



**HAL**  
open science

# Ensemble controllability of parabolic type equations

Baparou Danhane, Jérôme Lohéac

► **To cite this version:**

Baparou Danhane, Jérôme Lohéac. Ensemble controllability of parabolic type equations. 2023. hal-04174446v1

**HAL Id: hal-04174446**

**<https://hal.science/hal-04174446v1>**

Preprint submitted on 31 Jul 2023 (v1), last revised 16 Nov 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Ensemble controllability of parabolic type equations

Baparou Danhane\*      Jérôme Lohéac†

July 31, 2023

## Abstract

In this paper, we extend the concept of ensemble controllability introduced in 2009 by Jr-Shin Li and Navin Khaneja to a class of linear partial differential equation. More precisely, we consider some abstract parabolic equation, where the system depends on some unknown parameter which is assumed to belong to a compact interval. We investigate the possibility of approximatively reaching (in  $L^2$ -norm) any target state from any initial state, with an open loop control. Here the initial and target states might depend on the unknown parameter, but the control is assumed to be parameter independent.

*Keywords:* Ensemble control, parabolic equations, average control.

## 1 Introduction

The modeling of physical systems often involves in their dynamics some parameters that are inherent to them. These parameters are generally estimated but not known with precision. For instance, the dynamic of a car depends on its mass, the evolution of the temperature depends on the diffusion coefficient of the material. . .

In the context of classical control theory, it is customary to assume that these parameters are known and fixed. Thus, if a system is to be sent from its initial state to a prescribed final state, the constructed control implicitly depends on the fixed parameters. As a consequence, when these parameters are modified, the built control no longer solves the control problems. Hence, each time a system parameter is modified, a new control has to be designed. It therefore seems reasonable to ask whether it is possible to build controls that are independent of the parameters of the system. This is the aim of *Ensemble controllability*, and this notion is also named (*approximate*) *simultaneous controllability*.

The notion of ensemble controllability was first introduced in [1], where the authors studied the ensemble controllability of Bloch equations with a dispersion

---

\*Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

†Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France

in their natural frequencies. A much more elaborate study of the topic discussed in the above reference, using functional analysis approach, was undertaken in [2].

For linear control systems many results exist. From the generic concept of ensemble controllability, many other notions emerged such as *Uniform ensemble controllability* and  *$L^q$ -ensemble controllability*. One can cite for instance [3, 4, 5] where the authors gave necessary and sufficient conditions for  $L^q$ -ensemble controllability and uniform ensemble controllability for linear control systems. In [6, 7], the authors gave some obstructions to  $L^q$  and uniform ensemble controllability, and in [8], the author explore how the application of feedback methods enlarges the class of parameter-dependent linear systems that are uniformly ensemble reachable. In [9, 10], the authors gave some numerical tools to compute optimal controls. We also quote [11], where the authors investigate ensemble controllability for non-linear systems using the geometrical point of view.

We mention [12], where the authors investigate the ensemble feedback stabilization of linear systems using a Riccati based feedback mechanism. It should be mentioned that in their paper, the set of parametrization is a finite subset of the real line. More importantly, the feedback control proposed, itself, depends on the parameter but the gain does not. The fact that the gain does not depend on the parameter makes the analysis more delicate. The question of feedback stabilization of parameter depends system with feedback controls that do not depend on the parameter is still, at our knowledge, an open question, even for linear systems. Still in the context of stabilization, we also cite this recent paper [13] where the authors use the backstepping method to stabilize an ensemble of hyperbolic partial differential equations.

To our knowledge, the concept of ensemble controllability has not yet been investigated in the case of partial differential equations, especially when the set of parametrization is a continuum. It is worth noting, however, that in [12], as an application of their results, the authors considered a Galerkin discretization of a parameterized one dimensional heat equation with internal controls, where the uncertain parameter is the reaction coefficient and belongs to a finite subset of the real line. They have shown, under some conditions, that any finite truncation of the discretized system solution is ensemble controllable.

As mentioned above, the ensemble controllability of partial differential equations when the set of parametrization is a continuum is open, and in this paper, we investigate the  $L^2$ -ensemble controllability of linear parabolic type equations with internal controls.

The paper is organized as follows. In Section 2, we state the controllability problem and give the main result of the paper. This section is followed by Section 3 where we apply our main result to the heat equation, and give some relations between ensemble controllability and *average controllability*. The proof of the main result is given in Section 4. This section is followed by Section 5 where we made a digression on finite dimensional control systems. A numerical example is given in Section 6. Concluding remarks and open problems are provided in Section 7.

The following notations are used.

We set  $\mathbb{R}_+$  the nonnegative real line,  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ , and for  $\kappa \in \mathbb{N}$ ,  $\mathbb{N}_{\leq \kappa} = \{0, 1, \dots, \kappa\}$ .

Let  $\mathcal{H}$  be a Hilbert space. The notation  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  denotes the scalar product of elements of  $\mathcal{H}$ .

For two Banach spaces  $\mathcal{X}$  and  $\mathcal{Y}$ , the space  $\mathcal{X}_{n_1}^{n_2}$ , with  $n_1, n_2 \in \mathbb{N}^*$ , stands for the set of matrices with  $n_2$  rows and  $n_1$  columns with entries in  $\mathcal{X}$ . The notation  $\|\cdot\|_{\mathcal{X}}$  is used for the norm on  $\mathcal{X}$ . The space  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$  stands for the Banach space of linear and continuous map from  $\mathcal{X}$  into  $\mathcal{Y}$ , and we set  $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$ . We identify  $\mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$  to  $\mathbb{R}_{n_1}^{n_2}$ . For  $G \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ ,  $\text{Ran}(G)$  and  $\text{Ker}(G)$  respectively denote the range and the null space of  $G$ ,  $\text{rk}(G)$  is the dimension of  $\text{Ran}(G)$ , and  $G^*$  is the adjoint of  $G$ . For  $\mathcal{X} \subset \mathcal{Y}$ ,  $\bar{\mathcal{X}}$  is the closure of  $\mathcal{X}$  in  $\mathcal{Y}$ . For an operator  $A$ ,  $\mathcal{D}(A)$  denotes its domain. The matrix  $A \in \mathbb{R}_n^n$  is said Hurwitz if all its eigenvalues have a negative real part.

For a family of vectors  $(e_k)_{k \in \mathbb{N}}$ , the notation  $\text{Span}_{\mathbb{K}}\{e_k, k \in \mathbb{N}\}$  stands for the set of linear combinations of these vectors over the field  $\mathbb{K}$ .

For a Borel measure space  $\mathcal{M} = (\mathcal{M}, \mathcal{T}, \nu)$ , the space  $L^2(\mathcal{M}, \mathcal{X})$  which is the space of  $\nu$ -measurable functions  $f$  satisfying  $\int_{\mathcal{M}} \|f(\sigma)\|_{\mathcal{X}}^2 d\nu(\sigma) < \infty$  is simply denoted by  $L^2(\mathcal{M}, \mathcal{X})$ . Moreover, if  $\mathcal{X} = \mathbb{R}^k$  for some  $k \in \mathbb{N}^*$  then  $L^2(\mathcal{M}, \mathbb{R}^k)$  is identified to  $L^2(\mathcal{M}, \mathbb{R})^k$ , and denoted by  $L^2(\mathcal{M})^k$ . The space  $L_{\text{loc}}^2(\mathbb{R}_+, \mathcal{X})$  stands for the one of measurable functions from  $\mathbb{R}_+$  to  $\mathcal{X}$  that are square integrable on any bounded interval of  $\mathbb{R}_+$ . We similarly define the sets of continuous function,  $\mathcal{C}(\mathcal{M}, \mathcal{X})$  and  $\mathcal{C}(\mathcal{M})^k$ . The notation  $\text{supp}(f)$  stands for the support of the function  $f$ .

## 2 Problem statement and main result

Let  $(\Theta, \mathcal{B}, \mu)$  be a Borel measure space where  $\Theta$  is a compact subset of  $\mathbb{R}$ , and let  $X$ ,  $U$  and  $Y$  be three Hilbert spaces. For every  $\theta \in \Theta$ , we consider the Cauchy problem

$$\begin{aligned} \dot{y}(t, \theta) &= \alpha(\theta)Ay(t, \theta) + B(\theta)u(t) & (t > 0), \\ y(0, \theta) &= y^0(\theta). \end{aligned} \quad (2.1)$$

In the above equation and what follows,  $\dot{y}$  stands for the derivative of  $y$  with respect to the time variable.

We make the following assumptions on the above system.

- (hp<sub>1</sub>)  $(A, \mathcal{D}(A))$  generates an analytic exponentially stable semigroup  $(\mathbb{T}_t)_{t \geq 0}$  on  $X$ ;
- (hp<sub>2</sub>) For  $\mu$ -almost every  $\theta \in \Theta$ ,  $B(\theta) \in \mathcal{L}(U, X)$  and

$$\int_{\Theta} \|B(\theta)\|_{\mathcal{L}(U, X)}^2 d\mu(\theta) < \infty;$$

- (hp<sub>3</sub>) There exist  $L \in \mathcal{L}(U, Y)$  and  $C \in \mathcal{L}(X, Y)$  and  $\beta \in L^2(\Theta)$  such that for every  $\varphi \in L^2(\Theta, X)$ ;

$$LB^*(\theta)\varphi(\theta) = \beta(\theta)C\varphi(\theta) \quad (\theta \in \Theta, \mu - \text{a.e.});$$

- (hp<sub>4</sub>)  $A^*(\text{Ker}(C) \cap \mathcal{D}(A^*)) \subset \text{Ker}(C)$ ;
- (hp<sub>5</sub>)  $(A^*, C)$  is approximately observable;
- (hp<sub>6</sub>) The function  $\alpha \in \mathcal{C}(\Theta)$  is positive, injective and non vanishing, and  $\beta \in L^2(\Theta)$  is such that  $\beta(\theta) \neq 0$  for  $\mu$ -almost every  $\theta \in \Theta$ .

