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► **To cite this version:**

Ruikang Liang, Ugo Boscain, Mario Sigalotti. Controllability of quantum systems having weakly conically connected spectrum. 2023. hal-04174206v2

HAL Id: hal-04174206

<https://hal.science/hal-04174206v2>

Preprint submitted on 22 Oct 2023

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CONTROLLABILITY OF QUANTUM SYSTEMS HAVING WEAKLY CONICALLY CONNECTED SPECTRUM

RUIKANG LIANG*, UGO BOSCAIN*, AND MARIO SIGALOTTI*

Abstract. We explore the controllability of a closed multi-input control-affine quantum system. Previous studies have demonstrated that a spectrum connected by conical intersections which do not pile up yields exact controllability in finite dimension and approximate controllability in infinite dimension. Actually, the property that intersections between eigenvalues are conical and that they do not pile up is generic. However, in physical situations, due to symmetry of the system, the spectrum can exhibit intersections that are not conical and possibly pile up. We extend the controllability result to cover this type of situations under the hypothesis that the intersections have at least one conical direction and the piled-up intersections have “rationally unrelated germs”. Finally, we provide a testable first-order sufficient condition for controllability. Physically relevant examples are provided.

Key words. Conical eigenvalue intersections, quantum control, exact and approximate controllability

MSC codes. 81Q93, 93B05

1. Introduction. In this paper, we consider the controllability of the bilinear Schrödinger equation

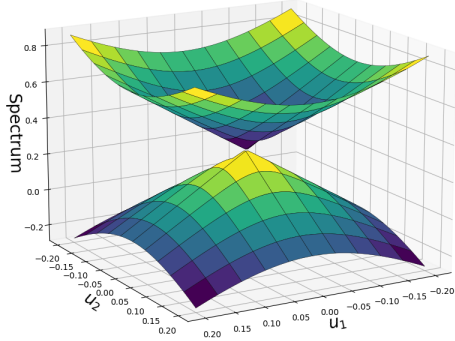
$$(1.1) \quad i\dot{\psi}(t) = H(u(t))\psi(t) = \left(H_0 + \sum_{\ell=1}^m u_{\ell}(t)H_{\ell} \right) \psi(t),$$

where ψ belongs to a complex Hilbert space, $H(u(t))$ is a self-adjoint operator on \mathcal{H} , and the control $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ takes values in an open and connected subset U of \mathbb{R}^m . The controllability properties of the system can be studied via different techniques [16, 6, 18, 13, 14, 7]. In this paper we approach the controllability problem by studying the spectrum of $H(u)$, where u is seen as a parameter. Such spectrum can exhibit eigenvalue intersections. Conical intersections can be defined in terms of the gap between eigenvalues in the vicinity of the intersection (see Figure 1.1a and Definition 2.4). It has been proven in [10] that if the spectrum of $u \mapsto H(u)$ is conically connected, i.e., there exists an eigenvalue intersection between every pair of subsequent eigenvalues and these intersections are not piled-up (see Figures 1.1c and 1.1d for examples), then

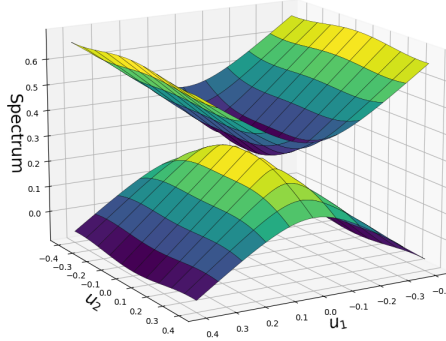
- when $n = \dim(\mathcal{H}) < \infty$, System (1.1) is exactly controllable in the unitary group, i.e., its lift on the unitary group $U(n)$ (or on $SU(n)$ in the case that $iH_0, \dots, iH_m \in su(n)$) has the property that for every choice of initial and finite state, there is an admissible trajectory of the system going from the former to the latter. In particular, System (1.1) is exactly controllable in the unit sphere of \mathcal{H} ;
- when $\dim(\mathcal{H}) = \infty$ and the operators H_1, \dots, H_m are bounded, System (1.1) is approximately controllable, i.e., for each ψ_0 on the unit sphere of \mathcal{H} , the set of states that are reachable from ψ_0 by an admissible trajectory of the system is a dense subset of the unit sphere of \mathcal{H} .

Although eigenvalue intersections are generically conical for $m = 2$ in the case of real Hamiltonians (as shown in [11]) and for $m = 3$ for complex Hamiltonians (as

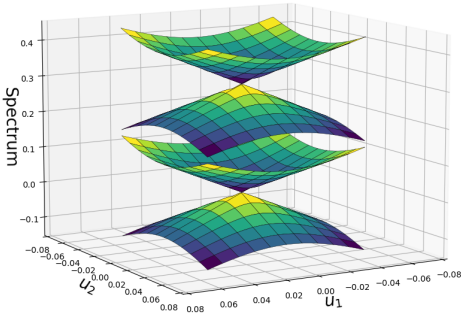
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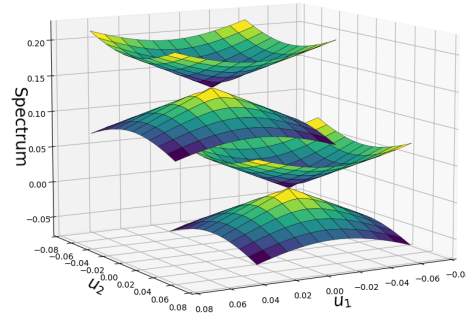
(a) A conical intersection



(b) A semi-conical intersection



(c) Piled-up intersections



(d) Non-piled-up intersections

Figure 1.1: Intersections between eigenvalues of $H(u)$ seen as functions of $u = (u_1, u_2)$, with $m = 2$: conical (a); semiconical (b); piled-up intersections (c); non-piled-up intersections (d).

shown in [15]), and generically they do not pile up, in physical situations, we can observe the following situations:

- Eigenvalue intersections may be non-conical. An example of a physical system that can exhibit non-conical intersections is the STIRAP process (Stimulated Raman Adiabatic Passage) (see [1, Example 6.1.2]). Notice that when $m = 2$, in most cases, the non-conical intersections are actually semi-conical, as studied in [3] (see Definition 2.7 and Figure 1.1b).
- In physical situations, due to symmetries in the system, eigenvalue intersections can pile up. For example, when controlling the rotation of a quan-

tum planar molecule (see [9]), the eigenvalues of H_0 are mostly double and $H_0 + \sum_{\ell=1}^m u_\ell H_\ell$ has many piled-up intersections. Similarly, in Eberly–Law type models (see, for example, [8]), conical intersections can pile up as well.

In this paper, we introduce the notion of weakly conical intersection, i.e., an isolated eigenvalue intersection for which there exists at least one conical direction (see [Definition 2.6](#)), which identifies a specific class of non-conical intersections that includes semi-conical ones. We extend the results of [10] to the case in which intersections can pile up and can be weakly conical. We say that such a spectrum is *weakly conically connected*. However, weak conical connectedness is not sufficient to obtain full controllability results. For example, the Hamiltonian

$$(1.2) \quad H(u, v) = \begin{pmatrix} -1 & -v & 0 & 0 & 0 \\ -v & -1 & -u & 0 & 0 \\ 0 & -u & 0 & u & 0 \\ 0 & 0 & u & 1 & v \\ 0 & 0 & 0 & v & 1 \end{pmatrix}$$

has a weakly conically connected spectrum. However, the dynamics

$$(1.3) \quad i\dot{\psi}(t) = H(u(t), v(t))\psi(t)$$

is not controllable, since we can easily verify that the set

$$\{(\psi_1, \psi_2, \psi_3, \bar{\psi}_2, \bar{\psi}_1) \mid \psi_1, \psi_2 \in \mathbb{C}, \psi_3 \in \mathbb{R} \text{ and } 2|\psi_1|^2 + 2|\psi_2|^2 + \psi_3^2 = 1\}$$

is invariant for (1.3).

Under the hypothesis that the piled-up intersections of the weakly connected spectrum have *rationally unrelated germs* (see [Definition 3.5](#)), we are able to prove a controllability result of the following type.

THEOREM 1.1. *Let $n = \dim(\mathcal{H}) < \infty$, $H_0, \dots, H_m (m \geq 2)$ be self-adjoint operators, and assume that the spectrum of $u \mapsto H(u)$ is weakly conically connected and that its eigenvalue intersections have rationally unrelated germs at each intersection point. Then $\text{Lie}(iH_0, \dots, iH_m) = \text{su}(n)$ if $iH_0, \dots, iH_m \in \text{su}(n)$ and $\text{Lie}(iH_0, \dots, iH_m) = \text{u}(n)$ otherwise, meaning that System (1.1) is exactly controllable in the unitary group.*

In the case where the Hilbert space \mathcal{H} is infinite-dimensional, some technical hypotheses are necessary (see [5, 12]) and only an approximate controllability result can be obtained (see [Theorem 3.8](#)). Our results extend [10, Theorem 14] in two ways. First, it applies to systems in which the spectrum is only weakly connected and, moreover, it does not require H_1, \dots, H_m to be bounded, but just relatively bounded with respect to H_0 .

The paper is organized as follows: In [Section 2](#) we recall the notion of conical and semi-conical intersections and we introduce the concept of weakly conical intersections. Characterizations of such intersections is given using spectral projectors. [Section 3](#) contains the proof of the main controllability results of the paper, namely [Theorem 1.1](#) and its counterpart for systems evolving in infinite-dimensional system ([Theorem 3.8](#)). In [Section 4](#) we provide testable conditions guaranteeing that the eigenvalues intersections have rationally unrelated germs, using first-order expansions of the spectral gaps near conical and semi-conical intersections. We conclude by some examples in [Section 5](#).

2. Eigenvalue intersections. We consider a control-affine Hamiltonian given by

$$(2.1) \quad H(u) = H_0 + \sum_{\ell=1}^m u_\ell H_\ell,$$

where $u = (u_1, \dots, u_m)$ belongs to an open and connected subset U of \mathbb{R}^m . We assume that $H(\cdot)$ satisfies

(A1) H_0 is a self-adjoint operator on a separable Hilbert space \mathcal{H} with domain $D(H_0)$.

For each $\ell \in \{1, \dots, m\}$, H_ℓ is a symmetric operator on $D(H_0)$.

We recall the following concept of relative boundedness (see, e.g., [17, sec. 4.1]).

DEFINITION 2.1. *Let A be a self-adjoint operator on a Hilbert space \mathcal{H} with domain $D(A)$ and B be a symmetric operator on the domain $D(A)$. We say that B is A -bounded if there exist $\alpha, \beta > 0$ such that $\|Bv\| \leq \alpha\|Av\| + \beta\|v\|$ for every $v \in D(A)$. The greatest lower bound α_0 of all possible constant α is called the relative bound of B with respect to A . We say that B is Kato-small with respect to A if it is A -bounded with relative bound 0.*

Let us introduce the following two possible assumptions for System (1.1):

(A2) For each $\ell \in \{1, \dots, m\}$, H_ℓ is H_0 -bounded.

(A2*) For each $\ell \in \{1, \dots, m\}$, H_ℓ is Kato-small with respect to H_0 .

Notice that **(A2*)** is a stronger assumption than **(A2)**.

