



**HAL**  
open science

# The Cauchy problem associated to the logarithmic Laplacian with an application to the fundamental solution

Huyuan Chen, Laurent Véron

► **To cite this version:**

Huyuan Chen, Laurent Véron. The Cauchy problem associated to the logarithmic Laplacian with an application to the fundamental solution. 2023. hal-04173733v1

**HAL Id: hal-04173733**

**<https://hal.science/hal-04173733v1>**

Preprint submitted on 30 Jul 2023 (v1), last revised 22 Apr 2024 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The Cauchy problem associated to the logarithmic Laplacian with an application to the fundamental solution

Huyuan Chen<sup>1</sup>

Department of Mathematics, Jiangxi Normal University,  
Nanchang, Jiangxi 330022, PR China

Laurent Véron<sup>2</sup>

Institut Denis Poisson, CNRS-UMR 7013  
Université de Tours, 37200 Tours, France

## Abstract

Let  $\mathcal{L}_\Delta$  be the logarithmic Laplacian operator with Fourier symbol  $2\ln|\zeta|$ , we study the expression of the diffusion kernel which is associated to the equation

$$\partial_t u + \mathcal{L}_\Delta u = 0 \quad \text{in } (0, \frac{N}{2}) \times \mathbb{R}^N, \quad u(0, \cdot) = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

We apply our results to give a classification of the solutions of

$$\begin{cases} \partial_t u + \mathcal{L}_\Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(0, \cdot) = f & \text{in } \mathbb{R}^N \end{cases}$$

and obtain an expression of the fundamental solution of the associated stationary equation in  $\mathbb{R}^N$ , and of the fundamental solution in a bounded domain, i.e.

$$\mathcal{L}_\Delta u = k\delta_0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{such that } u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Basic properties</b>	<b>8</b>
2.1	The logarithmic diffusion kernel . . . . .	8
2.2	Maximum principles . . . . .	16
<b>3</b>	<b>The Cauchy problem</b>	<b>17</b>
3.1	Uniqueness of weak Solutions . . . . .	17
3.2	Strong positive solutions . . . . .	21
3.3	Existence of weak solutions . . . . .	26
3.4	The initial trace . . . . .	28
<b>4</b>	<b>Fundamental solutions of <math>\mathcal{L}_\Delta</math></b>	<b>31</b>
4.1	Existence in $\mathbb{R}^N$ . . . . .	32
4.2	Existence in bounded domain . . . . .	40

---

<sup>1</sup>chenhuyuan@yeah.net

<sup>2</sup>veronl@univ-tours.fr

**Keywords:** Cauchy problem; logarithmic Laplacian; Fundamental solution.

**MSC2010:** 35K05; 35A08.

## 1 Introduction

The logarithmic Laplacian  $\mathcal{L}_\Delta$  in  $\mathbb{R}^N$  ( $N \geq 1$ ) is defined by the expression

$$\mathcal{L}_\Delta u(x) = c_N \int_{\mathbb{R}^N} \frac{u(x)1_{B_1(x)}(y) - u(y)}{|x - y|^N} dy + \rho_N u(x),$$

where

$$c_N := \pi^{-N/2} \Gamma\left(\frac{N}{2}\right) = \frac{2}{\omega_N}, \quad \rho_N := 2 \ln 2 + \psi\left(\frac{N}{2}\right) + \psi(1),$$

$\omega_N := |\mathbb{S}^N| = \int_{\mathbb{S}^N} dS$ ,  $\mathbb{S}^N$  is the unit sphere in  $\mathbb{R}^N$ ,  $\Gamma$  is the Gamma function,  $\psi = \frac{\Gamma'}{\Gamma}$  is the associated Digamma function. In this article our first aim is to study the Cauchy problem

$$\begin{cases} \partial_t u + \mathcal{L}_\Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, \cdot) = f & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $f$  is a measurable function in  $\mathbb{R}^N$  and  $T \in (0, \frac{N}{2})$ .

The historical model for diffusion equation is the heat equation

$$u_t(t, x) - \Delta u(t, x) = 0, \quad (x, t) \in (0, T) \times \mathbb{R}^N, \quad (1.2)$$

for which Widder proved in [35] the uniqueness among nonnegative classical solutions and provided the representation

$$u(t, x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u(0, y) dy.$$

By a classical solution  $u$  we intend a function  $u$  belonging to  $\mathcal{C}([0, T] \times \mathbb{R}^N)$ , such that  $u_t, u_{x_i x_j} \in \mathcal{C}((0, T) \times \mathbb{R}^N)$  satisfying (1.2) everywhere (see [21, 36] and the references therein).

In recent years, there has been a renewed and increasing interest in the study of diffusion equations involving linear and nonlinear integro-differential operators, and this growing interest is justified both by important progresses made in understanding nonlocal phenomena from a PDE or a probabilistic point of view, see e.g. [1, 4–9, 26–31] and the references therein, and by their wide range of applications. Among nonlocal differential order operators the simplest examples are the fractional powers of the Laplacian, which exhibit many phenomenological properties. Recall that for  $s \in (0, 1)$ , the fractional Laplacian  $(-\Delta)^s$  can be written as a singular integral operator defined in the principle value sense

$$(-\Delta)^s u(x) = c_{N,s} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where  $c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)} > 0$  is a normalized constant such that for a function  $u \in C_c^\infty(\mathbb{R}^N)$ ,

$$\mathcal{F}((-\Delta)^s u)(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-ix \cdot \xi} ((-\Delta)^s u)(x) dx = |\xi|^{2s} \widehat{u}(\xi) \quad \text{for all } \xi \in \mathbb{R}^N.$$

Here and in the sequel both  $\mathcal{F}$  and  $\widehat{\cdot}$  denote the Fourier transform. A representation formula for the  $s$ -fractional diffusion equation

$$u_t + (-\Delta)^s u = 0 \quad \text{in } (0, T) \times \mathbb{R}^N, \quad (1.3)$$

was recently proved in [2]. It is based upon the  $s$ -fractional diffusion kernel

$$\mathcal{P}_s(t, x) = t^{-\frac{N}{2s}} \mathbf{P} \left( t^{-\frac{1}{2s}} x \right), \quad (1.4)$$

where

$$\mathbf{P}(x) := \int_{\mathbb{R}^N} e^{ix \cdot \xi - |\xi|^{2s}} d\xi.$$

Note that  $\mathcal{P}_s \in \mathcal{C}^\infty((0, \infty) \times \mathbb{R}^N)$  (see e.g. [3, 17]). Under suitable conditions on  $f$ , a solution of (1.3) with initial data  $u(0, \cdot) = f$  is expressed by the formula

$$u_{s,f}(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_s(t, x - y) f(y) dy.$$

It is well-known that the following two limits hold when  $s$  tends 0 or 1:

$$\lim_{s \rightarrow 1^-} (-\Delta)^s u(x) = -\Delta u(x) \quad \text{and} \quad \lim_{s \rightarrow 0^+} (-\Delta)^s u(x) = u(x) \quad \text{for } u \in \mathcal{C}_c^2(\mathbb{R}^N),$$

see e.g. [15]. Furthermore the following surprising expansion at  $s = 0$  is proved in [11]

$$(-\Delta)^s u(x) = u(x) + s \mathcal{L}_\Delta u(x) + o(s) \quad \text{as } s \rightarrow 0^+ \quad \text{for all } u \in \mathcal{C}_c^2(\mathbb{R}^N) \text{ and } x \in \mathbb{R}^N$$

where the formal operator

$$\mathcal{L}_\Delta := \left. \frac{d}{ds} \right|_{s=0} (-\Delta)^s$$

is the *logarithmic Laplacian*; more precisely (see e.g. [11]),

- (i) for  $1 < p \leq \infty$ , there holds  $\mathcal{L}_\Delta u \in L^p(\mathbb{R}^N)$  and  $\frac{(-\Delta)^s u - u}{s} \rightarrow \mathcal{L}_\Delta u$  in  $L^p(\mathbb{R}^N)$  as  $s \rightarrow 0^+$ ;
- (ii)  $\mathcal{F}(\mathcal{L}_\Delta u)(\xi) = (2 \ln |\xi|) \widehat{u}(\xi)$  for a.e.  $\xi \in \mathbb{R}^N$ .

Recently, the related topics on the logarithmic laplacian has been investigated: the eigenvalues estimates [10, 23], log-Sobolev inequality [19], semilinear problems [13, 22], The domain of definition of  $\mathcal{L}_\Delta$  is the space  $Dom_{\mathcal{L}_\Delta}(\mathbb{R}^N) := \mathcal{L}(\mathbb{R}^N) \cap C_D(\mathbb{R}^N)$ , where

$$\mathcal{L}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^N} dx < +\infty \right\}.$$

and  $C_D(O)$  is the space of uniformly Dini continuous functions  $u$  in  $O$  such that for any  $x \in O$

$$\int_0^1 \frac{1}{r} \sup_{y \in O, |y-x| \leq r} |u(x) - u(y)| dr < +\infty.$$

If  $u \in \mathcal{L}(\mathbb{R}^N)$  then  $\mathcal{L}_\Delta u$  can be defined as a distribution with the duality product  $\langle \mathcal{L}_\Delta u, \varphi \rangle$  for every  $\varphi \in \mathcal{S}(\mathbb{R}^N)$ .

Using the Fourier transform, the formal diffusion kernel of the logarithmic Laplacian has the following expression (see e.g. [11, p. 2])

$$\mathcal{P}_{\ln}(t, x) := \int_{\mathbb{R}^N} |\xi|^{-2t} e^{ix \cdot \xi} d\xi. \quad (1.5)$$

From now this formula defines the logarithmic diffusion kernel.

**Proposition 1.1.** *Let the logarithmic diffusion kernel  $\mathcal{P}_{\ln}$  be defined by (1.5), then*

$$\mathcal{P}_{\ln}(t, x) = \mathcal{P}_0(t) |x|^{2t-N} \quad \text{for } (t, x) \in \left(0, \frac{N}{2}\right) \times (\mathbb{R}^N \setminus \{0\}), \quad (1.6)$$

where

$$\mathcal{P}_0(t) = \pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2})}{\Gamma(t)}. \quad (1.7)$$

Furthermore, we have that

(i)  $\mathcal{P}_{\ln}(t, \cdot) \notin L^1(\mathbb{R}^N)$ , but  $\mathcal{P}_{\ln}(t, \cdot) \in L^{\frac{N}{N-2t}, \infty}(\mathbb{R}^N)$  for every  $t \in (0, \frac{N}{2})$ , where  $L^{\frac{N}{N-2t}, \infty}(\mathbb{R}^N)$  is the Marcinkiewicz space with exponent  $\frac{N}{N-2t}$ ;

(ii)  $\mathcal{P}_{\ln}(t, \cdot)$  blows up uniformly in any compact set of  $\mathbb{R}^N$  as  $t \rightarrow (\frac{N}{2})^-$ . More precisely there holds for  $x \neq 0$

$$\lim_{t \rightarrow (\frac{N}{2})^-} (N - 2t) \mathcal{P}_{\ln}(t, x) = 2^{2-N} \omega_N.$$

(iii) For any  $x \neq 0$ ,

$$\lim_{t \rightarrow 0^+} \mathcal{P}_{\ln}(t, x) = 0,$$

and  $\lim_{t \rightarrow 0^+} \mathcal{P}_{\ln}(t, \cdot) = \delta_0$  in the sense of distributions, i.e.

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x) \phi(x) dx = \phi(0) \quad \text{for } \phi \in \mathcal{C}_c(\mathbb{R}^N).$$

In the next table we emphasise the striking differences between  $\mathcal{P}_{\ln}$  and  $\mathcal{P}_s$  for  $s \in (0, 1)$ ,

	Lifespan	Time asymptotics	Behaviour at $x = 0$	Decay as $ x  \rightarrow \infty$
$\mathcal{P}_s$	$(0, +\infty)$	decay rate $t^{-\frac{N}{2s}}$ at infinity	smooth	$ x ^{-2s-N}$
$\mathcal{P}_{\ln}$	$(0, \frac{N}{2})$	blow-up rate $\frac{1}{N-2t}$	singular $ x ^{2t-N}$	$ x ^{2t-N}$

Furthermore,  $\mathcal{P}_{\ln}(t, \cdot)$  is the fundamental solution of the fractional Laplacian  $(-\Delta)^t$  for  $t \in (0, 1)$ . In the sequel we call the expression below the *logarithmic diffusion equation*

$$\partial_t u + \mathcal{L}_{\Delta} u = 0, \quad (1.8)$$

without presuming in what sense it holds.

**Definition 1.2.** *We say that  $u$  is a strong solution of the logarithmic diffusion equation (1.8) with initial data  $f$ , if the following conditions hold:*

(i)  $\partial_t u \in \mathcal{C}((0, T) \times \mathbb{R}^N)$ ;

(ii)  $u \in \mathcal{C}([0, T) \times \mathbb{R}^N)$ ;

(iii) the equation is satisfied pointwise, i.e. for every  $(t, x) \in (0, T) \times \mathbb{R}^N$

$$\partial_t u(t, x) + \mathcal{L}_{\Delta} u(t, x) = 0, \quad (1.9)$$

and  $u(0, x) = f(x)$  for every  $x \in \mathbb{R}^N$ .

Note that in this definition we assume that  $\mathcal{L}_\Delta u(t, x)$  exists for all  $(t, x) \in (0, T) \times \mathbb{R}^N$ . This property needs a minimal regularity for  $x \mapsto u(t, x)$  such as the Dini continuity  $C_D$ . As a consequence of the equation, in this framework the function  $(t, x) \mapsto \mathcal{L}_\Delta u(t, x)$  is continuous in  $(0, T) \times \mathbb{R}^N$ .

We prove the following analogue of Widder's representation for the logarithmic Cauchy problem.

**Theorem 1.3.** *Let  $0 < T \leq \frac{N}{2}$  and  $f$  be a nonnegative function in  $\mathcal{C}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{2T-N} dx)$ . Then problem (1.1) admits a unique strong solution  $u = u_f$ , which is positive and has the form*

$$u_f(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x-y) f(y) dy = \mathcal{P}_0(t) I_{2t} * f(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}^N, \quad (1.10)$$

where  $I_{2t}(x) = |x|^{2t-N}$  is the Riesz kernel of order  $2t$ . Moreover

$$\lim_{t \rightarrow (\frac{N}{2})^-} (N-2t)u_f(t, x) = 2\|f\|_{L^1} \quad \text{for any } x \in \mathbb{R}^N. \quad (1.11)$$

Furthermore, the following implications hold:

(i) *If  $f$  satisfies for some  $M > 0$  and  $\tau < -N$*

$$0 \leq f(x) \leq M(1+|x|)^\tau \quad \text{for } x \in \mathbb{R}^N,$$

*then there exists  $c = c(N, f) > 1$  such that for any  $(t, x) \in (0, T) \times \mathbb{R}^N$*

$$\frac{\mathcal{P}_0(t)}{ct} (1+|x|)^{2t-N} \leq u_f(t, x) \leq c \frac{\mathcal{P}_0(t)}{t} (1+|x|)^{2t-N}, \quad (1.12)$$

*where  $\mathcal{P}_0$  is defined in (1.7).*

(ii) *If  $f$  satisfies for some  $M > 1$  and  $\tau \in (-N, -2T]$*

$$\frac{1}{M} (1+|x|)^\tau \leq f(x) \leq M(1+|x|)^\tau \quad \text{for } x \in \mathbb{R}^N,$$

*then there exists  $c = c(N, M, \tau) > 1$  such that for any  $(t, x) \in (0, T) \times \mathbb{R}^N$*

$$\frac{1}{c} \frac{\mathcal{P}_0(t)}{2t} (1+|x|)^{2t+\tau} \leq u_f(t, x) \leq c \frac{\mathcal{P}_0(t)}{2t} (1+|x|)^{2t+\tau}. \quad (1.13)$$

(iii) *If assume that for some  $M > 1$ ,*

$$\frac{1}{M} (1+|x|)^{-N} \leq f(x) \leq M(1+|x|)^{-N} \quad \text{for } x \in \mathbb{R}^N,$$

*then there exists  $c = c(N, M) > 1$  such that for any  $(t, x) \in (0, T) \times \mathbb{R}^N$*

$$\frac{1}{c} \frac{\mathcal{P}_0(t)}{2t} (1+|x|)^{2t-N} \ln(e+|x|) \leq u_f(t, x) \leq c \frac{\mathcal{P}_0(t)}{2t} (1+|x|)^{2t-N} \ln(e+|x|). \quad (1.14)$$

The counter part of this result is the following characterisation of the initial trace of any nonnegative solution of (1.9).

**Theorem 1.4.** *Let  $0 < T \leq \frac{N}{2}$  and  $f \in L^1(\mathbb{R}^N)$ . Then problem (1.1) admits a unique strong solution  $u = u_f$  expressed by*

$$u_f(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x-y) f(y) dy = \mathcal{P}_0(t) I_{2t} * f(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}^N, \quad (1.15)$$

where  $I_{2t}(x) = |x|^{2t-N}$  is the Riesz kernel of order  $2t$ . Moreover

$$\lim_{t \rightarrow (\frac{N}{2})^-} (N-2t)u_f(t, x) = \frac{2}{(4\pi)^{\frac{N}{2}} \Gamma(\frac{N}{2})} \int_{\mathbb{R}^N} f(y) dy \quad \text{for any } x \in \mathbb{R}^N. \quad (1.16)$$

Note that the previous inequalities show that the function  $u_f(t, \cdot)$  does not belong to  $L^1(\mathbb{R}^N)$ . Hence we propose a definition of weak solutions to (1.8) valid for more general initial data  $f$ .

**Definition 1.5.** We say that the function  $u$  is a weak solution of the logarithmic diffusion problem if the following conditions hold:

- (i)  $u \in L^1\left((0, T'], L^1\left(\mathbb{R}^N, (1 + |x|)^{2(T-T')-N} dx\right)\right)$  for every  $T' < T$ ;
- (ii)  $u \in \mathcal{C}\left((0, T), L^1_{\text{loc}}(\mathbb{R}^N)\right)$ ;
- (iii) For  $\varphi \in \mathbb{X}_0$  and any  $T' \in (0, T)$  there holds

$$\int_0^{T'} \int_{\mathbb{R}^N} [-\partial_t \varphi + \mathcal{L}_\Delta \varphi] u \, dt \, dx = \int_{\mathbb{R}^N} \varphi(0, \cdot) f \, dx - \int_{\mathbb{R}^N} u(T', x) \varphi(T', x) \, dx, \quad (1.17)$$

where  $\mathbb{X}_0 = \{\zeta \in \mathcal{C}^1([0, T] \times \mathbb{R}^N) : \zeta(t, \cdot) \in \mathcal{C}_c^1(\mathbb{R}^N) \text{ for all } t \in [0, T]\}$ .

When  $f = \mu$  is a Radon measure, identity (1.17) is replaced by

$$\int_0^{T'} \int_{\mathbb{R}^N} [-\partial_t \varphi + \mathcal{L}_\Delta \varphi] u \, dt \, dx = \int_{\mathbb{R}^N} \varphi(0, \cdot) d\mu - \int_{\mathbb{R}^N} u(T', x) \varphi(T', x) \, dx. \quad (1.18)$$

Our first existence and uniqueness result which shows the role of the Widder's type representation is as follows.

**Theorem 1.6.** Assume  $0 < T \leq \frac{N}{2}$  and  $f \in L^1(\mathbb{R}^N, (1 + |x|)^{2T-N} dx)$ , then problem (1.1) admits a unique weak solution  $u_f$  expressed by

$$u_f(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x - y) f(y) \, dy \quad \text{for all } t \in (0, T), \text{ a.e. } x \in \mathbb{R}^N. \quad (1.19)$$

The result is also valid if  $f$  is replaced by a Radon measure  $\mu$  in  $\mathbb{R}^N$  satisfying

$$\int_{\mathbb{R}^N} (1 + |x|)^{2T-N} d|\mu|(x) < \infty. \quad (1.20)$$

Of particular importance is the case where  $\mu$  is a Dirac mass.

**Corollary 1.7.** Let  $0 < T \leq \frac{N}{2}$ , then  $\mathcal{P}_{\text{In}}$  is the unique weak solution of

$$\begin{cases} \partial_t u + \mathcal{L}_\Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, \cdot) = \delta_0 & \text{in } \mathbb{R}^N. \end{cases} \quad (1.21)$$

Notice that,

- (a) from the uniqueness, the classical solution obtained in Theorem 1.3 is a weak solution of (1.1);
- (b) for a nonnegative function  $f \in L^1(\mathbb{R}^N)$ , the weak solution  $u_{s,f}$  of the fractional diffusion equation of (1.3) exists for  $(0, +\infty) \times \mathbb{R}^N$ , while the solution of the nonlocal diffusion equation (1.1) is exists only in  $(0, \frac{N}{2}) \times \mathbb{R}^N$ , and it blows up locally in  $\mathbb{R}^N$  as  $t \rightarrow (\frac{N}{2})^-$  in accordance with the expression (1.16).

The final aim of this study is to use the diffusion kernel  $\mathcal{P}_{\text{In}}$  to give an expression of the *fundamental solution* of the logarithmic Laplacian, that is the solution of

$$\mathcal{L}_\Delta u = k\delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (1.22)$$

**Theorem 1.8.** *Let  $N \geq 3$ , then the operator  $\mathcal{L}_\Delta$  has a radially symmetric fundamental function  $\Phi_{\ln}$  in  $\mathbb{R}^N$  and there exists  $c = c(N) > 0$  such that*

$$\left| \Phi_{\ln}(x) - c_0 \frac{|x|^{-N}}{\ln^2 |x|} \right| \leq c \frac{|x|^{-N}}{\ln^3 |x|} \quad \text{for all } x \in B_{\frac{1}{e}}(0) \setminus \{0\}, \quad (1.23)$$

where

$$c_0 = \frac{1}{4} \pi^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right).$$

Furthermore there exist a positive constants  $c_1$  such that

$$|\Phi_{\ln}(x)| \leq \frac{c_1 |x|^{\frac{3-N}{2}}}{\ln |x|} \quad \text{for } |x| \geq 2. \quad (1.24)$$

Our idea is to apply the parabolic operator  $\partial_t + \mathcal{L}_\Delta$  to connect the fundamental solutions of the logarithmic Laplacian with the one of the Helmholtz operator  $-\Delta - I$ . This requires the lifespan of  $\mathcal{P}_{\ln}$  to contain the interval  $(0, 1]$ . When  $N \geq 3$ , it is well defined since  $\mathcal{P}_0(t)$  for  $t = 1 < \frac{N}{2}$ , but it fails when  $N = 2$ , since in that case  $\mathcal{P}_0(1) = +\infty$ . However we conjecture that the fundamental solution  $\Phi_{\ln}$  exists for  $N = 1, 2$  and it decays like  $|x|^{\frac{1-N}{2}}$  as  $|x| \rightarrow +\infty$ .

In a bounded Lipschitz domain  $\Omega$  containing 0, we consider the problem below, the solution of which is the fundamental solution in  $\Omega$ ,

$$\begin{cases} \mathcal{L}_\Delta u = \delta_0 & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.25)$$

We denote by  $\sigma(\mathcal{L}_\Delta, \Omega)$  the spectrum of  $\mathcal{L}_\Delta$  in  $\Omega$  with zero condition in  $\Omega^c$ , that is the set of  $\lambda \in \mathbb{R}$  such that there exists a nontrivial solution to

$$\begin{cases} \mathcal{L}_\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.26)$$

It is proved in [11] that  $\sigma(\mathcal{L}_\Delta, \Omega) = \{\lambda_k^L(\Omega)\}_{k \geq 1}$  where  $k \mapsto \lambda_k^L(\Omega)$  is increasing and

$$\lim_{k \rightarrow +\infty} \lambda_k^L(\Omega) \rightarrow +\infty.$$

Furthermore  $\ker(\mathcal{L}_\Delta + \lambda_1^L(\Omega))$  is 1-dimensional and generated by a positive function  $\phi_1$ . First we characterise the fundamental solution of  $\mathcal{L}_\Delta$  in  $\Omega$ .

