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► **To cite this version:**

César-fernando Méndez-Barrios, Juan-diego Torres-García, Silviu-iulian Niculescu. Delay-Difference Approximations of PD-Controllers. Improperly-posed Systems in Multiple Delays Case. International Journal of Robust and Nonlinear Control, 2024, 10.1002/rnc.7217 . hal-04172009

**HAL Id: hal-04172009**

**<https://hal.science/hal-04172009>**

Submitted on 15 Jan 2024

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# Delay-Difference Approximations of PD-Controllers. Improperly-posed Systems in Multiple Delays Case

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## Summary

In this paper, we study the behavior of characteristic roots of a class of strictly proper LTI SISO systems that are subject to a PD controller for which the derivative action has been implemented by means of a delay-difference approximation scheme that uses multiple delays. To such an end, the *improperly-posed* case is discussed in depth, i.e. the situations where “*small delays*” induce instability in the closed-loop system. The effect of adding more delays on the *improperly-posed* case is analyzed and a methodology for determining the set of parameters that produces such a situation for a given number of delays is derived. Some illustrative examples complete the presentation.

## KEYWORDS:

properly-/improperly-posed systems; PD-control; quasi-polynomials; delay-dependent coefficients.

## 1 | INTRODUCTION

It is well known that Proportional-Integral-Derivative (PID) controllers are easy to implement on industrial processes and their tuning makes it possible to respond satisfactorily to the control of large classes of processes. Such features are at the origin of their large “*popularity*”<sup>1,2,3</sup>. It should be mentioned that the first analytical study of PID control dates back a century ago<sup>4</sup>. There exists a large literature on methods of tuning PID controllers (see, for instance<sup>5</sup> and the reference therein), and it has always been mentioned that the derivative action brings some undesired behaviors in closed-loop control systems. The reasoning behind this idea stems from a well-known property of derivative actions: *the potential to amplify high-frequency noise signals in the system*. In this regard, there are two main methodologies in order to deal with such a problem. The first is to use a derivative filter, while the second aims to reduce the noise amplification effect by *intentionally inducing delays* through a delay-difference approximation scheme of the derivative operator<sup>6</sup>. This last idea has been studied by adopting different approaches. Roughly speaking, in contrast to the derivative operator, the time-delay block exhibits a bounded magnitude with respect to the frequency. This means that regardless of the frequency content of the input signal, the inclusion of high-frequency noise does not result in signal amplification when processed through the time-delay block.

There are several attempts to substitute the derivative term and consider in its place delay-based schemes. Under such an idea we can mention the following works: the proportional minus delay (PMD) controller proposed in<sup>7</sup> carries out an *averaged derivative action*, producing fast responses, being *almost* insensitive to high-frequency noise; such an idea has been explored, and some *design rules* have been published for special cases<sup>8,9</sup>. In<sup>10</sup> an average derivative action is also implemented; however, the controller design considers a first-order Padé approximation to avoid the exponential term. A practical implementation of a PMD controller on a real plant is also analyzed and presented in<sup>11</sup> and a performance comparison between a classical PD controller and a delay-based one can be found in<sup>12</sup>. It should be noted that the PMD controller induces parameters that explicitly depend

on the delay term. However, such a dependency has not been analyzed in any of the above publications. Besides the previous works, it is worth mentioning that the derivative can be approximated by using the well-known method of finite differences<sup>13</sup>. Such a technique enhances the accuracy of the approximation by considering more than two points, bringing it closer to the ideal derivative. When explicitly using only two points, it yields the same response as the PMD. Moreover, this approach mitigates the amplification of noise, as mentioned in<sup>12</sup>. In this regard, a delay-difference approximation produces a characteristic equation whose coefficients depend explicitly on the delay *parameter*. While there are several studies of time-delay systems stability, such as the early results on differential-difference equations, presented in<sup>14</sup>, the study of the effects induced by the parameters on the (asymptotic) stability of the dynamical system is still a subject of recurring interest (see, e.g.,<sup>15</sup> and the references therein for an overview of the open literature). In particular,<sup>16,17</sup> proposed a simple characterization of the stability of the system with respect to the delay parameter, method that is at the origin of several developments well reported in the literature. If, in most of the cases, the delay is at the origin of bad behavior (oscillations, stability loss), the idea to use the delay as a *control parameter* emerged more recently<sup>18,19</sup>.

Nevertheless, the analysis of systems whose parameters depend explicitly on the delay parameter is less frequently reported in the literature; those that exist present a significantly increased difficulty of analysis. For instance<sup>20,21</sup>, focus on developing algorithms and less on analytical methods. In<sup>22</sup> delay *margins* are explicitly computed for linear systems with delay-dependent coefficients.

It should be pointed out that, under certain conditions, the approximation may lead to some unexpected behaviors in terms of sensitivity with respect to the delay parameter variation, and, in particular, in the case of unstable singular solutions of the characteristic function even if the system with the ideal derivative operator is stable. Such a situation is referred as *improperly-posed*, and to the best of the authors knowledge, was not sufficiently discussed in the open literature. The first ideas to address the problem were considered in<sup>23</sup>, where the authors used a first-order approximation of the derivative, namely:

$$\dot{y}(t) \approx \frac{y(t) - y(t - \tau)}{\tau},$$

where  $\tau > 0$  denotes the delay parameter. In<sup>24</sup>, the same discretization was studied and, although it was not fully characterized, the improperly-posed character of the system was mentioned and some sufficient conditions are explicitly derived. Additionally, the singular behavior of the unstable characteristic root was observed via a numerical example but without any further discussion of the observed phenomenon. Next, following a similar idea, an extension of the analysis to a second-order approximation has been proposed in<sup>25</sup>. In such a work, it is important to emphasize an interesting property known as the *locking real unstable roots mechanism*. This property highlights that the system's stability crossing exclusively happens through the origin when there is a change in the derivative "gain" parameter. Additionally, it confirms that any unstable (characteristic) root, if present, is real.

The interest in extending this study to a case with a more significant number of delays comes from the possibility of increasing the accuracy when approximating the derivative action, which at the same time allows *larger* delay values. Indeed, adding more points (delays) to the approximation yields a better derivative approximation. However, this also increases noise amplification as the operator approaches the ideal derivative action; additionally, the effects of adding  $\ell$ -delays to the system are not yet characterized and, as discussed in the rest of this note, the *locking real unstable roots mechanism* property is not valid for cases with more than two delays in the approximation scheme.

The main contribution of this note is to characterize certain unbounded solutions of the characteristic functions of the closed-loop LTI systems subjected to PD controllers for which the derivative action has been approximated by using delay-difference operators including  $\ell$  delays, assumed to be commensurate. The idea to use commensurate delays is consistent with the fact to conserve only one parameter for the approximation, that is a low-complexity controller (only three parameters: the gains and the delay). Furthermore, a methodology allowing to find an "*improperly-posed criteria*" based on parametric analysis and the number of delays considered on the approximation is derived. Despite producing a better approximation, the use of a larger number of delays represents a stability loss (*improperly-posedness*). Finally, the illustrative examples show the effectiveness of the results showing the stabilizing/destabilizing effect induced by the delay approximation.

**Notations:** Throughout this paper, the following standard notations will be adopted: the set of real (rational) numbers is denoted by  $\mathbb{R}$  ( $\mathbb{Q}$ ). In particular,  $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ . Similarly,  $\mathbb{N}$  denotes the set of natural numbers. Next,  $\mathbb{C}$  ( $\mathbb{C}_+$ ,  $\mathbb{C}_-$ ) represents the set of complex numbers (with strictly positive/negative real parts) and  $i := \sqrt{-1}$ . For a complex number  $z$ ,  $\Re(z)$  ( $\Im(z)$ ) represents its real (imaginary) component. For a given set  $S$ ,  $\text{card}\{S\}$  denotes the number of elements of the set  $S$ . In addition, we denote  $\bar{S} := S \cup \{0\}$  and  $S^* := S \setminus \{0\}$ . The  $j$ -th Chebyshev polynomials (for a precise definition, please refer to Appendix B) of the first and second kind are denoted by  $T_j$  and  $U_j$ , respectively. For a real number  $x$ ,  $\text{sign}(x)$  denotes the sign of the number  $x$ , where  $\text{sign}(x) \in \{0, \pm 1\}$ . Finally, the binomial coefficient is defined by  $\binom{n}{k} := \frac{n!}{(n-k)!k!}$ .

