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# Free boundary regularity and support propagation in mean field games and optimal transport

Pierre Cardaliaguet <sup>\*</sup>      Sebastian Munoz <sup>†</sup>      Alessio Porretta <sup>‡</sup>

## Abstract

We study the behavior of solutions to the first-order mean field games system with a local coupling, when the initial density is a compactly supported function on the real line. Our results show that the solution is smooth in regions where the density is strictly positive, and that the density itself is globally continuous. Additionally, the speed of propagation is determined by the behavior of the cost function near small values of the density. When the coupling is entropic, we demonstrate that the support of the density propagates with infinite speed. On the other hand, for a power-type coupling, we establish finite speed of propagation, leading to the formation of a free boundary. We prove that under a natural non-degeneracy assumption, the free boundary is strictly convex and enjoys  $C^{1,1}$  regularity. We also establish sharp estimates on the speed of support propagation and the rate of long time decay for the density. Moreover, the density and the gradient of the value function are both shown to be Hölder continuous up to the free boundary. Our methods are based on the analysis of a new elliptic equation satisfied by the flow of optimal trajectories. The results also apply to mean field planning problems, characterizing the structure of minimizers of a class of optimal transport problems with congestion.

**Keywords:** self-similar solutions; degenerate elliptic equations; intrinsic scaling; displacement convexity; Lagrangian coordinates; free boundary; mean field games; Hamilton-Jacobi equations; continuity equation.  
**MSC:** 35R35, 35Q89, 35B65, 35J70.

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# 1 Introduction

In this paper, we analyze the following first-order system of partial differential equations in the unknown scalar functions  $(u, m)$ :

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x, t) \in \mathbb{R} \times (0, T), \\ m_t - (mu_x)_x = 0 & (x, t) \in \mathbb{R} \times (0, T), \end{cases} \quad (1.1)$$

where  $f$  is an increasing function. Systems such as (1.1) appear in the theory of mean field games (MFG for short), which describes Nash equilibria in differential games with strategic interactions between a large number of agents ([19, 20]). In that context,  $m$  represents the distribution density of a population of infinitely many indistinguishable agents, while  $u$  is the value function of any generic agent, whose running costs depend on the density distribution  $m$ .

The system shall also be complemented with initial and terminal conditions, which can take the form of a given initial distribution  $m_0$  and a final pay-off for the agents, which may itself be dependent on the density of the population at the final time. In that case, the complete form of the system is as follows:

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x, t) \in \mathbb{R} \times (0, T) \\ m_t - (mu_x)_x = 0 & (x, t) \in \mathbb{R} \times (0, T) \\ m(x, 0) = m_0(x), u(x, T) = g(m(x, T)), & x \in \mathbb{R}. \end{cases} \quad (\text{MFG})$$

In (MFG), any such agent, typically represented by a dynamical state  $x(t)$ , optimizes some cost given by the kinetic energy plus a congestion cost depending on the distribution of mass, that is,

$$u(x, t) = \inf_{\gamma: [t, T] \rightarrow \mathbb{R}, \gamma(t) = x} \int_t^T \frac{1}{2} |\dot{\gamma}(s)|^2 + f(m(\gamma(s), s)) ds + g(m(\gamma(T), T)).$$

In turn, the density evolves according to the optimal feedback control of the agents,  $(-u_x)$ . At equilibrium, the value function of any single agent, and the density distribution of the population, satisfy the system (MFG) coupling the Hamilton–Jacobi (HJ) equation for  $u$  with the continuity equation for  $m$ .

Alternatively to prescribing a final cost, one may instead prescribe both the initial and final distributions of the agents. This is the so-called mean field optimal planning problem, which takes the complete form:

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = f(m) & (x, t) \in \mathbb{R} \times (0, T) \\ m_t - (mu_x)_x = 0 & (x, t) \in \mathbb{R} \times (0, T) \\ m(x, 0) = m_0(x), m(x, T) = m_T(x), & x \in \mathbb{R}, \end{cases} \quad (\text{MFGP})$$

for some prescribed mass distributions  $m_0, m_T$ .

In this latter case, the system (MFGP) can also be viewed as the first-order optimality conditions of an optimal transport problem on the Wasserstein space of measures. In this context, one seeks to minimize the functional

$$\mathcal{B}(m, v) := \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} |v|^2 dm dt + \int_0^T \int_{\mathbb{R}^d} F(m) dx dt \quad \text{subject to} \quad \begin{cases} m_t - (vm)_x = 0 \\ m(0) = m_0, m(T) = m_T \end{cases} \quad (1.2)$$

where  $F(s) = \int_0^s f(r)dr$ . This problem appears as a generalization of the dynamic formulation of the mass transport problem (cf. [4]). Indeed, the classical dynamic version of the Monge-Kantorovich problem corresponds to the case  $f = 0$ , whereas the resulting optimizer  $m(t)$  is the geodesic, in the Wasserstein space, that connects  $m_0$  and  $m_T$ . The function  $u$  appears from the dual formulation, and  $u(0), u(T)$  are the so-called Kantorovich potentials. Classical monographs on optimal transport theory are [1, 32, 36]. The additional cost term  $F(m)$ , which is convex whenever  $f$  is increasing, can be interpreted as an incentive to avoid congested areas. Similar optimal transport problems with congestion have been widely investigated in the literature, including their connection with MFG theory, since the solution of (MFGP) provides the minimizer  $(m, u_x)$  of  $\mathcal{B}$ ; see, for example, [2, 9, 18, 21, 30].

Since the courses given by P.-L. Lions at Collège de France, devoted to MFG theory, it was observed that the increasing character of  $f$  would produce a regularizing effect in the solutions  $(u, m)$  of system (1.1); indeed, the first order system can be reformulated as a single degenerate elliptic equation for  $u$  in the space-time variables ([25]). The approach suggested by Lions was recently developed and extended in [27, 28, 29, 31]. So far, in arbitrary dimensions, it is well understood that solutions are smooth under the blow-up assumption

$$\lim_{m \rightarrow 0^+} f(m) = -\infty, \quad (1.3)$$

provided that the marginals  $m_0, m_T$  are (strictly) positive, say for positive measures on a compact domain (e.g. on the flat torus) or for Gaussian-like measures on the whole space. It was also established in [27] that, for the one-dimensional case, assumption (1.3) may be removed, thus requiring only the positivity of the marginals.

Much less is known about what happens in case of compactly supported marginals. Regularizing effects of the type  $L^1 \rightarrow L^\infty$  have been proven to hold (see [21, 31]), but the propagation of the support of the solution, and even basic matters of regularity such as the continuity of the density, have largely remained open issues.

The purpose of this article is to investigate this question in the one-dimensional case of the space variable: we address both the case of finite and infinite speed of propagation of the support of initial measures. Roughly speaking, those two cases correspond to two model choices for the coupling function, namely  $f(m) = m^\theta$  for some  $\theta > 0$ , or  $f(m) = \log(m)$ . In the latter case, there is infinite speed of propagation, and the solution starting with compact support becomes instantaneously positive and smooth. By contrast, when  $f(m) = m^\theta$ , we observe finite speed of propagation, and the solution evolves with compact support. This leads to new interesting questions concerning the study of the free boundary  $\partial\{m(t) > 0\}$ , which is the main focus of our paper.

By way of analogy, which is also natural from the optimal transport viewpoint, for  $f = \log(m)$  the evolution of  $m$  is reminiscent of a nondegenerate diffusion, such as the heat equation. On the other hand, the case of a power nonlinearity  $f$  resembles the behavior of degenerate slow diffusions such as the flow through a porous medium (see [35]). This analogy becomes more compelling as in fact, when  $f(m) = m^\theta$ , we exhibit a family of self-similar solutions which evolves from a Dirac mass into a compactly supported measure. These solutions are given by the formula

$$m(x, t) = t^{-\bar{\alpha}} \left( R - \frac{\bar{\alpha}(1-\bar{\alpha})}{2} \left( \frac{x}{t^{\bar{\alpha}}} \right)^2 \right)_+^{1/\theta}, \quad \bar{\alpha} = \frac{2}{2+\theta}, \quad (1.4)$$

which is strongly reminiscent of the famous Barenblatt solution for the porous medium equation [3]. The behavior exhibited by this class of compactly supported solutions serves as a prototype for our analysis of problems (MFG) and (MFGP). In order to describe our main results on the propagation of the support and the characterization of the free boundary, we assume henceforth that

$$f(m) = m^\theta, \quad \theta > 0, \quad (1.5)$$

and that the initial measure  $m_0$  is a continuous, compactly supported, probability density, with a bump-like shape:

$$\{m_0 > 0\} = (a_0, b_0) \quad \text{and} \quad \frac{1}{C_0} \text{dist}(x, \{a_0, b_0\})^{\alpha_0} \leq m_0(x) \leq C_0 \text{dist}(x, \{a_0, b_0\})^{\alpha_0}, \quad (1.6)$$

for some  $\alpha_0, C_0 > 0$ . In order to keep our main statement in a simpler form, we will assume here a consistent condition on the terminal density, in case of problem (MFGP),

$$\{m_T > 0\} = (a_1, b_1), \quad \text{and} \quad \frac{1}{C_1} \text{dist}(x, \{a_1, b_1\})^{\alpha_0} \leq m_T(x) \leq C_1 \text{dist}(x, \{a_1, b_1\})^{\alpha_0}. \quad (1.7)$$

However, more general situations will be considered later, allowing for the behavior of  $m_T$  at the boundary of its support to differ from the behavior of  $m_0$ . Similarly, for problem (MFG), we will require here, for simplicity, consistency between  $f$  and the terminal cost coupling  $g$ , namely

$$g(s) = c_T s^\theta, \quad \text{for some } c_T \geq 0. \quad (1.8)$$

We may now state our main result, which proves that the unique solution of (MFG) or (MFGP) has a compactly supported density and the free boundary  $\partial\{m(t) > 0\}$  consists of two Lipschitz curves, which are  $C^{1,1}$  under a suitable non-degeneracy assumption at the initial time. Those curves can be characterized in terms of the flow of optimal trajectories for the agents' optimization problem.

In fact, we will show that  $u$  is smooth inside the support of  $m$ , and the characteristic flow

$$\begin{cases} \dot{\gamma}(x, \cdot) = -u_x(\gamma(x, \cdot), \cdot) \\ \gamma(x, 0) = x \end{cases}$$

is well defined starting from  $x$  in the support of  $m_0$ . Finally, we also show that the left and right free boundary curves are, respectively, convex and concave, and in problem (MFG) the support spreads outward in time.

**Theorem 1.1.** *Let  $f$  be given by (1.5), and let  $0 < \alpha < 1$ . Assume that  $m_0 : \mathbb{R} \rightarrow [0, \infty)$  satisfies (1.6),  $m_0^\theta \in C^{1,\alpha}(a_0, b_0)$ , and  $m_0^\theta$  is semi-convex. In case of problem (MFGP), assume also that  $m_T^\theta \in C^{1,\alpha}(a_0, b_0)$  satisfies (1.7). Let  $(u, m)$  be the solution to (MFGP), or to (MFG) with  $g$  satisfying (1.8). Then  $(u, m) \in C_{\text{loc}}^{2,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\}) \times C_{\text{loc}}^{1,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\})$ , and the following holds:*

1. *There exist two functions  $\gamma_L < \gamma_R \in W^{1,\infty}(0, T)$ , such that*

$$\{m > 0\} = \{(x, t) \in \mathbb{R} \times [0, T] : \gamma_L(t) < x < \gamma_R(t)\}. \quad (1.9)$$

*Moreover, the flow  $\gamma$  of optimal trajectories is well defined on  $(a_0, b_0) \times [0, T]$ , we have*

$$\gamma \in W^{1,\infty}((a_0, b_0) \times (0, T)) \cap C_{\text{loc}}^{2,\alpha}((a_0, b_0) \times [0, T]), \quad \gamma_x > 0, \quad \gamma_L(t) = \gamma(a_0, t), \quad \gamma_R(t) = \gamma(b_0, t),$$

*and  $\gamma$  is a classical solution in  $(a_0, b_0) \times (0, T)$  to the elliptic equation*

$$\gamma_{tt} + \frac{\theta m_0^\theta}{(\gamma_x)^{2+\theta}} \gamma_{xx} = \frac{(m_0^\theta)_x}{(\gamma_x)^{1+\theta}}. \quad (1.10)$$

2. *If we assume further the concavity condition*

$$(m_0^\theta)_{xx} \leq 0 \text{ in } \{x \in (a_0, b_0) : \text{dist}(x, \{a_0, b_0\}) < \delta\} \text{ for some } \delta > 0, \quad (1.11)$$

*then we have  $\gamma_L, \gamma_R \in W^{2,\infty}(0, T)$ , and there exists  $K > 0$  such that, for a.e.  $t \in [0, T]$ ,*

$$\frac{1}{K} \leq \ddot{\gamma}_L(t) \leq K, \quad \text{and} \quad -K \leq \ddot{\gamma}_R(t) \leq -\frac{1}{K},$$

*where  $K$  depends on  $T, C_0, \theta, \delta^{-1}, \|((m_0^\theta)_{xx})^-\|_\infty, |(m_0^\theta)_x(a_0^+)|, |(m_0^\theta)_x(b_0^-)|$  (and additionally on  $c_T$  for problem (MFG), and on  $C_1$  for problem (MFGP)).*

*Moreover, when  $(u, m)$  solves (MFG), we have, for  $t \in [0, T]$ ,*

$$-K(c_T + (T - t)) \leq \dot{\gamma}_L(t) \leq -\frac{1}{K}(c_T + (T - t)), \quad \text{and} \quad \frac{1}{K}(c_T + (T - t)) \leq \dot{\gamma}_R(t) \leq K(c_T + (T - t)).$$

In relation to the main text, Theorem 1.1 is a combination of Theorem 4.3 (for the existence of the solution  $(u, m)$  and its regularity in  $\{m > 0\}$ ), Theorem 4.10 (for the description of the free boundary) and Theorem 4.14 (for the regularity and convexity of the free boundary).

*Remark 1.2.* We now discuss the nondegeneracy conditions required on  $m_0$  at the boundary of its support. First, we note that the  $C^{1,\alpha}(a_0, b_0)$  (and therefore  $W^{1,\infty}(\mathbb{R})$ ) condition on  $m_0^\theta$ , together with (1.6), implies that  $\alpha_0 \geq \frac{1}{\theta}$  in (1.6). In turn, the concavity assumption (1.11) further restricts the behavior of  $m_0^\theta$ , forcing  $\alpha_0 = \frac{1}{\theta}$  in (1.6). However, this condition of a linear, nondegenerate behavior of  $m_0^\theta$  is natural (a case in point being the self-similar solution itself), and should be compared with standard nondegeneracy conditions on the initial data in other free boundary problems (e.g. in the study of the moving free boundary for the porous medium equation, see [35]).

We also wish to highlight that, when dealing with problem (MFGP), some asymmetry can be observed when requiring some conditions (e.g. concavity-type assumptions) on  $m_0$  but not on  $m_T$ . This kind of asymmetry arises because we are referring to the forward flow  $\gamma(x, t)$  in our statement. Of course, similar results will hold when reversing the time flow and exchanging the roles of  $m_0, m_T$ .

*Remark 1.3.* We stress that the first part of Theorem 1.1 remains true under more general conditions than (1.7) (respectively, (1.8)). We refer the reader to Theorem 4.10, which allows for the behavior of  $m_T$  at the boundary of its support to be different from the behavior of  $m_0$  (respectively, in case of problem (MFG), for the function  $g(s)$  to be a different power than  $f$ ).

The result in the second part of Theorem 1.1 corresponds exactly to the picture described by the self-similar solution (1.4). Indeed, our next result shows that the free boundary propagates with strictly convex (resp. concave) behavior at the left (resp. right) free boundary curve. In fact, if we strengthen the concavity assumption on  $m_0^\theta$ , we show that the free boundary evolves with the optimal speed given by the self-similar solution. Moreover, the long time decay of the density occurs with the same rate, as exhibited by (1.4).

**Theorem 1.4.** *Under the assumptions of Theorem 1.1, let  $(u, m)$  be the unique solution to (MFG) or (MFGP), and let  $\gamma$  be the associated flow of optimal trajectories. Assume in addition that  $-K \leq (m_0^\theta)_{xx} \leq -\frac{1}{K}$  in  $(a_0, b_0)$  for some  $K > 0$ , and, in case of problem (MFG), assume that  $c_T = \kappa_1 T$  in (1.8). If we define*

$$\bar{\alpha} = \frac{2}{2 + \theta}; \quad d(t) = \begin{cases} t & \text{if } u \text{ solves (MFG),} \\ \text{dist}(t, \{0, T\}) & \text{if } u \text{ solves (MFGP),} \end{cases}$$

then there exists a constant  $C > 0$  such that for every  $(x, t) \in [a_0, b_0] \times [0, T]$ ,

$$\frac{1}{C}(1 + d(t)^{\bar{\alpha}}) \leq |\text{supp}(m(\cdot, t))| \leq C(1 + d(t)^{\bar{\alpha}}), \quad |\gamma(x, t)| \leq C(1 + d(t)^{\bar{\alpha}}), \quad (1.12)$$

$$\frac{1}{C} \frac{m_0(x)}{(1 + d(t)^{\bar{\alpha}})} \leq m(\gamma(x, t), t) \leq C \frac{m_0(x)}{(1 + d(t)^{\bar{\alpha}})}, \quad (1.13)$$

where  $C = C(C_0, \kappa_1, \kappa_1^{-1}, |a_0|, |b_0|, K)$  in case of problem (MFG), and  $C = C(C_0, C_1, |a_0|, |b_0|, |a_1|, |b_1|, K)$  in case of (MFGP).

Theorem 1.4 is nothing but Theorem 4.15 below. Let us stress that a crucial role in the proof of the above results is played by the equation satisfied by the flow of optimal curves  $\gamma$ , namely (1.10). In particular, the Lipschitz regularity of  $\gamma$  is obtained by a maximum principle argument applied to  $\gamma_x$ , which is derived from (1.10). We obtain further insight by studying the equation of  $m$  in Lagrangian coordinates. Indeed, the function  $v = f(m(\gamma(x, t), t))$  satisfies the (degenerate) elliptic equation

$$-(\gamma_x(\theta v)^{-1} v_t)_t - (\gamma_x^{-1} v_x)_x = 0, \quad (1.14)$$

where one can prove, assuming (1.11), that the positive quantity  $\gamma_x$  is bounded below and above. This elucidates the key distinction between the present problem and slow diffusions of porous medium type; equation (1.14) is diffusive (rather than parabolic) in the time variable. Relying on this equation, we establish the regularity of  $m$  up to the free boundary. Namely, we prove that  $m$  is Hölder continuous, through an application of the intrinsic scaling regularity method (see [12, 13, 33]). In turn, we show that  $Du$  is Hölder continuous as well. We can summarize these regularity results, contained in Theorem 4.21 and 4.23 respectively, as follows:

**Theorem 1.5.** *Under the assumptions of Theorem 1.4, we have  $f(m) \in C_{\text{loc}}^\beta(\mathbb{R} \times (0, T))$  and  $u \in C_{\text{loc}}^{1, \frac{\beta}{2}}(\mathbb{R} \times (0, T))$  for some  $\beta \in (0, 1)$ .*

Finally, our last result shows that the solutions of (1.1) exhibit a different behavior when

$$f(m) = \log(m). \quad (1.15)$$

In contrast with the case of a power nonlinearity, the unbounded payoff as  $m \downarrow 0$  given by (1.15) implies that the support of the density propagates with infinite speed. This behavior is observable in both problem (MFG) (with  $g(s) = c_T \log(s)$ ,  $c_T \geq 0$ ) and in the planning problem (MFGP), which corresponds to the optimal transport functional (1.2) with the entropy term  $F(m) = m \log(m)$ . More specifically, under the assumption of  $m_0$  being continuous with compact support (and similarly for  $m_T$  in the case of (MFGP)), we establish the existence of classical solutions  $(u, m)$  with  $m > 0$  in  $(0, T)$ .

Compared to Theorem 1.1, the positivity of solutions on the whole space now makes it much more delicate to use the flow of optimal curves  $\gamma(x, t)$ , which are no longer confined in a bounded set. This difficulty leads us to require an extra symmetry and monotonicity assumption, namely that  $m_0$  is even and nonincreasing in  $(0, \infty)$  (and the same for the terminal density  $m_T$ ). We are able to take advantage of this assumption by showing that the solution  $m(\cdot, t)$  preserves this property for all  $t \in (0, T)$ , which is in itself a non-trivial feature of the MFG system (see Lemma 5.7). However, we no longer require any special behavior of  $m_0$  when vanishing at the boundary of its support, avoiding conditions such as (1.6), (1.7). In fact, the support now propagates instantly, regardless of the flatness of  $m_0$ .

**Theorem 1.6.** *Let  $f$  be given by (1.15), let  $\alpha \in (0, 1)$ , and assume that  $m_0$  is a continuous, compactly supported, density on  $\mathbb{R}$ , which is  $C_{\text{loc}}^{1, \alpha}$  in the set  $\{m_0 > 0\}$ , even and nonincreasing on  $[0, \infty)$ .*

1. *If  $g(s) = c_T \log(s)$ , for some  $c_T \geq 0$ , then there exists a unique classical solution  $(u, m) \in C^2(\mathbb{R} \times (0, T]) \times C^1(\mathbb{R} \times (0, T])$  of (MFG) such that  $m$  is continuous and bounded on  $\mathbb{R} \times [0, T]$ , positive on  $\mathbb{R} \times (0, T)$  with  $m(0) = m_0$ , and  $|x|^2 m(t) \in L^1(\mathbb{R})$ ,  $\frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$ , for every  $t \in (0, T)$ .*
2. *If  $m_T \in C_c(\mathbb{R})$  is even and nonincreasing on  $[0, \infty)$ , then there exists a unique (up to addition of a constant to  $u$ ) classical solution  $(u, m) \in C^2(\mathbb{R} \times (0, T)) \times C^1(\mathbb{R} \times (0, T))$  of (MFGP) such that  $m$  is continuous and bounded on  $\mathbb{R} \times [0, T]$ , positive on  $\mathbb{R} \times (0, T)$  with  $m(0) = m_0, m(T) = m_T$ , and  $|x|^2 m(t) \in L^1(\mathbb{R})$ ,  $\frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$ , for every  $t \in (0, T)$ .*

Theorem 1.6 is Theorem 5.9 below. As mentioned before, the symmetry and monotonicity assumption on  $m_0$ , which is required in Theorem 1.6, allows us to overcome certain difficulties in the obtention of classical positive solutions in the whole space. These difficulties disappear in compact domains, as is seen in Theorem 5.2, where we prove the existence of classical periodic solutions, with  $m > 0$  in  $(0, T)$ , under the only condition that  $m_0$  (and  $m_T$ ) are continuous and compactly supported.

**Outline of the paper.** The paper is organized as follows. In Section 2 we exhibit the class of self-similar solutions which will serve as a prototype for our main results. Section 3 presents the key features of smooth, periodic solutions with a positive density: structural properties, displacement convexity, Lipschitz estimates, and a modulus of continuity for the density. Section 4, dealing with compactly supported solutions, is the heart of the paper: starting with the existence of solutions (Subsection 4.1), it culminates with the regularity, geometric properties, and long time behavior of the free boundary (Subsection 4.2), and the Hölder regularity of  $m$  and  $Du$  (Subsection 4.3). Section 5 is devoted to the entropic coupling ( $f = \log$ ) and the infinite speed of propagation. Appendix A contains the computations for the self-similar solutions.

## 2 Self-similar solutions

We now exhibit a family of compactly supported, self-similar solutions of the system

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = m^\theta & (x, t) \in \mathbb{R} \times (0, \infty), \\ m_t - (mu_x)_x = 0 & (x, t) \in \mathbb{R} \times (0, \infty), \\ \int_{\mathbb{R}} m(t) dx = 1 & t \in (0, \infty), \end{cases} \quad (2.1)$$

where  $\theta > 0$ . By a solution of (2.1) we mean here that  $u$  is Lipschitz continuous and  $m$  continuous and nonnegative, the first equation being understood in the sense of viscosity solutions, while the second equation is satisfied in the sense of distributions. The solution is described in the following result.

**Proposition 2.1.** *For  $\theta > 0$ , let us set*

$$\bar{\alpha} = \frac{2}{2 + \theta}$$

and let  $R$  be the unique positive number such that  $\int_{\mathbb{R}} (R - \frac{1}{2}(\bar{\alpha} - \bar{\alpha}^2)y^2)_+^{1/\theta} dy = 1$ . A solution of (2.1) is given by  $(u, m)$ , with

$$m(x, t) = t^{-\bar{\alpha}} \phi(x/t^{\bar{\alpha}}), \quad \text{where } \phi(y) = \left( R - \frac{1}{2}(\bar{\alpha}(1 - \bar{\alpha}))y^2 \right)_+^{1/\theta}$$

and  $u$  defined as follows:

(i) either  $\theta = 2$  and

$$u(x, t) = \begin{cases} -\frac{1}{4t}x^2 - R \log t & \text{if } \Delta \leq 0 \\ -\frac{2R|x|}{|x| - \sqrt{\Delta}} - 2R \log\left(\frac{|x| - \sqrt{\Delta}}{\sqrt{8R}}\right) & \text{if } \Delta > 0 \end{cases} \quad (2.2)$$

where  $\Delta = x^2 - 8Rt$ ,

(ii) or  $\theta \neq 2$  and

$$u(x, t) = \begin{cases} -\bar{\alpha} \frac{x^2}{2t} - R \frac{1}{2\bar{\alpha}-1} t^{2\bar{\alpha}-1} & \text{if } \Delta \leq 0 \\ \frac{-R\bar{\alpha}}{(1-\bar{\alpha})(2\bar{\alpha}-1)} S^{2\bar{\alpha}-1} - \frac{\bar{\alpha}R}{1-\bar{\alpha}} S^{2\bar{\alpha}-2}(t-S) & \text{if } \Delta > 0, \end{cases} \quad (2.3)$$

where  $\Delta = |x| - \sqrt{\frac{2R}{\bar{\alpha}(1-\bar{\alpha})}} t^{\bar{\alpha}}$  and the function  $S = S(x, t)$  is defined implicitly by the equation:

$$S \sqrt{\frac{2R}{\bar{\alpha}(1-\bar{\alpha})}} - |x| S^{1-\bar{\alpha}} + \sqrt{\frac{2R\bar{\alpha}}{1-\bar{\alpha}}} (t-S) = 0. \quad (2.4)$$

The explicit construction of  $u$  can be understood by distinguishing two regions. First, one shows that, on the support of  $m$ ,  $u$  must be given by

$$u(x, t) = -\bar{\alpha} \frac{x^2}{2t} + c(t), \quad \text{with } c'(t) = -Rt^{-2\theta/(2+\theta)}. \quad (2.5)$$

Outside the support of  $m$ , the values of  $u$  are extended along the optimal curves, which are straight lines in the set  $\{m = 0\}$ . This leads to formula (2.2) if  $\theta = 2$ , or to formula (2.3) if  $\theta \neq 2$ . Note that  $\Delta \leq 0$  corresponds with the support of  $m$ . Finally, we point out that  $u$  is defined up to an additive constant. The proof of the statements made in Proposition 2.1 will be presented in Appendix A.

We note the following relevant facts about this solution, which will serve as a model for our later assumptions and results:

- At time  $t = 0$ , the measure corresponds to a Dirac mass at  $x = 0$ . For positive times  $t > 0$ , in general, the density  $m$  is merely Hölder continuous.
- For each  $t > 0$ , the function  $f(m)$  is always Lipschitz (away from  $t = 0$ ). Moreover,  $f(m(\cdot, t))$  is strictly concave within the support, and, in particular,  $f(m(\cdot, t))_x$  is non-zero at the endpoints. A weaker, local version of these conditions will serve as our non-degeneracy assumption on the initial distribution  $f(m_0)$  (see (4.53) and (4.54)), in order to prove our main regularity result for the free boundary (Theorem 4.14), and the Hölder continuity of  $m$  (Theorem 4.21). Moreover, the full strict concavity assumption on  $f(m_0)$  will yield our result on the optimal speed of support propagation and long time decay of  $m$  (Theorem 4.15).
- The value function  $u$  is smooth on the support of  $m$  but  $u_{xx}$  blows-up at the interface (see Remark A.4), at least when approaching from outside the support. In fact, it is shown in Proposition A.3 that  $u \in C^{1,s}$  for a certain  $0 < s < 1$ . Accordingly, our general results will show that the  $C^{1,s}$  regularity exhibited by this explicit solution in fact holds for arbitrary solutions of the MFG system, at least under the aforementioned non-degeneracy assumption (Theorem 4.23).



### 3 Structure and a priori estimates in the periodic setting

#### 3.1 Structural properties of the MFG system

The results of the paper are systematically obtained by establishing a priori estimates on a regularized system which has a smooth solution. In this subsection, we will explain the structural properties of the system (1.1), deriving the fundamental identities used throughout the paper. Here we assume that  $f' > 0$  on  $(0, \infty)$ , and  $(u, m)$  is a classical solution to (1.1), with  $m$  being positive, and  $u_x$  has at most a linear growth. Note that we only assume  $m(\cdot, 0) = m_0$  to be smooth and positive (no condition on the mass), so that the results of this part are valid for equations with periodic boundary conditions as well as in the whole space.

We begin with the elliptic equation satisfied by  $u$ , first derived by Lions in [25] (see also [28, 31]). It is obtained by simply eliminating  $m = f^{-1}(-u_t + u_x^2/2)$  from the system, thanks to the fact that  $f' > 0$ .

**Lemma 3.1.** *Let  $(u, m)$  be a classical solution to (1.1). The map  $u$  satisfies the quasilinear elliptic equation*

$$-u_{tt} + 2u_x u_{xt} - (u_x^2 + m f'(m)) u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, T) \quad (3.1)$$

with  $m = f^{-1}(-u_t + u_x^2/2)$ .

We observe that (3.1) is a degenerate elliptic equation for  $u$ ; the uniform ellipticity being lost when  $m$  vanishes. The study of (3.1) is the starting point of the regularity theory developed in [25, 28, 31] under conditions ensuring a positive control from below on  $m$ . In that case, equation (3.1) turns out to be equivalent to the system (1.1), at least for classical solutions. As it is customary for quasilinear problems, a key role is played by gradient estimates, which are obtained through the maximum principle. We will recall this approach in Subsection 3.2. For this purpose, it is convenient to introduce the linear second order operator

$$Q(v) := -v_{tt} + 2u_x v_{xt} - (u_x^2 + m f'(m)) v_{xx}, \quad (3.2)$$

and accordingly, the linearized operator generated from (3.1):

$$L(v) = Q(v) + 2(u_{xt} - u_x u_{xx}) v_x - \left( \frac{m f''(m)}{f'(m)} + 1 \right) u_{xx} (-v_t + u_x v_x). \quad (3.3)$$

For the rest of this section, the solutions will be tacitly assumed to be sufficiently smooth to justify the computations below.

**Lemma 3.2.** *Let  $u$  be a classical solution to (3.1), and  $Q, L$  be defined by (3.2) and (3.3). Then we have*

$$L(u_t) = 0, \quad L(u_x) = 0$$

and the function  $w := f(m)$  satisfies

$$Q(w) - w_x^2 + m(m f''(m) + 2f'(m)) u_{xx}^2 = 0. \quad (3.4)$$

*Proof.* The equations satisfied by  $u_t, u_x$  are obtained by differentiation of (3.1). From  $L(u_x) = 0$  we also obtain, by the chain rule,

$$L\left(\frac{1}{2}u_x^2\right) = -(u_{xt} - u_x u_{xx})^2 - m f'(m) u_{xx}^2. \quad (3.5)$$

Hence,

$$L(f(m)) = L(-u_t + \frac{1}{2}u_x^2) = -(u_{xt} - u_x u_{xx})^2 - m f'(m) u_{xx}^2 = -f(m)_x^2 - m f'(m) u_{xx}^2. \quad (3.6)$$

Now, using (3.3), we also have

$$\begin{aligned} L(f(m)) &= Q(f(m)) - 2f(m)_x^2 - \left( \frac{m f''(m)}{f'(m)} + 1 \right) u_{xx} (-f(m)_t + u_x f(m)_x) \\ &= Q(f(m)) - 2f(m)_x^2 + \left( \frac{m f''(m)}{f'(m)} + 1 \right) u_{xx} (m f'(m) u_{xx}), \end{aligned} \quad (3.7)$$

where we used the equation of  $m$  in the last step. Putting together (3.6) and (3.7) yields (3.4).  $\square$

As our goal is to understand what happens when  $m$  vanishes or gets close to 0, we need to introduce more geometric quantities related to the first order system (1.1). The first of these is the family of optimal trajectories associated to the HJ equation satisfied by  $u$ : we define  $\gamma : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$  as the solution to

$$\gamma_t(x, t) = -u_x(\gamma(x, t), t) \quad \text{in } \mathbb{R} \times [0, T], \quad \gamma(x, 0) = x \quad \text{in } \mathbb{R}. \quad (3.8)$$

By the standard theory of Hamilton-Jacobi equations, it is known that  $\gamma(x, \cdot)$  is the minimizer of the problem

$$\inf_{\alpha \in H^1, \alpha(t)=x} \int_t^T \frac{1}{2} |\dot{\alpha}|^2 + f(m(\alpha, s)) \, ds + u(\alpha(T), T).$$

The fundamental properties of  $\gamma$  are given next.

**Lemma 3.3.** *Let  $(u, m)$  be a classical solution to (1.1). One has*

$$\gamma_{tt}(x, t) = f'(m(\gamma(x, t), t))m_x(\gamma(x, t), t), \quad \forall (x, t) \in \mathbb{R} \times (0, T) \quad (\text{Euler equation}), \quad (3.9)$$

and

$$\gamma_x(x, t) = \frac{m_0(x)}{m(\gamma(x, t), t)} \quad \forall (x, t) \in \mathbb{R} \times (0, T) \quad (\text{conservation of mass}), \quad (3.10)$$

or, equivalently,

$$\int_{\gamma(x_1, t)}^{\gamma(x_2, t)} m(x, t) dx = \int_{x_1}^{x_2} m_0(x) dx \quad \forall x_1, x_2 \in \mathbb{R}, t \in [0, T]. \quad (3.11)$$

Moreover  $\gamma$  solves the quasilinear elliptic equation

$$-\frac{m_0 f'(m_0/\gamma_x)}{(\gamma_x)^3} \gamma_{xx} - \gamma_{tt} = -\frac{(m_0)_x f'(m_0/\gamma_x)}{(\gamma_x)^2} \quad \text{in } \mathbb{R} \times (0, T). \quad (3.12)$$

Before proving the lemma, it will be convenient to associate to  $(u, m)$  the solution  $M$  to the transport equation

$$M_t - u_x M_x = 0 \quad \text{in } \mathbb{R} \times [0, T], \quad M(x, 0) = -\int_0^x m_0(y) dy \quad \text{in } \mathbb{R}. \quad (3.13)$$

Notice that (3.8) defines  $\gamma$  as the curve of characteristics associated to this transport equation.

**Lemma 3.4.** *Let  $(u, m)$  be a classical solution to (1.1). One has  $M_x = -m < 0$  and  $M$  satisfies the quasilinear elliptic equation*

$$-\frac{M_t^2}{M_x^2} M_{xx} + 2\frac{M_t}{M_x} M_{xt} - M_{tt} + M_x f'(-M_x) M_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, T). \quad (3.14)$$

*Proof.* Differentiating (3.13) in space, we see that  $\mu := -M_x$  satisfies  $\mu_t - (u_x \mu)_x = 0$ , with initial condition  $\mu(0, x) = m_0$ : this is exactly the equation satisfied by  $m$ , so that  $M_x = -\mu = -m < 0$ . On the other hand, by definition  $u_x = M_t/M_x$ . Taking the derivative w.r.t.  $x$  of the equation for  $u$  (3.1), we obtain (3.14).  $\square$

*Proof of Lemma 3.3.* As  $\gamma$  solves (3.8), we have

$$\gamma_{tt} = -u_{xx}(\gamma, t)(-u_x(\gamma, t)) - u_{xt}(\gamma, t) = f'(m(\gamma, t))m_x(\gamma, t),$$

where the second equality comes from the derivation in space of the HJ equation. This is (3.9). As  $M_x = -m$ , (3.10) comes from the derivative in space of the transport equality  $M(\gamma(x, t), t) = M_0(x)$ . Then (3.11) follows by the integration in space of (3.10).

