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Gas-filled quasi-spherical resonators are excellent tools for the measurement of thermophysical properties of gas, and have also been retained for the determination of the Boltzmann constant with a low uncertainty, which can be derived from measurements of both the speed of sound in a noble gas and the volume of the resonator. To achieve this, a detailed modelling of the acoustic field in quasi-spherical resonators are of importance. Several phenomena and perturbations must be taken into account, including, among inertia and compressibility, heat conduction, viscosity, the shape of the resonator, small irregularities on the wall, and so on. The aim of this paper is to provide improvements to the current models of the acoustic field in such resonator. Namely, the model given here takes into account all the different perturbing elements together in a unique formalism, including the coupling between the different perturbing elements and the resulting modal coupling in a consistent manner. The first results obtained from this analytical model on a simple configuration show that the effect of modal coupling is small but should not be neglected regarding the accuracy required here, even if several improvement could still be provided to this new unified model.

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I. INTRODUCTION

The International Committee for Weights and Measures (CIPM) recently recommended the redefinition of the International System Units (SI), where the kelvin would be linked to an exactly defined value of the Boltzmann constant[1]. The advantage would be that the definition of the kelvin would not depend on any temperature nor on any method for its measurement. The value to be chosen for the Boltzmann constant must be known with a relative uncertainty of 10^{-6}. Gas-filled quasi-spherical resonators are excellent tools for the determination of the Boltzmann constant, which can be derived from measurements of both the speed of sound in a noble gas and the volume of the resonator. The choice of a quasi-spherical shape for the shell allows us to have a high quality factor for the acoustic resonances and non-degenerate electromagnetic modes.

Actually, an accurate design of the shell and a detailed modelling of the acoustic field in spherical or quasi-spherical resonators are of importance to measure thermophysical properties of gas and the Boltzmann constant. Several phenomena must be taken into account, including inertia, compressibility, heat conduction and shear viscosity (in the boundary layers and in the bulk of the fluid), bulk viscosity, the real shape of the resonator (in fact quasi-spherical shape), the acoustic input impedance of small acoustic elements flush-mounted on the wall that are necessary for the measurements (tubes, transducers), geometrical irregularities, roughness of the wall, shell motion due to external vibration sources and to the coupling between the fluid and the shell. All these phenomena, which act as perturbations to an idealized acoustic fluid filling a perfect rigid spherical shell, need to be taken into account to determine with a low uncertainty their influence on the resonance properties of the resonator, and then on the value obtained for the Boltzmann constant.

Groundbreaking experiments and a number of significant theoretical studies have already been made by Mehl, Moldover, Trusler et al.[2–7]. In particular, they suggested means to take into account several perturbations in the cavity. It is worth noting that these perturbations are taken into account separately, then neglecting the coupling between them. These perturbations are the thermal and viscous effects (in the boundary layers and the bulk of the gas), the deformation of the cavity, the influence of small elements located on the wall of the resonator (tubes, slits, and transducers), the vibroacoustic coupling between the fluid and the shell.

The aim of this paper is to suggest unified modelling able to gather all the different types of perturbing factors in an unique formalism, including the coupling between the different perturbation factors (through the modal coupling) neglected until now, which would be of interest to have a better interpretations of the measurement results in the quasi-spherical resonator. Here, the acoustic field is expressed by the coupling between Neumann modes of an ideal, unperturbed resonator that bounds outwardly the pertubated surface of the resonator (the coupling being due to energy transfer between modes induced by every perturbation). This model is then used on the most simple experimental conditions: a perfectly spherical cavity, filled with a dissipative gas (ar-

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gon), with only a receiver and a transmitter microphone flush-mounted on the wall. Assuming prefectly spherical cavity in the application considered in the present paper unables us to evaluate the effect of modal coupling, due to the viscous and thermal effects in the bulk and in the boundary layers (modeled at the lowest order), and to the microphones.

The fundamental equations of acoustic propagation in dissipative fluid are given in section II. In section III, the solution of the fundamental problem (propagation equation associated to the boundary conditions) for the acoustic pressure in the quasi-spherical resonator is expressed as an expansion on the spherical eigenfunctions, making use of the integral formulation. Finally in section IV, results are given and discussed. These results are obtained with this model applied on the most simple experimental configuration which has been carried out in 1988 at the NIST[2].

II. FUNDAMENTAL EQUATIONS OF ACOUSTICS IN DISSIPATIVE FLUID

The formalism used in the following lies on the works of Kirchhoff, gathered later by Rayleigh[8], Morse[9], Pierce[10] and Bruneau[11]. The presentation of the fundamental equations of acoustics chosen here refers to those of Bruneau.

The system considered is a quasi-spherical cavity because its shape retains certain advantages of spherical acoustic resonators (high quality factor) while simplifying the measurement of the resonator volume using microwave resonances (electromagnetic modes are not degenerate for the shape chosen here). The variables describing the dynamic and thermodynamic states of the fluid are the pressure variation p, the particle velocity $\vec{v}$, the density variation ρ', the entropy variation σ, and the temperature variation τ. The parameters which specify the properties and the nature of the fluid are the values of the density ρ_0, the static pressure P_0, the shear viscosity coefficient μ, the bulk viscosity coefficient η, the coefficient of thermal conductivity λ, the specific heat coefficient at constant pressure and constant volume per unit of mass C_P and C_V respectively, the specific heat ratio γ, and the increase in pressure per unit increase in temperature at constant density $\hat{\beta} = (\partial P/\partial T)_V$ ($\hat{\beta}\gamma = \alpha\rho_0 c_0^2 = 1/V(\partial V/\partial T)_P$, α being the volume thermal expansivity and c_0 the adiabatic speed of sound). A complete set of linearized homogeneous equations governing small amplitude disturbances of the fluid includes the following:

- the Navier–Stokes equation

$$\frac{1}{c_0}\frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho_0 c_0}\,\vec{\mathrm{grad}}\, p = \ell_v\, \vec{\mathrm{grad}}\,\mathrm{div}\,\vec{v} - \ell'_v\, \vec{\mathrm{rot}}\,\vec{\mathrm{rot}}\,\vec{v}\,, \quad (1)$$

where the characteristic lengths ℓ_v and ℓ'_v are defined as follows,

$$\ell_v = \frac{1}{\rho_0 c_0}\left(\frac{4}{3}\mu + \eta\right) \quad \text{and} \quad \ell'_v = \frac{\mu}{\rho_0 c_0}\,,$$

- the conservation of mass equation, taking into account the thermodynamic law expressing the density variation as function of the independent variables p and τ,

$$\rho_0 c_0\,\mathrm{div}\,\vec{v} + \frac{\gamma}{c_0}\frac{\partial}{\partial t}(p - \hat{\beta}\tau) = 0\,, \quad (2)$$

- the Fourier equation for heat conduction, taking into account the thermodynamic law expressing the entropy variation as function of the independent variables p and τ,

$$\left(\frac{1}{c_0}\frac{\partial}{\partial t} - \ell_h\,\Delta\right)\tau = \frac{\gamma - 1}{\hat{\beta}\gamma}\,\frac{1}{c_0}\frac{\partial p}{\partial t}\,, \quad (3)$$

the operator Δ being the laplacian, where the characteristic length ℓ_h is defined as

$$\ell_h = \frac{\lambda}{\rho_0 c_0 C_P}\,.$$

These equations (1), (2) and (3), associated to the boundary conditions, describe the acoustic field in the quasi-spherical cavity. The purpose of the following section is to find the propagation equations for the acoustic pressure, the temperature variation, and the particle velocity.

