New Lipschitz estimates and long-time asymptotic behavior for porous medium and fast diffusion equations

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Abstract

We obtain new estimates for the solution of both the porous medium and the fast diffusion equations by studying the evolution of suitable Lipschitz norms. Our results include instantaneous regularization for all positive times, long-time decay rates of the norms which are sharp and independent of the initial support, and new convergence results to the Barenblatt profile. Moreover, we address nonlinear diffusion equations including quadratic or bounded potentials as well. In the slow diffusion case, our strategy requires exponents close enough to 1, while in the fast diffusion case, our results cover any exponent for which the problem is well-posed and mass-preserving in the whole space.

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1 Introduction

The goal of this paper is to provide new estimates on the regularity of the solution $n(t,x)$ of the following nonlinear equation

$$\frac{\partial n}{\partial t} = \nabla \cdot (n \nabla (p + V)), \quad \text{in } (0,\infty) \times \Omega, \quad d \geq 2, \tag{1}$$

endowed with initial condition $n(0,x) = n_0(x) \geq 0, n_0 \in L^1(\Omega)$, where the domain is either the whole space $\Omega = \mathbb{R}^d$ or a convex bounded set, for which the problem is endowed with homogeneous Neumann boundary conditions. As constitutive law of the pressure we take the signed power law

$$p = P(n) := \operatorname{sign}(\gamma) n^\gamma,$$

while $V : \mathbb{R}^d \to \mathbb{R}$ is a potential whose regularity will be detailed later on. Equation (1) is a well-known example of a partial differential equation including convective effects and nonlinear diffusion whose theory is nowadays well established. For $\gamma > 0$ it is an equation of *porous medium* type, while for $\gamma < 0$ it is referred to as *fast diffusion*. In the latter case, we only consider the

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usual range of exponents $-2/d < \gamma < 0$. We are interested in proving decay estimates on the quantity

$$u(t) := \max_x |p(t,x)|^b |\nabla p(t,x) + \nabla V(x)|^2, \tag{2}$$

where the coefficient $b \in \mathbb{R}$ will be chosen in an appropriate range. These estimates will provide regularity and long-time asymptotic results. To this end, we exploit the fact that the evolution of the pressure is described by

$$\frac{\partial p}{\partial t} = \gamma p \Delta q + \nabla p \cdot \nabla q, \qquad \text{with } q := p + V. \tag{3}$$

Our analysis will focus on three cases: the trivial potential $V = 0$, the quadratic potential $V = |x|^2/2$, and a generic potential with bounded derivatives.

1.1 Historical remarks and recent results

The porous medium (*resp.* fast diffusion) equation, namely equation (1) with $V = 0$, is the simplest example of a nonlinear diffusion equation; it can be written as

$$\frac{\partial n}{\partial t} = \frac{|\gamma|}{\gamma + 1} \Delta n^{\gamma+1}, \tag{4}$$

for $\gamma > 0$ (*resp.* $-2/d < \gamma < 0$). The theory on these equations is nowadays well established, we refer the reader to the monographs [24, 25] for an overview. In this section, we recall some important properties of the solutions to these equations.

The porous medium equation. It is well known that, unlike solutions of the heat equation, solutions of the porous medium equation (PME) can exhibit a free boundary. In fact, since the equation is degenerate rather than uniformly parabolic, the speed of propagation is finite [19]. Consequently, if the initial data are compactly supported, the solution remains so for all times, and we may distinguish two regions $\Omega(t) := \{x;\ n(x,t) > 0\}$ and $\{x;\ n(x,t) = 0\}$, separated by a free interface. For compactly supported initial data, even if continuous, the porous medium equation does not admit a classical solution, since the solution's gradient is discontinuous on the free boundary [16]. However, in small enough neighborhoods of points in which $n(x,t) > 0$, solutions are smooth and satisfy the equation in the classical sense. Let us now recall a fundamental estimate for the porous medium equation [1, 2].

Lemma 1.1 (Aronson-Bénilan estimate)**.** *The pressure $p = n^\gamma$ satisfies*

$$\Delta p \geq -\frac{1}{\left(\gamma + \frac{2}{d}\right) t}, \qquad \textit{for all } t > 0. \tag{5}$$

This lower bound on the Laplacian of the pressure is used in [2] to prove that there exists a unique strong and continuous solution to the Cauchy problem with L^1-bounded initial data.

In [11], Caffarelli and Friedman prove that the solution $n(t,x)$ to the porous medium equation is actually Hölder continuous, uniformly in space and time. This result was further developed in [12] where the authors show that, after a certain *waiting time*, the pressure is actually Lipschitz. It is indeed known that due to the finite speed of propagation, singularities may appear in finite time if the support of the initial data contains "holes". While in the one-dimensional case the pressure is Lipschitz for all times $t > 0$, [1, 16], in higher dimensions the pressure gradient blows up at the *focusing time*, namely when the hole fully closes [3]. In particular, in [12] the authors prove that if B_{R_0} is the smallest ball containing the support of n_0, and

$t^* := \inf\{t > 0; \overline{B_{R_0}} \subset \Omega(t)\}$ is the focusing time, then $\partial_t p$ and ∇p are bounded for all $t > t^*$, and the bounds depend on n_0 and t.

Let us recall that the source solution of the porous medium equation is given by the following self-similar profile, usually referred to as Barenblatt solution

$$\begin{aligned} \mathcal{B}_{\mathrm{PME}}(t,x) &= t^{-\alpha d} F\big(x t^{-\alpha}\big), \qquad F(\xi) = \big(C - k|\xi|^2\big)_+^{\frac{1}{\gamma}}, \\ \alpha &= \frac{1}{d\gamma + 2}, \qquad k = \frac{\alpha\gamma}{2(\gamma+1)} > 0, \end{aligned} \tag{6}$$

where $C > 0$ depends on γ, d, and the mass of the initial data, $M = \|n_0\|_1$. It is nowadays well established that for $t \to \infty$ the solution of the PME converges to the self-similar profile. The quest for explicit rates of convergence has attracted a lot of attention over the last few decades, see for instance [22] and references therein. The solution satisfies the following convergence results

$$\lim_{t\to\infty} \|n(t) - \mathcal{B}_{\mathrm{PME}}(t)\|_{L^1(\mathbb{R}^d)} = 0,$$

$$\lim_{t\to\infty} t^{\alpha d} \|n(t) - \mathcal{B}_{\mathrm{PME}}(t)\|_{L^\infty(\mathbb{R}^d)} = 0,$$

with α defined as in (6). These rates are optimal for solutions with $L^1(\mathbb{R}^d)$ non-negative initial data. A related question that has been addressed by a vast literature is whether these rates of convergence can be improved for a different class of initial data. Although it is not the purpose of this paper to review in detail these results, let us mention that entropy methods have been adopted to find better rates for L^1-initial data with finite second moment, see for instance [13, 14, 20, 21]. Moreover, it has been shown that the Fisher information of equation (1) with $V = |x|^2/2$, namely

$$\mathcal{I}(n) = \int_{\mathbb{R}^d} n\, |x + \nabla p|^2 \,\mathrm{d}x,$$

decays exponentially

$$\mathcal{I}(n(t)) \le \mathcal{I}(n(t_0)) e^{-\lambda(t-t_0)}, \text{ for } t \ge t_0 > 0,$$

for some $\lambda > 0$. This result also holds for the fast diffusion equation for a more restrictive range of exponents [5, 14]. Let us notice that the functional $\mathcal{I}(n)$ is the integral counterpart of the quantity (2) for $b = 1/\gamma$, of which we study the asymptotic behavior, *cf.* Theorem 1.14.

Another interesting question is whether it is possible to obtain rates of convergence for the space derivative of the solution of equation (4). Here the main challenge arises due to the presence of solutions with compact support which does not coincide with the one of the self-similar profile. In [18], Lee and Vázquez show that after a certain time, the pressure is concave and converges to a truncated parabolic profile to all orders of differentiability. The authors assume the initial data to be compactly supported and satisfy a technical non-degeneracy condition.

An interesting estimate for the flux $\nabla n^{\gamma+1}$ that holds uniformly in space was proved in Bénilan's notes [4] where, for $n_0 \in L^\infty(\mathbb{R}^d)$, the author shows

$$\big|\nabla n^{\gamma+1}\big|^2 \le \frac{nK_1}{t}, \qquad K_1 = \frac{(\gamma+1)^2 C(\|n_0\|_\infty)}{\gamma(1-\gamma^2(d-1))},$$

under the condition that $\gamma^2(d-1) < 1$. Our method recovers this estimate. Indeed, the above inequality can be rewritten in terms of the quantity $u(t)$ in (2) for $b = 1/\gamma$, and gives

$$u(t) = \max_x n\big|\nabla n^\gamma\big|^2 \le \frac{K_2}{t}, \qquad K_2 = \frac{\gamma C(\|n_0\|_\infty)}{1-\gamma^2(d-1)},$$

which is the same estimate we provide in Theorem 1.12.

Fast diffusion equation. Let us now recall some properties of solutions to equation (4) for negative exponents, namely the standard fast diffusion equation (FDE). It is well known that, unlike the porous medium equation, the FDE admits classical solutions if the exponent satisfies $-2/d < \gamma < 0$. In particular, for any $n_0 \in L^1(\mathbb{R}^d)$ there exists a unique C^∞ solution which is always strictly positive. If γ is below the critical threshold $-2/d$, finite extinction phenomena arise and solutions may lose the mass-preservation property. Moreover, for $-2/d < \gamma < 0$, the Aronson-Bénilan estimate (5) still holds.