By (3.9) and then (3.10),  $m_x(\gamma, t) = \gamma_{tt}/f'(m(\gamma, t)) = \gamma_{tt}/f'(m_0/\gamma_x)$ . On the other hand, taking the derivative in space of (3.10) gives (using again (3.10) and the expression above for  $m_x(\gamma, t)$ ):

$$\gamma_{xx} = \frac{(m_0)_x}{m(\gamma, t)} - \frac{m_0 m_x(\gamma, t) \gamma_x}{(m(\gamma, t))^2} = \frac{(m_0)_x \gamma_x}{m_0} - \frac{\gamma_x^3}{f'(m_0/\gamma_x) m_0} \gamma_{tt}.$$

This is (3.12).  $\square$

We finally compute the equation satisfied by  $v(x, t) = f(m(\gamma(x, t), t))$ . The map  $v$  is the r.h.s. of the HJ equation viewed from the lens of the optimal trajectories.

**Lemma 3.5.** *Let  $(u, m)$  be a classical solution to (1.1). The map  $v(x, t) = f(m(\gamma(x, t), t))$  satisfies the quasilinear elliptic equation in divergence form:*

$$-\left(\frac{v_x}{\gamma_x}\right)_x - \left(\frac{\gamma_x^2}{m_0 f'(m_0/\gamma_x)} v_t\right)_t = 0. \quad (3.15)$$

If  $f(m) = m^\theta$ , this equation simplifies into

$$(i) \quad -\left(\frac{v_x}{\gamma_x}\right)_x - \left(\frac{\gamma_x}{\theta v} v_t\right)_t = 0, \quad \text{or} \quad (ii) \quad -v_{tt} - \frac{\theta v}{\gamma_x^2} v_{xx} + v_x \frac{\theta v}{\gamma_x^3} \gamma_{xx} + \frac{\theta + 1}{\theta} v^{-1} v_t^2 = 0, \quad (3.16)$$

while if  $f(m) = \log(m)$ , it becomes

$$-\left(\frac{v_x}{\gamma_x}\right)_x - (\gamma_x v_t)_t = 0. \quad (3.17)$$

*Proof.* Observe that (3.9) may be written as

$$\gamma_{tt} = \frac{v_x}{\gamma_x}. \quad (3.18)$$

We now compute the time derivative of  $v(x, t) = f(m_0(x))/\gamma_x(x, t)$  to find

$$v_t = f'(m_0/\gamma_x) \left(-\frac{m_0 \gamma_{xt}}{\gamma_x^2}\right). \quad (3.19)$$

Thus, putting together (3.18) and (3.19) we get

$$\left(\frac{v_x}{\gamma_x}\right)_x = \gamma_{xtt} = -\left(\frac{v_t \gamma_x^2}{m_0 f'(m_0/\gamma_x)}\right)_t,$$

which gives (3.15), and thus (3.16)-(i) and (3.17). Finally, if  $f(m) = m^\theta$ , developing (3.16)-(i) and using (3.19)—which implies that  $\gamma_{xt} = -(\gamma_x v_t)/(\theta v)$ —leads to (3.16)-(ii).  $\square$

### 3.2 Displacement convexity estimates on $m$ , Lipschitz estimates on $u$ and existence result

Before studying the problem with compactly supported marginals, we will begin obtaining some estimates for the simpler periodic setting with strictly positive marginals. We are mostly interested in a priori estimates that are independent of  $\min m_0$  and  $\min m_T$ . By approximation with positive densities  $m_0^\varepsilon, m_T^\varepsilon$ , these estimates will hold for the case in which  $m_0$  and  $m_T$  are compactly supported.

Throughout the remainder of this section,  $R \geq 1$  will denote a fixed constant, and we will analyze the MFG system on the one-dimensional torus of length  $R$ , denoted by  $R\mathbb{T}$ . Functions defined on  $R\mathbb{T}$  are meant to be  $R$ -periodic functions on  $\mathbb{R}$ . We will consider the problem

$$\begin{cases} -u_t + \frac{1}{2} u_x^2 = f(m) & (x, t) \in R\mathbb{T} \times (0, T), \\ m_t - (m u_x)_x = 0 & (x, t) \in R\mathbb{T} \times (0, T), \end{cases} \quad (3.20)$$

complemented either with the initial-terminal conditions

$$m(x, 0) = m_0(x), \quad u(x, T) = g(m(x, T)), \quad x \in R\mathbb{T}, \quad (3.21)$$

or with the prescribed marginal conditions of the planning problem:

$$m(x, 0) = m_0(x), \quad m(x, T) = m_T(x), \quad x \in R\mathbb{T}. \quad (3.22)$$

The functions  $f, g : (0, \infty) \rightarrow \mathbb{R}$  are assumed to satisfy  $f', g' > 0$ , with  $f, g \in C^2(0, \infty)$  and

$$\limsup_{m \rightarrow 0^+} m f'(m) < \infty, \quad \limsup_{m \rightarrow 0^+} \frac{m |f''(m)|}{f'(m)} < \infty. \quad (3.23)$$

The initial (and terminal) data  $m_0, m_T : R\mathbb{T} \rightarrow (0, \infty)$  are understood to be  $C^1$  functions satisfying

$$\int_{R\mathbb{T}} m_0 = \int_{R\mathbb{T}} m_T, \quad m_0, m_T > 0. \quad (3.24)$$

Moreover, we also assume that

$$f(m_0), f(m_T) \in C^{1,\alpha}(R\mathbb{T}). \quad (3.25)$$

The fact that the system (3.20) has smooth solutions when the marginals are strictly positive and the data are sufficiently smooth is already known (see [28, Theorem 1.1]). In fact, we will see later (Theorem 3.11) that, under the present assumptions, (3.20)–(3.21) and (3.20)–(3.22) both admit a classical solution  $(u, m) \in C^{2,\alpha}(R\mathbb{T} \times (0, T)) \times C^{1,\alpha}(R\mathbb{T} \times (0, T))$ .

We begin by recalling the so-called displacement convexity formula (see [16, 27]), as well as an identity which will later be useful to obtain energy estimates on the density.

**Proposition 3.6.** *Assume that  $f \in C^1(0, \infty)$ , and let  $(u, m)$  be a classical solution to (3.20). Then we have*

$$m f'(m) u_{xx}^2 + f(m)_x^2 = (f(m)_t u_x)_x - (f(m)_x u_x)_t. \quad (3.26)$$

Moreover, if  $h : (0, \infty) \rightarrow \mathbb{R}$  is twice differentiable, then

$$\frac{d^2}{dt^2} \int_{R\mathbb{T}} h(m) = \int_{R\mathbb{T}} m h''(m) (m u_{xx}^2 + f'(m) m_x^2). \quad (3.27)$$

*Proof.* We start by multiplying the continuity equation by  $u_{xx}$ , which yields

$$m u_{xx}^2 + m_x u_x u_{xx} - m_t u_{xx} = 0.$$

As a result, differentiating the HJ equation for the term  $u_x u_{xx}$  we obtain

$$m u_{xx}^2 + m_x (u_{xt} + f'(m) m_x) - m_t u_{xx} = 0, \quad (3.28)$$

which, after multiplying by  $f'(m)$ , yields

$$m f'(m) u_{xx}^2 + f'(m)^2 m_x^2 = (f(m)_t u_x)_x - (f(m)_x u_x)_t.$$

This proves (3.26). We note that (3.27) is merely special case of [27, Proposition 3.1], but we give a proof for the reader's convenience. Multiplying both sides of (3.28) by  $m h''(m)$ , we obtain

$$m h''(m) (m u_{xx}^2 + f'(m) m_x^2) = h''(m) m m_t u_{xx} - h''(m) m_x u_{xt} = H(m)_t u_{xx} - H(m)_x u_{xt},$$

where  $H(m) = m h'(m) - h(m)$ . This may be rewritten as

$$m h''(m) (m u_{xx}^2 + f'(m) m_x^2) = (H(m)_t u_x)_x - (H(m)_x u_x)_t \quad (3.29)$$

Now, from the continuity equation,

$$h(m)_t = h(m)_x u_x + m h'(m) u_{xx} = (h(m) - m h'(m))_x u_x + (m h'(m) u_x)_x = -H(m)_x u_x + (m h'(m) u_x)_x.$$

Therefore,

$$(H(m)_x u_x)_t = -h(m)_{tt} + (m h'(m) u_x)_{tx}.$$

Substituting in (3.29), we obtain

$$\begin{aligned} m h''(m) (m u_{xx}^2 + f'(m) m_x^2) &= h(m)_{tt} + (H(m) u_x)_x - (m h'(m) u_x)_{tx} \\ &= h(m)_{tt} - (h(m)_t u_x)_x - (m h'(m) u_{xt})_x = h(m)_{tt} - (h'(m) (m u_x)_t)_x, \end{aligned}$$

and (3.27) then follows by integrating both sides of this equation in space.  $\square$

We now note that the density attains its extremum values at the extremal times.

**Corollary 3.7.** *Assume that  $f \in C^1(0, \infty)$ , and let  $(u, m)$  be a classical solution to (3.20). Then*

$$\begin{aligned} \|m\|_\infty &\leq \max(\|m(\cdot, 0)\|_\infty, \|m(\cdot, T)\|_\infty), \\ \|m^{-1}\|_\infty &\leq \max(\|m(\cdot, 0)^{-1}\|_\infty, \|m(\cdot, T)^{-1}\|_\infty). \end{aligned}$$

*Proof.* One first observes that, for any convex function  $h : (0, \infty) \rightarrow \mathbb{R}$ , it follows from (3.27) that  $\int_{R\mathbb{T}} h(m)$  is convex in time, which yields

$$\int_{R\mathbb{T}} h(m) \leq \max\left(\int_{R\mathbb{T}} h(m(\cdot, 0)), \int_{R\mathbb{T}} h(m(\cdot, T))\right).$$

The upper bounds on  $m$  and  $m^{-1}$ , then follow by taking  $h(m) = m^p$  and letting  $p \rightarrow \pm\infty$ , respectively.  $\square$

*Remark 3.8.* These a priori estimates were proved in [16]. They could have also been derived without the displacement convexity formula, as an immediate consequence of Lemma 3.12 below.

We will now review the Lipschitz estimates which can be established on  $u$  following the approach suggested by P.-L. Lions in [25], and developed later in more generality in [28, 31]. The following  $L^\infty$  bounds on  $u$  and  $m(\cdot, T)$  are well-known consequences of the maximum principle and the Hopf-Lax formula (see [27, Propositions 4.1 and 4.2]).

**Proposition 3.9.** *Assume that  $f, g \in C^1(0, \infty)$ ,  $f' > 0, g' \geq 0$ , and  $m_0, m_T \in C(\mathbb{T})$ . If  $(u, m)$  is a classical solution to (3.20)–(3.21), then we have, for  $(x, t) \in R\mathbb{T} \times [0, T]$ ,*

$$\min m_0 \leq m(x, T) \leq \max m_0,$$

$$f(\min m_0)(T - t) + g(\min m_0) \leq u(x, t) \leq f(\max m_0)(T - t) + g(\max m_0).$$

Moreover, there exists a constant  $C = C(\|f(m_0)\|_\infty, \|f(m_T)\|_\infty)$  such that, if  $(u, m)$  solves (3.20)–(3.22), then

$$\text{osc}(u) \leq C(T + R^2 T^{-1}).$$

We may now obtain a gradient estimate which, crucially, is independent of  $\min m$ . Hereafter, we denote by  $Du$  the vector formed by the space and time first derivatives, that is,

$$Du := (u_x, u_t).$$

Moreover, we denote  $\kappa_0 > 0$  to be a constant such that

$$mf'(m), \frac{m|f''(m)|}{f'(m)} \leq \kappa_0 \quad \forall m \in (0, 2 \max(\|m_0\|_\infty, \|m_T\|_\infty)). \quad (3.30)$$

Such a constant exists by virtue of (3.23). It is understood that the term with  $\|m_T\|_\infty$  is treated as zero, in case of conditions (3.21).

**Proposition 3.10.** *Assume that (3.23), (3.24), and (3.25) hold true, and let  $(u, m)$  be a classical solution to (3.20)–(3.21) or (3.20)–(3.22). There exists a constant  $C$  such that*

$$\|Du\|_\infty \leq C.$$

where

$$C = \begin{cases} C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|g(m_0)\|_\infty) & \text{if } (u, m) \text{ solves (3.20)–(3.21) ,} \\ C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}) & \text{if } (u, m) \text{ solves (3.20)–(3.22),} \end{cases}$$

*Proof.* By rescaling, it is enough to consider the case  $R = 1$ . By approximation<sup>1</sup>, we may also assume that  $u \in C^3(\mathbb{T} \times [0, T])$ . We begin by noting that, using either Corollary 3.7 or Proposition 3.9, we have that  $\|m\|_\infty$  is controlled by  $\|m_0\|_\infty$ , and by  $\|m_T\|_\infty$  in case of (3.22). Hence we will use (3.30) for  $m(x, t)$  below. We let  $v(x, t) = \frac{1}{2}u_x^2 + \frac{1}{2T}\tilde{u}^2$ , where  $\tilde{u}$  is defined as

$$\tilde{u} = u - \min u + T - \frac{(\text{osc}(u) + 2T)}{T}(T - t),$$

so that  $\tilde{u}(x, 0) = u - \max u - T \leq -T$ ,  $\tilde{u}(x, T) = u - \min u + T \geq T$ , and  $\|\tilde{u}\|_\infty \leq (T + \text{osc}(u))$ . Let  $(x_0, t_0)$  be a point in which  $v$  achieves its maximum value. We note that, by the HJ equation, it is enough to bound  $v$ . If  $t_0 = 0$ , then we have

$$-v_t + u_x v_x = u_x f(m_0)_x + \frac{1}{T}\tilde{u}(f(m_0) + \frac{1}{2}u_x^2 - \frac{1}{T}(\text{osc}(u) + 2T)).$$

Now, we have  $v_x = 0$  and  $v_t \leq 0$ , and recall that  $\tilde{u}(x, 0) \leq -T$ ; so either  $f(m_0) + \frac{1}{2}u_x^2 - \frac{1}{T}(\text{osc}(u) + 2T) \leq 0$ , in which case there is nothing to prove, or we deduce

$$0 \leq u_x f(m_0)_x - (f(m_0) + \frac{1}{2}u_x^2 - \frac{1}{T}(\text{osc}(u) + 2T)).$$

This means

$$\frac{1}{2}u_x^2 \leq u_x f(m_0)_x - f(m_0) + \frac{1}{T}(\text{osc}(u) + 2T),$$

and yields the required estimate. The case in which  $t_0 = T$  is similar, so we assume now that  $0 < t_0 < T$ . We begin by noting that, since  $v_x = 0$ ,

$$u_x u_{xx} = -\frac{1}{T}u_x \tilde{u},$$

and, thus,

$$|u_{xx}| \leq \frac{C}{T}(T + \text{osc}(u)). \quad (3.31)$$

Now, recall the definitions of the linear operators  $Q, L$  in (3.2), (3.3). Using (3.5) we have

$$L\left(\frac{1}{2}u_x^2\right) \leq -(u_{xt} - u_x u_{xx})^2. \quad (3.32)$$

On the other hand, since  $Q(\tilde{u}) = 0$ , we have

$$Q\left(\frac{1}{2}\tilde{u}^2\right) = -(-\tilde{u}_t + u_x^2)^2 - m f'(m) \tilde{u}_x^2 \leq -(-\tilde{u}_t + u_x^2)^2 = -\left(f(m) + \frac{1}{2}u_x^2 - \frac{1}{T}(\text{osc}(u) + 2T)\right)^2.$$

Observe that, by Corollary 3.7 and Proposition 3.9,  $|f(m)|$  is bounded. Therefore, if  $f(m) + \frac{1}{4}u_x^2 \leq \frac{1}{T}(\text{osc}(u) + 2T)$ , there is nothing to prove. Else,

$$Q\left(\frac{1}{2}\tilde{u}^2\right) \leq -\frac{1}{32}u_x^4 - \frac{1}{2}(-\tilde{u}_t + u_x^2)^2,$$

and, thus, by definition of  $L$ , we obtain, using (3.30) and (3.31),

$$\begin{aligned} L\left(\frac{1}{2T}\tilde{u}^2\right) &\leq -\frac{1}{32T}u_x^4 - \frac{1}{2T}(-\tilde{u}_t + u_x \tilde{u}_x)^2 + \frac{2}{T}(u_{xt} - u_x u_{xx})u_x \tilde{u} - \frac{1}{T}\left(\frac{m f''(m)}{f'(m)} + 1\right)u_{xx}(-\tilde{u}_t + u_x \tilde{u}_x)\tilde{u} \\ &\leq -\frac{1}{32T}u_x^4 + \frac{1}{2T}(\kappa_0 + 1)^2(u_{xx})^2\tilde{u}^2 + \frac{4}{T^2}u_x^2\tilde{u}^2 + (u_{xt} - u_x u_{xx})^2 \\ &\leq -\frac{1}{64T}u_x^4 + \frac{C}{T^3}(T + \text{osc}(u))^4 + (u_{xt} - u_x u_{xx})^2. \end{aligned}$$

<sup>1</sup>See the proof of Theorem 3.11 for the details of such an approximation argument.

Putting together the above inequality with (3.32) we get

$$L(v) \leq -\frac{1}{64T}u_x^4 + \frac{C}{T^3}(T + \text{osc}(u))^4.$$

Now, since  $(x_0, t_0)$  is a maximum point for  $v$ , we have  $L(v) \geq 0$ , which yields

$$u_x^4 \leq \frac{C}{T^2}(T + \text{osc}(u))^4.$$

Recalling that  $\text{osc}(u)$  is estimated from Proposition 3.9, we conclude the estimate.  $\square$

We now show that, under the present assumptions, (3.20)–(3.21) and (3.20)–(3.22) may be solved classically.

**Theorem 3.11.** *Assume that conditions (3.23), (3.24), and (3.25) hold true. Then the systems (3.20)–(3.21) and (3.20)–(3.22) have a classical solution  $(u, m) \in C^{2,\alpha}(R\mathbb{T} \times [0, T]) \times C^{1,\alpha}(R\mathbb{T} \times [0, T])$ , with  $m$  being unique. In the case of (3.21),  $u$  is unique, and in the case of (3.22),  $u$  is unique up to a constant.*

*Proof.* The uniqueness is a standard result, proved through duality. We will do the proof of existence for (3.20)–(3.22); the alternative case of (3.20)–(3.21) requires only minor modifications. We may approximate  $f$  and the marginals with  $f^\epsilon \in C^4(0, \infty)$ ,  $m_0^\epsilon, m_T^\epsilon \in C^4(R\mathbb{T})$ , such that  $f^\epsilon(m_0^\epsilon), f^\epsilon(m_T^\epsilon)$  is uniformly bounded in  $C^{1,\alpha}(R\mathbb{T})$ . Indeed, we may simply take

$$f^\epsilon = f * \eta_\epsilon, \quad m_0^\epsilon = (f^\epsilon)^{-1}(f^\epsilon(m_0) * \eta_\epsilon + c_{0,\epsilon}), \quad m_T^\epsilon = (f^\epsilon)^{-1}(f^\epsilon(m_T) * \eta_\epsilon + c_{T,\epsilon}), \quad (3.33)$$

where  $\eta^\epsilon$  is the standard mollifier, and the non-negative constants  $c_{0,\epsilon}, c_{T,\epsilon}$  are adequately chosen such that  $c_{0,\epsilon} \cdot c_{T,\epsilon} = 0$ ,  $\int_{R\mathbb{T}} m_0^\epsilon = \int_{R\mathbb{T}} m_T^\epsilon$  and  $\lim_{\epsilon \rightarrow 0} c_{0,\epsilon} = \lim_{\epsilon \rightarrow 0} c_{T,\epsilon} = 0$ . With these regularized data, (3.20)–(3.22) has a unique classical solution  $(u^\epsilon, m^\epsilon) \in C^3(R\mathbb{T} \times [0, T]) \times C^2(R\mathbb{T} \times [0, T])$  satisfying  $\int_{R\mathbb{T}} \int_0^T u^\epsilon = 0$  (see [27, Thm 1.1]). Moreover, in view of Propositions 3.9 and 3.10, the solution is bounded in  $C^1 \times C^0$ , uniformly in  $\epsilon$ . The result will then follow by letting  $\epsilon \rightarrow 0$  and applying a version of R. Fiorenza's convergence theorem for elliptic oblique problems (see, for instance, [17, Lem. 17.29], [24, Lem. 2, Cor. 1], [14, 15]). For completeness, we sketch the details for this argument, which amounts to a proof of Fiorenza's result. From Lemma 3.1, the functions  $u^\epsilon$  solve the oblique quasilinear elliptic problem

$$\begin{cases} -\text{Tr}(A(u_x^\epsilon, u_t^\epsilon)D^2u^\epsilon) = 0 & (x, t) \in R\mathbb{T} \times [0, T], \\ -u_t^\epsilon(x, 0) + \frac{1}{2}(u_x^\epsilon)^2(x, 0) = f(m_0^\epsilon(x)) & x \in R\mathbb{T}, \\ -u_t^\epsilon(x, T) + \frac{1}{2}(u_x^\epsilon)^2(x, T) = f(m_T^\epsilon(x)) & x \in R\mathbb{T}, \end{cases}$$

where

$$A(u_x^\epsilon, u_t^\epsilon) = \begin{pmatrix} (u_x^\epsilon)^2 + m^\epsilon f'(m^\epsilon) & -u_x^\epsilon \\ -u_x^\epsilon & 1 \end{pmatrix} = \begin{pmatrix} (u_x^\epsilon)^2 + f^{-1}(-u_t^\epsilon + \frac{1}{2}(u_x^\epsilon)^2) f'(f^{-1}(-u_t^\epsilon + \frac{1}{2}(u_x^\epsilon)^2)) & -u_x^\epsilon \\ -u_x^\epsilon & 1 \end{pmatrix}.$$

Since  $\|u^\epsilon\|_{C^1}$ ,  $\|m^\epsilon\|_\infty \| (m^\epsilon)^{-1} \|_\infty$  are uniformly bounded, this equation is uniformly elliptic, uniformly in  $\epsilon$ . Thus, as a result of Lieberman's  $C^{1,s}$  estimate for oblique problems (see [23, Lem. 2.3]), there exists  $0 < s < 1$  and a constant  $C > 0$  such that

$$\|u^\epsilon\|_{C^{1,s}} \leq C.$$

Therefore if  $0 < \epsilon' < 1$ , the difference  $v = u^\epsilon - u^{\epsilon'}$  solves

$$\begin{cases} -\text{Tr}(A^\epsilon D^2v) = \text{Tr}((A^\epsilon - A^{\epsilon'})D^2u^{\epsilon'}) & (x, t) \in R\mathbb{T} \times [0, T], \\ -v_t(x, 0) + \frac{1}{2}(u_x^\epsilon(x, 0) + u_x^{\epsilon'}(x, 0))v_x(x, 0) = f(m_0^\epsilon(x)) - f(m_0^{\epsilon'}(x)) & x \in R\mathbb{T}, \\ -v_t(x, T) + \frac{1}{2}(u_x^\epsilon(x, T) + u_x^{\epsilon'}(x, T))v_x(x, 0) = f(m_T^\epsilon(x)) - f(m_T^{\epsilon'}(x)) & x \in R\mathbb{T}. \end{cases}$$

Let  $\alpha' = \min(\alpha, s)$ . The standard Schauder estimates for linear oblique problems (see, for instance, [24, Lem. 1]) imply that

$$\begin{aligned} \|v\|_{C^{2,\alpha'}} &\leq C(\|v\|_\infty + \|u^{\epsilon'}\|_{C^{2,\alpha'}}\|v\|_{C^1} + \|u^{\epsilon'}\|_{C^2}\|v^\epsilon\|_{C^{1,\alpha'}} + \|f(m_0^\epsilon) - f(m_0^{\epsilon'})\|_{C^{1,\alpha'}} \\ &\quad + \|f(m_T^\epsilon) - f(m_T^{\epsilon'})\|_{C^{1,\alpha'}} + \|v\|_{C^1}\|u^\epsilon + u^{\epsilon'}\|_{C^{2,\alpha'}}). \end{aligned}$$

Recalling the interpolation inequality for Hölder spaces,  $\|\cdot\|_{C^2} \leq \delta\|\cdot\|_{C^{2,\alpha}} + C_\delta\|\cdot\|_{C^0}$ , we deduce that

$$\|u^\varepsilon - u^{\varepsilon'}\|_{C^{2,\alpha'}} \leq C(o(1)(1 + \|u^\varepsilon\|_{C^{2,\alpha'}} + \|u^{\varepsilon'}\|_{C^{2,\alpha'}}) + \|u^{\varepsilon'}\|_{C^2} + 1) \leq \frac{1}{4}(\|u^\varepsilon\|_{C^{2,\alpha'}} + \|u^{\varepsilon'}\|_{C^{2,\alpha'}}) + C$$

as  $\varepsilon, \varepsilon' \rightarrow 0$ . Fixing a small  $\varepsilon'$ , and letting  $\varepsilon \rightarrow 0$ , we see that  $\|u^\varepsilon\|_{C^{2,\alpha'}}$  must be bounded. Now repeating the same argument but taking  $\alpha' = \alpha$ , we see that  $\|u^\varepsilon\|_{C^{2,\alpha}}$  must be bounded as well, which means  $u^\varepsilon$  converges to a solution  $u \in C^{2,\alpha}(R\mathbb{T} \times [0, T])$ . In turn, the  $C^{1,\alpha}$  regularity of  $m$  follows from the HJ equation and the  $C^2$  regularity of  $f$ .  $\square$

### 3.3 Continuity of the density

In this section, we will prove that the function  $f(m)$  satisfies a uniform modulus of continuity, independent of  $\min m$ . This estimate is a crucial step in treating the setting of compactly supported solutions, because it will allow us to prove that the density is globally continuous, despite the lack of a positive lower bound.

**Lemma 3.12.** *Under the assumptions of Theorem 3.11, let  $(u, m)$  be a classical solution to (3.20). Then the function  $v = f(m)$  satisfies the maximum principle and the minimum principle on each compact subset of  $R\mathbb{T} \times [0, T]$ .*

*Proof.* By approximation, we may assume that  $u$  and  $m$  are smooth. Hence  $v = f(m)$  satisfies (3.4). Now, from the continuity equation, we have

$$mf'(m)u_{xx} = f(m)_t - f(m)_x u_x = v_t - v_x u_x,$$

and, therefore, substituting in (3.4) yields

$$Q(v) - v_x^2 + m(mf''(m) + 2f'(m))(mf'(m))^{-2}(v_t - v_x u_x)^2 = 0. \quad (3.34)$$

We notice that  $Q$  is a purely second order linear elliptic operator, and the remaining terms of (3.34) can be written as first-order terms in  $v$ . Thus,  $v$  satisfies an elliptic equation with no zero-order terms. This implies that  $v$  satisfies the maximum and the minimum principle on every compact subset of  $R\mathbb{T} \times [0, T]$ , as wanted.  $\square$

We now compute an energy estimate for the function  $v$ . Recall that  $\kappa_0$  is given by (3.30).

**Proposition 3.13.** *Under the assumptions of Theorem 3.11, let  $(u, m)$  be a classical solution to (3.20)–(3.21) or (3.20)–(3.22), and let  $v = f(m)$ . Then there exists  $C$  such that*

$$\int_0^T \int_{R\mathbb{T}} |Dv|^2 \leq C$$

where

$$C = \begin{cases} C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|g(m_0)\|_\infty) & \text{if } (u, m) \text{ solves (3.20)–(3.21) ,} \\ C(\kappa_0, R, T, T^{-1}, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}) & \text{if } (u, m) \text{ solves (3.20)–(3.22),} \end{cases}$$

*Proof.* Integrating (3.26) in space-time yields

$$\int_0^T \int_{R\mathbb{T}} mf'(m)u_{xx}^2 + f(m)_x^2 = \int_{R\mathbb{T}} f(m)_x u_x(0) - f(m)_x u_x(T) \leq C - \int_{R\mathbb{T}} f(m)_x u_x(T),$$

where we used the gradient bound from Proposition 3.10. If  $(u, m)$  solves (3.22), then we use the bound on  $u_x$  and  $f(m_T)_x$ . If  $(u, m)$  solves (3.20), we have  $-f(m)_x u_x(T) = -f'(m)g'(m)^{-1}u_x(T)^2 < 0$ ; so, in any case, we obtain

$$\int_0^T \int_{R\mathbb{T}} mf'(m)u_{xx}^2 + f(m)_x^2 \leq C. \quad (3.35)$$

The bound on  $f(m)_t$  simply follows from the continuity equation:

$$f(m)_t^2 = (f(m)_x u_x + mf'(m)u_{xx})^2 \leq 2(f(m)_x^2 u_x^2 + (mf'(m))^2 u_{xx}^2) \leq C(f(m)_x^2 + mf'(m)u_{xx}^2),$$



where the bound on  $u_x$  and (3.30) were used in the last inequality. Integrating and using (3.35), we get the  $L^2$  bound for  $f(m)_t$ . Finally, we have proved that

$$\int_0^T \int_{R^{\mathbb{T}}} f(m)_t^2 + f(m)_x^2 \leq C,$$

where  $C$  depends on the same quantities as the bound of  $u_x$  in Proposition 3.10.  $\square$

The interior continuity now follows from a classical computation, originally attributed to H. Lebesgue [22], which implies that a sufficient condition for a  $W^{1,2}$  function in two variables to be continuous, is for it to satisfy the maximum and minimum principle.

**Proposition 3.14** (Interior modulus of continuity). *Under the assumptions of Theorem 3.11, let  $(u, m)$  be a classical solution to (3.20). Then the function  $v = f(m)$  has the following logarithmic modulus of continuity, valid for all concentric balls  $B_{r_1}$  and  $B_{r_2}$ ,  $r_1 \leq r_2$  contained in  $Q_T$ .*

$$(\text{osc}_{B_{r_1}}(v))^2 \log\left(\frac{r_2}{r_1}\right) \leq \pi \iint_{B_{r_2}} |Dv|^2.$$

*Proof.* Let  $r \in [r_1, r_2]$ , and let  $\theta_1, \theta_2 \in [0, 2\pi]$ . Then, using polar coordinates with origin at the center of the balls  $B_{r_i}$ ,

$$v(r, \theta_2) - v(r, \theta_1) = \int_{\theta_1}^{\theta_2} \frac{\partial v}{\partial \theta} d\theta.$$

Thus, integrating over a half circle and using the Cauchy-Schwarz inequality,

$$\text{osc}_{\partial B_r}(v) \leq \sqrt{\pi} \sqrt{\int_0^{2\pi} \left(\frac{\partial v}{\partial \theta}\right)^2 d\theta}.$$

Now, in view of Lemma 3.12,  $v$  satisfies the maximum and minimum principle, so

$$\text{osc}_{B_{r_1}}(v) \leq \text{osc}_{\partial B_r}(v).$$

and, thus,

$$(\text{osc}_{B_{r_1}}(v))^2 \leq \pi \int_0^{2\pi} \left(\frac{\partial v}{\partial \theta}\right)^2 d\theta.$$

On the other hand, we have

$$|Dv|^2 = \left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial v}{\partial \theta}\right)^2 \geq \frac{1}{r^2} \left(\frac{\partial v}{\partial \theta}\right)^2,$$

which then implies that

$$\frac{1}{r} (\text{osc}_{B_{r_1}}(v))^2 \leq \pi \int_0^{2\pi} r |Dv|^2 d\theta.$$

Integrating in  $r$  from  $r_1$  to  $r_2$  yields the result.  $\square$

A slight variant of this argument, by integrating over semi-disks instead, yields the continuity estimate up to the boundary.

**Proposition 3.15** (Boundary modulus of continuity). *Under the assumptions of Theorem 3.11, let  $x_0 \in \mathbb{T}$  and  $D_{r_1}$  and  $D_{r_2}$  be upper semi-disks of radii  $r_1 < r_2$ , centered at  $(x_0, 0)$ . Then we have*

$$(\text{osc}_{D_{r_1}}(v))^2 \log\left(\frac{r_2}{r_1}\right) \leq 2\pi \int \int_{D_{r_2}} |Dv|^2 + 4(r_2^2 - r_1^2) \|v_x(0)\|_\infty^2.$$

Similarly, if  $D_{r_1}$  and  $D_{r_2}$  are lower semi-disks centered at  $(x_0, T)$ , one has

$$(\text{osc}_{D_{r_1}}(v))^2 \log\left(\frac{r_2}{r_1}\right) \leq 2\pi \int \int_{D_{r_2}} |Dv|^2 + 4(r_2^2 - r_1^2) \|v_x(T)\|_\infty^2.$$

## 4 Finite speed of propagation and compactly supported solutions

This section, which is the main core of the paper, is dedicated to the study of solutions to problems (MFG) and (MFGP) with *compactly supported density*.

We will first obtain a preliminary existence (and uniqueness) result, under fairly general conditions on the coupling functions  $f, g$ , assuming the initial density  $m_0$  (and possibly the terminal density  $m_T$ ) to be compactly supported. By a suitable choice of approximation of  $m_0, m_T$ , we will show the property of finite speed of propagation of the support, and the existence of a unique continuous solution with compactly supported density  $m$ . This will allow us to build a rigorous framework for the study of the free boundary  $\partial\{(x, t) : m(x, t) > 0\}$ , carried out in Subsection 4.2, and will be tightly connected to the analysis of the flow of optimal trajectories associated to the optimization problem. The study of the regularity and geometric properties of the free boundary, as well as of its spreading speed, will be analyzed for the model case of  $f(m) = m^\theta$ ,  $\theta > 0$ . We conclude by establishing space-time Hölder regularity of the pair  $(m, Du)$  up to the free boundary.

### 4.1 Well-posedness results

Throughout this subsection, we will assume that

$$f, g \in C^2(0, \infty) \cap C[0, \infty), f' > 0, g' \geq 0, \text{ and (3.23) holds.} \quad (4.1)$$

For simplicity, and with no loss of generality, we will also normalize  $f$  so that

$$f(0) = 0. \quad (4.2)$$

As for the initial density (and possibly terminal density, in case of (MFGP)), we assume that  $m_0, m_T \in C_c(\mathbb{R})$  are compactly supported, non-negative functions, such that

$$\int_{\mathbb{R}} m_0 = \int_{\mathbb{R}} m_T = 1, \quad (4.3)$$

and

$$m_0, m_T \text{ vanish, respectively, outside the intervals } [a_0, b_0], [a_1, b_1]. \quad (4.4)$$

Rather than (3.25), we will instead assume that, for some  $\alpha \in (0, 1)$ ,

$$f(m_0) \in C^{1,\alpha}(\{m_0 > 0\}), f(m_T) \in C^{1,\alpha}(\{m_T > 0\}). \quad (4.5)$$

We remark that, in particular, this assumption implies that  $f(m_0), f(m_T) \in W^{1,\infty}(\mathbb{R})$ , but it allows for the possibility that  $f(m_0)_x$  or  $f(m_T)_x$  might be discontinuous at the boundary of the support, when considered as functions in  $\mathbb{R}$ .

Our first goal will be to show the well-posedness of systems (MFG) and (MFGP). Since we know, from the model case of Section 2 that, in general, the solution is not classical, we must work with a preliminary notion of generalized solution. We recall that  $C_b(\mathbb{R} \times [0, T])$  denotes the space of bounded continuous functions.