REMARK 2.2. *By Rellich–Kato theorem (see [17, Theorem V-4.3]), if **(A1)** and **(A2)** are satisfied, then there exists a neighborhood $W \subset \mathbb{R}^m$ of 0 such that for all $u \in W$, $H(u)$ is a self-adjoint operator with domain $D(H(u)) = D(H_0)$. If **(A1)** and **(A2*)** are satisfied, one can take $W = \mathbb{R}^m$ and, for every $u \in \mathbb{R}^m$ and $\ell \in \{1, \dots, m\}$, H_ℓ is Kato-small with respect to $H(u)$ (see [17, Problem IV-1.2]).*

The spectrum $\sigma(H(u))$ of $H(u)$ can be seen as a function of $u \in U$. Let us denote by $\{\lambda_j(u), \lambda_{j+1}(u)\}$ a pair of subsequent eigenvalues of $\sigma(H(u))$, meaning that they are the j th and $(j+1)$ th eigenvalues in a discrete sub-spectrum of $\sigma(H(u))$. To study the intersection between two eigenvalues, we will consider the following assumption:

(A3) There are two eigenvalues $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ of $H(\cdot)$, with $\lambda_j(\cdot) \leq \lambda_{j+1}(\cdot)$, that intersect at $\bar{u} = 0$ and $\lambda_j(\bar{u}) = \lambda_{j+1}(\bar{u}) = 0$. The subset $\{\lambda_j(u), \lambda_{j+1}(u)\}$ is separated from the rest of the spectrum by a positive gap on an open neighborhood w of $\bar{u} = 0$.

Notice that assumption **(A3)** can be generalized to any intersection $\bar{u} \in U$ between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ by replacing u with $u - \bar{u}$ and H_0 with $H(\bar{u}) - \lambda_j(\bar{u})$. This transformation does not affect Assumption **(A2*)**, in the sense that H_ℓ is Kato-small with respect to H_0 if and only if it is Kato-small with respect to $H(\bar{u}) - \lambda_j(\bar{u})$. Concerning assumptions **(A1)** and **(A2)**, if one deals with a general intersection $\bar{u} \in U$ between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$, the corresponding assumptions should be stated in terms of $H(\bar{u}) - \lambda_j(\bar{u})$ instead of H_0 .

REMARK 2.3. *Assumption **(A3)** implies that there exists a circle \mathcal{C} on \mathbb{C} that separates $\{\lambda_q(u) \mid q \in \{j, j+1\}, u \in \omega\}$ from $\bigcup_{u \in \omega} (\sigma(H(u)) \setminus \{\lambda_j(u), \lambda_{j+1}(u)\})$. Notice that \mathcal{C} forms the boundary of a closed neighborhood $D \subset \mathbb{C}$ of θ .*

In the following, let us define conical and weakly conical intersections.

DEFINITION 2.4 (Conical intersection). *An eigenvalue intersection $\bar{u} \in U$ between two eigenvalues $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ is said to be conical if there exists $C > 0$ such*

that

$$\frac{1}{C}|t| \leq |\lambda_{j+1}(\bar{u} + t\eta) - \lambda_j(\bar{u} + t\eta)| \leq C|t|$$

for every unit direction $\eta \in \mathbb{R}^m$ and t in a neighborhood of 0.

DEFINITION 2.5 (Conical direction). *Given an intersection \bar{u} between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$, a unit vector $\eta \in \mathbb{R}^m$ is said to be a conical direction at \bar{u} if there exists $C > 0$ such that*

$$\frac{1}{C}|t| \leq \lambda_{j+1}(t\eta) - \lambda_j(t\eta) \leq C|t|$$

for t in a neighborhood of 0.

DEFINITION 2.6 (Weakly conical intersection). *A non-conical eigenvalue intersection $\bar{u} \in U$ between two eigenvalues $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ is said to be weakly conical if \bar{u} is an isolated point in*

$$U_j = \{u \in U \mid \lambda_j(u) = \lambda_{j+1}(u)\}$$

and there exists at least one conical direction at \bar{u} .

In the case $m = 2$, let us recall the notion of semi-conical intersection introduced in [3]. Semi-conical intersections are automatically weakly conical.

DEFINITION 2.7 (Semi-conical intersection). *When $m = 2$, an intersection \bar{u} between two eigenvalues $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ of $H(\cdot)$ is said to be semi-conical if there exists a unit vector $\eta \in \mathbb{R}^2$, called the non-conical direction at \bar{u} , such that every unit vector $\mu \in \mathbb{R}^2$ transversal to η is a conical direction and, for every smooth curve $[0, 1] \ni t \mapsto \gamma(t) \in \mathbb{R}^2$ satisfying $\gamma(0) = \bar{u}$ and $\gamma'(0) = \eta$, there exists $C > 0$ such that*

$$\frac{1}{C}t^2 \leq |\lambda_{j+1}(\gamma(t)) - \lambda_j(\gamma(t))| \leq Ct^2$$

for t in a neighborhood of 0.

These definitions depend solely on the gap between eigenvalues in a neighborhood of the intersection \bar{u} . As we will see, the first-order expansion of $\lambda_{j+1}(u) - \lambda_j(u)$ in a neighborhood of \bar{u} is sufficient to determine whether the intersection is conical or not. In the case of $m = 2$, the second-order expansion of $\lambda_{j+1}(u) - \lambda_j(u)$ in a neighborhood of \bar{u} will determine whether the intersection is conical, semi-conical, or otherwise.

REMARK 2.8. *Notice that replacing H_0 with $H(\bar{u}) - \lambda_j(\bar{u})$ and u with $u - \bar{u}$ does not change the property of an intersection being conical or weakly conical.*

Under assumptions **(A1-A2-A3)** or **(A1-A2*-A3)**, we can define, for each u in a neighborhood of $\bar{u} = 0$, the spectral projector

$$(2.2) \quad P(u) = \frac{1}{2i\pi} \oint_C (H(u) - \xi)^{-1} d\xi$$

with respect to $\{\lambda_j(u), \lambda_{j+1}(u)\}$. By Kato (see [17, Remark II-4.2]), in a neighborhood of $\bar{u} = 0$ the operator-valued function $u \mapsto U(u)$ given by

$$(2.3) \quad U(u) := \left(1 - (P(u) - P(0))^2\right)^{-\frac{1}{2}} \left(P(u)P(0) + (1 - P(u))(1 - P(0))\right)$$

is well-defined and has the following properties: $U(\cdot)$ is holomorphic, $U^*(u)U(u) = U(u)U^*(u) = 1$, and $U(u)P(0) = P(u)U(u)$. Moreover, for u in a neighborhood of 0,

$$(2.4) \quad h(u) := U^*(u)P(u)H(u)P(u)U(u) = P(0)U^*(u)H(u)U(u)P(0)$$

is a self-adjoint operator on the two-dimensional subspace $P(0)\mathcal{H}$ and $\sigma(h(u)) = \{\lambda_j(u), \lambda_{j+1}(u)\}$.

REMARK 2.9. *Standard analyticity arguments (see, e.g., [17]) show that when (A1-A2-A3) are satisfied, there exists an open neighborhood $N \subset w$ of \bar{u} such that the mapping $N \ni u \mapsto P(u)$ is analytic. When (A2*) is satisfied we have $N = w$. Moreover, for every $u \in N$, there exists a neighborhood O of u such that*

$$(2.5) \quad P(v) = P(u) + \sum_{n \geq 1} P_u^{(n)}(v - u), \quad \forall v \in O,$$

with $P_u^{(n)}(v - u)$ the bounded self-adjoint operator given by

$$(2.6) \quad P_u^{(n)}(v - u) = (-1)^n \frac{1}{2i\pi} \oint_{\mathcal{C}} d\xi (H(u) - \xi)^{-1} \left(\sum_{\ell=1}^m (v_\ell - u_\ell) H_\ell (H(u) - \xi)^{-1} \right)^n.$$

In particular,

$$(2.7) \quad P(u) = P(0) + P_0^{(1)}(u) + \mathcal{O}(\|u\|^2),$$

where $P_0^{(1)}(u)$ is the bounded self-adjoint operator

$$(2.8) \quad P_0^{(1)}(u) = -\frac{1}{2i\pi} \oint_{\mathcal{C}} d\xi (H_0 - \xi)^{-1} \left(\sum_{\ell=1}^m u_\ell H_\ell \right) (H_0 - \xi)^{-1}.$$

The mapping $N \ni u \mapsto h(u)$ is also analytic. Since $H_0 P(0) = 0$ and $P(0) H_0 = 0$, the second-order expansion of $h(u)$ in a neighborhood of $\bar{u} = 0$ is

$$(2.9) \quad \begin{aligned} h(u) &= P(0)U^*(u)H(u)U(u)P(0) \\ &= P(0) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) P(0) + P(0) P_0^{(1)}(u) H_0 P_0^{(1)}(u) P(0) \\ &\quad + P(0) P_0^{(1)}(u) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) P(0) + P(0) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) P_0^{(1)}(u) P(0) + \mathcal{O}(\|u\|^3). \end{aligned}$$

We assume in the following that **A1-A2-A3** are satisfied. By adapting the results of [17] to the control-affine Hamiltonian $H(u)$ (see [17, II-(2.14)] for finite-dimensional systems and [17, Section VII-1.5] for infinite-dimensional systems), $P_0^{(1)}(u)$ is given by

$$(2.10) \quad P_0^{(1)}(u) = -P(0) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) A - A \left(\sum_{i=1}^m u_i H_i \right) P(0),$$

where A is the reduced resolvent of H_0 for the eigenvalue 0 (see [17, II-(2.11)] and [17, III-(6.31)]), such that

$$H_0 A = A H_0 = 1 - P(0), \quad P(0) A = A P(0) = 0.$$

PROPOSITION 2.10. *Let assumptions (A1-A2-A3) be satisfied and h be defined as in (2.4). Then, for u in a neighborhood of 0,*

$$(2.11) \quad \begin{aligned} h(u) = & P(0) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) P(0) \\ & - P(0) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) A \left(\sum_{\ell=1}^m u_\ell H_\ell \right) P(0) + \mathcal{O}(\|u\|^3). \end{aligned}$$

Proof. This is a direct result of equations (2.9) and (2.10). \square

LEMMA 2.11. *Let B be a self-adjoint operator on $P(0)\mathcal{H}$ and denote its eigenvalues by $\lambda_0 \leq \lambda_1$. Then*

$$(2.12) \quad \lambda_1 - \lambda_0 = 2\sqrt{-\det\left(B - \frac{1}{2}\text{tr}(B)\right)}.$$

Proof. Since B is a self-adjoint operator on a Hilbert space of dimension 2, we have

$$\det\left(B - \frac{1}{2}\text{tr}(B)\right) = \det(B) - \frac{1}{4}\text{tr}(B)^2,$$

and the eigenvalues of B are the roots of the quadratic equation $\lambda^2 - \text{tr}(B)\lambda + \det(B) = 0$. Therefore,

$$\lambda_1 - \lambda_0 = \sqrt{\text{tr}(B)^2 - 4\det(B)} = 2\sqrt{-\det\left(B - \frac{1}{2}\text{tr}(B)\right)}.$$

\square

Let us define the linear map that removes the trace of a linear operator on $P(0)\mathcal{H}$ by

$$(2.13) \quad \begin{aligned} \mathcal{R} : \mathcal{L}(P(0)\mathcal{H}) &\rightarrow \mathcal{L}(P(0)\mathcal{H}) \\ B &\mapsto B - \frac{1}{2}\text{tr}(B). \end{aligned}$$

Recall that $h(u) \in \mathcal{L}(P(0)\mathcal{H})$ is well defined for u in a neighborhood of 0 and $\sigma(h(u)) = \{\lambda_j(u), \lambda_{j+1}(u)\}$. By Lemma 2.11, we have

$$(2.14) \quad \lambda_{j+1}(u) - \lambda_j(u) = 2\sqrt{-\det\left(h(u) - \frac{1}{2}\text{tr}(h(u))\right)} = 2\sqrt{-\det\left(\mathcal{R}(h(u))\right)}.$$

Let us define for each $\ell \in \{1, \dots, m\}$

$$(2.15) \quad h_\ell^{(1)} = P(0)H_\ell P(0),$$

and, for $u \in \mathbb{R}^m$,

$$(2.16) \quad h^{(2)}(u) = -P(0) \left(\sum_{\ell=1}^m u_\ell H_\ell \right) A \left(\sum_{\ell=1}^m u_\ell H_\ell \right) P(0).$$

Therefore, we obtain from (2.11) that

$$(2.17) \quad h(u) = \sum_{\ell=1}^m u_\ell h_\ell^{(1)} + h^{(2)}(u) + \mathcal{O}(\|u\|^3).$$

We now use this second-order expansion to characterize conical and semi-conical intersections.