*Remark 2.1.* If Assumption (hp<sub>1</sub>) is fulfilled, we infer from [14, Theorem 5.2] that

$$\mathbb{T}_t^* z \in \mathcal{D}(A^*) \quad (z \in X, t > 0).$$

*Remark 2.2.* The Assumption (hp<sub>6</sub>) implies that

$$\overline{\text{Span}_{\mathbb{R}} \left\{ \theta \in \Theta \rightarrow \frac{\beta(\theta)}{\alpha(\theta)^k}, k \in \mathbb{N} \right\}} = L^2(\Theta). \quad (2.2)$$

This can be inferred from [3, Proposition 7] or from [15, Lemma 2.1]. One can also consult [16, Proposition 2.2] for unitary operators. If for instance,  $\Theta = [1, 2]$ ,  $\alpha(\theta) = \theta$  and  $\beta(\theta) = 1$ , (2.2) is automatically fulfilled thanks to the Weierstrass approximation theorem, see for instance [17, 18].

The first remark to be made is that since  $(A, \mathcal{D}(A))$  generate  $\mathbb{T}_t$  which is analytic and exponentially stable, it follows from (hp<sub>6</sub>) that for every  $\theta \in \Theta$ ,  $(\alpha(\theta)A, \mathcal{D}(A))$  also generate  $\mathbb{T}_{\alpha(\theta)t}$  which is analytic and exponentially stable. Therefore, from [19, Proposition 4.2.5] and (hp<sub>2</sub>), we deduce that for every  $u \in L^2_{\text{loc}}(\mathbb{R}_+, U)$ , every  $y^0 \in L^2(\Theta, X)$ , and almost every  $\theta \in \Theta$ , (2.1) admits a unique solution given by

$$y(t, \theta) = \mathbb{T}_{\alpha(\theta)t} y^0(\theta) + (\Phi_t u)(\theta) \quad (t > 0), \quad (2.3)$$

where

$$(\Phi_t u)(\theta) = \int_0^t \mathbb{T}_{\alpha(\theta)(t-\tau)} B(\theta) u(\tau) d\tau. \quad (2.4)$$

Defining  $y : (t, \theta) \in \mathbb{R}_+ \times \Theta \mapsto y(t, \theta)$  as a solution of (2.1), we can prove the following lemma.

**Lemma 2.3.** *Under Assumptions (hp<sub>1</sub>), (hp<sub>2</sub>) and (hp<sub>6</sub>), for every  $T > 0$ , the map  $\Phi_T$  defined by (2.4) belongs to  $\mathcal{L}(L^2(\mathbb{R}_+, U), L^2(\Theta, X))$ , and for every  $y^0 \in L^2(\Theta, X)$ , the solution  $y$  defined by (2.3) belongs to  $\mathcal{C}(\mathbb{R}_+, L^2(\Theta, X))$ .*

*Proof.* Using the fact that  $\mathbb{T}_t$  is exponentially stable, it follows from [14, Theorem 4.3] that there exist two constants  $M \geq 1$  and  $\delta > 0$  such that  $\|\mathbb{T}_t\|_{\mathcal{L}(X)} \leq M e^{-\delta t}$  for every  $t \geq 0$ . Since  $\alpha$  is a positive function of  $\theta$ , it therefore holds that for every  $t \geq 0$  and every  $\theta \in \Theta$ ,  $\|\mathbb{T}_{\alpha(\theta)t}\|_{\mathcal{L}(X)} \leq M e^{-\alpha(\theta)\delta t}$ . Setting  $\underline{\alpha} = \min_{\theta \in \Theta} \alpha(\theta)$ , we have

$$\|\mathbb{T}_{\alpha(\theta)t}\|_{\mathcal{L}(X)} \leq M e^{-\underline{\alpha}\delta t} \quad (t \geq 0, \theta \in \Theta). \quad (2.5)$$

Using (2.5), we have for every  $\tau \in [0, t]$

$$\|\mathbb{T}_{\alpha(\theta)(t-\tau)} B(\theta) u(\tau)\|_X \leq M e^{-\underline{\alpha}\delta(t-\tau)} \|B(\theta)\|_{\mathcal{L}(U, X)} \|u(\tau)\|_U.$$

From the above inequality, we obtain by integrating with respect to  $\tau$  on the time interval  $(0, t)$  and using Cauchy-Schwarz inequality

$$\int_0^t \left\| \mathbb{T}_{\alpha(\theta)(t-\tau)} B(\theta) u(\tau) \right\|_X d\tau \leq \frac{M}{\sqrt{2\delta\alpha}} \|B(\theta)\|_{\mathcal{L}(U, X)} \|u\|_{L^2((0, t), U)}.$$

From this observation, it follows that the map  $\tau \mapsto \mathbb{T}_{\alpha(\theta)(t-\tau)} B(\theta) u(\tau)$  belongs to  $L^1([0, t], X)$  and by Bochner integrability, we have for every  $t \geq 0$  and every  $\theta \in \Theta$ ,

$$\left\| \int_0^t \mathbb{T}_{\alpha(\theta)(t-\tau)} B(\theta) u(\tau) d\tau \right\|_X \leq \frac{M}{\sqrt{2\delta\alpha}} \|B(\theta)\|_{\mathcal{L}(U, X)} \|u\|_{L^2((0, t), U)}.$$

The preceding inequality leads to

$$\int_{\Theta} \left\| \int_0^t \mathbb{T}_{\alpha(\theta)(t-\tau)} B(\theta) u(\tau) d\tau \right\|_X^2 d\mu(\theta) \leq \frac{M^2 \gamma}{2\delta\alpha} \|u\|_{L^2((0, t), U)}^2. \quad (2.6)$$

where  $\gamma = \int_{\Theta} \|B(\theta)\|_{\mathcal{L}(U, X)}^2 d\mu(\theta) < \infty$ , thanks to Assumption **(hp<sub>2</sub>)**.

For every  $t \geq 0$ , the linearity of  $\Phi_t$  being obvious, we deduce from (2.6) that  $\Phi_t \in \mathcal{L}(L^2(\mathbb{R}_+, U), L^2(\Theta, X))$ . Moreover, using estimations (2.5) and (2.6), it follows that for every  $t \geq 0$ ,

$$\|y(t)\|_{L^2(\Theta, X)} \leq M e^{-\delta\alpha t} \|y^0\|_{L^2(\Theta, X)} + M \sqrt{\frac{\gamma}{2\delta\alpha}} \|u\|_{L^2((0, t), U)}.$$

Therefore, for any  $u \in L^2_{\text{loc}}(\mathbb{R}_+, U)$ ,  $y \in \mathcal{C}(\mathbb{R}_+, L^2(\Theta, X))$ .  $\square$

The Lemma 2.3 therefore lends legitimacy to the following definition.

**Definition 2.4.** The system (2.1) is said  $L^2(\Theta, X)$ -ensemble controllable in time  $T > 0$  if for every  $y^0, y^1 \in L^2(\Theta, X)$  and every  $\varepsilon > 0$ , there exists a parameter independent open loop control  $u \in L^2([0, T], U)$  such that the solution of (2.1) with  $y^0$  as initial state datum and  $u$  as input satisfies

$$\|y(T) - y^1\|_{L^2(\Theta, X)} \leq \varepsilon. \quad (2.7)$$

In this definition, functions of  $L^2([0, T], U)$  are seen as functions of  $L^2(\mathbb{R}_+, U)$  supported on  $[0, T]$ .

The main result of this paper is the following.

**Theorem 2.5.** Under the Assumptions **(hp<sub>1</sub>)** to **(hp<sub>6</sub>)**, the system (2.1) is  $L^2(\Theta, X)$ -ensemble controllable in any time  $T > 0$ .

Let us make the following remarks.

*Remark 2.6.* Observe that if the system (2.1) is  $L^2(\theta, X)$ -ensemble controllable in time  $T > 0$ , then for every  $\lambda \in \mathbb{R}$ , the system

$$\dot{z}(t, \theta) = (\alpha(\theta)A - \lambda I)z(t, \theta) + B(\theta)v(t) \quad (2.8)$$

is also  $L^2(\Theta, X)$ -ensemble controllable in time  $T > 0$ .