**Definition 4.1.** We say that  $(u, m) \in W^{1,\infty}(\mathbb{R} \times (0, T)) \times C_b(\mathbb{R} \times [0, T])$  is a solution to (MFG) (respectively, (MFGP)), if

- (i)  $u$  is a viscosity solution to the HJ equation

$$-u_t + \frac{1}{2}u_x^2 = f(m) \quad (x, t) \in \mathbb{R} \times (0, T),$$

- (ii)  $m$  satisfies the continuity equation

$$m_t - (mu_x)_x = 0 \quad (x, t) \in \mathbb{R} \times (0, T),$$

in the distributional sense, with  $m(\cdot, 0) = m_0$ .

(iii) We have  $u(\cdot, T) = g(m(\cdot, T))$  (respectively,  $m(\cdot, T) = m_T$ ).

*Remark 4.2.* Different notions of weak solutions could have been used, alternatively to Definition 4.1. In particular, distributional subsolutions of the HJ equation have been frequently used for MFG problems with a local coupling, both in case of final pay-off and in case of planning conditions (see e.g. [8], [9], [18], [30], and the survey [10]). That approach is tightly related to the concept of relaxed minima of variational problems, and avoids, for instance, any requirement of continuity and boundedness of  $m$  and  $u_x$ . Of course a similar approach would also apply to the present problems. However, since our primary goal in this article is the analysis of the free boundary for compactly supported solutions, it seems more natural to work from the beginning with the stronger notion of continuous solutions. It is also natural, in that context, to make use of the standard framework of viscosity solutions for HJ equations.

Our goal in this subsection will be to prove the following well-posedness result. In what follows,  $C_c(\mathbb{R})$  denotes the space of continuous, compactly supported functions on  $\mathbb{R}$ . Notice that, in particular, the result shows that the unique solution is such that  $m$  has compact support.

**Theorem 4.3.** *Assume that  $f, g$  satisfy (4.1),  $m_0, m_T \in C_c(\mathbb{R})$  satisfy (4.3)–(4.5), and  $\kappa_0$  is as in (3.30). Then the following holds.*

1. *There exists a unique solution  $(u, m)$  to (MFG). Moreover,  $(u, f(m)) \in C_{\text{loc}}^{2,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\}) \times C_{\text{loc}}^{1,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\})$ . There exists a constant*

$$C = C(T, T^{-1}, \kappa_0, \|f(m_0)\|_{W^{1,\infty}}, |\text{supp}(m_0)|, \|g(m_0)\|_{\infty})$$

such that

$$\|u\|_{\infty} \leq \|f(m_0)\|_{\infty} T + \|g(m_0)\|_{\infty}; \quad \|m\|_{\infty} \leq \|m_0\|_{\infty}; \quad \text{supp}(m) \subset [-C, C] \times [0, T],$$

$$\|Du\|_{L^{\infty}} \leq C, \tag{4.6}$$

$$\int_0^T \int_{\mathbb{R}} |D(f(m))|^2 \leq C, \tag{4.7}$$

and, for  $(x, t), (\bar{x}, \bar{t}) \in \mathbb{R} \times [0, T]$ ,

$$|f(m(x, t)) - f(m(\bar{x}, \bar{t}))| \leq \frac{C}{\sqrt{\log(|x - \bar{x}|^2 + |t - \bar{t}|^2)}}. \tag{4.8}$$

Finally, if  $g' > 0$  or  $g \equiv 0$ , then  $(u, f(m)) \in C_{\text{loc}}^{2,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\}) \times C_{\text{loc}}^{1,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\})$ .

2. *There exists a solution  $(u, m)$  to (MFGP). The function  $m$  is unique,  $u$  is unique up to a constant on each connected component of  $\{m > 0\}$ ,  $(u, f(m)) \in C_{\text{loc}}^{2,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\}) \times C_{\text{loc}}^{1,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\})$ . Moreover, there exist constants  $K, C > 0$ , with*

$$K = K(T, T^{-1}, \kappa_0, \|f(m_0)\|_{\infty}, \|f(m_T)\|_{\infty}, |\text{supp}(m_0)|, |\text{supp}(m_T)|, \text{dist}(\text{supp}(m_0), \text{supp}(m_T))),$$

$$C = C(K, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}),$$

such that

$$\text{osc}(u) \leq K; \quad \|m\|_{\infty} \leq \max(\|m_0\|_{\infty}, \|m_T\|_{\infty}); \quad \text{supp}(m) \subset [-C, C] \times [0, T],$$

and (4.6), (4.7), (4.8) hold.

*Remark 4.4.* We note that, due to lack of uniqueness for  $u$  in  $\{m = 0\}$  for the case of (MFGP), the explicit a priori estimate (4.6) for the solutions is limited to the support of  $m$ . However, the proof of Theorem 4.3 will show that there exists a solution to (MFGP) satisfying the global estimate

$$\|u\|_{W^{1,\infty}(\mathbb{R})} \leq C,$$

where  $C$  depends on the data as described above.

#### 4.1.1 From periodic to Neumann boundary conditions

As anticipated in Section 3, we intend to use the estimates for the periodic setting in order to build a solution to (MFG) and (MFGP). However, it will be convenient to switch from periodic to Neumann boundary conditions. Indeed, we will see later that this simplifies the analysis of the optimal trajectories, since they do not “wrap around” the domain as they do in the periodic case. The following result shows that we may switch to this point of view while preserving all of the estimates for the periodic setting.

**Proposition 4.5.** *Assume that  $f, g$  satisfy (4.1), and let  $0 < \alpha < 1$ . Assume that the positive functions  $m_0^\varepsilon, m_T^\varepsilon \in C(\mathbb{R})$  are such that  $m_0^\varepsilon, m_T^\varepsilon \equiv \varepsilon$  outside of the interval  $[-r, r]$ ,  $\int_{-r}^r m_0^\varepsilon = \int_{-r}^r m_T^\varepsilon$ , and  $f(m_0^\varepsilon), f(m_T^\varepsilon) \in C^{1,\alpha}(\mathbb{R})$ . Assume that  $g' > 0$ , and that (3.23) holds, and let  $R > r$ . Consider the system*

$$\begin{cases} -u_t^\varepsilon + \frac{1}{2}(u_x^\varepsilon)^2 = f(m^\varepsilon) & (x, t) \in [-R, R] \times (0, T) \\ m_t^\varepsilon - (m^\varepsilon u_x^\varepsilon)_x = 0 & (x, t) \in [-R, R] \times (0, T) \\ m^\varepsilon(x, 0) = m_0^\varepsilon(x), & x \in [-R, R] \\ u_x^\varepsilon(-R, t) = u_x^\varepsilon(R, t) = 0 & t \in [0, T], \end{cases} \quad (4.9)$$

where either  $u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T))$  or  $m^\varepsilon(\cdot, T) = m_T^\varepsilon$ .

- (i) There exists a unique solution  $(u^\varepsilon, m^\varepsilon) \in C^{2,\alpha}([-R, R] \times [0, T]) \times C^{1,\alpha}([-R, R] \times [0, T])$  to (4.9) satisfying  $u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T))$ . Moreover, there exists a constant  $C = C(R, T, T^{-1}, \kappa_0, \|f(m_0^\varepsilon)\|_{W^{1,\infty}}, \|g(m_0^\varepsilon)\|_\infty)$ , such that

$$\|u^\varepsilon\|_\infty \leq \|f(m_0^\varepsilon)\|_\infty T + \|g(m_0^\varepsilon)\|_\infty; \quad \min m_0^\varepsilon \leq m^\varepsilon \leq \max m_0^\varepsilon, \quad (4.10)$$

$$\|Du^\varepsilon\|_\infty \leq C; \quad \int_0^T \int_{-R}^R |D(f(m^\varepsilon))|^2 \leq C, \quad (4.11)$$

and, for each  $(x, t), (\bar{x}, \bar{t}) \in [-R, R] \times [0, T]$ ,

$$|f(m^\varepsilon(x, t)) - f(m^\varepsilon(\bar{x}, \bar{t}))| \leq C \left( \frac{\int_0^T \int_{-R}^R |D(f(m^\varepsilon))|^2}{\log(|x - \bar{x}|^2 + |t - \bar{t}|^2)} \right)^{\frac{1}{2}}. \quad (4.12)$$

- (ii) Up to an additive constant for  $u^\varepsilon$ , there exists a unique solution  $(u^\varepsilon, m^\varepsilon) \in C^{2,\alpha}([-R, R] \times [0, T]) \times C^{1,\alpha}([-R, R] \times [0, T])$  to (4.9) satisfying  $m^\varepsilon(\cdot, T) = m_T^\varepsilon$ . Moreover, there exist constants  $K, C > 0$ , with

$$K = K(R, T, T^{-1}, \kappa_0, \|f(m_0^\varepsilon)\|_\infty, \|f(m_T^\varepsilon)\|_\infty), \quad C = C(K, \|f(m_0^\varepsilon)\|_{W^{1,\infty}}, \|f(m_T^\varepsilon)\|_{W^{1,\infty}})$$

such that

$$\text{osc}(u^\varepsilon) \leq K, \quad \min(\min m_0^\varepsilon, \min m_T^\varepsilon) \leq m^\varepsilon \leq \max(\max m_0^\varepsilon, \max m_T^\varepsilon), \quad (4.13)$$

and (4.11), (4.12) hold.

*Proof.* For concreteness, we focus on the planning case,  $m^\varepsilon(\cdot, T) = m_T^\varepsilon$ . For each  $x \in [0, 2R]$ , define  $\tilde{m}_0(x) = m_0^\varepsilon(x - R)$ , and  $\tilde{m}_T(x) = m_T^\varepsilon(x - R)$ . Then these functions can be naturally extended to even, periodic functions  $\tilde{m}_0, \tilde{m}_T \in 4RT$ . With these marginals, the solution  $(\tilde{u}, \tilde{m})$  to (3.20)–(3.22), given by Theorem 3.11, is even, as well as symmetric with respect to  $x = \pm 2R$ . In particular, we have

$$\tilde{u}_x(0, t) = \tilde{u}_x(\pm 2R, t) \equiv 0.$$

As a result, the functions  $(u^\varepsilon, m^\varepsilon)$  defined by

$$u^\varepsilon(x, t) = \tilde{u}(x + R, t), \quad m^\varepsilon(x, t) = \tilde{m}(x + R, t)$$

are classical solutions to (4.9). The estimates on  $(u^\varepsilon, m^\varepsilon)$  readily follow by applying Corollary 3.7 and Propositions 3.9, 3.10, 3.13, 3.14, and 3.15 to the function  $(\tilde{u}, \tilde{m})$ . A similar discussion yields the result for the final cost problem,  $u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T))$ .  $\square$

### 4.1.2 An estimate on the flow of optimal trajectories

Given a solution  $(u^\varepsilon, m^\varepsilon)$  to (4.9), we may define the flow of optimal trajectories

$$\gamma^\varepsilon : [-R, R] \times [0, T] \rightarrow \mathbb{R}$$

according to (3.8). We remark that, when  $x = \pm R$ , the solution is the constant curve  $\gamma^\varepsilon(x, \cdot) \equiv x$ . Additionally, since  $u^\varepsilon \in C^{2,\alpha}([-R, R] \times [0, T])$ , we have  $\gamma^\varepsilon \in C^2([-R, R] \times [0, T])$ , and  $\gamma_x^\varepsilon > 0$ . We begin by showing that the trajectories starting in the support of  $m_0$  remain in a bounded set, independently of  $R$ .

**Proposition 4.6** (Finite propagation of the support). *Under the assumptions of Proposition 4.5, let  $R \geq 1$ ,  $0 < \varepsilon < 1$ , and let  $(u^\varepsilon, m^\varepsilon)$  be a solution to (4.9), and let  $\gamma^\varepsilon$  be the flow of trajectories associated to  $(u^\varepsilon, m^\varepsilon)$ . Then there exists a constant  $\bar{r} = \bar{r}(r, T, T^{-1}, \kappa_0, \|f(m_0^\varepsilon)\|_{W^{1,\infty}}, \|f(m_T^\varepsilon)\|_{W^{1,\infty}}, \|g(m_0^\varepsilon)\|_\infty)$ , such that*

$$\|\gamma^\varepsilon\|_{[-r,r] \times [0,T]} \leq \bar{r}.$$

*Proof.* For simplicity, we write  $\gamma^\varepsilon = \gamma$ . We will treat separately the planning problem and the final cost problem. First, assume that  $m^\varepsilon(\cdot, T) = m_T^\varepsilon$ . In view of Lemma 3.3, and the facts that  $\gamma(-R, \cdot) \equiv -R$ , and  $m_0 \equiv 0$  outside of  $[-r, r]$ , we have

$$\int_{-R}^y m_T^\varepsilon = \int_{-R}^{-r} m_0^\varepsilon = (R-r)\varepsilon,$$

where  $y = \gamma(-r, T)$ . Now, the left hand side is (strictly) increasing in  $y$ , and

$$\int_{-R}^{-r} m_T^\varepsilon = (R-r)\varepsilon,$$

thus  $\gamma(-r, T) = y = -r$ . Similarly  $\gamma(r, T) = r$ . Since  $\gamma_x > 0$ , this implies that  $\gamma([-r, r], T) \subset [-r, r]$ . Now, given  $x \in [-r, r]$ , we recall that

$$\gamma(x, \cdot) = \arg \min_{\alpha \in H^1(0,T), \alpha(0)=x} \int_0^T \frac{1}{2} |\dot{\alpha}|^2 + f(m^\varepsilon(\alpha(t), t)) dt + u^\varepsilon(\alpha(T), T).$$

Therefore, defining  $\alpha : [0, T] \rightarrow [-r, r]$  to be the straight line segment connecting  $x$  and  $\gamma(x, T)$ , we have

$$\begin{aligned} \int_0^T \left( \frac{1}{2} |\gamma_t(x, t)|^2 + f(m^\varepsilon(\gamma(x, t), t)) \right) dt + u^\varepsilon(\gamma(x, T), T) \\ \leq \int_0^T \left( \frac{1}{2} |\dot{\alpha}|^2 + f(m^\varepsilon(\alpha(x, t), t)) \right) dt + u^\varepsilon(\gamma(x, T), T), \end{aligned} \quad (4.14)$$

that is,

$$\int_0^T \frac{1}{2} |\gamma_t(x, t)|^2 dt \leq T \left( 2 \frac{r^2}{T^2} + f(\max m^\varepsilon) - f(\min m^\varepsilon) \right) \leq C.$$

In particular, given  $t \in [0, T]$ , we have

$$|\gamma(x, t)| \leq r + |\gamma(x, t) - \gamma(x, 0)| \leq r + \sqrt{2t} \sqrt{\int_0^T \frac{1}{2} |\gamma_t|^2} \leq r + \sqrt{C} \leq C.$$

This proves the result for the planning case. Next, we assume that  $u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T))$ . Observe that, in this case, while we do not know that  $\gamma([-r, r], T) \subset [-r, r]$ , we instead observe that  $u^\varepsilon$  is bounded independently of  $R$ , due to Proposition 4.5. Therefore, we have

$$\int_0^T \frac{1}{2} |\gamma_t(x, t)|^2 dt = u^\varepsilon(x, 0) - u^\varepsilon(\gamma(x, T), T) - \int_0^T f(m^\varepsilon(\gamma(x, t), t)) dt \leq C.$$

□

### 4.1.3 Compatible approximations and existence of solutions

In this section, we will apply Proposition 4.5 to prove the well-posedness result, Theorem 4.3. For that purpose, we will now build suitable  $C^{1,\alpha}(\mathbb{R})$  approximations  $m_0^\varepsilon > 0$  through a mild regularization procedure. For (MFGP), we must also build  $m_T^\varepsilon$ , while requiring that the two are suitably compatible, in the sense that they have the same mass. Furthermore, since these approximations will be needed later to prove finer results about the free boundary, we will ensure that they are also compatible in the sense that  $[a_0, b_0]$  is mapped bijectively onto  $[a_1, b_1]$  by the flow  $\gamma^\varepsilon(\cdot, T)$ .

**Definition 4.7.** Given  $r > 0$ , we say that  $(m_0^\varepsilon, m_T^\varepsilon)$  is a  $r$ -compatible approximation of  $(m_0, m_T)$  if  $(m_0^\varepsilon, m_T^\varepsilon)$  satisfies the assumptions of Proposition 4.5,  $m_0^\varepsilon \rightarrow m_0$ ,  $m_T^\varepsilon \rightarrow m_T$  uniformly, and, for every  $R > r$ ,

$$\int_{-R}^{a_0} m_0^\varepsilon = \int_{-R}^{a_1} m_T^\varepsilon \quad \text{and} \quad \int_{b_0}^R m_0^\varepsilon = \int_{b_0}^R m_T^\varepsilon. \quad (4.15)$$

**Lemma 4.8** (Construction of  $r$ -compatible approximations). *Under the assumption of Proposition 4.5, let  $r > 0$  be such that*

$$[a_0 - 2, b_0 + 2], [a_1 - 2, b_1 + 2] \subset [-r, r]. \quad (4.16)$$

*There exists a pair of vector-valued functions  $\eta_0, \eta_T \in C_c^\infty(\mathbb{R}, [0, \infty)^3)$  such that, for every  $\varepsilon \in (0, 1)$ , there exist constant vectors  $c_{0,\varepsilon}, c_{T,\varepsilon} \in [0, \infty)^3$  such that the pair  $(m_0^\varepsilon, m_T^\varepsilon)$  defined by*

$$m_0^\varepsilon = f^{-1}(\sqrt{f(m_0)^2 + f(\varepsilon)^2} + c_{0,\varepsilon} \cdot \eta_0), \quad m_T^\varepsilon = f^{-1}(\sqrt{f(m_T)^2 + f(\varepsilon)^2} + c_{T,\varepsilon} \cdot \eta_T) \quad (4.17)$$

*is a  $r$ -compatible approximation of  $(m_0, m_T)$ . Moreover, we have*

$$\lim_{\varepsilon \rightarrow 0^+} c_{0,\varepsilon} = \lim_{\varepsilon \rightarrow 0^+} c_{T,\varepsilon} = 0, \quad (4.18)$$

$$\begin{aligned} \eta_0 \equiv 0 \quad \text{in} \quad & \left[ a_0 - 1, \frac{1}{3}(2a_0 + b_0) \right] \cup \left[ \frac{1}{3}(a_0 + 2b_0), b_0 + 1 \right], \\ \eta_T \equiv 0 \quad \text{in} \quad & \left[ a_1 - 1, \frac{1}{3}(2a_1 + b_1) \right] \cup \left[ \frac{1}{3}(a_1 + 2b_1), b_1 + 1 \right], \end{aligned} \quad (4.19)$$

*and there exists a constant  $C = C(|\text{supp}(m_0)|, |\text{supp}(m_T)|, \text{dist}(\text{supp}(m_0), \text{supp}(m_T)))$  such that*

$$\|f(m_j^\varepsilon)\|_{W^{1,\infty}(\mathbb{R})} \leq C \|f(m_j)\|_{W^{1,\infty}(\mathbb{R})} + f(\varepsilon) + |c_{0,\varepsilon}| \cdot \|\eta_j\|_{C^1(\mathbb{R}, \mathbb{R}^3)}, \quad j \in \{0, T\}, \quad (4.20)$$

$$[f(m_j^\varepsilon)]_{C^\alpha(\mathbb{R})} \leq \frac{C}{f(\varepsilon)} (1 + [f(m_j)]_{C^\alpha(\{m_j > 0\})}) + |c_{0,\varepsilon}| [(\eta_j)_x]_{C^\alpha(\mathbb{R}, \mathbb{R}^3)}, \quad j \in \{0, T\} \quad (4.21)$$

$$[f(m_j^\varepsilon)]_{C^\alpha(\{m_j \geq \delta\})} \leq \frac{C}{f(\delta)} (1 + [f(m_j)]_{C^\alpha(\{m_j > 0\})}) + \frac{C}{f(\delta)} |c_{0,\varepsilon}| [(\eta_j)_x]_{C^\alpha(\mathbb{R}, \mathbb{R}^3)}, \quad j \in \{0, T\} \quad (4.22)$$

*Finally if  $j \in \{0, T\}$  and  $f(m_j)$  is semi-convex, then*

$$\|f(m_j^\varepsilon)_{xx}^-\|_{L^\infty(\mathbb{R})} \leq \|f(m_j)_{xx}^-\|_{L^\infty(\mathbb{R})} + |c_{0,\varepsilon}| \cdot \|(\eta_j)_{xx}^-\|_{L^\infty(\mathbb{R})}. \quad (4.23)$$

*Proof.* We let  $\eta_{0,1}, \eta_{0,2}, \eta_{0,3} \in C_c^\infty(\mathbb{R})$  be non-zero, non-negative bump functions supported, respectively, on

$$[a_0 - 2, a_0 - 1], \left[ \frac{1}{3}(2a_0 + b_0), \frac{1}{3}(a_0 + 2b_0) \right], \quad \text{and} \quad [b_0 + 1, b_0 + 2].$$

Similarly, we take  $\eta_{T,1}, \eta_{T,2}, \eta_{T,3} \in C_c^\infty(\mathbb{R})$  to be supported, respectively, on

$$[a_1 - 2, a_1 - 1], \left[ \frac{1}{3}(2a_1 + b_1), \frac{1}{3}(a_1 + 2b_1) \right], \quad \text{and} \quad [b_1 + 1, b_1 + 2].$$

With this, we can define

$$\eta_0 = (\eta_{0,1}, \eta_{0,2}, \eta_{0,3}), \quad \eta_T = (\eta_{T,1}, \eta_{T,2}, \eta_{T,3}).$$

We observe first that (4.19) holds by construction. Next, note that (4.16) guarantees that  $m_0^\varepsilon, m_T^\varepsilon \equiv \varepsilon$  outside of  $[-r, r]$ . For this reason, (4.15) will hold for all  $R > r$  as long as it holds for  $R = r$ . Now, we have

$$\int_{-r}^{a_0} m_0^\varepsilon = \int_{-r}^{a_0} f^{-1}(f(\varepsilon) + c_{0,\varepsilon}^1 \eta_{0,1}), \quad \text{and} \quad \int_{-r}^{a_1} m_T^\varepsilon = \int_{-r}^{a_1} f^{-1}(f(\varepsilon) + c_{T,\varepsilon}^1 \eta_{T,1}).$$

If, say,  $a_0 \leq a_1$ , then, taking  $c_{T,\varepsilon}^1 = 0$ , there exists a unique choice of  $c_{0,\varepsilon}^1 \geq 0$  that ensures

$$\int_{-r}^{a_0} m_0^\varepsilon = \int_{-r}^{a_1} m_T^\varepsilon.$$

Similarly, if  $a_1 < a_0$ , we may take  $c_{0,\varepsilon}^1 = 0$  and choose an adequate  $c_{T,\varepsilon}^1 > 0$ . By the same reasoning, there exists a choice of  $c_{0,\varepsilon}^3, c_{T,\varepsilon}^3$  such that

$$\int_{b_0}^r m_0^\varepsilon = \int_{b_0}^r m_T^\varepsilon.$$

We must also ensure that  $m_0^\varepsilon, m_T^\varepsilon$  have equal mass. For this purpose, we observe that

$$\int_{a_0}^{b_0} m_0^\varepsilon = \int_{a_0}^{b_0} f^{-1}(\sqrt{f(m_0)^2 + f(\varepsilon)^2} + c_{0,\varepsilon}^2 \eta_{0,2}), \quad \int_{a_1}^{b_1} m_T^\varepsilon = \int_{a_1}^{b_1} f^{-1}(\sqrt{f(m_T)^2 + f(\varepsilon)^2} + c_{T,\varepsilon}^2 \eta_{T,2}).$$

And, as before, depending on which of the quantities  $\int_{a_0}^{b_0} f^{-1}(\sqrt{f(m_0)^2 + f(\varepsilon)^2})$ ,  $\int_{a_1}^{b_1} f^{-1}(\sqrt{f(m_T)^2 + f(\varepsilon)^2})$  is larger, we may choose one of the numbers  $c_{0,\varepsilon}^2, c_{T,\varepsilon}^2$  to be zero, which leaves a unique way to choose the remaining one in such a way that

$$\int_{a_0}^{b_0} m_0^\varepsilon = \int_{a_1}^{b_1} m_T^\varepsilon.$$

This, together with (4.15), guarantees that the two functions have the same integral over  $[-R, R]$ . Now, (4.18) is a straightforward consequence of (4.3), (4.4), and the fact that the vectors  $c_{0,\varepsilon}, c_{T,\varepsilon}$  were chosen so that

$$c_{0,\varepsilon}^i c_{T,\varepsilon}^i = 0 \text{ for } i \in \{1, 2, 3\}.$$

It remains to show the Lipschitz,  $C^{1,\alpha}$ , and semi-convexity estimates on  $f(m_0^\varepsilon), f(m_T^\varepsilon)$ . For concreteness, we will only show these for  $f(m_0^\varepsilon)$ , since the arguments for  $f(m_T^\varepsilon)$  are identical. We have

$$f(m_0^\varepsilon) = \sqrt{f(m_0)^2 + f(\varepsilon)^2} + c_{0,\varepsilon} \cdot \eta_0; \quad f(m_0^\varepsilon)_x = \frac{f(m_0)f(m_0)_x}{\sqrt{f(m_0)^2 + f(\varepsilon)^2}} + c_{0,\varepsilon} \cdot (\eta_0)_x,$$

which readily yields (4.20). Now, given  $x, y \in [-r, r]$ , we may write

$$\begin{aligned} f(m_0^\varepsilon)_x(x) - f(m_0^\varepsilon)_x(y) &= c_{0,\varepsilon} \cdot ((\eta_0)_x(x) - (\eta_0)_x(y)) + \frac{(f(m_0)(x) - f(m_0)(y))f(m_0)_x(x)}{\sqrt{f(m_0)^2(x) + f(\varepsilon)^2}} \\ &+ f(m_0)(y) \frac{f(m_0)_x(x) - f(m_0)_x(y)}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2}} + f(m_0)(y)f(m_0)_x(y) \left( \frac{1}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2}} - \frac{1}{\sqrt{f(m_0)(y)^2 + f(\varepsilon)^2}} \right) \\ &= c_{0,\varepsilon} \cdot ((\eta_0)_x(x) - (\eta_0)_x(y)) + \frac{(f(m_0)(x) - f(m_0)(y))f(m_0)_x(x)}{\sqrt{f(m_0)^2(x) + f(\varepsilon)^2}} + f(m_0)(y) \frac{f(m_0)_x(x) - f(m_0)_x(y)}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2}} \\ &+ f(m_0)(y)f(m_0)_x(y) \left( \frac{(f(m_0)(y) - f(m_0)(x))(f(m_0)(y) + f(m_0)(x))}{\sqrt{f(m_0)(x)^2 + f(\varepsilon)^2} \sqrt{f(m_0)(y)^2 + f(\varepsilon)^2} (\sqrt{f(m_0)(y)^2 + f(\varepsilon)^2} + \sqrt{f(m_0)(x)^2 + f(\varepsilon)^2})} \right). \end{aligned} \tag{4.24}$$

Thus, we obtain

$$|f(m_0^\varepsilon)_x(x) - f(m_0^\varepsilon)_x(y)| \leq |c_{0,\varepsilon}| |\eta_0|_\alpha |x-y|^\alpha + \frac{C}{f(\varepsilon)} \|f(m_0)_x\|_\infty^2 |x-y|^\alpha + \frac{1}{f(\varepsilon)} \|f(m_0)\|_\infty [f(m_0)_x]_{C^\alpha(\{m_0>0\})} |x-y|^\alpha + \frac{C}{f(\varepsilon)} \|f(m_0)_x\|_\infty^2 |x-y|^\alpha,$$

which yields (4.21). Similarly, if  $x, y \in \{m_0 \geq \delta\}$ , we obtain (4.22). Finally, if  $m_0$  is smooth, then

$$\begin{aligned} f(m_0^\varepsilon)_{xx} &= \frac{f(m_0)f(m_0)_{xx}}{\sqrt{f(m_0)^2 + f(\varepsilon)^2}} + \frac{f(m_0)_x^2}{\sqrt{f(m_0)^2 + f(\varepsilon)^2}} - \frac{f(m_0)^2 f(m_0)_x^2}{(f(m_0)^2 + f(\varepsilon)^2)^{\frac{3}{2}}} + c_{0,\varepsilon} \cdot (\eta_0)_{xx} \\ &\geq -\|f(m_0)_{xx}\|_\infty + \frac{f(m_0)_x^2 f(\varepsilon)^2}{(f(m_0)^2 + f(\varepsilon)^2)^{\frac{3}{2}}} - |c_{0,\varepsilon}| \|(\eta_0)_{xx}\|_\infty \\ &\geq -\|f(m_0)_{xx}\|_\infty - |c_{0,\varepsilon}| \|(\eta_0)_{xx}\|_\infty, \end{aligned}$$

which shows (4.23). For a non-smooth  $m_0$ , the result then follows by standard approximation.  $\square$

Having constructed the necessary approximations, we are now ready to prove the well-posedness theorem.

*Proof of Theorem 4.3.* We begin with the case of (MFGP). We first prove existence of a solution satisfying the estimates. For this purpose, we choose the  $r$ -compatible approximations  $(m_0^\varepsilon, m_T^\varepsilon)$  as defined by (4.17), where  $r > 0$  is fixed and chosen large enough to satisfy (4.16), and take  $R > 2\bar{r}$ , where  $\bar{r}$  is the constant of Proposition 4.6. In view of (4.20) from Lemma 4.8, the constant  $C$  of Proposition 4.5 may be chosen independently of  $\varepsilon$ . In particular, from (4.13), we may choose the solutions  $(u^\varepsilon, m^\varepsilon)$  of Proposition 4.5 in such a way that  $\|u^\varepsilon\|_\infty \leq C$ . Therefore, due to the bounds (4.11) and (4.12), the family  $\{(u^\varepsilon, m^\varepsilon)\}_{\varepsilon \in (0,1)}$  is bounded and equicontinuous, and we may extract a subsequence and conclude that  $(u^\varepsilon, m^\varepsilon) \rightarrow (u, m) \in W^{1,\infty}([-R, R] \times [0, T]) \times C([-R, R] \times [0, T])$ , with  $(u, m)$  satisfying the required estimates.

Moreover, if  $(x, t) \in \{m > 0\}$ , by the equicontinuity of  $\{m^\varepsilon\}_{\varepsilon \in (0,1)}$ , there exists an open set  $V$  containing  $(x, t)$  such that  $f'(m^\varepsilon)m^\varepsilon > 0$ , so that  $u^\varepsilon$  satisfies a uniformly elliptic equation in  $V$  (recall (3.1)). As a result, if  $0 < t < T$ , the standard interior  $C^{1,\alpha}$  estimates followed by the Schauder estimates (see [17, Thm. 13.6] and [17, Thm. 6.2], respectively) imply that  $(u, m) \in C^{2,\alpha} \times C^{1,\alpha}$  in a neighborhood of  $(x, t)$ . On the other hand, if  $t \in \{0, T\}$ , by proceeding as in the proof of Theorem 3.11, through the  $C^{1,\alpha}$  estimates for oblique problems, followed by Fiorenza's convergence argument (as detailed in the proof of Theorem 3.11), we see that  $(u, m) \in C^{2,\alpha} \times C^{1,\alpha}$  in a neighborhood of  $(x, t)$ . In particular, this shows that  $u$  solves the HJ equation in the pointwise sense at  $(x, t)$ .

Now, we claim that  $m \in C_c(\mathbb{R} \times [0, T])$ , with  $\text{supp}(m) \subset [-\bar{r}, \bar{r}] \times [0, T]$ . Indeed, this follows from the fact that, for each  $t \in [0, T]$ ,  $m(\cdot, t)$  has mass 1 and, by Lemma 3.3 and Proposition 4.6,

$$\int_{-\bar{r}}^{\bar{r}} m(\cdot, t) = \lim_{\varepsilon \downarrow 0} \int_{-\bar{r}}^{\bar{r}} m^\varepsilon(\cdot, t) \geq \lim_{\varepsilon \downarrow 0} \int_{a_0}^{b_0} m_0^\varepsilon = \int_{a_0}^{b_0} m_0 = 1.$$

On the other hand, we can extend  $u$  to  $W^{1,\infty}(\mathbb{R} \times (0, T))$  in the following way:

$$u(x, t) = \begin{cases} u(-R, t) & \text{if } x < -R \\ u(R, t) & \text{if } x > R. \end{cases}$$

Notice that, since  $u_x^\varepsilon(\pm R, \cdot) \equiv 0$ ,

$$-u_t^\varepsilon(\pm R, t) = f(m^\varepsilon(\pm R, t)) \rightarrow f(0) \text{ uniformly in } t,$$

and, thus,

$$u_x^\varepsilon(\pm R, t) \rightarrow f(0)(T - t) + u(\pm R, T) \text{ in } C^{1,1}([0, T]).$$

In particular, we see that, for  $|x| > R$ ,

$$-u_t + \frac{1}{2}u_x^2 = f(0) = f(m).$$



It is then straightforward to verify, by using the basic stability property of viscosity solutions under uniform convergence (see [5]), that this extension  $(u, m)$  is a solution to (MFGP), in the sense of Definition 4.1, which satisfies all the necessary estimates.

Now, to prove uniqueness, we assume that  $(\tilde{u}, \tilde{m})$  is another solution. Since the function  $\tilde{u}$  is a viscosity solution to the HJ equation, and it is almost everywhere differentiable, it must also satisfy the equation pointwise almost everywhere. Therefore, it also solves the equation in the distributional sense. Noting that the pairs  $(u, m), (\tilde{u}, \tilde{m}) \in W^{1,\infty}(\mathbb{R} \times (0, T)) \times C_b(\mathbb{R} \times [0, T])$  are sufficiently regular to serve as test functions, we may apply the standard Lasry-Lions computation to the pair  $(u, m)$  and  $(\tilde{u}, \tilde{m})$ , obtaining

$$\int_0^T \int_{\mathbb{R}} \frac{1}{2} (m + \tilde{m}) |u_x - \tilde{u}_x|^2 + (m - \tilde{m})(f(m) - f(\tilde{m})) = 0.$$

Therefore, since the left hand side is non-negative, we conclude that  $m = \tilde{m}$ , and  $u_x = \tilde{u}_x$  in  $\{m > 0\}$ . Finally, since  $m$  is continuous,  $\{m > 0\}$  is an open set, and thus  $u$  and  $\tilde{u}$  at most differ by a constant on each connected component of  $\{m > 0\}$ . For the case of (MFG), the proof is completely analogous, noting that in applying Proposition 4.5, we take  $g^\varepsilon(m) = g(m) + \varepsilon m$ , to satisfy the strict monotonicity assumption. We note that, unless  $g' > 0$ , the Lieberman and oblique Schauder estimates may not be applied at  $t = T$ , hence the weaker conclusion  $(u, f(m)) \in C_{\text{loc}}^{2,\alpha}(\mathbb{R} \times [0, T]) \cap \{m > 0\} \times C_{\text{loc}}^{1,\alpha}(\mathbb{R} \times [0, T]) \cap \{m > 0\}$ . However, in the special case  $g \equiv 0$ ,  $u$  satisfies a smooth Dirichlet boundary condition at  $t = T$ , and thus we may apply the Ladyzhenskaya-Ural'tseva and Schauder estimates for the Dirichlet problem (see [17, Thm 13.7, Thm 6.19]) to the limiting function  $u$  to still obtain the  $C_{\text{loc}}^{2,\alpha}(\mathbb{R} \times [0, T]) \cap \{m > 0\}$  regularity. Note that global uniqueness follows because, since  $m$  is unique, the terminal value  $u(\cdot, T) = g(m(\cdot, T))$  is uniquely determined.  $\square$

## 4.2 Analysis of the free boundary

Having established the existence and uniqueness of solutions, with compactly supported density, we now study the set  $\partial(\{m > 0\})$ , where  $(u, m)$  is the solution to (MFG) or (MFGP). Henceforth, we restrict our analysis to the case that, for some given constant  $\theta > 0$ , we have

$$f(m) = m^\theta, \quad \theta > 0. \quad (4.25)$$

We will focus on the setting in which the initial distribution  $m_0$  (and possibly  $m_T$ , in case of problem (MFGP)) are each supported exactly on an interval. That is, we will assume that

$$\{m_0 > 0\} = (a_0, b_0) \text{ and } \{m_T > 0\} = (a_1, b_1), \quad (4.26)$$

for some finite  $a_0, b_0, a_1, b_1$ . Moreover, we will assume that  $m_0$  and  $m_T$  decay like powers near the endpoints of these intervals. In other words, there exist numbers  $\alpha_0, \alpha_1 > 0$  such that

$$\frac{1}{C_0} \text{dist}(x, \{a_0, b_0\})^{\alpha_0} \leq m_0(x) \leq C_0 \text{dist}(x, \{a_0, b_0\})^{\alpha_0}, \quad x \in [a_0, b_0], \quad (4.27)$$

$$\frac{1}{C_1} \text{dist}(x, \{a_1, b_1\})^{\alpha_1} \leq m_T(x) \leq C_1 \text{dist}(x, \{a_1, b_1\})^{\alpha_1}, \quad x \in [a_1, b_1]. \quad (4.28)$$

For concreteness, we will also assume here that

$$\alpha_0 \geq \alpha_1, \quad (4.29)$$

but we remark that the opposite case can be studied, with analogous results, by simply considering the time-reversed functions  $(-u(x, T-t), m(x, T-t))$ , which solve (MFGP), but with the roles of  $m_0$  and  $m_T$  reversed.