## 2 | MOTIVATING EXAMPLE AND PRELIMINARIES

Let us consider the following *strictly proper* LTI SISO system  $\Sigma = (A, b, c^T)$  with the following state-space representation:

$$\Sigma : \begin{cases} \dot{x}(t) = \mathbf{A}x(t) + bu(t), \\ y(t) = c^T x(t), \end{cases} \quad (1)$$

where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $c^T, b \in \mathbb{R}^n$ . The corresponding transfer function is given by:

$$H_{yu}(s) := \frac{P(s)}{Q(s)} \equiv c^T (s\mathbf{I} - \mathbf{A})^{-1} b, \quad (2)$$

where  $P, Q$  are polynomials in  $s$ , with real coefficients:

$$Q(s) := s^n + \sum_{j=0}^{n-1} q_j s^j, \quad P(s) := \sum_{j=0}^m p_j s^j, \quad q_j, p_j \in \mathbb{R}, \quad (3)$$

with  $n := \deg(Q) > m := \deg(P)$ ,  $p_m \neq 0$  and relative degree  $d_r = n - m \geq 1$ . As mentioned, this note addresses strictly proper LTI SISO systems subject to a classic PD-controller:

$$u(t) := k_p y(t) + k_d \dot{y}(t), \quad (4)$$

where  $(k_p, k_d) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  are the proportional and derivative gains, respectively. The transfer function gives the corresponding frequency-domain controller:

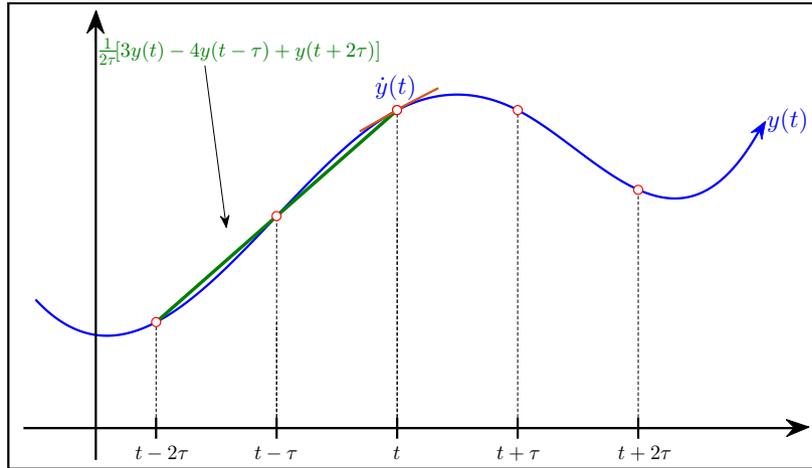
$$C_0(s) := k_p + k_d s. \quad (5)$$

In closed-loop, the characteristic function  $\Delta_0 : \mathbb{C} \rightarrow \mathbb{C}$  writes as:

$$\Delta_0(s) := Q(s) + (k_p + k_d s)P(s). \quad (6)$$

Denote by  $\sigma(\Delta_0)$  the spectrum of  $\Delta_0$ . In the sequel, we assume that (6) has all its characteristic roots located in  $\mathbb{C}_-$ , that is  $\sigma(\Delta_0) \subset \mathbb{C}_-$ . In other words,  $C_0$  is a stabilizing controller, and we will use the notation  $C_0 \in \text{Stab}(H_{yu})$ .

As outlined in the Introduction, our focus is the thorough analysis and comprehension of the effects resulting from the discretization of the derivative action. Several methodologies exist for such discretization; in this study, we adopt the well-established finite-difference method (see for instance<sup>13</sup> and<sup>26</sup>) to reconstruct the derivative. Specifically, we employ the  $\ell$ -th



**Figure 1** Three-points (or two-delays) approximation of the derivative of  $y(t)$  at  $t$ .

past values of the signal to achieve such a reconstruction. More precisely, the methodology can be stated as:

$$\dot{y}(t) \approx \sum_{k=0}^{\ell} y(t_k) L'_k(t), \quad (7)$$

where  $\ell \in \mathbb{N}$  represents the number of delays used on the approximation. In this context, we say that it is an  $(\ell + 1)$ –points approximation. The delay  $\tau \in \mathbb{R}_+$  is assumed strictly positive. The instants  $t_k$  are given by  $t_k = t - k\tau$ , and  $L_k$  denotes the  $k$ -th Lagrange coefficient polynomial for  $y$  at the time  $t_k$ , that is,

$$L'_k(t) = L_k(t) \sum_{j=0, j \neq k}^{\ell} \frac{1}{t - t_j}$$

where  $L_k$  is the polynomial given by

$$L_k(t) = \prod_{j=1, j \neq k}^{\ell} \frac{t - t_j}{t_k - t_j}.$$

Figure 1 illustrates the approximation when three points (two delays) are considered.

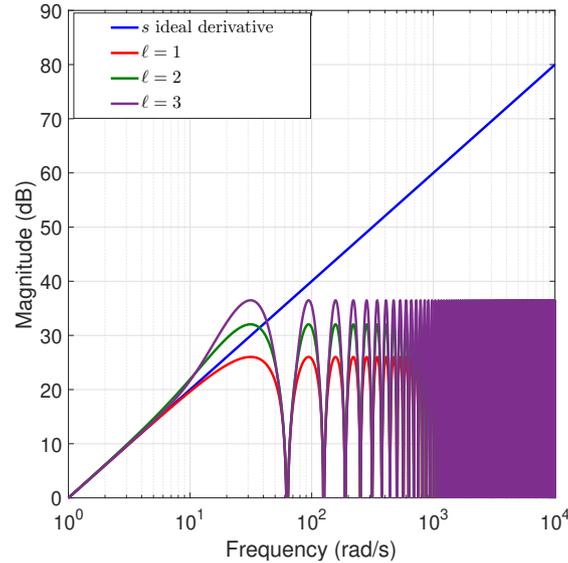
Inspired by this methodology, we aim to implement the derivative operator in  $C_0$ , given by (5) through such an approximation. After applying the Laplace transform, the discretized controller  $C_{\tau, \ell}$  reads:

$$C_{\tau, \ell}(s) := k_p + \frac{k_d}{\tau} \sum_{k=1}^{\ell} \frac{1}{k} (1 - e^{-\tau s})^k. \quad (8)$$

Taking this approximation, the characteristic function of the closed-loop system  $\Delta_{\ell} : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$  rewrites as:

$$\Delta_{\ell}(s; \tau) := Q(s) + \left( k_p + \frac{k_d}{\tau} \sum_{k=1}^{\ell} \frac{1}{k} (1 - e^{-\tau s})^k \right) P(s), \quad (9)$$

which is a *quasi-polynomial* with delay-dependent coefficients. As discussed in<sup>15</sup> into a different framework, it is sufficient to check the location of the *spectral abscissa*<sup>‡</sup>, given by  $\sigma_a(\Delta_{\ell}) = \max\{\Re(s_0) : s_0 \in \mathbb{C}, \Delta_{\ell}(s_0; \tau) = 0\}$ . Such a root depends continuously on the system's parameters and this property holds as long as  $\tau \in \mathbb{R}_+^*$  stays *strictly* positive (see, for instance,<sup>15</sup>).



**Figure 2** Bode diagrams of the ideal derivative and the corresponding versions with one, two, and three-delays approximations.

**Remark 1. Noise Reduction Effect.** As mentioned in the Introduction, an advantage of using delay-difference approximations is the reduction in the noise amplification produced when a derivative is used. Indeed, when we analyze the derivative operator's magnitude and phase response, given by  $M(s) = s$ , we observe the following relations:  $|M(i\omega)| = \omega$ ,  $\angle M(i\omega) = \frac{\pi}{2}$ . This indicates that as the frequency  $\omega$  of the input signal increases, the magnitude of its response will also increase. In other words, the higher the frequency, the more pronounced the effect of the derivative operator on the signal's magnitude. Next, let us analyze

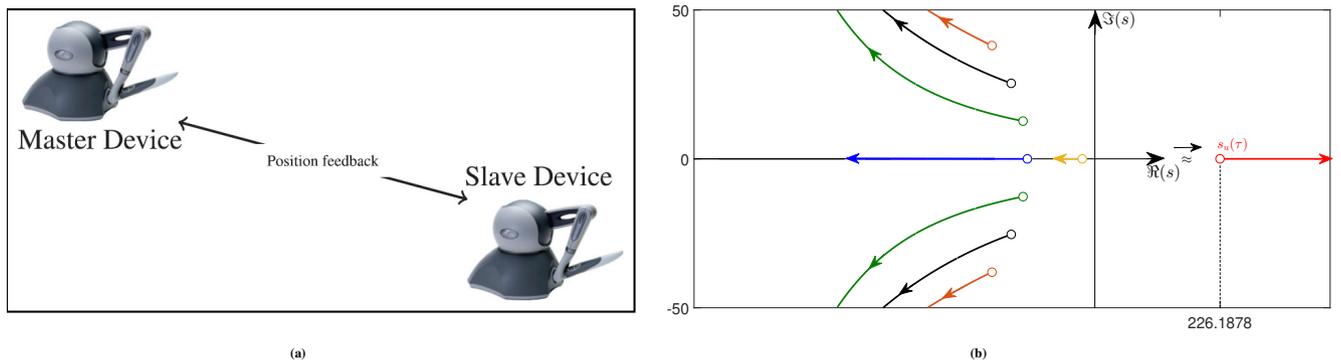
<sup>‡</sup>the spectral abscissa is the real part of the right-most root of the characteristic function.

the effect of the delay-based approximation. To perform a similar analysis on such an element, let us consider first the following:  $|e^{-i\omega\tau}| = |\cos(\omega\tau) - i\sin(\omega\tau)| = 1$ ,  $\angle e^{-i\omega\tau} = -\omega\tau$ . This relation allows us to conclude that, in contrast to the derivative operator, the time-delay block exhibits a bounded magnitude with respect to the frequency  $\omega$ . This means that regardless of the frequency content of the input signal, the inclusion of high-frequency noise does not result in signal amplification when processed through the time-delay block. To conclude, the frequency analysis of these two components highlights their main differences: while the derivative operator amplifies signals as frequencies increase, the time-delay block approximation maintains a bounded magnitude, making it less sensitive to the introduction of high-frequency noise. Figure 2 illustrates the Bode magnitude diagram of the ideal derivative and its one-, two-, and three-delay approximations.