III. THE HARMONIC FIELD IN A QUASI-SPHERICAL CAVITY

In this section a standard analytic procedure is developed whereby the acoustic field inside the cavity bounded by perturbed surface (irregular enclosure, domain D, surface S) is expressed as a sum over the eigenmodes (Neumann) of a cavity having separable geometry (regular enclosure, domain D_r, surface S_r) that bounds outwardly the perturbed enclosure considered (Figure 1).

The resulting acoustic modelling presented in section III.B thus includes in a coherent manner all types of perturbation in the resonator (quasi-spherical shape, dissipation in the bulk of the fluid, acoustic elements on the wall) and the resulting modal coupling.

A. The boundary problem with sources

The acoustic pressure in the perturbed enclosure (domain D bounded by surfaces S) is governed by the set of equations, including the propagation equation with harmonic source term $f(\vec{r})$ on the right hand side and

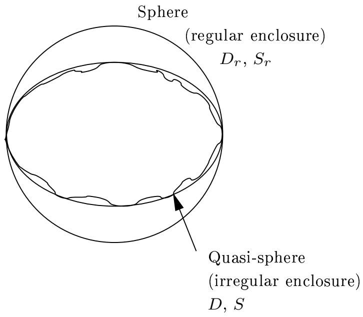

Figure 1. Sphere and quasi-sphere with irregular boundary conditions.

the boundary conditions with harmonic boundary source term $\sigma(\vec{r})$ on the right hand side, which takes the following form:

$$(\Delta + k_a^2)p(\vec{r}) = -f(\vec{r})\,, \quad \text{in } D\,, \tag{4a}$$

$$\left[\frac{\partial}{\partial n} + \mathrm{j}\, k_0 \beta(\vec{r})\right] p(\vec{r}) = \sigma(\vec{r})\,, \quad \text{on } S\,, \tag{4b}$$

where the symbol $\frac{\partial}{\partial n}$ implies a normal derivative outwardly directed, $k_0 = \omega / c_0$ is the adiabatic wavenumber, k_a the acoustic wavenumber including the dissipation in the bulk of the fluid (appendix A), and the parameter $\beta(\vec{r})$ is the specific admittance of the wall. The specific admittance $\beta(\vec{r})$ can include the viscous and thermal boundary layer effects, and any kind of discontinuitie and perturbation on the wall of the resonator (tubes, slits, transducers, roughness...). For the problem considered, there is no source in the domain D, then the term $f(\vec{r})$ is zero.

B. The associated eigenvalues problem

The modal wave functions of an ideal spherical cavity (domain D_r bounded by a surface S_r), which encloses the perturbed enclosure, are solutions of the homogeneous Helmholtz equation subject to the Neumann boundary condition at the wall, namely

$$(\Delta + k_m^2)\psi_m^t(\vec{r}) = 0\,, \quad \text{in } D_r\,, \tag{5a}$$

$$\frac{\partial}{\partial n}\psi_m^t(\vec{r}) = 0\,, \quad \text{on } S_r\,, \tag{5b}$$

where the subscript m stands for three quantum numbers (ν, μ, η) and the superscript t (denoted either c or s) represents the two degenerate eigenfunctions which depend on the azimuthal angle φ. The eigenfunctions ψ_m^t expressed in the spherical coordinates (r, θ, φ) are

$$\psi_m^c(r,\theta,\varphi) = \frac{1}{a_m} j_\nu(k_{\nu\eta} r) P_{\mu\nu}(\cos\theta)\cos\mu\varphi\,, \tag{6a}$$

$$\psi_m^s(r,\theta,\varphi) = \frac{1}{a_m} j_\nu(k_{\nu\eta} r) P_{\mu\nu}(\cos\theta)\sin\mu\varphi\,, \tag{6b}$$

where the functions $P_{\mu\nu}$ are Legendre functions, and the coefficients a_m are normalization coefficients (cf. appendix B).

Expanding the pressure field $p(\vec{r})$ on the eigenfunctions $\psi_p^{t'}(\vec{r})$, leads to

$$p(\vec{r}) = \sum_{p,t'} c_p^{t'} \psi_p^{t'}(\vec{r})\,. \tag{7}$$

Then, multiplying equation (4a) where $f = 0$, by the eigenfunction $\psi_m^t(\vec{r})$, and integrating all over the domain D, i.e.

$$\iiint_D \psi_m^t(\vec{r})(\Delta + k_a^2)p(\vec{r})\,\mathrm{d}D = 0\,, \tag{8}$$

solution of the posed problem for the acoustic pressure field is subsequently achieved with the help of Green's integral theorem which states

$$\begin{aligned}\iiint_D p(\vec{r})\Delta\psi_m^t(\vec{r})\,\mathrm{d}D - \iiint_D \psi_m^t(\vec{r})\Delta p(\vec{r})\,\mathrm{d}D \\ = \iint_S p(\vec{r})\frac{\partial}{\partial n}\psi_m^t(\vec{r})\,\mathrm{d}S - \iint_S \psi_m^t(\vec{r})\frac{\partial}{\partial n}p(\vec{r})\,\mathrm{d}S\,.\end{aligned} \tag{9}$$

The expression of $\Delta\psi_m^t$ and $\frac{\partial}{\partial n}\psi_m^t$ being required to satisfy equations (5a) and (5b) respectively, and the solutions p being expressed by equation (7), equation (9) gives straightforwardly the following equation

$$\sum_{p,t'} c_p^{t'}\left[(k_m^2 - k_a^2)N_{mp}^{tt'} + A_{mp}^{tt'} + E_{mp}^{tt'}\right] = S_m^t\,, \tag{10}$$

where

$$N_{mp}^{tt'} = \iiint_D \psi_p^{t'}(\vec{r})\psi_m^t(\vec{r})\,\mathrm{d}D\,, \tag{11a}$$

$$A_{mp}^{tt'} = \iint_S \psi_p^{t'}(\vec{r})\frac{\partial}{\partial n}\psi_m^t(\vec{r})\,\mathrm{d}S\,, \tag{11b}$$

$$E_{mp}^{tt'} = \iint_S \mathrm{j}\, k_0\beta(\vec{r})\psi_p^{t'}(\vec{r})\psi_m^t(\vec{r})\,\mathrm{d}S\,, \tag{11c}$$