**Theorem 1.9.** *Let  $N \geq 1$ ,  $\Omega$  be a bounded Lipschitz domain containing  $B_1$  and such that  $0 \notin \sigma(\mathcal{L}_\Delta, \Omega)$ . Then problem (1.25) possesses a fundamental solution  $\Phi_{\ln}^\Omega$  and there exists  $c > 0$  such that*

$$\left| \Phi_{\ln}^\Omega(x) - c_0 \frac{|x|^{-N}}{\ln^2 |x|} \right| \leq c \frac{|x|^{-N}}{\ln^3 |x|} \quad \text{for } x \in B_{\frac{1}{e}}(0) \setminus \{0\}. \quad (1.27)$$

Furthermore, there holds near  $\partial\Omega$

$$|\Phi_{\ln}^\Omega(x)| = O\left(\frac{1}{|\ln \rho(x)|^\tau}\right) \quad \text{for all } \tau \in \left(0, \frac{1}{2}\right), \quad (1.28)$$

where  $\rho(x) = \text{dist}(x, \partial\Omega)$ .



We also give some estimates of  $\Phi_{\ln}$  in some Orlicz classes and we prove at least if  $N \geq 4$  and under mild assumptions on  $f$  the function  $u = \Phi_{\ln} * f$  is Dini continuous and is therefore a strong solution of

$$-\mathcal{L}_{\Delta}u = f \quad \text{in } \mathbb{R}^N. \quad (1.29)$$

**Theorem 1.10.** *Let  $N \geq 4$ ,  $\theta > 1$  and  $f \in L^1(\mathbb{R}^N) \cap L^{\infty, \theta}(\mathbb{R}^N)$ , where*

$$L^{\infty, \theta}(\mathbb{R}^N) := \left\{ h \in L^{\infty}_{loc}(\mathbb{R}^N) : \operatorname{ess\,sup}_{y \in \mathbb{R}^N} (1 + |y|)^{\theta} |f(y)| < \infty \right\}. \quad (1.30)$$

*Then for any  $R > 0$  there exists  $c = c(N, R, \theta)$  such that*

$$|\Phi_{\ln} * f(x) - \Phi_{\ln} * f(x')| \leq \frac{c}{1 + \ln^2(|x - x'|)} (\|f\|_{L^1} + \|f\|_{L^{\infty, \theta}}) \quad \text{for all } x, x' \in B_R, \quad (1.31)$$

where

$$\|f\|_{L^{\infty, \theta}} = \operatorname{ess\,sup}_{y \in \mathbb{R}^N} (1 + |y|)^{\theta} |f(y)|. \quad (1.32)$$

The paper is organized as follows. Section 2 is devoted to the study of the diffusion kernel  $\mathcal{P}_{\ln}$  of  $\partial_t + \mathcal{L}_{\Delta}$  and to the existence and the representation of classical solutions of (1.1). Section 3 is devoted to the Cauchy problem associated to the logarithmic Laplacian, first in the weak sense and then in the classical sense, based upon the proof of the uniqueness of weak solutions. In section 4, we construct the fundamental solution of the logarithmic Laplacian using the diffusion kernel, representation formula for solutions of the Cauchy problem and precise estimates of the kernel of Helmholtz operator in  $\mathbb{R}^N$ . In Section 5 we use the estimates of  $\Phi_{\ln}$  to prove some imbedding properties in Orlicz spaces, to obtain point-wise estimate of  $\nabla \Phi_{\ln}$  in  $\mathbb{R}^N \setminus \{0\}$  and to prove that the solution of the Dirichlet problem in  $\mathbb{R}^N$  is Dini continuous.

## 2 Basic properties

In the course of this paper  $c$  will denote a generic positive constant, depending on different parameters (which can be specified in some cases), but not on variable, the value of which can vary from one occurrence to another. In some cases, in order to avoid misunderstanding, we introduce constants  $c_j$  or  $c_j$ ,  $j = 1, 2, \dots$

### 2.1 The logarithmic diffusion kernel

*Proof of Proposition 1.1.* We recall (see e.g. [32, Chapter V, p. 117]) that

$$\mathcal{F}(|\cdot|^{\tau}) = \sigma(\tau) |\cdot|^{-N-\tau} \quad \text{in } \mathcal{S}'(\mathbb{R}^N)$$

with

$$\sigma(\tau) := 2^{\tau+N} \pi^{\frac{N}{2}} \frac{\Gamma(\frac{\tau+N}{2})}{\Gamma(-\frac{\tau}{2})}.$$

Consequently, if  $\tau \neq -N$ , we have that

$$\mathcal{F}^{-1}(|\cdot|^{-N-\tau}) = \frac{1}{\sigma(\tau)} |\cdot|^{\tau} \quad \text{in } \mathcal{S}'(\mathbb{R}^N)$$

and

$$\mathcal{F}^{-1}(1) = \delta_0 \quad \text{in } \mathcal{S}'(\mathbb{R}^N).$$

From the properties of the Fourier transform there holds

$$\partial_t \widehat{\mathcal{P}}_{\ln}(t, \xi) + 2 \ln |\xi| \widehat{\mathcal{P}}_{\ln}(t, \xi) = 0,$$

which implies

$$\widehat{\mathcal{P}}_{\ln}(t, \xi) = e^{-2t \ln |\xi|} = |\xi|^{-2t} \quad \text{for } (t, \xi) \in (0, \frac{N}{2}) \times \mathbb{R}^N \setminus \{0\}.$$

By the inverse Fourier transform,

$$\mathcal{P}_{\ln}(t, x) = \mathcal{P}_0(t) |x|^{2t-N} \quad \text{for } (t, x) \in (0, \frac{N}{2}) \times \mathbb{R}^N,$$

where, we recall it,

$$\mathcal{P}_0(t) = \pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2})}{\Gamma(t)}.$$

Clearly  $\mathcal{P}_0$  is a positive and smooth function in  $(0, \frac{N}{2})$  and it vanishes at  $t = 0$ . The expression of  $\mathcal{P}_{\ln}$  shows that  $\mathcal{P}_{\ln}(t, \cdot)$  does not belong to  $L^1(\mathbb{R}^N)$  for any  $t \in (0, \frac{N}{2})$ , this is part (i). Since  $x\Gamma(x) \rightarrow 1$  as  $x \downarrow 1$  we obtain

$$\mathcal{P}_0(t) \cdot \frac{N-2t}{2} \rightarrow (4\pi)^{-\frac{N}{2}} \Gamma^{-1}(\frac{N}{2}) \quad \text{as } t \rightarrow (\frac{N}{2})^-.$$

This implies that for any  $x \neq 0$

$$\mathcal{P}_{\ln}(t, x) \cdot \frac{N-2t}{2} \rightarrow (4\pi)^{-\frac{N}{2}} \Gamma^{-1}(\frac{N}{2}) \quad \text{as } t \rightarrow (\frac{N}{2})^-.$$

Now (ii) follows by the fact that

$$\lim_{t \rightarrow (\frac{N}{2})^-} \Gamma(\frac{N-2t}{2})(\frac{N}{2} - t) = 1 \quad \text{and} \quad \lim_{t \rightarrow (\frac{N}{2})^-} |x|^{2t-N} = 1 \quad \text{for } x \neq 0.$$

Because  $\lim_{\tau \rightarrow 0} \Gamma(\tau) = \infty$ , we have for any  $x \neq 0$ ,

$$\lim_{t \rightarrow 0^+} \mathcal{P}_{\ln}(t, x) = 0.$$

For the proof of part (iii), if  $\phi \in \mathcal{C}_c(\mathbb{R}^N)$  has support in  $B_{\sigma_0}$ , for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|\phi(x) - \phi(0)| \leq \epsilon \quad \text{if } |x| \leq \delta.$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x) \phi(x) dx &= \pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2})}{\Gamma(1-t)} t \left( \int_{B_{\sigma_0}} |x|^{2t-N} dx \right) \phi(0) \\ &\quad + \pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2})}{\Gamma(1-t)} t \int_{B_{\sigma_0}} |x|^{2t-N} (\phi(x) - \phi(0)) dx. \end{aligned}$$

We have that

$$\pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2})}{\Gamma(1-t)} \rightarrow \pi^{-\frac{N}{2}} \Gamma(\frac{N}{2}) \quad \text{as } t \rightarrow 0^+,$$

$$t \int_{B_{\sigma_0}} |x|^{2t-N} dx = \frac{\sigma_0^{2t} \omega_N}{2} \rightarrow \frac{\omega_N}{2} \quad \text{as } t \rightarrow 0^+,$$

and

$$\begin{aligned} t \left| \int_{B_{\sigma_0}} |x|^{2t-N} (\phi(x) - \phi(0)) dx \right| &\leq t \int_{B_\delta} |x|^{2t-N} |\phi(x) - \phi(0)| dx \\ &\quad + t \int_{B_{\sigma_0} \setminus B_\delta} |x|^{2t-N} |\phi(x) - \phi(0)| dx \\ &\leq I + II. \end{aligned}$$

Then

$$I \leq \epsilon \omega_N \delta^{2t} \rightarrow \epsilon \omega_N \quad \text{as } t \rightarrow 0,$$

and

$$II \leq \|\phi\|_{L^\infty} \omega_N (\sigma_0^{2t} - \delta^{2t}) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Since  $\epsilon$  is arbitrary and

$$\pi^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right) \omega_N = 2,$$

we obtain

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x) \phi(x) dx = \phi(0).$$

□

**Lemma 2.1.** *Assume  $\tau < 0$  and  $(t, x) \in (0, -\frac{\tau}{2}) \times \mathbb{R}^N$ , then*

(i) *If  $-N < \tau < 0$  there exists  $c = c(N, \tau) > 1$  such that*

$$\frac{1}{ct} \min \{ |x|^{\tau+2t}, t + |x|^{2t} \} \leq \int_{\mathbb{R}^N} |x-y|^{2t-N} (1+|y|)^\tau dy \leq \frac{c}{t} |x|^{2t+\tau}.$$

(ii) *If  $\tau = -N$  there exists  $c = c(N, \tau) > 1$  such that*

$$\frac{1}{ct} \min \{ |x|^{2t-N} \ln(e+|x|), t + |x|^{2t} \} \leq \int_{\mathbb{R}^N} |x-y|^{2t-N} (1+|y|)^{-N} dy \leq \frac{c}{t} |x|^{2t-N} \ln(e+|x|).$$

(iii) *If  $\tau < -N$  there exists  $c = c(N, \tau) > 1$  such that*

$$\frac{1}{ct} \min \{ |x|^{2t-N}, t + |x|^{2t} \} \leq \int_{\mathbb{R}^N} |x-y|^{2t-N} (1+|y|)^\tau dy \leq \frac{c}{t} \max \{ |x|^{2t+\tau}, |x|^{2t-N} \}.$$

*Proof.* Assume  $x \neq 0$ . The integral over  $\mathbb{R}^N$  is the sum of the integrals over  $B_{\frac{|x|}{2}}(0)$ , over  $B_{\frac{|x|}{2}}(x)$  and over  $\mathbb{R}^N \setminus (B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x))$ . Clearly

$$\int_{B_{\frac{|x|}{2}}(x)} |x-y|^{2t-N} (1+|y|)^\tau dy \leq \frac{(2+|x|)^\tau}{2^\tau} \int_{B_{\frac{|x|}{2}}(x)} |x-y|^{2t-N} dy = \frac{\omega_N}{2^{2t+\tau+1}} \frac{|x|^{2t}(2+|x|)^\tau}{t} \quad (2.1)$$

and

$$\int_{B_{\frac{|x|}{2}}(x)} |x-y|^{2t-N} (1+|y|)^\tau dy \geq \frac{(2+3|x|)^\tau}{2^\tau} \int_{B_{\frac{|x|}{2}}(x)} |x-y|^{2t-N} dy \geq \frac{\omega_N}{2^{2t+\tau+1}} \frac{|x|^{2t}(2+3|x|)^\tau}{t}. \quad (2.2)$$

Concerning the integral over  $B_{\frac{|x|}{2}}(0)$ , one has

$$\begin{aligned} \frac{|3x|^{2t-N}\omega_N}{2^{2t-N}} \int_0^{\frac{|x|}{2}} (1+r)^\tau r^{N-1} dr &\leq \int_{B_{\frac{|x|}{2}}(0)} |x-y|^{2t-N} (1+|y|)^\tau dy \\ &\leq \frac{|x|^{2t-N}\omega_N}{2^{2t-N}} \int_0^{\frac{|x|}{2}} (1+r)^\tau r^{N-1} dr. \end{aligned}$$

If  $|x| \rightarrow \infty$ , there holds

$$\int_0^{\frac{|x|}{2}} (1+r)^\tau r^{N-1} dr = \begin{cases} \frac{|x|^{N+\tau}}{2^{N+\tau}(N+\tau)}(1+o(1)) & \text{if } -N < \tau < 0, \\ \ln|x|(1+o(1)) & \text{if } \tau = -N, \\ c^*(1+o(1)) & \text{if } \tau < -N, \end{cases}$$

for some explicit value  $c^*$ , and if  $x \rightarrow 0$ ,

$$\int_0^{\frac{|x|}{2}} (1+r)^\tau r^{N-1} dr = \frac{|x|^N}{2^N N} (1+o(1)).$$

This implies

$$\int_{B_{\frac{|x|}{2}}(0)} |x-y|^{2t-N} (1+|y|)^\tau dy \leq c \begin{cases} \inf\{|x|^{2t+\tau}, |x|^{2t}\} & \text{if } -N < \tau < 0, \\ \inf\{|x|^{2t-N} \ln(|x|+e), |x|^{2t}\} & \text{if } \tau = -N, \\ \inf\{|x|^{2t-N}, |x|^{2t}\} & \text{if } \tau < -N, \end{cases} \quad (2.3)$$

and

$$\int_{B_{\frac{|x|}{2}}(0)} |x-y|^{2t-N} (1+|y|)^\tau dy \geq \frac{1}{c} \begin{cases} \inf\{|x|^{2t+\tau}, |x|^{2t}\} & \text{if } -N < \tau < 0, \\ \inf\{|x|^{2t-N} \ln(|x|+e), |x|^{2t}\} & \text{if } -N = \tau, \\ \inf\{|x|^{2t-N}, |x|^{2t}\} & \text{if } -N > \tau, \end{cases} \quad (2.4)$$

where the constant  $c$  depends only on  $N$ .

Finally, we consider the integral over  $\mathbb{R}^N \setminus (B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x))$ ,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus (B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x))} |x-y|^{2t-N} (1+|y|)^\tau dy \\ = |x|^{2t+\tau} \int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\mathbf{e}_x))} |\mathbf{e}_x - z|^{2t-N} (|x|^{-1} + |z|)^\tau dz, \end{aligned}$$

where  $\mathbf{e}_x = \frac{x}{|x|}$ . This integral is invariant by rotation, thus we can assume that  $\mathbf{e}_N = (0, 0, \dots, 1) \in \mathbb{R}^N$ . The function

$$|x| \mapsto H(|x|) := \int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\mathbf{e}_N))} |\mathbf{e}_N - z|^{2t-N} (|x|^{-1} + |z|)^\tau dz$$

is increasing, therefore

$$0 = \lim_{|y| \rightarrow 0} H(|y|) \leq H(|x|) \leq \int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\mathbf{e}_N))} |\mathbf{e}_N - z|^{2t-N} |z|^\tau dz := H(\infty) < \infty.$$

This yields

$$\int_{\mathbb{R}^N \setminus (B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x))} |x - y|^{2t-N} (1 + |y|)^\tau dy \leq |x|^{2t+\tau} H(\infty). \quad (2.5)$$

For the lower bound,  $B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\mathbf{e}_N) \subset B_2(0)$ , therefore

$$\begin{aligned} \int_{\mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\mathbf{e}_N))} |\mathbf{e}_N - z|^{2t-N} (|x|^{-1} + |z|)^\tau dz &\geq \int_{\mathbb{R}^N \setminus B_2(0)} |\mathbf{e}_N - z|^{2t-N} (|x|^{-1} + |z|)^\tau dz \\ &\geq 2^{2t-N} \int_{\mathbb{R}^N \setminus B_2(0)} |z|^{2t-N} (|x|^{-1} + |z|)^\tau dz \\ &\geq 2^{2t-N} \omega_N \int_2^\infty r^{2t-1} (|x|^{-1} + r)^\tau dr. \end{aligned}$$

But the following relation holds

$$\int_2^\infty r^{2t-1} (|x|^{-1} + r)^\tau dr = |x|^{-2t-\tau} \int_{2|x|}^\infty s^{2t-1} (s + 1)^\tau ds \geq c \min \{1, |x|^{-\tau-2t}\}.$$

This implies

$$\int_{\mathbb{R}^N \setminus (B_{\frac{|x|}{2}}(0) \cup B_{\frac{|x|}{2}}(x))} |x - y|^{2t-N} (1 + |y|)^\tau dy \geq c \min \{|x|^{2t+\tau}, 1\}. \quad (2.6)$$

Since  $0 < t < -\frac{\tau}{2}$  the proof follows by combining (2.1)–(2.6).  $\square$

**Remark 2.2.** When  $|x| \geq 1$ , the estimates in Lemma 2.1 with possibly a new constant  $c = c(N, \tau) > 1$  can be expressed under the following form:

$$\begin{aligned} (i) \quad \frac{1}{ct} |x|^{2t+\tau} &\leq \int_{\mathbb{R}^N} |x - y|^{2t-N} (1 + |y|)^\tau dy \leq \frac{c}{t} |x|^{2t+\tau} && \text{if } -N < \tau < 0; \\ (ii) \quad \frac{1}{ct} |x|^{2t-N} \ln(e + |x|) &\leq \int_{\mathbb{R}^N} |x - y|^{2t-N} (1 + |y|)^{-N} dy \leq \frac{c}{t} |x|^{2t-N} \ln(e + |x|) && \text{if } \tau = -N; \\ (iii) \quad \frac{1}{ct} |x|^{2t-N} &\leq \int_{\mathbb{R}^N} |x - y|^{2t-N} (1 + |y|)^\tau dy \leq \frac{c}{t} |x|^{2t-N} && \text{if } \tau < -N. \end{aligned} \quad (2.7)$$

**Proposition 2.3.** Let  $T \in (0, \frac{N}{2})$  and  $f \in C^1(\mathbb{R}^N)$  such that  $f, |\nabla f| \in L^1(\mathbb{R}^N)$ . Then the function  $u_f$  defined by

$$u_f(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x - y) f(y) dy, \quad (2.8)$$

is a strong solution of (1.1).

(i) Since the mapping:  $f \mapsto \mathcal{P}_{\ln}(t, \cdot) * f$  is increasing and  $\mathcal{P}_0(t) > 0$ , if assume furthermore that  $f \neq 0$  is nonnegative and

$$f(x) \leq M(1 + |x|)^\tau \quad \text{for } x \in \mathbb{R}^N$$

for some  $M > 0$  and  $\tau < -N$ , then there exists  $c > 2$  such that for any  $(t, x) \in (0, T) \times \mathbb{R}^N$ ,

$$\frac{1}{ct} \mathcal{P}_0(t) (1 + |x|)^{2t-N} \leq u_f(t, x) \leq \frac{c}{t} \mathcal{P}_0(t) (1 + |x|)^{2t-N}, \quad (2.9)$$

where  $\mathcal{P}_0$  is defined in (1.7).

(ii) Moreover, if assume that

$$\frac{1}{M}(1 + |x|)^\tau \leq f(x) \leq M(1 + |x|)^\tau \quad \text{for } x \in \mathbb{R}^N,$$

for some  $\tau \in (-N, -2T]$  and some  $M > 1$ , then there exists  $c > 1$  such that for any  $(t, x) \in (0, T) \times \mathbb{R}^N$ ,

$$\frac{1}{ct}\mathcal{P}_0(t)(1 + |x|)^{2t+\tau} \leq u_f(t, x) \leq \frac{c}{t}\mathcal{P}_0(t)(1 + |x|)^{2t+\tau}. \quad (2.10)$$

(iii) At last, if assume also that, for some  $M > 1$ ,

$$\frac{1}{M}(1 + |x|)^{-N} \leq f(x) \leq M(1 + |x|)^{-N} \quad \text{for } x \in \mathbb{R}^N,$$

then there exists  $c > 1$  such that for any  $(t, x) \in (0, T) \times \mathbb{R}^N$ ,

$$\frac{1}{ct}\mathcal{P}_0(t)(1 + |x|)^{2t-N} \ln(e + |x|) \leq u_f(t, x) \leq \frac{c}{t}\mathcal{P}_0(t)(1 + |x|)^{2t-N} \ln(e + |x|). \quad (2.11)$$

*Proof.* First we note that if

$$u_f(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, z) f(x - z) dz = \mathcal{P}_0(t) I_{2t} * f(x), \quad (2.12)$$

then

$$\widehat{u}_f(t, \xi) = \mathcal{P}_0(t) \widehat{I_{2t} * f}(\xi) = \mathcal{P}_0(t) 4^t \pi^{\frac{N}{2}} \frac{\Gamma(t)}{\Gamma(\frac{N}{2} - t)} |\xi|^{-2t} \widehat{f}(\xi) = |\xi|^{-2t} \widehat{f}(\xi),$$

because of the identity (1.7). Therefore

$$\partial_t \widehat{u}_f(t, \xi) = -2 \ln |\xi| \widehat{f}(\xi) \implies \mathcal{F}(\partial_t u_f + \mathcal{L}_\Delta u_f) = 0. \quad (2.13)$$

Thus the equation is verified in  $\mathcal{S}'(\mathbb{R}^N)$ .

(i) Since  $u_f$  is defined by (2.8) and  $\nabla f \in L^1(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ , for all integers  $1 \leq i \leq N$  the expressions

$$\partial_{x_i} u_f(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, z) \partial_{x_i} f(x - z) dz,$$

are well-defined and are continuous functions of  $(x, t)$ . As a consequence,  $u_f(t, \cdot) \in \mathcal{C}^1(\mathbb{R}^N)$ . For the time-derivative, we note that

$$\partial_t \mathcal{P}_{\ln}(t, x) = \mathcal{P}'_0(t) |x|^{2t-N} + 2\mathcal{P}_0(t) |x|^{2t-N} \ln |x|,$$

then

$$\begin{aligned} \partial_t u_f(t, x) &= \int_{\mathbb{R}^N} \partial_t \mathcal{P}_{\ln}(t, z) f(x - z) dz \\ &= \mathcal{P}'_0(t) \int_{\mathbb{R}^N} |z|^{2t-N} f(x - z) dz + 2\mathcal{P}_0(t) \int_{\mathbb{R}^N} |z|^{2t-N} (\ln |z|) f(x - z) dz, \end{aligned}$$

which is continuous at any point  $(t, x) \in (0, T) \times \mathbb{R}^N$ . Moreover, from Proposition 1.1,

$$\lim_{t \rightarrow 0^+} u_f(t, x) = f(x) \quad \text{for any } x \in \mathbb{R}^N,$$

which implies that

$$u_f \in \mathcal{C}\left(\left[0, \frac{2}{N}\right) \times \mathbb{R}^N\right) \cap \mathcal{C}^1\left(\left(0, \frac{2}{N}\right) \times \mathbb{R}^N\right).$$

Since (2.13) holds, it follows that  $u_f$  satisfies (1.8); as a consequence it is a strong solution of (1.1).

Let  $R > 1$  such that  $\text{supp}(f) \subset \bar{B}_R$ , then if  $|x| > 2R$ ,

$$\frac{1}{2}|x| \leq |x - y| \leq 3|x| \quad \text{for } y \in B_R,$$

and if  $f$  is a nonnegative non identically zero function,

$$\frac{|x|^{2t-N}}{2^{N-2t}} \|f\|_{L^1(\mathbb{R}^N)} < \int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) dy = \int_{B_R} |x - y|^{2t-N} f(y) dy \leq \frac{|x|^{2t-N}}{3^{2t-N}} \|f\|_{L^1(\mathbb{R}^N)},$$

and if  $|x| \leq 2R$ , we have that

$$\int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) dy \leq \|f\|_{L^\infty(\mathbb{R}^N)} \int_{B_{3R}} |z|^{2t-N} dz = \frac{cR^{2t}}{t} \|f\|_{L^\infty(\mathbb{R}^N)}$$

for some  $c = c(N) > 0$ .

If we assume that  $0 \leq f(x) \leq M(1 + |x|)^\tau$  and  $\tau < -N$ , then the right-hand side part of (2.9) follows directly from Lemma2.1-(iii). For the left-hand side, we can suppose that  $\|f\mathbf{1}_{B_R}\|_{L^1} > 0$  for some  $R > 0$  and we have for  $|x| \geq 2R$ ,

$$\frac{|x|^{2t-N}}{2^{N-2t}} \|f\mathbf{1}_{B_R}\|_{L^1(\mathbb{R}^N)} \leq \int_{B_R} |x - y|^{2t-N} f(y) dy.$$

This implies (2.9) up to changing the constant  $c$ .