## 2.1 | Motivating example

As it has already been pointed out, there might exist cases where a system loses its stability after replacing the derivative term with its delay-difference approximation. In fact, any digital implementation will always require a discretization process. In this regard, in order to emphasize the importance of considering the limits of the discretization, let us consider the Phantom-Omni Haptic device (see, Figure 3a) studied in<sup>27</sup>, described by the transfer function:

$$H_{yu}(s) := \frac{L(T_z s + 1)}{s(T_p s + 1)}. \quad (10)$$



**Figure 3** (a) Phantom omni haptic system. (b) Singular behavior of the solution  $s_u(\tau)$  for decreasing values of  $\tau$ .

Considering a classical PD controller, with gains  $(k_p, k_d) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , the characteristic function  $\Delta_0 : \mathbb{C} \rightarrow \mathbb{C}$  of the closed-loop system can be expressed as:

$$\Delta_0(s) := s^2 + \hat{T}_p s + (LT_z \hat{T}_p s + L\hat{T}_p)(k_p + k_d s), \quad (11)$$

where  $\hat{T}_p := 1/T_p$ , with  $T_p \neq 0$ , and  $L, T_z \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . According to the Stodola condition<sup>28</sup>, in the case of a polynomial of degree two, having the same sign in all coefficients is a necessary and sufficient condition to ensure exponential stability. Thus, under the aforementioned assumptions,  $\Delta_0$  corresponds to a stable closed-loop system if and only if the following conditions hold:

$$\text{sign}(1 + k_d LT_z \hat{T}_p) = \text{sign}(\hat{T}_p (1 + k_d L + k_p LT_z)) = \text{sign}(k_p L \hat{T}_p) \neq 0.$$

Now, to consider a realistic scenario, let us borrow the experimental data set from<sup>27</sup>:  $L = 0.38123$ ,  $T_z = 63.031$  and  $T_p = 0.63304$ <sup>‡</sup>. Under such considerations, it is clear to see that choosing  $(k_p, k_d) = (-2, -2)$  rends the closed-loop system asymptotically stable. Now, let us replace the derivative action by its delay-difference approximation with a single time delay, leading to the following characteristic function  $\Delta_1 : \mathbb{C} \times \mathbb{R}_+^* \rightarrow \mathbb{C}$ :

$$\Delta_1(s; \tau) := s^2 + \hat{T}_p s + (LT_z \hat{T}_p s + L\hat{T}_p) \left( k_p + k_d \frac{1 - e^{-s\tau}}{\tau} \right). \quad (12)$$

<sup>‡</sup>This specific set of parameters was obtained using an identification procedure (see<sup>27</sup> for further details).

A common practice suggests that the smaller the time delay, the better the approximation is; in this spirit, consider  $\tau \in [0.1, 0.5]$ . With this particular selection of gains, the characteristic function  $\Delta_1$  has an unstable real solution  $s_u$  that exhibits a singular behavior. Specifically, as the delay value  $\tau$  decreases,  $s_u$  moves further to the right, as can be observed in Fig. 3b. Surprisingly, the system conserves its asymptotic stability by fixing the gain values to  $(k_p, k_d) = (2, 1)$  instead of the previous values. Motivated by the previous discussion, in the remaining part of the paper, we will disclose the mechanism behind such a singular phenomenon (for further examples, we refer to<sup>29</sup> (second-degree case) and<sup>23</sup> (scalar case)). Furthermore, we will explore how such a mechanism could be modified if we take several delays in the derivative approximation instead of only one delay value.

### 3 | IMPROPERLY-POSED SYSTEMS ANALYSIS

#### 3.1 | Definitions and problem formulation

With the above discussion in mind, consider the following definition, borrowed from<sup>23</sup>:

**Definition 1** (Improperly-posed system). Consider the LTI SISO system (1) with transfer function (2). Suppose that  $C_0$  of the form (5) is a stabilizing controller  $C_0 \in \text{Stab}(H_{yu})$  and is replaced by  $C_{\tau, \ell}$  given in (8). If there exists a sequence of real numbers  $(\tau_n)_{n \in \mathbb{N}}$ ,  $\tau_n \rightarrow 0_+$  when  $n \rightarrow \infty$  such that for all  $\epsilon > 0$ , there exists some positive integer  $n_u$ , with  $\tau_{n_u} < \epsilon$  and  $C_{\tau_{n_u}, \ell} \notin \text{Stab}(H_{yu})$  the controller  $C_{\tau, \ell}$  is called an *improperly-posed* controller for “*small*” delays. In this case, the closed-loop system is *improperly-posed*<sup>§</sup>.

*Remark 2.* Observe that, since  $C_0 \in \text{Stab}(H_{yu})$ , all of the zeros of  $\Delta_0$  are located in  $\mathbb{C}_-$ . Definition 1 simply tells us that the spectral abscissa of the system is located in  $\mathbb{C}_+$  for some “*small*” delay values, even though the solutions of the delay-free ( $\tau = 0$ ) case are stable. In other words, improperly-posed closed-loop systems correspond to some loss of continuity of the spectral abscissa for “*small*” delays.

As one may expect from the definition above, the properly-posed for “*small*” delays case refers to the situation where the system remains stable after approximating the derivative action via an infinitesimal delay.

*Remark 3.* Although a properly-posed system remains stable after applying the discretization for “*small*” delay values, there may be a delay margin (an upper bound) to guarantee the system’s closed-loop stability.

In the light of the above definition (see also the motivating example), we will focus on the following problem: *Find the improperly/properly-posed criteria with respect to the closed-loop system parameters and the number of delays considered in the derivative approximation for  $\ell \in \mathbb{N}$ .*

The above problem can be summarized as finding the structure of an unstable-unbounded solution of the characteristic function of an improperly-posed closed-loop system, as well as finding the corresponding region in the considered parameter space (including the gains, the delay, and the number of delays).

#### 3.2 | Improperly-Posed Case: Asymptotic Behavior Analysis

To analyze the effects of small delays on the discretization of the derivative action, we will use the so-called Newton’s Diagram Method<sup>30</sup> to find an appropriate change of variable, such an approach follows similar steps to those presented in<sup>31, 23</sup> and<sup>25</sup>. In fact, the change of variable aims to simplify the asymptotic behavior analysis with respect to the delay parameter.

**Proposition 1.** Consider the system with transfer function (2) in closed-loop with the delay-based control scheme (8). Let  $\ell \in \mathbb{N}$ , and consider the closed-loop characteristic function  $\Delta_\ell$  given by (9). Then, when  $\tau \rightarrow 0^+$  the quasi-polynomial (8) has an unbounded root  $s : \mathbb{R}_+ \rightarrow \mathbb{C}$  with the following structure:

$$s(\tau) = \frac{s_0}{\tau} + \varphi(\tau), \quad (13)$$

where  $s_0 \in \mathbb{C} \setminus \{0\}$  and  $\varphi$  is a holomorphic function satisfying  $\varphi(0) \neq 0$ .

*Proof.* The proof is based on the Newton-diagram method and will be omitted, as it closely follows the steps outlined in<sup>23</sup>.  $\square$

<sup>§</sup>for “small” delays.

Proposition 1 gives essential information about the structure of particular solutions of the characteristic function. Under such a framework, in the sequel, we aim to complete its characterization by analyzing a way to obtain " $s_0$ " and its location in the complex plane. It is worth mentioning that the location of " $s_0$ " in  $\mathbb{C}_+$  ( $\mathbb{C}_-$ ) is essential for concluding if the corresponding closed-loop system is improperly-(properly-) posed.

Now that the singular behavior of certain characteristic roots has been identified, we aim to find their *location* in the complex plane. Note that, a direct application of the previously mentioned Newton's Polygon method produces the auxiliary quasi-polynomial  $P_\ell^{aux} : \mathbb{C} \rightarrow \mathbb{C}$  given by:

$$P_\ell^{aux}(w) = \lim_{\tau \rightarrow 0^+} \tau^n \Delta(w\tau^{-\beta}; \tau) = \lim_{\tau \rightarrow 0^+} \tau^n \Delta\left(\frac{w}{\tau}; \tau\right), \quad (14)$$

which after computing reads:

$$P_\ell^{aux}(w) := \begin{cases} w^n & \text{if } m < n - 1 \\ w^{n-1} f_\ell(w; k_d p_{n-1}) & \text{if } m = n - 1 \end{cases}, \quad (15)$$

where  $n$  and  $m$  are the degree<sup>‡</sup> of the denominator and numerator in (2), respectively,  $\ell$  is the number of delays considered on the approximation and  $f_\ell : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$  is a complex analytic function defined as

$$f_\ell(w; \alpha) := w + \alpha \sum_{k=1}^{\ell} \frac{(1 - e^{-w})^k}{k}. \quad (16)$$

*Remark 4.* Note that, regardless of the number of delays,  $f_\ell(\cdot; \alpha)$  always presents at least a single root at  $w = 0$ , which is *invariant* with respect to the parameter  $\alpha$ ; this implies that  $P_\ell^{aux}$  has always a root of multiplicity  $n$  at  $w = 0$ .

With this in mind, we have the following result, which is a direct consequence of Proposition 1:

**Corollary 1.** A singular characteristic root  $s^*$  of the characteristic function (9) is given by:

$$s^*(\tau) = \frac{1}{\tau} z^* + \mathcal{O}(1),$$

where  $z^*$  is a nontrivial solution of the auxiliary quasi-polynomial  $P_\ell^{aux}(w)$ .

