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MATHEMATICAL ANALYSIS OF THE MOTION OF A PISTON IN A FLUID WITH DENSITY DEPENDENT VISCOSITY

VAIBHAV KUMAR JENA, DEBAYAN MAITY*, AND ABU SUFIAN

ABSTRACT. We study a free boundary value problem modelling the motion of a piston in a viscous compressible fluid. The fluid is modelled by 1D compressible Navier-Stokes equations with possibly degenerate viscosity coefficient, and the motion of the piston is described by Newton's second law. We show that the initial boundary value problem has a unique global in time solution, and we also determine the large time behaviour of the system. Finally, we show how our methodology may be adapted to the motion of several pistons.

1. INTRODUCTION AND MAIN RESULTS

In this article, we study the motion of a piston (point particle) inside a cylinder containing a viscous compressible fluid, with a density dependent viscosity. The fluid is modelled by the one-dimensional compressible Navier-Stokes equations, while the motion of the piston is described by Newton's second law. We assume that the problem is posed in a bounded domain $(-1, 1)$ and we denote by $h(t)$ the position of the piston at instant t . The domain occupied by fluid at instant t is denoted by $\mathcal{F}_{h(t)} := (-1, 1) \setminus \{h(t)\}$. With the above notations, the coupled motion of the piston and of the fluid is given by :

$$\tilde{\rho}_t + (\tilde{\rho}\tilde{u})_x = 0, \quad (t \geq 0, x \in \mathcal{F}_{h(t)}), \quad (1.1)$$

$$\tilde{\rho}(\tilde{u}_t + \tilde{u}\tilde{u}_x) - (\mu(\tilde{\rho})\tilde{u}_x)_x + (\tilde{\rho}^\gamma)_x = 0, \quad (t \geq 0, x \in \mathcal{F}_{h(t)}), \quad (1.2)$$

$$\tilde{u}(t, h(t)) = \dot{h}(t), \quad (t \geq 0), \quad (1.3)$$

$$m\ddot{h}(t) = [\mu(\tilde{\rho})\tilde{u}_x - \tilde{\rho}^\gamma](t, h(t)), \quad (t \geq 0), \quad (1.4)$$

with initial conditions

$$\begin{cases} h(0) = h_0, & \dot{h}(0) = \ell_0, \\ \tilde{u}(0, x) = \tilde{u}_0(x), & \tilde{\rho}(0, x) = \tilde{\rho}_0(x), \end{cases} \quad (x \in \mathcal{F}_{h_0}), \quad (1.5)$$

and boundary conditions

$$\tilde{u}(t, 1) = 0, \quad \tilde{u}(t, -1) = 0, \quad (t \geq 0). \quad (1.6)$$

In the above equations, $\gamma > 1$ is a constant, $\tilde{\rho}(t, x)$ denotes the density, and $\tilde{u}(t, x)$ denotes the velocity of the fluid (both in Eulerian coordinates). The positive constant m stands for the mass of the piston. The symbol $[f](x)$ denotes the jump of a function at the point x , i.e.,

$$[f](x) := f(x^+) - f(x^-).$$

In general, the viscosity coefficient $\mu(\tilde{\rho})$ in (1.2) is assumed to be a positive constant. In this article, we assume that $\mu(\tilde{\rho})$ depends on the density field $\tilde{\rho}$. More precisely, we consider the viscosity

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of the form

$$\mu(\tilde{\rho}) = \tilde{\rho}^\theta, \quad 0 < \theta < \frac{1}{2}. \quad (1.7)$$

Remark 1.1. Note that, in (1.6), we have homogeneous boundary condition for the velocity field, which means that there is no inflow or outflow of fluid from the cylinder. In particular, we do not need to impose any boundary condition for the density field.

For many years, the initial boundary value problem (1.1)–(1.6), at least when the viscosity coefficient is a positive constant, has piqued the interest of many researchers. Let us briefly review some relevant results from the literature. As far as we know, the problem was first studied by Shelukhin [22, 23], in a functional setup more regular than the ones which appear below. Later, Maity, Takahashi and Tucsnak [17] proved global in time existence and uniqueness of strong solutions in the same functional framework as ours using a monolithic approach. In fact, they studied the problem with inflow boundary conditions. We also mention the work of Shelukhin [24], which addresses the piston problem where the viscous gas and the piston are supposed to be heat conducting. The adiabatic piston problem was studied by Feireisl et. al. [7]. For the existence of weak solutions we refer to the articles by Plotnikov and Sokolowski [20] and Lequeurre [15]. We also refer to the article by Antman and Wilber [1], where the authors study the asymptotic problem as the ratio of the mass of fluid and that of the piston approaches zero.

The aforementioned results concern the existence and long-time behavior of the initial boundary value problem. There are few results also available for the Cauchy problem. We refer to the articles by Vázquez and Zuazua [25, 26] for results on the piston problem where the fluid is modelled by the viscous Burgers' equation. They showed that the piston escapes to the spatial infinity as time goes to infinity. Koike [12, 14, 13] studied the Cauchy problem for compressible fluid with constant viscosity. In contrast to the viscous Burgers' case, it was shown that the piston stabilizes to a finite distance as time approaches infinity.

The piston problem was also studied from a control theoretic point of view. For instance, the works of Liu, Takahashi and Tucsnak [16], Cîndea et. al. [4] and Ramaswamy, Roy and Takahashi [21] address the control problem for a piston in a fluid modelled by the viscous Burgers' equation. An optimal control problem for the compressible piston problem with constant viscosity was studied in [20]. Karafyllis and Krstic [11] studied the global feedback stabilisation of a system consisting of a viscous compressible fluid between two moving pistons, with density dependent viscosity.

Another set of references relevant to our work is the motion of a rigid body in a viscous compressible fluid, which is set up in \mathbb{R}^3 . In this context, we refer to articles by Boulakia and Guerrero [2], Feireisl [6], Hiber and Murata [10], Haak et. al. [8], and the references therein.

In this work, we extend the results of [22, 23, 17] to the case where viscosity depends on density, possibly in degenerate manner (see(1.7)). We prove global in time existence in the functional framework same as [17]. Moreover, we also determine the large time behaviour of the system. There are several results regarding global existence of one dimensional compressible fluid, without any piston, with viscosity depending on density of the form $\mu(\tilde{\rho}) = \tilde{\rho}^\theta$. The first result was due to Mellet and Vasseur [19] where they considered the case $0 < \theta < \frac{1}{2}$. Subsequently, the case $\theta \geq \frac{1}{2}$ has been studied by Haspot [9], Constantin et. al. [5] and Burtea and Haspot [3]. In this paper we consider the case $0 < \theta < \frac{1}{2}$, and we combine the approach of [19] together with [23, 17]. We follow a standard multiplier approach, to pass from local in time existence to global in time existence. The main step is to show that, for all time, the piston remains away from the extremities of the cylinder and the density is bonded below by some positive constant. We also show that our methodology also adapts to the case of several pistons. Similar to the single piston case for the several piston problem, we obtain the global (in time) existence of solution and we determine the large time behaviour of the system. For the Cauchy problem this has been studied by Koike [13] in the constant viscosity case.

Now we will present our main results. For this purpose, we will first provide the definition of solution to system (1.1)-(1.7). We look for solutions $(h, \tilde{\rho}, \tilde{u})$ to system (1.1)-(1.7) satisfying

$$\begin{aligned} h &\in H^2(0, T), \quad -1 < h(t) < 1 \text{ for all } t \in [0, T], \\ \tilde{\rho} &\in C([0, T]; H^1(\mathcal{F}_{h(\cdot)})) \cap H^1(0, T; L^2(\mathcal{F}_{h(\cdot)})), \\ \tilde{\rho}(t, x) &> 0 \quad \text{for all } t \in [0, T], x \in \mathcal{F}_{h(t)}, \\ \tilde{u} &\in L^2(0, T; H^2(\mathcal{F}_{h(\cdot)})) \cap H^1(0, T; L^2(\mathcal{F}_{h(\cdot)})) \cap C([0, T]; H_0^1(-1, 1)). \end{aligned} \quad (1.8)$$

We introduce below the concept of strong solution of (1.1)-(1.7), to be used in the remaining part of this work.

Definition 1.2. A triplet $(h, \tilde{\rho}, \tilde{u})$ is said to be a strong solution to the problem (1.1)-(1.7) on the interval $[0, T]$ when it satisfies (1.8), equations (1.1) and (1.2) a.e. in $(0, T) \times \mathcal{F}_{h(\cdot)}$, equation (1.4) a.e. in $(0, T)$, equations (1.3) and (1.6) in the sense of traces, and the initial conditions stated in (1.5).

We are now in a position to state the global existence result

Theorem 1.3. Let us assume that μ satisfies (1.7) for some $\theta \in \left(0, \frac{1}{2}\right)$, $h_0 \in (-1, 1)$, $\ell_0 \in \mathbb{R}$, and $(\tilde{\rho}_0, \tilde{u}_0)$ belongs to $H^1(\mathcal{F}_{h_0}) \times H_0^1(-1, 1)$ and satisfy

$$\tilde{u}_0(h_0) = \ell_0, \quad \tilde{\rho}_0(x) > 0 \text{ for } x \in [-1, 1] \setminus \{h_0\}.$$

Then, for any $T > 0$ the system (1.1)-(1.7) admits a unique strong solution on $[0, T]$, in the sense of Definition 1.2.

Our next goal is to determine the large time behaviour of the global solution. More precisely, we show that the global solution converges to the equilibrium. First of all, a simple calculation gives, for the global solution to the system (1.1)-(1.7), we have the mass conservation, i.e.,

$$M_L := \int_{-1}^{h(t)} \tilde{\rho}(t, x) dx = \int_{-1}^{h_0} \tilde{\rho}_0(x) dx, \quad M_R := \int_{h(t)}^1 \tilde{\rho}(t, x) dx = \int_{h_0}^1 \tilde{\rho}_0(x) dx, \quad (t \geq 0). \quad (1.9)$$

Lemma 1.4. Let $(h_\infty, \rho_\infty, u_\infty)$ be an equilibrium solution to the system (1.1)-(1.7). Then

$$h_\infty = \frac{M_L - M_R}{M_L + M_R}, \quad \rho_\infty = \frac{1}{2}(M_L + M_R), \quad u_\infty = 0,$$

where M_L and M_R are defined in (1.9).

Proof. From (1.1), (1.2), and (1.6), we first infer that

$$u_\infty = 0, \quad \rho_\infty = \begin{cases} \rho_{\infty,L}, & x \in (-1, h_\infty), \\ \rho_{\infty,R}, & x \in (h_\infty, 1), \end{cases}$$

where $\rho_{\infty,L}$ and $\rho_{\infty,R}$ are positive constants. Then using the equation of piston (1.4), we get $\rho_\infty = \rho_{\infty,L} = \rho_{\infty,R}$. From (1.9), we deduce that

$$M_L = (1 + h_\infty)\rho_\infty, \quad M_R = (1 - h_\infty)\rho_\infty,$$

from which we conclude the proof of the lemma. \square

We have the following result concerning the large time behaviour of the system:

Theorem 1.5. The global strong solution $(h, \tilde{\rho}, \tilde{u})$ to the system (1.1)-(1.7), satisfies the following

$$\|\tilde{\rho}(t, \cdot) - \rho^\infty\|_{H^1(\mathcal{F}_{h(t)})} + \|\tilde{u}(t, \cdot)\|_{H^1(-1, 1)} + |h(t) - h^\infty| \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

where ρ^∞ and h^∞ are given in Lemma 1.4.