(ii) If we assume  $\frac{1}{M}(1 + |x|)^\tau \leq f(x) \leq M(1 + |x|)^\tau$ , then (2.10) follows from Lemma2.1-(i).

(iii)- If we assume  $\frac{1}{M}(1 + |x|)^{-N} \leq f(x) \leq M(1 + |x|)^{-N}$ , then (2.11) follows from Lemma2.1-(ii).  $\square$

The next results will be used later on

**Lemma 2.4.** *There holds*

$$\lim_{t \rightarrow 0} t\Gamma(t) = 1, \tag{2.14}$$

$$\lim_{t \rightarrow 0} t^2\Gamma'(t) = -1 \tag{2.15}$$

and

$$\lim_{t \rightarrow 0} t^3\Gamma''(t) = 2. \tag{2.16}$$

In particular

$$\mathcal{P}'_0(0) = \pi^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right) \quad \text{and} \quad \mathcal{P}''_0(0) = 2\pi^{-\frac{N}{2}} \left(\Gamma\left(\frac{N}{2}\right) - \Gamma'\left(\frac{N}{2}\right)\right). \tag{2.17}$$

*Proof.* Since we have

$$\Gamma(t) = \int_0^\infty z^{t-1} e^{-z} dz = \left[ \frac{z^t}{t} e^{-z} \right]_{z=0}^\infty + \frac{1}{t} \int_0^\infty z^t e^{-z} dz,$$

we obtain (2.14). For  $\Gamma'$  there holds

$$\begin{aligned}\Gamma'(t) &= \int_0^\infty z^{t-1} \ln z e^{-z} dz = \left[ \left( \frac{z^t}{t} \ln z - \frac{z^t}{t^2} \right) e^{-z} \right]_{z=0}^\infty + \int_0^\infty \left( \frac{z^t}{t} \ln z - \frac{z^t}{t^2} \right) e^{-z} dz \\ &= \frac{1}{t^2} \int_0^\infty (tz^t \ln z - z^t) e^{-z} dz.\end{aligned}$$

Then (2.15) follows. For  $\Gamma''$  we also have

$$\begin{aligned}\Gamma''(t) &= \int_0^\infty z^{t-1} \ln^2 z e^{-z} dz = \left[ \left( \frac{z^t}{t} \ln^2 z - 2 \frac{z^t}{t^2} \ln z + \frac{2z^t}{t^3} \right) e^{-z} \right]_{z=0}^\infty \\ &\quad + \int_0^\infty \left( \frac{z^t}{t} \ln^2 z - 2 \frac{z^t}{t^2} \ln z + \frac{2z^t}{t^3} \right) e^{-z} dz \\ &= \frac{1}{t^3} \int_0^\infty (t^2 z^t \ln^2 z - 2tz^t \ln z + 2z^t) e^{-z} dz.\end{aligned}$$

Hence (2.16) follows by integration by parts. The first formula in (2.17) follows directly from (2.15). The second is a little more involved and left to the reader.  $\square$

Given  $t_0 \in (0, \frac{N}{2})$ , we consider the backward problem

$$\begin{cases} \partial_t u - \mathcal{L}_\Delta u = 0 & \text{in } [0, t_0) \times \mathbb{R}^N, \\ u(t_0, \cdot) = g & \text{in } \mathbb{R}^N. \end{cases} \quad (2.18)$$

The existence of strong solutions of (2.18) is established below.

**Corollary 2.5.** *Let  $t_0 \in (0, \frac{N}{2})$  and for  $g \in C_c^1(\mathbb{R}^N)$ , we denote*

$$v_g(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t_0 - t, x - y) g(y) dy, \quad (2.19)$$

then  $v_g$  is a strong solution of (2.18),

$$v_g \in \mathcal{C}([0, t_0] \times \mathbb{R}^N) \cap \mathcal{C}^1((0, t_0) \times \mathbb{R}^N),$$

and we have, for some  $c > 0$  dependent of  $\|f\|_{L^1(\mathbb{R}^N)}$  and  $\|f\|_{L^\infty(\mathbb{R}^N)}$

$$|v_g(t, x)| \leq c \frac{\mathcal{P}_0(t_0 - t)}{t_0 - t} (1 + |x|)^{2(t_0 - t) - N} \quad \text{for } (t, x) \in (0, t_0) \times \mathbb{R}^N. \quad (2.20)$$

*Proof.* Note that the backward diffusion kernel  $\mathcal{Q}_{\ln}$  is

$$\mathcal{Q}_{\ln}(t, x) = \mathcal{P}_{\ln}(t_0 - t, x) \quad \text{in } [0, t_0) \times \mathbb{R}^N.$$

Then the proof is the same as the one of Proposition 2.3.  $\square$



## 2.2 Maximum principles

We first recall some maximum principles for the logarithmic Laplacian valid in bounded domains [11, corollary 1.9]:

**Lemma 2.6.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain. Then  $\mathcal{L}_\Delta$  satisfies the maximum principle in  $\Omega$  if one of the following conditions is satisfied:*

(i)  $h_\Omega(x) + \rho_N \geq 0$  on  $\Omega$ , where

$$h_\Omega(x) = c_N \left( \int_{B_1(x) \setminus \Omega} \frac{1}{|x-y|^N} dy - \int_{\Omega \setminus B_1(x)} \frac{1}{|x-y|^N} dy \right); \quad (2.21)$$

(ii)  $|\Omega| \leq 2^N \exp\left(\frac{N}{2} \left(\psi\left(\frac{N}{2}\right) - \gamma\right)\right) |B_1(0)|$ .

The role of  $h_\Omega(x) + \rho_N$  is clear since the following expression,

$$\mathcal{L}_\Delta u(x) = c_N \int_\Omega \frac{u(x) - u(y)}{|x-y|^N} dy - c_N \int_{\mathbb{R}^N \setminus \Omega} \frac{u(y)}{|x-y|^N} dy + \left(h_\Omega(x) + \rho_N\right) u(x) \quad (2.22)$$

is an alternative definition of the logarithmic Laplacian.

For the associated diffusion operator, we have the following maximum principle:

**Corollary 2.7.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain. Assume  $v \in \mathcal{C}([0, T] \times \bar{\Omega})$  such that  $\partial_t v \in \mathcal{C}((0, T) \times \bar{\Omega})$  satisfies pointwise*

$$\begin{cases} \partial_t v + \mathcal{L}_\Delta v \geq 0 & \text{in } (0, T) \times \Omega, \\ v \geq 0 & \text{in } \left((0, T) \times (\mathbb{R}^N \setminus \Omega)\right) \cup \left(\{0\} \times \mathbb{R}^N\right). \end{cases} \quad (2.23)$$

If  $h_\Omega + \rho_N \geq 0$  on  $\Omega$ , then

$$v \geq 0 \quad \text{in } [0, T] \times \Omega.$$

*Proof.* We proceed by contradiction, assuming that for any  $T' \in (0, T)$ , we have that

$$v(t_0, x_0) := \min_{[0, T'] \times \bar{\Omega}} v(t, x) < 0. \quad (2.24)$$

Clearly  $(t_0, x_0)$  lies in  $(0, T) \times \Omega$ . Therefore  $v_t(x_0, t_0) \leq 0$  and thus

$$\mathcal{L}_\Delta v(t_0, x_0) \geq 0. \quad (2.25)$$

This yields a contradiction by using the expression (2.22), since

$$\mathcal{L}_\Delta v(t_0, x_0) = c_N \int_\Omega \frac{v(t_0, x_0) - v(t_0, y)}{|x_0 - y|^N} dy - c_N \int_{\mathbb{R}^N \setminus \Omega} \frac{v(t_0, y)}{|x_0 - y|^N} dy + \left(h_\Omega(x_0) + \rho_N\right) v(t_0, x_0) < 0,$$

from (2.24) and  $\left(h_\Omega(x_0) + \rho_N\right) v(t_0, x_0) \geq 0$  from the assumption on  $h_\Omega + \rho_N$ .  $\square$

In the following result we get rid of the restriction  $h_\Omega + \rho_N \geq 0$  provided the boundary value of  $v$  is bounded from below.

**Lemma 2.8.** Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain and  $v \in \mathcal{C}([0, T) \times \bar{\Omega})$  such that  $\partial_t v \in \mathcal{C}([0, T) \times \bar{\Omega})$  which satisfies pointwise the following inequalities

$$\begin{cases} \partial_t v + \mathcal{L}_\Delta v \geq 0 & \text{in } (0, T) \times \Omega, \\ v \geq 0 & \text{in } (0, T) \times (\mathbb{R}^N \setminus \Omega), \\ v(0, \cdot) \geq c & \text{in } \Omega, \end{cases} \quad (2.26)$$

for some  $c > 0$ . Then

$$v > 0 \quad \text{in } [0, T) \times \Omega.$$

*Proof.* We proceed by contradiction. Let  $t_0 > 0$  be the smallest  $t > 0$  such that  $v(t_0, x_0) = 0$  for some  $x_0 \in \Omega$ . Then  $v(t, x) > 0$  for all  $(t, x) \in (0, t_0) \times \Omega$ . Thus  $\partial_t v(t_0, x_0) \leq 0$  and

$$\mathcal{L}_\Delta v(t_0, x_0) = -c_N \int_{\mathbb{R}^N} \frac{v(t_0, y)}{|x_0 - y|^N} dy \geq 0,$$

where the equality holds only if  $u(t_0, \cdot) \equiv 0$  a.e. in  $\mathbb{R}^N$ . Since  $v$  is continuous in  $([0, T) \times \bar{\Omega})$  nonnegative and not identically 0 on the boundary, we are led to a contradiction.  $\square$

### 3 The Cauchy problem

#### 3.1 Uniqueness of weak Solutions

Our uniqueness result for weak solutions is the following.

**Theorem 3.1.** Let  $T \in (0, \frac{N}{2})$  and  $u$  be a weak solution of the equation with zero initial data

$$\begin{cases} \partial_t u + \mathcal{L}_\Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, \cdot) = 0 & \text{in } \mathbb{R}^N. \end{cases} \quad (3.1)$$

Then  $u(t, x) = 0$  for every  $t \in (0, T)$  and for almost everywhere  $x \in \mathbb{R}^N$ .

*Proof. Step 1.* We fix  $\theta \in \mathcal{C}_c^1(\mathbb{R}^N)$  with compact support in  $\bar{B}_{R_0}$  for some  $R_0 > 0$ . We aim to prove that

$$\int_{\mathbb{R}^N} u(t_0, x) \theta(x) dx = 0.$$

For  $t \in [0, t_0)$ , we define

$$\varphi_\theta = \mathcal{P}_{\text{In}}(t_0 - t, \cdot) * \theta.$$

By Corollary 2.5  $\varphi_\theta$  is a strong solution of

$$\begin{cases} \partial_t v - \mathcal{L}_\Delta v = 0 & \text{in } [0, t_0) \times \mathbb{R}^N, \\ v(t_0, \cdot) = \theta & \text{in } \mathbb{R}^N. \end{cases} \quad (3.2)$$

Then

$$|\varphi_\theta(t, x)| = \left| \left( \mathcal{P}_{\text{In}}(t_0 - t, \cdot) * \theta \right)(x) \right| \leq c \frac{\mathcal{P}_0(t_0 - t)}{t_0 - t} (1 + |x|)^{2(t_0 - t) - N},$$

where  $\frac{\mathcal{P}_0(t_0 - t)}{t_0 - t}$  is bounded as  $t \rightarrow t_0$  and  $c > 0$  depends on  $R_0, t_0$  and  $\|\theta\|_{L^1(\mathbb{R}^N)}, \|\theta\|_{L^\infty(\mathbb{R}^N)}$ .

Applying (2.19) to the derivatives of  $\theta \in \mathcal{C}_c^1(B_{R_0})$ , we also have

$$\begin{aligned} |\nabla \varphi_\theta(t, x)| &= \left| \left( (\nabla \theta) * \mathcal{P}_{\ln}(t_0 - t, \cdot) \right) (t, x) \right| \\ &= \left| \left( \mathcal{P}_{\ln}(t_0 - t, \cdot) * (\nabla \theta) \right) (t, x) \right| \\ &\leq c \frac{\mathcal{P}_0(t_0 - t)}{t_0 - t} (1 + |x|)^{2(t_0 - t) - N}, \end{aligned}$$

where  $c > 0$  depends on  $R_0$ ,  $t_0$  and  $\|\nabla \theta\|_{L^1(\mathbb{R}^N)}$ ,  $\|\nabla \theta\|_{L^\infty(\mathbb{R}^N)}$ . For  $R > 2R_0$  we define

$$\eta_R(x) = \eta_0\left(\frac{|x|}{R}\right) \quad \text{and} \quad \psi_R(x, t) := \varphi_\theta(x, t)\eta_R(x),$$

where  $\eta_0 : [0, +\infty) \rightarrow [0, 1]$  is a smooth, non-increasing function such that

$$\eta_0 = 1 \quad \text{in} \quad [0, \frac{1}{2}] \quad \text{and} \quad \eta_0 = 0 \quad \text{in} \quad [1, +\infty).$$

As a test function in (3.1) we take  $\psi_R$ . Since  $\varphi_\theta$  is a strong solution, we have that

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u(t_0, x) \theta(x) \psi_R(x) dx \right| &= \left| \int_0^{t_0} \int_{\mathbb{R}^N} [u \eta_R(x) \partial_t \varphi_\theta - u \mathcal{L}_\Delta(\psi_R(t, x))] dx dt \right| \\ &= \left| \int_0^{t_0} \int_{\mathbb{R}^N} u \mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R)) dx dt \right| \\ &\leq \int_0^{t_0} \int_{\mathbb{R}^N} |u| \left| \mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R)) \right| dx dt. \end{aligned}$$

The main point is to show that

$$\lim_{R \rightarrow \infty} \int_0^{t_0} \int_{\mathbb{R}^N} |u| \left| \mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R)) \right| dx dt = 0. \quad (3.3)$$

*Step 2.* We claim that there exists a positive constant  $c$  independent of  $t$  such that for  $R \gg 1$

$$\left| \mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x) \right| \leq c \min \left\{ R^{2(t_0 - t) - N} \ln R, |x|^{2(t_0 - t) - N} \ln |x| \right\}, \quad (3.4)$$

for any  $(t, x) \in (0, T) \times \mathbb{R}^N$ .

(i) In the sequel we take  $R \gg 1$ . If  $x \in B_{\frac{R}{4}}$ , then  $B_1(x) \subset B_{\frac{R}{4}}(x) \subset B_{\frac{R}{2}}$ , thus

$$|x - y| > \frac{1 + |y|}{4} \quad \text{for} \quad y \in \mathbb{R}^N \setminus B_{\frac{R}{2}}.$$

Since  $1 - \eta_R = 0$  in  $B_{\frac{R}{2}}$ , we have that

$$\begin{aligned} \left| \mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x) \right| &= c_N \left| \int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{-\varphi_\theta(t, y)(1 - \eta_R(y))}{|x - y|^N} dy \right| \\ &\leq 4^N c_N \int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} \frac{|\varphi_\theta(t, y)|}{(1 + |y|)^N} dy \\ &\leq c_1 \frac{\mathcal{P}_0(t_0 - t)}{t_0 - t} \int_{\mathbb{R}^N \setminus B_{\frac{R}{2}}} (1 + |y|)^{2(t_0 - t) - 2N} dy \\ &\leq c_2 R^{2(t_0 - t) - N}, \end{aligned}$$

the last inequality, being a consequence of the fact that  $\lim_{t \rightarrow t_0} \frac{\mathcal{P}_0(t_0 - t)}{t_0 - t}$  exists. Therefore, for any  $t \in [0, t_0]$  and  $x \in B_{\frac{R}{4}}$ ,

$$|\mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x)| \leq c_2 R^{2(t_0 - t) - N}. \quad (3.5)$$

(ii) Next, if  $x \in B_{2R} \setminus B_{\frac{R}{4}}$ , we note that  $\varphi_\theta(1 - \eta_R)$  is  $C^1$ , then

$$\begin{aligned} |\mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x)| &\leq c_N \left| \int_{B_1(x)} \frac{(\varphi_\theta(1 - \eta_R))(t, x) - (\varphi_\theta(1 - \eta_R))(t, y)}{|x - y|^N} dy \right| \\ &\quad + c_N \left| \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{-\varphi_\theta(1 - \eta_R)(t, y)}{|x - y|^N} dy \right| + |\rho_N \varphi_\theta(1 - \eta_R)|. \end{aligned}$$

From the estimate on  $\phi_\theta$  there holds if  $\frac{R}{4} \leq |x| \leq 2R$ ,

$$|(\varphi_\theta(1 - \eta_R))(t, x)| \leq cR^{2(t_0 - t) - N}.$$

Furthermore

$$\begin{aligned} \left| \int_{B_1(x)} \frac{(\varphi_\theta(1 - \eta_R))(t, x) - (\varphi_\theta(1 - \eta_R))(t, y)}{|x - y|^N} dy \right| &\leq \sup_{y \in B_1(x)} |\nabla \varphi_\theta(t, y)| |\nabla \eta_R(y)| \\ &\leq \frac{c_1}{R} (1 + |x|)^{2(t_0 - t) - N} \chi_{[\frac{R}{2}, R]}(|x|) \\ &\leq c_1 R^{2(t_0 - t) - N - 1}, \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{-\varphi_\theta(1 - \eta_R)(t, y)}{|x - y|^N} dy \right| &\leq \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{|\varphi_\theta(t, y)|}{|x - y|^N} dy \\ &\leq c_1 \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{|y|^{2(t_0 - t) - N}}{|x - y|^N} dy. \end{aligned}$$

Since  $R \gg 1$ , we have

$$\begin{aligned} \int_{B_{R/8}(x) \setminus B_1(x)} \frac{|y|^{2(t_0 - t) - N}}{|x - y|^N} dy &\leq (R/2)^{2(t_0 - t) - N} \int_{B_{R/8}(x) \setminus B_1(x)} \frac{1}{|x - y|^N} dy \\ &\leq cR^{2(t_0 - t) - N} \ln R. \end{aligned}$$

Clearly

$$\int_{B_{4R} \setminus (B_{\frac{R}{2}} \cup B_{R/8}(x))} \frac{|y|^{2(t_0 - t) - N}}{|x - y|^N} dy \leq cR^{2(t_0 - t) - N},$$

and

$$\int_{\mathbb{R}^N \setminus B_{4R}} \frac{|y|^{2(t_0 - t) - N}}{|x - y|^N} dy \leq \int_{\mathbb{R}^N \setminus B_{4R}} |y|^{2(t_0 - t) - 2N} dy = cR^{2(t_0 - t) - N}.$$

Thus, for any  $t \in [0, t_0]$  and  $x \in B_{2R} \setminus B_{\frac{R}{4}}$ , we have that

$$|\mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x)| \leq cR^{2(t_0-t)-N} \ln R. \quad (3.6)$$

(iii) Finally, for  $x \in \mathbb{R}^N \setminus B_{2R}$ ,

$$\begin{aligned} |\mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x)| &\leq c_N \left| \int_{B_1(x)} \frac{\varphi_\theta(t, x) - \varphi_\theta(t, y)}{|x - y|^N} dy \right| \\ &\quad + c_N \left| \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{-(\varphi_\theta(1 - \eta_R))(t, y)}{|x - y|^N} dy \right| + |\rho_N \varphi(t, x)|. \end{aligned}$$

We estimate the three terms on the right-hand side of this inequality as follows. Obviously

$$|\rho_N \varphi(t, x)| \leq c|x|^{2(t_0-t)-N},$$

next

$$\left| \int_{B_1(x)} \frac{\varphi_\theta(t, x) - \varphi_\theta(t, y)}{|x - y|^N} dy \right| \leq \sup_{y \in B_1(x)} |\nabla \varphi_\theta(t, y)| \leq c_1(1 + |x|)^{2(t_0-t)-N},$$

and finally

$$\begin{aligned} \left| \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{-(\varphi_\theta(1 - \eta_R))(t, y)}{|x - y|^N} dy \right| &\leq \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{|\varphi_\theta(t, y)|}{|x - y|^N} dy \\ &\leq c_1 \int_{\mathbb{R}^N \setminus (B_{\frac{R}{2}} \cup B_1(x))} \frac{|y|^{2(t_0-t)-N}}{|x - y|^N} dy. \end{aligned}$$

From these estimates, we obtain the following series of inequalities:

$$\begin{aligned} \int_{B_{\frac{|x|}{4}}(x) \setminus B_1(x)} \frac{|y|^{2(t_0-t)-N}}{|x - y|^N} dy &\leq c|x|^{2(t_0-t)-N} \int_{B_{\frac{|x|}{4}}(x) \setminus B_1(x)} \frac{1}{|x - y|^N} dy \\ &\leq c|x|^{2(t_0-t)-N} \ln \frac{|x|}{4}, \end{aligned}$$

$$\begin{aligned} \int_{B_{\frac{|x|}{4}}(0) \setminus B_{\frac{R}{2}}(0)} \frac{|y|^{2(t_0-t)-N}}{|x - y|^N} dy &\leq c|x|^{-N} \int_{B_{\frac{|x|}{4}}(0) \setminus B_{\frac{R}{2}}(0)} |y|^{2(t_0-t)-N} dy \\ &\leq c|x|^{2(t_0-t)-N}, \end{aligned}$$

$$\begin{aligned} &\int_{B_{2|x|}(x) \setminus (B_{\frac{|x|}{4}}(0) \cup B_{\frac{|x|}{4}}(x))} \frac{|y|^{2(t_0-t)-N}}{|x - y|^N} dy \\ &= |x|^{2(t_0-t)-N} \int_{B_{2(\mathbf{e}_x)} \setminus (B_{\frac{1}{4}}(0) \cup B_{\frac{1}{4}}(\mathbf{e}_x))} \frac{|z|^{2(t_0-t)-N}}{|\mathbf{e}_x - z|^N} dz \\ &= c|x|^{2(t_0-t)-N}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{2|x|}(x)} \frac{|y|^{2(t_0-t)-N}}{|x-y|^N} dy &= |x|^{2(t_0-t)-N} \int_{\mathbb{R}^N \setminus B_2} \frac{|z|^{2(t_0-t)-N}}{|e_x - z|^N} dz \\ &= c|x|^{2(t_0-t)-N}. \end{aligned}$$

Therefore, for any  $t \in [0, t_0]$  and  $x \in \mathbb{R}^N \setminus B_{2R}$

$$\left| \mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))(t, x) \right| \leq c|x|^{2(t_0-t)-N} \ln |x|. \quad (3.7)$$

*Step 3. End of the proof.* Assuming that (3.4) holds, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u| |\mathcal{L}_\Delta(\varphi_\theta(1 - \eta_R))| dx &\leq c \int_{\mathbb{R}^N} |u| \min \left\{ R^{2(t_0-t)-N} \ln R, |x|^{2(t_0-t)-N} \ln |x| \right\} dx \\ &\leq \int_{\mathbb{R}^N} |u(t, x)| (1 + |x|)^{2T-N} V_R(x) dx, \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} V_R(x) &= (1 + |x|)^{N-2T} \min \left\{ R^{2(t_0-t)-N} \ln R, |x|^{2(t_0-t)-N} \ln |x| \right\} \\ &\leq \min \left\{ R^{2(t_0-t)-2T} \ln R, |x|^{2(t_0-t)-2T} \ln |x| \right\} \\ &\leq R^{2(t_0-t)-2T} \ln R. \end{aligned} \quad (3.9)$$

Clearly the right-hand side of (3.9) tends to 0 uniformly in  $\mathbb{R}^N$  as  $R \rightarrow \infty$ . This implies that right-hand side of (3.8) shares this property. It follows that

$$\int_{\mathbb{R}^N} u(t_0, x) \theta(x) dx = 0,$$

by the dominated convergence theorem, thus  $u(t_0, x) = 0$ . Because  $t_0 < T$  is arbitrary,  $u$  is identically zero.  $\square$

### 3.2 Strong positive solutions

In this subsection we give a representation of the positive strong solutions of (1.1) as a Poisson type integral.