Correspondingly, in case of problem (MFG), we assume that the coupling function  $g$  at final time satisfies, for some given constants  $\theta_1 > 0$ ,  $c_1 \geq 0$ , with  $\theta_1 \geq \theta$ ,

$$g(m) = c_1 T m^{\theta_1}. \quad (4.30)$$

*Remark 4.9.* The factor of  $T$  in the definition of  $g$  is made explicit in order to adequately state the sharp long time behavior result, Theorem 4.15. This is the natural scaling for the final pay-off which is consistent with the behavior of the self-similar solution, see Section 2. However we also note that our assumptions include, in particular, the case of terminal condition  $u(\cdot, T) = 0$ .

### 4.2.1 Characterization of the free boundary through the equation satisfied by the flow

In this subsection, we properly establish the existence of the free boundary curves together with their basic characterization. Our main tool will be the elliptic equation satisfied by the flow  $\gamma$  of optimal characteristics (see (3.12)). We recall from Section 2 that  $\bar{\alpha} \in (0, 1)$  is defined by

$$\bar{\alpha} = \frac{2}{2 + \theta}.$$

**Theorem 4.10** (Characterization of the free boundary and the flow). *Let  $f$  be given by (4.25). Assume that (4.3)–(4.5) and (4.26)–(4.30) hold, and that  $f(m_0)$  is semi-convex. Let  $(u, m)$  be a solution to (MFG) or (MFGP). Then there exist two functions  $\gamma_L < \gamma_R \in W^{1,\infty}(0, T)$ , such that*

$$\{m > 0\} = \{(x, t) \in \mathbb{R} \times [0, T] : \gamma_L(t) < x < \gamma_R(t)\}. \quad (4.31)$$

Moreover, the flow  $\gamma$  of optimal trajectories is well defined on  $(a_0, b_0) \times [0, T]$ , we have

$$\gamma \in W^{1,\infty}((a_0, b_0) \times (0, T)) \cap C_{\text{loc}}^{2,\alpha}((a_0, b_0) \times [0, T]), \quad \gamma_x > 0, \quad \gamma_L(t) = \gamma(a_0, t), \quad \gamma_R(t) = \gamma(b_0, t),$$

and  $\gamma$  is a classical solution in  $(a_0, b_0) \times (0, T)$  to the elliptic equation

$$\gamma_{tt} + \frac{\theta f(m_0)}{(\gamma_x)^{2+\theta}} \gamma_{xx} = \frac{f(m_0)_x}{(\gamma_x)^{1+\theta}}. \quad (4.32)$$

Moreover, there exists a constant

$$C = \begin{cases} C(T, T^{-1}, C_0, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_0)_{xx}^-\|_{\infty}, \theta, \frac{\theta_1}{\theta}, c_1) & \text{if } (u, m) \text{ solves (MFG),} \\ C(T, T^{-1}, C_0, C_1, \|f(m_0)\|_{W^{1,\infty}}, \|f(m_T)\|_{W^{1,\infty}}, \|f(m_0)_{xx}^-\|_{\infty}, \theta) & \text{if } (u, m) \text{ solves (MFGP),} \end{cases}$$

such that

$$\|\gamma\|_{W^{1,\infty}([a_0, b_0] \times [0, T])} \leq C, \quad (4.33)$$

and, for each  $(x, t) \in (a_0, b_0) \times [0, T]$ ,

$$\frac{1}{C} m_0(x) \leq m(\gamma(x, t), t). \quad (4.34)$$

*Proof.* Throughout this proof, as usual, the constant  $C > 0$  may increase at each step. We will first treat the case in which  $(u, m)$  solves (MFGP). As in the proof of Theorem 4.3, we let  $(u^\varepsilon, m^\varepsilon)$  be the solution to (4.9) given by Proposition 4.5, corresponding to the choices of  $m_0^\varepsilon, m_T^\varepsilon$  given by (4.17). We may normalize the solution to satisfy  $\int_{\mathbb{R}^d} u^\varepsilon(\cdot, T) = 0$ . We also fix  $R$  large enough that  $(-R, R)$  contains both  $[a_0 - 2, b_0 + 2]$  and  $[a_1 - 2, b_1 + 2]$ . With this choice, we see from (4.17) and (4.19) that

$$\int_{-R}^{a_0} m_0^\varepsilon = \int_{-R}^{a_1} m_T^\varepsilon, \quad \text{and} \quad \int_{b_0}^R m_0^\varepsilon = \int_{b_1}^R m_T^\varepsilon, \quad (4.35)$$

which guarantees that  $\gamma^\varepsilon(a_0, T) = a_1$ , and  $\gamma^\varepsilon(b_0, T) = b_1$ , or, equivalently, that  $\gamma^\varepsilon(\cdot, T)$  is a bijection between  $\text{supp } m_0$  and  $\text{supp } m_T$ . We are now interested in a Lipschitz bound for  $\gamma^\varepsilon$ . Recall that  $\gamma_t^\varepsilon = -u_x^\varepsilon(\gamma^\varepsilon(x, t), t)$ , so  $\|\gamma_t^\varepsilon\|_{L^\infty((a_0, b_0) \times [0, T])}$  is bounded due to Proposition 4.5 (we emphasize that our choice of  $R$  is already fixed). Moreover, we have  $\gamma^\varepsilon \in C^2([-R, R] \times [0, T])$  and, by Lemma 3.3 we know that it satisfies (3.12); for the case  $f(m) = m^\theta$ , this equation may be rewritten as

$$\gamma_{tt}^\varepsilon + \frac{\theta f(m_0^\varepsilon)}{(\gamma_x^\varepsilon)^{2+\theta}} \gamma_{xx}^\varepsilon = \frac{f(m_0^\varepsilon)_x}{(\gamma_x^\varepsilon)^{1+\theta}}. \quad (4.36)$$

For  $K > 0$ , we set

$$v = \gamma_x^\varepsilon, \quad w = v - Kt^{\bar{\alpha}}, \quad \text{and} \quad k_1 = \|f(m_0)_{xx}^-\|_{\infty}.$$

We now want to show that, if  $K$  is chosen sufficiently large,

$$\max_{[a_0, b_0] \times [0, T]} w \leq \max_{\partial([a_0, b_0] \times [0, T])} w. \quad (4.37)$$

For this purpose we assume first that  $m_0^\varepsilon, m_T^\varepsilon$  are smooth. Then  $\gamma^\varepsilon$  is smooth as well, and differentiating (4.36) with respect to  $x$ , we get, for some function  $b(x, t)$ , and sufficiently small  $\varepsilon > 0$ ,

$$v_{tt} + \frac{\theta f(m_0^\varepsilon)}{v^{2+\theta}} v_{xx} + b(x, t) v_x = \frac{f(m_0^\varepsilon)_{xx}}{v^{1+\theta}} \geq \frac{-2k_1}{v^{1+\theta}}, \quad (4.38)$$

where equation (4.23) from Lemma 4.8 was used in the last inequality. Recalling the definition of  $w$ , this yields

$$w_{tt} + \frac{\theta f(m_0^\varepsilon)}{v^{2+\theta}} w_{xx} + b(x, t) w_x \geq \frac{-2k_1}{v^{1+\theta}} + K\bar{\alpha}(1 - \bar{\alpha})t^{\bar{\alpha}-2}. \quad (4.39)$$

Let  $(x_0, t_0)$  be an interior maximum point of  $w$ . Then the right hand side of (4.39) must be non-positive, which may be rewritten as

$$v(x_0, t_0) \leq \left( \frac{2k_1}{K\bar{\alpha}(1 - \bar{\alpha})} \right)^{\frac{1}{1+\theta}} t_0^{\frac{2-\bar{\alpha}}{1+\theta}} = \left( \frac{2k_1}{K\bar{\alpha}(1 - \bar{\alpha})} \right)^{\frac{1}{1+\theta}} t_0^{\bar{\alpha}}. \quad (4.40)$$

Since  $\max_{[a_0, b_0] \times [0, T]} w \geq 0$ , we have  $v(x_0, t_0) \geq Kt_0^{\bar{\alpha}}$ . Hence inequality (4.40) is impossible if we choose  $K$  sufficiently large, namely if  $K > \left( \frac{2k_1}{\bar{\alpha}(1 - \bar{\alpha})} \right)^{\frac{1+\theta}{2-\bar{\alpha}}}$ . This shows (4.37). To remove the assumption that  $m_0^\varepsilon, m_T^\varepsilon$  are sufficiently smooth to perform the above computations, we approximate them first with convolutions, as was done in (3.33) (but without any need to regularize  $f$ ), and we simply note that (4.37) is stable under such an approximation.

In view of (4.37), it now suffices to bound  $w$  at the boundary points. When  $t = 0$ , we have  $w = v \equiv 1$ , and when  $x = a_0$  or  $x = b_0$ ,  $m_0^\varepsilon(x) = \varepsilon = \min m^\varepsilon$ , so that, recalling (3.10),

$$w(x, t) \leq v(x, t) = \frac{m_0^\varepsilon(x)}{m^\varepsilon(\gamma^\varepsilon(x, t), t)} \leq 1.$$

Therefore, we are only left with estimating  $v(x, T)$ , for  $x \in (a_0, b_0)$ . We will assume that  $x \in (a_0, \frac{1}{2}(a_0 + b_0))$ , since the converse case is completely symmetric. We first observe that the explicit form of the approximations (4.17), where  $f(s) = s^\theta$ , together with (4.27) and (4.28), imply, for some constants  $c_0, c_1$ :

$$\frac{1}{c_0} (\text{dist}(x, \{a_0, b_0\})^{\alpha_0} + \varepsilon) \leq m_0^\varepsilon(x) \leq c_0 (\text{dist}(x, \{a_0, b_0\})^{\alpha_0} + \varepsilon), \quad x \in (a_0, b_0), \quad (4.41)$$

and

$$\frac{1}{c_1} (\text{dist}(x, \{a_1, b_1\})^{\alpha_1} + \varepsilon) \leq m_T^\varepsilon(x) \leq c_1 (\text{dist}(x, \{a_1, b_1\})^{\alpha_1} + \varepsilon), \quad x \in (a_1, b_1). \quad (4.42)$$

Let us set  $\bar{x} = \gamma^\varepsilon(x, T)$ ; since we have

$$\int_{a_0}^x m_0^\varepsilon = \int_{a_1}^{\bar{x}} m_T^\varepsilon \quad (4.43)$$

we deduce from (4.41)–(4.42) that  $x \rightarrow a_0$  if and only if  $\bar{x} \rightarrow a_1$ . Hence we can assume that  $\bar{x} \leq \frac{a_1 + b_1}{2}$  and we estimate

$$\frac{1}{c_0} ((x - a_0)^{\alpha_0 + 1} + \varepsilon(x - a_0)) \leq \int_{a_0}^x m_0^\varepsilon = \int_{a_1}^{\bar{x}} m_T^\varepsilon \leq c_1 ((\bar{x} - a_1)^{\alpha_1 + 1} + \varepsilon(\bar{x} - a_1)).$$

In particular, this implies that

$$\frac{1}{c_0} (x - a_0)^{\alpha_0 + 1} \leq c_1 (\bar{x} - a_1)^{\alpha_1 + 1} \quad \text{or} \quad \frac{1}{c_0} \varepsilon(x - a_0) \leq c_1 \varepsilon(\bar{x} - a_1),$$

and since  $\alpha_0 \geq \alpha_1$  we deduce, for some constant  $C$ , that

$$(x - a_0)^{\alpha_0} \leq C(\bar{x} - a_1)^{\alpha_1}.$$

As a result, using (4.41)–(4.42), we have, for  $\varepsilon$  small enough,

$$v(x, T) = \frac{m_0^\varepsilon(x)}{m^\varepsilon(\gamma^\varepsilon(x, T), T)} \leq \frac{c_0(x - a_0)^{\alpha_0} + \varepsilon}{\frac{1}{c_1}(\bar{x} - a_1)^{\alpha_1} + \varepsilon} \leq C.$$

We thus conclude that  $w \leq C$ , hence

$$\gamma_x^\varepsilon(x, t) \leq C(1 + t^\alpha). \quad (4.44)$$

This finally establishes that

$$\|\gamma^\varepsilon\|_{W^{1,\infty}([a_0, b_0] \times [0, T])} \leq C, \quad (4.45)$$

and, in particular,

$$\frac{1}{C}m_0^\varepsilon(x) \leq m^\varepsilon(\gamma^\varepsilon(x, t), t). \quad (4.46)$$

Now, recalling the proof of Theorem 4.3, we see that  $m^\varepsilon$  converges uniformly to the unique weak solution  $m$  of (MFGP). Moreover,  $\gamma^\varepsilon \rightarrow \gamma$  uniformly in  $[a_0, b_0] \times [0, T]$ , and (4.33), (4.34) follow. Now, assume that  $x_0 < \gamma(a_0, t_0)$ . We have, for  $\varepsilon$  small enough,  $x_0 < \gamma^\varepsilon(a_0, t_0)$ , so Lemma 3.3 yields

$$\int_{-R}^{x_0} m^\varepsilon \leq \int_{-R}^{a_0} m_0^\varepsilon, \implies \int_{-R}^{x_0} m \leq \int_{-R}^{a_0} m_0 = 0,$$

and, thus,  $m(x_0, t_0) = 0$ . Similarly,  $m(x_0, t_0) = 0$  whenever  $x_0 > \gamma(b_0, t_0)$ . On the other hand, if  $\gamma(a_0, t_0) < x_0 < \gamma(b_0, t_0)$ , we have, by continuity,  $\gamma(a_0, t_0) = \gamma(c_0, t_0)$  for some  $a_0 < c_0 < b_0$ . As a result, (4.34) yields

$$m(x_0, t_0) = m(\gamma(c_0, t_0), t_0) \geq \frac{1}{C}m_0(c_0) > 0,$$

and this proves (4.31). We now recall from the proof of Theorem 4.3 that  $(u^\varepsilon, f(m^\varepsilon))$  is bounded in  $C_{\text{loc}}^{2,\alpha}([-R, R] \times [0, T]) \cap \{m > 0\} \times C_{\text{loc}}^{1,\alpha}([-R, R] \times [0, T]) \cap \{m > 0\}$ , independently of  $\varepsilon$ . Thus, for  $(x, t) \in (a_0, b_0) \times [0, T]$ , letting  $\varepsilon \rightarrow 0$  in the relations

$$\gamma_t^\varepsilon = -u_x^\varepsilon(\gamma^\varepsilon(x, t), t), \quad \gamma_x^\varepsilon = \frac{m_0^\varepsilon(x)}{m^\varepsilon(\gamma^\varepsilon(x, t), t)}$$

shows that  $\gamma^\varepsilon \rightarrow \gamma$  in  $C_{\text{loc}}^2((a_0, b_0) \times [0, T])$ ,

$$\gamma_t = -u_x(\gamma(x, t), t), \quad \gamma_x = \frac{m_0(x)}{m(\gamma(x, t), t)}, \quad (4.47)$$

and, in particular,  $\gamma \in C_{\text{loc}}^{2,\alpha}((a_0, b_0) \times [0, T])$ . Finally, letting  $\varepsilon \rightarrow 0$  in (4.36) yields (4.32).

We now explain the necessary changes in the above proof to deal with the case that  $(u, m)$  solves (MFG). We initially assume that  $c_1 > 0$ . The first modification lies in the proof of (4.44), since our previous argument to estimate  $\gamma_x^\varepsilon(\cdot, T)$  does not apply if  $m^\varepsilon(\cdot, T) \neq m_T^\varepsilon$ . Assume then that the function  $w$  achieves its maximum value at a point  $(x_0, T)$ , with  $a_0 < x_0 < b_0$ . We first observe that

$$u^\varepsilon(\cdot, T) = g(m^\varepsilon(\cdot, T)),$$

so that

$$-\gamma_t^\varepsilon(\cdot, T) = u_x^\varepsilon(\gamma^\varepsilon(\cdot, T), T) = g'(\gamma^\varepsilon(\cdot, T))m_x^\varepsilon(\gamma^\varepsilon(\cdot, T), T).$$

Since  $m_x^\varepsilon = \frac{1}{f'}f(m^\varepsilon)_x = \frac{1}{f'}\gamma_{tt}^\varepsilon$ , we get

$$-\gamma_t^\varepsilon(\cdot, T) = \frac{g'}{f'}\gamma_{tt}^\varepsilon(\cdot, T) = Tc_1 \frac{\theta_1}{\theta}(m^\varepsilon)^{\theta_1 - \theta} \gamma_{tt}^\varepsilon.$$

Differentiating this relation with respect to  $x$  once more yields

$$T c_1 \frac{\theta_1}{\theta} (m^\varepsilon)^{\theta_1 - \theta} v_{tt} = -v_t - T c_1 \frac{\theta_1}{\theta} (\theta_1 - \theta) (m^\varepsilon)^{\theta_1 - \theta - 1} (m_x^\varepsilon)^2 \gamma_x f'(m^\varepsilon) \leq -v_t.$$

Now, evaluating this at  $x_0$ , and using the fact that  $w_t \geq 0$  at  $(x_0, T)$ , we get

$$T c_1 \frac{\theta_1}{\theta} (m^\varepsilon)^{\theta_1 - \theta} v_{tt} \leq -K \bar{\alpha} T^{\bar{\alpha} - 1}.$$

On the other hand, using  $w_x = v_x = 0$  and  $w_{xx} = v_{xx} \leq 0$ , (4.38) yields  $v_{tt} \geq -\frac{2k_1}{v^{1+\theta}}$ . Hence we get

$$-\frac{2k_1}{v^{1+\theta}} \frac{\theta_1}{\theta} (m^\varepsilon)^{\theta_1 - \theta} \leq -K \frac{\bar{\alpha} T^{\bar{\alpha} - 2}}{c_1}.$$

Using the uniform bound on  $m$  (see (4.10)) we obtain, for  $\varepsilon$  sufficiently small,

$$v \leq \left( \frac{2k_1 c_1 \theta_1}{\bar{\alpha} \theta} \|m_0\|_\infty^{\theta_1 - \theta} \right)^{\frac{1}{1+\theta}} \frac{T^{\frac{2-\bar{\alpha}}{1+\theta}}}{K^{\frac{1}{1+\theta}}} := C \frac{T^{\frac{2-\bar{\alpha}}{1+\theta}}}{K^{\frac{1}{1+\theta}}} = C \frac{T^{\bar{\alpha}}}{K^{\frac{1}{1+\theta}}}. \quad (4.48)$$

If we choose  $K > C^{\frac{1+\theta}{2-\bar{\alpha}}}$ , then the right hand side of (4.48) is bounded above by  $KT^{\bar{\alpha}}$ , which yields  $w(x, T) \leq 0$ , completing the proof of (4.44). We also observe that  $g = c_1 T m^\theta$  satisfies  $g' > 0$ , which, as seen in the proof of Theorem 4.3, is still sufficient to obtain uniform bounds of  $(u^\varepsilon, f(m^\varepsilon))$  in

$$C_{\text{loc}}^{2,\alpha}([-R, R] \times [0, T]) \cap \{m > 0\} \times C_{\text{loc}}^{1,\alpha}([-R, R] \times [0, T]) \cap \{m > 0\}. \quad (4.49)$$

When  $c_1 = 0$ , we can repeat the present proof for  $g^\varepsilon(m) = \varepsilon m^\theta$ . The functions  $(u^\varepsilon, f(m^\varepsilon))$  may still be estimated in  $C_{\text{loc}}^{2,\alpha} \times C_{\text{loc}}^{1,\alpha}$  away from  $t = T$ , and we conclude as before, except that we are only able to obtain that  $\gamma \in C_{\text{loc}}^{2,\alpha}((a_0, b_0) \times [0, T])$ . However, since we know from Theorem 4.3 that  $u \in C_{\text{loc}}^{2,\alpha}((\mathbb{R} \times [0, T]) \cap \{m > 0\})$ , the regularity of  $\gamma$  up to  $t = T$  follows from (4.47).  $\square$

We now obtain the optimal upper bound for the time evolution of the quantity  $\gamma_x$ , which is attained by the self-similar solutions of Section 2.

**Corollary 4.11** (Upper bound on  $\gamma_x$ ). *Under the assumptions of Theorem 4.10, let  $(u, m)$  be a solution to (MFG) or (MFGP), let  $\gamma$  be the flow of optimal trajectories for  $(u, m)$ , and define*

$$d(t) = \begin{cases} t & \text{if } u \text{ solves (MFG),} \\ \text{dist}(t, \{0, T\}) & \text{if } u \text{ solves (MFGP).} \end{cases} \quad (4.50)$$

Then there exists a constant  $C > 0$ , with

$$C = \begin{cases} C(C_0, \|f(m_0)_{xx}\|_{L^\infty(\{m_0 > 0\})}, \frac{\theta_1}{\theta}, c_1) & \text{if } u \text{ solves (MFG),} \\ C(C_0, C_1, \|f(m_0)_{xx}\|_{L^\infty(\{m_0 > 0\})}) & \text{if } u \text{ solves (MFGP),} \end{cases} \quad (4.51)$$

independent of  $T$ , such that

$$\gamma_x(x, t) \leq C(1 + d(t)^{\bar{\alpha}}).$$

*Proof.* In the proof of Theorem 4.10, we showed (4.44). In fact, by simply following the proof, one readily sees that the constant  $C$  in (4.44) may be chosen independently of  $T$ , and depending only on the quantities specified in (4.51). This observation applies for both (MFG) and (MFGP). Thus, for (MFG) there is nothing left to prove. As for (MFGP), repeating exactly the same argument for the function  $w = v - K(T - t)^{\bar{\alpha}}$  yields

$$\gamma_x(x, t) \leq C(1 + (T - t)^{\bar{\alpha}}).$$

Thus, combining this with (4.44), we conclude that, for  $(u, m)$  solving (MFGP)

$$\gamma_x(x, t) \leq C(1 + \min(t^{\bar{\alpha}}, (T - t)^{\bar{\alpha}})) = C(1 + d(t)^{\bar{\alpha}}).$$

$\square$

### 4.2.2 $C^{1,1}$ regularity, strict convexity, strict monotonicity, and long time behavior

In this subsection we obtain, under adequate compatibility and non-degeneracy assumptions on the data, uniform  $W^{2,\infty}(0, T)$  estimates for the free boundary. Additionally, we obtain strict convexity and strict concavity for the left and right free boundary curves, respectively, and prove that for the terminal cost problem, (MFG), the boundary is spreading outward. Finally, we quantify the exact rate of propagation of the support and the exact rate of decay in time for the density, which are the ones exhibited by the self-similar solutions of Proposition 2.1.

To obtain these extra properties on the free boundary, we strengthen the assumptions of the previous subsection. In particular, we will require the following compatibility condition between terminal and initial data, namely that

$$\alpha_0 = \alpha_1, \theta_1 = \theta. \quad (4.52)$$

where  $\alpha_0, \alpha_1, \theta_1$  are defined in (4.27)–(4.30). This kind of assumption will guarantee that the function  $\gamma_x$  is well-behaved at  $t = T$ . We will also strengthen the nondegeneracy assumption on  $m_0$  by requiring

$$f(m_0)_{xx} \leq 0 \text{ in } \{x \in (a_0, b_0) : \text{dist}(x, \{a_0, b_0\}) < \delta\} \text{ for some } \delta > 0. \quad (4.53)$$

We observe that, since  $f(m_0)$  is Lipschitz, (4.53) necessarily implies that  $\alpha_0 = \frac{1}{\theta}$  in (4.27), and

$$f(m_0)_x(b_0^-) < 0 < f(m_0)_x(a_0^+). \quad (4.54)$$

We begin by obtaining a uniform lower bound on  $\gamma_x(\cdot, T)$  for solutions to (MFGP).

**Lemma 4.12.** *Under the assumptions of Theorem 4.10, let  $(u, m)$  be a solution to (MFGP), and assume that  $\alpha_0 = \alpha_1$ . Then there exists a constant  $C = C(C_0, C_1)$ , such that, for  $x \in (a_0, b_0)$ ,*

$$\gamma_x(x, T) \geq \frac{1}{C}.$$

*Proof.* With no loss of generality, we assume that  $x \in (a_0, \frac{1}{2}(a_0 + b_0))$ , and that  $x$  is close enough to  $a_0$  to guarantee that  $\gamma(x, T) \in (a_1, \frac{1}{2}(a_1 + b_1))$ . We have, by conservation of mass,

$$\int_{a_0}^x m_0 = \int_{a_1}^{\gamma(x, T)} m_T(\gamma(\cdot, T)).$$

Thus, in view of (4.27) and (4.28),

$$C_0(x - a_0)^{\alpha_0+1} \geq \frac{1}{C_1}(\gamma(x, T) - a_1)^{\alpha_0+1} \implies \frac{(\gamma(x, T) - a_1)^{\alpha_0}}{(x - a_0)^{\alpha_0}} \leq (C_0 C_1)^{\frac{\alpha_0}{\alpha_0+1}}.$$

This, combined with (4.27) and (4.28) once more, implies that

$$\gamma_x(x, T) = \frac{m_0(x)}{m_T(\gamma(x, T), T)} \geq (C_0 C_1)^{-1} \frac{(x - a_0)^{\alpha_0}}{(\gamma(x, T) - a_1)^{\alpha_0}} \geq (C_0 C_1)^{-(1 + \frac{\alpha_0}{\alpha_0+1})}.$$

□

Next we obtain a global lower bound on  $\gamma_x$ . We also show that when (4.53) is strengthened to be both strict and global, as occurs in the self-similar solution, this bound can be improved to yield the optimal rate at which  $\gamma_x$  may grow in time, complementing the upper bound of Corollary 4.11.

**Proposition 4.13.** *(Lower bounds on  $\gamma_x$ ) Under the assumptions of Theorem 4.10, let  $(u, m)$  be a solution to (MFG) or (MFGP). Assume that (4.52) and (4.53) hold. Then we have:*

(i) *There exists a constant  $C > 0$  such that, for  $(x, t) \in (a_0, b_0) \times [0, T]$ ,  $m(\gamma(x, t), t) \leq C m_0(x)$ , that is,*

$$\gamma_x(x, t) \geq \frac{1}{C}, \quad (4.55)$$

and

$$C = \begin{cases} C(C_0, \delta^{-1}) & \text{if } u \text{ solves (MFG),} \\ C(C_0, C_1, \delta^{-1}) & \text{if } u \text{ solves (MFGP).} \end{cases} \quad (4.56)$$

(ii) Assume, in addition, that  $f(m_0)_{xx} \leq -\frac{1}{K} < 0$  in  $(a_0, b_0)$ , and that  $c_1 > 0$  in (4.30), and let  $d$  be defined as in (4.50). Then there exists a constant  $C > 0$  such that the sharp estimate

$$\gamma_x(x, t) \geq \frac{1}{C}(1 + d(t)^{\bar{\alpha}}) \quad (4.57)$$

holds, and

$$C = \begin{cases} C(C_0, c_1^{-1}) & \text{if } u \text{ solves (MFG),} \\ C(C_0, C_1, K) & \text{if } u \text{ solves (MFGP).} \end{cases} \quad (4.58)$$

*Proof.* We first treat the case in which  $(u, m)$  solves (MFGP). Observe that, since  $f \in C^\infty(0, \infty)$ , the interior Schauder estimates applied to (3.1) imply that the solution  $(u, m)$  is  $C^\infty$  in the set  $\{m > 0\} \cap \{0 < t < T\}$ , and so by Lemma 3.5 the function

$$v(x, t) = f(m(\gamma(x, t), t))$$

solves, for  $(x, t) \in [a_0, b_0] \times [0, T]$ ,

$$-v_{tt} - \frac{\theta v}{\gamma_x^2} v_{xx} + v_x \frac{\theta v}{\gamma_x^3} \gamma_{xx} + \frac{\theta + 1}{\theta} v^{-1} v_t^2 = 0. \quad (4.59)$$

Since  $v = \frac{f(m_0)}{\gamma_x^\theta}$  (by (3.10)), using (4.32) and  $\gamma_{tt} = -\frac{v_x}{\gamma_x}$  (by (3.9)) we deduce

$$\begin{aligned} -v_{tt} - \frac{\theta v}{\gamma_x^2} v_{xx} &= -\frac{v_x}{\gamma_x^2} \left( \frac{\theta f(m_0) \gamma_{xx}}{\gamma_x^{1+\theta}} \right) - \frac{\theta + 1}{\theta} v^{-1} v_t^2 = -\frac{v_x}{\gamma_x^2} \left( \frac{f(m_0)_x}{\gamma_x^\theta} - \gamma_x \gamma_{tt} \right) - \frac{\theta + 1}{\theta} v^{-1} v_t^2 \\ &\leq -\frac{v_x}{\gamma_x^2} \left( \frac{f(m_0)_x}{\gamma_x^\theta} - v_x \right). \end{aligned} \quad (4.60)$$

Now, given  $0 < \varepsilon < 1$ , for each  $(x, t) \in [a_0, b_0] \times [0, T]$ , we let

$$w(x, t) = v(x, t) - Cf(m_0)(x) - \varepsilon t,$$

where  $C \geq 1$  is a constant large enough to guarantee that  $w(x, T) \leq 0$ , and  $w \leq 0$  on the set  $\{(x, t) : \text{dist}(x, \{a_0, b_0\}) \geq \delta\}$ . Such a constant exists because of Lemma 4.12, and because, by (4.27),  $m_0 \geq \frac{1}{C_0} \delta^{\alpha_0}$  on  $\{x : \text{dist}(x, \{a_0, b_0\}) \geq \delta\}$ . Observe that, as a result of (4.53) and (4.54), we must have

$$f(m_0)_x \neq 0 \text{ on } \{x : 0 < \text{dist}(x, \{a_0, b_0\}) < \delta\}. \quad (4.61)$$

Let  $(x_0, t_0)$  be an interior maximum of  $w$  on the set

$$S := \{w > -\varepsilon t\} \cap \{\text{dist}(x, \{a_0, b_0\}) < \delta\}.$$

We note that, in  $S$ ,  $v(x, t) > Cf(m_0)$ . Moreover, at  $(x_0, t_0)$ ,  $v_x = Cf(m_0)_x$  and  $D^2 w \leq 0$ . Therefore, in view of (4.53), (4.60), and (4.61), we infer that

$$0 \leq -w_{tt} - \frac{\theta v}{\gamma_x^2} w_{xx} \leq \frac{-1}{\gamma_x^2} C \frac{f(m_0)_x^2}{f(m_0)} (v - Cf(m_0)) + \frac{C\theta v}{\gamma_x^2} f(m_0)_{xx} < 0. \quad (4.62)$$

This is a contradiction, so the maximum of  $w$  in  $S$  is achieved at  $\partial S$ . At such a point, we have either  $w = -\varepsilon t$ ,  $t = 0$ ,  $t = T$ , or  $\text{dist}(x, \{a_0, b_0\}) = \delta$ . By our choice of  $C$ , in each of these cases we have  $w \leq 0$ . Therefore,  $w \leq 0$  on all of  $[a_0, b_0] \times [0, T]$ , that is,

$$f(m(\gamma(x, t), t)) \leq Cf(m_0(x)) + \varepsilon t,$$

and the result follows by letting  $\varepsilon \rightarrow 0$ .

Now, to modify this for the terminal cost problem  $u(\cdot, T) = c_1 T f(m(\cdot, T))$  (recall that (4.52) holds), the only issue we must address is that we do not know a priori that  $w \leq 0$  at  $t = T$ . It is therefore enough to

prove that  $w$  cannot achieve a maximum in  $S$  at some  $(x_0, T)$ , where  $x_0 \in (a_0, b_0)$ . Assume otherwise. On one hand, we have, by (3.8) and the continuity equation,

$$\begin{aligned} v_t(x, T) &= f'(m)(m_t + m_x \gamma_t)(\gamma(x, T), T) = f'(m)(m_t - m_x u_x)(\gamma(x, T)) \\ &= f'(m) m u_{xx}(\gamma(x, T), T) = \theta v(x, T) u_{xx}(\gamma(x, T), T). \end{aligned} \quad (4.63)$$

On the other hand, by the definition of  $v$  and the chain rule,

$$f(m)_{xx}(\gamma(x, T), T) = \left( \frac{v_x}{\gamma_x} \right)_x(x, T) \frac{1}{\gamma_x(x, T)}. \quad (4.64)$$

Differentiating the terminal condition implies that  $u_{xx}(\cdot, T) = c_1 T f(m)_{xx}(\cdot, T)$ , so that, in view of (4.63) and (4.64), we obtain

$$v_t = \theta v u_{xx}(\gamma(x_0, T), T) = \theta c_1 T v f(m)_{xx}(\gamma(x_0, T), T) = \theta c_1 T v \left( \frac{1}{\gamma_x^2} v_{xx} - v_x \frac{1}{\gamma_x^3} \gamma_{xx} \right). \quad (4.65)$$

Now, notice that  $w_{xx}(x_0, T) = v_{xx} - C f(m_0)_{xx} \leq 0$  and  $f(m_0)_{xx}(x_0) \leq 0$ , so  $v_{xx} \leq 0$ . Hence, in view of (4.32) and (3.9), we obtain

$$v_t \leq -\theta c_1 T v_x \frac{v}{\gamma_x^3} \gamma_{xx} = -c_1 T v_x \left( \frac{f(m_0)_x}{\gamma_x^{2+\theta}} - \frac{v_x}{\gamma_x^2} \right). \quad (4.66)$$

As a result, since  $w_t \geq 0$  and  $w_x = 0$ , that is,  $v_t \geq \varepsilon$  and  $v_x = C f(m_0)_x$ , recalling (4.61) and the definition of  $S$ , we get

$$\varepsilon \leq v_t \leq -c_1 T C \frac{1}{\gamma_x^2} \frac{f(m_0)_x^2}{f(m_0)} (v - C f(m_0)) \leq 0.$$

This is a contradiction, which proves (4.55).