A self-similar solution with finite mass exists in the range $-2/d < \gamma < 0$, and it exhibits so-called *fat tails* for large values of $|x|$

$$\mathcal{B}_{\mathrm{FDE}}(t,x) = t^{-\alpha d} F\big(x t^{-\alpha}\big), \qquad F(\xi) = \big(C - k|\xi|^2\big)^{\frac{1}{\gamma}},$$

$$\alpha = \frac{1}{d\gamma + 2}, \qquad k = \frac{\alpha\gamma}{2(\gamma+1)} < 0.$$

Let us point out that since the exponent belongs to the range $-2/d < \gamma < 0$, the constant k is negative, while $\alpha > 0$. Therefore, while $\mathcal{B}^\gamma_{\mathrm{FDE}}$ is now convex, the signed pressure $-\mathcal{B}^\gamma_{\mathrm{FDE}}$ is again concave, though negative.

In the last decades, several results on the convergence of the solution of the FDE to the Barenblatt profile as $t \to \infty$ have been established under different assumptions on the initial data, see for instance [5, 10, 15, 17, 25]. In the very recent work [9], the authors find a necessary and sufficient condition on the initial data such that the solution convergences to the Barenblatt profile uniformly in *relative error*, or *weighted convergence*, namely

$$\lim_{t\to\infty} \left\| \frac{n(t) - \mathcal{B}_{\mathrm{FDE}}(t)}{\mathcal{B}_{\mathrm{FDE}}(t)} \right\|_{L^\infty(\mathbb{R}^d)} = 0. \tag{7}$$

To achieve this result, they first prove that if the initial data decays for large $|x|$ in a similar way as the source solution (see (11) for the definition of this class of data), then the solution can always be bounded from below and above by two Barenblatt profiles of masses $\underline{M}$ and $\overline{M}$, *cf.* [9, Theorem 1.1]. In particular, under appropriate conditions on n_0, for any $t_0 > 0$, there exist $\underline{\tau}, \bar\tau > 0$, and $\underline{M}, \overline{M} > 0$ such that

$$\mathcal{B}_{\mathrm{FDE}}(t - \underline{\tau}, x; \underline{M}) \le n(t,x) \le \mathcal{B}_{\mathrm{FDE}}(t + \bar\tau, x; \overline{M}), \qquad \forall x \in \mathbb{R}^d, t \ge t_0. \tag{8}$$

A rate of convergence for the relative error has been recently found in [6, Theorem 4.1].

Convergence rates of the solution of the fast diffusion to the Barenblatt profile were found in [5] for any C^k-seminorm with $k \in \mathbb{N}$, under the assumption that the initial data is bounded from above and below by two Barenblatt profiles. To this end, the authors use convenient Hölder interpolation inequalities between $L^2(\mathbb{R}^d)$ and $C^{k+1}(\mathbb{R}^d)$.

Our contribution. Our current contribution is to provide a new approach to the study of suitable Lipschitz bounds on the pressure, namely estimating the quantity defined in (2). This strategy has the advantage to work both for the PME and the FDE in a unified way, and it is essentially independent of assumptions on the initial data. It yields new results on regularity (valid for $t > 0$ and not only after some focusing time) and asymptotic behavior. However, for $\gamma > 0$ our method only works for very small γ, namely when the diffusion is almost linear. Under mild assumptions on the potential V, we provide results on equation (1) at least for smooth and suitably decaying solutions, *cf.* Proposition 1.11. For the most standard cases $V = 0$ and $V = |x|^2/2$, our results actually hold for general solutions and general initial data. In particular, for the standard equation (4) we consider as initial data any L^1-non-negative function, *cf.* Theorems 1.12 and 1.14. Moreover, using a proper time-dependent scaling, we also show a weighted convergence result for the pressure gradient, *cf.* Theorem 1.16. In the fast diffusion case, when (7) holds, such weight - which actually depends on n - can be replaced by an explicit function of time and space, leading to a new convergence rate in the C^1-seminorm.

The estimates we present in this paper are mainly applied to the study of the asymptotic behavior or the instantaneous regularization of the solutions of the PME or FDE. Yet, one of the main interests of the present paper is, in our opinion, that it shows that, despite the huge literature existing on these equations, it is still possible to find new and simple estimates with quite elementary techniques.

1.2 Preliminaries and assumptions

Time-dependent scaling. A fundamental remark which has been extensively used in the literature to study the properties of the standard PME and FDE consists in observing that solutions to equation (4) can actually be seen as solutions of a nonlinear Fokker-Planck equation with quadratic potential $V = |x|^2/2$ through the following time-dependent change of variables

$$\hat{n}(t,x) := \varphi(t)^d n(\psi(t), \varphi(t)x), \qquad \varphi(t) = e^t,\ \psi(t) = e^{(d\gamma+2)t}. \tag{9}$$

In fact, if $n(t,x)$ is a solution of equation (4), $\hat{n}(t,x)$ satisfies

$$\frac{\partial \hat{n}}{\partial t} = \frac{|\gamma|}{\gamma+1}\Delta \hat{n}^{\gamma+1} + \nabla \cdot (\hat{n}x). \tag{10}$$

Unlike the drift-less case, equation (10) has a unique compactly supported stationary state, which coincides with the Barenblatt profile evaluated at an appropriate time $t = t(d,\gamma)$. This property allows us to infer long-time behavior results on the porous medium/fast diffusion equation from the long-time behavior of solutions of this Fokker-Planck equation.

Tail behavior of the FDE solution. We now review some particular properties of the FDE that will be used throughout the paper. It is known that if γ belongs to the range $-2/d < \gamma < 0$, the solution exhibits polynomial tails. In [9, Theorem 1.1], the authors give a necessary and sufficient condition for the solution to satisfy (8), namely to be bounded from below and above by two Barenblatt profiles, which is $n_0 \in \mathcal{X} \setminus \{0\}$, where

$$\mathcal{X} := \{u \in L^1(\mathbb{R}^d), u \geq 0, |u|_{\mathcal{X}} < \infty\}, \text{ with } |u|_{\mathcal{X}} := \sup_{R>0} R^{-\frac{2}{\gamma}-d} \int_{B_R^c} |u| \, \mathrm{d}x < \infty. \tag{11}$$

Moreover, as shown in the proof of [5, Theorem 4], the gradient of the solution can be bounded uniformly by $|x|^{2/\gamma-1}$. We collect these properties as follows: for all t_0, there exist $k(t), K(t), c(t) \in L^\infty_{loc}(0,\infty)$ and $R > 0$ such that

$$k(t)(1+|x|^2)^{1/\gamma} \leq n(t,x) \leq K(t)(1+|x|^2)^{1/\gamma}, \quad |\nabla n(t,x)| \leq c(t)|x|^{2/\gamma-1}, \quad \forall |x| > R, t \geq t_0. \tag{12}$$

This tail behavior will be essential for us in order to rigorously justify the formal computations. However, this behavior is only known to hold in the drift-less case (4) and, through the change of variables (9), for equation (10). In order to provide a rigorous justification for the more general case, namely equation (1) for a generic potential V, we will impose (12) as an assumption on the solution, see Definition 1.9. In the trivial and quadratic potential cases, this condition will be removed later on using approximation arguments.

Local L^∞-bound on $n(t)$. An important consequence of the *semi-subharmonicity* of the pressure given by the fundamental estimate (5) is to provide a local bound on $\|n(t)\|_\infty$. In the porous medium case, this translates into a bound on $\|p(t)\|_\infty$, while for the fast diffusion, we infer a uniform (in space) lower bound.

Lemma 1.2. *Let $n(t,x)$ be the solution of equation (4) with $|\gamma| < 1$. There exists a positive constant C such that the L^∞-norm of $n(t)$ satisfies*

$$\|n(t)\|_\infty \le Ct^{-\frac{d}{d\gamma+2}}, \tag{13}$$

which is equivalent to the following bounds on the pressure

- $\max_x |p(t)| \le Ct^{-\frac{d\gamma}{d\gamma+2}}$, *for $\gamma > 0$,*
- $\min_x |p(t)| \ge Ct^{-\frac{d\gamma}{d\gamma+2}}$, *for $\gamma < 0$.*

Proof. Let $\gamma > 0$, and let $\bar{x}$ be a point in the support of $n(t)$. We denote $B_r(\bar{x})$ the ball with radius $r > 0$ centered at $\bar{x}$. Thanks to the Aronson-Bénilan estimate (5), we know that the function $f(x) := p(x) + \frac{K}{2d}|x-\bar{x}|^2$ is subharmonic for all $\bar{x} \in \mathbb{R}^d$, where $K = \frac{1}{(\gamma+2/d)t}$. Therefore, from

$$f(\bar{x}) \le \fint_{B_r(\bar{x})} f(x)\,\mathrm{d}x,$$

we find

$$\fint_{B_r(\bar{x})} p\,\mathrm{d}x \ge p(\bar{x}) - \frac{Kr^2}{2(d+2)}.$$

Let us choose a radius $R > 0$ such that $R^2 = cp(\bar{x})t$, with $c > 0$ small enough. Then, for all $0 < r \le R$ we have

$$\fint_{B_r(\bar{x})} p\,\mathrm{d}x \ge Cp(\bar{x}),$$

where from now on $C > 0$ denotes a constant that may change value from line to line. Since $p = n^\gamma$ and we are considering a range in which $\gamma < 1$, by Jensen's inequality, we have

$$\fint_{B_r(\bar{x})} n\,\mathrm{d}x = \fint_{B_r(\bar{x})} p^{1/\gamma}\,\mathrm{d}x \ge \left(\fint_{B_r(\bar{x})} p\,\mathrm{d}x\right)^{1/\gamma} \ge Cp(\bar{x})^{1/\gamma}.$$

Using the fact that n has constant mass at all times, integrating between 0 and R we find

$$M \ge \int_{B_R(\bar{x})} n\,\mathrm{d}x \ge R^d p(\bar{x})^{1/\gamma}.$$

By definition $R^2 = cp(\bar{x})t$, and we may finally establish the following bound

$$p(\bar{x}) \le Ct^{-\frac{d\gamma}{d\gamma+2}}.$$

Since $\bar{x}$ was chosen arbitrarily, we conclude that $p \in L^\infty_{loc}(0,\infty; L^\infty(\mathbb{R}^d))$ and the same upper-bound hols for the L^∞-norm of the pressure, while equation (13) holds for $n(t)$.