LEMMA 2.12. *For a unit direction $\eta \in \mathbb{R}^m$, there exists $C > 0$ such that*

$$(2.18) \quad \frac{1}{C}|t| \leq \lambda_{j+1}(t\eta) - \lambda_j(t\eta) \leq C|t|$$

for t in a neighborhood of 0 if and only if

$$(2.19) \quad \sum_{\ell=1}^m \eta_\ell \mathcal{R}(h_\ell^{(1)}) \neq 0.$$

Proof. By equations (2.14) and (2.17), we have

$$\lambda_{j+1}(t\eta) - \lambda_j(t\eta) = 2\sqrt{-\det(\mathcal{R}(h(t\eta)))} = 2t\sqrt{-\det\left(\sum_{\ell=1}^m \eta_\ell \mathcal{R}(h_\ell^{(1)})\right)} + o(t).$$

Therefore, (2.18) is satisfied for some $C > 0$ if and only if

$$\det\left(\sum_{\ell=1}^m \eta_\ell \mathcal{R}(h_\ell^{(1)})\right) \neq 0,$$

which is equivalent to (2.19) since $\sum_{\ell=1}^m \eta_\ell \mathcal{R}(h_\ell^{(1)})$ is a zero-trace self-adjoint operator on a two-dimensional space. \square

REMARK 2.13. *Notice that the existence of a conical direction implies that there exists $\ell \in \{1, \dots, m\}$ such that $\mathcal{R}(h_\ell^{(1)}) \neq 0$ and that the subset*

$$\left\{ \mu \in \mathbb{R}^m \mid \sum_{\ell=1}^m \mu_\ell \mathcal{R}(h_\ell^{(1)}) \neq 0 \right\}$$

is dense and has zero-measure complement in \mathbb{R}^m .

PROPOSITION 2.14. *Assume that (A1-A2-A3) are satisfied. The intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ at $\bar{u} = 0$ is conical if and only if the family of zero-trace operators $\{\mathcal{R}(h_\ell^{(1)})\}_{\ell \in \{1, \dots, m\}}$ is \mathbb{R} -linearly independent, where \mathcal{R} is the trace-removing function introduced in (2.13).*

Proof. According to Definition 2.4 and Lemma 2.12, we deduce that the intersection at $\bar{u} = 0$ is conical if and only if, for every unit direction $\eta \in \mathbb{R}^m$, we have

$$\sum_{\ell=1}^m \eta_\ell \mathcal{R}(h_\ell^{(1)}) \neq 0,$$

which is equivalent to the family $\{\mathcal{R}(h_\ell^{(1)})\}_{\ell \in \{1, \dots, m\}}$ being \mathbb{R} -linearly independent. \square

REMARK 2.15. *Since zero-trace symmetric operators on $P(0)\mathcal{H}$ form a real vector space of dimension 3, a necessary condition for an intersection to be conical is that $m \leq 3$. Moreover, in the case of real Hamiltonians, i.e., when there exists an orthonormal basis $\{\chi_j\}_j$ of the Hilbert space \mathcal{H} such that $\langle \chi_j, H_q \chi_k \rangle$ is real for all j, k and each $q \in \{0, \dots, m\}$, then a necessary condition is that $m \leq 2$.*

REMARK 2.16. *When $m = 2$, the intersection is conical if and only if $\mathcal{R}(h_1^{(1)})$ and $\mathcal{R}(h_2^{(1)})$ are \mathbb{R} -linearly independent. If we choose an orthonormal basis $\{\phi_j, \phi_{j+1}\}$ of $P(0)\mathcal{H}$, the intersection is conical if and only if the two matrices*

$$\begin{pmatrix} \frac{1}{2}\langle \phi_j, H_s \phi_j \rangle - \frac{1}{2}\langle \phi_{j+1}, H_s \phi_{j+1} \rangle & \langle \phi_j, H_s \phi_{j+1} \rangle \\ \langle \phi_{j+1}, H_s \phi_j \rangle & \frac{1}{2}\langle \phi_{j+1}, H_s \phi_{j+1} \rangle - \frac{1}{2}\langle \phi_j, H_s \phi_j \rangle \end{pmatrix}, \quad s \in \{1, 2\},$$

are \mathbb{R} -linearly independent. The latter is equivalent to the inequality

$$(2.20) \quad \det \begin{pmatrix} \frac{1}{2}\langle \phi_j, H_1 \phi_j \rangle - \frac{1}{2}\langle \phi_{j+1}, H_1 \phi_{j+1} \rangle & \langle \phi_j, H_1 \phi_{j+1} \rangle \\ \frac{1}{2}\langle \phi_j, H_2 \phi_j \rangle - \frac{1}{2}\langle \phi_{j+1}, H_2 \phi_{j+1} \rangle & \langle \phi_j, H_2 \phi_{j+1} \rangle \end{pmatrix} \neq 0.$$

The 2×2 matrix appearing in (2.20) has been introduced under the name of conicity matrix in [11]. Therefore, Proposition 2.14 generalizes the characterization of conical intersection by the conicity matrix given in [11, Proposition 4.4].

LEMMA 2.17. *When $m = 2$, for a smooth curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ with $\gamma(0) = 0$, $\dot{\gamma}(0) = \eta$ and $\ddot{\gamma}(0) = v$, there exists $C > 0$ such that*

$$(2.21) \quad \frac{t^2}{C} \leq \lambda_{j+1}(\gamma(t)) - \lambda_j(\gamma(t)) \leq Ct^2$$

if and only if

$$(2.22) \quad \eta_1 \mathcal{R}(h_1^{(1)}) + \eta_2 \mathcal{R}(h_2^{(1)}) = 0, \quad \frac{1}{2}v_1 \mathcal{R}(h_2^{(1)}) + \frac{1}{2}v_2 \mathcal{R}(h_2^{(1)}) + \mathcal{R}(h^{(2)}(\eta)) \neq 0.$$

Proof. For t in a neighborhood of 0, $\gamma(t) = t\eta + \frac{t^2}{2}v + \mathcal{O}(t^3)$. By (2.14) and (2.17), we have

$$\lambda_{j+1}(\gamma(t)) - \lambda_j(\gamma(t)) = \sqrt{\det \left(\sum_{\ell=1}^2 (t\eta_\ell + \frac{t^2}{2}v_\ell) \mathcal{R}(h_\ell^{(1)}) + t^2 \mathcal{R}(h^{(2)}(\eta)) \right)} + o(t^2).$$

By Lemma 2.12, we conclude that (2.21) is satisfied for some $C > 0$ if and only if (2.22) is satisfied and

$$\det \left(\sum_{\ell=1}^2 \frac{1}{2}v_\ell \mathcal{R}(h_\ell^{(1)}) + \mathcal{R}(h^{(2)}(\eta)) \right) \neq 0,$$

which is equivalent to the second equation of (2.22) since $\sum_{\ell=1}^2 \frac{1}{2}v_\ell \mathcal{R}(h_\ell^{(1)}) + \mathcal{R}(h^{(2)}(\eta))$ is a zero-trace self-adjoint operator on a two-dimensional space. \square

PROPOSITION 2.18. *Let $m = 2$. The intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ at $\bar{u} = 0$ is semi-conical if and only if $\mathcal{R}(h_1^{(1)})$ and $\mathcal{R}(h_2^{(1)})$ are colinear, $\mathcal{R}(h_s^{(1)}) \neq 0$ for some $s \in \{1, 2\}$, and, setting*

$$(2.23) \quad \eta = \left(-\frac{\|\mathcal{R}(h_2^{(1)})\|}{\sqrt{\|\mathcal{R}(h_1^{(1)})\|^2 + \|\mathcal{R}(h_2^{(1)})\|^2}}, \frac{\|\mathcal{R}(h_1^{(1)})\|}{\sqrt{\|\mathcal{R}(h_1^{(1)})\|^2 + \|\mathcal{R}(h_2^{(1)})\|^2}} \right),$$

$\mathcal{R}(h^{(2)}(\eta))$ is not colinear to $\mathcal{R}(h_s^{(1)})$.

Proof. By Definition 2.7 and Lemmas 2.12 and 2.17, $\bar{u} = 0$ is semi-conical if and only if there exists a unit direction $\eta \in \mathbb{R}^2$ such that

$$\begin{aligned} \eta_1 \mathcal{R}(h_1^{(1)}) + \eta_2 \mathcal{R}(h_2^{(1)}) &= 0, \\ \frac{1}{2}v_1 \mathcal{R}(h_2^{(1)}) + \frac{1}{2}v_2 \mathcal{R}(h_2^{(1)}) + \mathcal{R}(h^{(2)}(\eta)) &\neq 0, \quad \forall v \in \mathbb{R}^2, \end{aligned}$$

and, for all $\mu \in \mathbb{R}^2$ transversal to η ,

$$\mu_1 \mathcal{R}(h_1^{(1)}) + \mu_2 \mathcal{R}(h_2^{(1)}) \neq 0.$$

Equivalently, $\mathcal{R}(h_1^{(1)})$ and $\mathcal{R}(h_2^{(1)})$ are colinear and not both zero, and, setting η as in (2.23), $\mathcal{R}(h^{(2)}(\eta))$ is not colinear to $\mathcal{R}(h_1^{(1)})$ nor to $\mathcal{R}(h_2^{(1)})$. \square

REMARK 2.19. Assume that $\bar{u} \in U$ (not necessarily 0) is an eigenvalue intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ and that $\lambda_j(\bar{u}) = \lambda_{j+1}(\bar{u}) = \lambda \in \mathbb{R}$. The assumptions (A1-A2-A2*-A3) can be easily generalized to this intersection \bar{u} by replacing u with $u - \bar{u}$ and H_0 with $H(\bar{u}) - \lambda_j(\bar{u})$. Therefore, the characterization of conical and semi-conical intersections (Proposition 2.14 and Proposition 2.18) can still be applied. In particular, for u in a neighborhood of \bar{u} , we have

$$\lambda_{j+1}(u) - \lambda_j(u) = 2 \sqrt{-\det \left(\sum_{\ell=1}^m (u_\ell - \bar{u}_\ell) \mathcal{R}(P_{j,j+1}(\bar{u}) H_\ell P_{j,j+1}(\bar{u})) \right)} + o(\|u\|),$$

where $P_{j,j+1}(\bar{u})$ is the spectral projector with respect to $\{\lambda_j(\bar{u}), \lambda_{j+1}(\bar{u})\}$.