The rest of the article is organised as follows. In Section 2, we transform system (1.1)-(1.6) into mass Lagrangian coordinates, and present the main results in the transformed coordinate. We also present the local in time existence in this section. In Section 3, we prove the global in time existence of solutions, and in Section 4, we address the large time behaviour of the solution. In Section 5, we explain how the results can be extended to case of several piston. Finally, in Section 6 we make some comments about related open problems.

2. CHANGE OF COORDINATES

In this section, we rewrite the system in a fixed domain using Lagrangian mass change of coordinates. One of the advantages of this change of coordinates is that the position of the piston becomes fixed, see for instance [17, 7]. In Lagrangian mass coordinate, we replace the physical variable x by the signed mass of the fluid between $h(t)$ and x . More precisely, we set

$$y = X(t, x), \quad X(t, x) = \int_{h(t)}^x \tilde{\rho}(t, \eta) d\eta \quad (t \geq 0, x \in [-1, 1]). \quad (2.1)$$

Then using (1.1) and (1.9),

$$X(t, -1) = -r_-, \quad X(t, 1) = r_+, \quad \text{and } X(t, h(t)) = 0, \quad (t \geq 0), \quad (2.2)$$

where

$$r_- = M_L = \int_{-1}^{h_0} \tilde{\rho}_0(\eta) d\eta, \quad r_+ = M_R = \int_{h_0}^1 \tilde{\rho}_0(\eta) d\eta. \quad (2.3)$$

Assume that $\tilde{\rho}(t, \cdot)$ is sufficiently regular, positive and bounded away from zero. Then one can easily verify that X defined in (2.1) is a C^1 -diffeomorphism from $[-1, 1]$ to $[-r_-, r_+]$. For every $t \geq 0$, we denote by $Y(t, \cdot) = [X(t, \cdot)]^{-1}$, the inverse of $X(t, \cdot)$. In what follows, we set

$$\mathcal{F} := (-r_-, r_+) \setminus \{0\}, \quad \mathcal{F}_- = (-r_-, 0), \quad \mathcal{F}_+ = (0, r_+). \quad (2.4)$$

We consider the following change of variables

$$\rho(t, y) = \tilde{\rho}(t, Y(t, y)), \quad u(t, y) = \tilde{u}(t, Y(t, y)), \quad (t \geq 0, y \in \mathcal{F}). \quad (2.5)$$

In particular,

$$\tilde{\rho}(t, x) = \rho(t, X(t, x)), \quad \tilde{u}(t, x) = u(t, X(t, x)), \quad (t \geq 0, x \in \mathcal{F}_{h(t)}). \quad (2.6)$$

According to [17, 7], using the above change of variables, the system (1.1)-(1.6) can be written as

$$\rho_t + \rho^2 u_y = 0, \quad (t \geq 0, y \in \mathcal{F}), \quad (2.7)$$

$$u_t - (\rho \mu(\rho) u_y)_y + (\rho^\gamma)_y = 0, \quad (t \geq 0, y \in \mathcal{F}), \quad (2.8)$$

$$u(t, 0) = \dot{h}(t), \quad (t \geq 0), \quad (2.9)$$

$$m \ddot{h}(t) = [\rho \mu(\rho) u_y - \rho^\gamma](t, 0), \quad (t \geq 0), \quad (2.10)$$

$$u(t, -r_-) = u(t, r_+) = 0, \quad (t \geq 0), \quad (2.11)$$

$$\rho(0, y) = \rho_0(y), \quad u(0, y) = u_0(y), \quad (y \in \mathcal{F}), \quad (2.12)$$

$$h(0) = h_0, \quad \dot{h}(0) = \ell_0, \quad (2.13)$$

where

$$\rho_0(y) = \tilde{\rho}_0(Y(0, y)), \quad u_0(y) = \tilde{u}_0(Y(0, y)), \quad (y \in \mathcal{F}). \quad (2.14)$$

From the definition of X in (2.1), the following lemma is obvious.

Lemma 2.1. Let $(h_0, \ell_0, \tilde{\rho}_0, \tilde{u}_0)$ satisfy the hypothesis of Theorem 1.3. Then (ρ_0, u_0) defined by (2.14) satisfies

$$\rho_0 \in H^1(\mathcal{F}), \quad u_0 \in H_0^1(-r_-, r_+), \quad u_0(0) = \ell_0, \quad \rho_0(y) > 0, \quad \text{for } y \in \mathcal{F}. \quad (2.15)$$

We give the definition of strong solution of the system (2.7)–(2.14).

Definition 2.2. A triplet (h, ρ, u) is said to be a strong solution to the system (2.7)–(2.14) on the interval $[0, T]$ when it satisfies

$$\begin{aligned} h &\in H^2(0, T), \quad -1 < h(t) < 1 \text{ for all } t \in [0, T], \\ \rho &\in C([0, T]; H^1(\mathcal{F})) \cap H^1(0, T; L^2(\mathcal{F})), \\ \rho(t, y) &> 0, \quad \text{for all } t \in [0, T], y \in \mathcal{F}, \\ u &\in L^2(0, T; H^2(\mathcal{F})) \cap H^1(0, T; L^2(\mathcal{F})) \cap C([0, T]; H_0^1(-r_-, r_+)), \end{aligned}$$

equations (2.7) and (2.8) a.e. in $(0, T) \times \mathcal{F}$, equation (2.10) a.e. in $(0, T)$, equations (2.9) and (2.11) in the sense of traces, and the initial conditions stated in (2.12), (2.13).

Using the above change of variables Theorem 1.3 and Theorem 1.5 can be rephrased as follows.

Theorem 2.3. Let us assume that μ satisfies (1.7) for some $\theta \in \left(0, \frac{1}{2}\right)$, $h_0 \in (-1, 1)$, $\ell_0 \in \mathbb{R}$, and (ρ_0, u_0) satisfy (2.15). Then, for any $T > 0$ the system (2.7)–(2.14) admits a unique strong solution on $[0, T]$, in the sense of Definition 2.2.

Theorem 2.4. The global strong solution (h, ρ, u) to the system (2.7)–(2.13), satisfies the following

$$\|\rho(t, \cdot) - \rho^\infty\|_{H^1(\mathcal{F})} + \|u(t, \cdot)\|_{H^1(-r_-, r_+)} + |h(t) - h^\infty| \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

where ρ^∞ and h^∞ are given in Theorem 1.5.

The remaining part of this paper is devoted towards the proof of Theorem 2.3 and Theorem 2.4. The proof of Theorem 2.3 relies on a classical argument. We first show that the system (2.7)–(2.14) admits a unique local in time solution. Next we show that the solution can be continued for all time. Regarding the local in time existence, we have the following result.

Theorem 2.5. Let us assume that μ satisfies (1.7) for some $\theta \geq 0$. Let $C > 0$ be such that

$$\begin{aligned} \|u_0\|_{H_0^1(-r_-, r_+)} + \|\rho_0\|_{H^1(\mathcal{F})} &\leq C, \\ \frac{1}{C} &\leq \rho_0(y) \leq C \quad (y \in \mathcal{F}), \\ -1 + \frac{1}{C} &\leq h_0 \leq 1 - \frac{1}{C}. \end{aligned}$$

Then, there exists a T depending only on C , such that the system (2.7)–(2.13) admits a unique strong solution on $[0, T]$ in the sense of Definition 2.2.

The proof of the above result relies on maximal regularity result of a monolithic linear system and the Banach fixed point theorem. We refer to the proof of [17, Theorem 3.1] for a detailed presentation of the method (see also [18]). Indeed the proof of Theorem 2.5 can be directly adapted from that of [17, Theorem 3.1], with some slight modifications. Since the calculations are almost identical and too much of a repetition, we omit the details here.

3. GLOBAL IN TIME EXISTENCE AND UNIQUENESS

In this section, we are going to prove Theorem 2.3. More precisely, we are going to show that the local solution constructed in Theorem 2.5 can be extended to a solution defined on $[0, T]$ for any $0 < T < \infty$. Throughout this section, we assume that $(h_0, \ell_0, \rho_0, u_0)$ satisfy the hypothesis of (2.15), and (h, ρ, u) is the maximal strong solution to the system (2.7)–(2.13) associated with this initial data. This solution is defined on some time interval $[0, \tau)$, where $\tau > 0$.

Throughout this section, C will be a positive constant independent of τ . The constants may vary from line to line.

The proof is divided into several parts. First, we obtain the standard energy estimate as well as a modified energy-type estimate, inspired by [19], for the system (2.7)-(2.13). Then we will show that the piston remains away from the extremities of the cylinder. Next, we will prove the required estimates on the density field. The most important step is to show that the density remains bounded away from zero. Finally, we will obtain the regularity estimates on the velocity field, ensuring that the strong solution can be extended to any given time interval.

In what follows, to simplify the presentation, we set

$$\mathcal{J} := \|u_0\|_{H_0^1(-r_-, r_+)}^2 + \|\rho_0\|_{H^1(\mathcal{F})}^2 + \left\| \frac{1}{\rho^0} \right\|_{H^1(\mathcal{F})}^2 + m|\ell_0|^2.$$

3.1. Energy estimates. In this section, we will prove energy estimates satisfied by the solution to the system (2.7)-(2.13). We start with the following result which follows easily from the change of coordinates defined in (2.1).

Lemma 3.1. For $t \geq 0$, we have

$$\int_{\mathcal{F}_+} \frac{1}{\rho(t, y)} dy = 1 - h(t) \quad \text{and} \quad \int_{\mathcal{F}_-} \frac{1}{\rho(t, y)} dy = 1 + h(t). \quad (3.1)$$

Proof. Let us prove the first identity. The second one can be proved in a similar manner. Using (2.1) and (2.6), we easily get

$$\int_{\mathcal{F}_+} \frac{1}{\rho(t, y)} dy = \int_{h(t)}^1 dx = 1 - h(t). \quad \square$$

In view of the above lemma, we define the average function $\bar{\rho}$ as follows

$$\frac{1}{\bar{\rho}(t, y)} = \begin{cases} \frac{1}{r_-} \int_{\mathcal{F}_-} \frac{1}{\rho(t, \eta)} d\eta = \frac{1 + h(t)}{r_-} = \frac{1 + h(t)}{M_L}, & t \geq 0, y \in \mathcal{F}_-, \\ \frac{1}{r_+} \int_{\mathcal{F}_+} \frac{1}{\rho(t, \eta)} d\eta = \frac{1 - h(t)}{r_+} = \frac{1 - h(t)}{M_R}, & t \geq 0, y \in \mathcal{F}_+. \end{cases} \quad (3.2)$$

We define the kinetic and potential energy of the system, respectively, by

$$\mathcal{E}_k(t) := \frac{1}{2} \int_{\mathcal{F}} |u(t, y)|^2 dy + \frac{m}{2} |\dot{h}(t)|^2, \quad (3.3)$$

$$\mathcal{E}_p(t) := \frac{1}{\gamma - 1} \int_{\mathcal{F}} \left(\rho^{\gamma-1}(t, y) - \bar{\rho}^{\gamma-1}(t, y) \right) dy. \quad (3.4)$$

We also define

$$P(t) := \frac{1}{\gamma - 1} \int_{\mathcal{F}} \bar{\rho}^{\gamma-1}(t, y) dy = \frac{1}{\gamma - 1} \left[\frac{(r_+)^{\gamma}}{(1 - h(t))^{\gamma-1}} + \frac{(r_-)^{\gamma}}{(1 + h(t))^{\gamma-1}} \right], \quad (3.5)$$

so that

$$\mathcal{E}_p(t) + P(t) = \frac{1}{\gamma - 1} \int_{\mathcal{F}} \rho^{\gamma-1}(t, y) dy, \quad (t \geq 0). \quad (3.6)$$

Note that we have both $\mathcal{E}_k(t) \geq 0$ and $P(t) \geq 0$. We now show that the potential energy $\mathcal{E}_p(t)$ is also non-negative for any $t \geq 0$.