For $-2/d < \gamma < 0$ the Aronson-Bénilan estimate still holds for $p = -n^\gamma$. Therefore, we may argue in the same way choosing $R^2 = c|p(\bar{x})|t$ to obtain

$$\fint_{B_r(\bar{x})} p\,\mathrm{d}x \ge Cp(\bar{x}).$$

Since $\gamma < 0$, the function $s \mapsto s^{1/\gamma}$ is convex and decreasing on $s > 0$, so that we can use Jensen's inequality and obtain, for $p = -n^\gamma$:

$$\fint_{B_r(\bar{x})} n \, \mathrm{d}x = \fint_{B_r(\bar{x})} (-p)^{1/\gamma} \, \mathrm{d}x \geq \left(- \fint_{B_r(\bar{x})} p \, \mathrm{d}x \right)^{1/\gamma} \geq C |p(\bar{x})|^{1/\gamma}.$$

Using again that the mass is preserved, we find

$$M \geq \int_{B_R(\bar{x})} n \, \mathrm{d}x \geq R^d |p(\bar{x})|^{1/\gamma},$$

from which we conclude

$$\min_x |p(x)| \geq C t^{-\frac{d\gamma}{d\gamma+2}},$$

which is the claim. □

Assumptions. We now state the assumptions that we will alternatively impose on the potential V and the coefficients b, γ.

Assumption 1.3. The potential V satisfies $|\nabla V|, D^2 V \in L^\infty(\mathbb{R}^d)$.

Assumption 1.4. The potential V is the quadratic one: $V(x) = |x|^2/2$.

Assumption 1.5. For $\gamma, b > 0$, *resp.* $\gamma, b < 0$, we assume

(i) $\gamma \leq \min\left(\frac{1}{\sqrt{d}}, \frac{2}{d}, \frac{1}{2}\right)$, *resp.* $|\gamma| < \min\left(\frac{2}{d}, \frac{4}{3+d}\right)$,

(ii) $1 - \sqrt{1-\gamma^2(d-1)} < \gamma b < 1 + \sqrt{1-\gamma^2(d-1)}$,

(iii) $\gamma \leq \gamma b \leq 1 - \gamma$, *resp.* $|\gamma| < \gamma b \leq \min(1+|\gamma|, 2|\gamma|)$.

Assumption 1.6. For $\gamma, b > 0$, *resp.* $\gamma, b < 0$, we assume

(i) $\gamma < \min\left(\frac{1}{\sqrt{d}}, \frac{2}{d}\right)$, *resp.* $|\gamma| < \frac{2}{d}$,

(ii) $1 - \sqrt{1-\gamma^2(d-1)} \leq \gamma b \leq 1 + \sqrt{1-\gamma^2(d-1)}$,

(iii) $\gamma b < \min\left(1-\gamma, \frac{2}{d}\right)$, *resp.* $|\gamma| < \gamma b < \min\left(1+|\gamma|, \frac{2}{d}\right)$.

Assumption 1.7. For $\gamma, b > 0$, *resp.* $\gamma, b < 0$, we assume

(i) $\gamma < \frac{1}{\sqrt{d-1}}$, *resp.* $|\gamma| < \frac{2}{d}$,

(ii) $1 - \sqrt{1-\gamma^2(d-1)} < \gamma b < 1 + \sqrt{1-\gamma^2(d-1)}$, *resp.* $|\gamma| < \gamma b < 1 + \sqrt{1-\gamma^2(d-1)}$.

Remark 1.8. For assumptions (1.5, 1.6) condition *(i)* ensure that there exists b that satisfies both *(ii)* and *(iii)*. For the sake of completeness, we here report the proof of this statement. For assumption 1.5, $\gamma > 0$, condition *(i)* ensures that:

$$\begin{aligned} \gamma < 1/\sqrt{d} &\Rightarrow 1 - \gamma > 1 - \sqrt{1-\gamma^2(d-1)}, \\ \gamma < 2/d &\Rightarrow \gamma < 1 + \sqrt{1-\gamma^2(d-1)}, \\ \gamma < 1/2 &\Rightarrow \gamma < 1 - \gamma. \end{aligned}$$

This guarantees that under condition *(i)* we have

$$\max\{\gamma, 1-\sqrt{1-\gamma^2(d-1)}\} < \min\{1-\gamma, 1+\sqrt{1-\gamma^2(d-1)}\}.$$

For $\gamma < 0$, condition *(i)* gives

$$|\gamma| < 2/d \Rightarrow |\gamma| < 1+\sqrt{1-\gamma^2(d-1)},$$
$$|\gamma| < 4/(3+d) \Rightarrow 2|\gamma| > 1-\sqrt{1-\gamma^2(d-1)},$$

which guarantees in this case

$$\max\{|\gamma|, 1-\sqrt{1-\gamma^2(d-1)}\} < \min\{1+|\gamma|, 2|\gamma|, 1+\sqrt{1-\gamma^2(d-1)}\}.$$

Therefore, in both cases, there exists at least one $b \in \mathbb{R}$, $\gamma b > 0$, that satisfies both *(ii)* and *(iii)*.

For assumption 1.6, $\gamma > 0$, we first note that we have

$$2/d > 1-\sqrt{1-\gamma^2(d-1)},$$

and then, for $\gamma > 0$

$$\gamma < 1/\sqrt{d} \Rightarrow 1-\gamma > 1-\sqrt{1-\gamma^2(d-1)},$$

while for $\gamma < 0$,

$$|\gamma| < 2/d \Rightarrow |\gamma| < 1+\sqrt{1-\gamma^2(d-1)},$$

and the same conclusion holds in the same way.

1.3 Main results

Decay estimates on Lipschitz norms. We may now state the main results of the paper. First of all, let us give the definition of solutions that allows us to justify our computations. We later prove that in the most relevant cases, these assumptions can be removed by approximation arguments.

Definition 1.9. Let $n(t,x)$ be a strong solution of equation (1). We say that $n(t,x)$ is a *well-behaved* solution if

- *(for $\gamma > 0$)* n is continuous on $(0,\infty)\times\mathbb{R}^d$, and for each $t > 0$ the the support of $n(t)$ is compact and contained in a ball $B(0,R(t))$ where R is locally bounded on $(0,\infty)$; moreover, $n \in C^\infty(\{(t,x);\ n(t,x) > 0\})$, and $p \in L^\infty_{loc}((0,\infty); W^{1,\infty}(\mathbb{R}^d))$,
- *(for $\gamma < 0$)* $n \in C^\infty((0,\infty)\times\mathbb{R}^d)$, $n > 0$ on $(0,\infty)\times\mathbb{R}^d$, and n satisfies (12).

Remark 1.10. In order to justify the formal computation carried on in the proof of Proposition 1.11 we assume to deal with a solution satisfying the conditions of definition 1.9. As will be discussed in Section 2, for the cases $V = 0$, $V = |x|^2/2$ (which are equivalent one to the other thanks to the time-dependent scaling (9)), a solution that satisfies definition 1.9 exists for a large class of initial data which is, in fact, dense in $L^1(\mathbb{R}^d)$. Therefore, we will show that the same conclusions hold assuming only $n_0 \in L^1(\mathbb{R}^d)$ upon using an approximation argument and proving the convergence of the approximating sequence.