Indeed, for every  $\theta \in \Theta$ ,  $(\alpha(\theta)A, \mathcal{D}(A))$  generate the semigroup  $\mathbb{T}_{\alpha(\theta)t}$ , then  $(\alpha(\theta)A - \lambda I, \mathcal{D}(A))$  generates the semigroup  $e^{-\lambda t}\mathbb{T}_{\alpha(\theta)t}$ . Therefore, for every  $u \in L^2([0, T], U)$ , the solution of (2.8) with initial condition  $z^0(\theta)$  is given by

$$z(t, \theta) = e^{-\lambda t}(\mathbb{T}_{\alpha(\theta)t}z^0(\theta) + (\Phi_t u_\lambda)(\theta)) = e^{-\lambda t}y(T)$$

where  $u_\lambda(t) = e^{\lambda t}u(t)$ , and where  $y$  solves (2.1) with control  $u_\lambda$  and initial condition  $z^0(\theta)$ . If (2.1) is  $L^2(\theta, X)$ -ensemble controllable in time  $T > 0$ , then for every  $\varepsilon > 0$  there exists  $u_\lambda \in L^2([0, T], U)$  such that,

$$\|y(T) - e^{\lambda T}z^1\|_{L^2(\Theta, X)} \leq e^{\lambda T}\varepsilon,$$

that is to say that

$$\|z(T) - z^1\|_{L^2(\Theta, X)} \leq \varepsilon,$$

i.e., (2.8) is  $L^2(\Theta, X)$ -ensemble controllable.

*Remark 2.7.* As it will be seen in the proof of Theorem 2.5 in Section 4, this theorem is still valid for unbounded operators  $B(\theta) \in \mathcal{L}(U, X_{-1})$  provided that we replace Assumption (hp<sub>1</sub>) by the fact that for  $\mu$ -almost every  $\theta \in \Theta$ ,  $B(\theta)$  is admissible for  $\mathbb{T}_t$  and that the admissibility constant, which depends on  $\theta$ , belongs to  $L^2(\Theta)$ . These hypotheses ensure that Lemma 2.3 still holds with an unbounded control operator. In this case, we should have in Assumption (hp<sub>3</sub>)  $C \in \mathcal{L}(X_1^\kappa, Y)$  where  $X_1^\kappa$  is the dual space of  $X_{-1}$  with respect to the pivot space  $X$  (see [19]), and  $C$  should be required to be admissible for the semigroup  $\mathbb{T}_t^*$ . It goes without saying that in the proof, one should adjust what should be adjusted.

## 3 Applications

### 3.1 Application to the heat equation

Let  $\Omega$  be a smooth domain of  $\mathbb{R}^d$  ( $d \geq 1$ ),  $X = U = L^2(\Omega)$  and consider the system

$$\begin{aligned} \dot{y}(t, x, \theta) &= \alpha(\theta)\Delta y(t, x, \theta) + \beta(\theta)\chi_{\omega(\theta)}(x)u(t, x), & (t, x) &\in \mathbb{R}_+ \times \Omega, \\ y(t, x, \theta) &= 0, & (t, x) &\in \mathbb{R}_+ \times \partial\Omega, \\ y(0, x, \theta) &= y^0(x, \theta), & x &\in \Omega, \end{aligned} \tag{3.1}$$

where  $\theta \in \Theta$ , with  $\Theta$  a compact interval of  $\mathbb{R}$ , and  $\mu$  the Lebesgue measure on  $\Theta$ . In this system, the control is acting in a domain  $\omega(\theta) \subset \Omega$  that might depend on the parameter  $\theta$ . We also assume that there exists a nonempty open subset  $\omega_0$  such that  $\omega_0 \subset \bigcap_{\theta \in \Theta} \omega(\theta)$ .

System (3.1) correspond to (2.1) with

$$A = \Delta, \quad \mathcal{D}(A) = H^2(\Omega) \times H_0^1(\Omega) \quad \text{and} \quad B(\theta) = \beta(\theta)\chi_{\omega(\theta)}(x).$$

Obviously, in this context, Assumption (hp<sub>1</sub>) is satisfied, and Assumption (hp<sub>2</sub>) is satisfied, if  $\beta \in L^2(\Theta)$ . Also,  $A^* = A$  and for every  $\theta \in \Theta$ ,  $B(\theta)^* = B(\theta)$ .

Choosing  $Y = L^2(\omega_0)$ ,  $L = C = \chi_{\omega_0}$ , Assumptions **(hp<sub>3</sub>)** and **(hp<sub>4</sub>)** are satisfied. Since the heat equation is approximatively controllable in any time  $T > 0$  from any non-empty open subset of  $\Omega$ , it follows by duality that  $(A^*, C)$  is also approximately observable in any time, that is to say that Assumption **(hp<sub>5</sub>)** is also satisfied. It therefore follows from Theorem 2.5 that system (3.1) is  $L^2(\Theta, X)$ -ensemble controllable in any time  $T > 0$ . As a consequence, we have the following result.

**Proposition 3.1.** *If  $\alpha$  and  $\beta$  satisfy Assumption **(hp<sub>6</sub>)**, and if there exists a non-empty open subset  $\omega_0$  such that  $\omega_0 \subset \omega(\theta)$  for every  $\theta \in \Theta$ , then the system (3.1) is  $L^2(\Theta, L^2(\Omega))$ -ensemble controllable.*

### 3.2 Application to average controllability

In this paragraph, we link the concept of ensemble controllability to the one of *Average controllability*. As far as we know, the concept of average controllability has been introduced in [20] with the aim of controlling in average the dynamics of systems that depend on parameters with parameter independent open loop controls. Many results on partial differential equations have been obtained and can be found for instance in [21, 22, 23, 24]. Dealing with the numerical aspect of averaged controllability, we refer to [9, 25].

In this paper, the following definition borrowed from [20] will be considered.

**Definition 3.2.** Let  $\mathcal{M} = (\Theta, \mathcal{T}, \mu)$  be a probability measure space. The system (2.1) is *Approximately controllable in average in time  $T > 0$*  if for every  $y^0, y^1 \in L^2(\Theta, X)$  and every  $\varepsilon > 0$ , there exists a parameter independent open loop control  $u \in L^2([0, T], U)$  such that the solution of (2.1) with  $y^0$  as initial state datum and  $u$  as input satisfies

$$\left\| \int_{\Theta} (y(T, \theta) - y^1(\theta)) \, d\mu(\theta) \right\|_X \leq \varepsilon. \quad (3.2)$$

The next result follows directly from Theorem 2.5.

**Corollary 3.3.** *Under the Assumptions **(hp<sub>1</sub>)** to **(hp<sub>6</sub>)**, the system (2.1) is approximately controllable in average in any time  $T > 0$ .*

It is well-known that average controllability does not imply ensemble controllability. Indeed, consider, for instance, the system (2.1) with

$$\mathcal{M} = ([0, 1], \mathcal{T}, \mu), \quad \alpha(\theta) = 1, \quad A = 1, \quad B(\theta) = \theta.$$

In this example, we take  $\mu$  to be the Lebesgue measure and  $X = U = \mathbb{R}$ . It is easily seen through [20, Theorem 1] that this system is approximatively controllable in average in any time, but, according to [3, Proposition 7], it cannot be  $L^2(\Theta)$ -ensemble controllable in any time  $T > 0$ .

*Proof.* We aim to show that for every  $y^0, y^1 \in L^2(\Theta, X)$  and every  $\varepsilon > 0$ , there exists a parameter independent open loop control  $u \in L^2([0, T], U)$  such



that (3.2) is satisfied.

Let  $y^0, y^1 \in L^2(\Theta, X)$  and take any  $\varepsilon > 0$ .

According to Theorem 2.5, the system is  $L^2(\Theta, X)$ -ensemble controllable in any time  $T > 0$ . Hence, writing down Definition 2.4, there exists an open loop control  $u \in L^2([0, T], U)$  such that

$$\|y(T) - y^1\|_{L^2(\Theta, X)} \leq \varepsilon. \quad (3.3)$$

Using Cauchy-Schwarz inequality (recall that  $\mu(\Theta) = 1$ ), we get

$$\begin{aligned} \left\| \int_{\Theta} (y(T, \theta) - y^1(\theta)) \, d\mu(\theta) \right\|_X &\leq \int_{\Theta} \|y(T, \theta) - y^1(\theta)\|_X \, d\mu(\theta), \\ &\leq \left( \int_{\Theta} \|y(T, \theta) - y^1(\theta)\|_X^2 \, d\mu(\theta) \right)^{\frac{1}{2}} \\ &\leq \|y(T) - y^1\|_{L^2(\Theta, X)} \leq \varepsilon. \end{aligned}$$

□

As a consequence of Corollary 3.3, the heat equation considered in (3.1) is approximately controllable in average in any time  $T > 0$ . This result can be found in [22, Section 3].

## 4 Proof of Theorem 2.5

In this section, we provide a proof to our main result.

By Definition 2.4, system (2.1) is  $L^2(\Theta, X)$ -ensemble controllable in a time  $T > 0$  if and only if

$$\overline{\text{Ran}(\Phi_T)} = L^2(\Theta, X), \quad (4.1)$$

relation which is equivalent to

$$\text{Ker}(\Phi_T^*) = \{0\}. \quad (4.2)$$

We now compute the adjoint of  $\Phi_T$ .