Lemma 3.2. The potential energy $\mathcal{E}_p(t) \geq 0$, for any $t \geq 0$.

Proof. It is enough to prove for one side of the domain, say \mathcal{F}_+ . Using the definition of $\bar{\rho}$ given in (3.2) and identity (3.1), we get

$$\int_{\mathcal{F}_+} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) dy = \int_{\mathcal{F}_+} \left(\rho^{\gamma-1} - \frac{(r_+)^{\gamma-1}}{(1 - h(t))^{\gamma-1}} \right) dy = \int_{\mathcal{F}_+} \rho^{\gamma-1} dy - \frac{r_+}{\left(\frac{1}{r_+} \int_{\mathcal{F}_+} \frac{1}{\rho} dy \right)^{\gamma-1}}$$

$$= \frac{\left(\frac{1}{r_+} \int_{\mathcal{F}_+} \frac{1}{\rho} dy\right)^{\gamma-1} \left(\frac{1}{r_+} \int_{\mathcal{F}_+} \rho^{\gamma-1} dy\right) - 1}{\frac{1}{r_+} \left(\frac{1}{r_+} \int_{\mathcal{F}_+} \frac{1}{\rho} dy\right)^{\gamma-1}}.$$

Thus to show $\mathcal{E}_p(t) \geq 0$ we need to show that

$$r_+^\gamma \leq \left(\int_{\mathcal{F}_+} \frac{1}{\rho} dy\right)^{\gamma-1} \left(\int_{\mathcal{F}_+} \rho^{\gamma-1} dy\right). \quad (3.7)$$

Note that, for any $\alpha \in (0, 1]$

$$r_+^2 = \left(\int_{\mathcal{F}_+} \sqrt{\rho^\alpha} \frac{1}{\sqrt{\rho^\alpha}} dy\right)^2 \leq \left(\int_{\mathcal{F}_+} \rho^\alpha dy\right) \left(\int_{\mathcal{F}_+} \frac{1}{\rho^\alpha} dy\right).$$

If $\gamma - 1 = 1$, then we take $\alpha = 1$. If $\gamma - 1 < 1$, then we take $\alpha = \gamma - 1$ and apply Hölder's inequality

$$r_+^2 \leq \left(\int_{\mathcal{F}_+} \rho^{\gamma-1} dy\right) \left(\int_{\mathcal{F}_+} \frac{1}{\rho^{\gamma-1}} dy\right) \leq \left(\int_{\mathcal{F}_+} \rho^{\gamma-1} dy\right) \left(\int_{\mathcal{F}_+} \frac{1}{\rho} dy\right)^{\gamma-1} r_+^{2-\gamma}.$$

If $\gamma - 1 > 1$ we take $\alpha = 2$, apply Hölder's inequality to obtain

$$r_+^2 \leq \left(\int_{\mathcal{F}_+} \rho dy\right) \left(\int_{\mathcal{F}_+} \frac{1}{\rho} dy\right) \leq \left(\int_{\mathcal{F}_+} \rho^{\gamma-1} dy\right)^{\frac{1}{\gamma-1}} \left(\int_{\mathcal{F}_+} \frac{1}{\rho} dy\right)^{\frac{\gamma-2}{\gamma-1}}.$$

From the above inequalities we obtain (3.7) which completes the proof of the lemma. \square

We have the following energy identity for system (2.7)-(2.13).

Proposition 3.3. Let $\mu(\rho) = \rho^\theta$ for some $\theta \geq 0$. Let us set

$$\mathcal{E}(t) := \mathcal{E}_k(t) + \mathcal{E}_p(t) + P(t), \quad (t \geq 0).$$

Then the function $t \mapsto \mathcal{E}(t)$ is C^1 on $[0, T]$, and for any $t \geq 0$ we have

$$\dot{\mathcal{E}}(t) = - \int_{\mathcal{F}} \rho(t, y) \mu(\rho(t, y)) u_y(t, y)^2 dy, \quad (t \geq 0). \quad (3.8)$$

More precisely, for any $t \geq 0$

$$\mathcal{E}(t) + \int_0^t \int_{\mathcal{F}} \rho(t, y) \mu(\rho(t, y)) u_y(t, y)^2 dy ds = \mathcal{E}(0). \quad (3.9)$$

Proof. The fact that \mathcal{E} is C^1 on $[0, T]$ is a direct consequence of the regularity of the strong solution. Moreover, using (3.6) and (2.7), we deduce

$$\dot{\mathcal{E}}(t) = \int_{\mathcal{F}} uu_t dy + m\dot{h}(t)\ddot{h}(t) + \int_{\mathcal{F}} \rho^{\gamma-2} \rho_t dy = \int_{\mathcal{F}} uu_t dy + m\dot{h}(t)\ddot{h}(t) - \int_{\mathcal{F}} \rho^\gamma u_y dy. \quad (3.10)$$

Multiplying the momentum equation (2.8) with u , and integrating over \mathcal{F}_- with respect to y , we have

$$\int_{\mathcal{F}_-} u_t u dy - \int_{\mathcal{F}_-} \rho^\gamma u_y dy = - \int_{\mathcal{F}_-} \rho \mu(\rho) u_y^2 dy + (\rho \mu(\rho) u_y - \rho^\gamma)(t, 0^-) \dot{h}(t).$$

Similarly, over the region \mathcal{F}_+ , we have

$$\int_{\mathcal{F}_+} u_t u dy - \int_{\mathcal{F}_+} \rho^\gamma u_y dy = - \int_{\mathcal{F}_+} \rho \mu(\rho) u_y^2 dy - (\rho \mu(\rho) u_y - \rho^\gamma)(t, 0^+) \dot{h}(t).$$

By adding the above two identities and using (2.10), we get

$$\int_{\mathcal{F}} u_t u dy - \int_{\mathcal{F}} \rho^\gamma u_y dy = - \int_{\mathcal{F}} \rho \mu(\rho) u_y^2 dy - [\rho \mu(\rho) u_y - \rho^\gamma](t, 0) \dot{h}(t) = - \int_{\mathcal{F}} \rho \mu(\rho) u_y^2 dy - m\dot{h}(t)\ddot{h}(t).$$

Combining the above with (3.10), we deduce (3.8). \square

As a corollary of the above proposition we have the following energy estimate.

Corollary 3.4. There exists a positive constant C such that

$$\begin{aligned} \int_{\mathcal{F}} \left(\frac{1}{2} u^2 + \frac{1}{\gamma-1} \rho^{\gamma-1} \right) dy + \int_0^t \int_{\mathcal{F}} \rho \mu(\rho) u_y^2 dy ds + \frac{m}{2} \dot{h}^2(t) \\ = \mathcal{E}(0) = \int_{\mathcal{F}} \left(\frac{1}{2} u_0^2 + \frac{1}{\gamma-1} \rho_0^{\gamma-1} \right) dy + \frac{m}{2} \ell_0^2 \leq C \mathcal{J}, \quad (t \geq 0). \end{aligned}$$

Remark 3.5. In Section 3.2, we will show that the above results are enough to show that the piston remains away from the extremities of the cylinder for all time.

We now derive a modified energy type estimate for the system (2.7)-(2.13). This result is inspired from [19, Lemma 3.2] and [23]. It will give additional information about ρ , namely $L^\infty(L^2)$ bound of $(\rho^\theta)_y$, for $\theta > 0$. This and the fact that piston remains away from the extremities of the cylinder gives us lower and upper bound of ρ^θ .

Proposition 3.6. Let $\mu(\rho) = \rho^\theta$ for some $\theta > 0$. Then there exists a positive constant C such that

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{F}} \left(u + \frac{1}{\theta} (\rho^\theta)_y \right)^2 dy + \gamma \int_0^t \int_{\mathcal{F}} \rho^{\theta+\gamma-2} (\rho_y)^2 dy ds + \frac{1}{2m} \left(m \dot{h}(t) + \frac{1}{\theta} [\rho^\theta](t, 0) \right)^2 \\ + \int_0^t \frac{1}{m\theta} [\rho^\theta](s, 0) [\rho^\gamma](s, 0) ds \leq C \mathcal{J}, \quad (t \geq 0). \end{aligned} \quad (3.11)$$

Proof. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$\varphi'(\rho) = \frac{\mu(\rho)}{\rho}. \quad (3.12)$$

For instance, we may take $\varphi(\rho)$ as

$$\varphi(\rho) := \int_0^\rho \frac{\mu(\sigma)}{\sigma} d\sigma = \frac{\rho^\theta}{\theta}. \quad (3.13)$$

We multiply (2.8) by $(\varphi(\rho))_y = \varphi'(\rho) \rho_y$ and integrate over \mathcal{F}_- :

$$\int_{\mathcal{F}_-} u_t (\varphi(\rho))_y dy = \int_{\mathcal{F}_-} (\rho \mu(\rho) u_y)_y (\varphi(\rho))_y dy - \int_{\mathcal{F}_-} (\rho^\gamma)_y (\varphi(\rho))_y dy. \quad (3.14)$$

Using the equation for density (2.7), we can write the first term in the right hand side of the above equation as

$$\begin{aligned} \int_{\mathcal{F}_-} (\rho \mu(\rho) u_y)_y (\varphi(\rho))_y dy &\stackrel{(3.12)}{=} \int_{\mathcal{F}_-} (\varphi'(\rho) \rho^2 u_y)_y (\varphi(\rho))_y dy \stackrel{(2.7)}{=} - \int_{\mathcal{F}_-} (\varphi'(\rho) \rho_t)_y (\varphi(\rho))_y dy \\ &= - \int_{\mathcal{F}_-} (\varphi(\rho))_{ty} (\varphi(\rho))_y dy = - \frac{d}{dt} \int_{\mathcal{F}_-} \frac{((\varphi(\rho))_y)^2}{2} dy. \end{aligned}$$

Substituting the above in (3.14), we get

$$\int_{\mathcal{F}_-} u_t (\varphi(\rho))_y dy = - \frac{d}{dt} \int_{\mathcal{F}_-} \frac{((\varphi(\rho))_y)^2}{2} dy - \int_{\mathcal{F}_-} (\rho^\gamma)_y (\varphi(\rho))_y dy. \quad (3.15)$$

On the other hand, note that we also have

$$\frac{d}{dt} \int_{\mathcal{F}_-} u (\varphi(\rho))_y dy = \int_{\mathcal{F}_-} u_t (\varphi(\rho))_y dy + \int_{\mathcal{F}_-} u (\varphi(\rho))_{yt} dy. \quad (3.16)$$

For the second term in the right hand side of (3.16), we see that

$$\begin{aligned}
\int_{\mathcal{F}_-} u(\varphi(\rho))_{yt} \, dy &= \int_{\mathcal{F}_-} u(\varphi'(\rho)\rho_t)_y \, dy \stackrel{(2.7)}{=} - \int_{\mathcal{F}_-} u(\varphi'(\rho)\rho^2 u_y)_y \, dy = - \int_{\mathcal{F}_-} u(\rho\mu(\rho)u_y)_y \, dy \\
&= \int_{\mathcal{F}_-} \rho\mu(\rho)u_y^2 \, dy - (\rho\mu(\rho)u_y u)(t, 0^-) \stackrel{(2.11)}{=} \int_{\mathcal{F}_-} \rho\mu(\rho)u_y^2 \, dy - (\rho\mu(\rho)u_y)(t, 0^-)\dot{h}(t). \quad (3.17)
\end{aligned}$$