3. Systems connected by conical and weakly conical intersections. We say that System (1.1) satisfies **(H)** if

- H_0, \dots, H_m ($m \geq 2$) are self-adjoint operators on \mathcal{H} with $\dim(\mathcal{H}) < \infty$, and that System (1.1) satisfies **(H $^\infty$)** if the following assumptions hold true:
- \mathcal{H} is a separable Hilbert space with $\dim(\mathcal{H}) = \infty$,
- H_0 is a self-adjoint operator on \mathcal{H} with domain $D(H_0)$ and H_1, \dots, H_m ($m \geq 2$) are symmetric operators on $D(H_0)$,
- H_0 is an operator with compact resolvent, i.e., $(H_0 + i)^{-1}$ is a compact operator, and H_1, \dots, H_m are Kato-small with respect to H_0 .

REMARK 3.1. Notice that H_0 is an operator with compact resolvent if and only if the spectrum of H_0 is discrete, i.e., each point of $\sigma(H_0)$ is an isolated eigenvalue of finite multiplicity. H_0 -boundedness implies that for all $u \in U$, $H(u)$ is a self-adjoint operator on \mathcal{H} with domain $D(H_0)$, H_1, \dots, H_m are Kato-small with respect to $H(u)$ and that the spectrum of $H(u)$ is discrete. Moreover, both assumptions **(H)** and **(H $^\infty$)** imply **(A1-A2*)**, and therefore the conicity and weak conicity of the intersection points of $\sigma(H(u))$ can be characterized using the results of the previous section.

Under assumption **(H)** or **(H $^\infty$)**, the spectrum of $H(\cdot)$ is discrete. Let \mathcal{I} denote $\{1, \dots, \dim(\mathcal{H})\}$ (respectively, \mathbb{N}^*) when **(H)** (respectively, **(H $^\infty$)**) is satisfied. For every $u \in U$, we denote by $\{\lambda_j(u)\}_{j \in \mathcal{I}}$ the increasing sequence of eigenvalues of $H(u)$, counted according to their multiplicities, and by $\{\phi_j(u)\}_{j \in \mathcal{I}}$ a corresponding sequence of eigenvectors.

REMARK 3.2. Assume that System (1.1) satisfies (\mathbf{H}^∞) . Let $u : [0, T] \rightarrow U$ be a piecewise constant control function. Then the propagator of System (1.1) associated with u is given by

$$(3.1) \quad \Upsilon_t^u = e^{i(t-t_l)H(u_l)} \circ e^{i(t_l-t_{l-1})H(u_{l-1})} \circ \dots \circ e^{it_1H(u_1)}, \quad \text{for } t_l < t \leq t_{l+1}.$$

For any initial state $\psi_0 \in D(H_0)$, the solution $\psi(t) = \Upsilon_t^u(\psi_0)$ satisfies the Schrödinger equation

$$i\dot{\psi}(t) = H(u(t))\psi(t) \quad \text{for a.e. } t.$$

Similarly, for initial states $\psi_0 \in \mathcal{H} \setminus D(H_0)$, the solution $\psi(t) = \Upsilon_t^u(\psi_0)$ satisfies, for every $v \in D(H_0)$, the equation

$$\frac{d}{dt}\langle v, \psi(t) \rangle = -i\langle H(u(t))v, \psi(t) \rangle \quad \text{for a.e. } t.$$

Let us define the approximate controllability of System (1.1).

DEFINITION 3.3 (Approximate controllability). Let (\mathbf{H}^∞) be satisfied. We say that System (1.1) is approximately controllable if for every ψ_0, ψ_1 in the unit sphere of \mathcal{H} and every $\epsilon > 0$, there exists a piecewise constant control function $u : [0, T] \rightarrow U$ such that $\|\Upsilon_T^u(\psi_0) - \psi_1\| < \epsilon$.

It has been proven in [10] that for a control-affine Hamiltonian $H(u) = H_0 + \sum_{\ell=1}^m u_\ell H_\ell$ with u in an open and connected subset U of \mathbb{R}^m , if H_ℓ is bounded for each $\ell \in \{1, \dots, m\}$ and its spectrum is connected by conical intersections that do not pile up, then exact controllability can be established in finite dimension and approximate controllability in infinite dimension. We raise the question of whether it is possible to extend controllability results to the case where the control Hamiltonian H_ℓ may be unbounded and the spectrum is connected by intersections that are not necessarily conical and can pile up. We start by defining weak conical connectedness and rationally unrelated germs.

DEFINITION 3.4 (Weakly conical connected spectrum). Let (\mathbf{H}) or (\mathbf{H}^∞) be satisfied. We say that the spectrum $\sigma(H(\cdot))$ is weakly conically connected if each of its eigenvalue intersections is either conical or weakly conical, and, moreover, for every $(j, j+1) \in \mathcal{I}^2$, there exists $\bar{u}_j \in U$ such that $\lambda_j(\bar{u}_j) = \lambda_{j+1}(\bar{u}_j)$.

DEFINITION 3.5 (Rationally unrelated germs). Let (\mathbf{H}) or (\mathbf{H}^∞) be satisfied and the spectrum $\sigma(H(\cdot))$ be weakly conically connected. For a given intersection point $\bar{u} \in U$ of $\sigma(H(\cdot))$, define

$$(3.2) \quad \mathcal{I}(\bar{u}) := \{j \in \mathcal{I} \mid \lambda_j(\bar{u}) = \lambda_{j+1}(\bar{u})\}.$$

We say that intersections have rationally unrelated germs at \bar{u} if for every finite subset \mathcal{J} of $\mathcal{I}(\bar{u})$, and for every neighborhood V of \bar{u} , the family of functions

$$\left(V \ni u \mapsto \lambda_{j+1}(u) - \lambda_j(u) \right)_{j \in \mathcal{J}}$$

is rationally independent, meaning that if $\{\alpha_j\}_{j \in \mathcal{J}} \in \mathbb{Q}^{\mathcal{J}}$ is such that

$$\sum_{j \in \mathcal{J}} \alpha_j (\lambda_{j+1}(u) - \lambda_j(u)) = 0, \quad \forall u \in V,$$

then $\alpha_j = 0$ for all $j \in \mathcal{J}$.

LEMMA 3.6. *If System (1.1) satisfies (\mathbf{H}) or (\mathbf{H}^∞) and the spectrum $\sigma(H(\cdot))$ is weakly conically connected, then for every $\bar{u} \in U$ such that the spectrum $\sigma(H(\bar{u}))$ is simple, the graph having \mathcal{I} as nodes and having*

$$S(\bar{u}) := \{(j, k) \in \mathcal{I}^2 \mid \langle \phi_j(\bar{u}), H_l \phi_k(\bar{u}) \rangle \neq 0 \text{ for some } l = 1, \dots, m\}$$

as the set of edges is connected.

Proof. Assume by contradiction that there exists $\bar{u} \in U$ such that $\sigma(H(\bar{u}))$ is simple and the graph having \mathcal{I} as nodes and $S(\bar{u})$ as the set of edges is not connected. Then there exists a proper subset \mathcal{J} of \mathcal{I} such that $V = \overline{\text{span}\{\phi_j(\bar{u})\}_{j \in \mathcal{J}}}$ is a proper subspace of \mathcal{H} invariant for the evolution of the system. Let us set $V^\perp = \overline{\text{span}\{\phi_j(\bar{u})\}_{j \in \mathcal{I} \setminus \mathcal{J}}}$, which is also an invariant subspace, and we have $\mathcal{H} = V \oplus V^\perp$.

For each $j \in \mathcal{I}$, let us define

$$U_j := \{u \in U \mid \lambda_j(u) \text{ is simple}\},$$

which is an open and connected subset of U with a discrete complement. For each $u \in U_j$, the corresponding eigenvector $\phi_j(u)$ can be written as $\phi_j(u) = v + w$, with $v \in V$ and $w \in V^\perp$. We have

$$H(u)\phi_j(u) = \lambda_j(u)\phi_j(u) = \lambda_j(u)v + \lambda_j(u)w.$$

On the other hand, we have

$$H(u)\phi_j(u) = H(u)v + H(u)w,$$

with $H(u)v \in V$ and $H(u)w \in V^\perp$, since V and V^\perp are invariant spaces for $H(u)$. The uniqueness of decomposition implies that

$$H(u)v = \lambda_j(u)v, \quad H(u)w = \lambda_j(u)w.$$

Since $\lambda_j(u)$ is a simple eigenvalue, we conclude that $v = 0$ or $w = 0$ and that for all $u \in U_j$, $\phi_j(u)$ can only be in V or in V^\perp . Denoting by P_V the orthogonal projection on V , we have, for every $j \in \mathcal{I}$,

$$(3.3) \quad \|P_V \phi_j(u)\| \in \{0, 1\}, \quad \forall u \in U_j.$$

Since \mathcal{J} is a proper subset of \mathcal{I} , we can find $j \in \mathcal{J}$ such that $j+1 \in \mathcal{I} \setminus \mathcal{J}$ (or $j \in \mathcal{I} \setminus \mathcal{J}$ and $j+1 \in \mathcal{I}$). In other words, $\phi_j(\bar{u}) \in V$ and $\phi_{j+1}(\bar{u}) \in V^\perp$. Since the spectrum is simple at \bar{u} , we have that \bar{u} is in $U_j \cap U_{j+1}$. Let us consider an intersection $\bar{u}_{j,j+1} \in U$ between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$. Notice that there exists a neighborhood \mathcal{O} of $\bar{u}_{j,j+1}$ such that $\mathcal{O} \setminus \{\bar{u}_{j,j+1}\} \subset U_j \cap U_{j+1}$. Let us take a smooth curve $\gamma : [-1, 1] \rightarrow U$ such that $\gamma(-1) = \gamma(1) = \bar{u}$, $\gamma(t) \in U_j \cap U_{j+1}$ for every $t \in [-1, 0) \cup (0, 1]$, $\gamma(0) = \bar{u}_{j,j+1}$, and $\dot{\gamma}(0)$ is a non-zero vector tangent to a conical direction.

By Lemma 3.2 in [2] (which can be generalized to infinite-dimensional systems with holomorphic spectral projectors), we can choose $\phi_j(\gamma(\cdot))$ and $\phi_{j+1}(\gamma(\cdot))$ such that

$$(3.4) \quad \psi : [-1, 1] \ni t \mapsto \begin{cases} \phi_j(\gamma(t)), & \text{if } t \in [-1, 0] \\ \phi_{j+1}(\gamma(t)), & \text{if } t \in (0, 1] \end{cases}$$

is in $\mathcal{C}^\infty([-1, 1], \mathcal{H})$. Notice that for all $t \in [-1, 0) \cap (0, 1]$, $\gamma(t)$ is in $U_j \cap U_{j+1}$. By (3.3) and (3.4), we have $\|P_V \psi(t)\| \in \{0, 1\}$ for all $t \in [-1, 0) \cap (0, 1]$. Since

$\|P_V\psi(\cdot)\|$ is continuous on $[-1, 1]$, $\|P_V\psi(\cdot)\|$ is constant on $[-1, 1]$. However, we have $\|P_V\psi(-1)\| = \|P_V\phi_j(\bar{u})\| = 1$ and $\|P_V\psi(1)\| = \|P_V\phi_{j+1}(\bar{u})\| = 0$. The contradiction is reached. \square

The following result generalizes Lemmas 9 and 14 in [10] to the case of weakly connected spectra.