Proposition 1.11. *Let $n(t,x)$ be a solution to (1) that satisfies Definition 1.9. Then,*

- ***Generic V:*** *under assumptions* (1.3) *and* (1.5)*, if* $n \in L^\infty((0,\infty)\times\mathbb{R}^d)$*, there exist positive constants* C_1, C_2 *such that*

$$\max_x |p(t)|^b |\nabla q(t)|^2 \le \max\bigl(C_1, C_2 t^{-1}\bigr), \text{ for all } t > 0.$$

  If moreover, $\sup_x |p_0|^b |\nabla p_0 + \nabla V|^2 < \infty$*, we have*

$$\max_x |p(t)|^b |\nabla q(t)|^2 \le C, \text{ for all } t > 0.$$

- ***Quadratic V:*** *under assumptions* (1.4) *and* (1.6)*, and* $C_0 := \sup_x |p_0|^b |\nabla p_0 + x|^2 < \infty$*, there exists a positive constant* C *such that*

$$\max_x |p(t)|^b |\nabla q(t)|^2 \le C_0 e^{-Ct}, \text{ for all } t > 0.$$

- ***Trivial V:*** *assuming* $V = 0$*, under assumption* (1.7) *there exists a positive constant* C *such that*

$$\max_x |p(t)|^b |\nabla p(t)|^2 \le C t^{-1-\gamma d(b+1)\alpha}, \text{ for all } t > 0,$$

  where α *is given in* (6)*. Moreover, the above exponent is sharp.*
  Assuming $n_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$*, there exists a positive constant* C *such that*

$$\max_x |p(t)|^b |\nabla p(t)|^2 \le C t^{-1}, \text{ for all } t > 0.$$

For both trivial and quadratic potentials, the same estimates hold for a much larger class of solutions.

Theorem 1.12 (Trivial potential)**.** *Let* $V = 0$*,* $n_0 \in L^1(\mathbb{R}^d)$ *and* $n(t,x)$ *be the solution of equation* (1)*. Under assumptions (1.7), there exists a positive constant* C *such that*

$$\max_x |p(t)|^b |\nabla p(t)|^2 \le C t^{-1-\gamma d(b+1)\alpha}, \text{ for all } t > 0.$$

Moreover, this exponent is sharp in that equality holds for the Barenblatt solution.
Finally, assuming $n_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$*, there exists a positive constant* C *such that*

$$\max_x |p(t)|^b |\nabla p(t)|^2 \le C t^{-1}, \text{ for all } t > 0.$$

Remark 1.13. One can actually show that given the L^∞-norm of the initial data, $N := \|n_0\|_\infty$, t^{-1} is the sharpest polynomial decay of the quantity $\max_x |p(t)|^b |\nabla p(t)|^2$. This property can be easily verified by considering the solution whose initial data is the Barenblatt profile at some given time t_0, with L^∞-norm equal to N. Then, computing the quantity $|p(t,x)|^b |\nabla p(t,x)|^2$ in $|x|^2 = t_0$ and $t = \varepsilon t_0$ one can show that its maximum satisfies $u(t) = \mathcal{O}(1/t)$.

Theorem 1.14 (Quadratic potential)**.** *Let* $V = |x|^2/2$*,* $n_0 \in L^1(\mathbb{R}^d)$ *and* $n(t,x)$ *be the solution of equation* (1)*. If* $C_0 := \sup_x |p_0|^b |\nabla p_0 + x|^2 < \infty$*, under assumptions* (1.4) *and* (1.6) *the following holds*

$$\max_x |p(t)|^b |\nabla q(t)|^2 \le C_0 e^{-Ct}, \text{ for all } t > 0,$$

with $C = 1 - \gamma b d/2 > 0$*.*

Remark 1.15. Under a weaker assumption on the initial data, we may state that the above result holds away from $t = 0$. If we call $\hat{n}$ a solution of equation (1) (and $\hat{p}$ the corresponding pressure) with $V = |x|^2/2$ then, using the change of variables in (9) the corresponding n solves equation (4). This means that we have instantaneous regularization for n and hence also for $\hat{n}$. In particular, $\max_x |p(\tau)|^b |\nabla p(\tau)|^2 \le C(\tau)$. Thus, assuming only $\sup_x |\hat{p}(\tau)|^b |x|^2 < \infty$, we obtain that Theorem 1.14 holds for $\hat{n}(t,x)$ for all $t > \tau$.

Long-time convergence to the Barenblatt profile. As already mentioned, through the change of variables (9), results for the solution to the drift-less equation can be translated into results for the Fokker-Planck equation and viceversa. However, let us remark that the results stated above, namely Theorem 1.12 and Theorem 1.14, actually provide two different information, since the quantity for which we find decay rates is not the same for the two equations. Therefore, the latter result gives additional insights into the asymptotics of the solution to the standard equation, in particular, it provides convergence results of its gradient to the gradient of the self-similar profile. Applying Theorem 1.14 to the rescaled solution allows us to state the following result.

Theorem 1.16 (Weighted convergence to the Barenblatt gradient)**.** *Let $n(t,x)$ be the solution of equation* (4)*. If $C_0 := \sup_x |p_0|^b|\nabla p_0 + x|^2 < \infty$, under assumptions* (1.6) *we have*

$$\max_x |p(t,x)|^b \left|\nabla p(t,x) + xt^{-1}\right|^2 \le C_0 t^{\beta} t^{-\frac{C}{d\gamma+2}}, \text{ for all } t > 0, \tag{14}$$

with $C = 1 - \gamma bd/2$ and

$$\beta := -\alpha\gamma db - 2 + 2\alpha.$$

The above inequality provides a convergence result of the pressure gradient towards the gradient of the source solution, although weighted by $|p|^b$. Let us recall that the pressure of the Barenblatt profile is

$$\mathcal{P}(t,x) := t^{-\alpha\gamma d}\big(C - k|x|^2 t^{-2\alpha}\big)_+. \tag{15}$$

and thus its gradient is proportional to $-xt^{-1}$. Moreover, the exponent β is exactly the sharp exponent of Theorem 1.12, which means $|\mathcal{P}|^b|\nabla\mathcal{P}|^2 \simeq t^{\beta}$ in cylinders of the form $\{(t,x);\ k|x| = t^{\alpha}\}$. Therefore, in the fast diffusion case, assuming to be in a regime in which the solution $n(t,x)$ can actually be approximated by a Barenblatt profile, we may remove the weight from (14) and infer a new convergence result of the pressure gradient.

Structure of the paper. In Section 2, we study the decay of the Lipschitz-like norm defined in (2) and prove Proposition 1.11. Then, using an approximation argument we prove Theorems 1.12 and 1.14, namely the results for any L^1-bounded initial data in the drift-less and quadratic potential cases. Section 3 deals with the proof of Theorem 1.16 from which we infer the long-time asymptotic behavior of the solution to the fast diffusion equation. Finally, in Section 4 we show that the same asymptotic results obtained in the whole space also hold for the problem set in a convex bounded domain with homogeneous Neumann boundary conditions and also discuss the homogeneous Dirichlet case.

2 Decay estimates on Lipschitz norms

First, we show the main formal computation, from which we will deduce the decay results of Proposition 1.11 which will be fully justified later on in this section. Then, we prove Theorems 1.12 and 1.14 through a regularization argument.

Given $t_0 > 0$, let us assume that there exists $x_0 \in \mathbb{R}^d$ such that

$$\max_x e^{\alpha(p(t_0,x))}|\nabla q(t_0,x)|^2 = e^{\alpha(p(t_0,x_0))}|\nabla q(t_0,x_0)|^2, \tag{16}$$

where $\alpha = \alpha(p)$ is a continuously differentiable function of the pressure, which will be later chosen as $\alpha(p) = b\ln|p|$, $b \in \mathbb{R}$. We denote

$$v(t) := e^{\alpha(p(t,x_0))}\frac{|\nabla q(t,x_0)|^2}{2},$$

We now compute $\left(\frac{d}{dt}v(t,x_0)\right)_{|t=t_0}$. For the sake of simplicity, from now on we omit the dependency upon t_0 and x_0, and we write α, α', and α'' for $\alpha(p), \alpha'(p)$, and $\alpha''(p)$. Since x_0 is a maximum point, the following conditions hold

$$\nabla\left(e^{\alpha(p)}\frac{|\nabla q|^2}{2}\right) = 0, \qquad \Delta\left(e^{\alpha(p)}\frac{|\nabla q|^2}{2}\right) \leq 0.$$

The first condition gives

$$D^2 q \nabla q = -\frac{\alpha'}{2}|\nabla q|^2 \nabla p, \tag{17}$$

while from the second we infer

$$\Delta\left(e^{\alpha(p)}\frac{|\nabla q|^2}{2}\right) = e^{\alpha}\left(\frac{\alpha'' + |\alpha'|^2}{2}|\nabla q|^2|\nabla p|^2 + 2\alpha'\nabla p D^2 q\nabla q + \alpha'\frac{|\nabla q|^2}{2}\Delta p + \Delta\left(\frac{|\nabla q|^2}{2}\right)\right) \leq 0.$$

Using (17), we find

$$\Delta\left(\frac{|\nabla q|^2}{2}\right) \leq \frac{|\nabla q|^2|\nabla p|^2}{2}\left(|\alpha'|^2 - \alpha''\right) - \alpha'\frac{|\nabla q|^2}{2}\Delta p. \tag{18}$$

We compute the time derivative using equation (3)

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{\alpha}\frac{|\nabla q|^2}{2}\right) &= e^{\alpha}\left(\alpha'\frac{|\nabla q|^2}{2}\left(\gamma p\Delta q + \nabla p\cdot\nabla q\right) + \nabla q\cdot\nabla\left(\gamma p\Delta q + \nabla p\cdot\nabla q\right)\right) \\
&= e^{\alpha}\Big(\gamma p\frac{\alpha'}{2}|\nabla q|^2\Delta q + \frac{\alpha'}{2}|\nabla q|^2\nabla p\cdot\nabla q + \gamma\nabla q\cdot\nabla p\Delta q + \gamma p\nabla q\cdot\nabla\Delta q \\
&\qquad + \nabla q D^2 p\nabla q + \nabla q D^2 q\nabla p\Big).
\end{aligned}$$