To prove part (ii), we now assume that  $f(m_0)_{xx} \leq -\frac{1}{K}$ , where  $K > 0$ , and again we focus first on the case where  $(u, m)$  solves (MFGP). We repeat the above argument, but with a different choice for the function  $w$ , namely

$$w(x, t) = v(x, t) - \zeta(t) f(m_0)(x), \quad \text{where } \zeta(t) = C \left( \frac{1}{t^{\bar{\alpha}\theta}} + \frac{1}{(T-t)^{\bar{\alpha}\theta}} \right). \quad (4.67)$$

Since  $f(m_0)$  is now globally concave, we may also redefine  $S$  to be simply

$$S := \{w > 0\}.$$

Then, instead of (4.62), we obtain

$$\begin{aligned} 0 \leq -w_{tt} - \frac{\theta v}{\gamma_x^2} w_{xx} &\leq \frac{-1}{\gamma_x^2} \zeta(t_0) \frac{f(m_0)_x^2}{f(m_0)} w + \zeta''(t_0) f(m_0) + \frac{\zeta(t_0) \theta v}{\gamma_x^2} f(m_0)_{xx} \\ &\leq f(m_0)^{-\frac{2}{\theta}} \left( f(m_0)^{1+\frac{2}{\theta}} \zeta''(t_0) + \zeta(t_0) \theta f(m_0)_{xx} v^{1+\frac{2}{\theta}} \right) \end{aligned} \quad (4.68)$$

where we used that  $\gamma_x = \left( \frac{f(m_0)}{v} \right)^{\frac{1}{\theta}}$  due to (3.10). Since  $f(m_0)_{xx} \leq -\frac{1}{K}$ , and  $v > \zeta(t) f(m_0)$  because  $w > 0$ , we estimate the right hand side of (4.68) obtaining

$$0 \leq -f(m_0) \left( -\zeta''(t_0) + \theta \frac{1}{K} \zeta(t_0)^{2+\frac{2}{\theta}} \right),$$

It is straightforward to check that, choosing  $C$  sufficiently large, independently of  $T$ ,  $\zeta$  satisfies

$$-\zeta''(t) + \theta \frac{1}{K} \zeta(t)^{2+\frac{2}{\theta}} > 0.$$



Therefore, with this choice of  $\zeta$ , we obtain a contradiction. Moreover, since  $w(\cdot, 0) \equiv w(\cdot, T) \equiv -\infty$ , we conclude that the set  $S$  must be empty. That is,

$$v(x, t) \leq \zeta(t)f(m_0)(x), \quad (x, t) \in (a_0, b_0) \times (0, T),$$

which, combined with (4.55), readily implies (4.57). Finally, we prove (4.57) for the case in which  $(u, m)$  solves (MFG). We again define  $w$  according to (4.67), but this time  $\zeta$  is defined by

$$\zeta(t) = \frac{C}{t^{\bar{\alpha}\theta}}, \quad t \in (0, T).$$

We may now follow the same proof as for (MFGP), with the only issue being that we no longer have  $w(\cdot, T) \equiv -\infty$ , and, thus, we must consider the case of a maximum point  $(x_0, T)$  of  $w$ . We begin by noticing that, since  $w_{xx} = v_{xx} - \zeta(T)f(m_0)_{xx} \leq 0$ ,  $w_x = 0$ , and  $w_t \geq 0$ , (4.65) implies

$$\zeta'(T)f(m_0) \leq v_t \leq -\frac{\theta c_1 T v}{\gamma_x^2} \frac{1}{K} \zeta(T) - c_1 T C \frac{1}{\gamma_x^2} \frac{f(m_0)_x^2}{f(m_0)} w \leq -\frac{\theta c_1 T v}{\gamma_x^2} \frac{1}{K} \zeta(T).$$

This may be rearranged as

$$v(x_0, T) \leq \left( \frac{\bar{\alpha}K}{c_1} \right)^{\frac{\theta}{2+\theta}} T^{-\bar{\alpha}\theta} f(m_0)(x_0) := C_2 T^{-\bar{\alpha}\theta} f(m_0)(x_0).$$

Therefore, if we choose  $C > C_2$ , we conclude that  $v(x_0, T) \leq \zeta(T)f(m_0)(x_0)$ , as wanted.  $\square$

We can now establish our main regularity result for the free boundary.

**Theorem 4.14** (Regularity and convexity of the free boundary). *Under the assumptions of Theorem 4.10, let  $(u, m)$  be a solution to (MFG) or (MFGP). Assume that (4.52) and (4.53) hold. Let  $\gamma_L = \gamma(a_0, \cdot)$ ,  $\gamma_R = \gamma(b_0, \cdot)$  be, respectively, the left and right free boundary curves. Then  $\gamma_L, \gamma_R \in W^{2,\infty}(0, T)$ , and there exist constants  $K_1, K_2$ , with*

$$K_1 = K_1(C_0, \|\gamma_x\|_\infty, |f(m_0)_x(a_0^+)|^{-1}, |f(m_0)_x(b_0^-)|^{-1}) \quad \text{and} \quad K_2 = K_2(C_0, \|\gamma_x^{-1}\|_\infty, |f(m_0)_x(a_0^+)|, |f(m_0)_x(b_0^-)|)$$

such that, for a.e.  $t \in [0, T]$ ,

$$\frac{1}{K_1} \leq \dot{\gamma}_L(t) \leq K_2, \quad \text{and} \quad -K_2 \leq \dot{\gamma}_R(t) \leq -\frac{1}{K_1}. \quad (4.69)$$

Moreover, when  $(u, m)$  solves (MFG), we have, for  $t \in [0, T]$ ,

$$-K_2(c_1 T + (T-t)) \leq \dot{\gamma}_L(t) \leq -\frac{1}{K_1}(c_1 T + (T-t)), \quad \text{and} \quad \frac{1}{K_1}(c_1 T + (T-t)) \leq \dot{\gamma}_R(t) \leq K_2(c_1 T + (T-t)). \quad (4.70)$$

*Proof.* By symmetry, it is enough to show the estimates for  $\gamma_L$ . Let  $t \in (0, T)$ , and let  $h > 0$  be such that  $(t-h, t+h) \subset (0, T)$ . We begin by noting that (4.32) may be written as

$$\gamma_{tt} = \frac{1}{\gamma_x} \left( \frac{f(m_0)}{\gamma_x^\theta} \right)_x.$$

We therefore have, for  $(x_0, \tau) \in (a_0, \frac{a_0+b_0}{2}) \times [0, T]$ ,

$$\int_{a_0}^{x_0} \gamma_{tt}(x, \tau) dx = \int_{a_0}^{x_0} \frac{1}{\gamma_x} \left( \frac{f(m_0)}{\gamma_x^\theta} \right)_x dx = \frac{f(m_0)}{\gamma_x^{1+\theta}}(x_0) - \int_{a_0}^{x_0} \left( \frac{1}{\gamma_x} \right)_x \frac{f(m_0)}{\gamma_x^\theta} dx, \quad (4.71)$$

where in the last step we integrated by parts and used the fact that  $\gamma_x$  is bounded below and  $f(m_0)(a_0) = 0$ . Using the identity

$$\left( \frac{1}{\gamma_x} \right)_x \frac{f(m_0)}{\gamma_x^\theta} = \frac{1}{\theta} \left( \frac{1}{\gamma_x} \left( \frac{f(m_0)}{\gamma_x^\theta} \right)_x - \frac{f(m_0)_x}{\gamma_x^{1+\theta}} \right),$$

we infer from (4.71) that

$$\int_{a_0}^{x_0} \gamma_{tt}(x, \tau) dx = \left(1 - \frac{1}{\theta}\right) \frac{f(m_0)}{\gamma_x^{1+\theta}}(x_0) + \frac{1}{\theta} \int_{a_0}^{x_0} \left(\frac{1}{\gamma_x}\right)_x \frac{f(m_0)}{\gamma_x^\theta} dx + \frac{1}{\theta} \int_{a_0}^{x_0} \frac{f(m_0)_x}{\gamma_x^{1+\theta}} dx.$$

Multiplying this equality by  $\theta$  and adding to (4.71) yields

$$\int_{a_0}^{x_0} \gamma_{tt}(x, \tau) dx = \frac{\theta}{\theta+1} \frac{f(m_0)}{\gamma_x^{1+\theta}}(x_0) + \frac{1}{\theta+1} \int_{a_0}^{x_0} \frac{f(m_0)_x}{\gamma_x^{1+\theta}} dx.$$

Recalling that  $f(m_0)$  is Lipschitz, with  $f(m_0)(a_0) = 0$ , and that  $\gamma_x$  is bounded above and below, we conclude that, for  $x_0$  sufficiently close to  $a_0$ ,

$$\frac{1}{C_1}(x_0 - a_0) \leq \int_{a_0}^{x_0} \gamma_{tt}(x, \tau) dx \leq C_2(x_0 - a_0). \quad (4.72)$$

Next, observe that, for  $x \in (a_0, x_0)$ ,

$$\gamma(x, t+h) + \gamma(x, t-h) - 2\gamma(x, t) = \int_0^h \int_{t-s}^{t+s} \gamma_{tt}(x, \tau) d\tau ds,$$

and, therefore, integrating both sides,

$$\int_{a_0}^{x_0} \gamma(x, t+h) + \gamma(x, t-h) - 2\gamma(x, t) dx = \int_0^h \int_{t-s}^{t+s} \int_{a_0}^{x_0} \gamma_{tt}(x, \tau) dx d\tau ds.$$

Using (4.72), we see that this yields

$$\frac{1}{C}(x_0 - a_0)h^2 \leq \int_{a_0}^{x_0} \gamma(x, t+h) + \gamma(x, t-h) - 2\gamma(x, t) dx \leq C(x_0 - a_0)h^2,$$

so, dividing by  $(x_0 - a_0)$  and letting  $x_0 \rightarrow a_0^+$ , we obtain

$$\frac{1}{C}h^2 \leq \gamma_L(t+h) + \gamma_L(t-h) - 2\gamma_L(t) \leq Ch^2,$$

which yields (4.69).

Now assume that  $(u, m)$  solves (MFG). Recall that, since  $u(\cdot, T) = c_1 T f(m(\cdot, T))$ , we have

$$\gamma_t(\cdot, T) = -c_1 T \gamma_{tt}(\cdot, T). \quad (4.73)$$

We observe that, by Taylor's theorem, for  $x \in (a_0, b_0)$  and small  $h > 0$ , we have

$$\gamma(x, T-h) = \gamma(x, T) - h\gamma_t(x, T) + \int_{T-h}^T (s - (T-h))\gamma_{tt}(x, s) ds.$$

Thus, integrating from  $a_0$  to  $x_0$  and using (4.73), we obtain

$$\int_{a_0}^{x_0} \gamma(x, T-h) - \gamma(x, T) dx = hc_1 T \int_{a_0}^{x_0} \gamma_{tt}(x, T) dx + \int_{T-h}^T \int_{a_0}^{x_0} (s - (T-h))\gamma_{tt}(x, s) ds dx,$$

so we infer from (4.72) that, for  $x_0$  sufficiently close to  $a_0$ ,

$$\frac{c_1 T}{C} h(x_0 - a_0) + \frac{1}{2C} h^2(x_0 - a_0) \leq \int_{a_0}^{x_0} \gamma(x, T-h) - \gamma(x, T) dx \leq Cc_1 T h(x_0 - a_0) + \frac{1}{2} Ch^2(x_0 - a_0).$$

Dividing by  $(x_0 - a_0)$  and letting  $x_0 \rightarrow a_0^+$ , we see that

$$\frac{c_1 T}{C} h + \frac{1}{2C} h^2 \leq \gamma_L(T-h) - \gamma_L(T) \leq Cc_1 T h + \frac{1}{2} Ch^2.$$

Finally, dividing by  $h$  and letting  $h \rightarrow 0^+$  yields

$$-Cc_1T \leq \dot{\gamma}_L(T) \leq \frac{-c_1T}{C}.$$

Thus, in view of (4.69), and noting that  $\dot{\gamma}_L(t) = \dot{\gamma}_L(T) - \int_t^T \ddot{\gamma}_L(s)ds$ , we obtain (4.70).  $\square$

Finally, we show that the support grows with algebraic rate  $\bar{\alpha} = \frac{2}{2+\theta}$ , and the density decays to 0 with algebraic rate  $-\bar{\alpha}$ , as is expected from the model case of Section 2.

**Theorem 4.15** (Optimal rate of propagation and long time decay). *Under the assumptions of Theorem 4.10, let  $(u, m)$  be a solution to (MFG) or (MFGP), let  $\gamma$  be the associated flow of optimal trajectories, and let  $\mathcal{d} : [0, T] \rightarrow [0, \infty)$  be defined by (4.50). Assume also that  $-K \leq f(m_0)_{xx} \leq -\frac{1}{K}$  in  $(a_0, b_0)$  for some  $K > 0$ . Then there exists a constant  $C > 0$ , with*

$$C = \begin{cases} C(C_0, c_1, c_1^{-1}, |a_0|, |b_0|, K) & \text{if } u \text{ solves (MFG),} \\ C(C_0, C_1, |a_0|, |b_0|, |a_1|, |b_1|, K) & \text{if } u \text{ solves (MFGP),} \end{cases} \quad (4.74)$$

such that, for every  $(x, t) \in [a_0, b_0] \times [0, T]$ ,

$$\frac{1}{C}(1 + \mathcal{d}(t)^{\bar{\alpha}}) \leq |\text{supp}(m(\cdot, t))| \leq C(1 + \mathcal{d}(t)^{\bar{\alpha}}), \quad |\gamma(x, t)| \leq C(1 + \mathcal{d}(t)^{\bar{\alpha}}), \quad (4.75)$$

$$\frac{1}{C} \frac{m_0(x)}{(1 + \mathcal{d}(t)^{\bar{\alpha}})} \leq m(\gamma(x, t), t) \leq C \frac{m_0(x)}{(1 + \mathcal{d}(t)^{\bar{\alpha}})}. \quad (4.76)$$

*Proof.* Recalling (3.10), we observe that (4.76) is simply obtained from combining the upper bound on  $\gamma_x$  from Corollary 4.11 and the lower bound on  $\gamma_x$  from Proposition 4.13,

$$\frac{1}{C}(1 + \mathcal{d}(t)^{\bar{\alpha}}) \leq \gamma_x(x, t) \leq C(1 + \mathcal{d}(t)^{\bar{\alpha}}). \quad (4.77)$$

Now, integrating (4.77) between  $a_0$  and  $b_0$  immediately yields the first equation of (4.75). Moreover, for the second equation of (4.75), it suffices to show that, for each  $t_0 \in [0, T]$ , there exists at least one  $x_0 \in [a_0, b_0]$  such that  $|\gamma(x_0, t_0)| \leq C$ . When  $(u, m)$  solves (MFG), this follows from the fact that, by Theorem 4.14,  $\text{supp}(m(\cdot, t))$  is expanding, and, thus, in particular,  $\gamma(x_0, t_0) = a_0$  for some  $x_0 \in [a_0, b_0]$ . On the other hand, if  $(u, m)$  solves (MFGP), Theorem 4.14 implies that  $\text{supp}(m)$  is a convex set. Therefore, we may choose  $\gamma(x_0, t_0)$  to be the first coordinate of the intersection between  $\mathbb{R} \times \{t_0\}$  and the line segment  $\{(1-s)(a_0, 0) + s(a_1, T) : 0 \leq s \leq 1\}$ .  $\square$

### 4.3 Regularity of the solution up to the free boundary

#### 4.3.1 Intrinsic scaling and Hölder continuity of $m$

In this subsection, we show how the bounds on  $\gamma_x$  and the intrinsic scaling of the problem allow us to improve the logarithmic modulus of continuity for  $m$  to a Hölder one. Throughout the section, we continue to assume that  $f$  is given by (4.25), and we will assume the conditions of Theorem 4.14, namely (4.26)–(4.30) together with (4.52) and (4.53). We will focus on obtaining Hölder regularity estimates for the function

$$v(x, t) := f(m(\gamma(x, t), t)), \quad (x, t) \in [a_0, b_0] \times [0, T]. \quad (4.78)$$

This is equivalent to obtaining Hölder estimates for  $m(x, t)$ , in view of (4.33) and (4.55).

Our first result is a simple corollary of the bounds on  $\gamma_x$ , stated in the form of a Harnack inequality (cf. [13, Thm 11.1]).

**Proposition 4.16** (Harnack inequality). *Let the assumptions of Theorem 4.14 be in place, and let  $(u, m)$  be a solution to (MFG) or (MFGP), and let  $v$  be defined by (4.78). There exists a constant*

$$C = C(C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty, \theta)$$

such that the following alternative holds. Let  $x_0 \in (a_0, b_0)$ , and let  $\rho > 0$  be such that  $(x_0 - \rho, x_0 + \rho) \subset (a_0, b_0)$ . Then either  $\rho \geq \frac{1}{2} \text{dist}(x_0, \{a_0, b_0\})$  and  $\sup_{(x_0 - \rho, x_0 + \rho) \times [0, T]} v \leq C\rho$ , or  $\rho < \frac{1}{2} \text{dist}(x_0, \partial(a_0, b_0))$  and

$$\sup_{(x_0 - \rho, x_0 + \rho) \times [0, T]} v \leq C \inf_{(x_0 - \rho, x_0 + \rho) \times [0, T]} v. \quad (4.79)$$

*Proof.* For simplicity, we normalize  $a_0 = 0$ , and by symmetry we may assume that  $x_0 < \frac{b_0}{2}$ . Then, in view of (4.27) (where, we recall,  $\alpha_0 = \frac{1}{\theta}$  under the current assumptions), and the fact that  $\gamma_x = m_0 v^{-\frac{1}{\theta}}$  is bounded above and below, we have, for some constant  $C$ , and every  $(x, t) \in (x_0 - \rho, x_0 + \rho) \times [0, T]$ ,

$$\frac{1}{C}x \leq v(x, t) \leq Cx. \quad (4.80)$$

**Case 1:**  $\rho < \frac{1}{2}x_0$ . Then (4.80) implies

$$\sup_{(x_0 - \rho, x_0 + \rho) \times [0, T]} v \leq C \left( \frac{3x_0}{2} \right) = 3C^2 \left( \frac{1}{C} \left( \frac{1}{2}x_0 \right) \right) \leq 3C^2 \inf_{(x_0 - \rho, x_0 + \rho) \times [0, T]} v.$$

**Case 2:**  $\frac{1}{2}x_0 \leq \rho$ . Then, in view of (4.80)

$$\sup_{(x_0 - \rho, x_0 + \rho) \times [0, T]} v \leq C(x_0 + \rho) \leq 3C\rho.$$

□

We note that, unlike in the linear theory, the above Harnack inequality is not yet sufficient to obtain Hölder regularity, because the result does not hold for, say, translations of the solution. Instead, we will proceed by obtaining analogues of the Caccioppoli inequality and De Giorgi type lemmas, adapted to the scaling of the equation satisfied by  $v$  (recall (3.16)), which is much more diffusive in time than in space near the free boundary.

**Lemma 4.17** (Intrinsic Caccioppoli inequality). *Let the assumptions of Theorem 4.14 hold, and let  $(u, m)$  be a solution to (MFG) or (MFGP). There exists a positive constant  $C = C(\|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty)$  such that, for each  $\zeta \in C_c^\infty((a_0, b_0) \times (0, T))$ , and each  $k \geq 0$ , we have*

$$\int_{a_0}^{b_0} \int_0^T \zeta^2 ((\psi^\pm(v))_x^2 + (\theta v)^{-1} (\psi^\pm(v))_t^2) dt dx \leq C \int_{a_0}^{b_0} \int_0^T \psi^\pm(v)^2 (\zeta_x^2 + (\theta v)^{-1} \zeta_t^2) dt dx, \quad (4.81)$$

where  $\psi^\pm(v) = (v - k)^\pm$ .

*Proof.* In view of Lemma 3.5, and recalling that  $v$  is smooth in  $(a_0, b_0) \times (0, T)$ , we know that  $v$  satisfies

$$-(\gamma_x^{-1} v_x)_x - (\gamma_x (\theta v)^{-1} v_t)_t = 0. \quad (4.82)$$

Testing this equation against the function  $\psi(v)^\pm \zeta^2$  yields

$$\int_{a_0}^{b_0} \int_0^T \zeta^2 (\gamma_x^{-1} v_x^2 + \gamma_x (\theta v)^{-1} v_t^2) (\psi^\pm)'(v) = - \int_{a_0}^{b_0} \int_0^T 2\zeta \psi^\pm(v) (\gamma_x^{-1} v_x \zeta_x + \gamma_x (\theta v)^{-1} v_t \zeta_t). \quad (4.83)$$

We may estimate the right hand side as follows:

$$\begin{aligned} \left| \int_{a_0}^{b_0} \int_0^T 2\zeta \psi^\pm(v) (\gamma_x^{-1} v_x \zeta_x + \gamma_x (\theta v)^{-1} v_t \zeta_t) \right| &\leq \frac{1}{2} \int_{a_0}^{b_0} \int_0^T \zeta^2 (\gamma_x^{-1} v_x^2 + \gamma_x (\theta v)^{-1} v_t^2) \\ &\quad + 2 \int_{a_0}^{b_0} \int_0^T \psi^\pm(v)^2 (\gamma_x^{-1} \zeta_x^2 + \gamma_x (\theta v)^{-1} \zeta_t^2). \end{aligned} \quad (4.84)$$

We also notice that  $(\psi^\pm)'(v) = \pm \chi_{\pm(v-k) \geq 0}$ , so that, as a result of (4.83) and (4.84),

$$\int_{a_0}^{b_0} \int_0^T \zeta^2 (\gamma_x^{-1} (\psi^\pm(v))_x^2 + \gamma_x (\theta v)^{-1} (\psi^\pm(v))_t^2) \leq 4 \int_{a_0}^{b_0} \int_0^T \psi^\pm(v)^2 (\gamma_x^{-1} \zeta_x^2 + \gamma_x (\theta v)^{-1} \zeta_t^2).$$

The result now follows from the fact that  $\gamma_x$  is bounded above and below by positive constants.  $\square$

Our De Giorgi type lemma will be proved for the following special domains, adapted to the scaling of the equation (4.82).

**Definition 4.18.** Given  $(x_0, t_0) \in (a_0, b_0) \times (0, T)$  and  $\rho > 0$ , we define the intrinsic rectangle  $R_\rho(x_0, t_0)$  of radius  $\rho$  centered at  $(x_0, t_0)$  by

$$R_\rho(x_0, t_0) := (x_0 - \rho, x_0 + \rho) \times (t_0 - (\theta v(x_0, t_0))^{-\frac{1}{2}} \rho, t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}} \rho).$$

**Lemma 4.19** (De Giorgi type lemma on intrinsic rectangles). *Let the assumptions of Theorem 4.14 hold, and let  $(u, m)$  be a solution to (MFG) or (MFGP). There exists a positive constant  $\nu$ , with*

$$\nu^{-1} = \nu^{-1}(C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty),$$

such that the following holds. Let  $(x_0, t_0) \in (a_0, b_0) \times (0, T)$ ,  $0 < r < \frac{1}{4}$ ,

$$\mu^- = \min_{R_{4\rho}(x_0, t_0)} v, \quad \mu^+ = \max_{R_{4\rho}(x_0, t_0)} v, \quad \omega = \mu^+ - \mu^- = \text{osc}_{R_{4\rho}(x_0, t_0)}(v),$$

and

$$\rho \leq \frac{1}{8} \min \left( \text{dist}(x_0, \{a_0, b_0\}), (\theta v(x_0, t_0))^{-\frac{1}{2}} \text{dist}(t_0, \{0, T\}) \right). \quad (4.85)$$

Then

$$|\{\pm(v - (\mu^\pm \mp 2r\omega)) > 0\} \cap R_{2\rho}(x_0, t_0)| \leq \nu |R_{2\rho}(x_0, t_0)| \text{ implies that } \pm v \leq \pm(\mu^\pm \mp r\omega) \text{ in } R_\rho(x_0, t_0).$$

*Proof.* We begin by noting that, in view of (4.85) and Proposition 4.16,  $R_{4\rho}(x_0, t_0) \subset (a_0, b_0) \times (0, T)$  and, for some constant  $k_1 > 0$ ,

$$\max_{R_{4\rho}(x_0, t_0)} v \leq k_1 \min_{R_{4\rho}(x_0, t_0)} v. \quad (4.86)$$

We now define, for  $n \geq 0$ ,

$$r_n = r + \frac{1}{2^n} r, \quad \rho_n = \rho + \frac{1}{2^n} \rho, \quad k_n^\pm = \mu^\pm \mp r_n \omega.$$

and we choose non-negative functions  $\zeta_n \in C_c^\infty((a_0, b_0) \times [0, T])$  such that  $\zeta_n \equiv 1$  on  $R_{\rho_{n+1}}(x_0, t_0)$ ,  $\zeta_n \equiv 0$  outside of  $R_{\rho_n}(x_0, t_0)$ , and

$$|(\zeta_n)_x| \leq \frac{C2^n}{\rho}, \quad |(\zeta_n)_t| \leq \frac{C2^n (\theta v(x_0, t_0))^{-\frac{1}{2}}}{\rho}.$$

As a result of Lemma 4.17, (4.81) holds when taking  $\zeta = \zeta_n$  and

$$\psi(v) = \psi_n^\pm(v) := (v - k_n^\pm)_\pm.$$

On  $(x, t) \in Q_{2\rho} := (x_0 - 2\rho, x_0 + 2\rho) \times (t_0 - 2\rho, t_0 + 2\rho)$ , we now define the rescaled functions  $w_n, \bar{\zeta}_n$  by

$$w_n^\pm(x, t) = \psi_n^\pm(v(x, t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}}(t - t_0))), \quad \bar{\zeta}_n(x, t) = \zeta_n(x, t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}}(t - t_0)).$$

We see that (4.81) may be written as

$$\int_{Q_{2\rho}} \bar{\zeta}_n^2 ((w_n^\pm)_x^2 + v(x_0, t_0) v^{-1} (w_n^\pm)_t^2) \leq C \int_{Q_{2\rho}} (w_n^\pm)^2 ((\bar{\zeta}_n)_x^2 + v(x_0, t_0) v^{-1} (\bar{\zeta}_n)_t^2).$$

In view of (4.86), up to increasing the constant  $C$ , we thus obtain

$$\int_{Q_{2\rho}} \bar{\zeta}_n^2 |Dw_n^\pm|^2 \leq C \int_{Q_{2\rho}} |D\bar{\zeta}_n|^2 (w_n^\pm)^2 \leq \frac{C4^n}{\rho^2} \int_{Q_{2\rho}} (w_n^\pm)^2. \quad (4.87)$$

This is now the usual (rather than intrinsic) Caccioppoli inequality, and thus by the standard De Giorgi iteration argument (e.g. see [34, Lem. 5]), writing  $w_\infty^\pm = \lim_{n \rightarrow \infty} w_n^\pm$ , we see that there exists  $\nu > 0$  such that

$$|\{w_0^\pm > 0\} \cap Q_{2\rho}| \leq \nu |Q_{2\rho}| \text{ implies that } w_\infty^\pm = 0 \text{ in } Q_\rho. \quad (4.88)$$

Now, writing

$$\psi_\infty^\pm(v) = (v - (\mu^\pm \mp r\omega))_\pm,$$

we have  $w_\infty^\pm = \psi_\infty^\pm(v(x, t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}}(t - t_0)))$ . Scaling back the time variable we have  $|Q_{2\rho}| = (\theta v(x_0, t_0))^{\frac{1}{2}} |R_{2\rho}(x_0, t_0)|$  and

$$|\{w_0^\pm > 0\} \cap Q_{2\rho}| = (\theta v(x_0, t_0))^{\frac{1}{2}} |\{\psi_0^\pm(v) > 0\} \cap R_{2\rho}(x_0, t_0)|.$$

Thus, we conclude the proof by noticing that (4.88) may be equivalently written as:

$$|\{\psi_0^\pm(v) > 0\} \cap R_{2\rho}(x_0, t_0)| \leq \nu |R_{2\rho}(x_0, t_0)| \text{ implies that } \psi_\infty^\pm(v) = 0 \text{ in } R_\rho(x_0, t_0).$$

□

**Corollary 4.20** (Reduction of oscillation). *Let the assumptions of Theorem 4.14 be in place. Assume that the intrinsic rectangle  $R_\rho(x_0, t_0)$  satisfies (4.85). There exists a constant  $0 < \sigma < 1$ , independent of the choice of  $x_0, t_0$  and  $\rho$ , such that*

$$\text{osc}_{R_\rho(x_0, t_0)} v \leq \sigma \text{osc}_{R_{4\rho}(x_0, t_0)} v. \quad (4.89)$$

*Proof.* The proof of this corollary, included for the reader's convenience, will be a standard application of the classical arguments that yield interior Hölder continuity for functions that satisfy the Caccioppoli inequality (originally due to E. De Giorgi [11]. See also, for instance, [34]). Let  $\nu, r, \mu^+, \mu^-$ , and  $\omega$  be as in Lemma 4.19. To simplify notation, we write  $R_\rho := R_\rho(x_0, t_0)$ .

We begin by defining  $z_0 : [-2, 2] \times [-2, 2] \rightarrow [-1, 1]$  as

$$z_0(x, t) := 2\omega^{-1} \left( v(x_0 + \rho(x - x_0), t_0 + (\theta v(x_0, t_0))^{-\frac{1}{2}} \rho(t - t_0)) - \mu^- \right) - 1.$$

Notice that, since  $16 = |[-2, 2] \times [-2, 2]|$ , we must have

$$|z_0 \leq 0| \geq 8 \text{ or } |z_0 \geq 0| \geq 8. \quad (4.90)$$

We assume the former, and remark that the proof in the alternative case is completely analogous. We define, for  $n \geq 1$ ,  $z_n : [-2, 2] \times [-2, 2] \rightarrow (-\infty, 1]$  by

$$z_n(x, t) := \frac{1}{4r^n} (z_0 - (1 - 4r^n)). \quad (4.91)$$

Notice that  $z_n$  is nonincreasing, and we have

$$z_n \geq 0 \iff v \geq \mu^+ - 2r^n \omega,$$

and, thus, by a change of variables,

$$\begin{aligned} \int_{[-2, 2] \times [-2, 2]} |Dz_n^+|^2 &= \int_{R_{2\rho} \cap \{v \geq \mu^+ - 2r^n \omega\}} \frac{\rho^2}{4r^{2n}\omega^2} (v_x^2 + (\theta v(x_0, t_0))^{-1} v_t^2) \\ &= \frac{\rho^2}{4r^{2n}\omega^2} \int_{R_{2\rho}} (\psi(v)_x^2 + (\theta v(x_0, t_0))^{-1} \psi(v)_t^2), \end{aligned} \quad (4.92)$$

where  $\psi(v) = (v - \mu^+ + 2r^n\omega)^+$ . By Proposition 4.16, we thus infer that

$$\int_{[-2,2] \times [-2,2]} |Dz_n^+|^2 \leq \frac{C\rho^2}{r^{2n}\omega^2} \int_{R_{2\rho}} (\psi(v)_x^2 + (\theta v)^{-1}\psi(v)_t^2) \quad (4.93)$$

where, as usual,  $C$  is a generic constant that could be increased line by line. On the other hand, Proposition 4.16 and Lemma 4.17 imply that

$$\int_{R_{2\rho}} (\psi(v)_x^2 + (\theta v)^{-1}\psi(v)_t^2) \leq \frac{C}{\rho^2} \int_{R_{4\rho}} \psi(v)^2 \leq C\omega^2 r^{2n} (\theta v(x_0, t_0))^{-1/2}. \quad (4.94)$$

Thus, we deduce from the last two inequalities that

$$\int_{[-2,2] \times [-2,2]} |Dz_n^+|^2 \leq C, \quad (4.95)$$

where  $C$  is independent of  $n$ . Now, assume that, for some  $\delta > 0$  and some  $n \geq 1$ , we have

$$|z_{n-1} \leq 0| \geq 8 + (n-1)\delta. \quad (4.96)$$

We will show that, as long as

$$|z_n > 0| \geq \nu, \quad (4.97)$$

then (4.96) must also hold when  $n$  is replaced by  $n+1$ . Observe that, since we have

$$z_n = \frac{z_{n-1} - (1-r)}{r}, \quad (4.98)$$

then (4.97) may be rewritten as

$$|z_{n-1} > (1-r)| \geq \nu. \quad (4.99)$$

On the other hand, since the first alternative in (4.90) was assumed to hold, we have

$$|z_{n-1} \leq 0| \geq |z_0 \leq 0| \geq 8. \quad (4.100)$$

We recall that  $r < \frac{1}{4}$ . A straightforward compactness argument<sup>2</sup> then shows that, in view of (4.95), (4.99), and (4.100), if  $\delta > 0$  is chosen sufficiently small, depending only on  $\nu^{-1}$  and  $C$ , then

$$|0 < z_{n-1} < 1-r| \geq \delta.$$

Thus, we deduce from (4.96) and (4.98) that

$$|z_n \leq 0| \geq |z_{n-1} \leq 0| + |0 < z_{n-1} < 1-r| \geq 8 + n\delta.$$

Since the left hand side is bounded above by 16, this recursive process must fail after finitely many steps, and, therefore, there exists  $n \geq 1$ , depending only on  $\nu^{-1}$  and  $C$ , such that (4.97) does not hold. Rewritten in terms of  $v$ , this means that

$$|\{v > \mu^+ - 2r^n\omega\} \cap R_{2\rho}| < \nu |R_{2\rho}|.$$

Thus, by Lemma 4.19,  $v \leq \mu^+ - r^n\omega$  in  $R_\rho$ . In particular,

$$\text{osc}_{R_\rho}(v) \leq \mu^+ - r^n\omega - \mu^- = (1-r^n)\omega = (1-r^n)\text{osc}_{R_{4\rho}}(v).$$

□

We may now prove our main regularity result for the density.

<sup>2</sup>The explicit form of this estimate is known as the De Giorgi isoperimetric inequality (see [34, Lem. 10]).

**Theorem 4.21** (Hölder continuity of the density up to the free boundary). *Let the assumptions of Theorem 4.14 be in place, and let  $(u, m)$  be a solution to (MFG) or (MFGP). Let  $0 < \delta_0 < \min(1, \frac{1}{4}T)$ , and let  $\sigma$  be the constant of Corollary 4.20. There exist constants  $C > 0$ ,  $0 < s < 1$ , with*

$$C = C(\delta_0^{-1}, C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty, \theta, (1 - \sigma)^{-1}), \quad s^{-1} = s^{-1}(C_0, \|\gamma_x\|_\infty, \|\gamma_x^{-1}\|_\infty, (1 - \sigma)^{-1}),$$

such that

$$\|f(m(\gamma(\cdot, \cdot), \cdot))\|_{C^s([a_0, b_0] \times [\delta_0, T - \delta_0])} \leq C.$$

In particular,  $m$  is Hölder continuous on  $\mathbb{R} \times [\delta_0, T - \delta_0]$ .

*Proof.* Let  $(x_0, t_0), (x_1, t_1) \in (a_0, b_0) \times [\delta_0, T - \delta_0]$ , where  $x_1 < x_0$ . As usual, we will consider intrinsic rectangles centered at  $(x_0, t_0)$ , so we abbreviate  $R_\rho := R_\rho(x_0, t_0)$ . As in the proof of Proposition 4.16, we may assume that  $a_0 = 0$  and  $x_0 < \frac{1}{2} \min(b_0, 1)$ , and write, for some constant  $k_1 > 1$ , and each  $(x, t) \in R_{4\rho}$ ,

$$\frac{1}{k_1}x \leq v(x, t) \leq k_1x. \quad (4.101)$$

Therefore,  $x_0 \leq \sqrt{x_0} \leq \sqrt{k_1 v(x_0, t_0)}$ . Hence, letting

$$\rho_0 = \frac{1}{8\sqrt{k_1}(1 + \sqrt{\theta})} \delta_0 x_0, \quad a = \max(|x_1 - x_0|, \sqrt{\theta v(x_0, t_0)} |t_1 - t_0|),$$

we see that, in particular,  $\rho_0$  satisfies (4.85). Setting now

$$a = \max(|x_1 - x_0|, \sqrt{\theta v(x_0, t_0)} |t_1 - t_0|)$$

we distinguish two alternative cases.