LEMMA 3.7. *Let System (1.1) satisfy (\mathbf{H}) or (\mathbf{H}^∞) . Assume that the spectrum $\sigma(H(\cdot))$ is weakly conically connected and that the eigenvalue intersections have rationally unrelated germs at every intersection point $\bar{u} \in U$. Then there exists $\bar{U} \subset U$ which is dense and with zero-measure complement in U such that if $\sum_{j=1}^N \alpha_j \lambda_j(\bar{u}) = 0$ with $\bar{u} \in \bar{U}$, $N \in \mathbb{N}^*$, $N \leq \dim \mathcal{H}$, and $\{\alpha_1, \dots, \alpha_N\} \in \mathbb{Q}^N$ then $\alpha_1 = \dots = \alpha_N$.*

Proof. Let Ξ be the set of all $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Q}^N$ with $N \in \mathbb{N}^*$, $N \leq \dim \mathcal{H}$, and $\alpha_j \neq \alpha_k$ for some $j, k \in \{1, \dots, N\}$. For every $N \in \mathbb{N}^*$ and $\alpha \in \Xi \cap \mathbb{Q}^N$, define the set

$$U_\alpha = \left\{ u \in U \mid \sum_{j=1}^N \alpha_j \lambda_j(u) = 0 \right\}.$$

Let \bar{U} be the complement in U of the union of all U_α such that $\alpha \in \Xi$. Notice that a countable union of subsets of \mathbb{R}^m with empty interior and zero measure has empty interior and zero measure. Therefore, to prove that \bar{U} is dense and with zero-measure complement in U , we only need to prove that U_α has empty interior and zero measure for every $\alpha \in \Xi$.

Assume by contradiction that there exists $\alpha \in \Xi$ such that U_α has positive measure. Let $N \in \mathbb{N}^*$ be such that $\alpha \in \mathbb{Q}^N$. Notice that the sub-spectrum $\sigma_N(H(\cdot)) = \{\lambda_1(\cdot), \dots, \lambda_N(\cdot)\}$ is analytic on the open and connected set

$$\{u \in U \mid \lambda_1(u), \dots, \lambda_N(u) \text{ are simple}\},$$

which is dense and with zero-measure complement in U . Hence $U_\alpha = U$.

Fix an eigenvalue intersection point \bar{u} . Then for all $j \in \mathcal{I}(\bar{u})$, the intersection between $\{\lambda_j(\cdot), \lambda_{j+1}(\cdot)\}$ is either conical or weakly conical. Consider an analytic path $\gamma : (-1, 1) \rightarrow U$ such that $\gamma(0) = \bar{u}$, $\gamma(t)$ is not an eigenvalue intersection for $t \in (-1, 0) \cup (0, 1)$, and $\dot{\gamma}(0)$ is a (non-zero) conical direction at \bar{u} for the intersection between λ_j and λ_{j+1} for every $j \in \mathcal{I}(\bar{u})$. Then, we have

$$\sum_{l=1}^N \alpha_l \lambda_l(\gamma(t)) = 0, \quad \forall t \in (-1, 1).$$

By analytic dependence of the spectrum along γ in a neighbourhood of $\gamma(0)$ (see [19]), for all $j \in \mathcal{I}(\bar{u})$, the functions

$$t \mapsto \begin{cases} \lambda_j(\gamma(t)) & \text{if } t < 0, \\ \lambda_{j+1}(\gamma(t)) & \text{if } t \geq 0, \end{cases} \quad t \mapsto \begin{cases} \lambda_{j+1}(\gamma(t)) & \text{if } t < 0, \\ \lambda_j(\gamma(t)) & \text{if } t \geq 0, \end{cases}$$

and the functions $t \mapsto \lambda_l(\gamma(t))$, $l \notin \{j, j+1 \mid j \in \mathcal{I}(\bar{u})\}$, are analytic on $(-1, 1)$. Hence, setting $\alpha_l = 0$ for every $l \in \mathcal{I} \setminus \{1, \dots, N\}$, we have, for all $t \in (-1, 1)$,

$$\sum_{j \in \mathcal{I}(\bar{u})} \alpha_j \lambda_{j+1}(\gamma(t)) + \sum_{j \in \mathcal{I}(\bar{u})} \alpha_{j+1} \lambda_j(\gamma(t)) + \sum_{l \neq j, j+1, \forall j \in \mathcal{I}(\bar{u})} \alpha_l \lambda_l(\gamma(t)) = 0.$$

As a consequence,

$$\sum_{j \in \mathcal{I}(\bar{u})} (\alpha_j - \alpha_{j+1}) \left(\lambda_j(\gamma(t)) - \lambda_{j+1}(\gamma(t)) \right) = 0, \quad \forall t \in (-1, 1).$$

This is true for every analytic path γ as above. Since the intersections have rationally unrelated germs at \bar{u} , we have $\alpha_j - \alpha_{j+1} = 0$ for all $j \in \mathcal{I}(\bar{u})$. Since this is true for every intersection of $\sigma(H(\cdot))$, we have $\alpha_1 = \dots = \alpha_N$. The contradiction is reached. We conclude that \bar{U} is dense and with zero-measure complement in U . \square

THEOREM 3.8. *If System (1.1) satisfies (\mathbf{H}^∞) , the spectrum is weakly conically connected and the eigenvalue intersections have rationally unrelated germs at each intersection point, then System (1.1) is approximately controllable.*

Proof of Theorems 1.1 and 3.8. Notice that the assumption of **Theorem 1.1** imply **(H)**. By applying **Lemma 3.7**, we deduce that there exists $\bar{u} \in U$, such that if $\sum_{j=1}^N \alpha_j \lambda_j(\bar{u}) = 0$ with $N \in \mathbb{N}^*$, $N \leq \dim \mathcal{H}$, and $\{\alpha_1, \dots, \alpha_N\} \in \mathbb{Q}^N$, then $\alpha_1 = \dots = \alpha_N$. In particular, the spectrum $\sigma(H(\bar{u}))$ is simple and two spectral gaps $\lambda_j(\bar{u}) - \lambda_k(\bar{u})$ and $\lambda_r(\bar{u}) - \lambda_s(\bar{u})$ are different if $(j, k) \neq (r, s)$ and $j \neq k$, $r \neq s$. By **Lemma 3.6**, we have that the graph having \mathcal{I} as nodes and having

$$\{(j, k) \in \mathcal{I}^2 \mid \langle \phi_j(\bar{u}), H_l \phi_k(\bar{u}) \rangle \neq 0 \text{ for some } l = 1, \dots, m\}$$

as the set of edges is connected.

When $\dim \mathcal{H} = n < \infty$, we deduce from [10, Proposition 11] that the Lie algebra $\text{Lie}(iH_0, \dots, iH_m)$ is equal to $su(n)$ if $iH_0, \dots, iH_m \in su(n)$ and to $u(n)$ otherwise, and we conclude that System (1.1) is exactly controllable in the unitary group. When $\dim \mathcal{H} = \infty$, the approximate controllability of System (1.1) forllows from [10, Proposition 15]. \square

REMARK 3.9. *Let us notice that [10, Proposition 15] is only stated for bounded H_1, \dots, H_m . However, since it is an adaption of Theorem 2.6 in [9], it can be easily extended to the case where H_1, \dots, H_m are Kato-small with respect to H_0 . Following [9], a stronger version of the previous theorem can be stated as an infinite-dimensional counterpart of the exact controllability in the unitary group. With the same assumptions, System (1.1) is approximately simultaneously controllable, i.e., for every $r > 0$, $\psi_1, \dots, \psi_r \in \mathcal{H}$, $\epsilon > 0$, and every unitary operator $\hat{\Upsilon}$ on \mathcal{H} there exists a piecewise constant control $u : [0, T] \rightarrow U$ such that, for every $1 \leq k \leq r$,*

$$\left\| \hat{\Upsilon} \psi_k - \Upsilon_T^u \psi_k \right\| < \epsilon.$$

REMARK 3.10. *Notice that if we can choose an open and connected subset U' of U such that the spectrum $\sigma(H(u))$ is weakly conically connected and the eigenvalue intersections have rationally unrelated germs at every intersection point on U' , we can still apply the spectral conditions above and obtain controllability results.*

4. First-order condition for rational independence in the case $m = 2$.

Assume that $m = 2$, that System (1.1) satisfies **(H)** or **(H $^\infty$)**, and that the spectrum $H(u(\cdot))$ is weakly conically connected. Let us consider $\bar{u} \in U$ such that

$$(4.1) \quad \mathcal{I}(\bar{u}) = \{j \in \mathcal{I} \mid \lambda_{j+1}(\bar{u}) = \lambda_j(\bar{u})\}$$

has a cardinality greater than one. Our goal is to provide testable conditions guaranteeing that the eigenvalues intersections at \bar{u} have rationally unrelated germs. Let

us take $j \in \mathcal{I}(\bar{u})$ and denote by $P_{j,j+1}(\cdot)$ the spectral projector with respect to $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ in a neighborhood of \bar{u} . By [Lemma 2.12](#), for all unit direction $\eta \in \mathbb{R}^2$, we have

$$\left. \frac{d}{dt} \right|_{t=0^+} \left(\lambda_{j+1}(\bar{u} + t\eta) - \lambda_j(\bar{u} + t\eta) \right) = 2\sqrt{-\det(\eta_1 h_{j,1} + \eta_2 h_{j,2})},$$

where $h_{j,1}, h_{j,2}$ are the self-adjoint operators on $P_{j,j+1}(\bar{u})\mathcal{H}$ defined by

$$(4.2) \quad h_{j,s} = P_{j,j+1}(\bar{u})H_s P_{j,j+1}(\bar{u}) - \frac{1}{2} \operatorname{tr} \left(P_{j,j+1}(\bar{u})H_s P_{j,j+1}(\bar{u}) \right), \quad s \in \{1, 2\}.$$

For $\theta \in [0, 2\pi)$, set $\eta(\theta) = (\cos(\theta), \sin(\theta))^\top$ and define the family of functions

$$(4.3) \quad \mathcal{F}(\bar{u}) = \left([0, 2\pi) \ni \theta \mapsto g_j(\theta) := \left. \frac{d}{dt} \right|_{t=0^+} \left(\lambda_{j+1}(\bar{u} + t\eta(\theta)) - \lambda_j(\bar{u} + t\eta(\theta)) \right) \right)_{j \in \mathcal{I}(\bar{u})}.$$

PROPOSITION 4.1. *Assume that System [\(1.1\)](#) satisfies **(H)** or **(H[∞])** and that the spectrum $\sigma(H(\cdot))$ is weakly conically connected. Let $\bar{u} \in U$ be such that $\mathcal{I}(\bar{u})$ is nonempty. If the family of functions $\mathcal{F}(\bar{u})$ is rationally independent then the eigenvalue intersections have rationally unrelated germs at \bar{u} .*