**Lemma 3.2.** *Let  $T \in (0, \frac{N}{2})$ ,  $u$  be a strong solution of (1.1) and  $\{\sigma_\delta\}_{\delta>0} \subset \mathbb{C}_c^\infty(\mathbb{R}^N)$  a sequence of mollifiers with support in  $B_\delta$ . Then  $u * \sigma_\delta$  is a strong solution of (1.1), which converges locally uniformly to  $u$  in  $[0, T] \times \mathbb{R}^N$ .*

*Proof.* Since  $u$  is a strong solution, there holds

$$\partial_t \hat{u} + 2 \ln |\xi| \hat{u} = 0.$$

Multiplying by  $\hat{\sigma}_\delta$  and using the fact that  $\hat{\sigma}_\delta \hat{u} = \widehat{\sigma_\delta * u}$ , we obtain

$$\partial_t \widehat{\sigma_\delta * u} + 2 \ln |\xi| \widehat{\sigma_\delta * u} = 0.$$

Since  $\widehat{\sigma_\delta * u}$  is  $C^\infty$ , it follows that  $\sigma_\delta * u$  is a strong solution of (1.8). As for the convergence of  $\sigma_\delta * u$  to  $u$ , it is a classical result since  $u \in \mathcal{C}([0, T] \times \mathbb{R}^N)$ .  $\square$

**Lemma 3.3.** *Let  $T \in (0, \frac{N}{2})$  and  $f$  be a nonnegative function in  $\mathcal{C}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{2T-N} dx)$ . If  $u \geq 0$  is a strong solution of (1.1), then*

$$u(t, x) \geq \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x-y) f(y) dy \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^N.$$

*Proof.* For  $r > 2$  we denote by  $\phi_r \in C_c^{0,1}(\mathbb{R}^N)$  the following cut-off function,

$$\phi_r(x) = \begin{cases} 1 & \text{for } |x| \leq r-1, \\ r-|x| & \text{for } r-1 < |x| \leq r, \\ 0 & \text{for } |x| \geq r, \end{cases}$$

and by  $\{\sigma_\delta\}_{\delta>0} \subset C_c^\infty(\mathbb{R}^N)$  a sequence of mollifiers with support in  $B_\delta$ . We set  $f_{r,\delta} = \phi_r f * \sigma_\delta$  and define the function  $v_{r,\delta}$  by the expression

$$v_{r,\delta}(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x-y) f_{r,\delta}(y) dy = \mathcal{P}_{\text{In}}(t, \cdot) * (f_{r,\delta})(x).$$

By Proposition 2.3  $v_{r,\delta}$  is a positive strong solution of

$$\begin{cases} \partial_t v_{r,\delta} + \mathcal{L}_\Delta v_{r,\delta} = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ v_{r,\delta}(0, \cdot) = f_{r,\delta} & \text{in } \mathbb{R}^N. \end{cases} \quad (3.10)$$

By (2.9) we have for  $(t, x) \in (0, T) \times \mathbb{R}^N$  and independently of  $\delta$  and  $r$ ,

$$0 < v_{r,\delta}(t, x) \leq c(\|f\|_{L^\infty} + \|f\phi_r\|_{L^1}) \frac{\mathcal{P}_0(t)}{2t} (1+|x|)^{2t-N}, \quad (3.11)$$

where, we recall it,  $\frac{\mathcal{P}_0(t)}{2t}$  is bounded as  $t \rightarrow 0^+$ .

Let  $f_1(x) = (1+|x|)^{\tau_0}$  with  $\tau_0 \in (-\frac{N}{2}, -T)$  and

$$w_0 = \mathcal{P}_{\text{In}}(t, \cdot) * f_1.$$

By Proposition 2.3-(ii) there holds

$$w_0(t, x) \geq c(1+|x|)^{2t+\tau_0} \quad \text{for } (t, x) \in (0, T) \times \mathbb{R}^N,$$

where  $0 > 2t + \tau_0 > 2t - N$ . From (3.11) for every  $\varepsilon > 0$  there exists  $\rho = \rho(\varepsilon) > r$  such that

$$0 < v_{r,\delta}(t, x) \leq \varepsilon w_0(t, x) \quad \text{for any } |x| \geq \rho, t \in [0, T),$$

and since  $u$  is nonnegative in  $[0, T) \times \mathbb{R}^N$ ,  $u_\delta := u * \sigma_\delta$  is also nonnegative, thus

$$0 < v_{r,\delta}(t, x) \leq \varepsilon w_0(t, x) \leq \varepsilon w_0(t, x) + u_\delta(t, x) \quad \text{for any } |x| \geq \rho, t \in [0, T).$$

Clearly

$$v_{r,\delta}(0, x) = (\phi_r u * \sigma_\delta)(0, x) \leq u_\delta(0, x) \leq \varepsilon w_0(0, x) + u_\delta(0, x), \quad \text{for any } |x| \leq \rho.$$

Next we set

$$w_\delta = u_\delta + \varepsilon w_0 - v_{r,\delta} \quad \text{in } (0, T) \times \mathbb{R}^N.$$

Then

$$w_\delta(0, \cdot) = \varepsilon w_0 + ((1 - \phi_r)f) * \sigma_\delta \quad \text{in } \mathbb{R}^N,$$

and  $w_\delta(0, \cdot) \geq \varepsilon(1 + \rho)^{\tau_0} > 0$  in  $\bar{B}_\rho(0)$ . By Lemma 2.8  $w_\delta > 0$  in  $(0, T) \times \bar{B}_\rho(0)$ , that is,

$$v_{r,\delta} < \varepsilon w_0 + u \quad \text{for } |x| \leq \rho \text{ and } t \in [0, T).$$

Because  $\varepsilon > 0$  could be chosen arbitrary, the previous inequality implies that,

$$v_{r,\delta}(t, x) \leq u_\delta(t, x) \quad \text{for all } (x, t) \in \mathbb{R}^N \times [0, T),$$

and this is independent of  $\rho$ . Therefore

$$u_\delta(t, x) \geq \mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} ((f\phi_r) * \sigma_\delta)(y) dy.$$

When  $\delta \rightarrow 0$ ,  $u_\delta \rightarrow u$  locally uniformly in  $[0, T) \times \mathbb{R}^N$  by the previous lemma and  $(f\phi_r) * \sigma_\delta \rightarrow f\phi_r$  uniformly in  $B_r$ , thus

$$\int_{\mathbb{R}^N} |x - y|^{2t-N} ((f\phi_r) * \sigma_\delta)(y) dy \rightarrow \int_{\mathbb{R}^N} |x - y|^{2t-N} f\phi_r(y) dy,$$

by the monotone convergence theorem. Therefore

$$u(t, x) \geq \mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f\phi_r(y) dy.$$

Because  $r \mapsto v_r$  is increasing as  $\lim_{r \rightarrow \infty} \phi_r = 1$ , we conclude that

$$\lim_{r \rightarrow \infty} \mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f\phi_r(y) dy = \mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) dy \leq u(t, x),$$

by the monotone convergence theorem. This ends the proof.  $\square$

The next easy to prove result is fundamental in the sequel.

**Corollary 3.4.** *Let  $0 < \tau < T < \frac{N}{2}$  and  $u$  be a strong nonnegative solution of*

$$\partial_t u + \mathcal{L}_\Delta u = 0 \quad \text{in } (0, T) \times \mathbb{R}^N \tag{3.12}$$

*such that  $u(t, \cdot) \in L^1(\mathbb{R}^N, (1 + |x|)^{2T-N} dx)$  for all  $t \in [0, T]$ . Then for any  $(t, x) \in (0, T - \tau) \times \mathbb{R}^N$ ,*

$$\int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x - y) u(\tau, y) dy \leq u(t + \tau, x). \tag{3.13}$$

*As a consequence we have for every  $x \in \mathbb{R}^N$  and  $t \in (0, T - \tau)$ ,*

$$\int_0^{T-t} \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x - y) u(\tau, y) dy d\tau \leq \int_0^{T-t} u(t + \tau, x) d\tau. \tag{3.14}$$

*Proof.* Let  $\tau \in (0, T)$ . Then the function  $t \mapsto u_\tau(t, \cdot) = u(t + \tau, \cdot)$  is a strong solution of

$$\partial_t u + \mathcal{L}_\Delta u = 0 \quad \text{in } (0, T - \tau) \times \mathbb{R}^N \tag{3.15}$$



satisfying  $u_\tau(0, x) = u(\tau, x)$  for all  $x \in \mathbb{R}^N$ . Hence it is larger than  $u_{u_\tau(0, \cdot)}$  in  $(0, T - \tau) \times \mathbb{R}^N$ . It follows that

$$u(t + \tau, x) = u_\tau(t, x) \geq u_{u_\tau(0, \cdot)}(t, x) = \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x - y) u(\tau, y) dy \quad \text{for all } (t, x) \in (0, T - \tau) \times \mathbb{R}^N, \quad (3.16)$$

which ends the proof.  $\square$

Now we prove our main result:

*Proof of Theorem 1.3. Proof of (1.15).* For  $0 < T \leq \frac{N}{2}$ , let  $u \geq 0$  be a strong solution of equation (1.1) with initial data  $f \in \mathcal{C}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1 + |x|)^{2T - 2N} dx)$ . Our aim is prove to that  $u$  is a weak solution of (1.1).

We first prove that for  $T' < \frac{N}{2}$ ,

$$u \in L^1([0, T'], L^1(\mathbb{R}^N, (1 + |x|)^{2(T - T') - N} dx)).$$

Take an arbitrary  $0 < T' < T < \frac{N}{2}$ ; then (3.14) with  $x = 0$  and  $t = T - T'$  shows that

$$\begin{aligned} \int_0^{T'} \int_{\mathbb{R}^N} u(\tau, y) (1 + |y|)^{2t - N} dy d\tau &= \int_0^{T'} \mathcal{P}_0(\tau) \int_{\mathbb{R}^N} u(\tau, y) (1 + |y|)^{2t - N} dy d\tau \\ &\leq \int_0^{T'} u(t + \tau, 0) d\tau \\ &< \int_t^T u(\tau, 0) d\tau < +\infty, \end{aligned}$$

where  $\mathcal{P}_0$  is bounded in  $[0, T]$  because to  $T < \frac{N}{2}$ .

We claim that  $u \in \mathcal{C}((0, T), L^1_{\text{loc}}(\mathbb{R}^N))$ .

For given  $\epsilon > 0$  and any  $R > 0$ , if  $|t_1 - t_2|$  small enough, there holds that

$$\int_{B_R(0)} |u(t_1, y) - u(t_2, y)| dy \leq \epsilon \int_{B_R(0)} dy$$

by the continuity of  $u$ . As a consequence, the strong solution of (1.1) is a weak solution.

Define now the function

$$w = u - \mathcal{P}_{\text{In}} * f,$$

then  $w \geq 0$  by Lemma 3.3.

It is clear that  $w$  is a strong solution of (3.1), so it is a weak solution by above arguments. Now we apply Theorem 3.1 to obtain that

$$w(x, t) = 0 \quad \text{for almost every } (t, x) \in (0, T) \times \mathbb{R}^N.$$

By the continuity of strong solutions, we have that

$$u = \mathcal{P}_{\text{In}}(t, \cdot) * f.$$

*Proof of (1.16).* Observe that

$$\begin{aligned}\mathcal{P}_0(t) &= \pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2})}{\Gamma(t)} \\ &= \pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2} + 1)}{\Gamma(t)} \frac{2}{N-2t},\end{aligned}$$

where

$$\pi^{-\frac{N}{2}} 4^{-t} \frac{\Gamma(\frac{N-2t}{2} + 1)}{\Gamma(t)} \rightarrow \pi^{-\frac{N}{2}} 4^{-\frac{N}{2}} \Gamma\left(\frac{N}{2}\right)^{-1} \quad \text{as } t \rightarrow \left(\frac{N}{2}\right)^-.$$

Given an arbitrary  $\epsilon \in (0, 1)$  and  $x \in \mathbb{R}^N$ , there exists  $\varrho \in (0, \frac{1}{4})$  such that

$$0 < \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} f(y) \, dy \leq \epsilon \quad \text{and} \quad \int_{B_{\varrho}(0)} |x-y|^{2t-N} \, dy \leq \epsilon,$$

and for fixed  $\varrho$ , there is  $s_0 \in (\frac{N}{4}, \frac{N}{2})$  such that for  $t \in [s_0, \frac{N}{2})$

$$||x-y|^{2t-N} - 1| \leq \epsilon,$$

uniformly in  $\bar{B}_{\frac{1}{2}}(0) \setminus B_{\varrho}(x)$ . Therefore, we have that

$$\begin{aligned}& \left| \frac{N-2t}{2} (\mathcal{P}_{\text{In}}(t, \cdot) * f)(x) - \|f\|_{L^1(\mathbb{R}^N)} \right| \\ & \leq \frac{N-2t}{2} \mathcal{P}_0(t) \left( \int_{B_{\frac{1}{2}}(0) \setminus B_{\varrho}(x)} ||x-y|^{2t-N} - 1| f(y) \, dy \right. \\ & \quad \left. + \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} ||x-y|^{2t-N} - 1| f(y) \, dy + \int_{B_{\varrho}(x)} ||x-y|^{2t-N} - 1| f(y) \, dy \right) \\ & \leq c \left( \int_{B_{\frac{1}{2}}(0) \setminus B_{\varrho}(x)} f(y) \, dy \epsilon + \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} f(y) \, dy + \|f\|_{L^\infty(\mathbb{R}^N)} \int_{B_{\varrho}(x)} |x-y|^{2t-N} \, dy \right) \\ & \leq c(\|f\|_{L^1(\mathbb{R}^N)} + \|f\|_{L^\infty(\mathbb{R}^N)} + 1)\epsilon,\end{aligned}$$

which implies (1.16) since  $\epsilon$  is arbitrary.

The decays at infinity in (i) and (ii) follows by Proposition 2.3.  $\square$

**Corollary 3.5.** *Let  $0 < T \leq \frac{N}{2}$ ,  $f \in \mathcal{C}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N, (1+|x|)^{2T-N} dx)$  and  $u(t, x) \geq 0$  be a strong solution of (3.12). Then:*

(i) *the strong solution of (1.1) is a weak solution;*

(ii) *for  $t_1, t_2 > 0$ ,  $t_1 + t_2 < T$ ,*

$$\mathcal{P}_{\text{In}}(t_1, \cdot) * (\mathcal{P}_{\text{In}}(t_2, \cdot) * f) = \mathcal{P}_{\text{In}}(t_1 + t_2, \cdot) * f. \quad (3.17)$$

*Proof.* The main point in the proof of Theorem 1.3 is to show that the strong solution of (1.1) is a weak solution and  $\mathcal{P}_{\text{In}}(t, \cdot) * f$  is the unique solution of (1.1). By the uniqueness, Corollary 3.4 could be improved in the sense that

$$\mathcal{P}_{\text{In}}(t, \cdot) * u(\tau, \cdot) = u(t + \tau, \cdot)$$

and

$$\mathcal{P}_{\text{In}}(t_1, \cdot) * (\mathcal{P}_{\text{In}}(t_2, \cdot) * f) = \mathcal{P}_{\text{In}}(t_1 + t_2, \cdot) * f$$

for  $t_1 + t_2 < T$ . We complete the proof.  $\square$

### 3.3 Existence of weak solutions

*Proof of Theorem 1.6.* Let  $f$  be nonnegative function in  $L^1(\mathbb{R}^N)$ ; there exists a sequence of nonnegative functions  $\{f_n\}_{n \in \mathbb{N}}$  in  $\mathcal{C}_c(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  such that

$$0 \leq f_n \leq f_{n+1} \leq \dots \leq f \quad \text{in } \mathbb{R}^N$$

and

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} (f(x) - f_n(x)) dx = 0.$$

The Cauchy problem

$$\begin{cases} \partial_t u + \mathcal{L}_\Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\ u(0, \cdot) = f_n & \text{in } \mathbb{R}^N, \end{cases} \quad (3.18)$$

has a unique strong solution

$$u_n(t, x) = \mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f_n(y) dy,$$

and the mapping  $\mathbb{N} \ni n \rightarrow u_n$  is nondecreasing and

$$\mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f_n(y) dy \leq \mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) dy := u_f(t, x),$$

where  $u_f$  is well defined since  $f \in L^1(\mathbb{R}^N)$ . From Corollary 3.5,  $u_n$  is a weak solution of (3.18) and then

$$\int_0^{T'} \int_{\mathbb{R}^N} [-u_n \partial_t \varphi + u_n \mathcal{L}_\Delta \varphi] dt dx = \int_{\mathbb{R}^N} f(x) \varphi(0, x) dx - \int_{\mathbb{R}^N} u_n(T', x) \varphi(T', x) dx \quad (3.19)$$

for every function  $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}^N)$  such that  $\varphi(t, \cdot)$  has compact support in  $\mathbb{R}^N$  and  $t \in [0, T)$ .

Note that  $u_f$  is the limit of  $\{u_n\}$ , i.e.

$$\mathcal{P}_0(t) \int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) dy = \mathcal{P}_0(t) \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |x - y|^{2t-N} f_n(y) dy.$$

*Step 1: we prove that  $u_f \in L^1((0, T'), L^1(\mathbb{R}^N, (1 + |x|)^{2(T-T')-N} dx))$  for any  $T' \in (0, T)$ .* Observe that  $\mathcal{P}_0(t) \leq ct$  for  $t \in (0, T']$  and then

$$\begin{aligned} t \int_{\mathbb{R}^N} u_f(t, x) (1 + |x|)^{2(T-T')-N} dx &\leq ct \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) (1 + |x|)^{2(T-T')-N} dy dx \\ &= t \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |x - y|^{2t-N} (1 + |x|)^{2(T-T')-N} dx \right) f(y) dy \\ &\leq c \int_{\mathbb{R}^N} (1 + |y|)^{2t-2T'+2T-N} f(y) dy \\ &\leq c \int_{\mathbb{R}^N} (1 + |x|)^{2T-N} f(y) dy, \end{aligned}$$

where we used the fact, by Lemma 2.1,

$$\int_{\mathbb{R}^N} |x - y|^{2t-N} (1 + |x|)^{2(T'-T)} dx \leq \frac{c}{t} (1 + |y|)^{2t+2(T'-T)}.$$

Step 2: we prove that  $u_f \in \mathcal{C}((0, T), L^1_{\text{loc}}(\mathbb{R}^N))$ .

Fixed  $t_0 \in (0, T)$  and  $\varrho_0 \gg 1$ , for an arbitrary  $\epsilon \in (0, 1)$ , there exist  $\varrho \in (0, \frac{1}{4})$  and  $s_0 \in (0, \frac{\min\{t_0, T-t_0\}}{4})$  such that for  $|t - t_0| < s_0$

$$|\mathcal{P}_0(t) - \mathcal{P}_0(t_0)| < \epsilon, \quad \int_{B_\varrho(0)} f(y) dy \leq \epsilon,$$

and for  $x \in \bar{B}_{\varrho_0}(0)$

$$||x - y|^{2t-N} - |x - y|^{2t_0-N}| \leq \epsilon \quad \text{uniformly for } y \in \mathbb{R}^N \setminus B_\varrho(x).$$

Then

$$\begin{aligned} & \int_{B_{\varrho_0}(0)} |u_f(t, x) - u_f(t_0, x)| dx \\ &= \left| \mathcal{P}_0(t) \int_{B_{\varrho_0}(0)} \int_{\mathbb{R}^N} |x - y|^{2t-N} f(y) dy dx - \mathcal{P}_0(t_0) \int_{B_{\varrho_0}(0)} \int_{\mathbb{R}^N} f(y) |x - y|^{2t_0-N} dy dx \right| \\ &\leq \int_{B_{\varrho_0}(0)} \int_{\mathbb{R}^N} |\mathcal{P}_0(t) |x - y|^{2t-N} - \mathcal{P}_0(t_0) |x - y|^{2t_0-N}| f(y) dy dx \\ &\leq |\mathcal{P}_0(t) - \mathcal{P}_0(t_0)| \int_{B_{\varrho_0}(0)} \int_{\mathbb{R}^N} f(y) |x - y|^{2t-N} dy dx \\ &\quad + \mathcal{P}_0(t_0) \int_{B_{\varrho_0}(0)} \int_{\mathbb{R}^N \setminus B_\varrho(x)} ||x - y|^{2t-N} - |x - y|^{2t_0-N}| f(y) dy dx \\ &\quad + \mathcal{P}_0(t_0) \int_{B_{\varrho_0}(0)} \int_{B_\varrho(x)} (|x - y|^{2t-N} + |x - y|^{2t_0-N}) f(y) dy dx \\ &\leq c\epsilon + \mathcal{P}_0(t_0) \left( \int_{B_{\varrho_0}(0)} f(y) dy \epsilon + \epsilon \varrho_0^N \int_{\mathbb{R}^N} f(y) dy + c(\varrho_0^{-2t} + \varrho_0^{-2t_0}) \int_{B_\varrho(x)} f(y) dy \right) \\ &\leq c\epsilon, \end{aligned}$$

which implies that  $u_f \in \mathcal{C}((0, T), L^1_{\text{loc}}(\mathbb{R}^N))$ .