$$S_m^t = \iint_S \sigma(\vec{r})\psi_m^t(\vec{r})\,\mathrm{d}S\,, \tag{11d}$$

which is equivalent to the matrix equation, using the notation $[k^2N]$ for the matrix whose elements are $k_m^2 N_{mp}^{tt'}$,

$$\left([k^2N] - k_a^2[N] + [A] + [E]\right)[C] = [S]\,. \tag{12}$$

The matrix $[N]$ expresses the influence on the acoustic field of the depth of the deformation: the radial coordinate of the cavity depends on the angles θ and φ in spherical coordinates (the effect expressed by the matrix $[N]$ can be called "bulk" or "global"[12] modal coupling, corresponding to energy transfer between the mode labelled m and the mode labelled p). The matrix $[A]$ expresses the influence on the acoustic field of the slope of the deformation: the normal to the wall of the resonator depends on θ and φ too. The matrix $[E]$ expresses the effects on the acoustic field of the perturbations on the wall, taking also into account the shape of the resonator (the effects expressed by the matrices $[A]$ and $[E]$ can be called "boundary" or "local"[12] modal coupling, also corresponding to energy transfer between the mode labelled m and the mode labelled p). The vector $[S]$ expresses the effect of the acoustic source set on the wall of the resonator. Finally, the matrix $[k^2N] - k_a^2[N] + [A] + [E]$ in equation (12) involves coupling terms between the different modes (in the cross products $\psi_p^{t'}(\vec{r})\psi_m^t(\vec{r})$, weighed or not by the admittance β, and $\psi_p^{t'}(\vec{r})\frac{\partial}{\partial n}\psi_m^t(\vec{r})$), whereas the right hand side involves the modes created by the source.

The eigenvalues of the matrix in the left hand side of equation (12) includes then together all the perturbations we would take into account in the calculation of the resonance characteristics (resonance frequencies and half-widths) of the cavity: the effect of the deformation is included in the matrices $[N]$ and $[A]$, and all the perturbing factors on the wall of the cavity are included in the specific admittance β of the matrix $[E]$ for the passive acoustic effects and elements (boundary layer effects, tubes, transducers, irregularities, roughness, vibroacoustic coupling between the fluid and the shell, ...) and in the matrix $[S]$ for the active acoustic effects and elements (acoustic source, shell motion due to external vibration sources).

REMARK.– In previous works by Mehl[3,4], using the general formalism of Morse and Feshbach[13] and focusing only on the effects of the small deformations of the cavity (quasi-spherical shape) from the regular shape (perfect spherical shape), all the other perturbations on the wall of the resonator being taken into account by other means, the terms E_{mp} and S_m in equation (10) are then zero, which leads to, neglecting terms $N_{mp}^{tt'}$ for $m \neq p$ and $t \neq t'$ as made by Mehl[3] (eq. 8),

$$\left[(k_a^2 - k_m^2)N_{mm}^{tt} - A_{mm}^{tt}\right] c_m^t = \sum_{p\neq m, t\neq t'} c_p^{t'} A_{mp}^{tt'} .$$

The coefficients c_m^t are then given by

$$c_m^t = \frac{\sum\limits_{p\neq m, t\neq t'} A_{mp}^{tt'} c_p^{t'}}{(k_a^2 - k_m^2)N_{mm}^{tt} - A_{mm}^{tt}} , \tag{13}$$

and the difference $(k_a^2 - k_m^2)$ by

$$k_a^2 - k_m^2 = \frac{1}{N_{mm}^{tt} c_m^t} \sum_{p\neq m, t\neq t'} A_{mp}^{tt'} c_p^{t'} + \frac{A_{mm}^{tt}}{N_{mm}^{tt}} ,$$

that is

$$k_a^2 - k_m^2 = \frac{A_{mm}^{tt}}{N_{mm}^{tt}} + \frac{1}{N_{mm}^{tt} c_m^t} \sum_{p\neq m, t\neq t'} A_{mp}^{tt'} \frac{\sum\limits_{q\neq p, t''\neq t'} A_{pq}^{t't''} c_q^{t''}}{(k_a^2 - k_p^2)N_{pp}^{t't'} - A_{pp}^{t't'}} , \tag{14}$$

then, neglecting the non-diagonal terms in the sum over q and assuming that $(k_a^2 - k_p^2)N_{pp}^{t't'} >> A_{pp}^{t't'}$ and $A_{mp}^{tt'} = A_{pm}^{t't}$,

$$k_a^2 - k_m^2 \approx \frac{A_{mm}^{tt}}{N_{mm}^{tt}} + \sum_{p\neq m, t\neq t'} \frac{A_{mp}^{tt'\,2}}{(k_a^2 - k_p^2)N_{pp}^{t't'} N_{mm}^{tt}} . \tag{15}$$

In this expression (15) of the frequency shifts due to the deformation of the cavity, derived from Morse and Feshbach formalism[13], the coupling terms $N_{mp}^{tt'}$ ($m \neq p$ and $t \neq t'$) are neglected (the bulk modal coupling is neglected, assuming quasi-orthogonality) whereas the terms $A_{mp}^{tt'}$ are kept, i.e. the boundary coupling is taken into account (due to the boundary slope). Furthermore, in these previous works, the symmetry of the matrix $[A]$ is assumed but not demonstrated in the general case considered (every shape of the cavity). It is the aim of the remainder of the paper to keep all the coupling terms that was not taken into account in previous works in order to know more accurately the influence of each one.

IV. APPLICATION: ACOUSTIC FIELD IN A SPHERICAL RESONATOR

The case studied here is the spherical cavity used at the NIST to measure the gas constant[2]. This spherical cavity (mean radius $R = 8.890143 \times 10^{-2}$m) filled with a dissipative gas (argon) with a receiving microphone (specific admittance β_r, surface ΔS_r) and a transmitting microphone (sound source, vibration velocity V_σ, surface ΔS_σ). The dissipation in the bulk of the fluid is taken into account in the complex wavenumber k_a. The bounday layer effects are expressed in term of specific admittance β_{vh} at the lowest order (order 1/2 of the characteristic lengths ℓ_h and ℓ'_v). This specific admittance is uniform on the total area of the wall S.

The shell motion can be taken into account in the matrices E and S in an appropriate manner for the determination of the Boltzmann constant only if the spatial repartition of this shell motion is known or modelled with a great accuracy. Previous works[14] allow us to determine the influence of the shell motion on the acoustic resonance frequencies of a spherical cavity with a simple shape. However, regarding the complexity of the real shape of the resonator's shell, it doesn't seem possible, at short or mid-term, to carry out a vibroacoustical model with the accuracy required for this application. Then, currently, the acoustical measurements are limited to the frequency ranges for which the effect of the shell motion is not significant in the experimental results (this corresponds to frequencies far from the structural resonance frequencies of the shell). The shell motion is thus neglected here in the global modelling of the acoustic field.