Let us also introduce the following definitions:

**Definition 2** (Frequency crossing set). Consider the function  $f_\ell$  in (16) and a value  $\alpha^* \in \mathbb{R}^*$ . The *frequency crossing set*  $\Omega_{\alpha^*} \subset \mathbb{R}_+^*$  is the set of all  $\omega \in \mathbb{R}_+^*$  such that  $f_\ell(i\omega; \alpha^*) = 0$ .

**Definition 3** (Critical parameter/value). We say that  $\alpha \in \mathbb{R}^*$  is a *critical parameter* if there exists at least one real  $\omega^* \in \Omega_{\alpha^*}$  satisfying  $f_\ell(i\omega^*; \alpha^*) = 0$ . Similarly, a crossing frequency  $\omega^* \in \Omega_{\alpha^*}$  is called a *critical value* if there exists  $\alpha^* \in \mathbb{R}^*$  such that  $f_\ell(i\omega^*; \alpha^*) = 0$ .

*Remark 5.* Given our focus on nontrivial solutions, it becomes essential to verify the presence of an unstable unbounded root only when the relative degree of the system equals one. This condition is expressed as  $n := \deg Q \equiv 1 + m := \deg P$ . According to (15), it is clear to see that in cases where this condition is not fulfilled, the auxiliary quasi-polynomial takes the form of  $P_\ell^{aux}(w) = w^n$ , which exclusively yields trivial solutions.

### 3.3 | Auxiliary quasi-polynomial and its properties

In the following, let us study the solutions of the *auxiliary quasi-polynomial*  $f_\ell$  given by (16), with respect to the real parameter  $\alpha$ . According to the results reported in<sup>23</sup>, for  $\ell \equiv 1$ , there is always a *real root* located at  $\mathbb{C}_+$  ( $\mathbb{C}_-$ ) if and only if  $\alpha < -1$  ( $\alpha > -1$ ). Moreover, in the improperly-posed case, there are no other unstable roots. This property translated into a properly-posed interval defined by  $k_d p_{n-1} > -1$ . Under such an observation, as a first hypothesis, one may assume that such a property remains valid for any given number of delays. Surprisingly, such a property does not hold for any number of delays, and thus it is necessary to find a properly-posed interval given the value of  $\ell$ . In this regard, we have the following result:

**Proposition 2.** Let  $\ell \in \mathbb{N}$ , and consider the quasi-polynomial  $f_\ell(w; \alpha)$  given by (16). Then, the following properties hold:

- (i) For a given  $\alpha_0 \in \mathbb{R}^* \setminus \{-1\}$  the set  $\mathcal{Z}_\ell := \{r \in \mathbb{R} : f_\ell(r; \alpha_0) = 0\}$  has always  $\text{card} \{\mathcal{Z}_\ell\} \leq 2$ .

<sup>‡</sup>Observe that  $p_{n-1}$  correspond to the leading coefficient of  $P(s)$  in (2) when  $m = n - 1$ .

- (ii) For  $\alpha_0 := -1$  the set  $\mathcal{Z}_\ell$  has always  $\text{card}\{\mathcal{Z}_\ell\} = \ell + 1$ .
- (iii) There exists a sufficiently small value  $0 < |\alpha| \ll 1$ , such that all roots of  $\widehat{f}_\ell(w; \alpha) := f_\ell(w; \alpha)/w$  lie in  $\mathbb{C}_-$ .
- (iv) Let  $\alpha = \alpha^* \in \mathbb{R}^* \setminus \{-1\}$  be a critical value. Then, the non-zero characteristic roots of the quasi-polynomial  $f_\ell$  located on the imaginary axis are simple.

*Proof.*

- (i) Let  $\alpha_0 \in \mathbb{R}^* \setminus \{-1\}$ . Under such an assumption it is clear to see that  $0 \notin \mathcal{Z}_\ell$ . Now, to understand the behavior of  $f_\ell$ , let us analyze its derivative:

$$\begin{aligned} f'_\ell(r; \alpha_0) &:= \left. \frac{d}{dw} f_\ell(w; \alpha_0) \right|_{w=r} = 1 + \alpha_0(1 - \rho) \sum_{k=1}^{\ell} \rho^{k-1} \equiv 1 + \alpha_0 \left( \sum_{k=1}^{\ell} \rho^{k-1} - \sum_{k=1}^{\ell} \rho^k \right), \\ &\Rightarrow f'_\ell(r; \alpha_0) = 1 + \alpha_0(1 - \rho^\ell), \end{aligned} \quad (17)$$

where  $\rho := 1 - e^{-r}$ . From (17), it is evident that  $f'_\ell$  can have a maximum of two solutions in  $\mathbb{R}$ , implying that  $\text{card}\{\mathcal{Z}_\ell\} \leq 2$ , thereby proving (i).

- (ii) For  $\alpha_0 = -1$ , it is evident that  $w = 0$  is a root of  $f_\ell$  for any  $\ell \in \mathbb{N}$ . In addition, we have that

$$f'_\ell(w; \alpha_0) = 1 + \alpha_0 e^{-w} \sum_{k=0}^{\ell} (1 - e^{-w}). \quad (18)$$

Hence, by considering the change of variable  $z := 1 - e^{-w}$ , it results that (18) can be rewritten as

$$f'_\ell(z; \alpha_0) = 1 + \alpha_0(1 - z) \sum_{k=0}^{\ell} z^k \equiv (1 + \alpha_0) - \alpha_0 z^\ell. \quad (19)$$

Now, the change of variable suggests that  $w = 0$  is a root of  $f'_\ell$  if and only if  $z = 0$  is a root of (19). Therefore, the proof of (ii) is complete by noticing that for  $\alpha_0 = -1$ ,  $z = 0$  is a root of  $f'_\ell$  of multiplicity  $\ell$ .

- (iii) First, from the structure of  $f_\ell$ , it can be noted that for any value of  $\alpha$ ,  $f_\ell$  has an invariant root at the origin of the complex plane; with such an observation in mind, let us define  $\varphi : \mathbb{C} \rightarrow \mathbb{C}$  by  $\varphi(w) := \frac{1 - e^{-w}}{w}$ , which clearly is an analytic and bounded function in  $\overline{\mathbb{C}_+}$ , furthermore, it has the following limits:

$$\lim_{w \rightarrow 0} \varphi(w) = 1 \quad \wedge \quad \lim_{|w| \rightarrow \infty} \varphi(w) = 0.$$

In fact, to see that  $\varphi$  is bounded in  $\overline{\mathbb{C}_+}$ , let us first assume that  $|w| > 1$ . Under such a consideration, it is easy to observe that,

$$\left| \frac{1 - e^{-w}}{w} \right| \leq \frac{1 + |e^{-w}|}{|w|} \leq e^{-\Re\{w\}} + 1,$$

which is clearly bounded. For  $w \in \mathcal{B} := \{w \in \mathbb{C} : |w| \leq 1\}$ , the boundedness of  $\varphi$  follows straightforwardly from its analyticity and the compactness of  $\mathcal{B}$ . Next, from its definition we see that  $\widehat{f}_\ell$  is analytic in  $\mathbb{C}$ , moreover, it can be expressed as:

$$\widehat{f}_\ell(w; \alpha) = 1 + \alpha \varphi(w) \sum_{k=1}^{\ell} \frac{(1 - e^{-w})^{k-1}}{k}.$$

Defining  $M := \sum_{k=1}^{\ell} 2^{k-1}/k$  yields the inequality:

$$\left| \sum_{k=1}^{\ell} \frac{(1 - e^{-w})^{k-1}}{k} \right| \leq M.$$

Hence, by choosing  $\alpha_0$  such that  $M |\alpha_0| < 1$ , yields

$$|\alpha_0| |\varphi(w)| \left| \sum_{k=1}^{\ell} \frac{(1 - e^{-w})^{k-1}}{k} \right| < 1,$$

which allows concluding that  $\widehat{f}_\ell(w; \alpha) \neq 0$ ,  $\forall w \in \overline{\mathbb{C}_+}$  and all  $|\alpha| \leq |\alpha_0|$ , proving the property.

(iv) The proof follows straightforwardly, by noticing that the equations

$$f_\ell(i\omega; \alpha) = 0 \quad \wedge \quad \left. \frac{d}{ds} f_\ell(s; \alpha) \right|_{s=i\omega} = 0,$$

are incompatible for all  $\omega \in \mathbb{R}$  and all  $\alpha \in \mathbb{R}^*$ .

□

*Remark 6.* Observe that, as long as  $\alpha_0 \in \mathbb{R}^*$ , the real root of  $f'_\ell(r; \alpha_0)$  (if any) will be given by:

$$r = \log \left[ \left( 1 + \sqrt{\frac{1 + \alpha_0}{\alpha_0}} \right)^{-1} \right].$$

*Remark 7.* Observe that  $w = 0$  is an invariant root of  $f_\ell(w; \alpha)$  for any values of  $\alpha$  and  $\ell$ . Thus, such a solution is not of interest.