Now we can pass to the limit in (3.19) as  $n \rightarrow +\infty$  and we obtain that

$$\int_0^{T'} \int_{\mathbb{R}^N} [-u_f \partial_t \varphi_t + u_f \mathcal{L}_\Delta \varphi] dt dx = \int_{\mathbb{R}^N} f(x) \varphi(0, x) dx - \int_{\mathbb{R}^N} u_f(T', x) \varphi(T', x) dx.$$

As a conclusion, we get  $\mathcal{P}_{\text{In}}(t, \cdot) * f$  is the unique weak solution by Theorem 3.1.  $\square$

*Proof of Corollary 1.7.* Given  $T \in (0, \frac{N}{2})$ , let  $t_n \in (0, \frac{N-2T}{4})$  with  $n \in \mathbb{N}$  such that

$$t_n \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

then  $u_n := \mathcal{P}_{\text{In}}(t_n + \cdot, \cdot)$  is the strong and the weak solution of the Cauchy problem

$$\begin{cases} \partial_t u + \mathcal{L}_\Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(0, \cdot) = \mathcal{P}_{\text{In}}(t_n, \cdot) & \text{in } \mathbb{R}^N, \end{cases} \quad (3.20)$$

therefore

$$\int_0^{T'} \int_{\mathbb{R}^N} [-u_n \partial_t \varphi_t + u_n \mathcal{L}_\Delta \varphi] dt dx = \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t_n, x) \varphi(0, x) dx - \int_{\mathbb{R}^N} u_n(T', x) \varphi(T', x) dx. \quad (3.21)$$

We note the the limit of  $\{\mathcal{P}_{\text{In}}(t_n + \cdot, \cdot)\}_n$  is  $\mathcal{P}_{\text{In}}$  in related space and  $\mathcal{P}_{\text{In}}(t_n, \cdot) \rightarrow \delta_0$ , then passing to the limit of (3.21), we obtain that  $\mathcal{P}_{\text{In}}$  is a weak solution of (1.21).  $\square$

### 3.4 The initial trace

Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^N$ . Following [11] we define

$$\mathbb{H}(\Omega) = \left\{ u \in L^1_{loc}(\mathbb{R}^N) : u \mathbf{1}_{\Omega^c} = 0, \int \int_{|x-y| \leq 1} \frac{(u(x) - u(y))^2}{|x-y|^N} dx dy < \infty \right\}, \quad (3.22)$$

and  $\mathcal{C}_{c,D}(\Omega)$  the space of uniformly Dini continuous functions with support in  $\overline{\Omega}$ . Then  $\mathbb{H}(\Omega)$  contains  $\mathcal{C}_{c,D}(\Omega)$  and it is an Hilbert space for the norm

$$\|u\|_{\mathbb{H}(\Omega)} = \left( \int \int_{|x-y| \leq 1} \frac{(u(x) - u(y))^2}{|x-y|^N} dx dy \right)^{\frac{1}{2}}, \quad (3.23)$$

and the associated inner product

$$\langle u, v \rangle_{\mathbb{H}(\Omega)} = \int \int_{|x-y| \leq 1} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^N} dx dy. \quad (3.24)$$

The following fundamental result is proved in [11]

**Theorem 3.6.** *The imbedding of  $\mathbb{H}(\Omega)$  into  $L^2(\Omega)$  is compact and the operator  $\mathcal{L}_\Delta$  admits an increasing sequence of eigenvalues  $\{\lambda_k^L(\Omega)\}$  tending to infinity with corresponding eigenfunctions  $\phi_{k,\Omega}$ ,  $k \in \mathbb{N}^*$ . Furthermore  $\lambda_1^L(\Omega)$  is simple and  $\phi_{1,\Omega}$  has constant sign. Finally,  $\mathcal{L}_\Delta$  satisfies the maximum principle if and only if  $\lambda_1^L(\Omega) > 0$ , which in turn is satisfied if one of the two conditions (i) or (ii) of Lemma 2.6 is satisfied.*

**Lemma 3.7.** *Let  $\rho \in \mathcal{C}(\mathbb{R}^N)$  be a positive uniformly continuous function such that for any  $\theta > 0$  there exists  $m_\theta > 0$  such that*

$$\sup_{|x-z| \leq \theta} \frac{\rho(x)}{\rho(z)} \leq m_\theta. \quad (3.25)$$

*If  $\{\sigma_\delta\}_{\delta \in (0,1]}$  is a sequence of smooth mollifiers with support in  $B_\delta$  and  $u \in L^1_\rho(\mathbb{R}^N)$ , then  $u * \sigma_\delta \rightarrow u$  in  $L^1_\rho(\mathbb{R}^N)$ .*

*Proof.* Let  $\epsilon > 0$  and  $u \in L^1_\rho(\mathbb{R}^N)$ . Then there exists  $R > 0$  such that  $\|u \mathbf{1}_{B_R^c}\|_{L^1_\rho} < \frac{\epsilon}{4}$ . There exists  $u_n \in C_c(B_R)$  such that  $\|u \mathbf{1}_{B_R} - u_n\|_{L^1_\rho} < \frac{\epsilon}{4}$ . Then

$$u - u * \sigma_\delta = u - u_n + u_n - u_n * \sigma_\delta + u_n * \sigma_\delta - u * \sigma_\delta.$$

There holds

$$\begin{aligned} \|u - u * \sigma_\delta\|_{L^1_\rho} &\leq \|u - u_n\|_{L^1_\rho} + \|u_n - u_n * \sigma_\delta\|_{L^1_\rho} + \|u_n * \sigma_\delta - u * \sigma_\delta\|_{L^1_\rho} \\ &\leq I + II + III. \end{aligned}$$

Now  $I \leq \frac{\epsilon}{2}$ .

$$\begin{aligned} II &= \int_{B_{R+\delta}} \left| \int_{B_\delta} u_n(x) \sigma_\delta(y) dy - \int_{B_\delta} u_n(x-y) \sigma_\delta(y) dy \right| \rho(x) dx \\ &\leq \int_{B_{R+\delta}} \int_{B_\delta} |u_n(x) - u_n(x-y)| \sigma_\delta(y) \rho(x) dy dx \\ &\leq \sup_{|x| \leq R+\delta} \rho(x) \sup_{|y| \leq \delta} |u_n(x) - u_n(x-y)|. \end{aligned}$$

For  $n$  fixed, there exists  $\delta_0$  such that for  $\delta \leq \delta_0$ , one has  $II \leq \frac{\epsilon}{4}$ . Finally,

$$\begin{aligned}
III &= \int_{\mathbb{R}^N} |(u_n - u) * \sigma_\delta(x)| \rho(x) dx \\
&\leq \int_{B_{R+\delta}} \int_{B_\delta} |(u_n(x-y) - u(x-y)) \sigma_\delta(y)| dy \rho(x) dx + \int_{B_{R+\delta}} \int_{B_\delta} |u(x-y)| \sigma_\delta(y) dy \rho(x) dx \\
&\leq \sup_{|x| \leq R+\delta} \rho(x) \|u_n - u\|_{L^1(B_{R+\delta})} + m_\delta \int_{B_{R+\delta}} \int_{B_\delta} |u(x-y)| \sigma_\delta(y) dy \rho(x-y) dx \\
&\leq \left( \sup_{|x| \leq R+\delta} \rho(x) + m_\delta \right) \frac{\epsilon}{4}.
\end{aligned}$$

Combining the three estimates we complete the proof.  $\square$

**Theorem 3.8.** *Let  $u$  be a nonnegative strong solution of (1.9) in  $(0, T) \times \mathbb{R}^N$ . Then*

$$u \in L^1(0, T'; L^1(\mathbb{R}^N, (1 + |x|)^{-N})), \quad (3.26)$$

for any  $T' < T$ . There exists a nonnegative Radon measure  $\mu$  in  $\mathbb{R}^N$  satisfying (1.20) such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(t, x) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x), \quad (3.27)$$

and there holds

$$u(t, x) \geq \int_{\mathbb{R}^N} \mathcal{P}_{\text{In}}(t, x-y) d\mu(y) \quad \text{for all } (t, x) \in (0, T) \times \mathbb{R}^N. \quad (3.28)$$

*Proof. Step 1: Let  $u \in \text{Dom}_{\mathcal{L}_\Delta}(\mathbb{R}^N)$ , then given bounded Lipschitz domain  $\Omega$ , the following identity holds*

$$\int_{\Omega} \phi_{1,\Omega} \mathcal{L}_\Delta u dx = \lambda_1^L(\Omega) \int_{\Omega} u \phi_{1,\Omega} dx - c_N \int_{\Omega} \phi_{1,\Omega}(x) \int_{\Omega^c} \frac{u(y)}{|x-y|^N} dy dx. \quad (3.29)$$

Indeed, we have

$$\begin{aligned}
\int_{\Omega} \phi_{1,\Omega} \mathcal{L}_\Delta u dx &= \int_{\Omega} \phi_{1,\Omega}(x) \left( c_N \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^N} dy - c_N \int_{\Omega^c} \frac{u(y)}{|x-y|^N} dy + (h_\Omega(x) + \rho_N) u(x) \right) dx \\
&= \int_{\Omega} u(x) \left( c_N \int_{\Omega} \frac{\phi_{1,\Omega}(x) - \phi_{1,\Omega}(y)}{|x-y|^N} dy + (h_\Omega(x) + \rho_N) \phi_{1,\Omega}(x) \right) dx \\
&\quad - c_N \int_{\Omega} \phi_{1,\Omega} \int_{\Omega^c} \frac{u(y)}{|x-y|^N} dy dx \\
&= \lambda_1^L(\Omega) \int_{\Omega} u \phi_{1,\Omega} dx - c_N \int_{\Omega} \phi_{1,\Omega}(x) \int_{\Omega^c} \frac{u(y)}{|x-y|^N} dy dx.
\end{aligned}$$

*Step 2: Construction of the initial trace.* We multiply the equation by  $\phi_{1,\Omega}$  and get

$$\frac{d}{dt} \int_{\Omega} u(t, x) \phi_{1,\Omega}(x) dx + \lambda_1^L(\Omega) \int_{\Omega} u(t, x) \phi_{1,\Omega}(x) dx = c_N \int_{\Omega} \phi_{1,\Omega}(x) \int_{\Omega^c} \frac{u(y)}{|x-y|^N} dy dx \geq 0. \quad (3.30)$$

Hence

$$t \mapsto e^{-\lambda_1^L(\Omega)t} \int_{\Omega} u(t, x) \phi_{1,\Omega}(x) dx$$

is an increasing nonnegative function, which therefore admits a nonnegative limit  $\ell(\Omega)$  when  $t \rightarrow 0$ . This implies in particular that for any compact set  $G \subset \Omega$  and  $T' \in (0, T)$ ,

$$\int_0^{T'} \int_G u(t, x) dx dt < \infty. \quad (3.31)$$

Let  $\zeta \in C_c^\infty(\Omega)$  which vanishes outside the open set  $G \subset \bar{G} \subset \Omega$ ,  $\zeta \geq 0$ , then

$$\mathcal{L}_\Delta \zeta(x) = c_N \int_G \frac{\zeta(x) - \zeta(y)}{|x - y|^N} dy + (h_G(x) + \rho_N) \zeta(x),$$

thus

$$\int_G \zeta(x) \mathcal{L}_\Delta u(t, x) dx = \int_G u(t, x) \mathcal{L}_\Delta \zeta(x) dx - c_N \int_G \zeta(x) \int_{G^c} \frac{u(t, y)}{|x - y|^N} dy dx. \quad (3.32)$$

It follows that

$$\frac{d}{dt} \left( \int_G u(t, x) \zeta(x) dx - \|\mathcal{L}_\Delta \zeta\|_{L^\infty(G)} \int_t^{T'} \int_G u(s, x) dx ds \right) \geq c_N \int_G \zeta(x) \int_{G^c} \frac{u(t, y)}{|x - y|^N} dy dx \geq 0. \quad (3.33)$$

Hence  $t \mapsto \int_G u(t, x) \zeta(x) dx$  admits a nonnegative limit  $\ell(\zeta)$  when  $t \rightarrow 0^+$ , and

$$0 \leq \ell(\zeta) \leq \|\mathcal{L}_\Delta \zeta\|_{L^\infty(G)} \int_0^{T'} \int_G u(t, x) dx dt + \int_G u(T', x) \zeta(x) dx. \quad (3.34)$$

Now  $\|\mathcal{L}_\Delta \zeta\|_{L^\infty(G)} \leq \|\zeta\|_{C^2(G)}$ , therefore the mapping  $\zeta \mapsto \ell(\zeta)$  is a positive distribution, hence a measure. By a partition of unity, we construct the nonnegative Radon measure  $\mu$  such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(t, x) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta d\mu(x) \quad \text{for all } \zeta \in C_c(\mathbb{R}^N). \quad (3.35)$$

This measure is uniquely defined and is called the *initial trace* of  $u$ . Finally there holds from (3.32),

$$\int_G u(T', x) \zeta(x) dx - \int_{\mathbb{R}^N} \zeta d\mu(x) = - \int_0^{T'} \int_G u(t, x) \mathcal{L}_\Delta \zeta(x) dx + c_N \int_0^{T'} \int_G \zeta(x) \int_{G^c} \frac{u(t, y)}{|x - y|^N} dy dx dt. \quad (3.36)$$

*Step 3: the relation (3.26) holds.* We derive from (3.33) and (3.35)

$$\int_G u(T', x) \zeta(x) dx \geq \|\mathcal{L}_\Delta \zeta\|_{L^\infty(G)} \int_0^{T'} \int_G u(s, x) dx ds + \int_{\mathbb{R}^N} \zeta d\mu(x) + c_N \int_0^{T'} \int_G \zeta(x) \int_{G^c} \frac{u(t, y)}{|x - y|^N} dy dx dt.$$

Since (3.31) holds we obtain that

$$\int_0^{T'} \int_G \zeta(x) \int_{G^c} \frac{u(t, y)}{|x - y|^N} dy dx < \infty.$$

We take  $G = B_2$  and  $\zeta = 1$  on  $B_1$ , then

$$\int_0^{T'} \int_{B_1} \int_{B_2^c} \frac{u(t, y)}{|x - y|^N} dy dx dt < \infty.$$

If  $x \in B_1$  and  $y \in B_2^c$ ,  $|x - y| \geq \frac{1}{3}(|y| + 1)$ , hence

$$\int_0^{T'} \int_{B_2^c} \frac{u(t, y)}{(|y| + 1)^N} dy dt < \infty.$$

Combined with (3.31), this inequality implies (3.26).

*Step 4: the inequality (3.28) is verified.* Let  $\{\sigma_\delta\}_{\delta>0}$  be a sequence of smooth mollifiers with support in  $B_\delta$ ,  $\mu_\delta = \mu * \sigma_\delta$  and  $u_\delta = u * \sigma_\delta$ . Then

$$\begin{aligned} \int_G u_\delta(T', x) \zeta(x) dx &= \int_0^{T'} \int_G \mathcal{L}_\Delta \zeta(x) u_\delta(t, x) dx dt + \int_{\mathbb{R}^N} \zeta(x) u_\delta(0, x) dx \\ &\quad + c_N \int_0^{T'} \int_G \zeta(x) \int_{G^c} \frac{u_\delta(t, y)}{|x - y|^N} dy dx dt. \end{aligned}$$

When  $\delta \rightarrow 0$ ,  $u_\delta$  converges to  $u$  uniformly on compact subsets of  $(0, T'] \times \mathbb{R}^N$  and in  $L^1((0, T') \times B_R)$  for any  $R > 0$ . As in Step 3 we have that  $u_\delta \in L^1(0, T'; L^1(\mathbb{R}^N, (1 + |x|)^{-N}))$ , and by Lemma 3.7 it converges to  $u$  in  $L^1(0, T'; L^1(\mathbb{R}^N, (1 + |x|)^{-N}))$ . Consequently, it follows from (3.36) that the following limit exists with explicit value,

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N} \zeta(x) u_\delta(0, x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x).$$

Furthermore,

$$\begin{aligned} \int_{\mathbb{R}^N} u_\delta(t, x) \zeta(x) dx &= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} u(t, y) \sigma_\delta(x - y) dy \right) \zeta(x) dx \\ &= \int_{\mathbb{R}^N} u(t, y) \left( \int_{\mathbb{R}^N} \sigma_\delta(x - y) \zeta(x) dx \right) dy \\ &= \int_{\mathbb{R}^N} u(t, y) (\zeta * \hat{\sigma}_\delta)(y) dy \end{aligned}$$

where  $\hat{\sigma}_\delta(y) = \sigma_\delta(-y)$ . Therefore

$$\int_{\mathbb{R}^N} u_\delta(0, x) \zeta(x) dx = \lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u_\delta(t, x) \zeta(x) dx = \int_{\mathbb{R}^N} (\zeta * \hat{\sigma}_\delta)(y) d\mu(y) = \int_{\mathbb{R}^N} \zeta(y) \mu_\delta(y) dy,$$

where  $\mu_\delta = \mu * \sigma_\delta$ . This implies that  $u_\delta$  is a strong solution with initial data  $\mu_\delta$ . We apply Lemma 3.3 and get

$$u_\delta \geq u_{\mu * \sigma_\delta},$$

which yields to (3.28) by letting  $\delta \rightarrow 0$ . □

## 4 Fundamental solutions of $\mathcal{L}_\Delta$

We first give the proof of the existence and asymptotic properties of the solutions of the Helmholtz equation in  $\mathbb{R}^N$ .



## 4.1 Existence in $\mathbb{R}^N$

**Lemma 4.1.** *If  $N \geq 3$ , the fundamental solution  $\Phi_1$  to the equation*

$$-\Delta u - u = \delta_0 \quad \text{in } \mathbb{R}^N. \quad (4.1)$$

satisfies

$$\Phi_1(x) = c_{N,1}|x|^{2-N}(1 + o(1)) \quad \text{as } x \rightarrow 0, \quad (4.2)$$

and

$$\Phi_1(x) = -\frac{\sin(|x| - \frac{(N-1)\pi}{4})}{2(2\pi|x|)^{\frac{N-1}{2}}}(1 + O(|x|^{-1})) \quad \text{as } |x| \rightarrow \infty \quad (4.3)$$

*Proof.* The equation reduces to the classical Helmholtz equation

$$-\Delta u - u = \delta_0 \quad \text{in } \mathbb{R}^N. \quad (4.4)$$

The method for obtaining  $\Phi_1$  is standard but for the sake of completeness, we recall some elements. The function  $\Phi_1$  is radial, thus, if we define  $w$  by  $\Phi_1(r) = r^{1-\frac{N}{2}}w(r)$ , it satisfies the Bessel equation of order  $\frac{N}{2} - 1$

$$w''(r) + \frac{w'(r)}{r} + \left(1 - \frac{(N-2)^2}{4r^2}\right)w(r) = 0 \quad \text{on } (0, \infty). \quad (4.5)$$

The solution of (4.5) is a linear combination of two Hankel functions respectively of the first kind  $H_{\frac{N}{2}-1}^{(1)}$  and of the second kind  $H_{\frac{N}{2}-1}^{(2)}$ , that is

$$w(r) = AH_{\frac{N}{2}-1}^{(1)}(r) + BH_{\frac{N}{2}-1}^{(2)}(r). \quad (4.6)$$

We find  $A$  and  $B$  if we use the relation

$$|S^{N-1}| \lim_{r \rightarrow 0} r^{N-1} \Phi_1'(r) = -|S^{N-1}|,$$

which yields with  $B = 0$ ,

$$\lim_{r \rightarrow 0} \left( \left(1 - \frac{N}{2}\right) e^{-\frac{N}{2}} H_{\frac{N}{2}-1}^{(1)}(r) + r^{1-\frac{N}{2}} \left( H_{\frac{N}{2}-1}^{(1)} \right)'(r) \right) = -|S^{N-1}|.$$

The Hankel functions have the following behaviours when  $r \rightarrow 0$  (see e.g. [34]),

$$\begin{aligned} H_{\frac{N}{2}-1}^{(1)}(r) &= -i \frac{\Gamma(\frac{N}{2}-1)}{\pi} \left(\frac{2}{r}\right)^{\frac{N}{2}-1} (1 + o(1)), \\ \left( H_{\frac{N}{2}-1}^{(1)} \right)'(r) &= i \frac{(N-2)\Gamma(\frac{N}{2}-1)}{4\pi} \left(\frac{2}{r}\right)^{\frac{N}{2}} (1 + o(1)). \end{aligned} \quad (4.7)$$

Putting altogether these estimates we find

$$A = \frac{i}{4} \left(\frac{1}{2\pi}\right)^{\frac{N}{2}-1},$$

and

$$\Phi_1(r) = \frac{i}{4} \left(\frac{1}{2\pi r}\right)^{\frac{N}{2}-1} H_{\frac{N}{2}-1}^{(1)}(r), \quad (4.8)$$

which implies

$$\Phi_1(r) = \frac{\Gamma(\frac{N}{2} - 1)}{4\pi^{\frac{N}{2}}} r^{2-N} (1 + o(1)) \quad \text{as } r \rightarrow 0. \quad (4.9)$$

The Hankel function is oscillatory at infinity with asymptotic frequency  $2\pi$ . Expression (4.3) can be found in [24, see (5.16.1)-(5.16.3)]. Note in particular the following rough a priori estimate,

$$|\Phi_1(x)| \leq C_0 \max\{|x|^{2-N}, |x|^{\frac{1-N}{2}}\} \quad (4.10)$$

for  $x \in \mathbb{R}^N \setminus \{0\}$ . This ends the proof.  $\square$

The next estimate will be used several times in the sequel.

**Lemma 4.2.** *If  $N \geq 3$  and  $\alpha \in (0, 1)$ , then there holds for  $x \in B_{\frac{1}{e}} \setminus \{0\}$  and some  $c > 0$ ,*

$$|E_\alpha * \Phi_1(x)| \leq \begin{cases} c|x|^{2\alpha+2-N} & \text{if } N \geq 4 \text{ and } 0 < \alpha < 1, \text{ or } N = 3 \text{ and } 0 < \alpha < \frac{1}{2} \\ c \ln \frac{1}{|x|} & \text{if } N = 3. \text{ and } \alpha = \frac{1}{2} \\ c & \text{if } N = 3 \text{ and } \frac{1}{2} < \alpha < 1. \end{cases} \quad (4.11)$$

There also holds if  $x \in B_1^c$ ,

$$|E_\alpha * \Phi_1|(x) \leq \begin{cases} c|x|^{2\alpha-\frac{N+1}{2}} & \text{if } \frac{1}{2} < \alpha < 1 \\ C_\epsilon |x|^{2\alpha-\frac{N+1}{2}+\epsilon} & \text{if } 0 < \alpha \leq \frac{1}{2}, \end{cases} \quad (4.12)$$

*Proof.* We have

$$\begin{aligned} |E_\alpha * \Phi_1|(x) &= c_N \left| \int_{\mathbb{R}^N} |x-y|^{2\alpha-N} \Phi_1(y) dy \right| \\ &\leq c_1 \int_{B_1} |x-y|^{2\alpha-N} |y|^{2-N} dy + c_1 \left| \int_{B_1^c} |x-y|^{2\alpha-N} \Phi_1(y) dy \right| \\ &\leq I_\alpha(x) + II_\alpha(x). \end{aligned} \quad (4.13)$$

*Step I.* We have near  $x = 0$ ,

$$I_\alpha(x) = c_1 |x|^{2\alpha+2-N} \int_{B_{\frac{1}{|x|}}} |z|^{2-N} |\mathbf{e}_N - z|^{2\alpha-N} dz,$$

where  $\mathbf{e}_N = (0, 0, \dots, 1)$ . If  $N \geq 4$  and for any  $\alpha \in (0, 1)$ , or if  $N = 3$  and  $0 < \alpha < \frac{1}{2}$ , we have,

$$I_\alpha(x) = c_1 |x|^{2\alpha+2-N} \left( \int_{\mathbb{R}^N} |z|^{2-N} |\mathbf{e}_N - z|^{2\alpha-N} dz \right) (1 + o(1)) \quad \text{when } x \rightarrow 0. \quad (4.14)$$

If  $N = 3$  and  $\alpha = \frac{1}{2}$ , we have

$$I_\alpha(x) = c_1 \left( \ln \frac{1}{|x|} \right) (1 + o(1)) \quad \text{when } x \rightarrow 0. \quad (4.15)$$

Finally, if  $N = 3$  and  $\frac{1}{2} \leq \alpha < 1$ , we have

$$I_\alpha(x) = c_1' (1 + o(1)) \quad \text{when } x \rightarrow 0. \quad (4.16)$$

Concerning the second term  $II_\alpha$ , we use the oscillations of  $\Phi_1(x)$  when  $|x| \rightarrow \infty$ . There holds

$$|x - y|^{2\alpha - N} = |y|^{2\alpha - N} + O(|y|^{2\alpha - N - 1}) \quad \text{when } |y| \rightarrow \infty,$$

and

$$\Phi_1(y) = c_N^* |y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) + O\left(|y|^{-\frac{1+N}{2}}\right) \quad \text{when } |y| \rightarrow \infty.$$

Furthermore these two estimates hold uniformly with respect to  $x \in B_1$ . Therefore, for  $x \in B_1$  small enough and  $A > 1$  large enough, we have

$$\begin{aligned} \left| \int_{B_1^c} |x - y|^{2\alpha - N} \Phi_1(y) dy \right| &\leq \left| \int_{B_A \setminus B_1} |x - y|^{2\alpha - N} \Phi_1(y) dy \right| + \left| \int_{B_A \setminus B_1} |x - y|^{2\alpha - N} \Phi_1(y) dy \right| \\ &= c' + c \int_A^\infty r^{2\alpha - \frac{N+1}{2}} \sin\left(r - \frac{(N-1)\pi}{4}\right) dr + O\left(\int_A^\infty r^{2\alpha - \frac{N+3}{2}} dr\right). \end{aligned}$$

By integration by parts,

$$\begin{aligned} \int_A^\infty r^{2\alpha - \frac{N+1}{2}} \sin\left(r - \frac{(N-1)\pi}{4}\right) dr &= - \left[ r^{2\alpha - \frac{N+1}{2}} \cos\left(r - \frac{(N-1)\pi}{4}\right) \right]_{r=A}^\infty \\ &\quad + \left(2\alpha - \frac{N+1}{2}\right) \int_A^\infty r^{2\alpha - \frac{N+3}{2}} \cos\left(r - \frac{(N-1)\pi}{4}\right) dr. \end{aligned}$$

This implies

$$II_\alpha(x) = \left| \int_{B_1^c} |x - y|^{2\alpha - N} \Phi_1(y) dy \right| \leq c' + c'' A^{2\alpha - \frac{N+1}{2}}, \quad (4.17)$$

where  $c' = c'(A) > 0$  and this term is bounded whenever  $A$  is fixed. Combining (4.13) and (4.17) we obtain estimate (4.11) when  $0 < |x| \leq 1$  for some  $c > 0$ .