Using (17) and the equality $2\nabla q\cdot\nabla(\Delta q) = \Delta(|\nabla q|^2) - 2|D^2 q|^2$, where $|D^2 q|^2 := \sum_{i,j=1}^{d}(\partial_{i,j}q)^2$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{\alpha}\frac{|\nabla q|^2}{2}\right) &= e^{\alpha}\Big(\gamma p\frac{\alpha'}{2}|\nabla q|^2\Delta q + \frac{\alpha'}{2}|\nabla q|^2\nabla p\cdot\nabla q + \gamma\nabla q\cdot\nabla p\Delta q + \gamma p\Delta\left(\frac{|\nabla q|^2}{2}\right) - \gamma p|D^2 q|^2 \\
&\qquad - \frac{\alpha'}{2}|\nabla q|^2\nabla q\cdot\nabla p - \nabla q D^2 V\nabla q - \frac{\alpha'}{2}|\nabla q|^2|\nabla p|^2\Big) \\
&\leq e^{\alpha}\Big(\gamma p\frac{\alpha'}{2}|\nabla q|^2\Delta q + \gamma\nabla q\cdot\nabla p\Delta q + \gamma p\frac{|\nabla q|^2|\nabla p|^2}{2}(|\alpha'|^2 - \alpha'') - \gamma p\frac{\alpha'}{2}|\nabla q|^2\Delta p \\
&\qquad - \gamma p|D^2 q|^2 - \nabla q D^2 V\nabla q - \frac{\alpha'}{2}|\nabla q|^2|\nabla p|^2\Big) \\
&= e^{\alpha}\Big(\gamma p\frac{\alpha'}{2}|\nabla q|^2\Delta V + \gamma p\frac{|\nabla q|^2|\nabla p|^2}{2}(|\alpha'|^2 - \alpha'') \\
&\qquad + \underbrace{\gamma\nabla q\cdot\nabla p\Delta q - \gamma p|D^2 q|^2}_{\mathcal{A}} - \nabla q D^2 V\nabla q - \frac{\alpha'}{2}|\nabla q|^2|\nabla p|^2\Big),
\end{aligned}$$

where in the above inequality we used (18) and the fact that $\gamma p > 0$.

Let us now treat the term $\mathcal{A}$. We choose some coordinates where the fist vector of the basis is oriented as $\nabla q/|\nabla q|$. With this choice, we notice that from (17) we know that the value of the element $(D^2 q)_{1,1}$ of the Hessian matrix is

$$(D^2 q)_{1,1} = \left(\frac{\nabla q}{|\nabla q|}\right)^T D^2 q \left(\frac{\nabla q}{|\nabla q|}\right) = -\frac{\alpha'}{2}\nabla q \cdot \nabla p.$$

Let us denote

$$\lambda := -\frac{\alpha'}{2}\nabla q \cdot \nabla p, \quad \delta_i = (D^2 q)_{i,i} \text{ for } i = 2, \dots, d, \quad \delta := \sum_{i=2}^{d} \delta_i.$$

Since $\gamma p > 0$, we have

$$\begin{aligned}
\mathcal{A} = \gamma \nabla q \cdot \nabla p \Delta q - \gamma p |D^2 q|^2 &\le \gamma \nabla q \cdot \nabla p(\lambda + \delta) - \gamma p\left(\lambda^2 + \sum_{i=2}^{d} \delta_i^2\right) \\
&\le \gamma \nabla q \cdot \nabla p(\lambda + \delta) - \gamma p\left(\lambda^2 + \frac{\delta^2}{d-1}\right) \\
&= -\gamma\frac{\alpha'}{2}|\nabla q \cdot \nabla p|^2 + \gamma\delta\nabla q \cdot \nabla p - \gamma p\frac{|\alpha'|^2}{4}|\nabla q \cdot \nabla p|^2 - \gamma p\frac{\delta^2}{d-1}.
\end{aligned}$$

Using Young's inequality, we have

$$\gamma\delta\nabla q \cdot \nabla p \le \gamma p\frac{\delta^2}{d-1} + \gamma\frac{|\nabla q \cdot \nabla p|^2}{4p}(d-1),$$

and we finally find

$$\mathcal{A} \le \left(-\gamma\frac{\alpha'}{2} + \frac{\gamma(d-1)}{4p} - \gamma p\frac{|\alpha'|^2}{4}\right)|\nabla q \cdot \nabla p|^2.$$

Coming back to the estimate of the time derivative, and using the above inequality, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(e^{\alpha}\frac{|\nabla q|^2}{2}\right) \le e^{\alpha}\Bigg(&|\nabla q|^2|\nabla p|^2\left(-\frac{\alpha'}{2} + \gamma p\frac{|\alpha'|^2}{2} - \gamma p\frac{\alpha''}{2}\right) \\
&+ |\nabla q \cdot \nabla p|^2\left(-\gamma\frac{\alpha'}{2} + \frac{\gamma(d-1)}{4p} - \gamma p\frac{|\alpha'|^2}{4}\right) \\
&+ \gamma p\frac{\alpha'}{2}\Delta V|\nabla q|^2 - \nabla q D^2 V \nabla q\Bigg).
\end{aligned} \tag{19}$$

Now let us take $\alpha(p) = b\ln|p|$, with $b \in \mathbb{R}$ satisfying the assumptions of Proposition 1.11. In particular, $b\gamma > 0$. Substituting in (19) we get

$$\begin{aligned}
\frac{\partial}{\partial t}\left(|p|^b\frac{|\nabla q|^2}{2}\right) \le |p|^b\Bigg(&|\nabla q|^2|\nabla p|^2\left(-\frac{b}{2p} + \frac{\gamma b^2}{2p} + \frac{\gamma b}{2p}\right) + |\nabla q \cdot \nabla p|^2\left(-\frac{\gamma b}{2p} + \frac{\gamma(d-1)}{4p} - \frac{\gamma b^2}{4p}\right) \\
&+ \frac{\gamma b}{2}\Delta V|\nabla q|^2 - \nabla q D^2 V \nabla q\Bigg).
\end{aligned} \tag{20}$$

We denote

$$c_1 = -\frac{|b|}{2} + \frac{|\gamma|b^2}{2} + \frac{\gamma|b|}{2},$$
$$c_2 = -\frac{\gamma|b|}{2} + \frac{|\gamma|(d-1)}{4} - \frac{|\gamma|b^2}{4},$$

and

$$c_0 := c_1 + c_2 = -\frac{|b|}{2} + \frac{|\gamma|b^2}{4} + \frac{|\gamma|(d-1)}{4}.$$

We rewrite (20) as follows

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) \leq |p|^{b-1}\left(c_1|\nabla q|^2|\nabla p|^2 + c_2|\nabla q \cdot \nabla p|^2\right) + |p|^b\left(\frac{\gamma b}{2}\Delta V|\nabla q|^2 - \nabla q D^2 V \nabla q\right). \quad (21)$$

Note that, because of $|\nabla q \cdot \nabla p| \leq |\nabla q||\nabla p|$, whenever we have $c_1 \leq 0$, we can use

$$c_1|\nabla q|^2|\nabla p|^2 + c_2|\nabla q \cdot \nabla p|^2 \leq c_0|\nabla q \cdot \nabla p|^2. \quad (22)$$

Now we treat each different choice of potential separately.

2.1 Generic potential with bounded derivatives

Let V satisfy assumption (1.3), and let $n \in L^\infty((0,\infty) \times \mathbb{R}^d)$. By assumption (1.5) - points *(ii)* and *(iii)* -, we have $c_0 < 0, c_1 \leq 0$, so that we can use (22).

Since $p = q - V$, from equation (21) we find

$$\begin{aligned}\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) &\leq c_0|p|^{b-1}|\nabla q \cdot \nabla p|^2 + |p|^b\left(\frac{\gamma b}{2}\Delta V|\nabla q|^2 - \nabla q D^2 V \nabla q\right)\\
&\leq c_0|p|^{b-1}|\nabla q|^4 + c_0|p|^{b-1}|\nabla q \cdot \nabla V|^2 - 2c_0|p|^{b-1}|\nabla q|^2\nabla q \cdot \nabla V\\
&\quad + \frac{\gamma b}{2}|p|^b|\nabla q|^2||\Delta V||_\infty + |p|^b||D^2 V||_\infty|\nabla q|^2\\
&\leq c_0|p|^{b-1}|\nabla q|^4 + 2|c_0||p|^{b-1}|\nabla V|_\infty|\nabla q|^3 + C(\gamma, b, ||D^2V||_\infty)|p|^b|\nabla q|^2.\end{aligned}$$

By Young's inequality, we have

$$2|c_0||p|^{b-1}|\nabla V|_\infty|\nabla q|^3 \leq \frac{3}{4}|c_0||p|^{b-1}|\nabla q|^4 + 4|c_0||p|^{b-1}|\nabla V|_\infty^4,$$

hence

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) \leq \frac{c_0}{4}|p|^{b-1}|\nabla q|^4 + 4|c_0||p|^{b-1}||\nabla V||_\infty^4 + C|p|^b|\nabla q|^2, \quad (23)$$

for some $C > 0$.