Since  $\Phi_T$  is a linear and bounded operator from  $L^2(\mathbb{R}_+, U)$  into  $L^2(\Theta, X)$ ,  $\Phi_T^*$  is a linear and bounded operator from  $L^2(\Theta, X)$  into  $L^2(\mathbb{R}_+, U)$ . Therefore, for every  $\varphi^1 \in L^2(\Theta, X)$  and every  $u \in L^2(\mathbb{R}_+, U)$ , we have, by definition

$$\begin{aligned} \langle \Phi_T^* \varphi^1, u \rangle_{L^2(\mathbb{R}_+, U)} &= \int_{\Theta} \langle \varphi^1(\theta), (\Phi_T u)(\theta) \rangle_X \, d\mu(\theta) \\ &= \int_{\Theta} \left\langle \varphi^1(\theta), \int_0^T \mathbb{T}_{\alpha(\theta)(T-\tau)} B(\theta) u(\tau) \, d\tau \right\rangle_X \, d\mu(\theta) \\ &= \int_0^T \left\langle \int_{\Theta} B(\theta)^* \mathbb{T}_{\alpha(\theta)(T-\tau)}^* \varphi^1(\theta) \, d\mu(\theta), u(\tau) \right\rangle_U \, d\tau. \end{aligned}$$

It therefore follows that for every  $\varphi^1 \in L^2(\Theta, X)$ ,

$$(\Phi_T^* \varphi^1)(t) = \begin{cases} \int_{\Theta} B(\theta)^* \mathbb{T}_{\alpha(\theta)(T-t)}^* \varphi^1(\theta) \, d\mu(\theta), & \text{if } t \in (0, T), \\ 0, & \text{if } t \geq T. \end{cases} \quad (4.3)$$

Due to (4.2), we now aim to show the following unique continuation property,

$$\left( \int_{\Theta} B(\theta)^* \mathbb{T}_{\alpha(\theta)(T-t)}^* \varphi^1(\theta) \, d\mu(\theta) = 0, \quad \forall t \in [0, T] \right) \Rightarrow \varphi^1 = 0. \quad (\text{UCP})$$

Let  $\varphi^1 \in L^2(\Theta, X)$  such that for every  $t \in [0, T]$ ,

$$\int_{\Theta} B(\theta)^* \mathbb{T}_{\alpha(\theta)(T-t)}^* \varphi^1(\theta) \, d\mu(\theta) = 0. \quad (4.4)$$

Setting  $\varphi(t, \theta) = \mathbb{T}_{\alpha(\theta)(T-t)}^* \varphi^1(\theta)$ ,  $\varphi$  solves

$$\begin{cases} \dot{\varphi}(t, \theta) = -\alpha(\theta) A^* \varphi(t, \theta) & (t \in (0, T)), \\ \varphi(T, \theta) = \varphi^1(\theta). \end{cases} \quad (4.5)$$

Setting again  $\tilde{\varphi}(t, \theta) = \varphi(T-t, \theta)$ , it follows that  $\tilde{\varphi}$  solves

$$\begin{cases} \dot{\tilde{\varphi}}(t, \theta) = \alpha(\theta) A^* \tilde{\varphi}(t, \theta) & (t \in (0, T)), \\ \tilde{\varphi}(0, \theta) = \varphi^1(\theta). \end{cases} \quad (4.6)$$

Equation (4.6) shows that  $\tilde{\varphi}(t, \theta) = \mathbb{T}_{\alpha(\theta)t}^* \varphi^1(\theta)$ . Since  $\mathbb{T}_t$  is an analytic semi-group, (UCP) is equivalent to

$$\left( \int_{\Theta} B(\theta)^* \tilde{\varphi}(t, \theta) \, d\mu(\theta) = 0, \quad \forall t \geq 0 \right) \Rightarrow \varphi^1 = 0, \quad (\text{UCP}')$$

where  $\tilde{\varphi}$  solves (4.6).

Using (hp<sub>1</sub>) and applying the operator  $L$  defined in (hp<sub>3</sub>) to the above equation, we get

$$\int_{\Theta} LB(\theta)^* \tilde{\varphi}(t, \theta) \, d\mu(\theta) = 0 \quad (t \geq 0).$$

Assumption (hp<sub>3</sub>) ensures that  $L(B^* \varphi)(\theta) = \beta(\theta) C \varphi(\theta)$  for every  $\varphi \in L^2(\Theta, X)$ . Hence, it follows that

$$C \int_{\Theta} \beta(\theta) \tilde{\varphi}(t, \theta) \, d\mu(\theta) = 0 \quad (t \geq 0), \quad (4.7)$$

that is to say,

$$\int_{\Theta} \beta(\theta) \tilde{\varphi}(t, \theta) \, d\mu(\theta) \in \text{Ker}(C) \quad (t \geq 0).$$

Therefore, using (hp<sub>1</sub>), we have, according to Remark 2.1,

$$\int_{\Theta} \beta(\theta) \tilde{\varphi}(t, \theta) \, d\mu(\theta) \in \text{Ker}(C) \cap \mathcal{D}(A^*) \quad (t > 0).$$

and Assumption **(hp<sub>4</sub>)** leads to

$$C \int_{\Theta} \beta(\theta) A^* \tilde{\varphi}(t, \theta) \, d\mu(\theta) = 0 \quad (t > 0). \quad (4.8)$$

But  $\tilde{\varphi}$  is solution of (4.6), hence, we have

$$A^* \tilde{\varphi}(t, \theta) = \frac{1}{\alpha(\theta)} \dot{\tilde{\varphi}}(t, \theta) \quad (t > 0),$$

this combined with (4.8) gives us

$$C \int_{\Theta} \frac{\beta(\theta)}{\alpha(\theta)} \dot{\tilde{\varphi}}(t, \theta) \, d\mu(\theta) = 0 \quad (t > 0).$$

For every  $t > 0$ , let us now integrate the above equation on  $[t, +\infty)$ , to obtain

$$C \int_{\Theta} \frac{\beta(\theta)}{\alpha(\theta)} \left( \int_t^{+\infty} \dot{\tilde{\varphi}}(\tau, \theta) \, d\tau \right) \, d\mu(\theta) = 0 \quad (t > 0). \quad (4.9)$$

Using the fact that for every  $\theta \in \Theta$ ,  $\mathbb{T}_{\alpha(\theta)t}^*$  is exponentially stable, we deduce that

$$\int_t^{+\infty} \dot{\tilde{\varphi}}(\tau, \theta) \, d\tau = -\tilde{\varphi}(t, \theta) \quad (t > 0, \theta \in \Theta).$$

Equation (4.9) therefore reads

$$\int_{\Theta} \frac{\beta(\theta)}{\alpha(\theta)} C \tilde{\varphi}(t, \theta) \, d\mu(\theta) = 0 \quad (t > 0). \quad (4.10)$$

Applying the above process  $k$  times, we end up with

$$\int_{\Theta} \frac{\beta(\theta)}{\alpha(\theta)^k} C \tilde{\varphi}(t, \theta) \, d\mu(\theta) = 0 \quad (t > 0, k \in \mathbb{N}).$$

Using Assumption **(hp<sub>6</sub>)** and Remark 2.2, we deduce that for  $\mu$ -almost every  $\theta$

$$C \tilde{\varphi}(t, \theta) = 0, \quad \forall t > 0.$$

Finally, we use Assumption **(hp<sub>5</sub>)** to deduce that for  $\mu$ -almost every  $\theta$ ,

$$\varphi^1(\theta) = 0.$$

This proves **(UCP)**, and hence (2.1) is  $L^2(\Theta, X)$ -ensemble controllable in time  $T$ .

## 5 Digression on ordinary differential equations

In this section we consider the case where  $X = \mathbb{R}^n$ ,  $U = \mathbb{R}^m$  and  $Y = \mathbb{R}^q$ , with  $n$ ,  $m$  and  $q$  three integers. In this framework (2.1) is just an ordinary differential equation and the Assumptions **(hp<sub>1</sub>)** to **(hp<sub>6</sub>)** read:

- (as<sub>1</sub>) The matrix  $A$  belonging to  $\mathbb{R}_n^n$  is Hurwitz;
- (as<sub>2</sub>) The control operator  $B$  belongs to  $L^2(\Theta)_m^n$ ;
- (as<sub>3</sub>) There exist  $L \in \mathbb{R}_m^q$ ,  $C \in \mathbb{R}_n^q$  and  $\beta \in L^2(\Theta)$  such that for every  $\varphi \in L^2(\Theta)^n$ ,

$$LB^*(\theta)\varphi(\theta) = \beta(\theta)C\varphi(\theta);$$

- (as<sub>4</sub>)  $\text{Ker}(C)$  is stable by  $A^*$ ;
- (as<sub>5</sub>)  $(A^*, C)$  is observable;
- (as<sub>6</sub>) The function  $\alpha \in \mathcal{C}(\Theta)$  is positive, injective and non vanishing, and  $\beta \in L^2(\Theta)$  is such that  $\beta(\theta) \neq 0$  for  $\mu$ -almost every  $\theta \in \Theta$ .