Using (3.15) and (3.17) in (3.16), we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{F}_-} u(\varphi(\rho))_y \, dy + \frac{d}{dt} \int_{\mathcal{F}_-} \frac{((\varphi(\rho))_y)^2}{2} \, dy + \int_{\mathcal{F}_-} (\rho^\gamma)_y (\varphi(\rho))_y \, dy \\
= \int_{\mathcal{F}_-} \rho\mu(\rho)u_y^2 \, dy - (\rho\mu(\rho)u_y)(t, 0^-)\dot{h}(t).
\end{aligned}$$

We obtain an analogous expression over the region \mathcal{F}_+ :

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{F}_+} u(\varphi(\rho))_y \, dy + \frac{d}{dt} \int_{\mathcal{F}_+} \frac{((\varphi(\rho))_y)^2}{2} \, dy + \int_{\mathcal{F}_+} (\rho^\gamma)_y (\varphi(\rho))_y \, dy \\
= \int_{\mathcal{F}_+} \rho\mu(\rho)u_y^2 \, dy + (\rho\mu(\rho)u_y)(t, 0^+)\dot{h}(t).
\end{aligned}$$

Combining the last two equations, we get

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{F}} u(\varphi(\rho))_y \, dy + \frac{d}{dt} \int_{\mathcal{F}} \frac{((\varphi(\rho))_y)^2}{2} \, dy + \int_{\mathcal{F}} (\rho^\gamma)_y (\varphi(\rho))_y \, dy \\
= \int_{\mathcal{F}} \rho\mu(\rho)u_y^2 \, dy + [\rho\mu(\rho)u_y](t, 0)\dot{h}(t) \stackrel{(2.10)}{=} \int_{\mathcal{F}} \rho\mu(\rho)u_y^2 \, dy + m\ddot{h}(t)\dot{h}(t) + [\rho^\gamma](t, 0)\dot{h}(t).
\end{aligned}$$

This implies that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{F}} (u + (\varphi(\rho))_y)^2 \, dy + \int_{\mathcal{F}} (\rho^\gamma)_y (\varphi(\rho))_y \, dy \stackrel{(3.3)}{=} \dot{\mathcal{E}}_k(t) + \int_{\mathcal{F}} \rho\mu(\rho)u_y^2 \, dy + [\rho^\gamma](t, 0)\dot{h}(t). \quad (3.18)$$

In the above expression, in view of Proposition 3.3, we have control over all the terms on the right hand side, except the last one. To eliminate this term we modify the energy type function on the left hand side. For convenience, we denote

$$\rho_- = \rho(t, 0^-), \quad \rho_+ = \rho(t, 0^+).$$

We claim that

$$\frac{1}{2m} \frac{d}{dt} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right)^2 = -\dot{h}(t)[\rho^\gamma](t, 0) + \frac{1}{m\theta} (\rho_-^\theta - \rho_+^\theta)(\rho_+^\gamma - \rho_-^\gamma). \quad (3.19)$$

Proof of claim: First of all, using (2.7) we get

$$\frac{d}{dt} \varphi(\rho_\pm) = \varphi'(\rho_\pm)\rho_t(t, 0^\pm) = -\rho_\pm\mu(\rho_\pm)u_y(t, 0^\pm).$$

Using the above expression we proceed as follows:

$$\begin{aligned}
&\frac{1}{2m} \frac{d}{dt} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right)^2 \\
&= \frac{1}{m} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right) \left(m\ddot{h}(t) + \rho_- \mu(\rho_-) u_y(t, 0^-) - \rho_+ \mu(\rho_+) u_y(t, 0^+) \right) \\
&= \frac{1}{m} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right) \left(m\ddot{h}(t) - [\rho\mu(\rho)u_y](t, 0) \right) \\
&\stackrel{(2.10)}{=} -\frac{1}{m} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right) [\rho^\gamma](t, 0) \\
&= -\dot{h}(t)[\rho^\gamma](t, 0) + \frac{1}{m} (\varphi(\rho_-) - \varphi(\rho_+)) [\rho^\gamma](t, 0)
\end{aligned}$$

$$(3.13) \quad = -\dot{h}(t)[\rho^\gamma](t, 0) + \frac{1}{m\theta}(\rho_-^\theta - \rho_+^\theta)(\rho_+^\gamma - \rho_-^\gamma).$$

This completes the proof of (3.19). Combining (3.19) and (3.18), we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathcal{F}} \left(u + (\varphi(\rho))_y \right)^2 dy + \int_{\mathcal{F}} (\rho^\gamma)_y (\varphi(\rho))_y dy + \frac{1}{2m} \frac{d}{dt} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right)^2 \\ = \dot{\mathcal{E}}_k(t) + \int_{\mathcal{F}} \rho\mu(\rho)u_y^2 dy + \frac{1}{m\theta}(\rho_-^\theta - \rho_+^\theta)(\rho_+^\gamma - \rho_-^\gamma). \end{aligned}$$

Note that the last term in the above expression is always negative, since $\rho_-, \rho_+ > 0$ for all $t \geq 0$, $\gamma > 1$ and $\theta > 0$. Therefore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathcal{F}} \left(u + (\varphi(\rho))_y \right)^2 dy + \int_{\mathcal{F}} (\rho^\gamma)_y (\varphi(\rho))_y dy + \frac{1}{2m} \frac{d}{dt} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right)^2 \\ + \frac{1}{m\theta}(\rho_+^\theta - \rho_-^\theta)(\rho_+^\gamma - \rho_-^\gamma) \leq \dot{\mathcal{E}}_k(t) + \int_{\mathcal{F}} \rho\mu(\rho)u_y^2 dy. \end{aligned}$$

Integrating the above with respect to t , we get

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{F}} \left(u + (\varphi(\rho))_y \right)^2 dy + \int_0^t \int_{\mathcal{F}} (\rho^\gamma)_y (\varphi(\rho))_y dy ds + \frac{1}{2m} \left(m\dot{h}(t) - \varphi(\rho_-) + \varphi(\rho_+) \right)^2 \\ + \int_0^t \frac{1}{m\theta}(\rho_+^\theta - \rho_-^\theta)(\rho_+^\gamma - \rho_-^\gamma) ds \leq \mathcal{E}_k(t) - \mathcal{E}_k(0) + \int_0^t \int_{\mathcal{F}} \rho\mu(\rho)u_y^2 dy ds + \frac{1}{2} \int_{\mathcal{F}} \left(u_0 + (\varphi(\rho_0))_y \right)^2 dy \\ + \frac{1}{2m} \left(m\ell_0^2 - \varphi(\rho_0(0^-)) + \varphi(\rho_0(0^+)) \right)^2 \leq C\mathcal{J}, \end{aligned}$$

Cor.(3.4)

where C is a constant independent of t . This completes the proof of the proposition. \square

As a consequence of the above result and Corollary 3.4 we obtain $L^\infty(L^2)$ bound of ρ^θ .

Corollary 3.7. There exists a positive constant C such that

$$\int_{\mathcal{F}} \left((\rho^\theta)_y(t, y) \right)^2 dy \leq C\mathcal{J}, \quad (t \geq 0).$$

3.2. No-contact. We shall now show that the piston does not come in contact with the extremities of the cylinder. This is a consequence of Lemma 3.2 and Proposition 3.3.

Lemma 3.8. Let $\mu(\rho) = \rho^\theta$ for some $\theta \geq 0$. For any $t \geq 0$, we have

$$-1 + \delta_0 \leq h(t) \leq 1 - \delta_0,$$

where

$$\delta_0 := \min \left\{ \frac{(M_R)^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}}, \frac{(M_L)^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}} \right\}.$$

Proof. In view of Lemma 3.2 and (3.9), we have $P(t) \leq \mathcal{E}(0)$ for all $t \geq 0$. By substituting the expression of $P(t)$ from (3.5), we infer that

$$1 - h(t) \geq \frac{(r_+)^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}}, \quad \text{and} \quad 1 + h(t) \geq \frac{(r_-)^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}}, \quad (t \geq 0).$$

From the above two relation and (2.3) the lemma flows easily. \square

3.3. Estimates on density. We have sufficient information to show that the density function is bounded below and above. In the proofs below we will see that for showing the upper bound of ρ we only need $\theta > 0$, but for proving the lower bound of ρ we need $0 < \theta < \frac{1}{2}$.

Lemma 3.9. Let $\mu(\rho) = \rho^\theta$ for some $\theta > 0$. There exists a constant $\rho^* > 0$, depending only on \mathcal{J} and independent of τ , such that

$$\sup_{(t,y) \in [0,\tau) \times \mathcal{F}} \rho(t,y) \leq \rho^*.$$

Proof. First we will show that ρ is upper bounded on $[0, \tau) \times \mathcal{F}_+$. From Lemma 3.1, we have

$$\frac{1}{r_+} \int_0^{r_+} \frac{1}{\rho(t,y)} dy = \frac{1-h(t)}{r_+}.$$

Hence, for each $t \in (0, \tau)$, there exists $y_0(t) \in (0, r_+)$, such that

$$\rho(t, y_0(t)) = \frac{r_+}{1-h(t)}. \quad (3.20)$$

By the fundamental theorem of calculus, we have

$$\rho^\theta(t, y) = \rho^\theta(t, y_0(t)) + \int_{y_0(t)}^y (\rho^\theta)_y(t, \eta) d\eta, \quad (t \in [0, \tau), y \in \mathcal{F}_+).$$

Applying Corollary 3.7 and Lemma 3.8 to the above relation, we get that, for any $t \in [0, \tau), y \in \mathcal{F}_+$

$$\rho^\theta(t, y) \leq \frac{r_+^\theta}{\delta_0^\theta} + \sqrt{r_+} \left(\int_{\mathcal{F}_+} (\rho_y^\theta(t, y))^2 dy \right)^{\frac{1}{2}} \leq C.$$

This implies that

$$\sup_{(t,y) \in [0,\tau) \times \mathcal{F}_+} \rho(t, y) \leq C.$$

In a similar manner, we get

$$\sup_{(t,y) \in [0,\tau) \times \mathcal{F}_-} \rho(t, y) \leq C.$$

This completes the proof of the lemma. \square

Next, we show the lower bound for the density.

Lemma 3.10. Let $\mu(\rho) = \rho^\theta$ for some $0 < \theta < \frac{1}{2}$. There exists a constant $\rho_* > 0$, depending only on \mathcal{J} and independent of τ , such that

$$\inf_{(t,y) \in [0,\tau) \times \mathcal{F}} \rho(t, y) \geq \rho_*.$$

Proof. In order to obtain the lower bound, we set $\zeta(t, y) := \frac{1}{\rho(t, y)}$, for $(t, y) \in (0, \tau) \times \mathcal{F}$. First we show the bound of ζ in $(0, \tau) \times \mathcal{F}_+$. Taking y_0 as in (3.20) and using the fundamental theorem of calculus, we get

$$\zeta(t, y) = \zeta(t, y_0) + \int_{y_0}^y \zeta_y(t, y) dy = \frac{1-h(t)}{r_+} - \int_{y_0}^y \zeta^2(t, \eta) \rho_y(t, \eta) d\eta.$$

Since $\zeta^{1+\theta}(\rho^\theta)_y = \theta \zeta^2 \rho_y$, the above estimate implies that, for any $(t, y) \in [0, \tau) \times \mathcal{F}_+$

$$\begin{aligned} \zeta(t, y) &\stackrel{\text{Lem.3.8}}{\leq} C + \frac{1}{\theta} \int_{\mathcal{F}_+} \zeta^{1+\theta} |(\rho^\theta)_y| dy \leq C + \frac{1}{\theta} \left(\sup_{y \in \mathcal{F}_+} \zeta^{\theta+\frac{1}{2}} \right) \int_{\mathcal{F}_+} \zeta^{\frac{1}{2}} |(\rho^\theta)_y| dy \\ &\leq C + \frac{1}{\theta} \left(\sup_{y \in \mathcal{F}_+} \zeta \right)^{\theta+\frac{1}{2}} \left(\int_{\mathcal{F}_+} (\rho_y^\theta(t, y))^2 dy \right)^{\frac{1}{2}} \left(\int_{\mathcal{F}_+} \zeta dy \right)^{\frac{1}{2}} \stackrel{\text{Lem.3.1, Lem.3.8, Cor.3.7}}{\leq} C + C \left(\sup_{y \in \mathcal{F}_+} \zeta \right)^{\theta+\frac{1}{2}}. \end{aligned}$$