Let us start from the case $\gamma > 0$. By the global boundedness of the density, there exists $\overline{p} > 0$ such that $|p| \leq \overline{p}$. By assumption (1.5) $b - 1 > 0$, thus we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) &\leq \frac{c_0}{4}|p|^{2b}|\overline{p}|^{-b-1}|\nabla q|^4 + 4|c_0||\overline{p}|^{b-1}||\nabla V||_\infty^4 + C|p|^b|\nabla q|^2\\
&\leq -C_1|p|^{2b}|\nabla q|^4 + C_2|p|^b|\nabla q|^2 + C,\end{aligned}$$

where C_1, C_2 denotes positive constants whose value may from now on change from line to line. Recalling the definition $v(t) := \frac{1}{2}|p(t,x_0)|^b|\nabla q(t,x_0)|^2$, we obtain the differential inequality

$$v'(t_0) \leq -C_1 v^2(t_0) + C_2 v(t_0) + C. \tag{24}$$

We now consider the fast diffusion case, $\gamma < 0$. From $n \in L^\infty((0,\infty)\times\mathbb{R}^d)$, there exists $\overline{p} < 0$ such that $|p(t,x)| \geq |\overline{p}|$ for all $(x,t) \in \mathbb{R}^d \times (0,\infty)$. We come back to (23) and write

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) \leq c_0|p|^{2b}|\overline{p}|^{-b-1}|\nabla q|^4 + 2|c_0||p|^{3b/2}|\overline{p}|^{-b/2-1}|\nabla V|_\infty|\nabla q|^3 + C|p|^b|\nabla q|^2,$$

where we used $-b-1 \geq 0$ and $-b/2-1 \leq 0$, by assumption (1.5). This can be written in the form

$$v'(t_0) \leq -C_1 v^2(t_0) + Cv(t_0)^{3/2} + C_2 v(t_0) + C,$$

and, after applying the Young inequality $v(t_0)^{3/2} \leq \varepsilon v(t_0)^2 + C(\varepsilon)$ for suitable $\varepsilon > 0$ small, we deduce again (24).

2.2 Quadratic potential

Under assumption (1.4) we have $V = |x|^2/2$ and equation (21) reads

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) \leq |p|^{b-1}\left(c_1|\nabla q|^2|\nabla p|^2 + c_2|\nabla q\cdot\nabla p|^2\right) + c_3|p|^b|\nabla q|^2,$$

where

$$c_3 = \frac{\gamma b d}{2} - 1.$$

Conditions *(ii)* and *(iii)* of assumption (1.6) ensure $c_0 \leq 0$, and $c_1, c_3 < 0$. Therefore, we establish

$$\begin{aligned}\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla q|^2}{2}\right) &\leq c_0|p|^b|\nabla q\cdot\nabla p|^2 + c_3|p|^b|\nabla q|^2\\ &\leq c_3|p|^b|\nabla q|^2,\end{aligned}$$

namely

$$v'(t_0) \leq -Cv(t_0). \tag{25}$$

Remark 2.1. Let us point out that the same conclusion holds for any potential V that satisfies

$$D^2V - \Delta V\frac{\gamma b}{2}I \geq cI,$$

for some positive constant $c = c(\gamma, b, d)$ where $I \in \mathbb{R}^d \times \mathbb{R}^d$ denotes the identity matrix.

2.3 Trivial potential

If $V = 0$, then $q = p$ and equation (21) reads

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla p|^2}{2}\right) \leq c_0|p|^{b-1}|\nabla p|^4. \tag{26}$$

We recall that c_0 is defined as

$$c_0 = -\frac{|b|}{2} + \frac{|\gamma|b^2}{4} + \frac{|\gamma|(d-1)}{4},$$

and is negative under assumption (1.7).

Let us first prove the claimed result for $n_0 \in L^\infty(\mathbb{R}^d)$. For $\gamma, b > 0$, we have $|p(t,x)|^{-b-1} \geq |\overline{p}|^{-b-1}$. For $\gamma < 0$, by assumption $b \leq -1$, and $|p(t,x)| \geq |\overline{p}|$. Therefore, in both cases we find

$$\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla p|^2}{2}\right) \leq c_0 |p|^{2b} |\overline{p}|^{-b-1} |\nabla p|^4,$$

from which we find

$$v'(t_0) \leq -Cv^2(t_0). \tag{27}$$

We now show that we may find a different differential inequality, which actually improves the rate of decay for large times, by using the L^∞-bounds provided by Lemma 1.2. For this, we no longer need the uniform boundedness assumption on the initial data.

For $\gamma > 0$, we apply (13) to (26) to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(p^b \frac{|\nabla p|^2}{2}\right) &\leq c_0 p^{2b} p^{-(b+1)} |\nabla p|^4 \\ &\leq -Cp^{2b} |\nabla p|^4 t_0^{(b+1)\frac{d\gamma}{d\gamma+2}},\end{aligned}$$

namely

$$v'(t_0) \leq -Cv^2(t_0) t_0^{\gamma d(b+1)\alpha}. \tag{28}$$

In the fast diffusion case, we apply the lower bound of Lemma 1.2 to (26) to infer

$$\begin{aligned}\frac{\partial}{\partial t}\left(|p|^b \frac{|\nabla p|^2}{2}\right) &\leq c_0 |p|^{2b} |p|^{-(b+1)} |\nabla p|^4 \\ &\leq -Cp^{2b} |\nabla p|^4 t_0^{(b+1)\frac{d\gamma}{d\gamma+2}},\end{aligned}$$

where we used $b + 1 < 0$ by assumption (1.7). Finally, we obtain again (28).

2.4 Proof of Proposition 1.11

We conclude the proof of Proposition 1.11 by showing that the differential inequalities (24),(25),(27) and (28) actually hold for the quantity $u(t)$

$$u(t) = \max_x |p(t,x)|^b \frac{|\nabla q(t,x)|^2}{2}.$$

In order to do so, we first need to show that for every $t_0 > 0$ there exists x_0 such that (16) holds, namely the maximum is attained. Secondly, we will prove that $u = u(t)$ is locally Lipschitz on $(0,\infty)$. Before proving both claims in the next paragraph, let us draw now draw the main conclusions.

Since u is locally Lipschitz it is also almost everywhere differentiable. Let $t_0 > 0$ be a point in which u is differentiable. Since $u(t) \geq v(t)$ for all $t > 0$ and $u(t_0) = v(t_0)$, we have $u'(t_0) = v'(t_0)$, and the differential inequalities proven for v at $t = t_0$ also hold for the function u. Now, the definition of the function v involved x_0 (and hence t_0), but this is no more the case for the function u. Hence, u is a locally Lipschitz function on $(0,\infty)$ solving a.e. an ODE, and we can use such an ODE to obtain estimates on u. In particular, we may conclude that

- for a generic potential V: (24) implies

  $$u(t) \leq \max(C_1, C_2 t^{-1}), \quad \forall t > 0,$$

  and if in addition we assume $\sup_x |p(0,x)|^b |\nabla p(0,x) + \nabla V(x)|^2 < \infty$, this implies $u(0) < \infty$, hence

  $$u(t) \leq C, \quad \forall t \geq 0,$$

- for the quadratic potential: (25) gives

$$u(t) \le C_0 e^{-Ct}, \quad \forall t > 0,$$

- for the trivial potential: assuming the uniform boundedness of the initial data, from (27) we have

$$u(t) \le Ct^{-1} \quad \forall t > 0,$$

while from (28) we conclude

$$u(t) \le Ct^{-1-\gamma d(b+1)\alpha}. \tag{29}$$

This last rate of decay is actually sharp in that equality holds for the Barenblatt solution (6). The pressure of the Barenblatt profile is given by (15) and its gradient is

$$\nabla \mathcal{P}(t,x) = -2k\frac{x}{t}\mathbb{1}_{\{k|x|^2 \le Ct^{2\alpha}\}}.$$

Let $t_0 > 0$ and $x \in \{x; 2Ct_0^{2\alpha} \le 4k|x|^2 \le 3Ct_0^{2\alpha}\}$. In this region the solution is positive and x is far from the origin. Therefore, we have

$$|x| \simeq t_0^{\alpha}, \qquad |\mathcal{P}_{\mathrm{PME}}| \simeq t_0^{-\gamma\alpha d}, \qquad |\nabla \mathcal{P}_{\mathrm{PME}}| \simeq t_0^{-1+\alpha},$$

hence

$$|\mathcal{P}_{\mathrm{PME}}|^b |\nabla \mathcal{P}_{\mathrm{PME}}|^2 \simeq t_0^{-\gamma\alpha d b - 2 + 2\alpha},$$

which is in fact the same exponent of equation (29).

Finally, we now show that the maximum is attained and that $u(t)$ is locally Lipschitz. The argument is different for the PME and FDE.

Porous medium equation. Since the solution satisfies Definition 1.9, the pressure is smooth inside its support, globally Lipschitz, and vanishes on the free boundary. Hence, the quantity $|p(t_0,x)|^b|\nabla p(t_0,x) + V(x)|^2$ attains its maximum value in a point $x_0 \in \mathbb{R}^d$ inside the support.