Proof. Let us take a finite subset $\mathcal{J} \subset \mathcal{I}(\bar{u})$, a neighborhood \mathcal{O} of \bar{u} , and $\{\alpha_j\}_{j \in \mathcal{J}} \in \mathbb{Q}^{\mathcal{J}}$ such that $\sum_{j \in \mathcal{J}} \alpha_j (\lambda_{j+1}(u) - \lambda_j(u)) = 0$ for all $u \in \mathcal{O}$. We deduce that, for all $\theta \in [0, 2\pi)$,

$$0 = \left. \frac{d}{dt} \right|_{t=0^+} \left(\sum_{j \in \mathcal{J}} \alpha_j (\lambda_{j+1}(\bar{u} + t\eta(\theta)) - \lambda_j(\bar{u} + t\eta(\theta))) \right) = \sum_{j \in \mathcal{J}} \alpha_j g_j(\theta).$$

Therefore, if there exist a finite subset $\mathcal{J} \subset \mathcal{I}(\bar{u})$ and a neighborhood \mathcal{O} of \bar{u} such that the family $\{u \mapsto \lambda_{j+1}(u) - \lambda_j(u)\}_{j \in \mathcal{J}}$ is rationally dependent, then the family of functions $\mathcal{F}(\bar{u})$ is rationally dependent. By [Definition 3.5](#), we deduce that if the family of functions $\mathcal{F}(\bar{u})$ is rationally independent then the intersections at \bar{u} have rationally unrelated germs. \square

PROPOSITION 4.2. *Assume that $j \in \mathcal{I}(\bar{u})$ and that \bar{u} is a conical intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$. Let $g_j : \theta \in [0, 2\pi) \rightarrow \mathbb{R}$ be the function defined in [\(4.3\)](#). Then there exists a unique (k_j, r_j, β_j) in*

$$(4.4) \quad D := \left(\mathbb{R}_+^* \times (0, 1) \times [0, 2\pi) \right) \cup \left(\mathbb{R}_+^* \times \{0\} \times \{0\} \right)$$

such that

$$(4.5) \quad g_j(\theta) = 2k_j \sqrt{1 + r_j \cos(2\theta - \beta_j)}, \quad \forall \theta \in [0, 2\pi).$$

Proof. Since $h_{j,1}$ and $h_{j,2}$ are self-adjoint zero-trace operators on $P_{j,j+1}(\bar{u})\mathcal{H}$, they can be represented by the 2×2 Hermitian matrices

$$M_{j,1} = \begin{pmatrix} a_j & 0 \\ 0 & -a_j \end{pmatrix}, \quad M_{j,2} = \begin{pmatrix} b_j & c_j \\ \bar{c}_j & -b_j \end{pmatrix},$$

with $a_j, b_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$. Since \bar{u} is a conical intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$, it follows from [Proposition 2.14](#) that $h_{j,1}$ and $h_{j,2}$ are not colinear, hence $a_j \neq 0$ and

$c_j \neq 0$. Then, for every $\theta \in [0, 2\pi)$, we have

$$\begin{aligned} g_j(\theta) &= \frac{d}{dt} \Big|_{t=0^+} \left(\lambda_{j+1}(\bar{u} + t\eta(\theta)) - \lambda_j(\bar{u} + t\eta(\theta)) \right) \\ &= 2\sqrt{-\det \left(\cos(\theta)M_{j,1} + \sin(\theta)M_{j,2} \right)} \\ &= 2\sqrt{\frac{a_j^2 + b_j^2 + |c_j|^2}{2} + \frac{a_j^2 - b_j^2 - |c_j|^2}{2} \cos(2\theta) + a_j b_j \sin(2\theta)}. \end{aligned}$$

If $b_j = 0$ and $|a_j| = |c_j|$, let us set

$$k_j = |a_j|, \quad r_j = 0, \quad \beta_j = 0.$$

Notice that (k_j, r_j, β_j) belongs to D and that, for every $\theta \in [0, 2\pi)$,

$$g_j(\theta) = 2\sqrt{\frac{a_j^2 + |c_j|^2}{2}} = 2|a_j| = 2k_j\sqrt{1 + r_j \cos(2\theta - \beta_j)}.$$

Otherwise, if $b_j \neq 0$ or $|a_j| \neq |c_j|$, let us set

$$k_j = \sqrt{\frac{a_j^2 + b_j^2 + |c_j|^2}{2}}, \quad \rho_j = \sqrt{\left(\frac{a_j^2 - b_j^2 - |c_j|^2}{2}\right)^2 + a_j^2 b_j^2}, \quad r_j = \frac{\rho_j^2}{k_j^2}.$$

There exists a unique $\beta_j \in [0, 2\pi)$ such that

$$\cos(\beta_j) = \frac{a_j^2 - b_j^2 - |c_j|^2}{2\rho_j}, \quad \sin(\beta_j) = \frac{a_j b_j}{\rho_j}.$$

Since $a_j, c_j \neq 0$, then

$$\rho_j^2 = \left(\frac{a_j^2 - b_j^2 - |c_j|^2}{2}\right)^2 + a_j^2 b_j^2 = \left(\frac{a_j^2 + b_j^2 + |c_j|^2}{2}\right)^2 - a_j^2 c_j^2 < k_j^2.$$

Since, moreover, $b_j \neq 0$ or $|a_j| \neq |c_j|$, we also have that

$$\rho_j^2 = \left(\frac{a_j^2 - b_j^2 - |c_j|^2}{2}\right)^2 + a_j^2 b_j^2 = \left(\frac{a_j^2 - |c_j|^2}{2}\right)^2 + \frac{1}{4}b_j^4 + c_j^2 b_j^2 > 0.$$

Thus, we have $r_j = \frac{\rho_j^2}{k_j^2} \in (0, 1)$ and $(k_j, r_j, \beta_j) \in D$. Moreover, for all $\theta \in [0, 2\pi)$,

$$\begin{aligned} g_j(\theta) &= 2\sqrt{k_j^2 + \rho_j^2 \cos(\beta) \cos(2\theta) + \rho_j^2 \sin(\beta) \cos(2\theta)} \\ &= 2k_j\sqrt{1 + \frac{\rho_j^2}{k_j^2} \cos(2\theta - \beta)} = 2k_j\sqrt{1 + r_j \cos(2\theta - \beta)}. \end{aligned}$$

We have proven that, once the intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$ is conical, there exists $(k_j, r_j, \beta_j) \in D$ such that (4.5) holds true. The uniqueness of a representative in D can be easily proven. \square

PROPOSITION 4.3. Assume that $j \in \mathcal{I}(\bar{u})$ and that \bar{u} is a weakly conical intersection between $\lambda_j(\cdot)$ and $\lambda_{j+1}(\cdot)$. Let $g_j : \theta \in [0, 2\pi) \rightarrow \mathbb{R}$ be the function defined in (4.3). Then there exists a unique (k_j, β_j) in

$$(4.6) \quad D' := \mathbb{R}_+^* \times [0, \pi)$$

such that

$$(4.7) \quad g_j(\theta) = 2k_j |\cos(\theta - \beta_j)|, \quad \forall \theta \in [0, 2\pi).$$

Proof. The proof is similar to that of Proposition 4.2. \square

Let

$$(4.8) \quad \bar{D} := \left((0, 1) \times [0, 2\pi) \right) \cup \{(0, 0)\}$$

and consider the family

$$(4.9) \quad \bar{\mathcal{F}} := \left(\mathbb{R} \ni x \mapsto \sqrt{1 + r \cos(x - \beta)} \right)_{(r, \beta) \in \bar{D}}$$

of 2π -periodic functions. We are going to prove that such a family is linearly independent. For each $(r, \beta) \in \bar{D}$,

$$(4.10) \quad g : \mathbb{R} \ni x \mapsto \sqrt{1 + r \cos(x - \beta)}$$

is a 2π -periodic continuously differentiable function. The Fourier series for g is given by

$$(4.11) \quad S_N(x) = \frac{1}{2} A_0(r, \beta) + \sum_{n=1}^N \left(A_n(r, \beta) \cos(nx) + B_n(r, \beta) \sin(nx) \right)$$

and converges uniformly to g as $N \rightarrow \infty$. Meanwhile, since $|r \cos(x - \beta)| \leq r < 1$ for all $x \in \mathbb{R}$, we can develop g by means of the power series

$$(4.12) \quad \sqrt{1 + r \cos(x - \beta)} = 1 + \sum_{n=1}^{\infty} C_n r^n \cos^n(x - \beta), \quad C_n = \frac{(-1)^{n-1} (2n-2)!}{2^{2n-1} (n-1)! n!}.$$

Notice that $|C_n| < 1$ for all $n \geq 1$, and the convergence of the series in (4.12) is uniform with respect to $x \in \mathbb{R}$. In the following lemma we use the development (4.12) to compute the coefficients of the Fourier series (4.11).

LEMMA 4.4. Let $(r, \beta) \in \bar{D}$, where the latter is defined in (4.8), and consider g as in (4.10). The coefficients of the Fourier series for g are given by

$$\begin{aligned} A_0 &= 2 + \sum_{m=1}^{\infty} r^{2m} \frac{C_{2m}}{2^{2m-1}} \binom{2m}{m} < \infty, \\ A_{2n} &= \sum_{m=n}^{\infty} r^{2m} \frac{C_{2m}}{2^{2m-1}} \binom{2m}{m-n} \cos(2n\beta) < \infty, \quad n > 0, \\ B_{2n} &= \sum_{m=n}^{\infty} r^{2m} \frac{C_{2m}}{2^{2m-1}} \binom{2m}{m-n} \sin(2n\beta) < \infty, \quad n > 0, \\ A_{2n+1} &= \sum_{m=n}^{\infty} r^{2m+1} \frac{C_{2m+1}}{2^{2m}} \binom{2m+1}{m-n} \cos((2n+1)\beta) < \infty, \quad n \geq 0, \\ B_{2n+1} &= \sum_{m=n}^{\infty} r^{2m+1} \frac{C_{2m+1}}{2^{2m}} \binom{2m+1}{m-n} \sin((2n+1)\beta) < \infty, \quad n \geq 0. \end{aligned}$$

Proof. Let us take $(r, \beta) \in \bar{D}$. By (4.12), we have

$$\begin{aligned} \sqrt{1 + r \cos(x - \beta)} &= 1 + \sum_{n=1}^{\infty} C_n r^n \cos^n(x - \beta) \\ &= 1 + \sum_{n=1}^{\infty} C_n r^n \left(\frac{e^{i(x-\beta)} + e^{-i(x-\beta)}}{2} \right)^n \\ &= 1 + \sum_{n=1}^{\infty} C_n \sum_{k=0}^n r^n \frac{1}{2^n} \binom{n}{k} e^{-i(n-2k)(x-\beta)}. \end{aligned}$$

Since $r \in (0, 1)$ and for all $n \in \mathbb{N}^*$, $|C_n| < 1$, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=0}^n \left| C_n r^n \frac{1}{2^n} \binom{n}{k} e^{-i(n-2k)(x-\beta)} \right| &\leq \sum_{n=1}^{\infty} \sum_{k=0}^n |C_n| r^n \frac{1}{2^n} \binom{n}{k} \\ &\leq \sum_{n=1}^{\infty} r^n \sum_{k=0}^n \frac{1}{2^n} \binom{n}{k} = \sum_{n=1}^{\infty} r^n < \infty. \end{aligned}$$