For the perfect sphere considered below, the domain D and the surface S are the same as the regular domain D_r and surface S_r respectively. There is no modal coupling due to the deformation of the domain and the enclosure. As a consequence, the matrix $[N]$ is the identity and $[A]$ is zero. Modal coupling in such case is then due to dissipation and perturbations on the wall of the cavity only (matrix $[E]$). The integral equation (10) leads then to

$$\sum_{p,t'} c_p^{t'} \left[(k_m^2 - k_a^2)\delta_{mp}\delta_{tt'} + E_{mp}^{tt'}\right] = S_m^t \,, \tag{16}$$

$\gamma_{\nu,\eta}$ being the ηth root of the first derivative of the spherical Bessel function j_ν, and where (eq. 11c,d)

$$S_m^t = \iint_S \sigma(\vec{r})\psi_m^t(\vec{r})\,\mathrm{d}S \,, \tag{17a}$$

$$E_{mp}^{tt'} = \iint_S \mathrm{j}\,k_0\beta(\vec{r})\psi_p^{t'}(\vec{r})\psi_m^t(\vec{r})\,\mathrm{d}S \,. \tag{17b}$$

The small surfaces ΔS_r and ΔS_σ are much smaller than the total area S, and sufficiently small to assume that the associated specific admittances, velocity and eigenfunctions are uniform on them, giving for equations (17a,b)

$$S_m^t = \sigma\psi_m^t(R,\theta_\sigma,\varphi_\sigma)\Delta S_\sigma \,, \tag{18a}$$

and

$$\begin{aligned} E_{mp}^{tt'} &= {}^{vh}E_{mp}^{tt'} + {}^{r}E_{mp}^{tt'} \\ &= \mathrm{j}\,k_0\left[\beta_{vh}\langle\psi_p^{t'}|\psi_m^t\rangle_S + \beta_r\Delta S_r\langle\psi_p^{t'}|\psi_m^t\rangle_c\right] \,, \end{aligned} \tag{18b}$$

where the notation $\langle\psi_p^{t'}|\psi_m^t\rangle_i$ stands for the integral of the product of the eigenfunctions $\psi_p^{t'}$ and ψ_m^t over the surface either S or ΔS_i.

The contributions in the modal coupling of the boundary layer effects (acting on the whole surface S of the wall) and of each small element flush-mounted on the wall are calculated in sections IV.A and IV.B respectively, when the resonator is excited by a local sound source, which influence (terms S_m) is calculated in section IV.C. The coordinates θ and φ of the small elements flush-mounted on the wall are given in Table I, corresponding to the design of the cavity used at INM.

A. Viscous and thermal boundary layer effects

The specific admittance on the surface S due to the viscous and the thermal boundary layer effects have the form[11] (at the order 1/2 of the characteristic lengths):

$$\beta_{vh} \approx \frac{1+\mathrm{j}}{\sqrt{2}}\sqrt{k_0}\left[\left(1-\frac{k_r^2}{k_a^2}\right)\sqrt{\ell'_v} + (\gamma-1)\sqrt{\ell_h}\right] \,, \tag{19}$$

	θ_i	φ_i
Transmitting microphone	π	0
Receiving microphone	$\pi/2$	$3\pi/4$

Table I. θ and φ coordinates of the small elements flush-mounted on the wall of the resonator.

where the wavenumber k_a is related to the radial and azimuthal wavenumbers k_r and k_w by $k_a^2 = k_r^2 + k_w^2$. In the expression of this admittance, which acts as a small correction on the acoustic field, the complex wavenumber k_a, including the dissipation in the bulk of the fluid (§ A), is replaced by the adiabatic wavenumber k_0, the dissipation in the bulk being much smaller than in the boundary layers.

The term $(1 - k_r^2/k_a^2)$ is related to the incidence of the acoustic wave on the wall, which depends on the acoustic modes that occur in the cavity. Then, because of the modal coupling in the cavity, while the acoustic source generates an acoustic field at the angular frequency ω, different acoustic modes are excited at this angular frequency in the cavity. The specific admittance associated to each acoustic mode in equation (19) is then a function of the wavenumber $k_0 = \omega/c_0$, and of the factor related to this mode, namely

$$\begin{aligned} \beta_{vh} &= \beta_{\nu\eta} \\ &\approx \frac{1+\mathrm{j}}{\sqrt{2}}\sqrt{k_0}\left[\frac{\nu(\nu+1)}{\gamma_{\nu\eta}^2}\sqrt{\ell'_v} + (\gamma-1)\sqrt{\ell_h}\right] \,. \end{aligned} \tag{20}$$

On another hand, the integral $\langle\psi_p^{t'}|\psi_m^t\rangle_S$ on the whole surface S is here given by

$$\langle\psi_p^{t'}|\psi_m^t\rangle_S = \int_0^{2\pi}\int_0^{\pi} \psi_p^{t'}(R,\theta,\varphi)\psi_m^t(R,\theta,\varphi) \, R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \,,$$

that gives straigthforwardly (because of the orthogonality of the Legendre's functions and of functions sin and cos), for the influence of the viscous and thermal boundary layers in the resonator:

$${}^{vh}E_{mp}^{tt'} = \mathrm{j}\,k_0\beta_{\nu\eta}R^2\frac{j_\nu(\gamma_{\nu\eta})j_\nu(\gamma_{\nu\eta'})}{\mathcal{J}_{\nu\eta}\mathcal{J}_{\nu\eta'}}\,\delta_{\nu\nu'}\delta_{\mu\mu'}\delta_{tt'} \,, \tag{21}$$

$\mathcal{J}_{\nu\eta}$ being the norm of the function $j_\nu(k_{\nu\eta}r)$ (cf. appendix B).

B. Receiving microphone flush-mounted on the wall

On the small surface ΔS_r, the integral $\langle\psi_p^{t'}|\psi_m^t\rangle_r$ reduces to

$$\langle\psi_p^{t'}|\psi_m^t\rangle_r = \psi_m^t(R,\theta_r,\varphi_r)\psi_p^{t'}(R,\theta_r,\varphi_m ic)\Delta S_r \,, \tag{22}$$

that gives, for the coordinates $\theta_r = \pi/2$ and $\varphi_r = 3\pi/4$ (Table I), making use of the relations $k_{\nu\eta}R = \gamma_{\nu\eta}$,

$$
\begin{aligned}
{}^{r}E_{mp}^{tt'} &= \mathrm{j}\, k_0 \frac{1}{a_m a_p} \Delta S_r \beta_r j_\nu(\gamma_{\nu\eta}) j_{\nu'}(\gamma_{\nu'\eta'}) \\
&\quad P_{\nu\mu}(\cos\theta_r) P_{\nu'\mu'}(\cos\theta_r) \phi^t(3\pi\mu/4) \phi^{t'}(3\pi\mu'/4)\,, \quad (23)
\end{aligned}
$$

the function ϕ^t standing for either cos or sin.

The expression chosen here for the specific admittance β_r of the receiving microphone is the expression used in the study done at the NIST[2]:

$$\beta_r = \mathrm{j}\,\omega\,\rho_0 c_0 \chi\,, \qquad (24)$$

χ being the compliance per unit area of the membrane of the microphone (nominal value of 1.5×10^{-10}).