In this case, the proof of Theorem 2.5 can be given by classical arguments. Since this classical arguments reveal that certain assumptions are not necessary for controllability we give the main lines. Indeed, thank to (UCP), let  $\varphi^1 \in L^2(\Theta)^n$  such that

$$\int_{\Theta} B(\theta)^* e^{\alpha(\theta)(T-t)A^*} \varphi^1(\theta) \, d\mu(\theta) = 0 \quad (t \in [0, T]). \quad (5.1)$$

Using Assumption (as<sub>3</sub>), we have

$$C \int_{\Theta} \beta(\theta) e^{\alpha(\theta)(T-t)A^*} \varphi^1(\theta) \, d\mu(\theta) = 0 \quad (t \in [0, T]). \quad (5.2)$$

Computing the derivatives of the above equation with respect to  $t$ , and evaluating the obtained equation at time  $t = T$ , we deduce

$$CA^{*k} \int_{\Theta} \beta(\theta) \alpha(\theta)^k \varphi^1(\theta) \, d\mu(\theta) = 0 \quad (k \in \mathbb{N}). \quad (5.3)$$

We now observe that Assumptions (as<sub>4</sub>) and (as<sub>5</sub>) imply  $\text{Ker}(C) = \{0\}$ . Indeed, Assumption (as<sub>5</sub>) (i.e., the pair  $(A^*, C)$  is observable) is equivalent to the pair  $(A, C^*)$  is controllable. Therefore, the space  $\mathcal{N} = \{x \in \mathbb{R}^n \mid \forall k \in \mathbb{N}, (A^*)^k x \in \text{Ker}(C)\}$  is reduced to the null subspace of  $\mathbb{R}^n$ . But Assumption (as<sub>4</sub>) (i.e.,  $\text{Ker}(C)$  is stable by  $A^*$ ) ensures that  $\mathcal{N} = \text{Ker}(C)$ , and hence,  $\text{Ker}(C) = \{0\}$ . Consequently, (5.3) implies

$$A^{*k} \int_{\Theta} \beta(\theta) \alpha(\theta)^k \varphi^1(\theta) \, d\mu(\theta) = 0 \quad (k \in \mathbb{N}). \quad (5.4)$$

Thanks to Assumption (as<sub>1</sub>),  $A$  is invertible, hence,

$$\int_{\Theta} \beta(\theta) \alpha(\theta)^k \varphi^1(\theta) \, d\mu(\theta) = 0 \quad (k \in \mathbb{N}). \quad (5.5)$$

Assumption (as<sub>6</sub>) together with Remark 2.2 leads to

$$\overline{\text{Span}\{\theta \in \Theta \rightarrow \beta(\theta)\alpha(\theta)^k, k \in \mathbb{N}\}} = L^2(\Theta),$$

and it follows that  $\varphi^1 = 0$   $\mu$ -almost everywhere on  $\Theta$ . This ends the proof of Theorem 2.5 in the case  $X = \mathbb{R}^n$ ,  $U = \mathbb{R}^m$  and  $Y = \mathbb{R}^q$ .

We now give some comments on these lines.

*Remark 5.1.* We have seen in the above lines that Assumption **(as<sub>1</sub>)** is not necessary in the case of ordinary differential equations. In fact, we only need  $A$  to be invertible. One can see from the proof we provided in Section 4 that the exponential stability character of the semigroup was used to pass from (4.9) to (4.10). Therefore, this constraint was imposed by our proof. Theorem 2.5 might hold without the exponential stability character of the operator  $(A, \mathcal{D}(A))$ . However, this has resisted to several attempts of proof.

*Remark 5.2.* Observe that since  $\text{Ker}(C) = \{0\}$ , it follows that  $q \geq n$ . Moreover, from this condition, it follows from **(as<sub>1</sub>)** that for  $\mu$ -almost every  $\theta \in \Theta$ ,  $\text{rk}(B(\theta)) = n$ . In particular, we have  $m \geq n$ . Note that unlike the state controllability of linear time invariant systems where it is natural to consider  $m \leq n$ , it makes fully sense here to consider systems with  $m \geq n$  since the state space is an infinite dimensional space, here  $L^2(\Theta)^n$ .

*Remark 5.3.* Assumption **(as<sub>3</sub>)** can be useful in practice when “the control operator does not look nice”. For instance, if one wants to check the  $L^2(\Theta)^2$ -ensemble controllability of system (2.1) with

$$\Theta = [0, 1], \quad \alpha(\theta) = \theta^2, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ B(\theta) = \begin{pmatrix} 0 & 2\sqrt{\theta+1} & 1 & 1+\theta \\ \sqrt{\theta+1} & 0 & 1+\theta^3 & \theta^2 \end{pmatrix}, \quad (5.6)$$

this assumption tells us that it is not necessary to use five sources (inputs or controls) in system (5.6) since only two sources are sufficient. Indeed, taking in Assumption **(as<sub>3</sub>)**

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \beta(\theta) = \sqrt{\theta+1},$$

one can infer from [5, Theorem 5] (or [4, Theorem 1]) that system (5.6) is  $L^2(\Theta)^2$ -ensemble controllable. One should first note that the multiplication operator  $\mathcal{A}_\beta : L^2(\Theta) \rightarrow L^2(\Theta)$  defined for every  $f \in L^2(\Theta)$  by  $(\mathcal{A}_\beta f)(\theta) = \beta(\theta)f(\theta)$  is an isomorphism of  $L^2(\Theta)$ , when  $\beta \in L^\infty(\Theta)$  satisfies Assumption **(as<sub>6</sub>)**. Using therefore the fact that the system (2.1) with

$$\Theta = [0, 1], \quad \alpha(\theta) = \theta^2, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(\theta) = C^* = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

is  $\mathcal{C}(\Theta)^2$ -ensemble controllable, here comes [5, Theorem 5], we deduce from the density of continuous functions in  $L^2(\Theta)$  that the system given by (5.6) is  $L^2(\Theta)^2$ -ensemble controllable.

## 6 Numerical example

The aim of this paragraph is to illustrate numerically the  $L^2(\Theta, X)$ -ensemble controllability of system (3.1) in one dimensional case, More precisely, we take

$\Theta$  a compact interval of  $\mathbb{R}_+^*$ ,  $\mu$  the Lebesgue measure,  $\Omega = (0, \pi)$ ,  $\omega(\theta) = \omega = [\underline{\omega}, \bar{\omega}] \subset (0, \pi)$  independent of  $\theta$ ,  $X = L^2(0, \pi)$ ,  $U = L^2(\omega)$ , and consider the system

$$\begin{cases} \dot{y}(t, x, \theta) = \alpha(\theta)\partial_x y(t, x, \theta) + \beta(\theta)\chi_\omega(x)u(t, x), & (t, x) \in \mathbb{R}_+ \times (0, \pi), \\ y(t, x, \theta) = 0, & (t, x) \in \mathbb{R}_+ \times \{0, \pi\}, \\ y(0, x, \theta) = y^0(x, \theta), & x \in (0, \pi), \end{cases} \quad (6.1)$$

where  $\alpha$  and  $\beta$  satisfy Assumption **(hp6)**.

We recall that there exist an increasing sequence  $(\lambda_k)_{k \geq 1} \in (\mathbb{R}_+)^{\mathbb{N}^*}$  such that  $\lambda_k \rightarrow \infty$  when  $k \rightarrow \infty$  and a Hilbert basis  $(\phi_k)_{k \geq 1}$  of  $L^2(0, \pi)$  such that  $(\phi_k)_{k \geq 1} \subset H_0^1(0, \pi)$  and  $-\partial_x^2 \phi_k = \lambda_k \phi_k$ . In the present case, we have

$$\lambda_k = k^2 \quad \text{and} \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx) \quad (k \in \mathbb{N}^*, x \in (0, \pi)).$$

## 6.1 System discretization

Given any  $y^0 \in L^2(\Theta, X)$  and every  $u \in L^2([0, T], L^2(0, \pi))$ , we write the solution  $y(t, \theta) = \sum_{k=1}^{\infty} y_k(t, \theta) \phi_k$  where  $y_k(t, \theta) = \langle y(t, \theta), \phi_k \rangle_{L^2(0, \pi)}$  solves, for every  $k \geq 1$ ,

$$\begin{cases} \dot{y}_k(t, \theta) = -\alpha(\theta)\lambda_k y_k(t, \theta) + \beta(\theta)u_k(t) & (t > 0), \\ y_k(0, \theta) = y_k^0(\theta). \end{cases} \quad (6.2)$$

where we have set  $u_k(t) = \langle \chi_\omega u(t), \phi_k \rangle_{L^2(0, \pi)}$  and  $y_k^0(\theta) = \langle y^0(\theta), \phi_k \rangle_{L^2(0, \pi)}$ .

We now use finite element method by subdividing the interval  $\omega$  in  $M + 2$  points  $(x_k)_{k \in \mathbb{N}_{\leq M+1}}$  where  $x_k = \underline{\omega} + kh$  with  $h = (\bar{\omega} - \underline{\omega}) / (M + 1)$ , to approximate the control  $u$  by

$$u^M(t, x) = \sum_{\ell=1}^M v_\ell(t) \psi_\ell(x), \quad (6.3)$$

where

$$\psi_\ell(x) = \begin{cases} \frac{x - x_{\ell-1}}{x_\ell - x_{\ell-1}}, & \text{if } x \in [x_{\ell-1}, x_\ell], \\ \frac{x_{\ell+1} - x}{x_{\ell+1} - x_\ell}, & \text{if } x \in [x_\ell, x_{\ell+1}], \\ 0 & \text{otherwise} \end{cases} \quad (\ell \in \mathbb{N}_{\leq M}^*),$$

Since by construction  $u^M$  is supported in  $\omega$ , we have, by setting  $u_k^M(t) = \langle \chi_\omega u^M(t), \phi_k \rangle_{L^2(0, \pi)}$ ,

$$u_k^M(t) = \langle \chi_\omega u^M(t), \phi_k \rangle_{L^2(0, \pi)} = \sum_{\ell=1}^M \Gamma_{k, \ell} v_\ell(t),$$

with  $\Gamma_{k, \ell} = \langle \phi_k, \psi_\ell \rangle_{L^2(0, \pi)}$ . A direct computation leads to

$$\Gamma_{k, \ell} = \frac{-1}{hk^2} (\phi_k(x_{\ell-1}) - 2\phi_k(x_\ell) + \phi_k(x_{\ell+1})) \quad (k \in \mathbb{N}_{\leq M}^*, \ell \in \mathbb{N}_{\leq M}^*).$$