Next, we are interested in computing the values of  $\alpha$  for which the auxiliary quasi-polynomial (16), that is, the one that characterizes the improperly-posed case, has zeros in the right-half plane. To such an end, since the roots of (16) are continuous with respect to their parameters<sup>15</sup>, we must detect all critical pairs  $(\alpha^*, \omega^*)$ . Hence, in order to give the exact computation of such values, let us introduce the sets  $\mathcal{X}_\ell$  and  $\mathcal{W}_\ell$  defined by:

$$\mathcal{X}_\ell := \{ \zeta^* \in [-1, 1] : \varphi_\ell(\zeta^*) = 0 \}, \quad (20)$$

and

$$\mathcal{W}_\ell := \{ \omega \in \mathbb{R}^* : z = e^{i\omega} \text{ where } z \text{ satisfy } z^2 - 2\zeta^*z + 1 = 0, \text{ and } \zeta^* \in \mathcal{X}_\ell \}, \quad (21)$$

where  $\varphi_\ell$  is a polynomial in  $\zeta$ , with  $\deg(\varphi_\ell) = \ell$  defined as

$$\varphi_\ell(\zeta) := \sum_{k=1}^{\ell} \sum_{j=0}^k \frac{(-1)^j}{k} \binom{k}{j} T_j(\zeta). \quad (22)$$

Bearing in mind the above definitions, the following result allows us to characterize the complete set of critical pairs  $(\alpha^*, \omega^*)$ .

**Proposition 3.** Let  $\ell \in \mathbb{N}$ , and consider the quasi-polynomial  $f_\ell(w; \alpha)$  given by (16). Then,  $(\alpha^*, \omega^*) \in \mathbb{R}^* \times \mathbb{R}^*$  is a critical pair of  $f_\ell(w; \alpha)$  if  $\omega^* \in \mathcal{W}_\ell$  and  $\alpha^*$  is given by:

$$\alpha^* := \frac{\omega^*}{\sin(\omega^*) \psi_\ell(\zeta^*)}, \quad (23)$$

where  $\zeta^* \in \mathcal{X}_\ell$ , and  $\psi_\ell$  is a polynomial in  $\zeta$  with  $\deg(\psi) = \ell - 1$  defined by:

$$\psi_\ell(\zeta) := \sum_{k=1}^{\ell} \sum_{j=1}^k \frac{(-1)^j}{k} \binom{k}{j} U_{j-1}(\zeta).$$

*Proof.* Let  $\ell \in \mathbb{N}$  and  $w = i\omega$  with  $\omega \in \mathbb{R}$ . Then, from the definition of  $f_\ell$ , it is clear to see that the following equations hold:

$$f_\ell(i\omega; \alpha) = 0 \quad \Leftrightarrow \quad f_\ell(-i\omega; \alpha) = 0. \quad (24)$$

Thus, according to (24), if  $(i\omega)$  is a critical root, the following equations must be satisfied:

$$f_\ell(i\omega; \alpha) + f_\ell(-i\omega; \alpha) = 0, \quad (25)$$

$$f_\ell(i\omega; \alpha) - f_\ell(-i\omega; \alpha) = 0. \quad (26)$$

Consequently, from (25) we have:

$$\begin{aligned} f_\ell(i\omega; \alpha) + f_\ell(-i\omega; \alpha) &= \left( i\omega + \alpha \sum_{k=1}^{\ell} \frac{(1 - e^{-i\omega})^k}{k} \right) + \left( -i\omega + \alpha \sum_{k=1}^{\ell} \frac{(1 - e^{i\omega})^k}{k} \right), \\ &= \alpha \sum_{k=1}^{\ell} \left( \frac{(1 - e^{-i\omega})^k + (1 - e^{i\omega})^k}{k} \right). \end{aligned} \quad (27)$$

By considering (27), it follows that  $(i\omega)$  is a critical root of  $f_\ell$  if the following equation holds:

$$\sum_{k=1}^{\ell} \frac{(1 - e^{-i\omega})^k + (1 - e^{i\omega})^k}{k} = 0. \quad (28)$$

Now, by introducing the change of variable  $z := e^{-i\omega}$ , it is straightforward to see that (28) can be rewritten as:

$$\sum_{k=1}^{\ell} \frac{(1 - z)^k + (1 - z^{-1})^k}{k} = 0. \quad (29)$$

It is clear to see that (29) only includes constant terms or elements of the form  $z^j + z^{-j}$ , for  $j = 1, \dots, \ell$ . Moreover, since  $|z| = 1$ , such observation suggests writing (29) in terms of Chebyshev polynomials of the first kind, that is, by considering  $\zeta := \frac{1}{2}(z + z^{-1})$ , equation (28) can be rewritten as:

$$\sum_{k=1}^{\ell} \sum_{j=0}^k \frac{(-1)^j}{k} \binom{k}{j} T_j(\zeta) = 0. \quad (30)$$

Since the solutions of (30) are given by the set  $\mathcal{X}_\ell$ , it follows that all critical roots  $\pm i\omega$  are given by the elements of  $\mathcal{W}_\ell$ , as stated in Proposition 3. Next, to compute the critical parameters  $\alpha^*$ , let us take  $\omega^* \in \mathcal{W}_\ell$ , its associate value  $z^*$ , and consider (26), which yields to:

$$\begin{aligned} f_\ell(i\omega^*; \alpha) - f_\ell(-i\omega^*; \alpha) &= \left( i\omega^* + \alpha \sum_{k=1}^{\ell} \frac{(1 - e^{-i\omega^*})^k}{k} \right) - \left( -i\omega^* + \alpha \sum_{k=1}^{\ell} \frac{(1 - e^{i\omega^*})^k}{k} \right), \\ &= i2\omega^* + \alpha \sum_{k=1}^{\ell} \left( \frac{(1 - z^*)^k - \left(1 - \frac{1}{z^*}\right)^k}{k} \right), \\ &= i2\omega^* + \alpha (z^* - (z^*)^{-1}) \sum_{k=1}^{\ell} \sum_{j=1}^k \frac{(-1)^j}{k} \binom{k}{j} U_{j-1}(\zeta^*). \end{aligned} \quad (31)$$

Finally, by recognizing that  $z^* - (z^*)^{-1} = i2 \sin(\omega^*)$  and referring to the definition of  $\Psi_\ell$ , from (31), it follows that  $\alpha^*$  is given by (23), which allows us to conclude the proof.  $\square$

**Corollary 2.** Let  $\ell \in \mathbb{N}$ , and consider the quasi-polynomial  $f_\ell(w; \alpha)$  given by (16). Then, all the roots of  $f_\ell(w; \alpha)$  will lie in  $\overline{\mathbb{C}}_-$  if  $\alpha \in (\alpha_-, \alpha_+) \setminus \{0\}$ , where  $\alpha_-$  and  $\alpha_+$  are defined as

$$\alpha_+ := \min_{\zeta^* \in \mathcal{X}_\ell, \omega^* \in \mathcal{W}_\ell} \left\{ \alpha \in \mathbb{R}_+ : \alpha = \frac{\omega^*}{\sin(\omega^*) \Psi_\ell(\zeta^*)} \right\} \quad \text{and} \quad \alpha_- := \max_{\zeta^* \in \mathcal{X}_\ell, \omega^* \in \mathcal{W}_\ell} \left\{ \alpha \in \mathbb{R}_- : \alpha = \frac{\omega^*}{\sin(\omega^*) \Psi_\ell(\zeta^*)} \right\}. \quad (32)$$

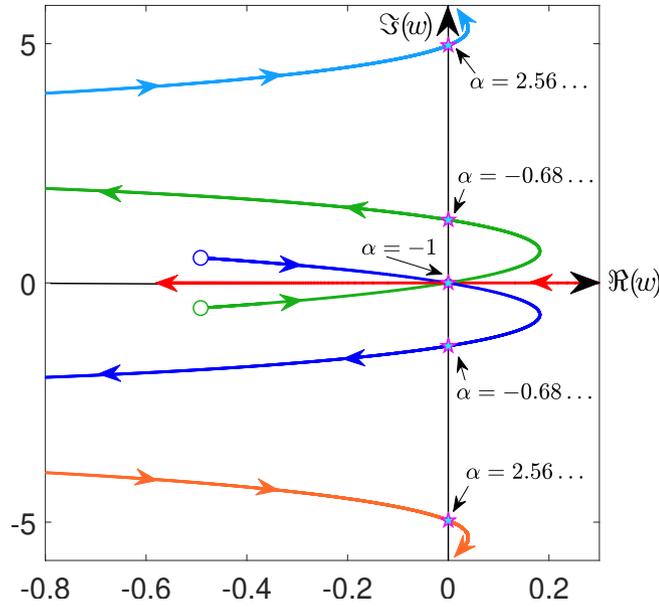
*Proof.* The proof follows straightforwardly from Proposition 2-(iii) and Proposition 3.  $\square$

**Example 3.1.** In order to illustrate the benefits of Proposition 3 to compute the critical parameters of  $f_\ell$ , let us consider the cases for which  $\ell = 3, 4, 5$ . In this regard, Table 1 summarizes a straightforward application of the previous results.

Now, to determine the stable interval  $\mathcal{I}$ , let us consider the case for which  $\ell = 3$ . Hence, by considering Corollary 2, it results that  $\mathcal{I} = (\alpha_-, \alpha_+) = \left( -\frac{2 \tan^{-1}(\sqrt{15})}{\sqrt{15}}, \frac{4\pi - 2 \tan^{-1}(\sqrt{15})}{\sqrt{15}} \right)$ . To illustrate the behavior of the solutions of  $f_\ell$  for  $\ell = 3$ , Fig.4 depicts the evolution of the rightmost roots when  $\alpha \in (-1.1, 3)$ .