*Step II.* When  $|x|$  is large, we write

$$\begin{aligned} |E_\alpha * \Phi_1|(x) &\leq c_1 \int_{B_1} |x - y|^{2\alpha - N} |y|^{\frac{1-N}{2}} dy + c_1 \left| \int_{B_1^c} |x - y|^{2\alpha - N} \Phi_1(y) dy \right| \\ &\leq \tilde{I}_\alpha(x) + \tilde{II}_\alpha(x). \end{aligned} \quad (4.18)$$

By a change of variable

$$\int_{B_1} |x - y|^{2\alpha - N} |y|^{\frac{1-N}{2}} dy = |x|^{2\alpha + \frac{1-N}{2}} \int_{B_{\frac{1}{|x|}}} |\mathbf{e}_N - z|^{2\alpha - N} |z|^{\frac{1-N}{2}} dz.$$

If  $|x| \geq 2$ , then  $|z| \leq \frac{1}{2}$  thus  $|\mathbf{e}_N - z| \geq \frac{1}{2}$ , hence

$$\int_{B_{\frac{1}{|x|}}} |\mathbf{e}_N - z|^{2\alpha - N} |z|^{\frac{1-N}{2}} dz \leq 2^{N-2\alpha} \omega_N \int_0^{\frac{1}{|x|}} r^{\frac{N-1}{2}} dr = \frac{2^{N+1-2\alpha} \omega_N}{N+1} |x|^{-\frac{N+1}{2}}.$$

This yields

$$\tilde{I}_\alpha(x) \leq \frac{2^{N+1-2\alpha} \omega_N}{N+1} |x|^{2\alpha - N}. \quad (4.19)$$

Concerning  $\tilde{I}_\alpha(x)$ , we have for  $|y| \geq 1$ ,

$$\Phi_1(y) = c|y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) + \beta(y)|y|^{-\frac{1+N}{2}},$$

where  $\beta$  is a bounded function. Then

$$\tilde{I}_\alpha(x) \leq c \left| \int_{B_1^c} |x-y|^{2\alpha-N} |y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) \right| + \|\beta\|_{L^\infty} \int_{B_1^c} |x-y|^{2\alpha-N} |y|^{-\frac{1+N}{2}} dy. \quad (4.20)$$

We first notice that

$$\begin{aligned} \int_{B_1^c} |x-y|^{2\alpha-N} |y|^{-\frac{1+N}{2}} dy &= |x|^{2\alpha-\frac{N+1}{2}} \int_{B_{\frac{1}{|x|}}^c} |\mathbf{e}_N - z|^{2\alpha-N} |z|^{-\frac{1+N}{2}} dz \\ &= |x|^{2\alpha-\frac{N+1}{2}} \left( \int_{\mathbb{R}^N} |\mathbf{e}_N - z|^{2\alpha-N} |z|^{-\frac{1+N}{2}} dz \right) (1 + o(1)). \end{aligned} \quad (4.21)$$

For the second term, its treatment depends on the value of  $\alpha$  with respect to  $\frac{1}{2}$ .

- If  $\frac{1}{2} < \alpha < 1$  we use spherical coordinates  $y = (r, \sigma)$  and  $x = (\rho, \theta)$ , then

$$\begin{aligned} \int_{B_1^c} |x-y|^{2\alpha-N} |y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) dy \\ &= \int_1^\infty \int_{S^{N-1}} r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha-\frac{N}{2}} \sin\left(r - \frac{(N-1)\pi}{4}\right) dS dr \\ &= - \int_{S^{N-1}} \left[ r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha-\frac{N}{2}} \cos\left(r - \frac{(N-1)\pi}{4}\right) \right]_{r=1}^\infty dS \\ &\quad + \int_1^\infty \int_{S^{N-1}} \frac{d}{dr} \left( r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha-\frac{N}{2}} \right) \cos\left(r - \frac{(N-1)\pi}{4}\right) dS dr \\ &:= -J_1 + J_2. \end{aligned}$$

We have that

$$|J_1| \leq c(1 + |x|^2)^{\alpha-\frac{N}{2}}.$$

Furthermore, since

$$\begin{aligned} \frac{d}{dr} \left( r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha-\frac{N}{2}} \right) &= \frac{N-1}{2} r^{\frac{N-3}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha-\frac{N}{2}} \\ &\quad + (2\alpha - N) r^{\frac{N-1}{2}} (r - \rho\langle\sigma, \theta\rangle) (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha-\frac{N+2}{2}}, \end{aligned} \quad (4.22)$$

uniformly with respect to  $\rho \leq 2r$  and  $(\sigma, \theta) \in S^{N-1} \times S^{N-1}$ , we obtain that  $J_2$  satisfies ,

$$|J_2| \leq c|x|^{2\alpha-\frac{N+1}{2}}.$$

Combining these estimates on  $J_1$  and  $J_2$  with (4.19), we obtain

$$|E_\alpha * \Phi_1|(x) \leq c'|x|^{2\alpha-\frac{N+1}{2}} \quad \text{if } |x| \geq 1. \quad (4.23)$$

- If  $0 < \alpha \leq \frac{1}{2}$ , we cannot use directly the integration by parts method since the function  $(r, \sigma) \mapsto (r - \rho\langle\sigma, \theta\rangle)(r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha - \frac{N+2}{2}}$  is not integrable near  $(\rho, \theta)$ . We have for  $|x| > 2$

$$\begin{aligned} \int_{B_1^c} |x-y|^{2\alpha-N} |y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) dy &= \int_{1 < |y| < 2|x|} |x-y|^{2\alpha-N} |y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) dy \\ &\quad + \int_{|y| > 2|x|} |x-y|^{2\alpha-N} |y|^{\frac{1-N}{2}} \sin\left(|y| - \frac{(N-1)\pi}{4}\right) dy \\ &= K_1 + K_2. \end{aligned}$$

We estimate the terms  $K_j$  by two different methods. Using again the spherical coordinates introduced above,

$$\begin{aligned} |K_2| &\leq \left| \int_{2|x|}^{\infty} \int_{S^{N-1}} r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha - \frac{N}{2}} \sin\left(r - \frac{(N-1)\pi}{4}\right) dS dr \right| \\ &\leq \left| \int_{S^{N-1}} \left[ r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha - \frac{N}{2}} \cos\left(r - \frac{(N-1)\pi}{4}\right) \right]_{r=2|x|}^{\infty} dS \right| \\ &\quad + \left| \int_{2|x|}^{\infty} \int_{S^{N-1}} \frac{d}{dr} \left( r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha - \frac{N}{2}} \right) \cos\left(r - \frac{(N-1)\pi}{4}\right) dS dr \right| \\ &= K_{2,1} + K_{2,2}. \end{aligned}$$

Clearly  $K_{2,1} \leq c|x|^{2\alpha - \frac{N+1}{2}}$ . Using (4.22) we have

$$\left| \frac{d}{dr} \left( r^{\frac{N-1}{2}} (r^2 + \rho^2 - 2r\rho\langle\sigma, \theta\rangle)^{\alpha - \frac{N}{2}} \right) \right| \leq cr^{2\alpha - \frac{N+3}{2}},$$

which yields  $K_{2,2} \leq c|x|^{2\alpha - \frac{N+1}{2}}$  by integration.

For estimating  $K_1$  we make the change of variable already but with a totally different point of view,

$$\begin{aligned} K_1 &= |x|^{2\alpha - \frac{N-1}{2}} \int_{\frac{1}{|x|} < |z| < 2} |\mathbf{e}_N - z|^{2\alpha-N} |z|^{\frac{1-N}{2}} \sin\left(|x||z| - \frac{(N-1)\pi}{4}\right) dz \\ &= |x|^{2\alpha - \frac{N-1}{2}} \int_{B_2} |\mathbf{e}_N - z|^{2\alpha-N} |z|^{\frac{1-N}{2}} \sin\left(|x||z| - \frac{(N-1)\pi}{4}\right) dz + O(|x|^{2\alpha-N}). \end{aligned} \tag{4.24}$$

We estimate the main term by real interpolation theory. Let  $T_\rho$  be the mapping from  $L^1(B_2)$  to  $\mathbb{R}$  defined by

$$f \mapsto T_\rho(f) := \int_{B_2} f(z) \sin\left(\rho|z| - \frac{(N-1)\pi}{4}\right) dz, \tag{4.25}$$

clearly

$$|T_\rho(f)| \leq \|f\|_{L^1(B_2)}. \tag{4.26}$$

Also, if  $f \in W^{1,1}(B_2)$ , then  $r \mapsto f(r, \cdot)$  is continuous from  $W^{1,1}(B_2)$  into  $L^1(S^{N-1})$ , therefore

$$\begin{aligned} \int_{B_2} f(z) \sin\left(\rho|z| - \frac{(N-1)\pi}{4}\right) dz &= -\frac{\cos\left(2\rho - \frac{(N-1)\pi}{4}\right)}{\rho} \int_{|z|=2} f(2, \cdot) dS \\ &\quad + \frac{1}{\rho} \int_{B_2} \langle \nabla f, \frac{z}{2} \rangle \cos\left(\rho|z| - \frac{(N-1)\pi}{4}\right) dz. \end{aligned}$$

This implies

$$|T_\rho(f)| \leq \frac{c}{\rho} \|f\|_{W^{1,1}(B_2)}, \quad (4.27)$$

where  $c > 0$  is independent of  $\rho$ . By Lions-Petree's real interpolation method theorem [25] and because

$$W^{s,1}(B_2) = [W^{1,1}(B_2), L^1(B_2)]_{s,1},$$

for  $0 < s < 1$ , with the notations therein, we derive

$$|T_\rho(f)| \leq \frac{c}{\rho^s} \|f\|_{W^{s,1}(B_2)}. \quad (4.28)$$

The function  $z \mapsto F(z) := |\mathbf{e}_N - z|^{2\alpha-N} |z|^{\frac{1-N}{2}}$  belongs to  $W^{1,1}(B_2)$  for any  $\alpha > \frac{1}{2}$ , to  $W^{2\alpha-\epsilon,1}(B_2)$  for any  $\epsilon > 0$  if  $\alpha \leq \frac{1}{2}$ . This implies that for any  $\epsilon > 0$  there exists  $C_\epsilon > 0$  such that

$$|K_1| \leq C_\epsilon |x|^{2\alpha+\epsilon-\frac{N+1}{2}} \quad \text{for all } |x| \geq 2, \quad (4.29)$$

and finally we prove

$$|E_\alpha * \Phi_1|(x) \leq c' |x|^{2\alpha+\epsilon-\frac{N+1}{2}} \quad \text{if } |x| \geq 1. \quad (4.30)$$

Combining (4.19), (4.23) and (4.30), we obtain the estimate (4.12), valid for any  $\epsilon > 0$  if  $\alpha \leq \frac{1}{2}$  and  $|x| \geq 1$ , which is the claim.  $\square$

*Proof of Theorem 1.8.* Let

$$u_*(x) = \int_0^1 \mathcal{P}_{\ln}(t, x) dt = \int_0^1 \mathcal{P}_0(t) |x|^{2t-N} dt \quad \text{for } x \neq 0. \quad (4.31)$$

Clearly  $u_*$  is radially symmetric and decreasing with respect to  $|x|$ . Since  $\mathcal{P}_{\ln}$  satisfies (1.9), we integrate in  $t$  over  $(0, 1)$  and get

$$\begin{aligned} \mathcal{L}_\Delta u_*(x) &= \int_0^1 \mathcal{L}_\Delta \mathcal{P}_{\ln}(t, x) dt = - \int_0^1 \partial_t \mathcal{P}_{\ln}(t, x) dt \\ &= \mathcal{P}_{\ln}(0, x) - \mathcal{P}_{\ln}(1, x) = - \frac{\Gamma(\frac{N}{2} - 1)}{\Gamma(1) \sqrt{2\pi}^{N/2}} |x|^{2-N}. \end{aligned} \quad (4.32)$$

It follows from [33, Theorem 5] that  $-\mathcal{L}_\Delta u_*(\cdot)$  is the fundamental solution of  $-\Delta$  in  $\mathbb{R}^N$  expressed by  $E_1(x) = c_{N,1} |x|^{2-N}$ .

*Step 1: Properties of the function  $u_*$ .*

We have that for  $0 < |x| < 1$ ,

$$\begin{aligned}
u_*(x) &= \int_0^1 \mathcal{P}_0(t) e^{(2t-N) \ln |x|} dt \\
&= \left[ \frac{1}{2 \ln |x|} \mathcal{P}_0(t) |x|^{2t-N} \right]_{t=0}^{t=1} - \frac{1}{2 \ln |x|} \int_0^1 \mathcal{P}'_0(t) e^{(2t-N) \ln |x|} dt \\
&= \left[ \frac{1}{2 \ln |x|} \mathcal{P}_0(t) |x|^{2t-N} \right]_{t=0}^{t=1} - \left[ \frac{1}{(2 \ln |x|)^2} \mathcal{P}'_0(t) |x|^{2t-N} \right]_{t=0}^{t=1} \\
&\quad + \frac{1}{(2 \ln |x|)^2} \int_0^1 \mathcal{P}''_0(t) e^{(2t-N) \ln |x|} dt \\
&= \frac{\mathcal{P}_0(1)}{2 \ln |x|} |x|^{2-N} + \frac{\mathcal{P}'_0(0)}{(2 \ln |x|)^2} |x|^{-N} - \frac{\mathcal{P}'_0(1)}{(2 \ln |x|)^2} |x|^{2-N} \\
&\quad + \frac{1}{(2 \ln |x|)^2} \int_0^1 \mathcal{P}''_0(t) e^{(2t-N) \ln |x|} dt.
\end{aligned}$$

From Lemma 2.4,  $\mathcal{P}'_0(0) = \pi^{-N/2} \Gamma(\frac{N}{2})$  and  $\mathcal{P}''_0$  is bounded on  $[0, 1]$ , hence

$$\left| \frac{1}{(2 \ln |x|)^2} \int_0^1 \mathcal{P}''_0(t) e^{(2t-N) \ln |x|} dt \right| \leq \frac{c}{(\ln |x|)^2} \int_0^1 e^{(2t-N) \ln |x|} dt \leq c \frac{|x|^{-N}}{(\ln |x|)^3}.$$

Therefore, we obtain that for  $0 < |x| < \frac{1}{e}$ ,

$$\left| u_*(x) - \frac{\Gamma(\frac{N}{2})}{4\pi^{N/2}} \frac{|x|^{-N}}{(\ln |x|)^2} \right| \leq c \frac{|x|^{-N}}{(-\ln |x|)^3}. \quad (4.33)$$

It is worth noting that  $|x| = 1$ , we have

$$u_*(x) = \int_0^1 \mathcal{P}_0(t) dt.$$

For the asymptotic behavior of  $u_*$  when  $|x| \rightarrow \infty$ , we have that for  $|x| > 1$ ,

$$\begin{aligned}
0 < u_*(x) &\leq c \int_0^1 t e^{(2t-N) \ln |x|} dt \\
&= c \left( \left[ \frac{1}{2 \ln |x|} t |x|^{2t-N} \right]_{t=0}^{t=1} - \left[ \frac{1}{(2 \ln |x|)^2} |x|^{2t-N} \right]_{t=0}^{t=1} \right) \\
&\leq \frac{c}{\ln |x|} |x|^{2-N} + \frac{c}{(\ln |x|)^2} |x|^{-N}.
\end{aligned} \quad (4.34)$$

Since  $\mathcal{P}_{\ln}$  is a weak solution of (1.21), then taking  $T' = 1$  and  $\varphi(t, \cdot) = \phi \in C_c^1(\mathbb{R}^N)$  in (1.18) we infer that

$$\begin{aligned}
0 &= \int_0^{\frac{1}{4}} \int_{\mathbb{R}^N} (-\partial_t \varphi + \mathcal{L}_\Delta \varphi) \mathcal{P}_{\ln} dx dt - \varphi(0, 0) + \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(1, x) \varphi(1, x) dx \\
&= \int_{\mathbb{R}^N} u_* \mathcal{L}_\Delta \phi dx - \phi(0) + \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(1, x) \phi dx.
\end{aligned}$$

As a consequence,  $u_*$  is a weak solution of

$$\mathcal{L}_\Delta u_* = \delta_0 - \mathcal{P}_{\ln}(1, \cdot) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad (4.35)$$

and a classical solution of

$$\mathcal{L}_\Delta u_* = -\mathcal{P}_{\ln}(1, \cdot) \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (4.36)$$

*Step 2: There exists  $v_1$  such that*

$$\mathcal{L}_\Delta v_1 = \mathcal{P}_{\ln}(1, \cdot) \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (4.37)$$

Given  $f \in L^1(\mathbb{R}^N, (1 + |x|)^{2T-N} dx)$ , denote

$$u_f(x) = \int_0^1 \mathcal{P}_{\ln}(t, \cdot) * f(t, x) dt.$$

Then

$$\begin{aligned} \mathcal{L}_\Delta u_f(x) &= \int_0^1 \int_{\mathbb{R}^N} \mathcal{L}_\Delta \mathcal{P}_{\ln}(t, x-y) f(y) dy dt \\ &= - \int_0^1 \partial_t \int_{\mathbb{R}^N} \mathcal{P}_{\ln}(t, x-y) f(y) dy dt = f(x) - \mathcal{P}_{\ln}(1, \cdot) * f(x). \end{aligned}$$

Now we aim to choose  $f$  such that

$$f(x) - \mathcal{P}_{\ln}(1, \cdot) * f(x) = \mathcal{P}_{\ln}(1, x).$$

Since  $\mathcal{P}_{\ln}(1, \cdot)$  is the fundamental solution of  $-\Delta$ ,  $f$  satisfies

$$-\Delta f - f = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (4.38)$$

Therefore  $\Phi_1$  is a solution of (4.38) and

$$v_1(x) = \int_0^1 (\mathcal{P}_{\ln}(t, \cdot) * \Phi_1)(x, t) dt = \int_0^1 \mathcal{P}_0(t) (E_t * \Phi_1)(x) dt, \quad (4.39)$$

satisfies (4.37).

*Step 3: Asymptotic estimates of  $v_1$*

*3-(i): Estimate near  $x = 0$ .* Since for  $0 \leq t \leq 1$  we have that  $\frac{1}{c}t \leq \mathcal{P}_0(t) \leq ct$ , we deduce from Lemma 4.2 that if  $|x| < \frac{1}{e}$  we have when  $N \geq 4$ ,

$$|v_1(x)| \leq c|x|^{2-N} \int_0^1 t|x|^{2t} dt = c|x|^{2-N} \left( \frac{|x|^2}{2 \ln |x|} - \frac{|x|^2}{4 \ln^2 |x|} + \frac{1}{4 \ln^2 |x|} \right) \leq \frac{c|x|^{2-N}}{4 \ln^2 |x|}, \quad (4.40)$$

and when  $N = 3$

$$|v_1(x)| \leq c|x|^{-1} \int_0^{\frac{1}{2}} t|x|^{2t} dt + \frac{c}{2} \leq \frac{c}{4|x| \ln^2 |x|} + \frac{c}{2}. \quad (4.41)$$

*3-(i): Estimate as  $|x| \rightarrow \infty$ .* If  $|x| > 2$  we have for any  $\epsilon > 0$

$$|v_1(x)| \leq C_\epsilon |x|^{-\frac{N+1}{2} + \epsilon} \int_0^{\frac{1}{2}} t|x|^{2t} dt + |x|^{-\frac{N+1}{2}} \int_{\frac{1}{2}}^1 t|x|^{2t} dt \leq \frac{c'}{|x|^{\frac{N-3}{2}} \ln |x|}. \quad (4.42)$$



*Step 4: Asymptotics of the fundamental solution of  $\mathcal{L}_\Delta$*

From (1.25)-(4.37) the function  $\Phi_{\ln} = u^* + v_1$  satisfies

$$\mathcal{L}_\Delta \Phi_{\ln} = \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Using (4.33), it follows from (4.40)-(4.41) that for  $0 < |x| \leq e^{-1}$  there holds for some  $c_1 > 0$

$$\left| \Phi_{\ln}(x) - \frac{\Gamma(\frac{N}{2})}{4\pi^{N/2}} \frac{|x|^{-N}}{(\ln|x|)^2} \right| \leq \frac{c|x|^{-N}}{\ln^3|x|}. \quad (4.43)$$

Using (4.42) we obtain, for  $|x|$  large enough and some positive constant  $c_2 > 0$ ,

$$|\Phi_{\ln}(x)| \leq \frac{c_2|x|^{\frac{3-N}{2}}}{\ln|x|}. \quad (4.44)$$

This ends the proof. □

**Remark 4.3.** *It is important to notice from (4.43) that  $\Phi_{\ln} \in L^1_{loc}(\mathbb{R}^N)$ .*

## 4.2 Existence in bounded domain

Let  $\Omega$  be a bounded Lipschitz domain and  $\mathbb{H}(\Omega)$  denote the space of all measurable functions  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  with  $u \equiv 0$  in  $\mathbb{R}^N \setminus \Omega$  and

$$\iint_{\substack{x,y \in \mathbb{R}^N \\ |x-y| \leq 1}} \frac{(u(x) - u(y))^2}{|x-y|^N} dx dy < +\infty.$$

Note that  $\mathbb{H}(\Omega)$  is a Hilbert space under the inner product

$$\mathcal{E}(u, w) = \frac{c_N}{2} \iint_{\substack{x,y \in \mathbb{R}^N \\ |x-y| \leq 1}} \frac{(u(x) - u(y))(w(x) - w(y))}{|x-y|^N} dx dy,$$

and with the associated norm  $\|u\|_{\mathbb{H}(\Omega)} = \sqrt{\mathcal{E}(u, u)}$ . By [12, Theorem 2.1], *the embedding:  $\mathbb{H}(\Omega) \hookrightarrow L^2(\Omega)$  is compact.* Let

$$\mathcal{E}_L : \mathbb{H}(\Omega) \times \mathbb{H}(\Omega) \rightarrow \mathbb{R}, \quad \mathcal{E}_L(u, w) = \mathcal{E}(u, w) - \int_{\mathbb{R}^N} (\mathbf{j} * u - \rho_N u) w dx.$$

where

$$\mathbf{j} : \mathbb{R}^N \rightarrow \mathbb{R}, \quad \mathbf{j}(z) = c_N 1_{\mathbb{R}^N \setminus B_1}(z) |z|^{-N}.$$

For additional properties of  $\mathcal{E}_L$  see Sections 3, 4 in [11]. If  $\lambda_1^L(\Omega) > 0$ , there is some  $c > 0$  such that

$$\mathcal{E}_L(u, u) \geq c\mathcal{E}(u, u).$$

We recall that  $\eta_0 \in C_c^\infty(\mathbb{R}^N)$  is the cut-off function defined in the proof of Theorem 3.1 and  $\eta(x) = \eta_0(|x|)$

**Lemma 4.4.** *The function  $\mathcal{L}_\Delta(\eta\Phi_{\ln})$  is uniformly bounded in  $\mathbb{R}^N \setminus \{0\}$ .*

*Proof.* Set  $v = \eta\Phi_{\ln}$  in  $\mathbb{R}^N \setminus \{0\}$ , then

$$\begin{aligned}
\mathcal{L}_\Delta v(x) &= c_N \int_{B_1(x)} \frac{v(x) - v(y)}{|x - y|^N} dy - c_N \int_{B_1^c(x)} \frac{v(y)}{|x - y|^N} dy + \rho_N v(x) \\
&= c_N \eta(x) \int_{B_1(x)} \frac{\Phi_{\ln}(x) - \Phi_{\ln}(y)}{|x - y|^N} dy + c_N \int_{B_1(x)} (\eta(x) - \eta(y)) \frac{\Phi_{\ln}(y)}{|x - y|^N} dy \\
&\quad - c_N \int_{B_1^c(x)} \frac{\eta(y)\Phi_{\ln}(y)}{|x - y|^N} dy + \rho_N v(x) \\
&= c_N \int_{\mathbb{R}^N} \frac{(\eta(x) - \eta(y))\Phi_{\ln}(y)}{|x - y|^N} dy.
\end{aligned} \tag{4.45}$$

Then

$$\begin{aligned}
\mathcal{L}_\Delta v(x) &= c_N \int_{B_{\frac{1}{2}}} \frac{(\eta(x) - 1)\Phi_{\ln}(y)}{|x - y|^N} dy + c_N \int_{B_1^c} \frac{\eta(x)\Phi_{\ln}(y)}{|x - y|^N} dy + c_N \int_{B_1 \setminus B_{\frac{1}{2}}} \frac{(\eta(x) - \eta(y))\Phi_{\ln}(y)}{|x - y|^N} dy \\
&= A_1(x) + A_2(x) + A_3(x).
\end{aligned}$$

(i) If  $y \in B_1 \setminus B_{\frac{1}{2}}$ , the function  $\Phi_{\ln}$  is bounded. If  $x \in B_2$ , then  $B_1 \setminus B_{\frac{1}{2}} \subset B_3(x)$ , hence

$$|A_3(x)| \leq c_N \|\Phi_{\ln}\|_{L^\infty(B_1 \setminus B_{\frac{1}{2}})} \|\nabla\eta\|_{L^\infty} \int_{B_3(x)} |x - y|^{1-N} dy \leq 3c_N \omega_N \|\Phi_{\ln}\|_{L^\infty(B_1 \setminus B_{\frac{1}{2}})} \|\nabla\eta\|_{L^\infty}.$$

We recall that  $\omega_N$  is the  $(N-1)$ -volume of the unit sphere  $S^{N-1}$ . If  $x \in B_2^c$ , then

$$\left| \frac{(\eta(x) - \eta(y))\Phi_{\ln}(y)}{|x - y|^N} \right| \leq 2\|\Phi_{\ln}\|_{L^\infty(B_1 \setminus B_{\frac{1}{2}})}.$$

Combining these two estimates we obtain

$$|A_3(x)| \leq c_1 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}$$

for some  $c_1 > 0$ .