Setting  $\mathbb{Y}(t, \theta) = (y_1(t, \theta), \dots, y_N(t, \theta))^\top$ ,  $\mathbb{U}(t) = (v_1(t), \dots, v_M(t))^\top$ , we deduce from (6.2) that

$$\dot{\mathbb{Y}}(t, \theta) = \alpha(\theta)\mathbb{A}\mathbb{Y}(t, \theta) + \beta(\theta)\mathbb{B}\mathbb{U}(t), \quad (6.4)$$

where

$$\mathbb{A} = \begin{pmatrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_N \end{pmatrix} \in \mathbb{R}_N^N \quad \text{and} \quad \mathbb{B} = \begin{pmatrix} \Gamma_{1,1} & \dots & \Gamma_{1,M} \\ \vdots & & \vdots \\ \Gamma_{N,1} & \dots & \Gamma_{N,M} \end{pmatrix} \in \mathbb{R}_M^N. \quad (6.5)$$

From the above, each solution  $y(t, \theta) = \sum_{k=1}^{\infty} y_k(t, \theta)\phi_k$  can be found as a limit of  $y^N(t, \theta) = \sum_{k=1}^N y_k(t, \theta)\phi_k$  when  $N$  tends to infinity, where  $(y_1(t, \theta), \dots, y_N(t, \theta))^\top = \mathbb{Y}(t, \theta)$  solves (6.4) with  $\mathbb{Y}(0, \theta) = (y_1^0(\theta), \dots, y_N^0(\theta))^\top$ .

Let us observe that the system (6.4) is  $L^2(\Theta)^N$ -ensemble controllable. More precisely, we have the following proposition.

**Proposition 6.1.** *Assume that (hp<sub>6</sub>) holds. For every  $N \in \mathbb{N}^*$  and every  $M \geq 3N - 1$ , the system (6.4) is  $L^2(\Theta)^N$ -ensemble controllable, where  $\mathbb{A}$  and  $\mathbb{B}$  are given by (6.5).*

*Proof.* Let us show that Assumptions (as<sub>1</sub>) to (as<sub>5</sub>) are satisfied. We clearly have that Assumptions (as<sub>1</sub>) and (as<sub>2</sub>) are satisfied. Assumption (as<sub>3</sub>) is also satisfied with  $C = \mathbb{B}^*$  and  $L = I_M$ .

We then aim to prove that for  $M$  large enough, we have  $\text{rk}(\mathbb{B}) = N$ . This will ensure Assumptions (as<sub>4</sub>) and (as<sub>5</sub>), and the proof will be concluded using Theorem 2.5 (see also Section 5).

Let us assume by contradiction that  $\text{rk}(\mathbb{B}) < N$ , then there exists  $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ , which are not all trivial, such that

$$0 = \left\langle \sum_{k=1}^N \alpha_k \phi_k, \psi_\ell \right\rangle_{L^2(0, \pi)} \quad (\ell \in \mathbb{N}_{\leq M}^*). \quad (6.6)$$

Let us then define  $f(x) = \sum_{k=1}^N \alpha_k \phi_k$ , there exists a polynomial  $P$  of degree at most  $N-1$  such that  $f(x) = \sin(x)P(\cos(x))$  for every  $x \in [0, \pi]$ . More precisely,  $P = \sum_{k=1}^N \alpha_k U_{k-1}$ , where  $U_k$  is the second kind Tchebychev polynomial of degree  $k$ . This ensures that  $f$  admits at most  $N-1$  zeros on  $(0, \pi)$ . Consequently, there exist an interval  $I \subset \omega$  of length greater than  $(\bar{\omega} - \underline{\omega})/N$  where  $f$  is of constant sign. But, for  $M \geq 3N - 1$  (i.e.,  $3\frac{\bar{\omega} - \underline{\omega}}{M+1} \leq \frac{\bar{\omega} - \underline{\omega}}{N}$ ), there exists  $\ell \in \mathbb{N}_{\leq M}^*$  such that  $x_{\ell-1}, x_\ell, x_{\ell+1} \in I$ , that is to say that  $\text{supp } \psi_\ell \subset I$ . This leads to a contradiction with (6.6), since  $\psi_\ell$  and  $f$  have a constant sign on  $\text{supp } \psi_\ell$ , and do not vanish in the interior of  $\text{supp } \psi_\ell$ .  $\square$

## 6.2 Minimal $L^2$ -norm control

Given  $T > 0$ ,  $y^0, y^1 \in L^2(\Theta, L^2(0, \pi))$  and  $\varepsilon > 0$ , we aim to minimize the following problem

$$\begin{aligned} \min \quad & \frac{1}{2} \int_0^T \int_\omega |u(t, x)|^2 dx dt \\ \text{s.t.} \quad & \sqrt{\int_\Theta \int_0^\pi |y(T, \theta, x) - y^1(\theta, x)|^2 dx d\theta} \leq \varepsilon, \end{aligned} \quad (6.7)$$

where  $y$  is the solution of (6.1) with control  $u$  and initial condition  $y^0$ . Using Fenchel-Rockafellar duality, see e.g. [26, 27], we obtain that the minimizer of the above problem is given by

$$u(t, x) = \int_\Theta \beta(\theta) \varphi(t, \theta, x) d\theta \quad (t \in (0, T), x \in \omega),$$

where  $\varphi$  solves the adjoint problem

$$\begin{aligned} -\dot{\varphi}(t, \theta, x) &= \alpha(\theta) \partial_x^2 \varphi(t, \theta, x) & (t \in (0, T), x \in (0, \pi), \theta \in \Theta), \\ \varphi(T, \theta, x) &= \varphi^1(\theta, x) & (x \in (0, \pi), \theta \in \Theta). \end{aligned}$$

with  $\varphi^1 \in L^2(\Theta, L^2(0, \pi))$  the minimizer of

$$\begin{aligned} J(\varphi^1) &= \frac{1}{2} \int_0^T \int_\omega \left| \int_\Theta \beta(\theta) \varphi(t, \theta, x) d\theta \right|^2 dx dt \\ &\quad - \int_\Theta \int_0^\pi y^1(\theta, x) \varphi^1(\theta, x) dx d\theta + \int_\Theta \int_0^\pi y^0(\theta, x) \varphi(T, \theta, x) dx d\theta \\ &\quad + \varepsilon \sqrt{\int_\Theta \int_0^\pi |\varphi^1(\theta, x)|^2 dx d\theta}. \end{aligned}$$

Considering the discretization proposed in Section 6.1, the minimization problem (6.7) becomes

$$\begin{aligned} \min \quad & \frac{1}{2} \int_0^T v(t)^\top \mathbb{M} v(t) dt \\ \text{s.t.} \quad & \sqrt{\int_\Theta |\mathbb{Y}(T, \theta) - \mathbb{Y}^1(\theta)|^2 d\theta} \leq \varepsilon, \end{aligned} \quad (6.8)$$

where  $\mathbb{Y}$  is solution of (6.4), with initial condition  $\mathbb{Y}^0(\theta)$ , and where for  $\iota \in \{0, 1\}$ ,  $\mathbb{Y}^\iota(\theta) = \left( \langle y^\iota(\theta), \phi_1 \rangle_{L^2(0, \pi)}, \dots, \langle y^\iota(\theta), \phi_N \rangle_{L^2(0, \pi)} \right)^\top$ , and where  $\mathbb{M} \in \mathbb{R}_M^M$  is given by  $\mathbb{M}_{k, \ell} = \langle \psi_k, \psi_\ell \rangle_{L^2(0, \pi)}$ . Note that one can also choose  $\mathbb{M} = hI_M$ . The minimizer is obtained through the adjoint system, i.e.,

$$v(t) = \mathbb{M}^{-1} \mathbb{B}^\top \int_\Theta \beta(\theta) p(t, \theta) d\theta \quad (t \in (0, T)),$$

with  $p$  solution of

$$\begin{aligned} -\dot{p}(t, \theta) &= \alpha(\theta) \mathbb{A}^* p(t, \theta) & (t \in (0, T), \theta \in \Theta), \\ p(T, \theta) &= p^1(\theta) & (\theta \in \Theta). \end{aligned}$$



with  $p^1 \in L^2(\Theta)^{\mathbb{N}}$  the minimizer of

$$\begin{aligned} \mathbb{J}(p^1) &= \frac{1}{2} \int_0^T \left( \mathbb{B}^\top \int_{\Theta} \beta(\theta) p(t, \theta) d\theta \right)^\top \mathbb{M}^{-1} \left( \mathbb{B}^\top \int_{\Theta} \beta(\theta) p(t, \theta) d\theta \right) dt \\ &\quad - \int_{\Theta} \langle \mathbb{Y}^1(\theta), p^1(\theta) \rangle d\theta + \int_{\Theta} \langle \mathbb{Y}^0(\theta), p(T, \theta) \rangle d\theta + \varepsilon \sqrt{\int_{\Theta} |p^1(\theta)|^2 d\theta}. \end{aligned}$$

*Remark 6.2.* It is expected (but not proved yet) that  $\varphi^{1, \mathbb{N}}$ , defined by  $\varphi^{1, \mathbb{N}}(\theta, x) = \sum_{k=1}^{\mathbb{N}} p^1(\theta) \sin(kx)$ , goes to  $\varphi^1$  as  $\mathbb{N}$  and  $\mathbb{M}$  goes to  $\infty$ .