**Case 1.**  $(x_1, t_1) \in R_{4\rho_0}$ . Equivalently, we have  $a < 4\rho_0$ . Let  $n \geq 0$  be the unique integer such that

$$\frac{1}{4^n} \rho_0 \leq a < \frac{1}{4^{n-1}} \rho_0.$$

Iterating Corollary 4.20, we see that, in view of (4.101),

$$\text{osc}_{R_{4^{-(n-1)}\rho_0}}(v) \leq \sigma^{n-1} \text{osc}_{R_{4\rho_0}}(v) \leq \frac{1}{\sigma} \sigma^n k_1 (x_0 + 4\rho_0) \leq C \sigma^n \rho_0 \delta_0^{-1}.$$

Moreover, by increasing the value of  $\sigma$  if necessary, we may assume that  $\sigma > \frac{1}{4}$ , so that  $s = -\log \sigma (\log 4)^{-1}$  satisfies  $0 < s < 1$ . Thus, observing that  $(x_1, t_1) \in R_{4^{-(n-1)}\rho_0}$  and  $n \geq -\log(\rho_0^{-1}a) (\log 4)^{-1}$ , we have

$$|v(x_1, t_1) - v(x_0, t_0)| \leq C(\rho_0^{-1}a)^s \rho_0 \delta_0^{-1} \leq C \rho_0^{1-s} \delta_0^{-1} (|x_1 - x_0|^s + (\theta v(x_0, t_0))^{\frac{s}{2}} |t_1 - t_0|^s).$$

**Case 2.**  $(x_1, t_1) \notin R_{4\rho_0}$ . Then  $a \geq 4\rho_0$ , and so, since  $x_1 < x_0$ , appealing to the lower bound on  $\gamma_x = (f(m_0(x))v(x, t)^{-1})^{\frac{1}{\theta}}$ , and allowing the constant  $C$  to increase at each step, we have

$$\begin{aligned} |v(x_1, t_1) - v(x_0, t_0)| &\leq C(f(m_0)(x_1) + f(m_0)(x_0)) \leq 2CC_0^\theta x_0 \leq C\delta_0^{-1} \rho_0 \\ &\leq C\delta_0^{-1} a \leq C\delta_0^{-1} (|x_1 - x_0| + \sqrt{\theta v(x_0, t_0)} |t_1 - t_0|). \end{aligned}$$

Finally, to see that  $m$  itself is also Hölder continuous, we simply observe that, by (4.33) and Proposition 4.13, the inverse of the map  $(x, t) \mapsto \Gamma(x, t) := (\gamma(x, t), t)$  is Lipschitz. Therefore, since  $f^{-1} : [0, \infty) \rightarrow [0, \infty)$  is Hölder continuous on bounded sets,

$$m = f^{-1} \circ v \circ \Gamma^{-1} : \text{supp}(m) \cap (\mathbb{R} \times [\delta_0, T - \delta_0]) \rightarrow \mathbb{R}$$

is the composition of Hölder continuous functions.  $\square$

*Remark 4.22.* The result of Theorem 4.21 may be improved to obtain Hölder continuity up to the initial time  $t = 0$ , by working with one-sided (in time) analogues of the intrinsic rectangles  $R_\rho$ . Moreover, in the case of (MFGP), the Hölder regularity may be established up to  $t = T$  as well, by simply imposing on  $m_T$  the same assumptions as those of  $m_0$ , and applying the continuity result at  $t = 0$  to the reflected functions  $(-u(x, T - t), m(x, T - t))$ .



### 4.3.2 Hölder continuity of $Du$

We now address the regularity of  $u$  in view of the Hölder regularity of  $m$ . For (MFGP), since the solution is not unique outside of  $\{m > 0\}$ , we choose to work with the specific  $u$  that was constructed in the proof of Theorem 4.3. With this choice, for both (MFG) and (MFGP), we observe that  $u$  is the unique  $\text{BUC}(\mathbb{R} \times [0, T])$  viscosity solution to the HJ equation with terminal condition  $u(\cdot, T)$ . This implies that  $u$  satisfies the representation formula

$$u(x, t) = \inf_{\gamma \in H^1((t, T)), \gamma(t)=x} \int_t^T \frac{1}{2} |\dot{\gamma}(s)|^2 + f(m(\gamma(s), s)) ds + u(\gamma(T), T).$$

We also note that, from the proof of Theorem 4.3, the function  $-u(x, T-t)$  is also the unique viscosity solution to the HJ equation with terminal condition  $-u(\cdot, 0)$ , and, therefore, also satisfies the representation formula.

**Theorem 4.23.** *Assume that  $f(m)$  is in  $C_{\text{loc}}^\beta$ . Then the map  $u$  is  $C_{\text{loc}}^{1, \frac{\beta}{2}}$  in  $\mathbb{R} \times (0, T)$ .*

*Proof.* In view of the proof of Theorem 4.3, we know that  $u_x$  is globally bounded. Let us first check that  $u$  is locally semiconcave with a semiconcavity modulus of the form  $\omega(r) = Cr^{\beta/2}$ . The argument is known and goes back to [6]. Fix  $(x, t) \in \mathbb{R} \times I$ , where  $I$  is a fixed compact sub-interval of  $(0, T)$ . Let  $\gamma$  be optimal for  $u(x, t)$  and  $0 < |h|, h' < \tau$  small. We set

$$\gamma_{\pm}(s) := \gamma(\theta_1^{\pm} s + \theta_2^{\pm}) + \theta_3^{\pm}(s - (t + \tau)) \quad \forall s \in [t \pm h', t + \tau],$$

where  $\theta_1^{\pm} = \frac{\tau}{\tau \pm h'}$ ,  $\theta_2^{\pm} = t - \theta_1^{\pm}(t \pm h')$ ,  $\theta_3^{\pm} = -\frac{\pm h}{\tau \pm h'}$ . Note that

$$\theta_1^{\pm}(t \pm h') + \theta_2^{\pm} = t, \quad \theta_1^{\pm}(t + \tau) + \theta_2^{\pm} = t + \tau, \quad \gamma_{\pm}(t \pm h') = x \pm h, \quad \gamma_{\pm}(t + \tau) = \gamma(t + \tau),$$

so that

$$\begin{aligned} & u(x + h, t + h') + u(x - h, t - h') - 2u(x, t) \\ & \leq \int_{t+h'}^{t+\tau} \frac{1}{2} |\dot{\gamma}_+|^2 + f(m(\gamma_+, s)) ds + \int_{t-h'}^{t+\tau} \frac{1}{2} |\dot{\gamma}_-|^2 + f(m(\gamma_-, s)) ds - 2 \int_t^{t+\tau} \frac{1}{2} |\dot{\gamma}|^2 + f(m(\gamma, s)) ds \\ & \leq \int_t^{t+\tau} \frac{\theta_1^+}{2} |\dot{\gamma} + \theta_3^+/\theta_1^+|^2 + \frac{1}{\theta_1^+} f(m(\gamma(s) + \theta_3^+((s - \theta_2^+)/\theta_1^+ - (t + \tau)), (s - \theta_2^+)/\theta_1^+)) ds \\ & \quad + \int_t^{t+\tau} \frac{\theta_1^-}{2} |\dot{\gamma} + \theta_3^-/\theta_1^-|^2 + \frac{1}{\theta_1^-} f(m(\gamma(s) + \theta_3^-((s - \theta_2^-)/\theta_1^- - (t + \tau)), (s - \theta_2^-)/\theta_1^-)) ds \\ & \quad - 2 \int_t^{t+\tau} \frac{1}{2} |\dot{\gamma}|^2 + f(m(\gamma(s), s)) ds. \end{aligned}$$

Using the Hölder regularity of  $f(m)$  and the fact that  $u$  is bounded we find

$$\begin{aligned} & u(x + h, t + h') + u(x - h, t - h') - 2u(x, t) \\ & \leq \int_t^{t+\tau} \frac{1}{2} (\theta_1^+ + \theta_1^- - 2) |\dot{\gamma}|^2 + \dot{\gamma}(\theta_3^+ + \theta_3^-) + \frac{1}{2} ((\theta_3^+)^2/\theta_1^+ + (\theta_3^-)^2/\theta_1^-) ds \\ & \quad + \int_t^{t+\tau} (1/\theta_1^+ + 1/\theta_1^- - 2) f(m(\gamma(s), s)) ds \\ & \quad + C_I \int_t^{t+\tau} \frac{1}{\theta_1^+} (|\theta_3^+((s - \theta_2^+)/\theta_1^+ - (t + \tau))|^\beta + |((s - \theta_2^+)/\theta_1^+) - s|^\beta) ds \\ & \quad + C_I \int_t^{t+\tau} \frac{1}{\theta_1^-} (|\theta_3^-((s - \theta_2^-)/\theta_1^- - (t + \tau))|^\beta + |((s - \theta_2^-)/\theta_1^-) - s|^\beta) ds \\ & \leq C \left( \frac{\tau(h')^2}{\tau^2 - (h')^2} + \frac{\tau(h)^2}{\tau^2 - (h)^2} + C_I(|h|^\beta \tau + |h'|^\beta \tau) \right) \end{aligned}$$

where  $C = C(\|u_x\|_\infty)$ , since  $|\dot{\gamma}| = |u_x|$ . We choose  $\tau = (|h| + h')^\delta$  with  $\delta = 1 - \beta/2$ , which leads to

$$u(x + h, t + h') + u(x - h, t - h') - 2u(x, t) \leq C(|h| + h')^{1 + \frac{\beta}{2}}.$$

This inequality implies the local semiconcavity of  $u$  with modulus  $\omega(r) = Cr^{\beta/2}$  [7, Thm. 2.1.10].

We now set  $w(x, t) = -u(x, T - t)$ . Then, as above,  $w$  is locally semiconcave with a modulus  $\omega(r) = Cr^{\beta/2}$  since the semiconcavity property does not rely on the regularity of the terminal value. Hence  $u$  is semiconcave and semiconvex with modulus  $\omega$ . Following [7, Thm. 3.3.7], the derivatives of  $u$  are therefore locally  $\beta/2$ -Hölder continuous.  $\square$

## 5 Infinite speed of propagation: the case of entropic coupling

This section is devoted to the special case of so-called entropic coupling, namely when  $f(m) = \log m$ . We will show that, in this case, the evolution of  $m$  has the property of infinite speed of propagation; marginals with compact support evolve into positive, smooth densities.

### 5.1 The periodic case

As before, we start by considering periodic solutions defined on the torus  $R\mathbb{T}$ , for arbitrarily large  $R$ :

$$\begin{cases} -u_t + \frac{1}{2}u_x^2 = \log(m) & (x, t) \in R\mathbb{T} \times (0, T), \\ m_t - (mu_x)_x = 0 & (x, t) \in R\mathbb{T} \times (0, T), \end{cases} \quad (5.1)$$

complemented either with prescribed marginals

$$m(x, 0) = m_0(x), \quad m(x, T) = m_T(x), \quad x \in R\mathbb{T}. \quad (5.2)$$

or with final pay-off condition

$$u(x, T) = g(m(x, T)), \quad x \in R\mathbb{T}. \quad (5.3)$$

We define solutions which are smooth in  $(0, T)$ , with traces at  $t = 0, t = T$  in the space of measures. For this purpose, we denote by  $\mathcal{P}(R\mathbb{T})$  the set of Borel probability measure on  $R\mathbb{T}$ , endowed with the weak-\* convergence.

**Definition 5.1.** We say that  $(u, m)$  is a (classical) solution of (5.1) if  $(u, m) \in C^2(R\mathbb{T} \times (0, T)) \times C^1(R\mathbb{T} \times (0, T))$ ,  $m > 0$  in  $R\mathbb{T} \times (0, T)$  and the equations are satisfied in a classical sense. For  $m_0, m_T \in \mathcal{P}(R\mathbb{T})$ , we say that (5.2) is satisfied if  $m \in C([0, T]; \mathcal{P}(R\mathbb{T}))$  and  $m(0) = m_0, m(T) = m_T$ . Respectively, we say that (5.3) is satisfied, if  $m \in C([0, T]; \mathcal{P}(R\mathbb{T}))$ ,  $m(T) \in L^1(R\mathbb{T})$  and  $\lim_{t \rightarrow T^-} u(x, t) = g(m(x, T))$  for every  $x \in R\mathbb{T}$ .

The main result of this subsection is the following theorem.

**Theorem 5.2.** *Assume that  $m_0, m_T \in C_c(R\mathbb{T})$ . Then the following holds:*

1. *There exists a unique (up to addition of a constant to  $u$ ) solution  $(u, m)$  of (5.1)–(5.2) such that  $m \in L^\infty(R\mathbb{T} \times (0, T))$ . In addition,  $u, m \in C^\infty(R\mathbb{T} \times (0, T))$ .*
2. *Assume that  $g(s) = c_T \log(s)$ , for some  $c_T \geq 0$ . Then there exists a unique solution  $(u, m)$  of (5.1)–(5.3) such that  $m \in L^\infty(R\mathbb{T} \times (0, T))$ . In addition,  $u, m \in C^\infty(R\mathbb{T} \times (0, T])$  and  $m > 0$  in  $(0, T]$ .*

We recall (see Theorem 3.11) that, if  $m_0, m_T \in C^{1,\alpha}(R\mathbb{T})$  and are strictly positive, then the problems (5.1)–(5.2) and (5.1)–(5.3) admit a unique classical solution  $(u, m)$  (up to addition of a constant to  $u$ , in case of planning conditions (5.2)). Therefore, for the proof of Theorem 5.2, we will proceed by approximating  $m_0, m_T$  with strictly positive smooth measures.

As a first step, we derive estimates which are independent of lower bounds of  $m_0, m_T$  as well as independent of  $R$ . We denote by  $W_2(\mu, \nu)$  the 2-Wasserstein distance between measures  $\mu, \nu \in \mathcal{P}(R\mathbb{T})$ , and, for a single measure  $\mu \in L^1(R\mathbb{T})$ , we denote the entropy of  $\mu$  by  $\mathcal{E}(\mu) = \int_{R\mathbb{T}} \mu \log \mu dx$  and by  $M_2(\mu) = \int_{R\mathbb{T}} |x|^2 d\mu$  its second order moment.

**Proposition 5.3.** *Let  $(u, m)$  be a solution of (5.1), and assume that  $u, m$  are continuous in  $R\mathbb{T} \times [0, T]$ .*

1. *If  $(u, m)$  solves (5.1)–(5.2), there exists a constant  $K$ , only depending on  $T, \mathcal{E}(m_0), \mathcal{E}(m_T), M_2(m_0), M_2(m_T)$  (and independent of  $R$ ) such that*

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dx dt + \int_0^T \int_{R\mathbb{T}} |m \log(m)| dx dt \leq K, \quad (5.4)$$

$$\sup_{t \in (0, T)} \int_{R\mathbb{T}} m(t) |x|^2 dx \leq K, \quad (5.5)$$

and

$$\sup_{t \in (0, T)} \int_{R\mathbb{T}} m(t) u_x^2 dx + \sup_{t \in (0, T)} \left| \int_{R\mathbb{T}} m(t) \log m(t) \right| \leq K. \quad (5.6)$$

As a consequence, it also holds that

$$W_2(m(t), m(s)) \leq K |t - s|^{\frac{1}{2}} \quad \forall t, s \in (0, T). \quad (5.7)$$

2. *If  $(u, m)$  solves (5.1)–(5.3) (with  $g(m) = c_T \log(m)$ ), then the estimates (5.4), (5.5), (5.7) hold, for some  $K$  only depending on  $T, \mathcal{E}(m_0), M_2(m_0)$ . If  $c_T > 0$ , then (5.6) holds true as well, with  $K$  depending also on  $(c_T)^{-1}$ .*

*Proof.* We first consider the case of problem (5.1)–(5.2). Using the equation of  $m$ , we compute

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{R\mathbb{T}} m |x|^2 dx &= \frac{1}{2} \int_{R\mathbb{T}} m_t |x|^2 dx = - \int_{R\mathbb{T}} x m u_x \\ &\leq \frac{1}{2} \int_{R\mathbb{T}} m |x|^2 dx + \frac{1}{2} \int_{R\mathbb{T}} m |u_x|^2 dx. \end{aligned}$$

Hence, Gronwall's lemma implies that, for some constant depending only on  $T$ ,

$$\int_{R\mathbb{T}} m(t) |x|^2 dx \leq C_T \int_0^T \int_{R\mathbb{T}} m |u_x|^2 dx + \int_{R\mathbb{T}} m_0 |x|^2 dx. \quad (5.8)$$

On the other hand, we know that  $(m, u_x)$  is the optimizer of the functional

$$\mathcal{B}(m, v) := \int_0^T \int_{R\mathbb{T}} \frac{1}{2} |v|^2 dm + \int_0^T \int_{R\mathbb{T}} m \log(m) dx dt, \quad \text{subject to } \begin{cases} m_t - (vm)_x = 0 \text{ in } R\mathbb{T} \times (0, T) \\ m(0) = m_0, m(T) = m_T. \end{cases}$$

This means that

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dx dt + \int_0^T \int_{R\mathbb{T}} m \log(m) dx dt \leq \int_0^T \int_{R\mathbb{T}} \frac{1}{2} |v|^2 d\mu + \int_0^T \int_{R\mathbb{T}} \mu \log(\mu) dx dt$$

for any  $(\mu, v)$  such that  $\mu_t = (\mu v)_x$ , with  $\mu(0) = m_0, \mu(T) = m_T$ . Let us take  $\mu$  as the Wasserstein geodesic connecting  $m_0, m_T$ . By McCann's classical displacement convexity result for Wasserstein geodesics [26], we know that  $\int_{R\mathbb{T}} \mu \log(\mu) dx$  is convex, hence

$$\int_{R\mathbb{T}} \mu \log(\mu) dx \leq \max(\mathcal{E}(m_0), \mathcal{E}(m_T)).$$

We deduce that

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dx dt + \int_0^T \int_{R\mathbb{T}} m \log(m) dx dt \leq C_T W_2(m_0, m_T) + \max(\mathcal{E}(m_0), \mathcal{E}(m_T)).$$

This yields, for some constant  $C$  independent of  $R$ ,

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + \int_0^T \int_{R\mathbb{T}} (m \log(m))_+ dxdt \leq C + \int_0^T \int_{R\mathbb{T}} (m \log m)_- dxdt. \quad (5.9)$$

Here and below, the constants will be independent of  $R$ , although they may depend on  $T$ ,  $\mathcal{E}(m_0)$ ,  $\mathcal{E}(m_T)$ ,  $M_2(m_0)$ ,  $M_2(m_T)$ . Since  $(s \log s)_- \leq c\sqrt{s}$  we have

$$\begin{aligned} \int_0^T \int_{R\mathbb{T}} (m \log m)_- dxdt &\leq c \int_0^T \int_{R\mathbb{T}} \sqrt{m} dxdt \\ &\leq C + \varepsilon \int_0^T \int_{R\mathbb{T}} m (|x|^2 + 1) dxdt + C_\varepsilon \int_0^T \int_{R\mathbb{T}} \frac{1}{1 + |x|^2} dxdt \end{aligned} \quad (5.10)$$

by Young's inequality. Then (5.9) yields

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + \int_0^T \int_{R\mathbb{T}} (m \log(m))_+ dxdt \leq C + \varepsilon \int_0^T \int_{R\mathbb{T}} m (|x|^2 + 1) dxdt + C_\varepsilon \int_0^T \int_{R\mathbb{T}} \frac{1}{1 + |x|^2} dxdt. \quad (5.11)$$

From (5.8), after integration we deduce

$$\begin{aligned} \int_0^T \int_{R\mathbb{T}} m (|x|^2 + 1) dxdt &\leq T C_T \int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + C_T M_2(m_0) \\ &\leq \varepsilon T C_T \int_0^T \int_{R\mathbb{T}} m (|x|^2 + 1) dxdt + C(\varepsilon, T, m_0). \end{aligned}$$

Choosing  $\varepsilon$  suitably small, we get an estimate for  $\int_0^T \int_{R\mathbb{T}} m (|x|^2 + 1) dxdt$ . In turn, from (5.10) and (5.11), we deduce (5.4). Now, the right-hand side in (5.8) is controlled, and we get the estimate (5.5). Moreover, from the equation satisfied by  $m$  and the estimate (5.4), we immediately deduce (5.7).

We are left with the pointwise estimate of  $\int_{R\mathbb{T}} m u_x^2 dx$ . For this purpose, we observe that

$$\frac{d}{dt} \left[ \int_{R\mathbb{T}} m u_x^2 dx - \int_{R\mathbb{T}} m \log m dx \right] = 0. \quad (5.12)$$

Therefore,

$$\int_{R\mathbb{T}} m u_x^2 dx - \int_{R\mathbb{T}} m \log m dx = \frac{1}{T} \int_0^T \left\{ \int_{R\mathbb{T}} m u_x^2 dx - \int_{R\mathbb{T}} m \log m dx \right\} dt.$$

We deduce that

$$\left| \int_{R\mathbb{T}} m u_x^2 dx - \int_{R\mathbb{T}} m \log m dx \right| \leq \frac{1}{T} \left\{ \int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + \int_0^T \int_{R\mathbb{T}} |m \log(m)| dxdt \right\} \leq C. \quad (5.13)$$

Since, by the displacement convexity formula (3.27), we have

$$\int_{R\mathbb{T}} m \log m dx \leq \max(\mathcal{E}(m_0), \mathcal{E}(m_T)),$$

we conclude by (5.13) that  $|\int_{R\mathbb{T}} m \log m dx|$  is bounded, so  $\int_{R\mathbb{T}} m u_x^2 dx$  is bounded above, uniformly in  $t$ . This yields (5.6).

In case of problem (5.1)–(5.3), the only difference is in the first estimate, (5.4). By the optimality condition, using that  $g(s) = c_T \log(s)$ , we know that

$$\int_{R\mathbb{T}} u(x, 0) m_0 dx \leq \frac{1}{2} \int_0^T \int_{R\mathbb{T}} \mu |v|^2 dxds + \int_0^T \int_{R\mathbb{T}} \mu \log \mu dxds + c_T \int_{R\mathbb{T}} \mu_T \log(\mu_T) dx$$

for any curve  $\mu(t)$  such that  $\mu(0) = m_0$  and  $\mu(T) = \mu_T$ , with  $\mu_t = (\mu v)_x$ .

It is enough to choose some  $\mu_T$  with finite entropy to obtain a global estimate of the right-hand side. Since we also have, by duality,

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + \int_0^T \int_{R\mathbb{T}} m \log(m) dxdt + c_T \int_{R\mathbb{T}} m(T) \log(m(T)) dx = \int_{R\mathbb{T}} u(x, 0) m_0 dx$$

we get immediately (5.9) if  $c_T = 0$ . If  $c_T > 0$ , we estimate the term at  $t = T$  in a similar way as in (5.10), (5.11), and we end up with

$$\begin{aligned} \int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + \int_0^T \int_{R\mathbb{T}} |m \log(m)| dxdt + c_T \int_{R\mathbb{T}} |m(T) \log(m(T))| dx \leq \\ C + \varepsilon \int_0^T \int_{R\mathbb{T}} m (|x|^2 + 1) dxdt + \varepsilon \int_{R\mathbb{T}} m(T) (|x|^2 + 1) dx + C_\varepsilon (1 + T) \int_{R\mathbb{T}} \frac{1}{1 + |x|^2} dxdt. \end{aligned}$$

Using (5.8) and choosing  $\varepsilon$  suitably small, the estimates (5.4), (5.5) and (5.7) follow as before. Notice that, if  $c_T > 0$ , we also estimate  $\mathcal{E}(m(T))$  and then we are back to the previous case, obtaining (5.6) as well.  $\square$

Now we show a local bound on the value function  $u$ , which is independent of the period  $R$ .

**Proposition 5.4.** *Let  $(u, m)$  be a solution of (5.1), and assume that  $u, m$  are continuous in  $R\mathbb{T} \times [0, T]$ .*

1. *If  $(u, m)$  solves (5.1)–(5.2), there exists a constant  $C > 0$ , depending on  $\|m_0\|_\infty, \|m_T\|_\infty, M_2(m_0), M_2(m_T)$ , but independent of  $R$ , such that, if we normalize  $u$  such that  $\int u(T) m_T = 0$ , then we have*

$$-\frac{C}{t}(1 + |x|^2) \leq u(x, t) \leq \frac{C}{T-t}(1 + |x|^2) \quad \forall t \in (0, T), x \in \mathbb{R}. \quad (5.14)$$

2. *If  $(u, m)$  solves (5.1)–(5.3) (with  $g(m) = c_T \log(m)$ ), then the estimate (5.14) holds for some  $C$  only depending on  $\|m_0\|_\infty, M_2(m_0), c_T$ .*

*Proof.* First we consider the case of (5.1)–(5.2), and we adapt a similar proof given in [31, Lemma 4.2] for bounded domains.

We observe that, from the standard duality between  $u, m$  we have

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dxdt + \int_0^T \int_{R\mathbb{T}} m \log(m) dxdt = \int_{R\mathbb{T}} u(x, 0) m_0 - \int_{R\mathbb{T}} u(x, T) m_T$$

which implies, using Proposition 5.3 and  $\int u(T) m_T = 0$ , that

$$\int_{R\mathbb{T}} u(x, 0) m_0 \geq -K. \quad (5.15)$$

Notice that the constant depends on  $m_0, m_T$  through the entropy, by Proposition 5.3; however, for bounded functions,  $\mathcal{E}(m)$  is itself estimated in terms of  $M_2(m)$  and  $\|m\|_\infty$ .

Now, let us consider the Wasserstein geodesic  $\mu(\cdot)$  which connects, in time  $(0, t)$ ,  $m_0$  with any measure  $\lambda$ , supposed to be compactly supported in  $(-R/2, R/2)$ . This means that  $\mu(t) = \lambda$ ,  $\mu(0) = m_0$  and  $\mu_s = (\mu v)$  in  $(0, t)$ , for some  $v$  such that  $\int_0^t \int_{R\mathbb{T}} |v|^2 d\mu < \infty$ . By duality with the equation satisfied by  $u$ , we have

$$\begin{aligned} - \int_{R\mathbb{T}} u(x, t) d\lambda + \int_{R\mathbb{T}} u(x, 0) m_0 dx + \frac{1}{2} \int_0^t \int_{R\mathbb{T}} |u_x|^2 \mu dxds = \int_0^t \int_{R\mathbb{T}} \mu \log m dxds + \int_0^t \int_{R\mathbb{T}} \mu v u_x dxds \\ \leq \frac{1}{2} \int_0^t \int_{R\mathbb{T}} |u_x|^2 \mu dxds + \frac{1}{2} \int_0^t \int_{R\mathbb{T}} \mu |v|^2 dxds + T \log(\|m\|_\infty \vee 1) \end{aligned}$$

which yields

$$\begin{aligned} \int_{R\mathbb{T}} u(x, t) d\lambda &\geq \int_{R\mathbb{T}} u(x, 0) m_0 dx - \frac{1}{2} \int_0^t \int_{R\mathbb{T}} \mu |v|^2 dx ds - T \log(\|m\|_\infty \vee 1) \\ &\geq -K - \frac{c}{t} W_2(\lambda, m_0)^2 - T \log(\|m\|_\infty \vee 1), \end{aligned}$$

where we used (5.15) and the scaling of Wasserstein geodesic. If we let  $\lambda$  converge (in the weak-\* topology) towards a Dirac mass  $\delta_{x_0}$ , we get

$$u(x_0, t) \geq -K - \frac{c}{t} \int_{\mathbb{R}} |x_0 - y|^2 dm_0(y) - T \log(\|m\|_\infty \vee 1).$$

We recall that, by Corollary 3.7,  $\|m(t)\|_\infty$  is controlled by the initial-terminal values. Hence, there exists a constant  $C$ , depending on  $\|m_0\|_\infty, \|m_T\|_\infty, T$  and  $M_2(m_0), M_2(m_T)$ , such that

$$u(x_0, t) \geq -\frac{C}{t} (1 + |x_0|^2).$$

A similar argument (instead of (5.15) we simply use that  $\int_{R\mathbb{T}} u(x, T) m_T = 0$ ) shows the upper bound  $u(x_0, t) \leq \frac{C}{T-t} (1 + |x_0|^2)$  and concludes the proof of (5.14).

In the case of (5.3), the duality equality takes the form

$$\int_0^T \int_{R\mathbb{T}} m |u_x|^2 dx dt + \int_0^T \int_{R\mathbb{T}} m \log(m) dx dt = \int_{R\mathbb{T}} u(x, 0) m_0 - c_T \int_{R\mathbb{T}} m(T) \log(m(T)) dx$$

and (5.15) follows again from Proposition 5.3. We also recall that the  $L^\infty$  bound on  $m$  is given by Proposition 3.9. Then we obtain as before the lower estimate of  $u$ . For the upper estimate, we just observe that  $\int_{R\mathbb{T}} u(T) m(T) dx = c_T \int_{R\mathbb{T}} m(T) \log(m(T)) dx$  and this is bounded above (uniformly with respect to  $R$ ) if either  $c_T = 0$  or  $c_T > 0$  (from Proposition 5.3). Hence we repeat the argument above using any geodesic connecting  $m(T)$  with a Dirac mass.  $\square$

In the next step we aim at showing that, if  $(u, m)$  is a solution of (5.1), then  $m(t)$  becomes positive for  $t > 0$  even if starting from a compactly supported initial measure. For this purpose, we use in a key way the displacement convexity estimates. However, we warn the reader that, while the estimates of Proposition 5.3 and 5.4 were all independent of the period  $R$ , this will not be the case for the following bounds on  $\log(m)$ .

**Lemma 5.5.** *Let  $(u, m)$  be a solution to (5.1), where  $m_0, m_T > 0$ . Set  $K = \max(\|m_0\|_\infty, \|m_T\|_\infty)$ . Then, for each integer  $p \geq 1$ ,*

$$\frac{d^2}{dt^2} \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right|^p \geq 0. \quad (5.16)$$

Moreover, there exists a constant  $C > 0$ , depending on  $\|m_0\|_\infty, \|m_T\|_\infty$ , such that, for each  $t \in [0, T]$ ,

$$\left\| \log \left( \frac{m(t)}{K} \right) \right\|_\infty^2 \leq R \frac{d^2}{dt^2} \int_{R\mathbb{T}} \left| \log \left( \frac{m(t)}{K} \right) \right| + C \quad (5.17)$$

*Proof.* Letting  $h(m) = \log\left(\frac{m}{K}\right)^p$ , we obtain

$$h'(m) = p \log \left( \frac{m}{K} \right)^{p-1} \frac{1}{m}, \quad h''(m) = p \left( p-1 - \log \left( \frac{m}{K} \right) \right) \log \left( \frac{m}{K} \right)^{p-2} \frac{1}{m^2}.$$

We observe that, by Corollary 3.7 and by definition of  $K$ , we have  $\frac{m}{K} \leq 1$ . Hence, in the range of  $m$ ,  $h$  is positive and convex when  $p$  is even, and negative and concave when  $p$  is odd. The displacement convexity formula (3.27) yields

$$\frac{d^2}{dt^2} \int_{R\mathbb{T}} h(m) = \int_{R\mathbb{T}} h''(m) (m^2 u_{xx}^2 + m_x^2),$$

which, in particular, shows (5.16), and, setting  $p = 1$ , we obtain

$$\int_{R\mathbb{T}} \left( \log \left( \frac{m}{K} \right)_x \right)^2 \leq \frac{d^2}{dt^2} \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right|.$$

On the other hand, by the fundamental theorem of calculus, and the fact that  $m$  is a density, we have

$$\left\| \log \left( \frac{m(t)}{K} \right) - \log \left( \frac{1}{K} \right) \right\|_{\infty}^2 \leq R \int_{R\mathbb{T}} \left( \log \left( \frac{m}{K} \right)_x \right)^2,$$

and (5.17) follows.  $\square$

We can now state the (local in time) uniform bound from below, which is independent of  $\|m_0^{-1}\|_{\infty}, \|m_T^{-1}\|_{\infty}$ .

**Proposition 5.6.** *Let  $(u, m)$  be a solution to (5.1), where  $m_0, m_T > 0$ . There exists a constant  $C_R > 0$ , only depending on  $\|m_0\|_{\infty}, \|m_T\|_{\infty}$  and  $R$ , such that, for each  $t \in (0, T)$ ,*

$$\| \log m(t) \|_{\infty} \leq C_R \left( \frac{1}{t^2} + \frac{1}{(T-t)^2} \right).$$

*Proof.* Let  $t_0 \in [0, \frac{T}{2}]$ . In view of (5.16), we have, for integers  $p \geq 1$  and  $s \in (0, \frac{t_0}{2})$ ,

$$\begin{aligned} \left( \frac{1}{(T-2t_0)} \int_{t_0}^{T-t_0} \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right|^{2p} \right)^{\frac{1}{p}} &\leq \left( \int_{R\mathbb{T}} \left| \log \left( \frac{m(t_0-s)}{K} \right) \right|^{2p} + \int_{R\mathbb{T}} \left| \log \left( \frac{m(T-t_0+s)}{K} \right) \right|^{2p} \right)^{\frac{1}{p}} \\ &\leq \left( \int_{R\mathbb{T}} \left| \log \left( \frac{m(t_0-s)}{K} \right) \right|^{2p} \right)^{\frac{1}{p}} + \left( \int_{R\mathbb{T}} \left| \log \left( \frac{m(T-t_0+s)}{K} \right) \right|^{2p} \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, integrating in  $s$ , we infer that

$$\left( \int_{t_0}^{T-t_0} \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right|^{2p} \right)^{\frac{1}{p}} \leq \frac{2(T-2t_0)^{\frac{1}{p}}}{t_0} \int_{\frac{t_0}{2}}^{T-\frac{t_0}{2}} \left( \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right|^{2p} \right)^{\frac{1}{p}},$$

and letting  $p \rightarrow \infty$  yields

$$\left\| \log \left( \frac{m}{K} \right) \right\|_{L^{\infty}(R\mathbb{T} \times [t_0, T-t_0])}^2 \leq \frac{2}{t_0} \int_{\frac{t_0}{2}}^{T-\frac{t_0}{2}} \left\| \log \left( \frac{m(t)}{K} \right) \right\|_{\infty}^2 dt.$$

Now, integrating (5.17) against a test function  $\zeta$  supported in  $[\frac{t_0}{3}, T - \frac{t_0}{3}]$ , satisfying  $0 \leq \zeta \leq 1$ ,  $\zeta \equiv 1$  in  $[\frac{t_0}{2}, T - \frac{t_0}{2}]$ , and  $|\zeta''| \leq \frac{C}{t_0^2}$ , we get

$$\left\| \log \left( \frac{m}{K} \right) \right\|_{L^{\infty}(R\mathbb{T} \times [t_0, T-t_0])}^2 \leq \frac{CR}{t_0^3} \int_{\frac{t_0}{3}}^{T-\frac{t_0}{3}} \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right| dt + \frac{C}{t_0}.$$

Finally, by Proposition 5.4,  $u$  is bounded by  $\frac{C(1+R^2)}{t_0}$  on  $[t_0, T-t_0]$ , and, hence, by the HJ equation, we estimate

$$\int_{\frac{t_0}{3}}^{T-\frac{t_0}{3}} \int_{R\mathbb{T}} \left| \log \left( \frac{m}{K} \right) \right| \leq \frac{CR^3}{t_0}.$$

This yields

$$\left\| \log \left( \frac{m}{K} \right) \right\|_{L^{\infty}(R\mathbb{T} \times [t_0, T-t_0])} \leq \frac{C(1+R^2)}{t_0^2},$$

which implies the result.  $\square$

Finally, we have all the ingredients for the proof of Theorem 5.2.