(ii) If  $y \in B_1^c$ , we write

$$\begin{aligned}
A_2(x) &= c_N \int_{B_1^c} \frac{\eta(x)\Phi_{\ln}(y)}{|x - y|^N} dy = c_N \int_{B_2 \setminus B_1} \frac{\eta(x)\Phi_{\ln}(y)}{|x - y|^N} dy + c_N \int_{B_2^c} \frac{\eta(x)\Phi_{\ln}(y)}{|x - y|^N} dy \\
&= A_{21}(x) + A_{22}(x).
\end{aligned}$$

Then  $A_{21}(x) = 0$  if  $|x| > 1$  and

$$A_{21}(x) \leq c_N \|\nabla\eta\|_{L^\infty} \|\Phi_{\ln}\|_{L^\infty(B_2 \setminus B_1)} \int_{B_2 \setminus B_1} \frac{dy}{|x - y|^{N-1}} \quad \text{for all } x \in B_1.$$

This implies that  $B_1(x)$  is uniformly bounded. Concerning  $B_2(x)$  which vanishes also in  $B_1^c$ , we have

$$A_{22}(x) \leq c_N \int_{B_2^c} \frac{|\Phi_{\ln}(y)|}{(|y| - 1)^N} dy \quad \text{for all } x \in B_1.$$

By (4.44) the integral is convergent and we obtain Combining these two estimates we obtain that

$$|A_2(x)| \leq c_2 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\},$$

for some  $c_2 > 0$ .

(iii) If  $x \in B_{\frac{1}{2}}$ ,  $A_1(x) = 0$ . If  $|x| \geq 1$ ,

$$|A_1(x)| \leq \frac{c_N}{(|x| - \frac{1}{2})^N} \int_{B_{\frac{1}{2}}} |\Phi_{\ln}(y)| dy,$$

a quantity which is uniformly bounded. If  $\frac{1}{2} < |x| < 1$  we write since  $\eta(y) = 1$  therein

$$\begin{aligned} |A_1(x)| &\leq c_N \int_{B_{\frac{1}{2}} \setminus B_{\frac{1}{3}}} \frac{|\eta(y) - \eta(x)| \phi_{\ln}(y)}{|x - y|^N} dy + c_N \int_{B_{\frac{1}{3}}} \frac{|\eta(y) - \eta(x)| \phi_{\ln}(y)}{|x - y|^N} dy \\ &\leq c_N \|\Phi_{\ln}\|_{L^\infty(B_{\frac{1}{2}} \setminus B_{\frac{1}{3}})} \|\nabla \eta\|_{L^\infty} \int_{B_{\frac{1}{2}} \setminus B_{\frac{1}{3}}} \frac{1}{|x - y|^{N-1}} dy + c_N 6^N \int_{B_{\frac{1}{3}}} |\phi_{\ln}(y)| dy \\ &\leq c_N \|\Phi_{\ln}\|_{L^\infty(B_{\frac{1}{2}} \setminus B_{\frac{1}{3}})} \|\nabla \eta\|_{L^\infty} \int_{B_{\frac{3}{2}}} \frac{dz}{|z|^{N-1}} + c_N 6^N \|\phi_{\ln}\|_{L^1(B_{\frac{1}{3}})}. \end{aligned}$$

Hence it follows that for some  $c_3 > 0$

$$|A_1(x)| \leq c_2 \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Combining the three estimates we obtain the claim.  $\square$

**Lemma 4.5.** *Set  $h = \mathcal{L}_\Delta(\eta\Phi_{\ln})\mathbf{1}_\Omega$ . If  $0 \notin \sigma(\mathcal{L}_\Delta, \Omega)$  there exists a unique function  $w \in \mathbb{H} \cap C(\overline{\Omega})$  such that*

$$\begin{cases} \mathcal{L}_\Delta w = h & \text{in } \Omega, \\ w = 0 & \text{in } \Omega^c. \end{cases} \quad (4.46)$$

*Proof.* The function  $h$  is bounded hence it belongs to  $L^2(\Omega)$ . By [12, Theorem 2.1] the embedding of  $\mathbb{H}(\Omega)$  into  $L^2(\Omega)$  is compact and by [11, Theorem 3.1]  $C_c^\infty(\Omega)$  is dense in  $\mathbb{H}(\Omega)$ . Then the operator  $\mathcal{L}_\Delta$  is a Fredholm operator and since 0 is not an eigenvalue, there exists a unique  $w \in \mathbb{H}(\Omega)$  such that (4.46) holds in the weak sense, and by density such that

$$\mathcal{E}_L(w, \zeta) = \int_\Omega h \zeta dx \quad \text{for all } \zeta \in \mathbb{H}(\Omega).$$

By [11, Theorem 11] the function  $w$  is continuous and it satisfies

$$|w(x)| = O\left(\frac{1}{|\ln \rho(x)|^\tau}\right) \quad \text{for all } \tau \in \left(0, \frac{1}{2}\right), \quad (4.47)$$

near  $\partial\Omega$  where we recall that  $\rho(x) = \text{dist}(x, \partial\Omega)$ . This ends the proof.  $\square$

*Proof of Theorem 1.9.* Set  $\Phi_{\ln}^\Omega = \eta\Phi_{\ln} - w$ . Then  $\Phi_{\ln}^\Omega$  vanishes in  $\Omega^c$  and satisfies

$$\mathcal{L}_\Delta \Phi_{\ln}^\Omega = 0 \quad \text{in } \Omega \setminus \{0\}.$$

Estimates (5.1) and (1.28) follow from (4.43) and (4.47). Now let  $\zeta \in C_c^\infty(\Omega)$ , then

$$\int_{\mathbb{R}^N} \Phi_{\ln}^\Omega \mathcal{L}_\Delta \zeta dx = \int_{\mathbb{R}^N} \eta \Phi_{\ln} \mathcal{L}_\Delta \zeta dx - \mathcal{E}_L(w, \zeta) = A_4 - A_5.$$

We have that

$$\begin{aligned}
A_4 &= c_N \int_{\mathbb{R}^N} \eta(x) \Phi_{\ln}(x) \int_{B_1(x)} \frac{\zeta(x) - \zeta(y)}{|x-y|^N} dy dx - c_N \int_{\mathbb{R}^N} \eta(x) \Phi_{\ln}(x) \int_{B_1^c(x)} \frac{\zeta(y)}{|x-y|^N} dy dx \\
&\quad + \rho_N \int_{\mathbb{R}^N} \eta(x) \Phi_{\ln}(x) \zeta(x) dx \\
&= c_N \int_{\mathbb{R}^N} \Phi_{\ln}(x) \int_{B_1(x)} \frac{(\eta\zeta)(x) - (\eta\zeta)(y)}{|x-y|^N} dy dx - c_N \int_{\mathbb{R}^N} \Phi_{\ln}(x) \int_{B_1^c(x)} \frac{(\eta\zeta)(y)}{|x-y|^N} dy dx \\
&\quad - c_N \int_{\mathbb{R}^N} \Phi_{\ln}(x) \int_{\mathbb{R}^N} \frac{\eta(x) - \eta(y)}{|x-y|^N} \zeta(y) dy dx + \rho_N \int_{\mathbb{R}^N} \eta(x) \Phi_{\ln}(x) \zeta(x) dx \\
&= \zeta(0) - c_N \int_{\mathbb{R}^N} \Phi_{\ln}(x) \int_{\mathbb{R}^N} \frac{\eta(x) - \eta(y)}{|x-y|^N} \zeta(y) dy dx,
\end{aligned}$$

and, by (4.45),

$$\begin{aligned}
A_5 &= \int_{\mathbb{R}^N} \zeta \mathcal{L}_\Delta(\eta \Phi_{\ln}) dx = c_N \int_{\mathbb{R}^N} \zeta(x) \int_{\mathbb{R}^N} \frac{\eta(x) - \eta(y)}{|x-y|^N} \Phi_{\ln}(y) dy dx \\
&= -c_N \int_{\mathbb{R}^N} \Phi_{\ln}(x) \int_{\mathbb{R}^N} \frac{\eta(x) - \eta(y)}{|x-y|^N} \zeta(y) dy dx.
\end{aligned}$$

This implies that  $\mathcal{L}_\Delta \Phi_{\ln}^\Omega = \delta_0$  in  $\mathcal{D}'(\Omega)$ . □

## 5 Regularity estimates

This section is devoted to the proof of some regularity estimates. We decompose the fundamental solution  $\Phi_{\ln}$  of  $\mathcal{L}_\Delta$  in  $\mathbb{R}^N$  under the form

$$\Phi_{\ln} = \Phi_{\ln} \mathbf{1}_{B_1} + \Phi_{\ln} \mathbf{1}_{B_1^c} = \Phi_{\ln,1} + \Phi_{\ln,2}. \quad (5.1)$$

**Lemma 5.1.** *Let  $N \geq 3$ . If  $f \in L^\infty(\mathbb{R}^N)$  then  $\Phi_{\ln,1} * f$  is uniformly continuous and there exists  $c > 0$  such that*

$$\|\Phi_{\ln,1} * f\|_\infty \leq c \|f\|_\infty. \quad (5.2)$$

*Proof.* By estimate (1.23)  $\Phi_{\ln,1} \in L^1(\mathbb{R}^N)$ . Estimate (5.2) follows and classically  $\Phi_{\ln,1} * f$  is uniformly continuous. □

Similarly we derive from (1.24),

**Lemma 5.2.** *Let  $N \geq 3$ , then  $\Phi_{\ln,2} \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ . If  $f \in L^{q'}(\mathbb{R}^N)$  for all  $q' \leq \frac{2N}{N+2}$ , then  $\Phi_{\ln,2} * f$  is uniformly continuous and*

$$\|\Phi_{\ln,2} * f\|_\infty \leq c \|f\|_{\frac{2N}{N+2}}. \quad (5.3)$$

In the next result we prove some regularity estimate of extended Marcinkiewicz type for  $\Phi_{\ln}$ .

**Proposition 5.3.** *Assume  $N \geq 3$ , then for some  $c = c(N) > 0$  there holds*

$$E_{\ln,1}(\lambda) := \{x \in B_1 : |\Phi_{\ln,1}(x)| > \lambda\} \subset \left\{x \in B_1 : |x| \leq c(\lambda \ln^2 \lambda)^{-\frac{1}{N}}\right\} \quad \text{for all } \lambda > 1, \quad (5.4)$$

hence

$$\text{meas}\{E_{\ln,1}(\lambda)\} \leq \frac{c}{\lambda \ln^2 \lambda}. \quad (5.5)$$

Furthermore, if  $N > 3$ ,

$$E_{\ln,2}(\lambda) := \{x \in B_1^c : |\Phi_{\ln,2}(x)| > \lambda\} \subset \left\{ x \in B_1^c : |x| \leq \frac{c}{\lambda^{\frac{2}{N-3}} \ln\left(\frac{1}{\lambda}\right)} \right\} \quad \text{for all } 0 < \lambda < 1 \quad (5.6)$$

which implies

$$\text{meas} \{E_{\ln,2}(\lambda)\} \leq \frac{c}{\lambda^{\frac{2N}{N-3}} |\ln \lambda|^N}, \quad (5.7)$$

and if  $N = 3$ ,

$$E_{\ln,2}(\lambda) := \{x \in B_1^c : |\Phi_{\ln,2}(x)| > \lambda\} \subset \left\{ x \in B_1^c : |x| \leq e^{\frac{c}{\lambda}-1} \right\} \quad \text{for all } 0 < \lambda < 1, \quad (5.8)$$

which yields

$$\text{meas} \{E_{\ln,2}(\lambda)\} \leq c_N e^{\frac{Nc}{\lambda}}. \quad (5.9)$$

*Proof.* If  $x \in E_{\ln,1}(\lambda)$ , then from (1.23),

$$(1 + \ln^2 |x|) |x|^N \leq \frac{c}{\lambda},$$

which implies in particular  $|x|^N \leq \frac{c}{\lambda}$ , thus  $1 + \ln^2 |x| \geq 1 + \frac{1}{N^2} \ln^2\left(\frac{c}{\lambda}\right)$ , which yields

$$|x|^N \leq \frac{cN^2}{\lambda(N^2 + \ln^2\left(\frac{c}{\lambda}\right))},$$

from what estimates inequalities (5.4) and (5.5) follow.

Next, if  $x \in E_{\ln,2}(\lambda)$ , then by (1.24),

$$\lambda \leq c_1 \frac{|x|^{\frac{3-N}{2}}}{1 + \ln |x|},$$

which implies  $(1 + \ln |x|) |x|^{\frac{N-3}{2}} \leq \frac{c_1}{\lambda}$ . If  $N = 3$  we deduce that (5.8) and (5.9) hold. If  $N \geq 4$ , we have in particular  $1 \leq |x| \leq \left(\frac{c_1}{\lambda}\right)^{\frac{2}{N-3}}$ . From the monotonicity properties of the function

$$r \mapsto (1 + \ln r) r^{\frac{N-3}{2}} := \mu(r) \quad \text{for } r > 1,$$

we can find some  $\theta > 0$  such that

$$\mu \left( \theta \frac{\left(\frac{c_1}{\lambda}\right)^{\frac{2}{N-3}}}{\ln\left(\frac{c_1}{\lambda}\right)} \right) \geq \frac{c_1}{\lambda}.$$

This implies

$$E_{\ln,2}(\lambda) \subset \left\{ x : 1 \leq |x| \leq \theta \frac{\left(\frac{c_1}{\lambda}\right)^{\frac{2}{N-3}}}{\ln\left(\frac{c_1}{\lambda}\right)} \right\}.$$

Hence we obtain (5.6) and (5.7).  $\square$

In the next result we derive estimate of  $\Phi_{\ln}$  in Orlicz classes  $L_M$  for some suitable functions  $M$ .

**Theorem 5.4.** Let  $M : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a  $C^1$  increasing function such that  $M(0) = 0$ .

1- If  $M$  satisfies

$$\int_2^\infty \frac{M(t)}{t^2 \ln^2 t} dt < \infty, \quad (5.10)$$

then  $M(|\phi_{\ln,1}|) \in L^1(B_1)$ .

2- If  $N > 3$  and  $M$  satisfies

$$\int_0^{\frac{1}{2}} \frac{M(t)}{t^{\frac{3(N-1)}{N-3}} |\ln t|^N} dt < \infty, \quad (5.11)$$

then  $M(|\phi_{\ln,2}|) \in L^1(B_1^c)$ .

3- If  $N = 3$  and  $M$  satisfies

$$\int_0^{\frac{1}{2}} \frac{M(t) e^{\frac{Nc}{\lambda}}}{\lambda^2} < \infty, \quad (5.12)$$

then  $M(|\phi_{\ln,2}|) \in L^1(B_1^c)$ .

*Proof.* 1- Set  $e_{\ln,1}(\lambda) = \text{meas}(E_{\ln,1}(\lambda))$  and let  $\alpha \in (0, 1)$  such that  $\phi_{\ln,1}(x) \geq 2$  if  $0 < |x| \leq \alpha$ . Then

$$\begin{aligned} \int_{B_1} M(|\phi_{\ln,1}(x)|) dx &= \int_{B_\alpha} M(|\phi_{\ln,1}(x)|) dx + \int_{B_1 \setminus B_\alpha} M(|\phi_{\ln,1}(x)|) dx \\ &\leq - \int_2^\infty M(t) de_{\ln,1}(t) + \int_{B_1 \setminus B_\alpha} M(|\phi_{\ln,1}(x)|) dx. \end{aligned}$$

By standard results on Stieljes integral

$$\begin{aligned} - \int_2^{t_n} M(t) de_{\ln,1}(t) &= - \left[ M(t) e_{\ln,1}(t) \right]_2^{t_n} + \int_2^{t_n} M'(t) e_{\ln,1}(t) dt \\ &\leq - \left[ M(t) e_{\ln,1}(t) \right]_2^{t_n} + \int_2^{t_n} \frac{M'(t)}{t \ln^2 t} dt \\ &\leq - \left[ M(t) e_{\ln,1}(t) \right]_2^{t_n} + \left[ \frac{M(t)}{t \ln^2 t} \right]_2^{t_n} + \int_2^{t_n} \frac{M(t)}{t^2 \ln^2 t} \left( 1 + \frac{2}{\ln t} \right) dt, \end{aligned}$$

where  $\{t_n\}$  is a sequence tending to infinity and such that

$$\lim_{n \rightarrow \infty} \int_2^{t_n} \frac{M(t_n)}{t_n \ln^2 t_n} dt = 0.$$

Hence

$$\lim_{n \rightarrow \infty} \int_2^{t_n} M(t) de_{\ln,1}(t) \leq \left( 1 + \frac{2}{\ln 2} \right) \int_2^\infty \frac{M(t)}{t^2 \ln^2 t} dt < \infty.$$

This implies 1. The proof of 2 and 3 is similar.  $\square$

**Lemma 5.5.** Let  $N \geq 3$ , then there holds

$$|\nabla \Phi_1(x)| \leq c \max \left\{ |x|^{1-N}, |x|^{-\frac{N-1}{2}} \right\} \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}. \quad (5.13)$$

More precisely, since  $\Phi_1$  is radial

$$\Phi_1'(r) = \begin{cases} cr^{1-N}(1 + o(1)) & \text{as } r \rightarrow 0, \\ c \frac{\sin\left(r - \frac{(N+1)\pi}{4}\right)}{r^{\frac{N-1}{2}}} (1 + O\left(\frac{1}{r}\right)) & \text{as } r \rightarrow \infty. \end{cases} \quad (5.14)$$

*Proof.* If  $\Phi_1(x) = \Phi_1(|x|)$  is the solution of the Helmholtz equation in  $\mathbb{R}^N$ , then  $Z = \Phi_1'$  satisfies

$$Z'' + \frac{N-1}{r}Z' + \left(1 - \frac{N-1}{r^2}\right)Z = 0 \quad \text{on } (0, \infty).$$

Set  $z(r) = r^{\frac{N}{2}-1}Z(r)$ , we have that

$$z'' + \frac{1}{r}z' + \left(1 - \frac{N^2}{4r^2}\right)z = 0 \quad \text{on } (0, \infty),$$

which is a classical Bessel equation. Using the asymptotic expressions of Hankel functions we obtain

$$Z(r) = cr^{1-N}(1 + o(1)) \quad \text{as } r \rightarrow 0,$$

and

$$Z(r) = c \frac{\sin\left(r - \frac{(N+1)\pi}{4}\right)}{r^{\frac{N-1}{2}}} \left(1 + O\left(\frac{1}{r}\right)\right) \quad \text{as } r \rightarrow \infty.$$

Thus

$$\Phi_1'(r) = \begin{cases} cr^{1-N}(1 + o(1)) & \text{as } r \rightarrow 0, \\ c \frac{\sin\left(r - \frac{(N+1)\pi}{4}\right)}{r^{\frac{N-1}{2}}} \left(1 + O\left(\frac{1}{r}\right)\right) & \text{as } r \rightarrow \infty, \end{cases} \quad (5.15)$$

which ends the proof.  $\square$

**Theorem 5.6.** *Let  $N \geq 4$ , then there exists  $c > 0$  such that*

$$|\nabla\Phi_{\ln}(x)| \leq c \max\left\{\frac{|x|^{-N-1}}{1 + \ln^2|x|}, \frac{1}{\sqrt{1 + \ln^2|x|}}\right\} \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}. \quad (5.16)$$

*Proof.* We recall that

$$\Phi_{\ln}(x) = u^*(x) + v_1(x) = \int_0^1 \mathcal{P}_0(t)|x|^{2t-N} dt + \int_0^1 \mathcal{P}_0(t)(|\cdot|^{2t-N} * \Phi_1(\cdot))(x) dt.$$

*Step 1: Estimate of  $\nabla u^*$ .* There holds

$$\begin{aligned} |\nabla u^*(x)| &= \int_0^1 \mathcal{P}_0(t)(N-2t)|x|^{2t-N-1} dt \\ &\leq c|x|^{-1-N} \int_0^1 t|x|^{2t} dt \\ &= \frac{c|x|^{1-N}}{2 \ln|x|} - \frac{c|x|^{1-N}}{4 \ln^2|x|} + \frac{cx|^{-1-N}}{4 \ln^2|x|}. \end{aligned}$$

if  $|x| \notin \{0, 1\}$ . One has

$$|\nabla u^*(x)| \leq \frac{cx|^{-1-N}}{4 \ln^2|x|}(1 + o(1)) \quad \text{as } x \rightarrow 0, \quad (5.17)$$

and

$$|\nabla u^*(x)| \leq \frac{cx|^{1-N}}{2 \ln |x|} (1 + o(1)) \quad \text{as } |x| \rightarrow \infty. \quad (5.18)$$

By standard asymptotic expansion near  $|x| = 1$ ,

$$\lim_{|x| \rightarrow 1} |\nabla u^*(x)| \leq c \lim_{|x| \rightarrow 1} \left( \frac{c|x|^{1-N}}{2 \ln |x|} - \frac{c|x|^{1-N}}{4 \ln^2 |x|} + \frac{cx|^{-1-N}}{4 \ln^2 |x|} \right) = c. \quad (5.19)$$

when  $|x| = 1$ . Therefore  $x \mapsto |\nabla u^*(x)|$  is uniformly bounded on  $B_\epsilon^c$  for any  $\epsilon > 0$ .