Therefore, by applying the Fubini–Tonelli theorem, we can change the order of summation. On the other hand, we notice that, for $m \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=0}^{2m} \binom{2m}{k} e^{-i(2m-2k)(x-\beta)} &= \binom{2m}{m} + 2 \sum_{k=0}^{m-1} \binom{2m}{k} \cos((2m-2k)(x-\beta)), \\ \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-i(2m-2k)(x-\beta)} &= 2 \sum_{k=0}^m \binom{2m+1}{k} \cos((2m+1-2k)(x-\beta)). \end{aligned}$$

Then we have

$$\begin{aligned} \sqrt{1 + r \cos(x - \beta)} &= 1 + \sum_{m=1}^{\infty} r^{2m} \frac{C_{2m}}{2^{2m}} \binom{2m}{m} \\ &\quad + \sum_{m=1}^{\infty} r^{2m} \frac{C_{2m}}{2^{2m-1}} \sum_{k=1}^m \binom{2m}{m-k} \cos(2k(x-\beta)) \\ &\quad + \sum_{m=0}^{\infty} r^{2m+1} \frac{C_{2m+1}}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{m-k} \cos((2k+1)(x-\beta)). \end{aligned}$$

We notice that $\cos(n(x-\beta)) = \cos(n\beta) \cos(nx) + \sin(n\beta) \sin(nx)$. By properly changing the order of summation and by term identification with the Fourier series (4.11), we get the formulas for the coefficients stated in the lemma. \square

LEMMA 4.5. *Fix $K \in \mathbb{N}^*$ and let β_1, \dots, β_K be pairwise distinct points in $[0, 2\pi)$. If $\{\alpha_k\}_{1 \leq k \leq K}$ is a set of non-zero coefficients in \mathbb{R} , then there exists $C > 0$ such that for every integer $A > 0$, there exists an integer $N > A$ such that*

$$\left| \sum_{k=1}^K \alpha_k \exp(iN\beta_k) \right| > C.$$

Proof. Since $n \mapsto (\exp(in\beta_1), \dots, \exp(in\beta_K))$ is recurrent, it is enough to prove that $m \mapsto \sum_{k=1}^K \alpha_k \exp(in\beta_k)$ is not identically zero. The case $K = 1$ is trivial, since $\alpha_1 \neq 0$.

In the case $K > 1$, let us assume by contradiction that

$$\sum_{k=2}^K \alpha_k \exp(in(\beta_k - \beta_1)) = -\alpha_1, \quad \forall n \in \mathbb{N}.$$

Then, for every $h \in \mathbb{N}^*$,

$$\frac{1}{h} \sum_{n=1}^h \sum_{k=2}^K \alpha_k \exp(in(\beta_k - \beta_1)) = -\alpha_1.$$

Therefore, by changing the order of summation, we obtain that

$$(4.13) \quad \sum_{k=2}^K \alpha_k \left(\frac{1}{h} \sum_{n=1}^h \exp(in(\beta_k - \beta_1)) \right) = -\alpha_1.$$

Since $\beta_k - \beta_1 \in (0, 2\pi)$ for every $k \in \{2, \dots, K\}$, we have

$$\frac{1}{h} \sum_{n=1}^h \exp(in(\beta_k - \beta_1)) \xrightarrow{h \rightarrow \infty} 0, \quad \forall k \in \{2, \dots, K\}.$$

We then deduce from (4.13) that $\alpha_1 = 0$. The contradiction is reached. \square

PROPOSITION 4.6. $\tilde{\mathcal{F}}$ is a linearly independent family of functions.

Proof. Assume by contradiction that there exists a positive integer K , K different points $(r_1, \beta_1), \dots, (r_K, \beta_K)$ in \bar{D} , and a set $\{\alpha_k\}_{1 \leq k \leq K}$ of non-zero coefficients in \mathbb{R} such that

$$\sum_{k=1}^K \alpha_k \sqrt{1 + r_k \cos(x - \beta_k)} = 0, \quad \forall x \in \mathbb{R}.$$

Therefore, by the Fourier series expansion (4.11), we would have

$$\sum_{k=1}^K \alpha_k A_n(r_k, \beta_k) = 0, \quad \sum_{k=1}^K \alpha_k B_n(r_k, \beta_k) = 0, \quad \forall n > 0.$$

By changing order of (r_k, β_k) , we can assume without loss of generality that $0 \leq r_1 \leq \dots \leq r_K < 1$. Let us define, for $n \in \mathbb{N}$,

$$(4.14) \quad \tilde{A}_{2n+1}(r) := \sum_{m=n}^{\infty} r^{2m+1} \frac{C_{2m+1}}{2^{2m}} \binom{2m+1}{m-n}, \quad \tilde{A}_{2n}(r) := \sum_{m=n}^{\infty} r^{2m} \frac{C_{2m}}{2^{2m}} \binom{2m}{m-n}.$$

Therefore \tilde{A}_n is defined for every $n \in \mathbb{N}$. We notice that for every $n > 0$, $A_n(r, \beta) + iB_n(r, \beta) = \tilde{A}_n(r)e^{in\beta}$. Thus, for every $n \in \mathbb{N}^*$,

$$(4.15) \quad \sum_{k=1}^K \alpha_k \tilde{A}_n(r_k) e^{in\beta_k} = 0.$$

Since $C_{2m+1} > 0$ for every $m \in \mathbb{N}$ and $C_{2m} < 0$ for every $m \in \mathbb{N}^*$ then, by (4.14), for $r, r' \in \mathbb{R}$ such that $0 \leq r < r' < 1$,

$$(4.16) \quad \frac{|\tilde{A}_n(r)|}{|\tilde{A}_n(r')|} < \left(\frac{r}{r'}\right)^n, \quad \forall n \in \mathbb{N}^*.$$

If $r_K = 0$, then $K = 1$ since there is only one point in \bar{D} for which $r = 0$. Since the function corresponding to $(r = 0, \beta = 0)$ is non-zero, we should have $\alpha_1 = 0$, which is a contradiction. Otherwise, if $r_K > 0$, there exists $1 \leq l \leq K$ such that $r_l = \dots = r_K$ and, if $l > 1$, $r_{l-1} < r_l$. Recall that $\alpha_k \neq 0$ for all $1 \leq k \leq K$. By [Lemma 4.5](#), there exists $C > 0$ such that for every $A \in \mathbb{N}$, there exists an integer $N > A$ such that

$$\left| \sum_{k=l}^K \alpha_k \exp(iN\beta_k) \right| > C.$$

Then by [\(4.16\)](#), there exists a big enough $M \in \mathbb{N}$ such that, for all $N > M$,

$$(4.17) \quad \left| \sum_{k=1}^{l-1} \alpha_k \tilde{A}_N(r_k) \right| < C |\tilde{A}_N(r_K)|.$$

Therefore, by [Lemma 4.5](#), there exists $\bar{N} > M$ such that

$$\begin{aligned} \left| \sum_{k=l}^K \alpha_k \exp(i\bar{N}\beta_k) \tilde{A}_{\bar{N}}(r_k) \right| &= \left| \sum_{k=l}^K \alpha_k \exp(i\bar{N}\beta_k) \right| |\tilde{A}_{\bar{N}}(r_K)| > C |\tilde{A}_{\bar{N}}(r_K)| \\ &> \left| \sum_{k=1}^{l-1} \alpha_k \tilde{A}_{\bar{N}}(r_k) \right|. \end{aligned}$$

With this inequality, equality [\(4.15\)](#) can no longer hold true for $n = \bar{N}$. Thus the contradiction is reached and the proof is concluded. \square

Now, let us define

$$(4.18) \quad \mathcal{I}^c(\bar{u}) = \{j \in \mathcal{I} \mid \lambda_j(\cdot) \text{ and } \lambda_{j+1}(\cdot) \text{ intersect conically at } \bar{u}\},$$

and, in accordance with [Proposition 4.2](#), the family of functions

$$(4.19) \quad \mathcal{F}_c(\bar{u}) = \left([0, 2\pi) \ni \theta \mapsto g_j(\theta) = 2k_j \sqrt{1 + r_j \cos(2\theta - \beta_j)} \right)_{j \in \mathcal{I}^c(\bar{u})}.$$

For each $(r, \beta) \in \bar{D}$, we furthermore define

$$(4.20) \quad \mathcal{I}_{r,\beta}^c(\bar{u}) = \{j \in \mathcal{I}^c(\bar{u}) \mid (r_j, \beta_j) = (r, \beta)\}.$$

PROPOSITION 4.7. *The family of functions $\mathcal{F}_c(\bar{u})$ is rationally independent if and only if for each nonempty $\mathcal{I}_{r,\beta}^c(\bar{u})$ the family $(k_n)_{n \in \mathcal{I}_{r,\beta}^c(\bar{u})}$ is rationally independent.*

Proof. First, we should notice that the rational independence of $\mathcal{F}_c(\bar{u})$ is equivalent to the rational independence of the family

$$(4.21) \quad \bar{\mathcal{F}}_c(\bar{u}) := \left(\mathbb{R} \ni x \mapsto k_j \sqrt{1 + r_j \cos(x - \beta_j)} \right)_{j \in \mathcal{I}^c(\bar{u})}.$$

Assume that there exists $(r, \beta) \in \bar{D}$ such that $\mathcal{I}_{r,\beta}^c(\bar{u})$ is nonempty and the family $(k_j)_{j \in \mathcal{I}_{r,\beta}^c(\bar{u})}$ is rationally dependent. Therefore, the sub-family

$$\left(x \mapsto k_j \sqrt{1 + r \cos(x - \beta)} \right)_{j \in \mathcal{I}_{r,\beta}^c(\bar{u})}$$

of $\bar{\mathcal{F}}_c(\bar{u})$ is rationally dependent. Hence $\mathcal{F}_c(\bar{u})$ is rationally dependent.

Assume that for all $(r, \beta) \in \bar{D}$, $\mathcal{I}_{r, \beta}^c(\bar{u})$ is either empty or $(k_j)_{j \in \mathcal{I}_{r, \beta}^c(\bar{u})}$ is rationally independent. For all finite subset \mathcal{J} of $\mathcal{I}_{r, \beta}^c(\bar{u})$, let us take $(\alpha_j)_{j \in \mathcal{J}} \in \mathbb{Q}^{\mathcal{J}}$ such that

$$\sum_{j \in \mathcal{J}} \alpha_j k_j \sqrt{1 + r_j \cos(x - \beta_j)} = 0, \quad \forall x \in \mathbb{R}.$$

Since \mathcal{J} is finite, there exist $M \in \mathbb{N}^*$ and $(\bar{r}_1, \bar{\beta}_1), \dots, (\bar{r}_M, \bar{\beta}_M) \in \bar{D}$ pairwise distinct such that, for each $j \in \mathcal{J}$,

$$(r_j, \beta_j) \in \{(\bar{r}_1, \bar{\beta}_1), \dots, (\bar{r}_M, \bar{\beta}_M)\},$$

and for each $1 \leq m \leq M$, the set

$$\mathcal{I}_m = \{j \in \mathcal{J} \mid (r_j, \beta_j) = (\bar{r}_m, \bar{\beta}_m)\}$$

is nonempty and finite. Then we have

$$\sum_{m=1}^M \left(\sum_{n \in \mathcal{I}_m} \alpha_n k_n \right) \sqrt{1 + \bar{r}_m \cos(x - \bar{\beta}_m)} = 0, \quad \forall x \in \mathbb{R}.$$

By [Proposition 4.6](#) we deduce that $\sum_{j \in \mathcal{I}_m} \alpha_j k_j = 0$ for all $m \in \{1, \dots, M\}$. Since $(k_n)_{n \in \mathcal{I}_m}$ is assumed to be rationally independent, we have $\alpha_n = 0$ for every $n \in \mathcal{I}_m$. Therefore $\alpha_j = 0$ for every $j \in \mathcal{J}$.