C. Sound source

The influence of the source on the acoustic field is expressed by (eq. 18a)

$$S_m^t = \sigma \psi_m^t(R, \theta_\sigma, \varphi_\sigma) \Delta S_\sigma\,, \qquad (25)$$

where $\sigma = \mathrm{j}\,\omega\,\rho_0 V_\sigma$, V_σ being the vibration velocity of the source, that gives

$$S_m^t = \mathrm{j}\, k_0 \rho_0 c_0 V_\sigma \Delta S_\sigma \psi_m^t(R, \theta_\sigma, \varphi_\sigma)\,. \qquad (26)$$

The sound source being an electrostatic microphone (used as a transmitter), the volume velocity $V_\sigma \Delta S_\sigma$ of the sound source is then derived from the following transduction equation

$$M_{p\sigma} i = Y_\sigma p(R, \theta_\sigma, \varphi_\sigma) + \Delta S_\sigma V_\sigma\,, \qquad (27)$$

where $M_{p\sigma}$ and Y_σ are respectively the pressure sensitivity and the acoustic admittance of the microphone, and i the electric current through the microphone. Making use of equations (27) and (7) in equation (26) gives

$$S_m^t = \mathrm{j}\, k_0 \rho_0 c_0 M_{p\sigma} i \psi_m^t(R, \theta_\sigma, \varphi_\sigma) - \sum_{p,t'} c_p^{t'\,\sigma} E_{mp}^{tt'}\,, \qquad (28a)$$

with

$${}^{\sigma}E_{mp}^{tt'} = \mathrm{j}\, k_0 \rho_0 c_0 Y_\sigma \langle \psi_p^{t'} | \psi_m^t \rangle_\sigma \qquad (28b)$$

and

$$\langle \psi_p^{t'} | \psi_m^t \rangle_\sigma = \psi_m^t(R, \theta_\sigma, \varphi_\sigma) \psi_p^{t'}(R, \theta_\sigma, \varphi_\sigma) \Delta S_\sigma\,. \qquad (28c)$$

The influence of the source in the acoustic field is then composed of an active part linked to the electrical excitation applied on the microphone, and a reactive part due to the acoustic input admittance of the microphone flush-mounted on the wall.

The expressions (23) and (28a-c) of the effects of small elements flush-mounted on the wall of the cavity shows that these effects are strongly dependant on the relative position of these elements on the wall (neglected in the previous works).

D. Results

1. Effect of modal coupling on the acoustic pressure

The theoretical effect of modal coupling on the bulk behaviour of the acoustic pressure in a spherical cavity filled with argon gas, at the triple point of water for different static pressures is shown on figures 2 to 5, representing the cartographic projection of the amplitude of the acoustic pressure on the wall of the cavity, with respect to the spherical coordinates φ and θ, when the transmitting microphone generates a harmonic signal at the resonance frequency of a radial mode. The cavity has only a transmitting and a receiving microphone on its wall.

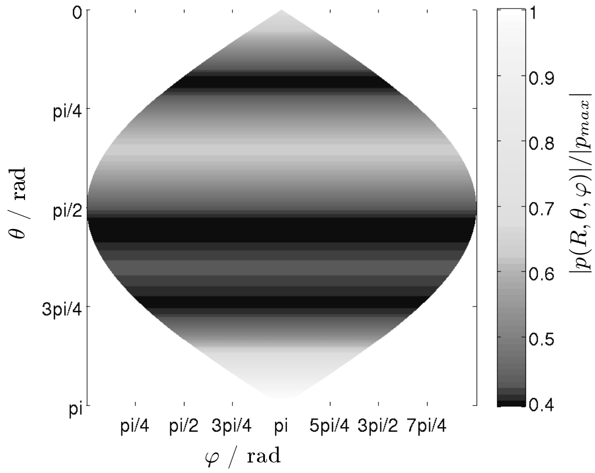

Figure 2. Acoustic pressure on the wall of a spherical cavity at the resonance frequency of the (0,2) radial mode, at the static pressure 25 kPa.

These almost qualitative representations in the lower static pressure range, on figures 2 and 3, show that the acoustic field is quasi-axisymmetric around the axis of the transmitting microphone because the influence of the receiving microphone is minimum. The maxima at $\theta = \pi$ and $\theta = 0$ are due to the transmitting microphone. The other deviations from a uniform acoustic pressure, that should be observed at the resonance frequency of a radial mode, are due to the influence of the viscous and thermal effects on non radial modes which are immediately close to the radial mode expected. Then, the (0,2) mode is mostly perturbed by the (3,1) mode, and the (0,5) mode by the (12,1) mode, all of them without azimuthal components because of the axisymmetry of the system for such low static pressures.

In the higher static pressure range, the viscous and thermal effects are less important, then the deviations from a uniform acoustic pressure on the wall of the cavity due to modal coupling are smaller, which can be seen

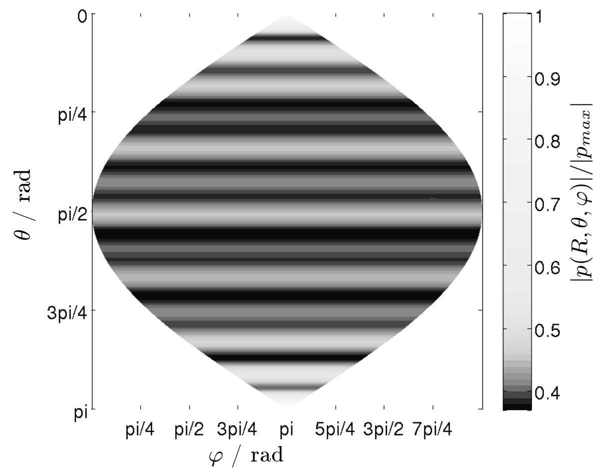

Figure 3. Acoustic pressure on the wall of a spherical cavity at the resonance frequency of the (0,5) radial mode, at the static pressure 25 kPa.

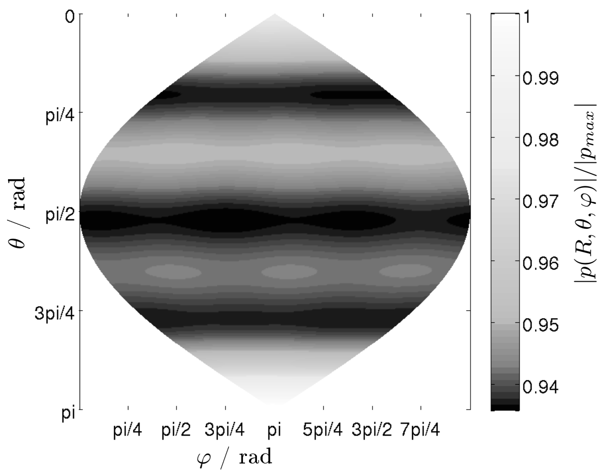

Figure 4. Acoustic pressure on the wall of a spherical cavity at the resonance frequency of the (0,2) radial mode, at the static pressure 500 kPa.

on figures 4 and 5. The impedance of the receiving microphone decreases, and then its influence on the acoustic pressure in the cavity increases (but remains small), generating also azimuthal modes by modal coupling.