*Step 2: Estimate of  $\nabla v_1$ .* We write

$$\begin{aligned} v_1(x) &= \int_0^1 \mathcal{P}_0(t) (|\cdot|^{2t-N} * \Phi_1(\cdot))(x) dt \\ &= \int_0^1 \mathcal{P}_0(t) \int_{B_1} |x-y|^{2t-N} \Phi_1(y) dy dt + \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x-y|^{2t-N} \Phi_1(y) dy dt, \end{aligned}$$

thus

$$\nabla v_1(x) = \int_0^1 \mathcal{P}_0(t) (|\cdot|^{2t-N} * \nabla \Phi_1(\cdot))(x) dt.$$

Setting  $\mathbf{e}_y = \frac{y}{|y|}$ , we have

$$\begin{aligned} |\nabla v_1(x)| &\leq c \int_0^1 \int_{B_1} \mathcal{P}_0(t) |x-y|^{2t-N} |y|^{1-N} dy dt + \left| \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x-y|^{2t-N} \langle \nabla \Phi_1(y), \mathbf{e}_y \rangle dy dt \right| \\ &= A_6(x) + A_7(x). \end{aligned}$$

*Step 2-1: Estimate of  $A_6(x)$ .* We have

$$\begin{aligned} A_6(x) &\leq c' |x|^{1-N} \int_0^1 t |x|^{2t} \int_{B_{\frac{1}{|x|}}} |\mathbf{e}_N - z|^{2t-N} |z|^{1-N} dz dt \\ &= c' |x|^{1-N} \int_{B_{\frac{1}{|x|}}} |\mathbf{e}_N - z|^{-N} |z|^{1-N} \left( \int_0^1 t |x|^{2t} |\mathbf{e}_N - z|^{2t} dt \right) dz \\ &= c' |x|^{1-N} \int_{B_{\frac{1}{|x|}}} |\mathbf{e}_N - z|^{-N} |z|^{1-N} \left( \frac{|x|^2 |\mathbf{e}_N - z|^2}{2 \ln(|x| |\mathbf{e}_N - z|)} + \frac{1 - |x|^2 |\mathbf{e}_N - z|^2}{4 \ln^2(|x| |\mathbf{e}_N - z|)} \right) dz. \end{aligned}$$

Set  $\zeta = |x|(z - \mathbf{e}_N)$ , then

$$A_6(x) \leq c' |x|^{1-N} \int_{B_1(-|x|\mathbf{e}_N)} |\zeta|^{-N} |\mathbf{e}_N + |x|^{-1}\zeta|^{1-N} \left( \frac{|\zeta|^2}{2 \ln(|\zeta|)} + \frac{1 - |\zeta|^2}{4 \ln^2(|\zeta|)} \right) d\zeta. \quad (5.20)$$

We set  $|\zeta| = 1 + m$ , then by standard expansion we have

$$\frac{|\zeta|^2}{2 \ln(|\zeta|)} + \frac{1 - |\zeta|^2}{4 \ln^2(|\zeta|)} = 2 + O(|\zeta| - 1) \quad \text{as } |\zeta| \rightarrow 1.$$

*Step 2-1-(i): Estimate of  $A_6(x)$  when  $x \rightarrow 0$ .* If  $|x| \leq \frac{1}{2}$ , then

$$\begin{aligned} A_6(x) &\leq 2^N c' |x|^{1-N} \int_{B_1(-|x|\mathbf{e}_N)} |\mathbf{e}_N + |x|^{-1}\zeta|^{1-N} \left( \frac{|\zeta|^2}{2 \ln(|\zeta|)} + \frac{1 - |\zeta|^2}{4 \ln^2(|\zeta|)} \right) d\zeta \\ &\leq 2^N c' \int_{B_1(-|x|\mathbf{e}_N)} ||x|\mathbf{e}_N + \zeta|^{1-N} (2 + O(|\zeta| - 1)) d\zeta, \end{aligned}$$



therefore

$$\limsup_{x \rightarrow 0} A_6(x) \leq 2^N c' \int_{B_1} |\zeta|^{1-N} (2 + O(|\zeta| - 1)) d\zeta. \quad (5.21)$$

*Step 2-1-(ii): Estimate of  $A_6(x)$  when  $|x| \rightarrow \infty$ .* We have directly from (5.20)

$$\begin{aligned} A_6(x) &\leq c' \frac{|x|^{2-N}}{2 \ln |x|} \left( \int_{B_1(-|x|\mathbf{e}_N)} ||x|\mathbf{e}_N + \zeta|^{1-N} d\zeta \right) (1 + o(1)) \\ &= c' \frac{|x|^{2-N}}{2 \ln |x|} \left( \int_{B_1} |z|^{1-N} dz \right) (1 + o(1)). \end{aligned} \quad (5.22)$$

Combining (5.21) and (5.22) we deduce that

$$A_6(x) \leq c \min \left\{ 1, \frac{|x|^{2-N}}{\ln(1 + |x|)} \right\} \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}. \quad (5.23)$$

*Step 2-2: Estimate of  $A_7(x)$ .* Using (5.15) we have as in (4.20)

$$\begin{aligned} A_7(x) &\leq \left| \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x - y|^{2t-N} |y|^{\frac{1-N}{2}} \sin \left( |y| - \frac{(N-1)\pi}{4} \right) dy dt \right| \\ &\quad + \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x - y|^{2t-N} |y|^{-\frac{N+1}{2}} dy dt \\ &\leq K_1(x) + K_2(x). \end{aligned}$$

Clearly

$$\begin{aligned} K_2(x) &\leq c \int_{B_1^c} |x - y|^{-N} |y|^{-\frac{N+1}{2}} \int_0^1 t |x - y|^{2t} dt dy \\ &\leq c \int_{B_1^c} |x - y|^{-N} |y|^{-\frac{N+1}{2}} \left( \frac{|x - y|^2}{2 \ln |x - y|} + \frac{1 - |x - y|^2}{4 \ln^2 |x - y|} \right) dy. \end{aligned}$$

We have

$$K_2(x) = c |x|^{-\frac{N+1}{2}} \int_{B_{\frac{1}{|x|}}^c} |\mathbf{e}_N - z|^{-N} |z|^{-\frac{N+1}{2}} \left( \frac{|x|^2 |\mathbf{e}_N - z|^2}{2(\ln |x| + \ln |\mathbf{e}_N - z|)} + \frac{1 - |x|^2 |\mathbf{e}_N - z|^2}{4(\ln |x| + \ln |\mathbf{e}_N - z|)^2} \right) dz.$$

*When  $|x|$  is small.* The integral defining  $K_2(x)$  is convergent since  $N \geq 4$  and we have by integration by part

$$K_2(x) = \frac{2cc_N |x|^2}{(N+1) |\ln |x||} (1 + o(1)) \quad \text{as } x \rightarrow 0.$$

The term  $K_1(x)$  has already been partially estimated in Lemma 4.2. By (4.17) we have

$$\left| \int_{B_1^c} |x - y|^{2t-N} |y|^{\frac{1-N}{2}} \sin \left( |y| - \frac{(N-1)\pi}{4} \right) dy \right| \leq c_1 \quad \text{if } |x| \leq 1,$$

uniformly with respect to  $\tau$ . This implies that

$$\left| \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x - y|^{2t-N} |y|^{\frac{1-N}{2}} \sin \left( |y| - \frac{(N-1)\pi}{4} \right) dy dt \right| \leq c_2,$$

and finally

$$A_7(x) \leq c_3 \quad \text{for all } x \in B_1 \setminus \{0\}.$$

When  $|x|$  is large, we use the results of Lemma 4.2, in *Step II-Estimate of  $\tilde{\Pi}_\alpha$* . By (4.21) we have

$$\begin{aligned} \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x-y|^{2t-N} |y|^{-\frac{N+1}{2}} dy dt &\leq \int_0^1 t |x|^{2t-\frac{N+1}{2}} \int_{B_1^c} (|\mathbf{e}_N - z|^{2t-N} |z|^{-\frac{N+1}{2}}) dz \\ &\leq |x|^{-\frac{N+1}{2}} \int_{B_{\frac{1}{|x|}}^c} |\mathbf{e}_N - z|^{-N} |z|^{-\frac{N+1}{2}} \int_0^1 t |x|^{2t} |\mathbf{e}_N - z|^{2t} dt dz \\ &\leq |x|^{-\frac{N+1}{2}} \int_{B_1^c(-|x|\mathbf{e}_N)} |\zeta|^{-N} |\mathbf{e}_N + |x|^{-1}\zeta|^{-\frac{N+1}{2}} \left( \frac{|\zeta|^2}{2 \ln |\zeta|} + \frac{1-|\zeta|^2}{4 \ln^2 |\zeta|} \right) d\zeta, \end{aligned}$$

since the last term has already been estimated in (5.20). Therefore

$$\begin{aligned} \int_0^1 \mathcal{P}_0(t) \int_{B_1^c} |x-y|^{2t-N} |y|^{-\frac{N+1}{2}} dy dt \\ \leq |x|^{-\frac{N+1}{2}} \int_{B_1^c(-|x|\mathbf{e}_N)} |\zeta|^{-N} |\mathbf{e}_N + |x|^{-1}\zeta|^{-\frac{N+1}{2}} \left( \frac{|\zeta|^2}{2 \ln |\zeta|} + \frac{1-|\zeta|^2}{4 \ln^2 |\zeta|} \right) d\zeta := M(x). \end{aligned}$$

The above integral is convergent, but the limit when  $|x| \rightarrow \infty$  is a little more tricky to obtain: Let  $a > 2$ , for  $|x| > 2a$ , we write

$$\begin{aligned} |x|^{-\frac{N+1}{2}} \int_{B_1^c(-|x|\mathbf{e}_N)} |\zeta|^{-N} |\mathbf{e}_N + |x|^{-1}\zeta|^{-\frac{N+1}{2}} \left( \frac{|\zeta|^2}{2 \ln |\zeta|} + \frac{1-|\zeta|^2}{4 \ln^2 |\zeta|} \right) d\zeta \\ = |x|^{-\frac{N+1}{2}} \left( \int_{B_a} + \int_{B_1^c(-|x|\mathbf{e}_N) \cap B_a^c} \right) |\zeta|^{-N} |\mathbf{e}_N + |x|^{-1}\zeta|^{-\frac{N+1}{2}} \left( \frac{|\zeta|^2}{2 \ln |\zeta|} + \frac{1-|\zeta|^2}{4 \ln^2 |\zeta|} \right) d\zeta \\ = M_1(x) + M_2(x). \end{aligned}$$

Then

$$\begin{aligned} M_1(x) &\leq |x|^{-\frac{N+1}{2}} \int_{B_a} |\zeta|^{-N} |\mathbf{e}_N + |x|^{-1}\zeta|^{-\frac{N+1}{2}} \left( \frac{|\zeta|^2}{2 \ln |\zeta|} + \frac{1-|\zeta|^2}{4 \ln^2 |\zeta|} \right) d\zeta \\ &\leq |x|^{-\frac{N+1}{2}} 2^{\frac{N+1}{2}} \int_{B_a} |\zeta|^{-N} \left( \frac{|\zeta|^2}{2 \ln |\zeta|} + \frac{1}{4 \ln^2 |\zeta|} \right) d\zeta \\ &\leq c(a) |x|^{-\frac{N+1}{2}}. \end{aligned}$$

Next, since  $a > 2$ ,

$$\begin{aligned} M_2(x) &\leq |x|^{-\frac{N+1}{2}} \int_{B_a^c} |\mathbf{e}_N + |x|^{-1}\zeta|^{-\frac{N+1}{2}} \frac{|\zeta|^{2-N}}{\ln |\zeta|} d\zeta \\ &= \int_{B_{\frac{a}{|x|}}^c} |\mathbf{e}_N + t|^{-\frac{N+1}{2}} \frac{|t|^{2-N}}{\ln |t| + \ln |x|} d\zeta \leq \frac{c'(a)}{\ln |x|}. \end{aligned}$$

Therefore,

$$M(x) \leq \frac{c}{1 + \ln |x|} \quad \text{for all } x \in B_1^c. \quad (5.24)$$

Combining (5.17), (5.18), (5.19), (5.23) and (5.24), we obtain (5.16), which ends the proof.  $\square$

**Remark** The restriction  $N \geq 4$  appears to be purely technical and we conjecture that the estimates holds whenever  $N \geq 3$ .

We recall that  $\Phi_{\ln} = u^* + v_1$  with the notations of Theorem 1.8

**Lemma 5.7.** *Let  $N \geq 4$  and  $f \in L^1(\mathbb{R}^N)$ , then  $v_1 * f$  is Lipschitz continuous and*

$$|v_1 * f(x) - v_1 * f(y)| \leq c \|f\|_{L^1} |x - y|. \quad (5.25)$$

*Proof.* By Theorem 5.6-Step 2 we have

$$|\nabla v_1(x)| \leq c_3 \mathbf{1}_{B_1}(x) + \frac{c}{1 + \ln|x|} \mathbf{1}_{B_1^c}(x).$$

The proof follows by Young's inequality for convolution.  $\square$

The regularity concerning  $u^* * f$  is more involved

**Lemma 5.8.** *Let  $\theta > 1$ , and  $f \in L^\infty(\mathbb{R}^N; (1 + |y|)^\theta)$ . Then  $u^* * f$  is Dini continuous and for any  $R > 0$  there exists for some constant  $c = c(N, R) > 0$ ,*

$$|u^* * f(x) - u^* * f(x')| \leq \frac{c \|f\|_{\infty, \theta}}{1 + \ln^2(|x - x'|)} \quad \text{for all } x, x' \in B_R, \quad (5.26)$$

where  $\|f\|_{\infty, \theta}$  is defined in (1.32).

*Proof.* Let  $R > 0$ , we have

$$\begin{aligned} |u^* * f(x) - u^* * f(x')| &= \left| \int_0^1 \mathcal{P}_0(t) \int_{\mathbb{R}^N} (|x - y|^{2t-N} - |x' - y|^{2t-N}) f(y) dy dt \right| \\ &\leq \|f\|_{\infty, \theta} \int_0^1 \mathcal{P}_0(t) \int_{\mathbb{R}^N} \left| |x - y|^{2t-N} - |x' - y|^{2t-N} \right| (1 + |y|)^{-\theta} dy dt \\ &\leq \|f\|_{\infty, \theta} \int_0^1 \mathcal{P}_0(t) \int_{\mathbb{R}^N} \left| |x - x' - y|^{2t-N} - |y|^{2t-N} \right| (1 + |y - x'|)^{-\theta} dy dt \\ &\leq c \|f\|_{\infty, \theta} \int_0^1 t \int_{B_R^c} \left| |x - x' - y|^{2t-N} - |y|^{2t-N} \right| (1 + |y - x'|)^{-\theta} dy dt \\ &\quad + c \|f\|_{\infty, \theta} \int_0^1 t \int_{B_R} \left| |x - x' - y|^{2t-N} - |y|^{2t-N} \right| (1 + |y - x'|)^{-\theta} dy dt \\ &\leq c \|f\|_{\infty, \theta} (A(|x - x'|) + B(|x - x'|)). \end{aligned}$$

We suppose that  $\max\{|x|, |x'|\} \leq \frac{R}{8}$ .

$$\begin{aligned} A(|x - x'|) &\leq c \int_0^1 t \int_{B_R^c} \left| |x - x' - y|^{2t-N} - |y|^{2t-N} \right| (1 + |y|)^{-\theta} dy dt \\ &\leq c \int_0^1 t |x - x'|^{2t-\theta} \int_{B^c_{\frac{R}{|x-x'|}}} \left| |\mathbf{e}_{x-x'} - z|^{2t-N} - |z|^{2t-N} \right| (|x - x'|^{-1} + |z|)^{-\theta} dy dt \\ &\leq c \int_0^1 t |x - x'|^{2t-\theta} \int_{B^c_{\frac{R}{|x-x'|}}} \left| |\mathbf{e}_{x-x'} - z|^{2t-N} - |z|^{2t-N} \right| (|x - x'|^{-1} + |z|)^{-\theta} dy dt \\ &\leq c \int_0^1 t |x - x'|^{2t-\theta} \int_{B^c_{\frac{R}{|x-x'|}}} |z|^{2t-N} \left| |z|^{-1} \mathbf{e}_{x-x'} - \mathbf{e}_z \right|^{2t-N} - 1 \left| (|x - x'|^{-1} + |z|)^{-\theta} dy dt. \end{aligned}$$

Since

$$\left| |z|^{-1} \mathbf{e}_{x-x'} - \mathbf{e}_z \right|^{2t-N} - 1 = \frac{N-2t}{|z|} (1 + o(1)) \quad \text{as } |z| \rightarrow \infty,$$

we deduce that

$$\begin{aligned} A(|x-x'|) &\leq c' \int_0^1 t |x-x'|^{2t-\theta} \int_{\frac{R}{|x-x'|}}^\infty r^{2t-2-\theta} dr dt \\ &\leq c'' |x-x'| R^{-1-\theta} \int_0^1 t R^{2t} dt \\ &\leq c'' \left( \frac{R^2}{2 \ln R} + \frac{1-R^2}{4 \ln^2 R} \right) \frac{|x-x'|}{R^{1+\theta}}. \end{aligned} \quad (5.27)$$

Since  $|x-x'| \leq \frac{R}{4}$ , we have for  $a < 4$ ,

$$\begin{aligned} B(|x-x'|) &\leq \int_0^1 t \int_{B_R} \left| |x-x'-y|^{2t-N} - |y|^{2t-N} \right| (1+|y-x'|)^{-\theta} dy dt \\ &\leq \int_0^1 t |x-x'|^{2t} \int_{B_{\frac{R}{|x-x'|}}} \left| |\mathbf{e}_{x-x'} - z|^{2t-N} - |z|^{2t-N} \right| dz dt \\ &\leq 2 \int_0^1 t |x-x'|^{2t} \int_{B_a} |z|^{2t-N} dz dt + \int_0^1 t |x-x'|^{2t} \int_{B_{\frac{R}{|x-x'|}} \setminus B_a} \left| |\mathbf{e}_{x-x'} - z|^{2t-N} - |z|^{2t-N} \right| dz dt \\ &\leq A_8 + A_9. \end{aligned}$$

Then

$$\begin{aligned} A_8 &\leq c_N \int_0^1 t |x-x'|^{2t} a^{2t} dt = c_N \left( \frac{|x-x'|^2 a^2}{2 \ln(|x-x'|a)} + \frac{1-|x-x'|^2 a^2}{4 \ln^2(|x-x'|a)} \right) \\ &\leq \frac{2c_N}{\ln^2(|x-x'|a)} (1 + o(|x-x'|)), \end{aligned} \quad (5.28)$$

and

$$\begin{aligned} A_9 &\leq \int_0^1 t |x-x'|^{2t} \int_{B_{\frac{R}{|x-x'|}} \setminus B_a} \left| |z|^{-1} \mathbf{e}_{x-x'} - \mathbf{e}_z \right|^{2t-N} - 1 \left| |z|^{2t-N} dz dt \right. \\ &\leq c \int_0^1 t |x-x'|^{2t} \int_a^{\frac{R}{|x-x'|}} r^{2t-2} dr dt \\ &\leq c \int_0^1 |x-x'|^{2t} \left( \frac{R^{2t-1}}{|x-x'|^{2t-1}} - a^{2t-1} \right) \frac{t dt}{2t-1}. \end{aligned} \quad (5.29)$$

We have that

$$\int_0^1 |x-x'|^{2t} \left( \frac{R^{2t-1}}{|x-x'|^{2t-1}} - a^{2t-1} \right) \frac{t dt}{2t-1} = |x-x'| \int_0^1 (R^{2t-1} - (a|x-x'|)^{2t-1}) \frac{t dt}{2t-1}$$

and

$$\begin{aligned} R^{2t-1} - (a|x-x'|)^{2t-1} &= e^{(2t-1) \ln R} - e^{(2t-1) (\ln a + \ln |x-x'|)} \\ &= (2t-1) \ln \left( \frac{R}{a|x-x'|} \right) + O((2t-1)^2 (\ln^2 R + \ln^2 a|x-x'|)). \end{aligned}$$

By integration,

$$\left| \int_0^1 (R^{2t-1} - (a|x-x'|)^{2t-1}) \frac{t dt}{2t-1} \right| \leq \ln \left( \frac{R}{a|x-x'|} \right) + O(\ln^2 R + \ln^2 a|x-x'|).$$

This finally yields to

$$A_9 \leq c|x - x'| \left[ \ln \left( \frac{R}{a|x - x'|} \right) + O(\ln^2 R + \ln^2 a|x - x'|) \right]. \quad (5.30)$$

We notice that the leading term when  $x - x' \rightarrow 0$  is coming from the estimate of  $B_1$  and it gives the Dini modulus of continuity expressed by (5.26). This ends the proof.  $\square$

*Proof of Theorem 1.10.* It follows directly from Lemmas 5.7 and 5.8.  $\square$

**Acknowledgements:** This work is supported by the Natural Science Foundation of China, No. 12071189, by Jiangxi Province Science Fund for Distinguished Young Scholars, No. 20212ACB211005.

## References

- [1] B. Barrios, A. Figalli, X. Ros-Oton: Free boundary regularity in the parabolic fractional obstacle problem. *Comm. Pure Appl. Math.* 71(10), 2129–2159 (2018).
- [2] B. Barrios, I. Peral, F. Soria, E. Valdinoci: A Widder’s Type Theorem for the Heat Equation with Nonlocal Diffusion. *Arch. Rational Mech. Anal.* 213 629–650 (2014).
- [3] K. Bogdan and T. Jakubowski: Estimates of the heat kernel of fractional Laplacian perturbed by gradient operators. *Commun. Math. Phys.* 271(1), 179–198 (2007).
- [4] X. Cabré, J. Roquejoffre: The influence of fractional diffusion in Fisher-KPP equations. *Commun. Math. Phys.* 320(3), 679–722 (2013).
- [5] L. Caffarelli, S. Salsa, L. Silvestre: Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian. *Invent. Math.* 171, 425–461 (2008).
- [6] L. Caffarelli, A. Figalli: Regularity of solutions to the parabolic fractional obstacle problem. *J. Reine Angew. Math.* 680, 191–233 (2013).
- [7] H. Chen: The Dirichlet elliptic problem involving regional fractional Laplacian. *J. Math. Phys.* 59(7), 071504, 19 pp (2018).
- [8] H. Chen, P. Felmer, A. Quaas: Large solution to elliptic equations involving fractional Laplacian. *Ann. l’Inst. Henri Poincaré-AN* 32, 1199–1228 (2015).
- [9] H. Chen, L. Véron: Initial trace of positive solutions to fractional diffusion equations with absorption. *J. Funct. Anal.* 276, 1145–1200 (2019).
- [10] H. Chen, L. Véron: Bounds for eigenvalues of the Dirichlet problem for the logarithmic Laplacian. *Adv. Calc. Var.* 16(3), 541–558 (2023).
- [11] H. Chen, T. Weth: The Dirichlet Problem for the logarithmic Laplacian. *Comm. Part. Diff. Eq.* 44, 1100–1139 (2019).
- [12] E. Correa, A. De Pablo: Nonlocal operators of order near zero. *J. Math. Anal. Appl.* 461, 837–867 (2018).

- [13] V. Crismale, L. De Luca, A. Kubin, A. Ninno, M. Ponsiglione: The variational approach to  $s$ -fractional heat flows and the limit cases  $s \rightarrow 0^+$  and  $s \rightarrow 1^-$ . *J. Funct. Anal.* **284**(8), No. 109851, 38 pp. (2023).
- [14] A. de Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez: A general fractional porous medium equation. *Commun. Pure Appl. Math.* **65**, 1242–1284 (2012).
- [15] E. Di Nezza, G. Palatucci, E. Valdinoci: Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136**(5), 521–573 (2012).
- [16] S. Djitte, M. Fall, T. Weth: A fractional Hadamard formula and applications. *Calc. Var. Part. Diff. Equ.* **60**(6), No. 231, 31 pp (2021).
- [17] J. Droniou, T. Galloüet, J. Vovelle: Global solution and smoothing effect for a non-local regularization of a hyperbolic equation. *J. Evol. Equ.* **3**(3), 499–521 (2003).
- [18] G. Evéquoz, T. Weth: Dual variational methods and nonvanishing for the nonlinear Helmholtz equation. *Adv. math.* **280**, 690–728 (2015).
- [19] R. Frank, T. König, H. Tang: Classification of solutions of an equation related to a conformal log Sobolev inequality. *Adv. Math.* **375**, 107395, 27 pp (2020).
- [20] I. M. Gel’fand, G. E. Šilov: Generalized functions. Vol. 2, Spaces of fundamental and generalized functions. *Academic Press* (1968).
- [21] D. Haimo: Widder temperature representations. *J. Math. Anal. Appl.* **41**, 170–178 (1973).
- [22] S. Jarohs, S. Saldaña, T. Weth: A new look at the fractional Poisson problem via the logarithmic Laplacian. *J. Funct. Anal.* **279**(11), 108732, 50 pp (2020).
- [23] A. Laptev, T. Weth: Spectral properties the logarithmic Laplacian. *Anal. Math. Phys.* **11** (3), No. 133, 24 pp (2021).
- [24] N. Lebedev: Special functions and their applications. *Dover Publications Inc., New York, 1972*.
- [25] J. L. Lions, J. Petree: Sur une classe d’espaces d’interpolation. *Publ. I.H.E.S* **19**, 5–68 (1964).
- [26] R. Musina, A.I. Nazarov: On fractional Laplacians. *Comm. Part. Diff. Eq.* **39**, 1780–1790 (2014).
- [27] X. Ros-Oton, J. Serra: The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.* **101**, 275–302 (2014).
- [28] X. Ros-Oton, J. Serra: The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.* **213**, 587–628 (2014).
- [29] Y. Sire, S. Terracini, G. Tortone: On the nodal set of solutions to degenerate or singular elliptic equations with an application to  $s$ -harmonic functions. *J. Math. Pures Appl.* (9) **143**, 376–441 (2020).
- [30] L. Silvestre: Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* **60**(1), 67–112 (2007).

- [31] S. Terracini, G. Tortone, S. Vita: On  $s$ -harmonic functions on cones. *Anal. PDE* 11(7), 1653–1691 (2018).
- [32] E. M. Stein: Singular integrals and differentiability properties of functions. *Princeton Mathematical Series*, no. 30, Princeton University Press, Princeton, N.J. 1970.
- [33] P.B. Stinga: User’s guide to the fractional Laplacian and the method of semigroups. *Handbook of fractional calculus with applications Vol. 2* De Gruyter, Berlin, 235265 (2019).
- [34] E. T. Whittaker, G. N. Watson: A course of Modern analysis. *Cambridge Univ. Pr.* (1902), 2nd ed. (1996).
- [35] D. Widder: Positive temperatures on an infinite rod. *Trans. Am. Math. Soc.* 55(1), 85–95 (1944)
- [36] D. Widder: The Heat Equation. *Academic Press, New York* (1975)