Since $0 < \theta < \frac{1}{2}$, applying Young's inequality to the above estimate we get, for any $(t, y) \in [0, \tau) \times \mathcal{F}_+$:

$$\zeta(t, y) \leq C + \frac{1}{2} \left(\sup_{y \times \mathcal{F}_+} \zeta \right),$$

which implies that

$$\frac{1}{2} \sup_{(t, y) \in (0, \tau) \times \mathcal{F}_+} \zeta(t, y) \leq C.$$

Hence, we have $\inf_{(t, y) \in [0, \tau) \times \mathcal{F}_+} \rho(t, y) \geq C$. Similarly, we can show that $\inf_{(t, y) \in (0, \tau) \times \mathcal{F}_-} \rho(t, y) \geq C$. \square

We can easily obtain the other bounds on ρ from Corollary 3.4 and Corollary 3.7 now that we have lower and upper bounds of ρ ,

Lemma 3.11. Let $\mu(\rho) = \rho^\theta$ for some $0 < \theta < \frac{1}{2}$. There is a constant $C > 0$, depending only on \mathcal{J} and independent of τ , such that

$$\sup_{t \in [0, \tau)} \int_{\mathcal{F}} \rho_y(t, y)^2 dy \leq C, \quad \int_0^t \int_{\mathcal{F}} \rho_y(s, y)^2 dy ds \leq C, \quad (t \geq 0).$$

Proof. The first conclusion follows easily from Corollary 3.7 and Lemma 3.10. Note that $L^2(L^2)$ bound of ρ_y follows from the $L^\infty(L^2)$ bound of ρ_y , with a bound that depends on τ . In order to get an estimate independent of τ we may proceed as follows. Using Proposition 3.6, Lemma 3.9 and Lemma 3.10, for any $t \geq 0$, we have

$$\int_0^t \int_{\mathcal{F}} \rho_y^2 dy ds \leq C(\rho^*, \rho_*) \int_0^t \int_{\mathcal{F}} \rho^{\theta+\gamma-2} \rho_y^2 dy ds \leq C. \quad \square$$

3.4. Estimates on velocity. In this subsection, we estimate the derivatives of u as proved in the results below. We first note that by combining Corollary 3.4 and Lemma 3.10 we obtain the following.

Lemma 3.12. Let $\mu(\rho) = \rho^\theta$ for some $0 < \theta < \frac{1}{2}$. There exists a positive constant C , depending only on \mathcal{J} and independent of τ , such that

$$\int_0^t \int_{\mathcal{F}} u_y^2 dy ds \leq C, \quad (t \geq 0).$$

Lemma 3.13. Let $\mu(\rho) = \rho^\theta$ for some $0 < \theta < \frac{1}{2}$. There exists a positive constant C , depending only on \mathcal{J} and independent of τ , such that

$$\int_0^t \int_{\mathcal{F}} u_t^2 dy ds + \int_{\mathcal{F}} u_y^2 dy + m \int_0^t \ddot{h}(s)^2 ds \leq C, \quad (t \geq 0).$$

Proof. The proof is divided into several steps.

Step 1: We multiply (2.8) by u_t and integrate over $(0, t) \times \mathcal{F}_-$:

$$\int_0^t \int_{\mathcal{F}_-} u_t^2 dy ds = \int_0^t \int_{\mathcal{F}_-} (\rho^{1+\theta} u_y)_y u_t dy ds - \int_{\mathcal{F}} (\rho^\gamma)_y u_t dy ds. \quad (3.21)$$

Integrating by parts and using (2.9), we obtain

$$\begin{aligned} \int_0^t \int_{\mathcal{F}_-} (\rho^{1+\theta} u_y)_y u_t dy ds &= \frac{1+\theta}{2} \int_0^t \int_{\mathcal{F}_-} \rho^\theta \rho_t u_y^2 dy ds - \frac{1}{2} \int_{\mathcal{F}_-} \rho^{1+\theta} u_y^2 dy \\ &\quad + \frac{1}{2} \int_{\mathcal{F}_-} \rho_0^{1+\theta} u_{0y}^2 dy + \int_0^t \rho^{1+\theta}(s, 0^-) u_y(s, 0^-) \ddot{h}(s) ds, \end{aligned} \quad (3.22)$$

and

$$\begin{aligned}
-\int_0^t \int_{\mathcal{F}_-} (\rho^\gamma)_y u_t dy ds &= -\gamma \int_0^t \int_{\mathcal{F}_-} \rho^{\gamma-1} \rho_t u_y dy ds + \int_{\mathcal{F}_-} \rho^\gamma u_y dy \\
&\quad - \int_{\mathcal{F}_-} \rho_0^\gamma u_{0y} dy - \int_0^t \rho^\gamma(s, 0^-) \ddot{h}(s) ds. \quad (3.23)
\end{aligned}$$

Substituting (3.22) and (3.23) in (3.21) we infer that

$$\begin{aligned}
\int_0^t \int_{\mathcal{F}_-} u_t^2 dy ds &= \frac{1+\theta}{2} \int_0^t \int_{\mathcal{F}_-} \rho^\theta \rho_t u_y^2 dy ds - \frac{1}{2} \int_{\mathcal{F}_-} \rho^{1+\theta} u_y^2 dy + \frac{1}{2} \int_{\mathcal{F}_-} \rho_0^{1+\theta} u_{0y}^2 dy \\
&\quad - \gamma \int_0^t \int_{\mathcal{F}_-} \rho^{\gamma-1} \rho_t u_y dy ds + \int_{\mathcal{F}_-} \rho^\gamma u_y dy - \int_{\mathcal{F}_-} \rho_0^\gamma u_{0y} dy \\
&\quad + \int_0^t (\rho^{1+\theta} u_y - \rho^\gamma)(s, 0^-) \ddot{h}(s) ds. \quad (3.24)
\end{aligned}$$

Analogously, we also have

$$\begin{aligned}
\int_0^t \int_{\mathcal{F}_+} u_t^2 dy ds &= \frac{1}{2} \int_0^t \int_{\mathcal{F}_+} (1+\theta) \rho^\theta \rho_t u_y^2 dy ds - \frac{1}{2} \int_{\mathcal{F}_+} \rho^{1+\theta} u_y^2 dy + \frac{1}{2} \int_{\mathcal{F}_+} \rho_0^{1+\theta} u_{0y}^2 dy \\
&\quad - \gamma \int_0^t \int_{\mathcal{F}_+} \rho^{\gamma-1} \rho_t u_y dy ds + \int_{\mathcal{F}_+} \rho^\gamma u_y dy - \int_{\mathcal{F}_+} \rho_0^\gamma u_{0y} dy \\
&\quad - \int_0^t (\rho^{1+\theta} u_y - \rho^\gamma)(s, 0^+) \ddot{h}(s) ds. \quad (3.25)
\end{aligned}$$

The above two estimates, along with (2.7) and (2.10), show that

$$\begin{aligned}
\int_0^t \int_{\mathcal{F}} u_t^2 dy ds + m \int_0^t \ddot{h}(s)^2 ds + \frac{1}{2} \int_{\mathcal{F}} \rho^{1+\theta} u_y^2 dy &= -\frac{1+\theta}{2} \int_0^t \int_{\mathcal{F}} \rho^{2+\theta} u_y^3 dy ds \\
&\quad + \gamma \int_0^t \int_{\mathcal{F}} \rho^{\gamma+1} u_y^2 dy ds + \int_{\mathcal{F}} \rho^\gamma u_y dy + \frac{1}{2} \int_{\mathcal{F}} \rho_0^{1+\theta} u_{0y}^2 dy - \int_{\mathcal{F}} \rho_0^\gamma u_{0y} dy. \quad (3.26)
\end{aligned}$$

The last two terms on the right hand side of (3.26) can be estimated due to the assumption on the initial data. The second term is uniformly bounded due to Corollary 3.4, Lemma 3.9, and Lemma 3.10. To estimate the third term we proceed as follows

$$\int_{\mathcal{F}} \rho^\gamma u_y dy \leq \frac{1}{4} \int_{\mathcal{F}} \rho^{1+\theta} u_y^2 dy + 4 \int_{\mathcal{F}} \rho^{2\gamma-(1+\theta)} dy \stackrel{\text{Lem.3.9}}{\leq} \int_{\mathcal{F}} \frac{\rho^{1+\theta}}{4} u_y^2 dy + 4(\rho^*)^{2\gamma-(1+\theta)}(r_+ - r_-).$$

Plugging the above estimate in (3.26) and using Corollary 3.4, Lemma 3.9, and Lemma 3.10, we deduce that, there exists a positive constant C independent of t such that

$$\int_0^t \int_{\mathcal{F}} u_t^2 dy ds + m \int_0^t (\ddot{h}(s))^2 ds + \frac{1}{4} \int_{\mathcal{F}} \rho^{1+\theta} u_y^2 dy \leq C \left(1 + \int_0^t \int_{\mathcal{F}} |\rho^{1+\theta} u_y^3| dy ds \right), \quad (t \geq 0). \quad (3.27)$$

Step 2: We claim that the following holds: for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_0^t \int_{\mathcal{F}} |\rho^{1+\theta} u_y^3| dy ds \leq \varepsilon \int_0^t \int_{\mathcal{F}} u_t^2 dy ds + C_\varepsilon \int_0^t \left(\int_{\mathcal{F}} u_y^2 dy \right)^2 ds + C_1, \quad (t \geq 0), \quad (3.28)$$

where the positive constant C_1 depends only on \mathcal{J} , ρ_* and ρ^* .

We only show the above estimate on \mathcal{F}_+ . The proof is similar for \mathcal{F}_- . We begin with the identity

$$\rho^{1+\theta} u_y^3 = (\rho^{1+\theta} u_y - \rho^\gamma) u_y^2 + \rho^\gamma u_y^2,$$

so that

$$\begin{aligned} \int_0^t \int_{\mathcal{F}_+} |\rho^{1+\theta} u_y^3| dy ds &\leq \int_0^t \left\| (\rho^{1+\theta} u_y - \rho^\gamma)(s, \cdot) \right\|_{L^\infty(\mathcal{F}_+)} \left(\int_{\mathcal{F}_+} u_y^2 dy \right) ds + \int_0^t \int_{\mathcal{F}_+} \rho^\gamma u_y^2 dy ds \\ &\stackrel{\text{Lem.3.12, Lem.3.9}}{\leq} \int_0^t \left\| (\rho^{1+\theta} u_y - \rho^\gamma)(s, \cdot) \right\|_{L^\infty(\mathcal{F}_+)} \left(\int_{\mathcal{F}_+} u_y^2 dy \right) ds + C_1, \end{aligned} \quad (3.29)$$

where C_1 depends only of \mathcal{J} , ρ_* , and ρ^* . For any $t \geq 0$, we have

$$\begin{aligned} (\rho^{1+\theta} u_y - \rho^\gamma)(t, y) &= (\rho^{1+\theta} u_y - \rho^\gamma)(t, z) + \int_z^y (\rho^{1+\theta} u_y - \rho^\gamma)_y(t, \eta) d\eta \\ &= (\rho^{1+\theta} u_y - \rho^\gamma)(t, z) + \int_z^y u_t(t, \eta) d\eta, \quad (y, z \in \mathcal{F}_+), \end{aligned}$$

which implies that

$$\begin{aligned} |(\rho^{1+\theta} u_y - \rho^\gamma)(y)| &\leq \frac{1}{r_+} \int_{\mathcal{F}_+} |(\rho^{1+\theta} u_y - \rho^\gamma)(t, z)| dz + \int_{\mathcal{F}_+} |u_t| dy \\ &\leq \frac{1}{2r_+^2} \int_{\mathcal{F}_+} \rho^{2(1+\theta)} dy + \frac{1}{2} \int_{\mathcal{F}_+} u_y^2 dy + \frac{1}{r_+} \int_{\mathcal{F}_+} \rho^\gamma dy + \int_{\mathcal{F}_+} |u_t| dy \\ &\stackrel{\text{Lem.3.9}}{\leq} \frac{1}{2} \int_{\mathcal{F}_+} u_y^2 dy + \int_{\mathcal{F}_+} |u_t| dy + C_1, \quad (t \geq 0, y \in \mathcal{F}_+). \end{aligned} \quad (3.30)$$