**Proof of Theorem 5.2.** We start with the case of problem (5.1)–(5.2). Let  $m_0^\varepsilon, m_T^\varepsilon$  be two sequences of functions such that  $m_0^\varepsilon, m_T^\varepsilon \in C^{1,\alpha}(R\mathbb{T})$ ,  $m_0^\varepsilon, m_T^\varepsilon > 0$  in  $R\mathbb{T} \times [0, T]$  and  $m_0^\varepsilon, m_T^\varepsilon$  converge uniformly to  $m_0, m_T$  respectively. Such an approximation can be readily built by convolution, for instance. By Theorem 3.11, there exists a smooth positive solution  $(u^\varepsilon, m^\varepsilon)$  of (5.1), where we normalize  $u^\varepsilon$  such that

$$\int_{R\mathbb{T}} u^\varepsilon(T) m_T^\varepsilon dx = 0.$$

By Corollary 3.7, we know that  $m^\varepsilon$  is uniformly bounded. Then, from Proposition 5.4, we deduce that  $u^\varepsilon$  is locally bounded in  $(0, T)$ . It also follows from Proposition 5.6 that  $\log(m^\varepsilon)$  is locally bounded in  $(0, T)$  (i.e.  $m^\varepsilon$  is locally uniformly bounded below). In turn, this implies that  $u_x$  is locally bounded in  $(0, T)$ ; one can use for example [31, Thm 6.5] which shows<sup>3</sup> that  $\frac{|u_x^\varepsilon|^2}{4} + \log(m^\varepsilon) \leq C_\delta$  for every  $x \in R\mathbb{T}, t \in (a + \delta, b - \delta)$ , where  $C_\delta$  only depends on  $\delta$  and the bound on  $u^\varepsilon$  in  $(a, b)$ . Since  $u^\varepsilon$  satisfies

$$-u_{tt} + 2u_x u_{xt} - (u_x^2 + 1)u_{xx} = 0, \quad (5.18)$$

once  $(u^\varepsilon)_x$  is locally bounded then the above equation becomes a quasilinear equation which has bounded uniformly elliptic coefficients in any compact subset of  $R\mathbb{T} \times (0, T)$ . By a standard bootstrap regularity from Schauder's estimates, we deduce that  $u^\varepsilon$  is locally bounded in  $C^{k,\alpha}$  (for every  $k \in \mathbb{N}, \alpha \in (0, 1)$ ). Hence,  $u^\varepsilon$  converges in  $R\mathbb{T} \times (0, T)$  towards a function  $u$  which is  $C^\infty$ . Since  $m^\varepsilon = \exp(-(u^\varepsilon)_t + |(u^\varepsilon)_x|^2/2)$ , we also have  $m^\varepsilon$  converging to some  $m \in C^\infty(R\mathbb{T} \times (0, T))$ . But the global estimates also imply that  $m \in L^\infty(Q_T)$ . In addition, by (5.7), we also have that  $m^\varepsilon(t)$  is equi-continuous in the Wasserstein space of measures, hence it uniformly converges in  $[0, T]$ . We deduce that  $m \in C^0([0, T]; \mathcal{P}(R\mathbb{T}))$  and  $m(0) = m_0, m(T) = m_T$ . This concludes the proof that  $(u, m)$  is a solution of (5.1), which is classical inside  $(0, T)$ .

In case of problem (5.1)–(5.3), the proof is similar, except that we only approximate  $m_0$ . The  $L^\infty$  bound on  $m^\varepsilon$  follows from Proposition 3.9, then we argue as before to deduce that  $m^\varepsilon$  is locally uniformly bounded below, and  $u^\varepsilon, m^\varepsilon$  are locally bounded in  $C^{k,\alpha}$ . Applying Propositions 3.9 and 3.10 to problem (5.1)–(5.3) for  $t \in (T/2, T)$ , we conclude that  $u_\varepsilon, Du^\varepsilon$  are uniformly bounded up to  $t = T$ , and  $m^\varepsilon$  is bounded below up to  $t = T$ . By regularity of equation (5.18) up to the boundary  $t = T$ , we conclude that  $u, m$  are smooth up to  $t = T$  and  $u(T) = c_T \log(m(T))$ .

For the uniqueness of solutions, we use some argument which was already developed for much weaker notions of solutions ([8], [10], [30]). Let  $(u, m)$  be a solution which is classical inside, as in Definition 5.1, and such that  $m \in L^\infty(R\mathbb{T} \times (0, T))$ . First we notice that, from (5.12), the bound on  $m$  implies that  $mu_x^2 \in L^1(R\mathbb{T} \times (0, T))$ . Next we observe that  $u$  satisfies  $-u_t + \frac{1}{2}u_x^2 \leq \log(\|m\|_\infty)$ , so (up to a time translation) we can assume that  $u(\cdot, x)$  is nondecreasing. this implies that  $u(t)$  admits one-sided traces at  $t = 0, t = T$ , namely two measurable functions (not necessarily finite) defined as  $u(x, 0^+) = \lim_{t \downarrow 0} u(x, t) \in \mathbb{R} \cup \{-\infty\}$ ,  $u(x, T^-) = \lim_{t \uparrow 0} u(x, t) \in \mathbb{R} \cup \{+\infty\}$ . We first show that  $u(0^+) \in L^1(dm_0)$ ; indeed, since  $(u, m)$  is smooth in  $R\mathbb{T} \times (0, T)$ , we know that

$$\int_{R\mathbb{T}} u(t_0)m(t_0) - \int_{R\mathbb{T}} u(t_1)m(t_1) = \int_{t_0}^{t_1} \int_{R\mathbb{T}} mu_x^2 + \int_{t_0}^{t_1} \int_{R\mathbb{T}} m \log(m), \quad \forall 0 < t_0 < t_1 < T. \quad (5.19)$$

Choosing, for instance,  $t_1 = \frac{T}{2}$ , it follows that  $\int_{R\mathbb{T}} u(t_0)m(t_0)$  is bounded below, for every arbitrarily small  $t_0$ . Since  $u$  is nondecreasing, we deduce  $\int_{R\mathbb{T}} u(s)m(t_0) \geq -C$  for any  $s > t_0$ ; letting  $t_0 \rightarrow 0$  (and using that  $m \in C([0, T]; \mathcal{P}(R\mathbb{T}))$ ) yields  $\int_{R\mathbb{T}} u(s)m_0 \geq -C$ . Then by monotone convergence we deduce, as  $s \rightarrow 0^+$ , that  $\int_{R\mathbb{T}} u(0^+)m_0 \geq -C$ . Since the opposite inequality is clear by monotonicity and Proposition 5.4, we find that  $u(0^+) \in L^1(dm_0)$ . Similarly we reason to show that  $u(T^-) \in L^1(dm_T)$  (in case of problem (5.2)) or that  $m(T) \log(m(T)) \in L^1(R\mathbb{T})$  (in case of (5.3) with  $c_H > 0$ ).

Now, with a truncation argument, from the equality (5.19) we will show that  $(u, m)$  satisfies

$$\int_{R\mathbb{T}} u(0^+)dm_0 - \int_{R\mathbb{T}} u(T^-)dm(T) = \int_0^T \int_{R\mathbb{T}} mu_x^2 + \int_0^T \int_{R\mathbb{T}} m \log(m). \quad (5.20)$$

<sup>3</sup>The proof in [31] is given for Neumann boundary conditions, but applies identically to periodic solutions



Indeed, we first replace  $u$  by truncations  $u_k := \min(k, \max(u, -k))$ , we multiply the HJ equation by  $m$  and we integrate in  $(t_0, t_1)$ ; next we can let  $t_0 \rightarrow 0^+, t_1 \rightarrow T^-$  (using the weak-\* convergence of  $m$  and the strong  $L^1$  convergence of  $u_k$  at  $t = 0, t = T$ ). Then we finally let  $k \rightarrow \infty$  (thanks to  $u(0^+) \in L^1(dm_0), u(T^-) \in L^1(dm(T))$ ) and we obtain (5.20).

In a similar way, one can prove that, for any couple of solutions  $(u, m), (\tilde{u}, \tilde{m})$ , it holds

$$\int_{R\mathbb{T}} u(0^+) dm_0 - \int_{R\mathbb{T}} u(T^-) d\tilde{m}(T) \leq \int_0^T \int_{R\mathbb{T}} \tilde{m} \left( \tilde{u}_x u_x - \frac{1}{2} u_x^2 \right) + \int_0^T \int_{R\mathbb{T}} \tilde{m} \log(m). \quad (5.21)$$

The proof of (5.21) can be done, as before, replacing first  $u$  with its truncation  $u_k$  and integrating in  $(t_0, t_1)$ :

$$\int_{R\mathbb{T}} u_k(t_0) \tilde{m}(t_0) - \int_{R\mathbb{T}} u_k(t_1) \tilde{m}(t_1) \leq \int_{t_0}^{t_1} \int_{R\mathbb{T}} \tilde{m} \left( (u_k)_x \tilde{u}_x - \frac{1}{2} (u_k)_x^2 \right) + \int_{t_0}^{t_1} \int_{R\mathbb{T}} \tilde{m} \log(m) 1_{\{|u| < k\}}.$$

The right-hand side integrand is easily dominated from above. Once more, we can let first  $t_0 \rightarrow 0, t_1 \rightarrow T$  and then  $k \rightarrow \infty$ , in order to get (5.21).

From (5.20), (5.21), the uniqueness follows as in the classical Lasry-Lions monotonicity argument. For problem (5.1)–(5.2), we take two solutions normalized such that  $\int_{R\mathbb{T}} u(T^-) dm_T = \int_{R\mathbb{T}} \tilde{u}(T^-) d\tilde{m}_T = 0$ . Using (5.20), (5.21) for both couples we obtain that

$$\begin{aligned} & \int_0^T \int_{R\mathbb{T}} \tilde{m} \left( \frac{1}{2} u_x^2 - \frac{1}{2} \tilde{u}_x^2 - (u_x - \tilde{u}_x) \tilde{u}_x \right) + \int_0^T \int_{R\mathbb{T}} m \left( \frac{1}{2} \tilde{u}_x^2 - \frac{1}{2} u_x^2 - (\tilde{u}_x - u_x) u_x \right) \\ & + \int_0^T \int_{R\mathbb{T}} (m \log(m) - \tilde{m} \log(\tilde{m}))(m - \tilde{m}) \leq 0, \end{aligned}$$

which implies  $m = \tilde{m}$  and  $u_x = \tilde{u}_x$ . From the HJ equation we deduce that  $u - \tilde{u} = C$ , and this concludes the proof. For the problem (5.1)–(5.3), we proceed similarly and we get uniqueness using the coupled condition  $u(T) = g(m(T))$ .  $\square$

## 5.2 Preservation of monotonicity of the solutions

In this subsection, we show that the MFG system preserves a certain monotonicity property. As the phenomenon does not depend on the specific form of the coupling functions  $f$  and  $g$ , we suppose here that  $f$  and  $g$  are smooth and nondecreasing on  $(0, \infty)$ . We work in the periodic setting and assume the structure condition:

$$\begin{aligned} & \text{the densities } m_0, m_T : R\mathbb{T} \rightarrow \mathbb{R} \text{ are even,} \\ & \text{nonincreasing on } [0, R/2] \text{ and nondecreasing on } [-R/2, 0]. \end{aligned} \quad (5.22)$$

**Lemma 5.7.** *Assume that  $(u, m) \in C^2(R\mathbb{T} \times [0, T]) \times C^1((R\mathbb{T} \times [0, T]))$  is the unique classical solution to (3.20)–(3.21) or (3.20)–(3.22), such that  $m$  is positive on  $R\mathbb{T} \times [0, T]$  and (5.22) holds. Then*

$$\forall t \in [0, T], m(\cdot, t) : R\mathbb{T} \rightarrow \mathbb{R} \text{ is even, nonincreasing on } [0, R/2] \text{ and nondecreasing on } [-R/2, 0]. \quad (5.23)$$

*In addition any optimal trajectory  $\gamma(x, \cdot)$  starting from  $x \in [0, R/2]$  is concave in time. Finally, for the MFG problem (3.20)–(3.21),  $\gamma(x, \cdot)$  is nondecreasing in time for any  $x \in [0, R/2]$  and  $u_x$  is nonpositive in  $[0, R/2] \times [0, T]$ .*

Of course, par approximation, this preservation of the structure also holds in the whole space: in the next subsection, we shall use it in the case  $f(m) = \log(m)$  to build classical solutions in the whole space. In the case  $f(m) = m^\theta$ , it shows that, if  $m_0, m_T$  are even on  $\mathbb{R}$  and nonincreasing on  $[0, +\infty)$ , then any trajectory starting from  $x \in [0, R/2]$  is concave in time: compare with Theorem 4.14.

*Proof.* We do the proof in the MFG case, i.e., when  $(u, m)$  solves (3.20)–(3.21), with the proof for the planning problem (3.20)–(3.22) being similar and simpler. By the symmetry assumption and the uniqueness of the solution,  $m(\cdot, t), u(\cdot, t)$  are even for any  $t \in [0, T]$ . Thus  $m_x(0, t) = u_x(0, t) = 0$ . Let us set

$$M(x, t) = 1/2 - \int_0^x m(y, t) dy, \quad (x, t) \in [0, R/2] \times [0, T].$$

We first note that  $M$  is a classical solution to

$$-\operatorname{Tr}\left(\begin{pmatrix} \frac{M_t^2}{M_x^2} & -\frac{M_t}{M_x} \\ -\frac{M_t}{M_x} & 1 \end{pmatrix} D_{x,t}^2 M\right) + M_x f'(-M_x) M_{xx} = 0 \quad \text{in } [0, R/2] \times (0, T) \quad (5.24)$$

with boundary condition, for  $(x, t) \in [0, R/2] \times [0, T]$ ,

$$M(0, t) = 1/2, \quad M(R/2, t) = 0, \quad M(x, 0) = M_0(x), \quad \text{where } M_0(x) = 1/2 - \int_0^x m_0(y) dy, \quad (5.25)$$

and

$$M_t(x, T) + M_x(x, T) g'(-M_x(x, T)) M_{xx}(x, T) = 0. \quad (5.26)$$

The elliptic equation (5.24) was proved in Lemma 3.4 (where we also explained that  $u_x = M_t/M_x$ ). The boundary conditions (5.25) at  $x = 0$  and  $t = 0$  hold by definition. For  $x = R/2$ , it comes from the fact that  $m$  is a probability measure and from the symmetry. The boundary condition (5.26) at  $t = T$  comes from the boundary condition for  $u$ , which implies that

$$(M_t/M_x)(x, T) = u_x(x, T) = (g'(m)m_x)(x, T) = g'(-M_x(x, T))(-M_{xx}(x, T)).$$

The main part of the proof consists in showing that  $x \rightarrow M(x, t)$  is convex on  $[0, R/2]$  for any  $t \in [0, T]$ . For this we consider the map

$$\tilde{M}(x, t) = \inf_{\substack{\lambda y + (1-\lambda)z = x, \\ y, z \in [0, R/2], \lambda \in [0, 1]}} \lambda M(y, t) + (1-\lambda)M(z, t).$$

Note that  $\tilde{M} \leq M$  and that  $\tilde{M}$  is continuous and satisfies the boundary condition (5.25) by our assumption on  $m_0$ . We now prove that  $\tilde{M}$  is a viscosity supersolution to the elliptic equation (5.24) and satisfies the boundary condition (5.26) in the viscosity sense.

Assume that  $\phi$  is a test function touching  $\tilde{M}$  from below at  $(x_0, t_0) \in (0, R/2) \times (0, T]$ . If  $t_0 < T$ , we have to check that  $\phi_x(x_0, t_0) \neq 0$  and that

$$-\operatorname{Tr}\left(\begin{pmatrix} \frac{\phi_t^2}{\phi_x^2} & -\frac{\phi_t}{\phi_x} \\ -\frac{\phi_t}{\phi_x} & 1 \end{pmatrix} D_{x,t}^2 \phi\right) + \phi_x f'(-\phi_x) \phi_{xx} \geq 0 \quad \text{at } (x_0, t_0). \quad (5.27)$$

If  $t_0 = T$ , we have to prove that

$$\phi_t(x_0, T) + \phi_x(x_0, T) g'(-\phi_x(x_0, T)) \phi_{xx}(x_0, T) \geq 0. \quad (5.28)$$

Note that, if  $\tilde{M}(x_0, t_0) = M(x_0, t_0)$ , these inequalities hold because  $M$  satisfies (5.24) and (5.26). Thus, we assume from now on that  $\tilde{M}(x_0, t_0) < M(x_0, t_0)$ . Let  $y_0 < x_0 < z_0$  and  $\lambda_0 \in (0, 1)$  be optimal in the definition of  $\tilde{M}(x_0, t_0)$ .

*In this step we assume that  $t_0 < T$ .* By optimality of  $(y_0, z_0, \lambda_0)$ , and the fact that, by symmetry,  $M_{xx}(0, t_0) = m_x(0, t_0) = M_{xx}(R/2, t_0) = m_x(R/2, t_0) = 0$ , we have that

$$\begin{aligned} \phi_x(x_0, t_0) &= \frac{M(z_0, t_0) - M(y_0, t_0)}{z_0 - y_0} < 0 \\ \phi_x(x_0, t_0) &= M_x(y_0, t_0) \text{ or } y_0 = 0, \quad \phi_x(x_0, t_0) = M_x(z_0, t_0) \text{ or } z_0 = R/2, \\ \phi_{xx}(x_0, t_0) &\leq 0, \quad M_{xx}(y_0, t_0) \geq 0, \quad M_{xx}(z_0, t_0) \geq 0. \end{aligned} \quad (5.29)$$

Fix  $\theta, \theta_1, \theta_2 \in \mathbb{R}$  such that  $\lambda_0 \theta_1 + (1 - \lambda_0) \theta_2 = \theta$ , with  $\theta_1 = 0$  if  $y_0 = 0$  and  $\theta_2 = 0$  if  $z_0 = R/2$ . For  $h$  and  $s$  small, we have

$$\phi(x_0 + h\theta, t_0 + s) \leq \tilde{M}(x_0 + h, t_0 + s) \leq \lambda_0 M(y_0 + \theta_1 h, t_0 + s) + (1 - \lambda_0) M(z_0 + \theta_2 h, t_0 + s),$$

with an equality at  $h = s = 0$ . This implies that

$$\phi_t(x_0, t_0) = \lambda_0 M_t(y_0, t_0) + (1 - \lambda_0) M_t(z_0, t_0), \quad (5.30)$$

and

$$\begin{pmatrix} \theta^2 \phi_{xx} & \theta \phi_{xt} \\ \theta \phi_{xt} & \phi_{tt} \end{pmatrix} (x_0, t_0) \leq \lambda_0 \begin{pmatrix} \theta_1^2 M_{xx} & \theta_1 M_{xt} \\ \theta_2 M_{xt} & M_{tt} \end{pmatrix} (y_0, t_0) + (1 - \lambda_0) \begin{pmatrix} \theta_2^2 M_{xx} & \theta_2 M_x \\ \theta_2 M_x & M_{tt} \end{pmatrix} (z_0, t_0).$$

Multiplying the previous inequality by  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0$  and taking the trace gives

$$\begin{aligned} & \left( \theta^2 \phi_{xx} - 2\theta \phi_{xt} + \phi_{tt} \right) (x_0, t_0) \\ & \leq \lambda_0 \left( \theta_1^2 M_{xx} - 2\theta_1 M_{xt} + M_{tt} \right) (y_0, t_0) + (1 - \lambda_0) \left( \theta_2^2 M_{xx} - 2\theta_2 M_{xt} + M_{tt} \right) (z_0, t_0). \end{aligned} \quad (5.31)$$

Let us choose  $\theta = \phi_t(x_0, t_0)/\phi_x(x_0, t_0)$ ,  $\theta_1 = M_t(y_0, t_0)/M_x(y_0, t_0)$  and  $\theta_2 = M_t(z_0, t_0)/M_x(z_0, t_0)$ : this choice is licit because, if  $y_0 = 0$ , the boundary conditions (5.25) imply that  $M_t(0, t_0) = 0$ , and thus  $\theta_1 = 0$ . We obtain in the same way that  $\theta_2 = 0$  if  $z_0 = R/2$ . Then (5.29) and (5.30) imply that  $\lambda_0 \theta_1 + (1 - \lambda_0) \theta_2 = \theta$  holds. With this choice of  $\theta$ ,  $\theta_1$  and  $\theta_2$ , (5.31) becomes

$$\begin{aligned} & \left( \frac{\phi_t^2}{\phi_x^2} \phi_{xx} - 2 \frac{\phi_t}{\phi_x} \phi_{xt} + \phi_{tt} \right) (x_0, t_0) \\ & \leq \lambda_0 \left( \frac{M_t^2}{M_x^2} M_{xx} - 2 \frac{M_t}{M_x} M_{xt} + M_{tt} \right) (y_0, t_0) + (1 - \lambda_0) \left( \frac{M_t^2}{M_x^2} M_{xx} - 2 \frac{M_t}{M_x} M_{xt} + M_{tt} \right) (z_0, t_0) \\ & = \lambda_0 (M_x f'(-M_x) M_{xx}) (y_0, t_0) + (1 - \lambda_0) (M_x f'(-M_x) M_{xx}) (z_0, t_0), \end{aligned}$$

where we used the equation satisfied by  $M$  for the last equality. Recalling that  $M_x < 0$ , that  $f' \geq 0$  and that  $M_{xx}(y_0, t_0) \geq 0$  and  $M_{xx}(z_0, t_0) \geq 0$  while  $\phi_{xx}(x_0, t_0) \leq 0$  (by (5.29)) gives (5.27).

We now assume that  $t_0 = T$  and check the boundary condition (5.28). To fix the ideas, we assume that  $y_0 > 0$  and  $z_0 < R/2$ , the other cases being similar. Note that (5.29) still holds in this case. Moreover, as, for any  $s \leq 0$  small we have

$$\phi(x_0, T + s) \leq \tilde{M}(x_0, T + s) \leq \lambda_0 M(y_0, T + s) + (1 - \lambda_0) M(z_0, T + s),$$

we get

$$\phi_t(x_0, T) \geq \lambda_0 M_t(y_0, T) + (1 - \lambda_0) M_t(z_0, T).$$

Thus, as  $\phi_x(x_0, T) < 0$ ,  $g' \geq 0$  and  $\phi_{xx}(x_0, T) \leq 0$  while  $M_{xx}(y_0, T) \geq 0$  and  $M_{xx}(z_0, T) \geq 0$ ,

$$\begin{aligned} & \phi_t(x_0, T) + \phi_x(x_0, T) g'(-\phi_x(x_0, T)) \phi_{xx}(x_0, T) \geq \lambda_0 M_t(y_0, T) + (1 - \lambda_0) M_t(z_0, T) \\ & \geq \lambda_0 (M_t(y_0, T) + M_x(y_0, T) g'(-M_x(y_0, T)) M_{xx}(y_0, T)) \\ & \quad + (1 - \lambda_0) (M_t(z_0, T) + M_x(z_0, T) g'(-M_x(z_0, T)) M_{xx}(z_0, T)) = 0, \end{aligned}$$

which is (5.28).

*Conclusion.* We have proved that  $\tilde{M}$  is a viscosity supersolution to the elliptic equation satisfied by  $M$  (including the boundary conditions). Using the regularity of  $M$ , we can choose  $\lambda > 0$  (large) and  $\epsilon > 0$  (arbitrarily small) such that the map  $\hat{M}_{\epsilon, \lambda}(x, t) = M(x, t) - 2\epsilon + \epsilon \exp\{-\lambda t\}$  is a classical strict subsolution of this equation (including the boundary conditions). This implies that  $\hat{M}_{\epsilon, \lambda} \leq \tilde{M}$ . Then, letting  $\epsilon \rightarrow 0$ , we get  $M \leq \tilde{M}$ . As by construction,  $\tilde{M} \leq M$ , we conclude that  $M = \tilde{M}$ ; hence  $M$  is convex with respect to the  $x$  variable, and then  $m = -M_x$  is nonincreasing with respect to the  $x$  variable on  $[0, R/2] \times [0, T]$ . Fix now  $x \in [0, R/2]$  and let  $\gamma(x, \cdot)$  be optimal solution starting from  $(x, 0)$ , i.e. the solution to

$$\gamma_t = -u_x(\gamma, t), \quad \gamma(x, 0) = x.$$

By symmetry and periodicity,  $u_x(0, t) = u_x(R/2, t) = 0$  for  $t \in [0, T]$ . Therefore  $\gamma(x, t) \in [0, R/2]$  for  $(x, t) \in [0, R/2] \times [0, T]$  and, in view of the Euler equation (3.9) and the monotonicity of  $m$ ,  $\gamma$  is concave in  $t$ . Differentiating the terminal condition (3.21) with respect to the space variable implies that

$$\gamma_t(x, T) = -g'(m(\gamma(x, T), T))m_x(\gamma(x, T), T) \geq 0.$$

As  $\gamma(x, \cdot)$  is concave, we infer that  $\gamma_t(x, \cdot)$  is nonnegative on  $[0, T]$ . As  $\gamma_t = -u_x(\gamma)$  and  $x \mapsto \gamma(x, t)$  is onto from  $[0, R/2]$  to itself, this implies that  $u_x$  is nonpositive.  $\square$

### 5.3 Solutions in the whole space

We now work in the whole space, returning to the entropic coupling function  $f(m) = \log(m)$  (and  $g(m) = c_T \log(m(T))$ ). Our main result is the existence of a classical solution to (MFG) or to (MFGP) under the structure condition (5.22). We adapt Definition 5.1 to the case that  $x$  belongs to the whole space. In what follows,  $\mathcal{P}_2(\mathbb{R})$  will denote the set of Borel probability measures on  $\mathbb{R}$  with a finite second order moment, equipped with the 2-Wasserstein distance.

**Definition 5.8.** We say that  $(u, m)$  is a (classical) solution to (MFG) if  $(u, m) \in C^2(\mathbb{R} \times (0, T)) \times C^1(\mathbb{R} \times (0, T))$ , with  $m > 0$  in  $\mathbb{R} \times (0, T]$ , if  $m \in C([0, T]; \mathcal{P}_2(\mathbb{R}))$  with  $m(0) = m_0$ , if the equations are satisfied in the classical sense for  $t \in (0, T)$  and, finally, if  $m(T) \in L^1(\mathbb{R})$  and  $\lim_{t \rightarrow T^-} u(x, t) = g(m(x, T))$  for every  $x \in \mathbb{R}$ .

Similarly, we say that  $(u, m)$  is a (classical) solution to (MFGP) if  $(u, m) \in C^2(\mathbb{R} \times (0, T)) \times C^1(\mathbb{R} \times (0, T))$ , with  $m > 0$  in  $\mathbb{R} \times (0, T)$ , if  $m \in C([0, T]; \mathcal{P}_2(\mathbb{R}))$  with  $m(0) = m_0$  and  $m(T) = m_T$ , and if the equations are satisfied in the classical sense for  $t \in (0, T)$ .

Let us notice that, whenever (5.22) holds, in view of the preservation of symmetry property of Lemma 5.7, the solutions to the MFG system with periodic and Neumann boundary conditions coincide. For this reason, we will not require the analysis of Subsection 4.1.1 in this case.

**Theorem 5.9.** *Assume that  $f(m) = \log(m)$ .*

1. *Assume that  $m_0, m_T$  are continuous, compactly supported densities on  $\mathbb{R}$ , even, nonincreasing on  $[0, \infty)$ , with  $m_0 \in C_{\text{loc}}^{1, \alpha}(\{m_0 > 0\})$ . Then there exists a unique (up to addition of a constant to  $u$ ) solution  $(u, m) \in C^2(\mathbb{R} \times (0, T)) \times C^1(\mathbb{R} \times (0, T))$  of (MFGP) such that  $m$  is continuous and bounded on  $\mathbb{R} \times [0, T]$  and  $\frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$ , for every  $t \in (0, T)$ .*
2. *Assume that  $m_0$  is a continuous, compactly supported density on  $\mathbb{R}$ , even, nonincreasing on  $[0, \infty)$ , with  $m_0 \in C_{\text{loc}}^{1, \alpha}(\{m_0 > 0\})$ , and  $g(s) = c_T \log(s)$ , for some  $c_T \geq 0$ . Then there exists a unique solution  $(u, m) \in C^2(\mathbb{R} \times (0, T]) \times C^1(\mathbb{R} \times (0, T])$  of (MFG) such that  $m$  is continuous and bounded on  $\mathbb{R} \times [0, T]$  and  $\frac{u(t)}{(1+|x|^2)} \in L^\infty(\mathbb{R})$ , for every  $t \in (0, T)$ .*

Let us start with a priori estimates for positive periodic solutions:

**Lemma 5.10.** *Suppose that (5.22) holds and that  $(u, m) \in C^2(R\mathbb{T} \times [0, T]) \times C^1(R\mathbb{T} \times [0, T])$  is a classical solution to (5.1)–(5.2) or to (5.1)–(5.3) on  $R\mathbb{T} \times (0, T)$  with  $m$  positive in  $R\mathbb{T} \times [0, T]$ . Let  $\gamma : R\mathbb{T} \times [0, T] \rightarrow R\mathbb{T}$  be the associated flow of optimal trajectories.*

1. *(Global estimates) There exists  $C_0 > 0$  depending only on  $T, \mathcal{E}(m_0), \mathcal{E}(m_T), M_2(m_0)$  (and  $M_2(m_T)$  for the planning problem), such that:*

$$\sup_{t \in [0, T]} \int_{R\mathbb{T}} x^2 m(x, t) dx \leq C_0, \quad W_2(m(t), m(s)) \leq C_0 |t - s|^{1/2} \quad \forall s, t \in [0, T]. \quad (5.32)$$

and

$$-\frac{C_0}{t}(x^2 + 1) \leq u(x, t) \leq \frac{C_0}{(T-t)}(x^2 + 1). \quad (5.33)$$

Moreover, for any  $(x, t) \in (-R/2, R/2) \times [0, T]$ ,

$$m(x, t) \leq \begin{cases} \max\{\|m_0\|_\infty, \|m_T\|_\infty\} & \text{if } (u, m) \text{ satisfies (5.1)–(5.2)} \\ \|m_0\|_\infty & \text{if } (u, m) \text{ satisfies (5.1)–(5.3)} \end{cases} \quad (5.34)$$

and

$$(8C_0)^{-1/2} \left( \int_{|x|}^{R/2} m_0(y) dy \right)^{3/2} \leq m(\gamma(x, t), t). \quad (5.35)$$

2. (Interior estimates) Fix any  $\delta \in (0, (1/2) \wedge (T/4))$  and  $a \in (1, (R/2 - 1))$ , with  $R > 4$ . For any  $\eta \in (0, R/2)$  and  $\theta \in (\eta, R/2)$  such that

$$\delta^{3/2} (8C_0)^{-1/2} \left| \log \left( \int_{\theta-\eta}^{R/2} m_0(y) dy \right) \right|^{1/2} > 2a, \quad (5.36)$$

one has

$$\min_{t \in [\delta, T-\delta]} \gamma(\theta - \eta, t) > a \quad (5.37)$$

and

$$\|m, 1/m\|_{C^{2,\alpha}((-a,a) \times (\delta, T-\delta))} \leq C(\eta^{-1}, \delta^{-1}, \|m_0\|_\infty, \|m_T\|_\infty, K_\theta, C_0),$$

where

$$K_\theta := \|m_0^{-1}\|_{L^\infty((-\theta, \theta))} + \|m_0\|_{C^{1,\alpha}((-\theta, \theta))} + \left( \int_\theta^{R/2} m_0(y) dy \right)^{-1}. \quad (5.38)$$

*Proof.* Estimates (5.32) and (5.33) are given in Proposition 5.3. By Lemma 5.7 the solution  $(u, m)$  satisfies (5.23), and  $x \rightarrow \gamma(x, t)$  is increasing on  $[0, R/2]$ , with  $\gamma(0, t) = 0$  and  $\gamma(R/2, t) = R/2$  for any  $t \in [0, T]$ .

*Step 1: Bounds on the density.* The upper bounds on  $m$  in (5.34) hold by Proposition 3.9. Let us now prove the lower bound (5.35). Let  $x \in [0, R/2)$  and  $k = (2C_0)^{1/2} \left( \int_x^{R/2} m_0(y) dy \right)^{-1/2}$ . We first assume that  $k < R/2$ . Then, by (5.23) (for the second inequality) and (5.32) and the choice of  $k$  (for the last one), we have

$$\begin{aligned} \int_x^{R/2} m_0(y) dy &= \int_{\gamma(x,t)}^{R/2} m(y, t) dy \leq \int_{\gamma(x,t) \wedge k}^k m(y, t) dy + \int_k^{R/2} m(y, t) dy \\ &\leq m(k \wedge \gamma(x, t), t)(k - k \wedge \gamma(x, t)) + k^{-2} \int_k^{R/2} x^2 m(y, t) dy \\ &\leq m(k \wedge \gamma(x, t), t)(k - k \wedge \gamma(x, t)) + \frac{1}{2} \int_x^{R/2} m_0(y) dy. \end{aligned}$$

This implies that  $\gamma(x, t) < k$  and (5.35) in this case. Next we suppose that  $k \geq R/2$ . Then the same computation shows that

$$m(\gamma(x, t), t) \geq (2/R) \int_x^{R/2} m_0(y) dy \geq (2C_0)^{-1/2} \left( \int_x^{R/2} m_0(y) dy \right)^{3/2},$$

where the last inequality holds because  $k \geq R/2$ . Thus we have proved (5.35) for  $x \in [0, R/2)$ . The result for negative  $x$  holds by symmetry.