*Remark 8.* Figure 4 confirms, as stated in Proposition 2-(ii), that for  $\alpha = -1$ ,  $w = 0$  is a root with multiplicity 4 (recall that there is an invariant root at  $w = 0$ ).

Computing more examples suggests that, as shown in Figure 5a, the larger the number of delays, the smaller the interval  $(\alpha_-, \alpha_+)$  for which the system is properly-posed. It can also observe the changes in the *properly-posed* region on the  $(k_d, p_{n-1})$  space by setting  $\alpha = k_d p_{n-1}$  and finding the region corresponding to  $\alpha_- < k_d p_{n-1} < \alpha_+$  (see Fig.5b).



**Figure 4** Evolution of the rightmost roots of  $f_\ell$  for increasing value of  $\alpha$ , where  $\alpha \in (-1.1, 3)$ .

**Table 1** Relations between  $f_\ell$ ,  $\varphi_\ell$  and  $\psi_\ell$  for different values of  $\ell$ .

$\ell$	Functions		Solutions		
	$f_\ell$	$\varphi_\ell$	$\zeta^*$	$\omega^{*\dagger}$	$\alpha^*$
3	$w + \alpha \sum_{k=1}^3 \frac{(1-e^{-w})^k}{k}$	$-4\zeta^3 + 9\zeta^2 - 6\zeta + 1$	$\frac{1}{4}$	$\pm \tan^{-1}(\sqrt{15}) + 2m\pi$	$\frac{\mp 2(2m\pi \pm \tan^{-1}(\sqrt{15}))}{\sqrt{15}}$
4	$w + \alpha \sum_{k=1}^4 \frac{(1-e^{-w})^k}{k}$	$6\zeta^4 - 16\zeta^3 + 12\zeta^2 - 2$	$-\frac{1}{3}$	$(2m+1)\pi \pm \tan^{-1}(2\sqrt{2})$	$\frac{\mp 3((2m+1)\pi \pm \tan^{-1}(2\sqrt{2}))}{10\sqrt{2}}$
5	$w + \alpha \sum_{k=1}^5 \frac{(1-e^{-w})^k}{k}$	$-\frac{48}{5}\zeta^5 + 30\zeta^4 - 28\zeta^3 + 12\zeta - \frac{22}{5}$	$\frac{3+\sqrt{1065}}{48}$	$2m\pi \pm \tan^{-1}\left(\frac{8}{\sqrt{\gamma_1}}\right)$	$\frac{\mp 3\sqrt{\delta_1}\left(2m\pi \pm \tan^{-1}\left(\frac{8}{\sqrt{\gamma_1}}\right)\right)}{152}$
			$\frac{3-\sqrt{1065}}{48}$	$(2m+1)\pi \pm \tan^{-1}\left(\frac{8}{\sqrt{\gamma_2}}\right)$	$\frac{\mp 3\sqrt{\delta_2}\left((2m+1)\pi \pm \tan^{-1}\left(\frac{8}{\sqrt{\gamma_2}}\right)\right)}{152}$

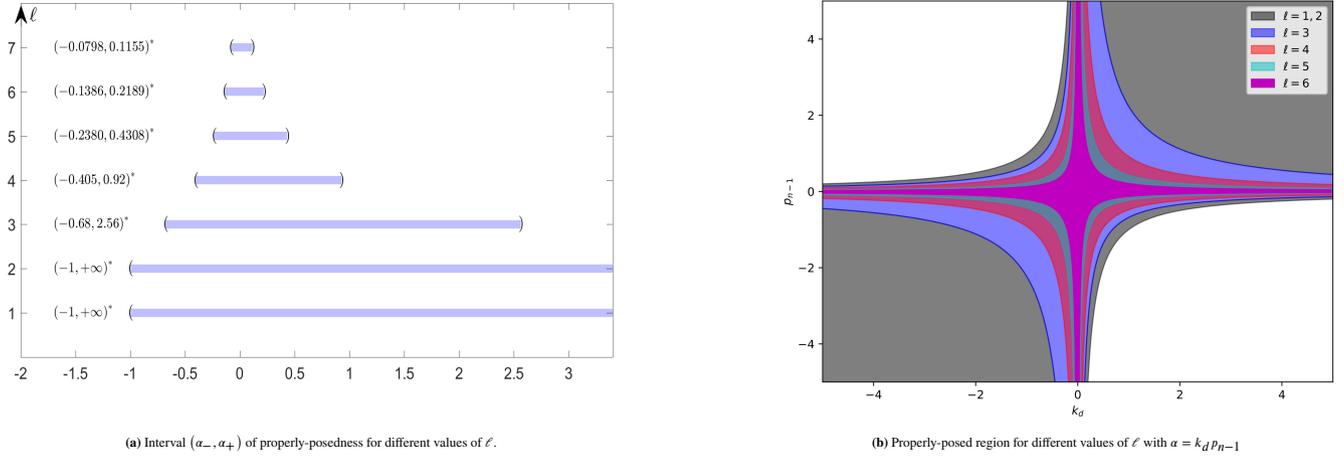
Notes:

$$\dagger m \in \mathbb{Z}; \quad \gamma_1 := 59 + 3\sqrt{\frac{213}{5}}; \quad \gamma_2 := 59 - 3\sqrt{\frac{213}{5}}; \quad \delta_1 := 2555 + 387\sqrt{\frac{213}{5}}; \quad \delta_2 := 2555 - 387\sqrt{\frac{213}{5}}.$$

### 3.4 | Characterizing properly/improperly-posed closed-loop systems\*

The previous example has shown a crucial property: *crossings from stability towards instability or vice versa may occur*. Therefore, to find the properly-posed interval it will be required the explicit computation of the crossing direction at the given critical value  $\alpha^*$ .

Let  $\mathcal{W}_\ell$  denote the set of *crossing frequencies* given by (21). It is worth noting that when  $\ell = 1$ , it implies that  $\text{card}\{\mathcal{W}_\ell\} = 0$ . Such an observation stems from the fact that, for  $\ell \in \{1, 2\}$ , any potential crossings take place exclusively at the origin. This distinctive characteristic has been extensively explored in previous works such as<sup>23</sup> and<sup>25</sup> to characterize regions of properly/improperly-posedness. Bearing in mind the previous observations, in the following we wish to express the crossing direction of the solution  $s^* = i\omega^*$  as a function of  $\alpha$ , seen as a *parameter*. More precisely, in the case of *simple* roots on the imaginary axis, we compute explicitly the sign of the real part of the derivative at the point of interest. Simple computations



**Figure 5** Effect of increasing the number of delays on stability.

lead to the following formula:

$$\text{sign} \left( \Re \left[ \frac{dw}{d\alpha} \right]_{w=i\omega^*, \alpha=\alpha^*}^{-1} \right) = \text{sign} \left( \Re \left[ \frac{\alpha}{w(\alpha)} + \alpha^2 \frac{e^{-w(\alpha)}}{w(\alpha)} \sum_{k=1}^{\ell} (1 - e^{-w(\alpha)})^{k-1} \right]_{w=i\omega^*, \alpha=\alpha^*} \right).$$

Further computations show that the sign is *invariant* with respect to the parameter  $\alpha$ , and it depends on the *crossing frequency*  $\omega^*$  and the *number of delays*  $\ell$ . More precisely, one gets:

$$\text{sign} \left( \Re \left[ \frac{dw}{d\alpha} \right]_{w=i\omega^*, \alpha=\alpha^*}^{-1} \right) = \text{sign} \left( -\Re \left[ \sum_{k=1}^{\ell} i^k 2^{k-1} e^{-i\omega^*(k+1)/2} [\sin(\omega^*/2)]^{k-1} \right] \right). \quad (33)$$

**Proposition 4.** Let  $\ell \in \mathbb{N}$ , and consider the auxiliary quasi-polynomial  $f_{\ell}(\cdot; \alpha)$  given by (16). Let  $\alpha = \alpha^* \in \mathbb{R}^* \setminus \{-1\}$  be a critical parameter for the corresponding critical value  $\omega^*$ . Then, for  $\alpha > \alpha^*$  sufficiently close to  $\alpha^*$ , the root  $w(\alpha)$  will *crossing* towards stability (instability) if

$$\Re \left[ \frac{dw}{d\alpha} \right]_{w=i\omega^*, \alpha=\alpha^*}^{-1} < 0 (> 0).$$

*Proof.* From Proposition 2-(iv) it is known that any crossing point occurring on  $i\mathbb{R}_+^*$  is a simple root, which implies that the *Implicit Function Theorem*<sup>32</sup> is valid and it is easy to observe that it suffices to examine in which direction the real part of the root is moving.  $\square$

Inspired by<sup>33,34</sup>, we consider the following stability indicator function  $\mathcal{I} : (-1, +\infty)^* \times \mathbb{R}^* \rightarrow \{0, \pm 1\}$ :

$$\mathcal{I}(\alpha^*) := \text{sign} \left( \Re \left[ \frac{dw}{d\alpha} \right]_{w=i\omega^*, \alpha=\alpha^*}^{-1} \right). \quad (34)$$