### 6.3 Numerical experiments

To numerically approximate a control, we finally consider a discretization of  $\Theta$  and numerically compute the integral over  $\Theta$ , similarly, we use a numerical scheme to approximate the solutions of (6.4) and of the adjoint system. In practice, we have used the mid point rule on a uniform discretization of  $\Theta$  to compute the integrals over  $\Theta$ , and the Crank-Nicolson method, with a uniform discretization of  $[0, T]$  to compute the solutions of the direct and adjoint systems.

For the numerical simulation below, we have considered  $\Theta = [1, 2]$ ,  $\alpha(\theta) = \theta$ ,  $\beta(\theta) = 1$  and  $\omega = (0.5, 0.8)$ . We have also considered  $\mathbb{N} = 21$ ,  $\mathbb{M} = 200$ , and the discretization step is  $5.10^{-3}$  for  $\Theta$  and  $5.10^{-4}$  for the time interval  $[0, T]$ . The final time is set to  $T = 1$ , and we have set  $\varepsilon = 10^{-1}$ . The initial condition is  $y^0(\theta, x) = 2\theta x(\pi - x)(x - \theta)$ , and the target is  $y^1(\theta, x) = 0$ .

On Figure 1, we display the obtained control, its  $L^2$ -norm is  $4.230773 \times 10^2$ . On Figure 2, we show the  $L^2$ -norm of the solution with respect to time, at the final time  $T$ , its norm is equal to  $\varepsilon$  (up to a numerical tolerance which as been set to  $10^{-6}$ ), and  $(\theta, x) \mapsto y(T, \theta, x)$  is plotted on Figure 3. Finally, we have shown on Figure 4 the time evolution of the  $\mathbb{N}$  Fourier coefficients used for the simulation.

## 7 Concluding remarks

In this paper, we have studied the  $L^2(\Theta, X)$ -ensemble controllability of some class of parabolic equations. But there still exist many practical questions to be addressed. Out of the one raised in Remark 5.1, we list below some of them.

- In Assumption (hp<sub>3</sub>) we have assumed a kind of decoupling condition on the control operator, mainly the separation between the space variable and the parameter one. What if this splitting is not possible? For instance, in (3.1), one might think of the control operator of the form  $B(\theta, x) = \chi_\omega(x)\rho(\theta, x)$  instead of  $B(\theta, x) = \chi_\omega(x)\rho(\theta)$ .
- Another question that is already suggested implicitly in Section 3, for the ensemble controllability of (3.1), is: what happen if we do not have the assumption that  $\bigcap_{\theta \in \Theta} \omega(\theta)$  contained a nontrivial open subset of  $\Omega$ ?

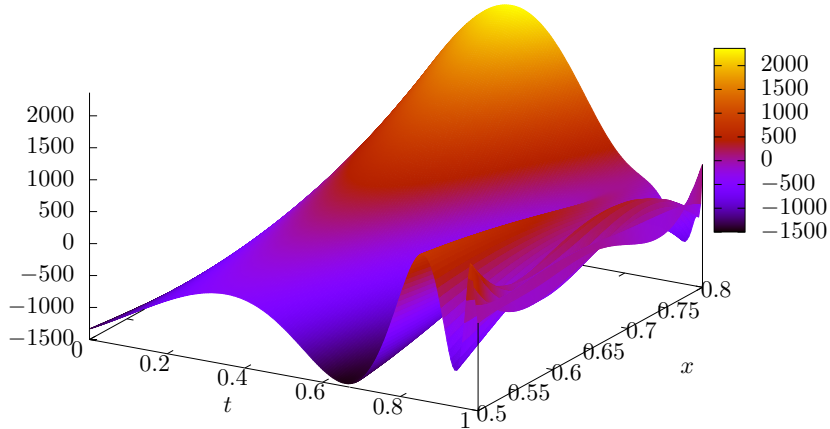


Figure 1: Control ( $L^2$ -norm :  $4.230773 \times 10^2$ ). See Section 6.3 for the parameters used for this simulation.

- Can we consider  $\alpha$  depending on the space variable in (2.1)? Similarly, can we consider an operator  $A$  that depend on  $\theta$ ? For instance the ensemble controllability of (3.1) is open if we replace  $\theta\Delta$  by  $\text{div}(M(\theta)\nabla)$ , with  $M(\theta)$  a positive matrix (which is not of the form  $\alpha(\theta)I$ ).
- The question of  $\mathcal{C}(\Theta, X)$ -ensemble controllability, that is *uniform ensemble controllability* of system (2.1) is open.
- Last but not least, in this paper we have considered the problem of internal controllability. The case of boundary controllability is open. In fact, even if, as stated in Remark 2.7, the results are still valid with unbounded control operators, the verification of Assumption (hp<sub>3</sub>) is not easy, and we do not have examples of unbounded control operators where Assumption (hp<sub>3</sub>) is fulfilled.  
For a one dimensional heat equation as considered in (3.1) but this time with boundary controls, we can prove using [3] that the  $\mathcal{C}(\Theta, X)$ -ensemble controllability cannot hold, and we believe that the  $L^2(\Theta, X)$ -ensemble controllability is also impossible. However, this is not clear in higher space dimension. In fact, we cannot end up with a contradiction with the Hautus test, as it is done in [3].
- Dealing with the numerical approach: what are the required relations between  $N$ ,  $M$  and the discretization of  $[0, T]$  and  $\Theta$ , to ensure that the

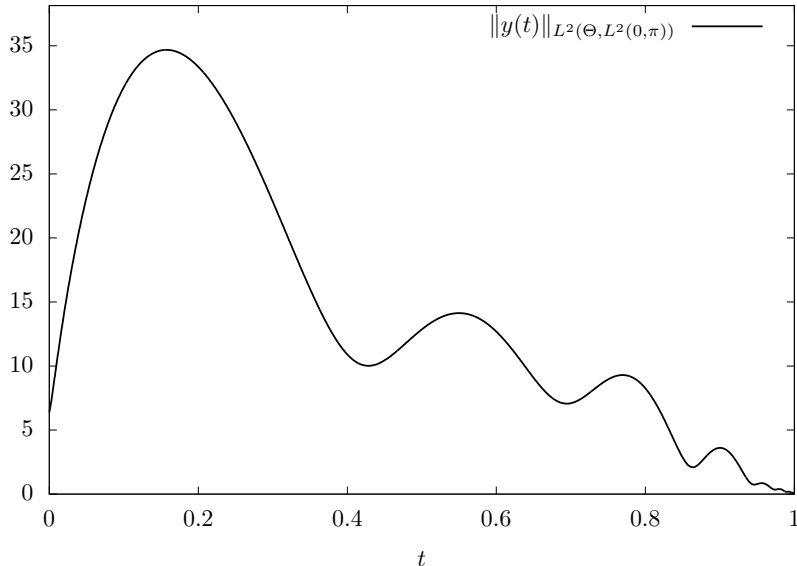


Figure 2: Time evolution of the  $L^2$ -norm of the solution. See Section 6.3 for the parameters used for this simulation.

discrete control  $u^M$  and the discrete adjoint  $\varphi^{1,N}$  goes to  $u$  and  $\varphi^1$  as  $N \rightarrow \infty$ ?

In addition, we chose here to approach  $y$  with a finite combination of eigenfunctions of the Dirichlet-Laplace operator. This was to ensure the ensemble controllability of the discretized system (see Proposition 6.1). Even if other discretization are possible, it is not clear that the discretized system will be ensemble controllable. For instance, if we use classical finite differences, we will necessarily have  $M < N$  when  $\omega$  is strictly included in  $\Omega$ , and hence  $\text{rk}(\mathbb{B}) < N$ . Thus, the controllability of the discretized system is no more a direct application of Theorem 2.5.

## Acknowledgements

The authors thank Martin Lazar for helpful discussions and comments on this topic. The authors are supported by the French ANR project TRECOS, ANR-20-CE40-0009. The second author is also supported by the ANR project NICETWEET, ANR-20-CE48-0009.

## References

- [1] J.-S. Li, N. Khaneja, Ensemble control of Bloch equations, IEEE Transactions on Automatic Control 54 (3) (2009) 528–536. doi:10.1109/TAC.

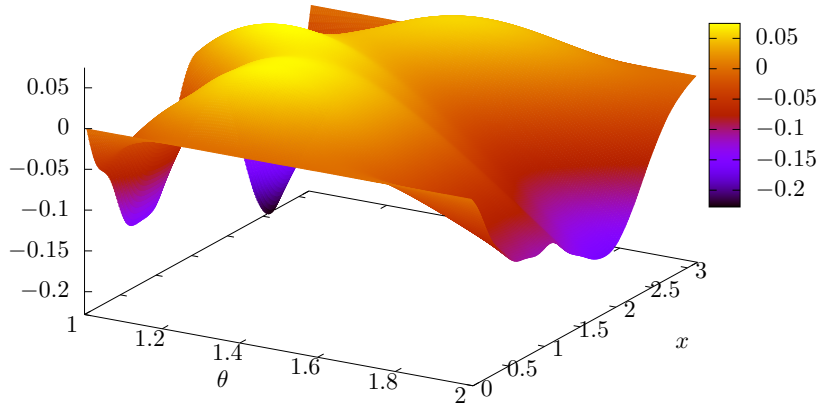


Figure 3: Parameter dependent state at final time. See Section 6.3 for the parameters used for this simulation.

2009.2012983.

