \frac{R_p^3}{r^3}$$

Plastic domain  $R_i, R_p$   $\epsilon_p = \alpha(r)(2 \mathbf{e}_r \otimes \mathbf{e}_r - \mathbf{e}_\theta \otimes \mathbf{e}_\theta - \mathbf{e}_\phi \otimes \mathbf{e}_\phi)$

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_o, \quad b = \frac{\sigma_o}{6\mu_o}, \quad \text{Tr}(\boldsymbol{\sigma}) = 3K_o \text{Tr}(\boldsymbol{\epsilon})$$

$$\frac{d\sigma_{rr}}{dr} - 2\frac{\sigma_o}{r} = 0, \quad \sigma_{rr} = \sigma_I + 2\sigma_o \ln\left(\frac{r}{R_i}\right)$$

$$\frac{d}{dr} \left( \frac{L_o}{r^2} \frac{d}{dr} (r^2 u) \right) = \frac{d}{dr} (L_o \text{Tr} \boldsymbol{\epsilon}) = 12\mu_o \frac{1}{r^2} \frac{d}{dr} (r^3 \alpha(r))$$

In the core, elastic properties  $K_2, \mu_2$

$$u(r) = A_2 r, \quad \sigma_{rr} = 3K_2 A_2 = \sigma_I$$

$$\alpha = \gamma_o \left(1 - \frac{R_p^3}{r^3}\right), \quad \gamma_o \frac{3K_o + 4\mu_o}{18\mu_o K_o}$$

$$\Sigma = 3K_o \left(E - \frac{\sigma_o}{6\mu_o} - \frac{2}{3}\sigma_o p\right)$$

$$E = \frac{\sigma_o}{6\mu_o} p + \frac{2\sigma_o}{9K_o} \log \frac{p}{c} + \frac{1}{3K_o} \left(\sigma_l + \frac{2\sigma_o}{3}\right)$$

$$\sigma_l \left(\frac{1}{K_2} - \frac{1}{K_1}\right) = \frac{p}{c} \frac{3K_o + 4\mu_o}{6K_o \mu_o}$$

Energy

$$2\mathcal{W} = \frac{1}{\Omega} \int_{\Omega} \sigma_{rr}(\varepsilon_{rr} - 2\alpha(r)) + 2\sigma_{\theta\theta}(\varepsilon_{\theta\theta} - \alpha(r)) \, d\Omega$$
$$3\Sigma(p, c)E(p, c) + \frac{\sigma_o \gamma_o}{6\mu_o K_o} (p - c - p \log \frac{p}{c})$$

Dissipation

$$D_m = \mathcal{G}\dot{c} + 2\dot{p} \ln \frac{p}{c} = \mathcal{G}\dot{c} + \int_{\Omega_o} \boldsymbol{\sigma} : \dot{\boldsymbol{\alpha}} \, d\Omega$$

with

$$\mathcal{G} = \sigma_o^2 \frac{3K_o + 4\mu_o}{18\mu_o K_o} + \frac{K_o - K_2}{6K_o K_2} \sigma_I^2 \left(\frac{p}{c}\right)$$

If we consider an hollow sphere  $\sigma_I = 0$

$$\|\gamma(r)\| \leq \varepsilon_m, \rightarrow \frac{R_p^3}{R_i^3} = f_c$$

gives the missing link. If not criterion  $\mathcal{G} \leq G_c$  gives an identical results

# Solution with transfer of internal state

An additional domain appears  $R_i, \Gamma$ , where the elastic properties are  $K_2, \mu_2$ , with uniform internal strain  $\gamma_2$   
For such internal strain, the radial displacement satisfies

$$\frac{\Lambda_2}{r^2} \frac{d}{dr} (r^2 u) = \int_{R_i}^{\Gamma} \gamma_2 \frac{12\mu_2}{t} dt + A$$

then

$$r^2 u(r) = \frac{4\mu_2}{\Lambda_2} \gamma_2 \left( r^3 \ln\left(\frac{r}{R_i}\right) - \frac{1}{3}(r^3 - R_i^3) \right) + Ar^3 + B$$

In the plastic domain the solution satisfies  $\gamma_p = \gamma_o \left(1 - \frac{R_p^3}{r^3}\right)$

$$\sigma_{rr} = \frac{2}{3} \sigma_o \ln \frac{r}{\Gamma} + \sigma_{rr}(\Gamma)_p$$

and displacement depends on two constants in plastic domains, and elastic domain. We have 6 constants to be determined, 2 boundaries conditions and 4 interface conditions.