**Proposition 5.** Let  $\ell, N \in \mathbb{N}$ , and consider the auxiliary quasi-polynomial  $f_{\ell}(\cdot; \alpha)$  given by (16). For a given  $(\alpha^*, \ell) \in (-1, +\infty)^* \times \mathbb{N}$ , let  $\alpha_h, h \in \{1, \dots, N\}$  be the solutions of Proposition 3, such that the following order relations hold:

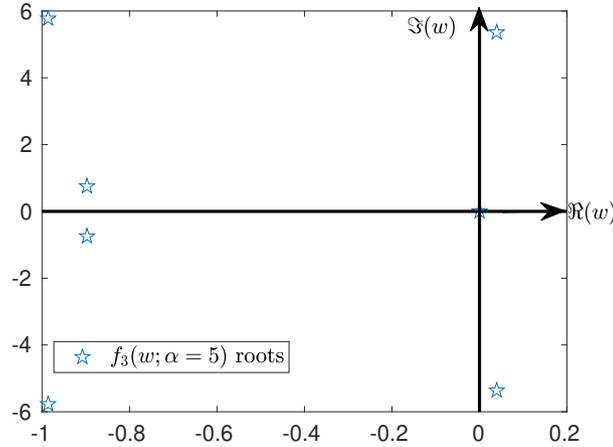
$$-1 < \alpha_1 < \alpha_2 < \dots < \alpha_{N-1} < \alpha^* \leq \alpha_N.$$

Then, the number of roots located in  $\mathbb{C}_+$  is given by:

$$n^{(\ell, \alpha_0)} = (1 + \text{sign}(\alpha^*)) \sum_{i=j+1}^{N-1} \mathcal{I}(\alpha_i) - (1 - \text{sign}(\alpha^*)) \sum_{i=N}^j \mathcal{I}(\alpha_i), \quad (35)$$

where  $j \in \mathbb{N}$  is chosen by considering that it satisfies  $\alpha_j = \alpha_-$ , with  $\alpha_-$  given by Corollary 2.

*Proof.* From Proposition 2 (iii) we know that, regardless of the number of delays, there exists some interval  $\alpha \in (\alpha_-, \alpha_+)^*$ , such that  $f_{\ell}/w$  is stable, in other words, there are no roots of  $f_{\ell}$  in  $\mathbb{C}_+$ . By continuity with respect to  $\alpha$  we know that roots can only cross from the left-half plane to the right-half plane by crossing the imaginary axis. Furthermore, from Proposition 2-(iv)



**Figure 6**  $f_3(w; \alpha = 5)$  roots.

whenever a root is located over the imaginary axis, that root is simple, which means that every crossing point corresponds to only a pair of complex-conjugate roots. From the same result, we also know that any real crossing may only occur for  $\alpha^* \in \{0, -1\}$ . Since  $\alpha^* > -1$ , the above observations allow us to conclude the proof.  $\square$

**Example 3.2.** To illustrate the mechanism provided by Proposition 5 to *count* the number of unstable roots for a given  $\alpha = \alpha_0 \in \mathbb{R}^*$ , let us consider  $\ell = 3$  and  $\alpha_0 = 5$ . From Proposition 5, we shall consider  $\alpha_i \in \{-0.68 \dots, 2.56 \dots, 5.80 \dots\}$ . Table 2 presents the crossing directions as well as the frequencies where they occur for each  $\alpha$  of interest. Figure 6 depicts the two unstable roots counted for  $\alpha = 5$ , demonstrating the efficiency of the methodology.

**Table 2** Counting of unstable roots for a given value of  $\alpha$ .

Number of unstable roots of $f_3(w; \alpha) _{\alpha=5}$			
$i$	$\alpha_i$	$\omega$	$I(\alpha_i, \omega)$
1	$-\frac{4 \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)}{\sqrt{15}}$	$2 \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)$	-1
2	$\frac{2\left(2\pi - 2 \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)\right)}{\sqrt{15}}$	$2\pi - 2 \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)$	1
3	$\frac{2\left(4\pi - 2 \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)\right)}{\sqrt{15}}$	$4\pi - 2 \tan^{-1}\left(\sqrt{\frac{3}{5}}\right)$	1
			$n^{(3,5)} = 2$

With the notations and results above, we state now our main result:

**Theorem 1.** Given a system  $\Sigma$  with corresponding transfer function  $H_{yu}$  given by (2) subjected to a PD controller with the derivative action approximated via a delay-difference approximation and frequency-domain representation (8), the following properties hold:

- (i) if the relative degree is greater than one ( $n > m + 1$ ), then the closed-loop system is always properly-posed independently of the number of delays in the delay-difference approximation scheme;

- (ii) if the relative degree of the system equals one ( $n \equiv m + 1$ ), then there exists an open interval  $(\gamma_1, \gamma_2) =: I_\ell \subset \mathbb{R}$ , such that the closed-loop system is properly-posed if the gain " $k_d$ " satisfies the condition  $\gamma_1 < k_d p_{n-1} < \gamma_2$ . Furthermore, as the number of delays  $\ell$  increases, the diameter of the interval  $I_\ell$  decreases.

*Proof.* (i) can be directly verified from Corollary 1 and the Newton's diagram method. For (ii), in order for the system to be properly-posed, all of the nontrivial solutions of the auxiliary quasi-polynomial  $P_\ell^{aux}(w)$ , given by (15), must be in  $\mathbb{C}_-$ . Since the trivial solutions are not of interest, consider the quasi-polynomial  $f_\ell(w; \alpha)$  given by (16). Thanks to Proposition 2 (ii) and Proposition 3 we can explicitly compute at least one interval of the parameter  $\alpha$  for which condition (ii) holds.  $\square$

*Remark 9.* The case corresponding to one-delay approximation is equivalent to the PMD controller presented in<sup>7</sup>, which as it can be intuitively derived from this result, can also be improperly-posed. Such an aspect was not considered in the PMD controller study.

*Remark 10.* A system subjected to a PD controller (8) with  $\ell \in \{1, 2\}$  is properly-posed if and only if  $k_d p_{n-1} > -1$ . In other words, for the one and two delay cases the interval  $I_1 = I_2 = (-1, \infty)$ .

## 4 | ILLUSTRATIVE EXAMPLES

All the numerical solutions of quasi-polynomials obtained in this section were computed using the Quasi-Polynomial Mapping Based Rootfinder (QPmR). Readers may refer to<sup>35</sup> for further details on such an algorithm.

### 4.1 | Destabilizing effect of the time-delay approximation

Examples 4.1 and 4.2 show how the derivative action discretization can lead to instability in closed-loop for "small" delay values.

**Example 4.1.** Consider as a first example the system described by the transfer function:

$$H_{yu}(s) = \frac{s}{5 - s^2}. \quad (36)$$

Considering a classical PD controller with gains  $(k_p, k_d) = (1.5, 1)$  the system is stable in closed-loop and has roots at  $s = -1 \pm 3i$ . However, it has relative degree one, and  $k_d p_{n-1} = -1.5 < -1$ ; thus, the closed-loop system is improperly-posed for such gains. Considering an approximation of the derivative action via three delays, the auxiliary quasi-polynomial reads:

$$f_3(w; -1.5) = w - 1.5 \left( \frac{11 - 18e^{-w} + 9e^{-2w} - 2e^{-3w}}{6} \right),$$

which has a solution at  $w = 2.335$ , which according to Corollary 1 leads to a singular solution of the characteristic equation with the following structure:

$$s^*(\tau) = \frac{2.335}{\tau} + \mathcal{O}(1).$$

If we consider  $\tau = 0.1$  in the discretization process, we should expect a solution located around  $s \approx 23.35$ . Indeed, computing the solutions of the characteristic equation, we find a root located at  $s = 25.23$ .

**Example 4.2.** As a second example, consider a system described by the transfer function:

$$H_{yu}(s) = \frac{\beta(s^2 + 1)}{s^3 - 2s^2 + s + 1}, \quad (37)$$

where  $\beta = -1$ . It is easy to observe that the system can be stabilized by a PD controller with  $(k_p, k_d) = (2, 3)$ ; nonetheless, similar to the previous example, the system's relative degree is one, and  $k_d p_{n-1} = -3 < -1$ . Now, if we consider an approximation with two delays, we obtain the following auxiliary quasi-polynomial

$$f_2(w; -3) = w - 3 \left( \frac{3 - 4e^{-w} + e^{-2w}}{2} \right),$$

which has a solution at  $w = 4.428$ . As in the previous example, such a solution indicates that the characteristic equation presents a solution of the form:

$$s^*(\tau) = \frac{4.428}{\tau} + \mathcal{O}(1).$$

Taking, for example,  $\tau = 0.2$  and computing the solutions of the characteristic equation, we find that  $s = 26.34$  is a solution. As we might expect, this solution is close to  $s \approx 22.14$ .

## 4.2 | Stabilizing effect of the time-delay approximation

The next example shows the benefits of using a delay-difference approximation of the PD-controller and the advantages that may be acquired by increasing the number of delays considered in the approximation.