It now remains to show that the function $u = u(t)$ is locally Lipschitz in $(0,\infty)$. Take an instant $t_0 > 0$. Unless $n(t_0)$ is a stationary solution to the equation (in which case there is essentially nothing to prove), we have $u(t_0) > 0$. By assumption n is continuous in (t,x) so that a simple semicontinuity argument shows that there exists $\delta > 0$ such that

$$u(t) \ge \frac{u(t_0)}{2}, \quad \forall t \in [t_0 - \delta, t_0 + \delta].$$

Let us denote $L := \sup_{t \in [t_0-\delta, t_0+\delta]} \mathrm{Lip}(q(t))$. For $t \in [t_0 - \delta, t_0 + \delta]$, we can restrict the set on which we take the maximum in the definition of the function $u(t)$. More precisely, we have for t in this interval

$$u(t) = \max_{x \in H(t)} |p(t,x)|^b \frac{|\nabla q(t,x)|^2}{2},$$

where $H(t) := \{x;\ |p(t,x)|^b \ge \frac{u(t_0)}{4L^2}\}$. It is now sufficient to prove that there exists a set $E \subset \mathbb{R}^d$, independent of time, such that $H(t) \subset E \subset \{x;\ p(t,x) > 0\}$ for all $t \in [t_0 - \delta, t_0 + \delta]$. Let us define

$$E := \{x;\ p(t_0,x) \ge \alpha\}, \quad \text{with } \alpha := \left(\frac{u(t_0)}{8L^2}\right)^{1/b}.$$

Then, let us suppose that there exist $t_n \in [t_0 - \delta, t_0 + \delta]$, and $x_n \in \mathbb{R}^d$ such that $p(t_n, x_n) \ge 2\alpha$ and $p(t_0, x_n) < \alpha$, namely, that $x_n \in H(t_n)$ and $x_n \notin E$. Since there exists a bounded set

$K \subset \mathbb{R}^d$ large enough such that $x_n \in \operatorname{supp} p(t_n) \subset K$, we can extract a subsequence x_{n_k} such that $x_{n_k} \to x \in K$, and therefore

$$p(t_0, x) \geq 2\alpha, \qquad p(t_0, x) < \alpha,$$

which is a contradiction. By a similar argument, one can show that $E \subset \{x;\ p(t,x) > 0\}$. Therefore, we have proven that we have

$$u(t) = \max_E |p(t,x)|^b \frac{|\nabla p(t,x) + \nabla V(x)|^2}{2},$$

where E is independent of time. Since $|p(t,x)|^b |\nabla p(t,x)|^2$ is C^∞ on the positivity set of the pressure, it is, therefore, Lipschitz continuous in t uniformly in x. Hence $u(t)$ is Lipschitz continuous on $[t_0 - \delta, t_0 + \delta]$, and this concludes the proof.

Fast diffusion equation. We now show that there exists $x_0 \in \mathbb{R}^d$ in which the quantity $|p(t_0)|^b |\nabla q(t_0)|^2$ attains its maximum. To this end, we show $\lim_{|x|\to\infty} |p|^b |\nabla q|^2 = 0$. This tail behavior follows from assumption (12) and the fact that we imposed $b > -1$. Since $n(t,x)$ satisfies (12), we have

$$|\nabla p(t,x)| \leq C(t)(1 + |x|),$$

where the constant $C(t)$ is locally bounded in time. Note moreover that in any of the three cases (trivial potential, quadratic potential, generic potential with bounded derivatives) we also have $|\nabla V(x)| \leq C(1 + |x|)$. Therefore, we get

$$|p(t,x)|^b |\nabla q(t,x)|^2 \leq C(t)(1 + |x|^2)^{b+1},$$

and we obtain the desired result since $b < -1$, which proves that the maximum is attained.

In order to conclude that u is differentiable almost everywhere and $u'(t_0) = v'(t_0)$, it remains to show that the function $u = u(t)$ is locally Lipschitz in $(0, \infty)$ and exactly as for the porous medium case, we only need to show that the maximum can be localy restricted to a fixed subset independent of time. Again it is enough to assume $u(t_0) > 0$, find $\delta > 0$ such that $u(t) > u(t_0)/2$ for $t \in [t_0 - \delta, t_0 + \delta]$, and restrict to a ball $B(0,R)$ such that $|p(t,x)|^b \frac{|\nabla q(t,x)|^2}{2} < \frac{u(t_0)}{2}$ for $t \in [t_0 - \delta, t_0 + \delta]$ and $x \notin B(0,R)$, which is possible because the above estimates on p and ∇p are supposed to be uniform in time.

2.5 Proof of the main results

To extend the results for $V = 0$ and $V = |x|^2/2$ to the class of solutions with L^1 initial data, we proceed by approximation and compactness arguments. In order to show that the approximating sequence is compact and that its initial data convergences to n_0, we need to show local equicontinuity in time. To this end, it is crucial to infer a bound on $n^{\gamma+1}$.

Lemma 2.2. *Let $n(t,x)$ be the solution of equation (4), with $|\gamma| < 1$. There exists $r > 1$ such that $n^{\gamma+1} \in L^r([0,T] \times \mathbb{R}^d))$ for every $T > 0$.*

Proof. For any $T > 0$, we have

$$\int_0^T \int_{\mathbb{R}^d} |n(t)|^{(\gamma+1)r} \,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \|n(t)\|_\infty^{(\gamma+1)r-1} \left(\int_{\mathbb{R}^d} n(t)\,\mathrm{d}x \right) \mathrm{d}t = M \int_0^T \|n(t)\|_\infty^{(\gamma+1)r-1} \,\mathrm{d}t.$$

By Lemma 1.2 we have

$$\|n(t)\|_\infty \leq C t^{-\frac{d}{d\gamma+2}},$$

which show that it is enough to choose $r > 1$ such that

$$[(\gamma+1)r-1]\frac{d}{d\gamma+2} < 1,$$

i.e. $1 < r < \frac{\gamma+2/d+1}{\gamma+1}$. $\square$

We note that in the case $\gamma < 0$ the same result could be obtained in a much simpler way, since we have $n \in L^1$ and $\frac{1}{\gamma+1} > 1$.

We have now all the elements to prove the main results.

Proof of Theorem 1.12. If $\gamma > 0$, let us take $n_{0,\varepsilon} \in C_c(\mathbb{R}^d)$ such that its support is a ball $B(0, R_\varepsilon)$ and $n_{0,\varepsilon} \to n_0$ strongly in $L^1(\mathbb{R}^d)$. Moreover, let us assume that the initial pressure $p_{0,\varepsilon} = (n_{0,\varepsilon})^\gamma$ is such that $|\nabla p_{0,\varepsilon}(x)|$ is bounded from below by a positive constant on $\partial B(0, R_\varepsilon)$ and that $n_{0,\varepsilon}$ is strictly positive inside $B(0, R_\varepsilon)$. It is known that the solution $n_\varepsilon(t, x)$ of (1) with initial data $n_{0,\varepsilon}$ satisfies Definition 1.9.

If $\gamma < 0$, let us take $n_{0,\varepsilon} \in \mathcal{X}\setminus\{0\}$. Then by [9, Theorem 1.1] and [5, Theorem 4], the solution $n_\varepsilon(t, x)$ of equation (4) satisfies property (12), and therefore Definition 1.9. We conclude that Proposition 1.11 holds for n_ε.

Because of the well-known contractivity in L^1 of the PME an FDE we deduce that $n_\varepsilon(t)$ is Cauchy in $L^1(\mathbb{R}^d)$ for every t and we call n its limit as $\varepsilon \to 0$.

Thanks to Proposition 1.11, given $\tau > 0$, $\left|\nabla n_\varepsilon^{\gamma(b/2+1)}(t)\right|$ is uniformly bounded for all $t > \tau$. The control of this quantity implies that the sequence of regularized solutions is equicontinuous in space. Equicontinuity in time follows, for instance, from Lemma 2.2, namely the fact that $\partial_t n_\varepsilon$ is uniformly bounded in $L^r_{\text{loc}}(0, \infty; W^{-2,r}(\mathbb{R}^d))$. Thus, for all $\tau > 0$ we have

$$n_\varepsilon \to n \text{ in } C_{\text{loc}}((\tau, \infty) \times \mathbb{R}^d),$$

and $n(t, x)$ satisfies equation (4) for all $t > 0$. Moreover, $n(t) \to n_0$ as $t \to 0$ thanks to the uniform bound on n in $W^{1,r}(0, T; W^{-2,r}(\mathbb{R}^d))$. Therefore, $n(t, x)$ is the unique solution of equation (4) with initial data n_0. Since $n \mapsto \max_x |\nabla n^{\gamma(b/2+1)}|^2$ is lower semi-continuous we conclude. $\square$

Proof of Theorem 1.14. We argue by approximation as before. In fact, through the time-scaling (9), equation (1) with $V = |x|^2/2$ is equivalent to equation (4). The equicontinuity properties still hold since they are not affected by the change of variable, and therefore we have uniform convergence of the sequence. The functional $p \mapsto \max_x |p|^b |\nabla p + x|^2$ is lower semi-continuous, and therefore we conclude. $\square$

3 Asymptotic behaviour at large times

We want to study the asymptotic behavior as $t \to \infty$ of the solution $n(t, x)$ to the drift-less porous medium and fast diffusion equations (4). In order to do so, we take advantage of the fact that $n(t, x)$ can be seen as the solution of a convective porous medium equation with quadratic potential through the change of variables (9).