Such maps of the acoustic field on the wall of the cavity could be used to support the design of an acoustic resonator for the determination of the Boltzmann constant, or for acoustic thermometry. Indeed, they show us the optimal places on the wall for the measurement devices to maximize the signal to noise ratio and reduce the perturbations effects. For example here, the best place for the receiving microphone would be in front of the transmitting microphone (maximum of acoustic pressure for all the acoustic modes and static pressures of interest, axisymmetric acoustic system).

Moreover, the particular results presented in this section show that to elucidate the individual physical phenomena in the formalism, it is necessary to consider separately the physical phenomena involved and to do simplifying assumptions.

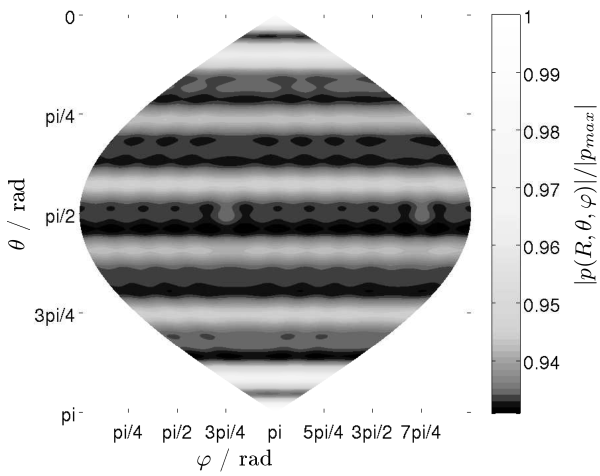

Figure 5. Acoustic pressure on the wall of a spherical cavity at the resonance frequency of the (0,5) radial mode, at the static pressure 500 kPa.

2. Influence of modal coupling on the acoustic resonance properties of the cavity

For the determination of the Boltzmann constant, the value of k is derived from the extrapolation at zero pressure of the speed of sound determined from measurements of the acoustic resonance frequencies of radial modes in the cavity, on which are applied correction terms calculated from the model of the acoustic field in the cavity. The comparison between the experimental and theoretical half-widths allows us to check the validity of the model used to calculate the corrections to be applied on the resonance frequencies.

The global model presented here allows us to calculate the acoustic frequency response of the resonator. Fitting a resonant function (lorenzian function) with this calculated transfer function around the resonance frequency of an acoustic mode gives us the resonance frequency and the half-width of this mode according to the global analytical model.

Since the global formalism presented here includes modal coupling (non-diagonal terms in the matrices), a direct analytic expression of the perturbations effects on the resonance properties of the cavity cannot be derived from this global modelling without doing several simplifying assumptions. Then, only a global and numerical comparison between the current and previous theoretical results is presented here. However, it is worth to notice that neglecting the terms $m \neq p$ in the matrix E in equation (16) leads finally to the analytical expressions of the frequency shifts and half-widths determined in previous works[2,6].

The effect of the acoustic perturbations in the cavity and of the resulting modal coupling on the resonance properties of radial modes from (0,2) to (0,6) as function of the static pressure is shown on figures 6 and 7. The major perturbation in the cavity being the viscous and thermal effects, the absolute value of the frequency shifts and the half-widths both increase when the static pressure and the quantum number of the modes decrease.

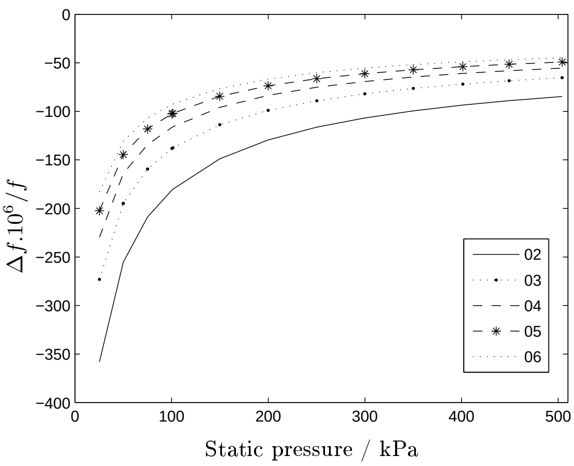

Figure 6. Frequency shifts for radial modes from (0,2) to (0,6) as function of the static pressure.

The comparison between the results obtained with this unified model and the model used previously at the NIST to determine the current value of the Boltzmann constant[2] (Fig. 8 and 9) shows the discrepancies due to the only modal coupling, the models used for each source of perturbation being the same in both methods.

According to the curves on Fig. 8, the contribution of modal coupling in the resonance frequencies of the radials modes considered here is significant in the higher static pressure range, but lower than 0.2×10^{-6} in relative values in the lower static pressure range, and then should not take a larger part in the extrapolation at zero static pressure of the speed of sound. However, this contribution should not be neglected regarding the final uncertainty required on the value of the Boltzmann constant. Here, the effect of modal coupling on the values of the resonance frequencies is nearly a linear function of the static pressure and does not depend significantly on the frequency. To interpret this, a few calculation on a simplified expression of equation (16) (neglecting the viscous

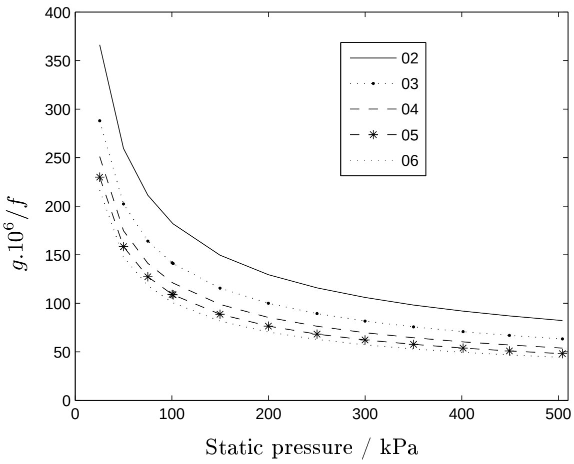

Figure 7. Theoretical half-widths for radial modes from (0,2) to (0,6) as function of the static pressure.

and thermal effects) leads to express directly the effect of modal coupling on the resonance frequencies as a term proportional to $\rho_0 c_0^2 \chi \Delta S_r$ (χ being the constant compliance of the receiving microphone), which is a linear function of the static pressure, and does not depend on the frequency.

The differences between the half-widths (Fig. 9) calculated from the two models confirm that the effect of the modal coupling in the resonance properties of radial modes should not be neglected in this spherical resonator.