The plastic strain satisfy  $\varepsilon_p = \gamma_o(1 - \frac{p}{c})$

Along the moving interface  $\sigma_\Gamma$  satisfies

$$\sigma_\Gamma = \sigma_o \frac{3K_o + 4\mu_o}{6\mu_o} \frac{K_2}{K_o - K_2} \frac{c}{p} = K_2 Q \frac{c}{p}$$

and the state  $\Gamma/R_p$  satisfies

$$E - \frac{\sigma_o}{6\mu_o}(1 - p) = \frac{\sigma_\Gamma}{3K_o} + \frac{2\sigma_o}{3} \log\left(\frac{p}{c}\right) + \sigma_o \frac{3K_o + 4\mu_o}{6\mu_o}$$

and finally

$$\mathcal{G} = Q \frac{\sigma_o}{3\mu_o K_o} \left( 4\mu_o(K_o - K_2) + K_2(3K_o + 4\mu_o) \left(\frac{p}{c}\right)^2 \right)$$

with

$$2K_2(A_2 + \frac{12\mu_2\gamma_2}{3K_2 + 4\mu_2}) = \sigma_\Gamma$$

$$c = \left(\frac{\Gamma}{R_e}\right)^3, p = \left(\frac{\Gamma}{R_p}\right)^3$$

# Conclusion on sharp interface

Criterion  $\|\varepsilon_p\| = \varepsilon_c$  and  $\mathcal{G} \leq G_c$  give same results.

The coupling between damage and plasticity is observed on the irreversible strain

$$E - \frac{1}{K}\Sigma = E_{ir}$$

$K$  depends on  $c$

Then  $\mathcal{G}$  contains two contributions : one associated to  $K(c)$  and another one to the variations of internal residual state



# Sharp or diffuse interface

- Sharp interface  $\Gamma$  : elasto-plastic quasi-brittle material  
Material 1  $\Phi_1(\varepsilon, \alpha_o)$  changes in material 2  $\Phi_2(\varepsilon, \alpha_1)$

$$[[d]]_{\Gamma} = -1$$

Dissipation

$$D_m = \int_{\Omega} A_i \cdot \dot{\alpha}_i \, d\Omega + \int_{\Gamma} \mathcal{G} \phi \, dS$$

$$\mathcal{G} = [[\Phi]]_{\Gamma} - \boldsymbol{\sigma} : [[\varepsilon]]_{\Gamma}$$

- Diffuse interface : moving layers with internal continuous evolution of variables and continuity conditions at boundary of the layer.  
For example : in the layer, steady state analysis for any mechanical quantity

$$\dot{f} + \phi \nabla f \cdot \mathbf{n} = 0, \xi = \mathbf{x} - \phi \mathbf{n} t$$

Dissipation

$$D_m = \int_{\Omega} A \cdot \dot{\alpha} \, d\Omega - \int_{\Gamma} \int_e (Y \nabla d + A \cdot \nabla \alpha) \cdot \mathbf{n} j \, dz \phi \, dS$$

# Initiation of defect

Consider now, the case of diffuse interface, introducing a damage variable  $d$ , and

$$\Phi(\boldsymbol{\varepsilon}, \alpha, d) = \frac{1}{2}(\boldsymbol{\varepsilon} - \alpha) : \mathbb{C}(d) : (\boldsymbol{\varepsilon} - \alpha) + \frac{1}{2}\alpha : \mathbb{H}.\alpha$$

$\mathbb{C}(d)$  is a continuous decreasing function of  $d$

Consider the motion of the surface  $d = 0+$ , to avoid a dissipation on this surface

$$\mathcal{G} = [[\Phi]]_r - \mathbf{n}.\boldsymbol{\sigma}.[[\boldsymbol{\varepsilon}]]_r = 0$$

then  $[[\alpha]]_r = 0$  and  $[[\boldsymbol{\varepsilon}]]_r = 0$

Hypothesis :  $\alpha$  remains constant in damaged materials as proposed for sharp interface.

- With this hypothesis, the solution required on change in the layer, replacing  $L_2$  uniform by  $L(r)$  in the equation of  $\text{Tr } \varepsilon$

$$\frac{1}{r^2} \frac{d}{dr} r^2 u = \frac{12\mu\gamma_2}{L(t)} \log\left(\frac{t}{R_i}\right)$$

This changes the value of  $\sigma_r$ .

- The main coupling is seen on the global irreversible strain. Part due to plasticity and contribution of internal state of residual stresses.
- More complex behaviour must be study : especially to define how the internal state (as hardening) changes during the motion of the layer
- Internal structure of such layer must be studied, with respect to condition of uniqueness and entropy production.  $\varepsilon_p$  must be increase as d does.