Therefore

$$\left\| (\rho^{1+\theta} u_y - \rho^\gamma)(t, \cdot) \right\|_{L^\infty(\mathcal{F}_+)} \leq \frac{1}{2} \int_{\mathcal{F}_+} u_y^2 dy + \int_{\mathcal{F}_+} |u_t| dy + C_1, \quad (t \geq 0).$$

Substituting the above expression in (3.29), we get for any $t \geq 0$,

$$\begin{aligned} \int_0^t \int_{\mathcal{F}_+} |\rho^{1+\theta} u_y^3| dy ds &\leq \int_0^t \left(\frac{1}{2} \int_{\mathcal{F}_+} u_y^2 dy + \int_{\mathcal{F}_+} |u_t| dy + C_1 \right) \int_{\mathcal{F}_+} u_y^2 dy ds + C_1 \\ &\stackrel{\text{Lem.3.12}}{\leq} \frac{1}{2} \int_0^t \left(\int_{\mathcal{F}_+} u_y^2 dy \right)^2 ds + \int_0^t \left(\int_{\mathcal{F}_+} |u_t| dy \right) \left(\int_{\mathcal{F}_+} u_y^2 dy \right) ds + C_1 \\ &\leq \frac{1}{2} \int_0^t \left(\int_{\mathcal{F}_+} u_y^2 dy \right)^2 ds + \varepsilon \int_0^t \int_{\mathcal{F}_+} u_t^2 dy ds + C_\varepsilon \int_0^t \left(\int_{\mathcal{F}_+} u_y^2 dy \right)^2 ds + C_1 \\ &\leq C_\varepsilon \int_0^t \left(\int_{\mathcal{F}_+} u_y^2 dy \right)^2 ds + \varepsilon \int_0^t \int_{\mathcal{F}_+} u_t^2 dy ds + C_1. \end{aligned}$$

This completes the proof of the estimate (3.28).

Step 3: Using (3.28) in (3.27), we get

$$\int_0^t \int_{\mathcal{F}} u_t^2 dy ds + m \int_0^t \ddot{h}(s)^2 ds + \int_{\mathcal{F}} \frac{\rho^{1+\theta}}{4} u_y^2 dy \leq CC_\varepsilon \int_0^t \left(\int_{\mathcal{F}} u_y^2 dy \right)^2 ds + C\varepsilon \int_0^t \int_{\mathcal{F}} u_t^2 dy ds + CC_1.$$

Then choosing small enough ε and using Lemma 3.10 we conclude that

$$\int_0^t \int_{\mathcal{F}} u_t^2 dy ds + \int_{\mathcal{F}} u_y^2 dy + m \int_0^t (\ddot{h}(s))^2 ds \leq C + C \int_0^t \left(\int_{\mathcal{F}} u_y^2 dy \right)^2 ds. \quad (3.31)$$

In particular, we have

$$\int_{\mathcal{F}} u_y^2 dy \leq C + C \int_0^t \left(\int_{\mathcal{F}} u_y^2 dy \right)^2 ds,$$

from which, applying Grönwall's inequality, we infer that

$$\int_{\mathcal{F}} u_y^2 dy \leq C \exp \left(\int_0^t \int_{\mathcal{F}} u_y^2 dy ds \right) \stackrel{\text{Lem.3.12}}{\leq} C, \quad (t \geq 0). \quad (3.32)$$

Using the above estimate on the right hand side of (3.31), we obtain, for any $t \geq 0$

$$\begin{aligned} \int_0^t \int_{\mathcal{F}} u_t^2 dy ds + \int_{\mathcal{F}} u_y^2 dy + m \int_0^t \ddot{h}(s)^2 ds &\leq C + \int_0^t \left(\int_{\mathcal{F}} u_y^2 dy \right) \left(\int_{\mathcal{F}} u_t^2 dy \right) ds \\ &\stackrel{(3.32)}{\leq} C + C \int_0^t \int_{\mathcal{F}} u_y^2 dy ds \stackrel{\text{Lem.3.12}}{\leq} C. \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 3.14. Let $\mu(\rho) = \rho^\theta$ for some $0 < \theta < \frac{1}{2}$. There exists a positive constant C , depending only on \mathcal{J} , such that

$$\int_0^t \int_{\mathcal{F}} u_{yy}^2 dy ds \leq C, \quad (t \geq 0).$$

Proof. We prove the estimate only on \mathcal{F}_+ . The estimate on \mathcal{F}_- will be similar. Let us fix $y_0 \in \mathcal{F}_+$. From the momentum equation (2.8), we have

$$-u_y(t, y) = -\rho^{\gamma-\theta-1} - \frac{(\rho^{1+\theta} u_y - \rho^\gamma)(y_0)}{\rho^{1+\theta}} - \frac{\int_{y_0}^y u_t(t, \eta) d\eta}{\rho^{1+\theta}}, \quad (t \geq 0, y \in \mathcal{F}_+).$$

Differentiating both sides with respect to y ,

$$u_{yy}(t, y) = -(\gamma-\theta-1)\rho^{\gamma-\theta-2} \rho_y + \frac{(\rho^{1+\theta} u_y - \rho^\gamma)(y_0)}{\rho^{\theta+2}} \rho_y + \frac{\int_{y_0}^y u_t(t, \eta) d\eta}{\rho^{\theta+2}} \rho_y - \frac{u_t}{\rho^{1+\theta}}, \quad (t \geq 0, y \in \mathcal{F}_+).$$

Therefore, using Lemma 3.10, Lemma 3.9 and (3.30) we obtain

$$|u_{yy}| \leq C \left(|\rho_y| + |u_t| + |\rho_y| \left(\int_{\mathcal{F}_+} |u_t|^2 dy \right)^{1/2} \right), \quad (t \geq 0, y \in \mathcal{F}_+), \quad (3.33)$$

which implies that

$$\begin{aligned} \int_0^t \int_{\mathcal{F}_+} u_{yy}^2 dy ds \\ \leq C \left(\int_0^t \int_{\mathcal{F}_+} (\rho_y^2 + u_t^2) dy ds + \int_0^t \left(\int_{\mathcal{F}_+} \rho_y^2 dy \right) \left(\int_{\mathcal{F}_+} u_t^2 dy \right) ds \right) \stackrel{\text{Lem.3.11, Lem.3.13}}{\leq} C. \quad \square \end{aligned}$$

3.5. Proof of Theorem 2.3. Combining Corollary 3.4, Lemma 3.8, Lemma 3.10, Lemma 3.9 and Lemma 3.13. it follows that there exists a constant $C \geq 1$, depending only on \mathcal{J} such that

$$\begin{aligned} \|u(t, \cdot)\|_{H^1(\mathcal{F})} + \|\rho(t, \cdot)\|_{H^1(\mathcal{F}_-)} + \|\rho(t, \cdot)\|_{H^1(\mathcal{F}_+)} &\leq C, \quad t \in [0, \tau), \\ \frac{1}{C} &\leq \rho(t, \cdot) \leq C, \quad t \in [0, \tau), y \in \mathcal{F}, \\ -1 + \frac{1}{C} &\leq h(t) \leq 1 - \frac{1}{C}, \quad t \in [0, \tau). \end{aligned}$$

Then, according to the local existence result Theorem 2.5, we can extend the solution beyond τ , and $[0, \tau)$ cannot be the maximal interval of existence. Hence, the solution is global. \square

4. LARGE-TIME BEHAVIOR OF STRONG SOLUTION

In this section we are going to prove Theorem 2.4. Throughout this section we assume that (h, ρ, u) is the global strong solution to the system (2.7)-(2.13). We start with the following result, which shows that, as t goes to infinity, velocity of the fluid and the piston goes to zero, and density of the fluid converges to the average density defined in (3.2).

Lemma 4.1. Let (h, ρ, u) be the global strong solution to the system (2.7)-(2.13). Then

$$\|u(t, \cdot)\|_{H_0^1(-r_-, r_+)} + |\dot{h}(t)| + \|\rho(t, \cdot) - \bar{\rho}(t, \cdot)\|_{H^1(\mathcal{F})} \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (4.1)$$

where $\bar{\rho}$ is defined in (3.2).

Proof. Let us set

$$Q(t) := \left\| \frac{\mu(\rho(t, \cdot))\rho_y(t, \cdot)}{\rho(t, \cdot)} \right\|_{L^2(\mathcal{F})}^2 + \|u_y(t, \cdot)\|_{L^2(\mathcal{F})}^2. \quad (4.2)$$

We show that $Q \in W^{1,1}(0, \infty)$. More precisely,

$$\int_0^\infty Q(s) ds \leq C \quad \text{and} \quad \int_0^\infty |Q'(s)| ds \leq C. \quad (4.3)$$

The fact that $Q \in L^1(0, \infty)$ follows from Corollary 3.4, Lemma 3.10 and Lemma 3.11. To show $Q' \in L^1(0, \infty)$, using the continuity equation (2.7), we can rewrite the momentum equation (2.8) as

$$u_t = (\rho\mu(\rho)u_y)_y - (\rho^\gamma)_y = - \left(\frac{\mu(\rho)}{\rho} \rho_t \right)_y - (\rho^\gamma)_y.$$

Multiplying the above identity with $\frac{\mu(\rho)\rho_y}{\rho}$ and integrating over \mathcal{F} , we get

$$\begin{aligned} \int_{\mathcal{F}} u_t \frac{\mu(\rho)\rho_y}{\rho} dy &= - \int_{\mathcal{F}} \left(\frac{\mu(\rho)}{\rho} \rho_t \right)_y \frac{\mu(\rho)\rho_y}{\rho} dy - \int_{\mathcal{F}} (\rho^\gamma)_y \frac{\mu(\rho)\rho_y}{\rho} dy \\ &= - \int_{\mathcal{F}} \left(\frac{\mu(\rho)\rho_y}{\rho} \right)_t \frac{\mu(\rho)\rho_y}{\rho} dy - \int_{\mathcal{F}} (\rho^\gamma)_y \frac{\mu(\rho)\rho_y}{\rho} dy = - \frac{1}{2} \frac{d}{dt} \left\| \frac{\mu(\rho)\rho_y}{\rho} \right\|_{L^2(\mathcal{F})}^2 - \int_{\mathcal{F}} (\rho^\gamma)_y \frac{\mu(\rho)\rho_y}{\rho} dy, \end{aligned}$$

which implies that

$$\frac{d}{dt} \left\| \frac{\mu(\rho)\rho_y}{\rho} \right\|_{L^2(\mathcal{F})}^2 \leq C \int_{\mathcal{F}} (u_t^2 + \rho_y^2) dy,$$

where C is independent of t . On the other hand, integrating by parts

$$\frac{d}{dt} \int_{\mathcal{F}} u_y^2 dy = -2\dot{h}(t)[u_y](t, 0) - 2 \int_{\mathcal{F}} u_{yy} u_t dy \leq C \left(|\dot{h}(t)|^2 + \int_{\mathcal{F}} (u_t^2 + u_y^2 + u_{yy}^2) dy \right).$$