Proof of Theorem 1.16. Since $\hat{n}$ defined in (9) satisfies equation (10), by Theorem 1.14 we have

$$\max_x |\hat{n}(t,x)|^{\gamma b} \left|\text{sign}(\gamma)\nabla\hat{n}^\gamma(t,x) + x\right|^2 \le C_0 e^{-Ct}.$$

Using the definition of $\hat{n}(t,x)$ we find

$$e^{\gamma bdt}\max_x |n(\psi(t),e^t x)|^{\gamma b}\ \Big||\gamma|e^{d(\gamma-1)t}e^{(d+1)t}n^{\gamma-1}(\psi(t),e^t x)\nabla n(\psi(t),e^t x)+x\Big|^2 \le C_0 e^{-Ct}.$$

Let us denote $s:=\psi(t)$ and $y=e^t x$. Substituting in the equation above, we obtain sequentially

$$s^{\frac{\gamma bd}{d\gamma+2}}\max_y |n(s,y)|^{\gamma b}\ \Big||\gamma|s^{\frac{d\gamma+1}{d\gamma+2}}n^{\gamma-1}(s,y)\nabla n(s,y)+ys^{-\frac{1}{d\gamma+2}}\Big|^2 \le C_0 s^{-\frac{C}{d\gamma+2}},$$

$$s^{\frac{\gamma bd}{d\gamma+2}}\max_y |n(s,y)|^{\gamma b}\ \Big|\mathrm{sign}(\gamma)s^{\frac{d\gamma+1}{d\gamma+2}}\nabla|n(s,y)|^{\gamma}+ys^{-\frac{1}{d\gamma+2}}\Big|^2 \le C_0 s^{-\frac{C}{d\gamma+2}},$$

$$s^{\frac{\gamma d(2+b)+2}{d\gamma+2}}\max_y |n(s,y)|^{\gamma b}\ \big|\mathrm{sign}(\gamma)\nabla n^{\gamma}(s,y)+ys^{-1}\big|^2 \le C_0 s^{-\frac{C}{d\gamma+2}}.$$

Therefore

$$\max_x |n(t,x)|^{\gamma b}\ \big|\mathrm{sign}(\gamma)\nabla n^{\gamma}(t,x)+xt^{-1}\big|^2 \le C_0 t^{\beta}t^{-\frac{C}{d\gamma+2}}, \tag{30}$$

with $\beta=-\alpha\gamma db-2+2\alpha$, and this concludes the proof. ☐

Weighted convergence of the pressure gradient. For $-2/d<\gamma<0$, since $n(t,x)$ is always positive and the gradient of the Barenblatt pressure is proportional to xt^{-1}, equation (30) gives an interesting insight on the rate of convergence of the gradient of n^γ for large times. Let us take $n_0\in\mathcal{X}\setminus\{0\}$. We recall that the pressure of the Barenblatt profile of the fast diffusion equation is

$$\mathcal{P}(t,x)=t^{-\alpha\gamma d}\big(C-k|x|^2t^{-2\alpha}\big).$$

Let us take $x\in C_{a,b}(t):=\{x;\ |x|t^{-\alpha}\in[a,b]\}$ for some constants $0<a<b$. Since by [9, Theorem 1.1] n satisfies (8) if and only if $n_0\in\mathcal{X}\setminus\{0\}$, we have, for $x\in C_{a,b}(t)$, a behavior of the form $n^{\gamma b}(t,x)\gtrsim t^{-\alpha\gamma db}$. Moreover, in such region we also have $|\nabla\mathcal{P}(t,x)|^2\simeq t^{-2+2\alpha}$. Let us recall $\beta=-\alpha\gamma db-2+2\alpha$. Finally, from (30) we find

$$\frac{\big\|\nabla p(t)-\nabla\mathcal{P}(t)\big\|^2_{L^\infty(C_{a,b}(t))}}{\big\|\nabla\mathcal{P}(t)\big\|^2_{L^\infty(C_{a,b}(t))}}\lesssim t^{-\frac{C}{d\gamma+2}},\quad \forall t>0.$$

4 Problem posed on a bounded domain

We now discuss the same problem posed on bounded domains, both for Neumann and Dirichlet homogeneous boundary conditions. For an overview of the asymptotic behavior of the PME and FDE in bounded domains, we refer the reader to [7, 8, 23] and references therein.

4.1 Neumann boundary conditions on a convex domain

Let us now consider the same equation on a bounded, smooth, and convex domain $\Omega\subset\mathbb{R}^d$

$$\begin{cases}\dfrac{\partial n}{\partial t} &= \nabla\cdot(n\nabla q), \quad \text{in } (0,\infty)\times\Omega,\\ n(0,x) &= n_0(x), \qquad \text{in } \Omega,\\ \nabla q\cdot\nu &= 0, \qquad\quad \text{on } (0,\infty)\times\partial\Omega,\end{cases}$$

where $q = p + V$ and ν represents the outward normal to $\partial\Omega$. Let us assume that

$$\partial_\nu V \geq 0 \text{ on } \partial\Omega.$$

The computation performed in Section 2 follows through in the same way. We only need to ensure that the maximum of the quantity $e^{\alpha(p)}|\nabla q|^2$ is not attained on $\partial\Omega$.

First of all, we denote $h(x)$ a convex function such that $\Omega = \{x; h(x) < 0\}$, $\partial\Omega = \{x; h(x) = 0\}$. We assume $\nabla h \neq 0$ and $D^2 h > 0$ (in the sense of positive-definite symmetric matrices) on $\partial\Omega$. We fix an instant t and write q for $q(t, \cdot)$. Therefore, for any curve $\omega : (0,1) \to \partial\Omega$ we have $\nabla q(\omega(s)) \cdot \nabla h(\omega(s)) = 0$. We differentiate (in s) and obtain

$$\omega'(s)(D^2(q(\omega(s)))\nabla h(\omega(s)) + D^2 h(\omega(s))\nabla q(\omega(s))) = 0.$$

We can choose $\omega'(s) = \nabla q(\omega(s))$ (which is a possible choice, since ∇q is a tangent vector) to obtain

$$\nabla q(\omega(s)) D^2(q(\omega(s)))\nabla h(\omega(s)) + \nabla q(\omega(s)) D^2 h(\omega(s))\nabla q(\omega(s))) = 0. \tag{31}$$

Let us assume that the maximum is attained on the boundary. This implies that there exists a positive constant μ such that

$$\alpha'(p)|\nabla q(\omega(s))|^2 \nabla p(\omega(s)) + 2D^2 q(\omega(s))\nabla q(\omega(s)) = \mu\nu,$$

which, taking the scalar product with $\nu = \nabla h$, gives

$$\nabla h(\omega(s)) D^2 q(\omega(s))\nabla q(\omega(s)) > 0,$$

since $\partial_\nu p = \partial_\nu q - \partial_\nu V \leq 0$ on $\partial\Omega$ and $\alpha' > 0$ (we recall that we use $\alpha(p) = b\log|p|$, so that $\alpha'(p) = b/p$ and the choice of the sign of b guarantees that b and p always have the same sign). Thus, from (31) we deduce

$$\nabla q(\omega(s)) D^2 h(\omega(s))\nabla q(\omega(s))) < 0,$$

which is a contradiction since h is convex.

4.2 Dirichlet boundary conditions

For the problem set in a bounded domain with homogeneous Dirichlet boundary conditions, the computations of Section 2 could still be performed at a formal level. However, the quantity under investigation, namely

$$u(t) = \max_x |p(t,x)|^b |\nabla p(t,x)|^2 = \max_x |\nabla n^{\gamma(b/2+1)}(t,x)|^2,$$

is actually not finite for our range of exponents b. Indeed, let us consider the drift-less problem

$$\begin{cases} \dfrac{\partial n}{\partial t} &= \dfrac{|\gamma|}{\gamma+1}\Delta n^{\gamma+1}, & \text{in } (0,\infty)\times\Omega \\ n(0,x) &= n_0(x), & \text{in } \Omega, \\ n &= 0, & \text{on } (0,\infty)\times\partial\Omega. \end{cases}$$

This problem admits a solution of the form $n(t,x) = a(x)b(t)$, where the functions $a(x), b(t)$ satisfy

$$b'(t) = -\frac{|\gamma|}{\gamma+1} b^{\gamma+1}(t), \quad \text{and} \quad a(x) = -\Delta a^{\gamma+1}(x),$$

for $t > 0$ and $x \in \Omega$. We denote $\tilde{a}(x) := a^{\gamma+1}(x)$. This function satisfies $-\Delta\tilde{a} = \tilde{a}^{1/(\gamma+1)} > 0$, in Ω and $\tilde{a} = 0$ on $\partial\Omega$. Thus, on the boundary we have $\partial_\nu \tilde{a} < 0$. Let us now consider the quantity

$$|\nabla a^{\gamma(b/2+1)}| = |\nabla \tilde{a}^\theta|,$$

with $\theta = \gamma(b/2+1)/(\gamma+1)$. Even for the maximum value of b allowed, the one such that we have $\gamma b = 1 + \sqrt{1-\gamma^2(d-1)}$, we obtain $\theta < 1$. Therefore the quantity $|\nabla a^{\gamma(b/2+1)}|$ blows up on the boundary, hence $u(t) = +\infty$.

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