- [2] K. Beauchard, J.-M. Coron, P. Rouchon, Controllability issues for continuous-spectrum systems and ensemble controllability of Bloch equations, *Commun. Math. Phys.* 296 (2) (2010) 525–557. doi:10.1007/s00220-010-1008-9.
- [3] G. Dirr, M. Schönlein, Uniform and  $L^q$ -ensemble reachability of parameter-dependent linear systems, *J. Differ. Equations* 283 (2021) 216–262. doi:10.1016/j.jde.2021.02.032.
- [4] J.-S. Li, J. Qi, Ensemble control of time-invariant linear systems with linear parameter variation, *IEEE Transactions on Automatic Control* 61 (10) (2016) 2808–2820. doi:10.1109/TAC.2015.2503698.
- [5] M. Schönlein, U. Helmke, Controllability of ensembles of linear dynamical systems, *Math. Comput. Simul.* 125 (2016) 3–14. doi:10.1016/j.matcom.2015.10.006.
- [6] X. Chen, Controllability issues of linear ensemble systems over multi-dimensional parameterization spaces, preprint (2022). arXiv:2003.04529.
- [7] B. Danhane, J. Lohéac, M. Jungers, Conditions for uniform ensemble output controllability, and obstruction to uniform ensemble controllability,

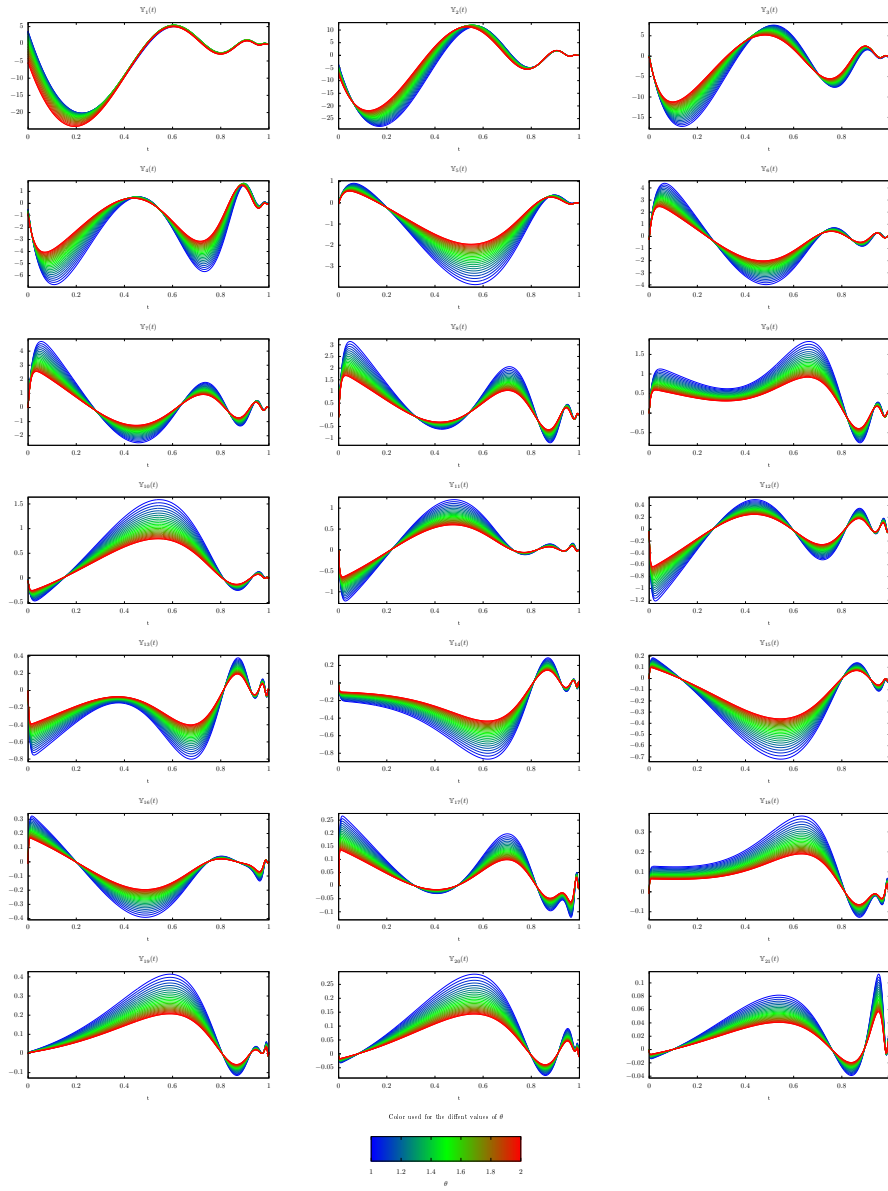


Figure 4: Time evolution of the 21 Fourier coefficients used in the simulation. See Section 6.3 for the parameters used for this simulation.

preprint (Oct. 2022).  
 URL <https://hal.science/hal-03824645>

- [8] M. Schönlein, Feedback equivalence and uniform ensemble reachability, Lin-

- ear Algebra Appl. 646 (2022) 175–194. doi:10.1016/j.laa.2022.03.026.
- [9] M. Lazar, J. Lohéac, Control of parameter dependent systems, in: Numerical control. Part A, Amsterdam: Elsevier/North Holland, 2022, pp. 265–306. doi:10.1016/bs.hna.2021.12.008.
- [10] M. Schönlein, Computation of open-loop inputs for uniformly ensemble controllable systems, Math. Control Relat. Fields 12 (3) (2022) 813–829. doi:10.3934/mcrf.2021046.
- [11] A. Agrachev, Y. Baryshnikov, A. Sarychev, Ensemble controllability by Lie algebraic methods, ESAIM, Control Optim. Calc. Var. 22 (4) (2016) 921–938. doi:10.1051/cocv/2016029.
- [12] P. A. Guth, K. Kunisch, S. S. Rodrigues, Ensemble feedback stabilization of linear systems, preprint (2023). arXiv:2306.01079.
- [13] V. Alleaume, M. Krstic, Ensembles of hyperbolic pdes: Stabilization by backstepping, preprint (2023). arXiv:2307.13195.
- [14] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Vol. 44 of Appl. Math. Sci., Springer, Cham, 1983. doi:10.1007/978-1-4612-5561-1.
- [15] H. A. Seid, Cyclic multiplication operators on  $L_p$ -spaces, Pac. J. Math. 51 (1974) 549–562. doi:10.2140/pjm.1974.51.549.
- [16] W. T. Ross, W. R. Wogen, Common cyclic vectors for unitary operators, J. Oper. Theory 62 (1) (2009) 65–81.
- [17] P. Borwein, T. Erdélyi, The full Müntz theorem in  $C[0, 1]$  and  $L_1[0, 1]$ , J. Lond. Math. Soc., II. Ser. 54 (1) (1996) 102–110. doi:10.1112/jlms/54.1.102.
- [18] T. Erdélyi, W. B. Johnson, The “full Müntz theorem” in  $L_p[0, 1]$  for  $0 < p < \infty$ , J. Anal. Math. 84 (2001) 145–172. doi:10.1007/BF02788108.
- [19] M. Tucsnak, G. Weiss, Observation and control for operator semigroups, Birkhäuser Adv. Texts, Basler Lehrbüch., Basel: Birkhäuser, 2009.
- [20] E. Zuazua, Averaged control, Automatica 50 (12) (2014) 3077–3087. doi:10.1016/j.automatica.2014.10.054.
- [21] A. Avetisyan, A. Z. Khurshudyan, Averaged controllability of Euler-Bernoulli beams with random material characteristics: the Green’s function approach, Mechanics. Proceedings of National Academy of Sciences of Armenia 72 (4) (2019) 7–18. doi:10.33018/72.4.1.
- [22] J. A. Bárcena-Petisco, E. Zuazua, Averaged dynamics and control for heat equations with random diffusion, Syst. Control Lett. 158 (2021) 15, id/No 105055. doi:10.1016/j.sysconle.2021.105055.

- [23] J. Coulson, B. Ghahesifard, A.-R. Mansouri, On average controllability of random heat equations with arbitrarily distributed diffusivity, *Automatica* 103 (2019) 46–52. doi:[10.1016/j.automatica.2019.01.014](https://doi.org/10.1016/j.automatica.2019.01.014).
- [24] F. J. Marín, F. Martínez-Frutos, Jesús and Periago, Robust averaged control of vibrations for the Bernoulli-Euler beam equation, *J. Optim. Theory Appl.* 174 (2) (2017) 428–454. doi:[10.1007/s10957-017-1128-x](https://doi.org/10.1007/s10957-017-1128-x).
- [25] M. Abdelli, C. Castro, Numerical approximation of the averaged controllability for the wave equation with unknown velocity of propagation, *ESAIM, Control Optim. Calc. Var.* 27 (2021) 26, id/No 64. doi:[10.1051/cocv/2021060](https://doi.org/10.1051/cocv/2021060).
- [26] J. M. Borwein, Q. J. Zhu, Techniques of variational analysis, Vol. 20 of CMS Books Math./Ouvrages Math. SMC, New York, NY: Springer, 2005. doi:[10.1007/0-387-28271-8](https://doi.org/10.1007/0-387-28271-8).
- [27] I. Ekeland, R. Témam, Convex analysis and variational problems., unabridged, corrected republication of the 1976 English original Edition, Vol. 28 of Classics Appl. Math., Philadelphia, PA: Society for Industrial and Applied Mathematics, 1999.