Combining the above two estimates with Lemma 3.11, Lemma 3.12, Lemma 3.13 and Lemma 3.14 we obtain (4.3). Thus $\lim_{t \rightarrow \infty} Q(t) = 0$. In particular,

$$\|\rho_y(t, \cdot)\|_{L^2(\mathcal{F})}^2 + \|u_y(t, \cdot)\|_{L^2(\mathcal{F})}^2 \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Finally, from the definition of $\bar{\rho}$ in (3.2) and Poincaré-Wirtinger inequality, we deduce

$$\begin{aligned} \|\rho - \bar{\rho}\|_{L^2(\mathcal{F}_-)} &\stackrel{\text{Lem.3.8, Lem.3.9}}{\leq} C \left\| \frac{1}{\rho} - \frac{1}{\bar{\rho}} \right\|_{L^2(\mathcal{F}_-)} \\ &\stackrel{(3.2)}{=} C \left\| \frac{1}{\rho} - \frac{1}{r_-} \int_{\mathcal{F}_-} \frac{1}{\rho} dy \right\|_{L^2(\mathcal{F}_-)} \stackrel{\text{Lem.3.10}}{\leq} C \|\rho_y(t, \cdot)\|_{L^2(\mathcal{F}_-)} \xrightarrow{t \rightarrow \infty} 0. \end{aligned}$$

Similarly, we get the analogous estimate over the region \mathcal{F}_+ . Hence $\lim_{t \rightarrow \infty} \|\rho(t, \cdot) - \bar{\rho}(t, \cdot)\|_{H^1(\mathcal{F})} = 0$. This completes the proof of the lemma. \square

The next result shows that the density becomes same on both sides of the piston as time goes to infinity.

Lemma 4.2. Let (h, ρ, u) be the global strong solution to the system (2.7)-(2.13). Then

- (i) $[\rho^\theta](t, 0) \rightarrow 0$, as $t \rightarrow \infty$.
- (ii) For all $(y_-, y_+) \in \mathcal{F}_- \times \mathcal{F}_+$, $|\rho^\theta(t, y_-) - \rho^\theta(t, y_+)| \rightarrow 0$ as $t \rightarrow \infty$.

Proof. The first conclusion follows if we show that $[\rho^\theta](\cdot, 0) \in H^1(0, \infty)$. Since $0 < \theta < \frac{1}{2}$ and $\gamma > 1$, we get

$$\int_0^\infty [\rho^\theta](t, 0)^2 dt \underset{\text{Lem.3.10}}{\leq} C \int_0^\infty [\rho^\theta](t, 0) [\rho^\gamma](t, 0) dt \underset{(3.11)}{\leq} C.$$

Note that

$$\frac{d}{dt} \rho^\theta(t, 0^\pm) = -\theta \rho^{\theta+1}(t, 0^\pm) u_y(t, 0^\pm).$$

Therefore

$$\int_0^\infty \left(\frac{d}{dt} [\rho^\theta](t, 0) \right)^2 ds \underset{\text{Lem.3.9}}{\leq} C \int_0^\infty \int_{\mathcal{F}} (u_y^2 + u_{yy}^2) dy dt \underset{\text{Lem.3.12, Lem.3.14}}{\leq} C.$$

The second conclusion follows by observing that, for any $t > 0$ and $(y_1, y_2) \in \mathcal{F}_- \times \mathcal{F}_+$

$$|\rho^\theta(t, y_1) - \rho^\theta(t, y_2)| \leq |[\rho^\theta](t, 0)| + C(\rho^*) \|\rho_y(t)\|_{L^2(\mathcal{F})} \xrightarrow[\text{Lem.4.1}]{} 0, \quad \text{as } t \rightarrow \infty. \quad \square$$

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. In view of Lemma 3.8, for every $t \geq 0$ we have that the set $\{h(t)\}_{t \geq 0}$ is relatively compact in \mathbb{R} . Let $\{t_n\}_{n > 0}$ be a sequence of positive numbers such that

$$t_n \rightarrow \infty, \quad \lim_{n \rightarrow \infty} h(t_n) = h_* \in [-1 + \delta_0, 1 - \delta_0]. \quad (4.4)$$

From (4.1) and Lemma 4.2 we infer that

$$\lim_{n \rightarrow \infty} \rho(t_n, y) = \frac{M_L}{1 + h_*} \quad (y \in \mathcal{F}_-), \quad \lim_{n \rightarrow \infty} \rho(t_n, y) = \frac{M_R}{1 - h_*} \quad (y \in \mathcal{F}_+),$$

and

$$\frac{M_L}{1 + h_*} = \frac{M_R}{1 - h_*}.$$

Therefore

$$h_* = \frac{M_L - M_R}{M_L + M_R} = h_\infty,$$

and

$$\lim_{n \rightarrow \infty} \rho(t_n, y) = \frac{M_L + M_R}{2}, \quad (y \in \mathcal{F}), \quad (4.5)$$

which ends the proof. \square

5. THE CASE OF SEVERAL PISTONS

In this section we briefly explain how to extend our main results Theorem 1.3 and Theorem 1.5 to the case of several pistons. Let $N \in \mathbb{N}$ be the number of pistons. We denote by $h_i(t)$ the position of the i^{th} piston at instant t . We suppose that initially there is no contact between the pistons, and without loss of generality, we may assume that the initial positions of the pistons are in increasing order, i.e.,

$$-1 < h_1(0) < h_2(0) < \dots < h_N(0) < 1. \quad (5.1)$$

The domain occupied by the fluid is denoted by

$$\mathcal{F}_{\{h_i(t)\}_{i=1}^N} := (-1, 1) \setminus \{h_i(t)\}_{i=1}^N. \quad (5.2)$$

Moreover, the density and velocity of the fluid is denoted by $\tilde{\rho}$ and \tilde{u} respectively, and the mass of the i^{th} piston is denoted by m_i . With the above notations, the equations modelling the motion of several pistons in a density dependent viscous compressible fluid is given by

$$\tilde{\rho}_t + (\tilde{\rho}\tilde{u})_x = 0, \quad (x \in \mathcal{F}_{\{h_i(t)\}_{i=1}^N}, t \geq 0), \quad (5.3)$$

$$\rho(\tilde{u}_t + \tilde{u}\tilde{u}_x) - (\mu(\tilde{\rho})\tilde{u}_x)_x + (\tilde{\rho}^\gamma)_x = 0, \quad (x \in \mathcal{F}_{\{h_i(t)\}_{i=1}^N}, t \geq 0), \quad (5.4)$$

$$\tilde{u}(t, h_i(t)) = \dot{h}_i(t), \quad i \in \{1, \dots, N\}, (t \geq 0), \quad (5.5)$$

$$m_i \ddot{h}_i = [\mu(\tilde{\rho})\tilde{u}_x - \tilde{\rho}^\gamma](t, h_i(t)), \quad i \in \{1, \dots, N\}, (t \geq 0), \quad (5.6)$$

$$\tilde{u}(t, 1) = 0, \quad \tilde{u}(t, -1) = 0, \quad (t \geq 0), \quad (5.7)$$

$$h_i(0) = h_{i_0}, \quad \dot{h}_i(0) = \ell_{i_0}, \quad i \in \{1, \dots, N\}, \quad (5.8)$$

$$\tilde{u}(0, x) = \tilde{u}_0(x), \quad \tilde{\rho}(0, x) = \tilde{\rho}_0(x), \quad x \in \mathcal{F}_{\{h_{i_0}\}_{i=1}^N}, \quad (5.9)$$

where

$$\mu(\tilde{\rho}) = \tilde{\rho}^\theta \quad 0 < \theta < \frac{1}{2}. \quad (5.10)$$

We introduce the definition of strong solutions to the system (5.3)–(5.10).

Definition 5.1. A tuple $(h_1, \dots, h_N, \tilde{\rho}, \tilde{u})$ is said to be a strong solution to the problem (5.3)–(5.10) on the interval $[0, T]$ when it satisfies

$$h_i \in H^2(0, T), \quad i \in \{1, \dots, N\}, \quad -1 < h_1(t) < \dots < h_N(t) < 1, \quad \text{for all } t \in [0, T],$$

$$\tilde{\rho} \in C([0, T]; H^1(\mathcal{F}_{\{h_i(t)\}_{i=1}^N})) \cap H^1(0, T; L^2(\mathcal{F}_{\{h_i(t)\}_{i=1}^N})),$$

$$\tilde{\rho}(t, x) > 0, \quad \text{for all } t \in [0, T], \quad x \in \mathcal{F}_{\{h_i(t)\}_{i=1}^N},$$

$$\tilde{u} \in L^2(0, T; H^2(\mathcal{F}_{\{h_i(t)\}_{i=1}^N})) \cap H^1(0, T; L^2(\mathcal{F}_{\{h_i(t)\}_{i=1}^N})) \cap C([0, T]; H_0^1(-1, 1)),$$

equations (5.3) and (5.4) a.e. in $(0, T) \times \mathcal{F}_{\{h_i(t)\}_{i=1}^N}$, equation (5.6) a.e. in $(0, T)$, equations (5.5) and (5.7) in the sense of traces, and the initial conditions stated in (5.8) and (5.9).

We now state the global existence result for system (5.3)–(5.9).

Theorem 5.2. Let us assume that μ satisfies (5.10), $h_{i_0} \in (-1, 1)$, $\ell_{i_0} \in \mathbb{R}$, for $i \in \{1, \dots, N\}$, and $(\tilde{\rho}_0, \tilde{u}_0)$ belongs to $H^1(\mathcal{F}_{\{h_{i_0}\}_{i=1}^N}) \times H_0^1(-1, 1)$ and satisfy

$$\begin{aligned} -1 < h_{1_0} < h_{2_0} < \dots < h_{N_0} < 1, \\ \tilde{u}_0(h_{i_0}) = \ell_{i_0}, \quad \text{for } i \in \{1, \dots, N\}, \quad \tilde{\rho}_0(x) > 0 \quad \text{for } x \in \mathcal{F}_{\{h_{i_0}\}_{i=1}^N}. \end{aligned}$$

Then, for any $T > 0$ the system (5.3)–(5.9) admits a unique strong solution on $[0, T]$, in the sense of Definition 5.1.