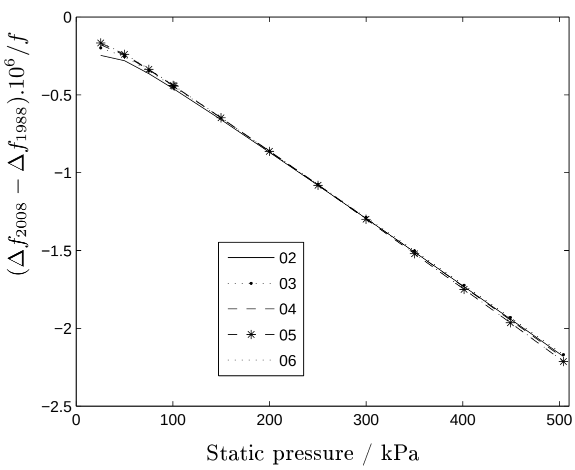

Figure 8. Differences between the frequency shifts calculated according to actual and previous theories for radial modes from (0,2) to (0,6) as function of the static pressure.

Finally, the comparison between the theoretical half-widths calculated from the unified modelling of the acous-

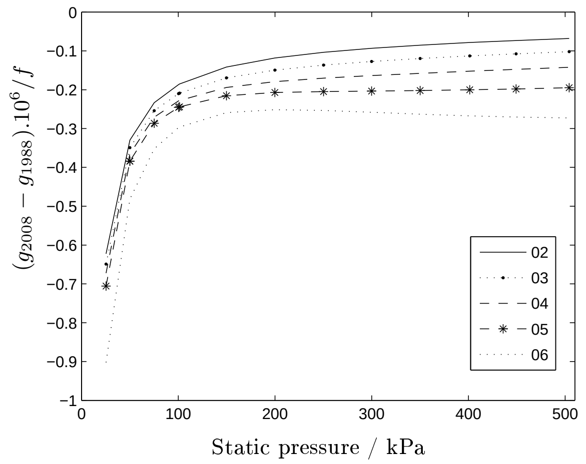

Figure 9. Differences between the half-widths calculated according to actual and previous theories for radial modes from (0,2) to (0,6) as function of the static pressure.

tic field and the half-widths measured twenty years ago at the NIST[2] is shown on Fig. 10 with the respective experimental deviations, as function of the static pressure for acoustic radial modes from (0,2) to (0,6). For static pressures higher than 50 kPa, the discrepancies between theory and experiment are larger than the experimental deviations, showing that this model is not sufficient yet to take into account all the phenomena that ocur in this spherical cavity. In the lower static pressure range, the experimental deviations are too large to conclude on the agreement between the analytical and the experimental results.

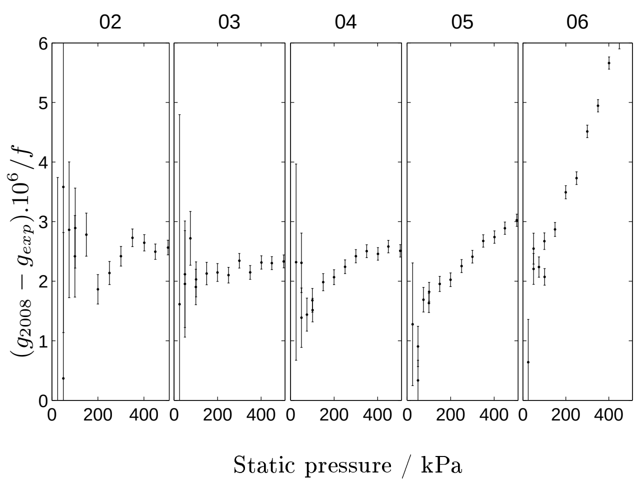

Figure 10. Excess half-widths and experimental deviations for radial modes from (0,2) to (0,6).

V. CONCLUSION

To sum up, the present paper provides some important improvements to the modelling of the acoustic field in a quasi-spherical cavity with irregular wall (§ III) because the model given here allows us to take into account all the different perturbing elements together: the viscous and thermal effects in the boundary layers and in the bulk of the gas, the shape of the cavity (quasi-spherical shape), small elements located on the wall of the cavity (ducts, acoustic transducers, small geometrical irregularities), the roughness of the wall, the coupling between the fluid and the shell. This unified model includes the coupling between the different perturbing elements, depending on the relative position of these elements on the wall for the small local components, with respect to the real shape of the cavity, and the resulting modal coupling in a consistent manner.

Regarding the first results obtained with this model on the particular case of the spherical cavity used at the NIST twenty years ago[2], it seems that the contribution of modal coupling in the resonance frequency is small but should not be neglected regarding the accuracy required here. Moreover, the discrepancies between the theoretical and experimental half-widths show that several phenomena are still missing in this unified model and should be taken into account in the future.

This work have to be carried out with the accurate modelling of each small element located on the wall of the cavity, especially the acoustic transducers, thin slits (undesirable but unavoidable in practice) which could be modelled as local roughness and the coupling between the fluid and the shell. All these perturbing elements would be taken into account in the global modelling suggested here in term of source or specific admittance on the wall.

Another improvement would be the modelling of the viscous and thermal effects at a higher order, which would avoid to separate the effects in the boundary layer and in the bulk, taking into account low density gas effects (thermal jump, slip conditions), and to integrate it in the unified modelling of the acoustic field in the cavity.

Appendix A: ACOUSTIC PROPAGATION IN DISSIPATIVE FLUID

1. Propagation equations

The particle velocity field $\vec{v}$ can be written as the sum of a vortical velocity $\vec{v}_v$ and a laminar velocity $\vec{v}_l$:

$$\vec{v} = \vec{v}_v + \vec{v}_l \,, \tag{A1a}$$

$$\vec{\mathrm{rot}}\, \vec{v}_l = \vec{0} \quad \text{and} \quad \mathrm{div}\, \vec{v}_l \neq 0 \,, \tag{A1b}$$

$$\mathrm{div}\, \vec{v}_v = 0 \quad \text{and} \quad \vec{\mathrm{rot}}\, \vec{v}_v \neq \vec{0} \,. \tag{A1c}$$

Neglecting the coupling between the vortical and laminar movements in the bulk of the fluid, Navier–Stokes equation (1) is written as two equations, and then the fundamental equations (1) to (3) lead straightforwardly to the propagation equations for p, τ, $\vec{v}_l$ and $\vec{v}_v$.

The calculation of the propagation equation for the temperature variation τ is given in reference[11] (§ 2.5.1), neglecting the terms of order greater than the first order of the viscous and thermal lenghts. The temperature variation τ can be written as the sum of an acoustic temperature τ_a and an entropic temperature τ_h, respectively solutions of the two following equations:

$$\left[\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \left(1 + \ell_{vh}\,\frac{1}{c_0}\frac{\partial}{\partial t}\right)\Delta\right]\tau_a \approx 0\,, \tag{A2}$$

where $\ell_{vh} = \ell_v + (\gamma - 1)\ell_h$,

$$\left[\frac{1}{c_0}\frac{\partial}{\partial t} - \ell_h\,\Delta\left(1 + (\gamma-1)(\ell_v - \ell_h)\frac{1}{c_0}\frac{\partial}{\partial t}\right)\right]\tau_h \approx 0\,. \tag{A3}$$

The pressure variation $p = p_a + p_h$ and the laminar particle velocity $\vec{v}_l = \vec{v}_a + \vec{v}_h$ satisfy the same equations (A2) and (A3) than the temperature variation τ.