*Step 2: Elliptic estimates.* We now prove  $C^{2,\alpha}$  estimates for  $\gamma_x$  and  $w = \log(m(\gamma))$ . Recalling from (3.10) that  $\gamma_x(x, t) = m_0(x)/m(\gamma(x, t), t)$ , we note that (5.34) and (5.35) imply that  $\gamma_x$  is locally bounded above and below:

$$\frac{m_0(x)}{\max\{\|m_0\|_\infty, \|m_T\|_\infty\}} \leq \gamma_x(x, t) \leq \frac{(8C_0)^{1/2} m_0(x)}{\left( \int_{|x|}^{R/2} m_0(y) dy \right)^{3/2}} \quad \forall (x, t) \in (-R/2, R/2) \times [0, T]. \quad (5.39)$$

Let  $w(x, t) = \log(m(\gamma(x, t), t))$ . Then  $w$  solves the elliptic equation in divergence form (see (3.17)):

$$-(\gamma_x w_t)_t - \left( \frac{1}{\gamma_x} w_x \right)_x = 0 \quad \text{in } (-R/2, R/2) \times (0, T). \quad (5.40)$$

Fix  $\eta \in (0, R/2)$  and  $\delta \in (0, T/4)$ . Let  $\theta \in (\eta, R/2)$  and  $K_\theta$  be defined by (5.38). As, by (5.34), (5.35) and (5.39),

$$|w| + |\gamma_x| + |1/\gamma_x| \leq C(K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0) \quad \text{on } [-\theta, \theta] \times [0, T],$$

we infer by elliptic regularity that

$$\|w\|_{C^{0,\alpha}([-\theta+\eta/3, \theta-\eta/3] \times [\delta/3, T-\delta/3])} \leq C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0).$$

Recalling that  $\gamma_x = m_0/m(\gamma) = m_0 e^{-w}$ , this implies that

$$\|\gamma_x, 1/\gamma_x\|_{C^{0,\alpha}([-\theta+\eta/3, \theta-\eta/3] \times [\delta/3, T-\delta/3])} \leq C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0).$$

On the other hand (see (3.12) with  $f(s) = \log s$ ),  $\gamma$  solves the elliptic equation

$$\gamma_{tt} + \frac{\gamma_{xx}}{\gamma_x^2} = \frac{1}{\gamma_x} \frac{(m_0)_x}{m_0} \quad \text{on } (-R/2, R/2) \times (0, T), \quad \gamma(x, 0) = x \text{ on } [-R/2, R/2].$$

Using the Schauder estimates, we therefore have

$$\|\gamma\|_{C^{2,\alpha}([-\theta+\eta/2, \theta-\eta/2] \times [\delta/2, T-\delta/2])} \leq C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0). \quad (5.41)$$

Returning to (5.40), we obtain, again by the Schauder estimates,

$$\|w\|_{C^{2,\alpha}([-\theta+\eta, \theta-\eta] \times [\delta, T-\delta])} \leq C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0). \quad (5.42)$$

*Step 3: Lower bound on  $\gamma$ .* We claim that, for any  $\delta \in (0, (1/2) \wedge (T/4))$  and any  $x \in [0, R/2]$ ,

$$\min_{t \in [\delta, T-\delta]} \gamma(x, t) \geq (1 - \delta) \left( (R/2) \wedge \left( \delta^{3/2} (C_0)^{-1/2} \left| \log \left( \int_x^{R/2} m_0(y) dy \right) \right|^{1/2} \right) - 1 \right). \quad (5.43)$$

*Proof of (5.43):* Fix  $x \in [0, R/2]$ ,  $t \in [\delta, T - \delta]$  and set

$$a = \max_{s \in [t-\delta^2, t+\delta^2]} \gamma(x, s) + 1.$$

Since we want a lower bound for  $a$ , we can assume that  $a \leq R/2$ . Then, recalling (3.11) and the fact that  $m(\cdot, t)$  is nonincreasing, we have for any  $s \in [t - \delta^2, t + \delta^2]$ ,

$$\int_x^{R/2} m_0(y) dy = \int_{\gamma(x,s)}^{R/2} m(y, s) dy \geq \int_{\gamma(x,s)}^a m(y, s) dy \geq (a - \gamma(x, s)) m(a, s) \geq m(a, s).$$

Using the previous inequality together with the HJ equation, and integrating in  $(t - \delta^2, t + \delta^2)$ , we get

$$\begin{aligned} 2\delta^2 \log \left( \int_x^{R/2} m_0(y) dy \right) &\geq \int_{t-\delta^2}^{t+\delta^2} \log(m(a, s)) ds \geq \int_{t-\delta^2}^{t+\delta^2} -u_t(a, s) ds \\ &\geq - \left( \frac{C_0}{t-\delta^2} + \frac{C_0}{T-t-\delta^2} \right) (a^2 + 1) \geq -8a^2 C_0 \delta^{-1} \end{aligned}$$

where we used (5.33) and the fact that  $a \geq 1$  and  $\delta/2 \leq t - \delta^2 \leq t + \delta^2 \leq T - \delta/2$ . Thus, up to increasing the value of  $C_0$ , we obtain

$$a \geq \delta^{3/2} C_0^{-1/2} \left| \log \left( \int_x^{R/2} m_0(y) dy \right) \right|^{1/2}.$$

This proves that

$$\max_{s \in [t-\delta^2, t+\delta^2]} \gamma(x, s) \geq (R/2) \wedge \left( \delta^{3/2} (C_0)^{-1/2} \left| \log \left( \int_x^{R/2} m_0(y) dy \right) \right|^{1/2} \right) - 1.$$

As  $\gamma(x, \cdot)$  is nonnegative and concave, letting  $s_0$  be a maximum point of  $\gamma(x, \cdot)$  in  $[t - \delta^2, t + \delta^2]$ ,

$$\gamma(x, t) \geq \left( \frac{t}{s_0} \mathbf{1}_{s_0 \in [t, t + \delta^2]} + \frac{T - t}{T - s_0} \mathbf{1}_{s_0 \in [t - \delta^2, t]} \right) \gamma(s_0) \geq \min \left\{ \frac{t}{t + \delta^2}, \frac{T - t}{T - t + \delta^2} \right\} \gamma(s_0) \geq (1 - \delta) \gamma(s_0).$$

Using our estimate on  $\gamma(x, s_0) = \max_{[t - \delta^2, t + \delta^2]} \gamma$  gives (5.43).

*Step 4: Interior estimate of  $m$ .* Fix  $\delta \in (0, (1/2) \wedge T/4)$  and  $a \in (1, (R/2 - 1))$ , with  $R > 4$ . Now assume that  $\eta \in (0, R/2)$  and  $\theta \in (\eta, R/2)$  are such that (5.36) holds. Then

$$(1 - \delta) \left( (R/2) \wedge \left( \delta^{3/2} (C_0)^{-1/2} \left| \log \left( \int_{\theta - \eta}^{R/2} m_0(y) dy \right) \right|^{1/2} \right) - 1 \right) > a + 1. \quad (5.44)$$

We claim that

$$\|m, m^{-1}\|_{C^{2,\alpha}((-a,a) \times (\delta, T - \delta))} \leq C(\eta^{-1}, \delta^{-1}, K_\theta, \|m_0\|_\infty, \|m_T\|_\infty, C_0).$$

Indeed, let  $\gamma^{-1}(\cdot, t) : (-R/2, R/2) \rightarrow (-R/2, R/2)$  denote the inverse in space of  $\gamma(\cdot, t)$ . We first claim that  $\gamma^{-1}([-a, a] \times [\delta, T - \delta]) \subset (-\theta + \eta, \theta - \eta) \times [\delta, T - \delta]$ . Indeed, otherwise, there exists  $t \in [\delta, T - \delta]$  and  $x \in [-a, a]$  such that  $(x, t) \notin \gamma((-\theta + \eta, \theta - \eta) \times [\delta, T - \delta])$ . This means that  $x \notin \gamma((-\theta + \eta, \theta - \eta), t)$ . As  $\gamma(\cdot, t)$  is continuous, vanishes at  $x = 0$  and is increasing, this implies that  $a \geq |x| \geq \gamma(\theta - \eta, t)$ , which, by (5.43), contradicts (5.44). In particular, (5.37) holds. Recalling (5.41),  $\gamma^{-1}$  is bounded in  $C^{2,\alpha}$  in  $[-a, a] \times [\delta, T - \delta]$  and then (5.42) implies the  $C^{2,\alpha}$  bound on  $\log(m) = w \circ \gamma^{-1}$ . This implies that  $m$  and  $1/m$  are bounded in  $C^{2,\alpha}$  in  $[-a, a] \times [\delta, T - \delta]$ .  $\square$

*Proof of Theorem 5.9.* (i) We prove the existence of a classical solution. We start with the planning problem (MFGP) and explain at the end of the proof the necessary changes for the (MFG) problem.

Let  $[-b_0, b_0]$  (resp.  $[-b_T, b_T]$ ) be the support of  $m_0$  (resp. of  $m_T$ ). For  $R \geq \bar{R} := \max\{b_0, b_T\}$ , let  $\tilde{m}_0^R$  and  $\tilde{m}_T^R$  be the continuous periodic map of period  $R$  such that  $\tilde{m}_0^R = m_0$  and  $\tilde{m}_T^R = m_T$  on  $(-R/2, R/2)$ . We let  $m_0^{R,\epsilon} = \tilde{m}_0^R * \eta_\epsilon$  and  $m_T^{R,\epsilon} = \tilde{m}_T^R * \eta_\epsilon$  where  $\eta_\epsilon$  is a standard mollifier, smooth, even and positive on  $\mathbb{R}$ . Then, for  $R \geq \bar{R}$  and  $\epsilon > 0$ ,  $m_0^{R,\epsilon}$  and  $m_T^{R,\epsilon}$  are smooth, positive and satisfy (5.22). Let  $(u^{R,\epsilon}, m^{R,\epsilon})$  be the classical solution to (3.20)–(3.22) given by Theorem 3.11. By Lemma 5.7,  $m^{R,\epsilon}$  satisfies (5.23).

We now prove an interior estimate for  $m^{R,\epsilon}$ . Fix  $a > 1$  and  $\delta \in (0, (1/2) \wedge (T/4))$ . Choose  $R_0 > 4$  such that  $(R_0/2 - 1) \geq a + 1$ . Fix also  $r_0 \in (0, b^0)$  large enough such that

$$\delta^{3/2} (C_0)^{-1/2} \left| \log \left( \int_{r_0}^{b_0} m_0(y) dy \right) \right|^{1/2} > 2a + 1.$$

This is possible as  $\int_{r_0}^{b_0} m_0(y) dy \rightarrow 0$  as  $r_0 \rightarrow (b^0)^-$ . We then set  $\theta = (r_0 + b_0)/2$  and  $\eta = (b_0 - r_0)/2$ . We finally choose  $\epsilon_0 \in (0, \eta/4)$  such that, for  $\epsilon \in (0, \epsilon_0)$ ,

$$\delta^{3/2} (8C_0)^{-1/2} \left| \log \left( \int_{r_0}^{b_0} m_0^{R,\epsilon}(y) dy \right) \right|^{1/2} > 2a \quad \text{and} \quad \int_{\theta}^{R/2} m_0^{R,\epsilon}(y) dy \geq \frac{1}{2} \int_{\theta}^{R/2} m_0(y) dy.$$

Then, for any  $R \geq R_0$ ,  $\epsilon \in (0, \epsilon_0)$ , condition (5.36) holds for  $m_0^\epsilon$ . By Lemma 5.10,

$$\begin{aligned} \|m^{R,\epsilon}, 1/m^{R,\epsilon}\|_{C^{2,\alpha}((-a,a) \times (\delta, T - \delta))} &\leq C(\eta^{-1}, \delta^{-1}, K_\theta^\epsilon, \|\tilde{m}_0^R\|_\infty, \|\tilde{m}_T^R\|_\infty, C_0^\epsilon) \\ &= C(\eta^{-1}, \delta^{-1}, K_\theta^\epsilon, \|m_0\|_\infty, \|m_T\|_\infty, C_0^\epsilon), \end{aligned} \quad (5.45)$$

where  $C_0^\epsilon$  depends only on  $T$ ,  $\mathcal{E}(\tilde{m}_0^R)$ ,  $\mathcal{E}(\tilde{m}_T^R)$ ,  $M_2(\tilde{m}_0^R)$ ,  $M_2(\tilde{m}_T^R)$ , and thus only on  $T$ ,  $\mathcal{E}(m_0)$ ,  $\mathcal{E}(m_T)$ ,  $M_2(m_0)$ ,  $M_2(m_T)$ , and

$$\begin{aligned} K_\theta^\epsilon &= \|(\tilde{m}_0^{R,\epsilon})^{-1}\|_{L^\infty((-\theta, \theta))} + \|\tilde{m}_0^{R,\epsilon}\|_{C^{1,\alpha}((-\theta, \theta))} + \left( \int_{\theta}^{R/2} \tilde{m}_0^{R,\epsilon}(y) dy \right)^{-1} \\ &\leq \|m_0^{-1}\|_{L^\infty((-\theta - \epsilon_0, \theta + \epsilon_0))} + \|m_0\|_{C^{1,\alpha}((-\theta - \epsilon_0, \theta + \epsilon_0))} + 2 \left( \int_{b_0 - \eta/2}^{b_0} m_0(y) dy \right)^{-1}, \end{aligned} \quad (5.46)$$

which is finite since  $\theta + \epsilon_0 = b_0 - \eta$ . This shows that

$$\|m^{R,\epsilon}, 1/m^{R,\epsilon}\|_{C^{2,\alpha}((-a,a)\times(\delta,T-\delta))} \leq C(a, \delta^{-1}, T, m_0, m_T).$$

Since  $\log(m^{R,\epsilon})$  is uniformly Lipschitz continuous in  $(-a, a) \times (\delta, T - \delta)$  and since the map  $u^{R,\epsilon}$  is a locally uniformly bounded solution of a HJ equation with r.h.s.  $\log(m^{R,\epsilon})$ ,  $u^{R,\epsilon}$  is uniformly Lipschitz continuous in  $(-a/2, a/2) \times (2\delta, T - 2\delta)$ . By (3.1) (with  $f = \log s$ ), we know that  $u^{R,\epsilon}$  satisfies the elliptic equation

$$-u_{tt} + 2u_x u_{xt} - (u_x^2 + 1)u_{xx} = 0. \quad (5.47)$$

Hence, by elliptic regularity, we obtain

$$\|u^{R,\epsilon}\|_{C^{2,\alpha}((-a/2,a/2)\times(2\delta,T-2\delta))} \leq C(a, \delta^{-1}, m_0, m_T, C_0).$$

We can now use the estimates above and the first part of Lemma 5.10 to find (a subsequence) of  $(u^{R,\epsilon}, m^{R,\epsilon})$  which converges, as  $\epsilon \rightarrow 0^+$  and then  $R \rightarrow \infty$ , to a pair  $(u, m)$  which is a  $C^{2,\alpha}$  solution of the MFG system (1.1) in  $\mathbb{R} \times (0, T)$  with  $f(m) = \log(m)$  and such that:

$$m \in C^0([0, T], \mathcal{P}_2(\mathbb{R})), \quad m(0) = m_0, \quad m(T) = m_T, .$$

Moreover, by construction,  $m(\cdot, t)$  is even and  $x \rightarrow m(x, t)$  is nonincreasing on  $[0, +\infty)$  for any  $t \in [0, T]$ .

Let us finally check the continuity of  $m$  at  $t = 0$  (the case  $t = T$  being symmetric). Let  $t_n \rightarrow 0^+$ . Then the maps  $m(\cdot, t_n)$  are nonincreasing on  $[0, \infty)$  and converge weakly-\* (as measures) to  $m_0$  which has a continuous density: this limit is therefore locally uniform.

We now consider the (MFG) problem. We regularize  $m_0^{R,\epsilon}$  as above and let  $(u^{R,\epsilon}, m^{R,\epsilon})$  be the classical solution to (5.1)–(5.3). Let  $\gamma^{R,\epsilon}$  be the associated flow of optimal trajectories. Let us recall that, under our structure condition,  $\gamma^{R,\epsilon}(x, \cdot)$  is concave and nondecreasing in time for any  $x \in [0, R/2]$ .

Fix  $a > 1$  and  $\delta \in (0, (1/2) \wedge (T/4))$ . We can choose  $R_0, \theta, \eta$  and  $\epsilon_0 > 0$  as in the first part of the proof such that, for  $R \geq R_0$  and  $\epsilon \in (0, \epsilon_0)$ ,

$$\int_{\theta}^{R/2} m_0^{R,\epsilon}(y) dy \geq \frac{1}{2} \int_{\theta}^{R/2} m_0(y) dy > 0$$

and

$$\|m^{R,\epsilon}, 1/m^{R,\epsilon}\|_{C^{2,\alpha}((-a,a)\times(\delta,T-\delta))} \leq C(a, \delta^{-1}, T, m_0, m_T).$$

With this choice we also have (estimate (5.43) in Lemma 5.10)

$$\min_{t \in [\delta, T-\delta]} \gamma(\theta - \eta, t) > a.$$

Recall also that, by (5.34) and (5.35), for any  $(x, t) \in (-R/2, R/2) \times [0, T]$ ,

$$(8C_0)^{-1/2} \left( \int_{|x|}^{R/2} m_0^{R,\epsilon}(y) dy \right)^{3/2} \leq m^{R,\epsilon}(\gamma(x, t), t) \leq \|m_0^{R,\epsilon}\|_{\infty} \leq \|m_0\|_{\infty}.$$

Hence for any  $(y, t) \in [0, a] \times [\delta, T]$ , there exists  $x \in [0, \theta - \eta]$  with  $\gamma^{R,\epsilon}(x, t) = y$  and therefore

$$\|m_0\|_{\infty} \geq m^{R,\epsilon}(y, t) = m^{R,\epsilon}(\gamma(x, t), t) \geq \frac{1}{2} (8C_0)^{-1/2} \left( \int_{\theta}^{R/2} m_0(y) dy \right)^{3/2},$$

which proves that  $m^{R,\epsilon}$  is bounded above and below in  $[-a, a] \times [\delta, T]$ . Next we show a Lipschitz bound for  $u^{R,\epsilon}$  in  $[0, a] \times [\delta, T]$ . For any  $(y, t) \in [0, a] \times [\delta, T]$ , there exists  $x \in [0, y]$  such that  $\gamma^{R,\epsilon}(x, t) = y$ . Recall that  $\gamma_t^{R,\epsilon}(x, t) = -u_x(y, t)$ . On the other hand, by concavity of  $\gamma^{R,\epsilon}(x, \cdot)$ ,

$$0 \leq \gamma^{R,\epsilon}(x, s) \leq y + \gamma_t^{R,\epsilon}(x, t)(s - t) \quad \forall s \in [0, T].$$



Thus (choosing  $s = 0$  and using that  $y \leq a$  and  $t \geq \delta$ )

$$0 \leq \gamma_t^{R,\epsilon}(x, t) = -u_x^{R,\epsilon}(y, t) \leq y/t \leq a/\delta.$$

By symmetry, this proves that

$$\|u_x^{R,\epsilon}\|_{L^\infty([-a,a] \times [\delta,T])} \leq a/\delta.$$

Let us finally check a local bound for  $u^{R,\epsilon}$ . We already have a bound below (Lemma 5.10). As  $u^{R,\epsilon}(\cdot, t)$  is nonincreasing on  $[0, R/2]$  and  $\gamma(0, \cdot) = 0$ , we have, for  $(x, t) \in [0, a] \times [\delta, T]$ :

$$\begin{aligned} u^{R,\epsilon}(x, t) &\leq u^{R,\epsilon}(0, t) = \int_t^T \frac{1}{2} |\gamma_s^{R,\epsilon}(0, s)|^2 + \ln(m^{R,\epsilon}(\gamma^{R,\epsilon}(0, s))) \, ds + c_T \ln(m^{R,\epsilon}(\gamma^{R,\epsilon}(0, T), T)) \\ &\leq C \|m^{R,\epsilon}, 1/m^{R,\epsilon}\|_{L^\infty(\{0\} \times [\delta, T])} \leq C(a, \delta^{-1}). \end{aligned}$$

We have proved positive upper and lower bounds for  $m^{R,\epsilon}$  and Lipschitz bounds for  $u^{R,\epsilon}$  independent of  $R, \epsilon$  on  $[-a, a] \times [\delta, T]$ . By the elliptic equation (5.47) satisfied by  $u^{R,\epsilon}$ , we can infer that

$$\|u^{R,\epsilon}\|_{C^{2,\alpha}([-a/2, a/2] \times [2\delta, T])} \leq C(a, \delta^{-1}).$$

We can then conclude as before.

(ii) The uniqueness of solutions is proved with the same kind of argument used in Theorem 5.2. First of all, we observe that, since  $m(t) \in \mathcal{P}_2(\mathbb{R})$  and  $u(t)/(1 + |x|^2) \in L^\infty(\mathbb{R})$ , then we have  $u(t)m(t) \in L^1(\mathbb{R})$  for any  $t \in (0, T)$ . This implies that a similar equality as (5.19) holds, namely

$$\int_{\mathbb{R}} u(t_0)m(t_0) - \int_{\mathbb{R}} u(t_1)m(t_1) = \int_{t_0}^{t_1} \int_{\mathbb{R}} m u_x^2 + \int_{t_0}^{t_1} \int_{\mathbb{R}} m \log(m) \quad \forall 0 < t_0 < t_1 < T. \quad (5.48)$$

As in the proof of Theorem 5.2, we deduce from (5.48) (and the time-monotonicity of  $u$ ) that  $u(0^+) \in L^1(dm_0)$  and  $u(T^-) \in L^1(dm(T))$ . Then, using a truncation argument for  $u$  and the continuity of  $m$ , inequality (5.48) is extended up to  $t_0 = 0$  and  $t_1 = T$ , obtaining the equivalent of (5.20) but integrated for  $x \in \mathbb{R}$ . Notice that the same truncation argument for  $u$  as in Theorem 5.2 works here, because  $m$  has finite moments (uniformly in time), so actually  $\int_{\mathbb{R}} u_k(t)m(t)$  ends up being continuous in  $[0, T]$  for fixed  $k$ . In a similar way, we obtain the equivalent of (5.21) for  $x \in \mathbb{R}$ , and we conclude from the Lasry-Lions monotonicity argument as in the compact case. □

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## A Construction of the self-similar solutions

The goal of this appendix is to prove the statements made in Proposition 2.1 regarding the construction of the self-similar solution, as well as to provide a precise analysis of the regularity of the value function  $u$ . We begin by showing that, inside the support of  $m$ , the system is solved in the classical sense.

**Lemma A.1.** *Let  $u, m$  be defined as in Proposition 2.1. Then  $m$  is a weak solution of the continuity equation,  $u$  is  $C^1$  in the support of  $m$ , and  $u$  satisfies, in the classical sense,*

$$-u_t + \frac{1}{2}|u_x|^2 = m^\theta \quad \text{in } \{m > 0\}.$$

*Proof.* We set  $\gamma^\pm(t) = \pm(2R/(\bar{\alpha} - \bar{\alpha}^2))^{1/2}t^{\bar{\alpha}}$ , and we note that

$$\{m(t, \cdot) > 0\} = \{(x, t), \gamma^-(t) < x < \gamma^+(t)\}.$$

Here we have, according to (2.5) and if  $m(x, t) > 0$ , :

$$-u_t + \frac{1}{2}|u_x|^2 = -\bar{\alpha}\frac{x^2}{2t^2} + Rt^{-2\theta/(2+\theta)} + \frac{1}{2}\bar{\alpha}^2\frac{x^2}{t^2} = -(\bar{\alpha} - \bar{\alpha}^2)\frac{x^2}{2t^2} + Rt^{-2\theta/(2+\theta)}$$

while

$$f(m(x, t)) = t^{-\bar{\alpha}\theta} \left( R - \frac{1}{2}(\bar{\alpha} - \bar{\alpha}^2)\left(\frac{x}{t^{\bar{\alpha}}}\right)^2 \right) = t^{-\bar{\alpha}\theta} R - \frac{1}{2}(\bar{\alpha} - \bar{\alpha}^2)\frac{x^2}{t^{\bar{\alpha}\theta+2\bar{\alpha}}}.$$

By the definition of  $\bar{\alpha}$ , we have  $\bar{\alpha}\theta = 2\theta/(2 + \theta)$ , and  $\bar{\alpha}\theta + 2\bar{\alpha} = 2$ . Thus we conclude that

$$-u_t + \frac{1}{2}u_x^2 = f(m(x, t)) \quad \text{in } \{m > 0\}.$$

On the other hand, for any test function  $\varphi \in C_c^\infty(\mathbb{R} \times (0, T))$ , we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(x, t)m(x, t)dx &= \frac{d}{dt} \int_{\gamma^-(t)}^{\gamma^+(t)} \varphi(x, t)t^{-\bar{\alpha}}\phi(x/t^{\bar{\alpha}})dx \\ &= \int_{\gamma^-(t)}^{\gamma^+(t)} (\varphi_t(x, t)t^{-\bar{\alpha}}\phi(x/t^{\bar{\alpha}}) + \varphi(x, t)(-\bar{\alpha}t^{-\bar{\alpha}-1})(\phi(x/t^{\bar{\alpha}}) + xt^{-\bar{\alpha}}\phi'(x/t^{\bar{\alpha}})))dx \end{aligned}$$

where

$$\int_{\gamma^-(t)}^{\gamma^+(t)} \varphi(x, t)x\phi'(x/t^{\bar{\alpha}})dx = -t^{\bar{\alpha}} \int_{\bar{\gamma}^-(t)}^{\bar{\gamma}^+(t)} (\varphi_x(x, t)x + \varphi(x, t))\phi(x/t^{\bar{\alpha}})dx.$$

So

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \varphi(x, t)m(x, t)dx &= \int_{\gamma^-(t)}^{\gamma^+(t)} (\varphi_t(x, t)t^{-\bar{\alpha}}\phi(x/t^{\bar{\alpha}}) - \varphi_x(x, t)(-\bar{\alpha}t^{-\bar{\alpha}-1})x\phi(x/t^{\bar{\alpha}}))dx \\ &= \int_{\mathbb{R}} (\varphi_t(x, t) - \varphi_x(x, t)u_x(x, t))m(x, t)dx. \end{aligned}$$

This shows that  $m$  solves the continuity equation. □

We now extend the definition of  $u$  outside the support of  $m$ , and analyze its behavior near the interface. We recall that the free boundary is the set  $\{\Delta = 0\}$ , where  $\Delta = |x| - \sqrt{\frac{2R}{\bar{\alpha}(1-\bar{\alpha})}}t^{\bar{\alpha}}$ .

**Lemma A.2.** *For each  $(x, t) \in \{\Delta > 0\}$ , the equation (2.4) has a unique positive solution  $S \in (0, t)$ , and  $S$  is a smooth function of  $(x, t)$ . Furthermore, the function  $S = S(x, t)$  extends continuously to  $\{\Delta \geq 0\}$ ,  $S(x, t) = t$  on  $\Delta = 0$ , and one has the estimates*

$$S(x, t) \geq c_0 \left( \frac{t}{|x| + \Delta} \right)^{\frac{1}{1-\bar{\alpha}}}, \tag{A.1}$$

$$|t - S(x, t)| \leq C_0 t^{1-\frac{\bar{\alpha}}{2}} \Delta^{\frac{1}{2}}. \tag{A.2}$$

where  $c_0 = \left( \frac{2R\bar{\alpha}}{1-\bar{\alpha}} \right)^{\frac{1}{2(1-\bar{\alpha})}}$  and  $C_0 = \left( \frac{2}{R\bar{\alpha}(1-\bar{\alpha})} \right)^{\frac{1}{4}}$ .

*Proof.* Let us set  $C_R := \sqrt{\frac{2R}{\bar{\alpha}(1-\bar{\alpha})}}$ ; hence the interface  $\Delta = 0$  is the curve  $|x| = C_R t^{\bar{\alpha}}$ . We define

$$F(x, t, s) = -|x|s^{1-\bar{\alpha}} + C_R(\bar{\alpha}t + (1-\bar{\alpha})s). \quad (\text{A.3})$$

Then, we have

$$\frac{\partial F}{\partial s}(x, t, s) = -(1-\bar{\alpha})s^{-\bar{\alpha}}(|x| - C_R s^{\bar{\alpha}}). \quad (\text{A.4})$$

Notice that, since  $\Delta(x, t) > 0$ ,  $\frac{\partial F}{\partial s}(x, t, s) < 0$  for  $s \in [0, t]$ . Moreover,  $F(x, t, 0) = \bar{\alpha}C_R t > 0$ , and  $F(x, t, t) = -t^{1-\bar{\alpha}}\Delta < 0$ . Hence there exists a unique  $S \in (0, t)$  with  $F(x, t, S) = 0$ . Now, since  $\frac{\partial F}{\partial s}(x, t, S) < 0$ , the implicit function theorem guarantees that the function  $S$  is smooth in  $\{\Delta > 0\}$ . We now show a lower bound on  $S$ , by taking  $s = \left(\frac{\bar{\alpha}C_R t}{|x| + \Delta}\right)^{\frac{1}{1-\bar{\alpha}}}$ . First we note that, since  $\bar{\alpha} < 1$ , we have

$$s < \left(\frac{C_R t}{|x|}\right)^{\frac{1}{1-\bar{\alpha}}} \leq t \quad \text{in } \{\Delta > 0\}.$$

Hence  $s \in (0, t)$ . Moreover, we have

$$F(x, t, s) = -\frac{|x|}{|x| + \Delta}\bar{\alpha}C_R t + C_R(\bar{\alpha}t + (1-\bar{\alpha})s) \geq 0.$$

Consequently, since  $\frac{\partial F}{\partial s} < 0$  on  $(0, t)$ , and  $F(x, t, S) = 0$ , we have  $S \geq s$ , that is,

$$S(x, t) \geq \left(\frac{\bar{\alpha}C_R t}{|x| + \Delta}\right)^{\frac{1}{1-\bar{\alpha}}}.$$

This is (A.1). Finally, we are now concerned with the continuous extension to  $\Delta = 0$ . First of all, we rewrite (2.4) as

$$\bar{\alpha}C_R(t - S) = S^{1-\bar{\alpha}}(|x| - C_R S^{\bar{\alpha}}) \quad (\text{A.5})$$

which yields

$$\bar{\alpha}C_R(t - S) \left(1 - \frac{1}{\bar{\alpha}}S^{1-\bar{\alpha}}\frac{t^{\bar{\alpha}} - S^{\bar{\alpha}}}{t - S}\right) = S^{1-\bar{\alpha}}\Delta. \quad (\text{A.6})$$

We can then use the elementary inequality

$$\frac{(1-\bar{\alpha})}{2}(1-w) \leq \left(1 - \frac{1}{\bar{\alpha}}w^{1-\bar{\alpha}}\frac{1-w^{\bar{\alpha}}}{1-w}\right),$$

valid for all real numbers  $w \in (0, 1)$ . Using this inequality with  $w = \frac{S}{t}$ , we deduce from (A.6)

$$\frac{\bar{\alpha}C_R(1-\bar{\alpha})}{2t}(t - S)^2 \leq S^{1-\bar{\alpha}}\Delta \leq t^{1-\bar{\alpha}}\Delta,$$

which reduces to

$$|t - S| \leq C_0 t^{1-\frac{\bar{\alpha}}{2}} \Delta^{\frac{1}{2}}$$

by setting  $C_0 = \left(\frac{2}{R\bar{\alpha}(1-\bar{\alpha})}\right)^{\frac{1}{4}}$ . From this estimate, one sees that if  $(x_n, t_n) \in \{\Delta > 0\}$  is such that  $(x_n, t_n) \rightarrow (x_0, t_0) \in \{\Delta = 0\}$ , then  $|S(x, t) - t| \rightarrow 0$ .  $\square$

We can now establish the Hölder regularity of  $Du$ .

**Proposition A.3.** *There exists  $0 < s < 1$  such that the function  $u$  (defined in (2.2) or (2.3)) is smooth away from  $\Delta = 0$ , and is uniformly  $C^{1,s}$  on compact subsets of  $(0, \infty) \times \mathbb{R}$ . Moreover, it is a viscosity solution of*

$$-u_t + \frac{1}{2}u_x^2 = m^\theta \quad \text{in } \mathbb{R} \times (0, T).$$

*Proof.* Given  $(x, t) \in \{\Delta > 0\}$ ,  $u(x, t)$  has been defined through the method of characteristics; outside the support of  $m$ , the characteristics are straight lines that join  $(x, t)$  to a unique point  $(\bar{x}, S)$  belonging to the curve  $\{\Delta = 0\}$ . This means that

$$u(x, t) = u(\bar{x}, S) + \frac{1}{2}(t - S)|\lambda|^2, \quad \begin{cases} x = \lambda(t - S) + \bar{x}, & \lambda = -u_x(\bar{x}, S) = \frac{\alpha\bar{x}}{S} \\ \bar{x} = C_R S^{\bar{\alpha}} \end{cases}$$

which leads to the value  $S = S(x, t)$  defined by (A.5) and, correspondingly to the formulas (2.2) or (2.3). By construction, relying on the method of characteristics, it follows that  $u$  satisfies  $-u_t + \frac{1}{2}u_x^2 = 0$  in  $\{\Delta > 0\}$ , and  $u$  is actually a viscosity solution in the whole  $\mathbb{R} \times (0, T)$ .

Let us now look at the regularity of  $u$ . Since  $S$  is smooth away from  $\Delta = 0$ , so is  $u$ . More precisely, recalling that  $S$  is given by the equation

$$\bar{\alpha}C_R t = S^{1-\bar{\alpha}}|x| - (1 - \bar{\alpha})C_R S \tag{A.7}$$

we have, by implicit differentiation,

$$S_x = \frac{-\text{sgn}(x)S}{(1 - \bar{\alpha})(|x| - C_R S^{\bar{\alpha}})}. \tag{A.8}$$

Suppose that  $\theta \neq 2$ , so  $u$  is given by (2.3), which simplifies into

$$u = -\frac{2\bar{\alpha}R}{2\bar{\alpha}-1}S^{2\bar{\alpha}-1} - \frac{\bar{\alpha}R}{1-\bar{\alpha}}tS^{2\bar{\alpha}-2}.$$

Therefore,

$$u_x = -2\bar{\alpha}R \left(1 - \frac{t}{S}\right) S^{2\bar{\alpha}-2} S_x,$$

and from (A.8) and (A.5) we see that this simplifies to:

$$u_x = -\frac{2R}{C_R(1-\bar{\alpha})}S^{\bar{\alpha}-1}\text{sgn}(x) \tag{A.9}$$

By Proposition A.2, we deduce that  $u \in C^1(\mathbb{R} \times (0, T))$ ; in fact, we can see that  $u \in C^{1,s}$ , if we prove a uniform  $C^s$  bound for  $S$  on compact subsets near  $\Delta = 0$ . To this purpose, let  $K$  be a compact subset of  $\Delta > 0$ , and let  $(x, t), (\bar{x}, \bar{t})$  be two points in  $K$ . We write  $\bar{S} = S(\bar{x}, \bar{t})$ . With no loss of generality, we may assume that  $S \geq \bar{S}$ . By formula (A.7), we have

$$S^{1-\bar{\alpha}}|x| - \bar{S}^{1-\bar{\alpha}}|\bar{x}| - (1 - \bar{\alpha})C_R(S - \bar{S}) = C_R(t - \bar{t})$$

which yields, by using  $S^{1-\bar{\alpha}} \geq \bar{S}^{1-\bar{\alpha}} + (1 - \bar{\alpha})S^{-\bar{\alpha}}(S - \bar{S})$

$$\frac{(1 - \bar{\alpha})}{S^{\bar{\alpha}}} (|x| - C_R S^{\bar{\alpha}}) (S - \bar{S}) \leq C_R |t - \bar{t}| + \bar{S}^{1-\bar{\alpha}} |x - \bar{x}|.$$

By definition of  $\Delta$ , and Proposition A.2, we deduce

$$|S - \bar{S}| \leq K_{t, \bar{t}} \frac{(|x - \bar{x}| + |t - \bar{t}|)}{\Delta}.$$

Suppose now that  $\Delta \geq (|x - \bar{x}| + |t - \bar{t}|)^\sigma$ , for some  $\sigma \in (0, 1)$ . Then we get

$$|S - \bar{S}| \leq K_{t, \bar{t}} (|x - \bar{x}| + |t - \bar{t}|)^{1-\sigma}.$$

On the other hand, if  $\Delta < (|x - \bar{x}| + |t - \bar{t}|)^\sigma$ , then we also have

$$\bar{\Delta} < (|x - \bar{x}| + |t - \bar{t}|)^\sigma + |\Delta - \bar{\Delta}| \leq C_{t, \bar{t}} (|x - \bar{x}| + |t - \bar{t}|)^\sigma$$

because  $\Delta$  is a Lipschitz function of  $(x, t)$  far from  $t = 0$  (it is also globally  $\bar{\alpha}$ -Hölder, as well). Hence we estimate, using (A.2)

$$|S - \bar{S}| \leq |S - t| + |t - \bar{t}| + |\bar{t} - \bar{S}| \leq C_0 \max(t, \bar{t})^{1-\frac{\bar{\alpha}}{2}} \left( \Delta^{\frac{1}{2}} + \bar{\Delta}^{\frac{1}{2}} \right) + |t - \bar{t}| \leq \tilde{C}_{t, \bar{t}} (|x - \bar{x}| + |t - \bar{t}|)^{\frac{\bar{\alpha}}{2}}.$$

This concludes with the Hölder bound of  $S(x, t)$ , and therefore with the  $C^{1,s}$  regularity of  $u$ . Finally, for the case  $\theta = 2$ , we argue in the same way by using formula (2.2); notice that  $u$  is explicit in this case, because (2.4) is a quadratic equation in  $\sqrt{S}$ . □

*Remark A.4.* We observe that the solution  $u$  found above is not  $W^{2,\infty}$ . In fact, by differentiating once more (A.9) one gets

$$u_{xx} = -\sqrt{\frac{2R\bar{\alpha}}{1-\bar{\alpha}}} \frac{S^{\bar{\alpha}-1}}{\left(|x| - \sqrt{\frac{2R}{\bar{\alpha}(1-\bar{\alpha})}} S^{\bar{\alpha}}\right)} < 0.$$

Now, as  $\Delta \rightarrow 0$ ,  $S \rightarrow t$ , and thus the denominator  $|x| - \sqrt{\frac{2R}{\bar{\alpha}(1-\bar{\alpha})}} S^{\bar{\alpha}} \rightarrow 0$ . Hence  $u_{xx}$  is unbounded.

The same holds for the case  $\theta = 2$ , since we have

$$u_x = -\frac{4R}{x - \sqrt{\Delta}}, u_{xx} = -\frac{4R}{\sqrt{\Delta}(x - \sqrt{\Delta})}.$$

So  $u_{xx}$  is again unbounded as  $\Delta \rightarrow 0$ .

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