Next, we determine the equilibrium solution to the system (5.3)-(5.10). Let us set

$$h_0 = h_0(t) = -1 \text{ and } h_{N+1} = h_{N+1}(t) = 1 \quad (t \geq 0). \quad (5.11)$$

The following mass conservation holds.

$$M_i := \int_{h_{i-1}(t)}^{h_i(t)} \tilde{\rho}(t, x) \, dx = \int_{h_{i-1}(0)}^{h_i(0)} \tilde{\rho}_0(x) \, dx, \quad (i = 1, 2, \dots, N+1, t \geq 0). \quad (5.12)$$

Lemma 5.3. Let $(h_{1\infty}, \dots, h_{N\infty}, \rho_\infty, u_\infty)$ be an equilibrium solution to the system (5.3)-(5.10). Then $u_\infty = 0$, and ρ_∞ and $h_{i\infty}$ are given by

$$\rho_\infty = \frac{1}{2} \sum_{i=1}^{N+1} M_i, \quad h_{i\infty} = \frac{\sum_{j=1}^i M_j - \sum_{j=i+1}^{N+1} M_j}{\sum_{j=1}^{N+1} M_j}, \quad i \in \{1, \dots, N\}. \quad (5.13)$$

Proof. From (5.3), (5.4) and (5.7) we first infer that

$$u_\infty = 0, \quad \rho_\infty = \begin{cases} \rho_{1\infty}, & x \in (h_0, h_{1\infty}), \\ \vdots & \vdots \\ \rho_{(N+1)\infty}, & x \in (h_{N\infty}, h_{N+1}), \end{cases}$$

where $\rho_{i\infty}$, $i \in \{1, \dots, N+1\}$, are positive constants. Then using (5.6), gives $\rho_\infty = \rho_{1\infty} = \dots = \rho_{(N+1)\infty}$. Using (5.12), we conclude that

$$\begin{aligned} M_1 &= (h_{1\infty} - h_0)\rho_\infty, \\ M_i &= (h_{i\infty} - h_{(i-1)\infty})\rho_\infty, \text{ for } i \in \{2, \dots, N\}, \\ M_{N+1} &= (h_{N+1} - h_{N\infty})\rho_\infty. \end{aligned} \quad (5.14)$$

Summing the above identities, we get

$$\sum_{i=1}^{N+1} M_i = (h_{N+1} - h_0)\rho_\infty = 2\rho_\infty,$$

from which we obtain the expression of ρ_∞ as in (5.13). Substituting this in (5.14), successively, we obtain the expressions of each of the h_i 's as given in (5.13). \square

The next result asserts the large time behaviour of the system (5.3)-(5.10).

Theorem 5.4. The global strong solution $(h_1, \dots, h_N, \tilde{\rho}, \tilde{u})$ to the system (5.3)-(5.10), satisfies the following

$$\|\tilde{\rho}(t, \cdot) - \rho_\infty\|_{H^1(\mathcal{F}_{\{h_i(t)\}_{i=1}^N})} + \|\tilde{u}(t, \cdot)\|_{H^1((-1,1))} + \sum_{i=1}^N |h_i(t) - h_{i\infty}| \rightarrow 0, \quad \text{as } t \rightarrow \infty,$$

In particular, as $t \rightarrow \infty$, each of the piston comes to rest, and the density becomes constant in the interval $(-1, 1)$.

Let us explain the main steps to prove Theorem 5.2 and Theorem 5.4. We begin by rewriting the system in the mass Lagrangian coordinate, as we did for the single piston. We define the transformation

$$y := X(t, x) = \int_{h_1(t)}^x \tilde{\rho}(t, \eta) \, d\eta, \quad (t \geq 0, x \in [-1, 1]).$$

Using (5.12), it is easy to see that, for any $t \geq 0$

$$\begin{aligned} r_0 &:= X(t, -1) = -M_1, & r_{N+1} &:= X(t, 1) = \sum_{j=2}^N M_j, \\ r_1 &:= X(t, h_1(t)) = 0, & r_i &:= X(t, h_i(t)) = \sum_{j=2}^i M_j \text{ for } i \in \{2, \dots, N\}, \end{aligned}$$

Moreover, if $\tilde{\rho}(t, \cdot)$ is sufficiently regular, positive and bounded away from zero, then X is a C^1 -diffeomorphism from $[-1, 1]$ to $[-r_0, r_{N+1}]$. For every $t \geq 0$, we denote by $Y(t, \cdot) = [X(t, \cdot)]^{-1}$, the inverse of $X(t, \cdot)$. Let us set

$$\mathcal{F}_N = (-r_0, r_{N+1}) \setminus \{r_1, r_2, \dots, r_N\},$$

and we consider the following change of variables

$$\rho(t, y) = \tilde{\rho}(t, Y(t, y)), \quad u(t, y) = \tilde{u}(t, Y(t, y)) \quad (t \geq 0, y \in \mathcal{F}_N).$$

Analogous to the single piston case, using the above change of variables, the system (5.3)-(5.9) can be written as

$$\rho_t + \rho^2 u_y = 0 \quad (y \in \mathcal{F}_N, t \geq 0), \quad (5.15)$$

$$u_t - (\rho \mu(\rho) u_y)_y + (\rho^\gamma)_y = 0 \quad (y \in \mathcal{F}_N, t \geq 0), \quad (5.16)$$

$$u(t, r_i) = \dot{h}_i(t) \quad i \in \{1, \dots, N\}, (t \geq 0), \quad (5.17)$$

$$m_i \ddot{h}_i(t) = [\rho \mu(\rho) u_y - \rho^\gamma](t, r_i) \quad i \in \{1, \dots, N\}, (t \geq 0), \quad (5.18)$$

$$u(t, -r_0) = u(t, r_{N+1}) = 0 \quad (t \geq 0), \quad (5.19)$$

$$h_i(0) = h_{i_0}, \quad \dot{h}_i(0) = \ell_{i_0} \quad i \in \{1, \dots, N\}, \quad (5.20)$$

$$\rho(0, y) := \rho_0(y), \quad u(0, y) := u_0(y) \quad (y \in \mathcal{F}_N). \quad (5.21)$$

where

$$\rho_0(y) = \tilde{\rho}_0(Y(0, y)), \quad u_0(y) = \tilde{u}_0(Y(0, y)).$$

The rest of the proof is similar to the single piston case. Similar to Theorem 2.5 local in time existence and uniqueness of the above system can be obtained. Regarding the global existence, we only state the analogous version of Proposition 3.3, Proposition 3.6 and Lemma 3.8 for the system (5.15)–(5.21). We define the average function $\bar{\rho}$ as follows,

$$\bar{\rho}(t, y) = \begin{cases} \frac{M_1}{1 + h_1(t)}, & \text{for } y \in (-r_0, r_1), t \geq 0, \\ \frac{M_i}{h_i(t) - h_{i-1}(t)}, & \text{for } i \in \{2, 3, \dots, N\}, y \in (r_{i-1}, r_i), t \geq 0, \\ \frac{M_{N+1}}{1 - h_N(t)}, & \text{for } y \in (r_N, r_{N+1}), t \geq 0. \end{cases}$$

Next, we define the kinetic and potential energy of the system, respectively, as follows

$$\mathcal{E}_k(t) := \frac{1}{2} \int_{\mathcal{F}_N} |u(t, y)|^2 dy + \frac{1}{2} \sum_{i=1}^N m_i \dot{h}_i^2(t),$$

$$\mathcal{E}_p(t) := \frac{1}{\gamma - 1} \int_{\mathcal{F}_N} (\rho^{\gamma-1}(t, y) - \bar{\rho}^{\gamma-1}(t, y)) dy.$$

We also define the quantity

$$\begin{aligned}
P(t) &:= \frac{1}{\gamma-1} \int_{\mathcal{F}_N} \bar{\rho}^{\gamma-1}(t, y) dy \\
&= \frac{1}{\gamma-1} \left[\frac{(M_1)^\gamma}{(1+h_1(t))^{\gamma-1}} + \sum_{i=2}^N \frac{(M_i)^\gamma}{(h_i(t)-h_{i-1}(t))^{\gamma-1}} + \frac{(M_{N+1})^\gamma}{(1-h_N(t))^{\gamma-1}} \right].
\end{aligned}$$

Note that, $\mathcal{E}_k(t), \mathcal{E}_p(t), P(t) \geq 0$ for any $t \geq 0$. We have the following energy identity for the system (5.15)–(5.21). The proof is similar to that of Proposition 3.3.

Proposition 5.5. Let $\mu(\rho) = \rho^\theta$ for some $\theta \geq 0$. Let us set

$$\mathcal{E}(t) := \mathcal{E}_k(t) + \mathcal{E}_p(t) + P(t), \quad (t \geq 0).$$

Then the function $t \mapsto \mathcal{E}(t)$ is C^1 on $[0, T]$, and for any $t \geq 0$ we have

$$\dot{\mathcal{E}}(t) = - \int_{\mathcal{F}_N} \rho(t, y) \mu(\rho(t, y)) u_y(t, y)^2 dy, \quad (t \geq 0). \quad (5.22)$$

The next result shows that the pistons do not come in contact with each other or with the extremities of the cylinder.

Lemma 5.6. Let us set

$$\delta_0 := \min \left\{ \frac{(M_1)^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}}, \frac{(M_2)^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}}, \dots, \frac{(M_{N+1})^{\frac{\gamma}{\gamma-1}}}{((\gamma-1)\mathcal{E}(0))^{\frac{1}{\gamma-1}}} \right\}.$$

Then for any $t \geq 0$

$$-h_{i-1}(t) + \delta_0 \leq h_i(t) \leq h_{i+1}(t) - \delta_0 \quad (i = 1, 2, \dots, N).$$

Finally, we state the analogous version of Proposition 3.6.

Proposition 5.7. Let $\mu(\rho) = \rho^\theta$ for some $\theta > 0$. Then there exists a positive constant C , depending only on the initial data, such that

$$\begin{aligned}
&\frac{1}{2} \int_{\mathcal{F}_N} \left(u + \frac{1}{\theta} (\rho^\theta)_y \right)^2 dy + \gamma \int_0^t \int_{\mathcal{F}_N} \rho^{\theta+\gamma-2} (\rho_y)^2 dy ds \\
&+ \sum_{i=1}^N \frac{1}{2m_i} \left(m_i \dot{h}(t) + \frac{1}{\theta} [\rho^\theta](t, r_i) \right)^2 + \sum_{i=1}^N \int_0^t \frac{1}{m_i \theta} [\rho^\theta](s, r_i) [\rho^\gamma](s, r_i) ds \leq C \quad (t \geq 0).
\end{aligned}$$

Once we have the above results, we can mimic the steps given in Section 3 and Section 4, to obtain the proof of Theorem 5.2 and Theorem 5.4. Since the calculations are almost identical and too much of a repetition, we omit the details here.

6. CONCLUDING REMARKS

The main results of this paper concern global existence and large time behaviour of the system modelling the motion of pistons in a viscous compressible fluid with density dependent viscosity. In view of our results several questions seem natural which we believe merit further attention.

Viscosity of the form $\mu(\rho) = \rho^\theta, \theta \geq \frac{1}{2}$: In this paper, we have crucially used the fact that $\theta < \frac{1}{2}$ to get the lower bound of density field (see Lemma 3.10). It seems natural to consider the piston problem with $\mu(\rho) = \rho^\theta$, with $\theta \geq \frac{1}{2}$. Perhaps one can follow the arguments of [3] to get the global existence in this case.

Large time behaviour for the Cauchy problem: One can also consider the Cauchy problem with density dependent viscosity. We think the global in time existence can be proved in a similar manner. However, the main challenge would be to show the decay rates of the solution, and determine the

large time behaviour of the piston. We remind that in the constant viscosity case this was studied in [12, 14, 13].

Piston problem with two different fluids: Another interesting problem is to consider the system with the two sides of the piston filled with two different fluids. More precisely, now the viscosities will be of the form ρ^{θ_1} and ρ^{θ_2} , respectively, on the left and right side of the piston with θ_1 different from θ_2 . Accordingly, the pressure laws will be changed to ρ^{γ_1} and ρ^{γ_2} , $\gamma_1 \neq \gamma_2$ respectively. The local in time existence can be proved easily. However, if we follow our method, we end up with a term like $\int_0^t \frac{1}{m} \left(\frac{\rho^{\theta_2}(s, 0^+)}{\theta_2} - \frac{\rho^{\theta_1}(s, 0^-)}{\theta_1} \right) (\rho^{\gamma_2}(s, 0^+) - \rho^{\gamma_1}(s, 0^-)) ds$ in Proposition 3.6, which doesn't have any sign.

Control problems: As far as we know, there is no controllability result available for the system (1.1)–(1.6) even when the viscosity is constant. The objective is to find controls, in terms of the boundary conditions, to drive the fluid to rest and the piston to a desired position. When the fluid is modelled by the viscous Burgers' equation the relevant control problem was studied in [16, 4, 21].

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