The propagation equation for the vortical particle velocity is obtained making use of equations (A1a-c) in the Navier–Stokes equation (1):

$$\left[\frac{1}{c_0}\frac{\partial}{\partial t} - \ell'_v\,\Delta\right]\vec{v}_v = \vec{0}\,. \tag{A4}$$

2. General solutions

The solutions for the pressure variation $p = p_a + p_h$ and the particle velocity $\vec{v}_l = \vec{v}_a + \vec{v}_h$ can be derived from the solution for the temperature variation τ. Writing out the solution $\tau = \tau_a + \tau_h$ in equation (3), making use of equations (A2) and (A3) leads to

$$p_a \approx \frac{\gamma\hat{\beta}}{\gamma - 1}\left(1 - \ell_h\,\frac{1}{c_0}\frac{\partial}{\partial t}\right)\tau_a\,, \tag{A5a}$$

$$p_h \approx \gamma\hat{\beta}(\ell_v - \ell_h)\frac{1}{c_0}\frac{\partial}{\partial t}\tau_h << p_a\,. \tag{A5b}$$

Writting out equations (A5a,b) in equation (1) and making use of equations (A1), (A2) and (A3), gives for the particle velocities $\vec{v}_a$ and $\vec{v}_h$:

$$\vec{v}_a \approx -\frac{1}{\rho_0 c_0}\frac{\gamma\hat{\beta}}{\gamma - 1}\left[\left(\frac{1}{c_0}\frac{\partial}{\partial t}\right)^{-1} + (\ell_v - \ell_h)\right]\vec{\mathrm{grad}}\,\tau_a\,, \tag{A6a}$$

$$\vec{v}_h \approx \frac{\gamma\hat{\beta}}{\gamma - 1}\ell_h\,\vec{\mathrm{grad}}\,\tau_h\,. \tag{A6b}$$

Equations (A2) to (A6b) can also be written for harmonic motion $e^{j\omega t}$ making use of the three following wavenumbers:

$$k_a^2 \approx \frac{k_0^2}{1 + j\,k_0\ell_{vh}} \approx k_0^2[1 - j\,k_0\ell_{vh}]\,, \tag{A7}$$

$$k_h^2 \approx -\frac{j\,k_0}{\ell_h}[1 - j\,k_0(\gamma-1)(\ell_v - \ell_h)]\,, \tag{A8}$$

$$k_v^2 = -\frac{j\,k_0}{\ell'_v}\,. \tag{A9}$$

Assuming that $p_a >> p_h$, the acoustic pressure p_a, denoted p in the present paper, is governed by the same propagation equation (A2) as the acoustic temperature variation τ_a. This propagation equation should be associated to adapted boundary conditions to describe the acoustic field in the cavity.

Appendix B: NORMALIZATION COEFFICIENTS

The normalization coefficients $\mathcal{J}_{\nu\eta}$ of the spherical bessel functions $j_\nu(k_{\eta\nu}r)$ (eq. 21) are given by the orthonormality condition on the bessel functions:

$$\mathcal{J}_{\nu\eta}^2 = \int_0^R [j_\nu(k_{\eta\nu}r)]^2\,r^2\,\mathrm{d}r\,,$$

yielding

$$\mathcal{J}_{\nu\eta}^2 = \frac{R^3}{2}\,, \tag{B1a}$$

for $\nu = 0$ and $\eta = 1$,

$$\mathcal{J}_{\nu\eta}^2 = \frac{R^3}{2}\,[j_\nu(\gamma_{\eta\nu})]^2\left[1 - \frac{\nu(\nu+1)}{\gamma_{\nu\eta}^2}\right]\,, \tag{B1b}$$

otherwise.

The normalization coefficients $\mathcal{J}_{\nu\eta}$ of the spherical bessel functions $j_\nu(k_{\eta\nu}r)$ (eq. 6) are given by the orthonormality condition on the eigenfunctions $\psi_m^t(\vec{r})$ in the regular domain D_r:

$$a_m^2 = \iiint_{D_r}\left[\psi_m^t(\vec{r})\right]^2\,\mathrm{d}\vec{r}\,,$$

leading to

$$a_m^2 = \frac{4}{3}\pi R^3\,, \tag{B2a}$$

for $\nu = 0$ and $\eta = 1$,

$$a_m^2 = (1 + \delta_{\mu 0})\pi R^3\,[j_\nu(\gamma_{\eta\nu})]^2\left[1 - \frac{\nu(\nu+1)}{\gamma_{\nu\eta}^2}\right]\frac{1}{2\nu+1}\frac{(\nu+\mu)!}{(\nu-\mu)!}\,, \tag{B2b}$$

otherwise.

[1] I. M. Mills, P. J. Mohr, T. J. Quinn, B. N. Taylor, and E. R. Williams, "Redefinition of the kilogramm, ampere, kelvin and mole: a proposed approach to implementing CIPM recommendation 1 (CI-2005)", Metrologia **43**, 227–246 (2006).
[2] M. R. Moldover, J. P. M. Trusler, T. J. Edwards, J. B. Mehl, and R. S. Davis, "Measurement of the universal gas constant R using a spherical acoustic resonator", Journal of Research of the National Bureau of Standards **93**, 85–144 (1988).
[3] J. B. Mehl, "Acoustic resonance frequencies of deformed spherical resonators", Journal of the Acoustical Society of America **71**, 1109–1113 (1982).
[4] J. B. Mehl, "Acoustic resonance frequencies of deformed spherical resonators 2", Journal of the Acoustical Society of America **79**, 278–285 (1986).
[5] M. R. Moldover, J. B. Mehl, and M. Greenspan, "Gas-filled spherical resonators: Theory and experiment", Journal of the Acoustical Society of America **79**, 253–272 (1986).
[6] J. B. Mehl, M. R. Moldover, and L. Pitre, "Designing quasi-spherical resonators for acoustic thermometry", Metrologia **41**, 295–304 (2004).
[7] J. P. M. Trusler, *Physical acoustics and metrology of fluids* (Adam Higler, Bristol) (1991).
[8] J. W. S. Rayleigh, *The Theory of Sound*, second edition (Dover, New York) (1896).
[9] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York) (1968).
[10] A. D. Pierce, *Acoustics: an Introduction to its Physical Principles and Applications* (McGraw-Hill, New York) (1981).
[11] M. Bruneau and T. Scelo (translator and contributor), *Fundamentals of Acoustics* (ISTE, London) (2006).
[12] K. S. Sum and J. Pan, "Effects of the inclination of a rigid wall on the free vibration characteristics of acoustic modes in a trapezoidal cavity", Journal of the Acoustical Society of America **119**, 2201–2210 (2006).
[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York) (1953).
[14] J. B. Mehl, "Spherical acoustic resonators: Effect of shell motion", Journal of the Acoustical Society of America **78**, 782–788 (1985).