We thus conclude that $\mathcal{F}_c(\bar{u})$ is rationally independent if and only if for each nonempty $\mathcal{I}_{r, \beta}(\bar{u})$ the family $(k_n)_{n \in \mathcal{I}_{r, \beta}(\bar{u})}$ is rationally independent. \square

Let us define

$$(4.22) \quad \mathcal{I}^{wc}(\bar{u}) = \{j \in \mathcal{I} \mid \lambda_j(\cdot) \text{ and } \lambda_{j+1}(\cdot) \text{ intersect weakly conically at } \bar{u}\},$$

and, in accordance with [Proposition 2.18](#),

$$(4.23) \quad \mathcal{F}_{wc}(\bar{u}) = \left([0, 2\pi) \ni \theta \mapsto g_j(\theta) = 2k_j |\cos(\theta - \beta_j)| \right)_{j \in \mathcal{I}^{wc}(\bar{u})}.$$

For each $\beta \in [0, \pi)$, we define

$$(4.24) \quad \mathcal{I}_\beta^{wc}(\bar{u}) = \{j \in \mathcal{I}^{wc}(\bar{u}) \mid \beta_j = \beta\}.$$

PROPOSITION 4.8. *The family of functions $\mathcal{F}_{wc}(\bar{u})$ is rationally independent if and only if for each nonempty $\mathcal{I}_\beta^{wc}(\bar{u})$ the family $(k_j)_{j \in \mathcal{I}_\beta^{wc}(\bar{u})}$ is rationally independent.*

Proof. We should notice that the rational independence of $\mathcal{F}_{wc}(\bar{u})$ is equivalent to the rational independence of the family of functions

$$(4.25) \quad \bar{\mathcal{F}}_{wc}(\bar{u}) := \left(\mathbb{R} \ni x \mapsto k_j |\cos(x - \beta_j)| \right)_{j \in \mathcal{I}^{wc}(\bar{u})}.$$

Notice that $x \mapsto |\cos(x - \beta)|$ is differentiable everywhere on \mathbb{R} except at $x = \beta + k\pi$, $k \in \mathbb{Z}$, and the family of functions

$$\left(\mathbb{R} \ni x \mapsto |\cos(x - \beta)| \right)_{\beta \in [0, \pi)}$$

is linearly independent. Then the proof can be done using similar arguments as those used for [Proposition 4.7](#). \square

PROPOSITION 4.9. *The family of functions $\mathcal{F}(\bar{u})$ (defined in [\(4.3\)](#)) is rationally independent if and only if $\mathcal{F}_c(\bar{u})$ (defined in [\(4.19\)](#)) and $\mathcal{F}_{wc}(\bar{u})$ (defined in [\(4.23\)](#)) are rationally independent.*

Proof. Assume that $\mathcal{F}(\bar{u})$ is rationally independent. Since $\mathcal{F}_c(\bar{u})$ and $\mathcal{F}_{wc}(\bar{u})$ are sub-families of $\mathcal{F}(\bar{u})$, they are also rationally independent.

Assume now that $\mathcal{F}_c(\bar{u})$ and $\mathcal{F}_{wc}(\bar{u})$ are rationally independent. Let us consider a finite subset \mathcal{J} of $\mathcal{I}(\bar{u})$ and $(\alpha_j)_{j \in \mathcal{J}} \in \mathbb{Q}^{\mathcal{J}}$ such that

$$\sum_{j \in \mathcal{J}} \alpha_j g_j(\theta) = 0, \quad \forall \theta \in [0, 2\pi).$$

Set

$$\mathcal{J}^c = \mathcal{J} \cap \mathcal{I}^c(\bar{u}), \quad \mathcal{J}^{wc} = \mathcal{J} \cap \mathcal{I}^{wc}(\bar{u}).$$

If $\mathcal{J} = \mathcal{J}^c$ or $\mathcal{J} = \mathcal{J}^{wc}$, by rational independence of $\mathcal{F}_c(\bar{u})$ and $\mathcal{F}_{wc}(\bar{u})$, we deduce that $\alpha_j = 0$ for every $j \in \mathcal{J}$. Otherwise, since all $g_j(\cdot)$ are π -periodic, we have

$$\sum_{j \in \mathcal{J}^c} \alpha_j k_j \sqrt{1 + r_j \cos(2\theta - \beta_j)} + \sum_{j \in \mathcal{J}^{wc}} \alpha_j k_j |\cos(\theta - \beta_j)| = 0, \quad \forall \theta \in \mathbb{R}.$$

Let us take $j \in \mathcal{J}^{wc}$ and $\beta = \beta_j$, and define $\mathcal{J}_\beta^{wc} = \mathcal{I}_\beta^{wc}(\bar{u}) \cap \mathcal{J}^{wc} \neq \emptyset$. Then, for all $\theta \in \mathbb{R}$, we have

$$(4.26) \quad \left(\sum_{p \in \mathcal{J}_\beta^{wc}} k_p \alpha_p \right) |\cos(\theta - \beta)| = - \sum_{p \in \mathcal{J}^{wc} \setminus \mathcal{J}_\beta^{wc}} \alpha_p k_p |\cos(\theta - \beta_p)| - \sum_{p \in \mathcal{J}^c} \alpha_p k_p \sqrt{1 + r_p \cos(2\theta - \beta_p)}.$$

Let us note that the right-hand side of the above equality is differentiable at $\theta = \beta$, while $|\cos(\theta - \beta)|$ is not. Hence,

$$\sum_{p \in \mathcal{J}_\beta^{wc}} \alpha_p k_p = 0.$$

By [Proposition 4.8](#) and the rational independence of $\mathcal{F}_{wc}(\bar{u})$, we obtain that $\alpha_p = 0$ for every $p \in \mathcal{J}_\beta^{wc}$. This is true for each $\beta = \beta_j$ with $j \in \mathcal{J}^{wc}$. Therefore, $\alpha_j = 0$ for every $j \in \mathcal{J}^{wc}$. Then

$$\sum_{j \in \mathcal{J}^c} \alpha_j k_j \sqrt{1 + r_j \cos(2\theta - \beta_j)} = 0, \quad \forall \theta \in \mathbb{R}.$$

By the rational independence of $\mathcal{F}_c(\bar{u})$, we deduce that $\alpha_j = 0$ for every $j \in \mathcal{J}^c$. Thus, $\alpha_j = 0$ for every $j \in \mathcal{J}$. We conclude that $\mathcal{F}(\bar{u})$ is rationally independent. \square

5. Examples. In the following, we illustrate how to apply the results of the paper on systems having piled-up intersections (Examples 1,2,3) and weakly conical intersections (Examples 2,3). Example 3 is for a system evolving in an infinite-dimensional space.

Example 5.1. Let us consider $\mathcal{H} = \mathbb{C}^4$ and

$$(5.1) \quad H(u_1, u_2) = \begin{pmatrix} 1 & u_1 & 0 & 0 \\ u_1 & 2 & u_2 & 0 \\ 0 & u_2 & 3 & \sqrt{2}u_1 \\ 0 & 0 & \sqrt{2}u_1 & 4 \end{pmatrix}, \quad (u_1, u_2) \in \mathbb{R}^2.$$

It is not difficult to compute that, at $\bar{u} = (0, \sqrt{2})$, λ_1 intersects conically with λ_2 and λ_3 intersects conically with λ_4 , with $(k_1, r_1, \beta_1) = (\frac{2}{3}, \frac{1}{2}, 0)$ and $(k_3, r_3, \beta_3) = (\frac{\sqrt{7}}{3}, \frac{5}{7}, 0)$. By [Proposition 4.7](#), the eigenvalue intersections have rationally unrelated germs at $(0, \sqrt{2})$. Moreover, λ_2 intersects conically with λ_3 at $\bar{u} = (\sqrt{12 - 2\sqrt{33}}, 0)$. Therefore, by [Remark 3.10](#) and by [Theorem 1.1](#), System (1.1) is exactly controllable in the unitary group, since, up to suitably choosing $U' \subset \mathbb{R}^2$, the spectrum of $H(\cdot)$ on U' is weakly conically connected and its eigenvalue intersections have rationally unrelated germs at every intersection point.

Example 5.2. Let us take $\mathcal{H} = \mathbb{C}^5$, $U = \mathbb{R}^2$, and

$$(5.2) \quad H(u_1, u_2) = \begin{pmatrix} -1 & u_1 & 0 & 0 & 0 \\ u_1 & -1 & u_2 & 0 & 0 \\ 0 & u_2 & 0 & u_1 & 0 \\ 0 & 0 & u_1 & 1 & 2u_2 \\ 0 & 0 & 0 & 2u_2 & 1 \end{pmatrix}, \quad (u_1, u_2) \in U.$$

It is not difficult to compute that, at $\bar{u} = (0, 0)$, λ_1 intersects semi-conically with λ_2 and λ_4 intersects semi-conically with λ_5 with $(k_1, \beta_1) = (1, 0)$ and $(k_4, \beta_4) = (2, \frac{\pi}{2})$. By [Proposition 4.8](#), the eigenvalue intersections have rationally unrelated germs at $(0, 0)$. Moreover, λ_2 intersects conically with λ_3 at $\bar{u} = (\frac{2}{3}, 0)$ and λ_3 intersects conically with λ_4 at $\bar{u} = (0, \frac{3-\sqrt{3}}{3})$. Therefore, by [Remark 3.10](#) and by [Theorem 1.1](#), System (1.1) is exactly controllable in the unitary group.

Example 5.3. Let us consider a Eberly–Law type model of a qubit coupled with a harmonic oscillator, described by the equation

$$(5.3) \quad i \frac{d\psi(t)}{dt} = H(u)\psi(t), \quad \psi_t \in \mathbb{C}^2 \times \mathcal{L}(\mathbb{R}, \mathbb{C}),$$

where the Hamiltonian $H(u) = H_0 + u_1 H_1 + u_2 H_2$ is described next. In absence of any coupling, the drift Hamiltonian H_0 is given by

$$H_0 = \frac{\omega_{eg}}{2} \sigma_z \otimes \mathbb{I}_c + \omega_c \mathbb{I}_q \otimes (a^\dagger a + \frac{1}{2} \mathbb{I}_c),$$

where \mathbb{I}_q and \mathbb{I}_c are identity operator in the qubit and harmonic oscillator, σ_z the Pauli matrix, a^\dagger the creation operator, a the annihilation operator, and $\omega_{eg} < \omega_c$. The eigenstates of H_0 can be represented as

$$\phi_{2n+1} = |g, n\rangle, \quad \phi_{2n+2} = |e, n\rangle, \quad n \geq 0,$$

with corresponding eigenvalues

$$\lambda_{2n+1} = \left(-\frac{\omega_{eg}}{2} + \left(n + \frac{1}{2} \right) \omega_c \right), \quad \lambda_{2n+2} = \left(\frac{\omega_{eg}}{2} + \left(n + \frac{1}{2} \right) \omega_c \right), \quad n \geq 0.$$

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