**Example 4.3.** Consider a system described by

$$H_{yu}(s) = \frac{1}{s^4 + s^2 + 4s - 31}. \quad (38)$$

The open-loop system is unstable, having roots located at  $s = \{-2.43, 2.07, 0.179, \pm 2, 48i\}$ . Observe that, unlike the previous examples, the system's relative degree is greater than one, therefore the closed-loop system is properly-posed when approximating the derivative by a delay-difference operator including  $\ell$  delays. In the ideal PD-controller, with gains  $(k_p, k_d) = (31.4, -3.4)$  the closed-loop system has the following unstable roots:

$$s = 0.5 \pm 1.02i.$$

Consider now the one-delay approximation of the derivative action, leading to the following closed-loop characteristic function:

$$\Delta_1(s; \tau) := s^4 + s^2 + 4s - 31 + k_p + k_d \frac{1 - e^{-\tau s}}{\tau}. \quad (39)$$

We are looking for a delay  $\tau$  such that the characteristic function (39) has all of its solutions located in the left-hand plane of  $\mathbb{C}$ . Figure 7a depicts the behavior of some characteristic roots of the closed-loop system as we increase the delay  $\tau$  from 0.01 to 0.85. It can be observed that the system does not stabilize for any value of the delay parameter.

Alternatively, consider the proposed PID-delayed controller proposed in<sup>10</sup>, which is defined as  $C_{PIDd}(s) = C_d(s) + C_i(s)$ , where

$$C_d(s) = K_p - k_d + k_d(1 - e^{-\tau_1 s}) \approx k_p - k_d + k_d \tau_1 \frac{s}{1 + \frac{1}{2} \tau_1 s},$$

which is a PMD-type controller where the delay is approximated using the Padé approximation, and

$$C_i(s) = \frac{k_i(a\tau_2 s + 1)}{a\tau_2 s + 1 - e^{-s\tau_2}} \approx \frac{k_i(1 + \frac{\tau_2}{2}s)(1 + a\tau_2 s)}{(a+1)\tau_2(1 + \frac{a\tau_2}{2(a+1)}s)},$$

which is a delay-based approximation of an integral action for which the delay has been approximated using the Padé approximation. Considering the controller  $C_{PIDd}(s)$  produces a configuration with at least five parameters to tune, which aims to emulate the behavior of a PID controller, which is out of the scope of this work. Nonetheless, the controller  $C_d(s)$  intends to produce a similar behavior in terms of the derivative action than the delay-difference approximation discussed in the paper, without dealing with an infinite-dimensional system. If we consider the system subjected to  $C_d(s)$  the characteristic function rewrites as follows:

$$\Delta_{PIDd}(s; \tau_1) := \frac{\tau_1}{2}s^5 + s^4 + \frac{\tau_1}{2}s^3 + (1 + 2\tau_1)s^2 + (4 - \frac{31\tau_1}{2} - \frac{kd\tau_1}{2} + \frac{k_p\tau_1}{2}),$$

where  $\tau_1 > 0$ . Direct computation of classic methods such as the Routh-Hurwitz criteria shows that the system will remain unstable for any  $(k_p, k_d, \tau_1) \in \mathbb{R}^2 \times \mathbb{R}_+^*$ .

Alternatively, one may choose to approximate the derivative action using two time-delays, obtaining the following characteristic function:

$$\Delta_2(s; \tau) := s^4 + s^2 + 4s - 31 + k_p + k_d \frac{3 - 4e^{-\tau s} + e^{-2\tau s}}{2\tau}. \quad (40)$$

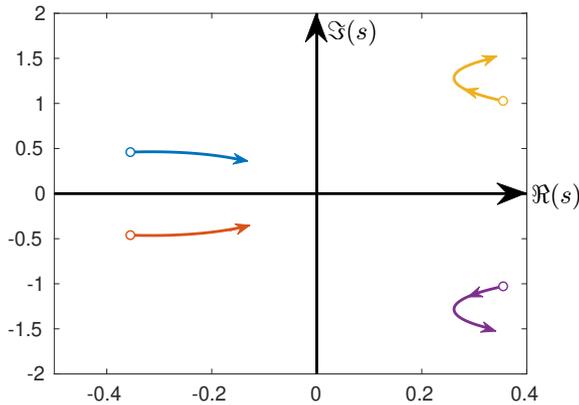
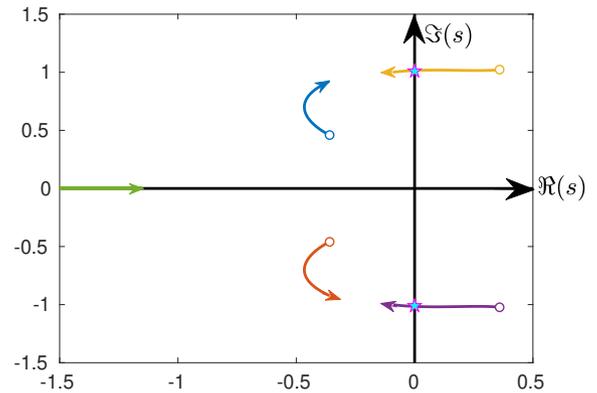
Figure 7b shows the evolution of the closed-loop system characteristic roots as the value of  $\tau$  changes. Contrary to what we observed in the one-delay case and when using a Padé approximation, there exist a certain delay  $\tau$  for which a stability shift occurs.

## 4.3 | Beneficial effect of the approximation in the properly-posed case

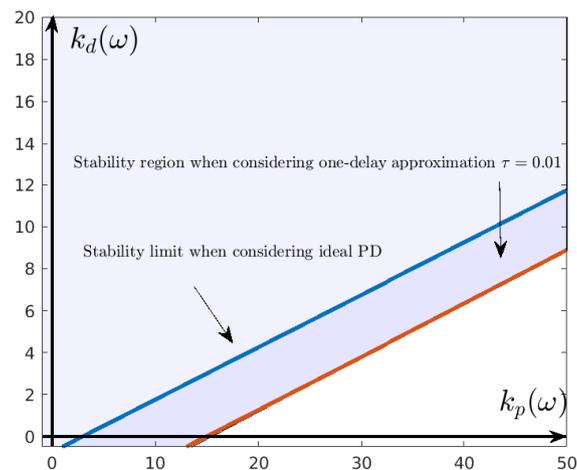
The following examples aim to demonstrate that, under specific conditions, in a properly-posed system, the stability region may "expand" when considering the time-delay approximation of the derivative action.

**Example 4.4.** Consider the system represented by the following transfer function:

$$H_{yu}(s) = \frac{1}{s^3 + 4s^2 + 4s + 1}. \quad (41)$$

(a) Behavior of the closed-loop characteristic equation (39) as the value of  $\tau$  changes in  $(0, 0.85]$ .(b) Behavior of the closed-loop characteristic equation (40) as the value of  $\tau$  changes in  $(0, 0.85]$ .

**Figure 7** Comparison of the behavior of closed-loop characteristic roots depending on the number of delays taken into account in the approximation. The crossing, indicated by a star, shows that there is a certain value of the delay  $\tau$  for which the system stabilizes when considering the two-delay approximation.



**Figure 8** Stability region of system (41) in closed-loop with classical PD controller and a one-delay approximation version.

The system's relative degree is greater than one, indicating that the system is properly-posed. Furthermore, as per the Routh-Hurwitz criterion, to maintain stability in a closed loop with a classical PD controller, it is sufficient to ensure:  $k_p > -1$  and  $k_d > (k_p - 3)/4$ . Subsequently, by replacing the derivative action with its corresponding one-delay approximation, the stability boundary of the system is illustrated in Figure 8. It is observed that there exists a range within which a lower value of the derivative gain  $k_d$  still maintains stability.

#### 4.4 | Robustness with Respect to the Delay Parameter

In this final example, we aim to demonstrate the effectiveness of our methodology in computing the delay margin for a properly-posed closed-loop system.

**Example 4.5.** Consider a system with transfer function representation given by:

$$H_{yu}(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}, \quad (42)$$

where  $w_n$  is the natural frequency of the system, and  $\zeta$  is its damping ratio. In closed-loop with a PD controller the closed-loop characteristic function reads:

$$\Delta(s) = s^2 + 2\zeta w_n s + w_n^2 + w_n^2(k_p + k_d s). \quad (43)$$

To warranty the closed-loop stability of the system one has to chose  $k_p > -1$  and  $k_d > -2\zeta/w_n$ . Consider now that  $w_n = \sqrt{2}$  and  $\zeta = 3/(2\sqrt{2})$ . Under such a configuration, the control gains  $(k_p, k_d) = (3, 2)$  place the closed-loop characteristic roots at  $s = \frac{1}{2}(-7 \pm \sqrt{17})$ . Since the relative degree of the system is greater than one, making it properly-posed for small delays, we can approximate the derivative action in the controller scheme. As previously mentioned, in the properly-case scenario there might still exist a *delay margin*  $\tau_{\max}$ , such that the closed-loop system loss its stability for any  $\tau \geq \tau_{\max}$ . In this vein, following a similar methodology to the one proposed in<sup>23</sup>, we can compute the so-called *margin* of the delay parameter when considering the system described by the transfer function (42) in closed-loop with the proposed PD controller considering different approximations of the derivative action, as shown in Table 3.

**Table 3** Delay margin for different approximations of the derivative action.

Delay margin			
$\ell$	$\tau(\omega)$	$\tau_{\max}$	$\omega$
3	$\frac{480}{-11\omega^2 + \sqrt{841\omega^4 - 6976\omega^2 + 53824} + 88}$	$\approx 0.49$	$\approx 7.42$
5	$\frac{137\omega^2 + \sqrt{786769\omega^4 - 19500304\omega^2 + 50353216} - 1096}{15(\omega^4 - 25\omega^2 + 64)}$	$\approx 0.15$	$\approx 21.21$
7	$\frac{1089\omega^2 + \sqrt{332661121\omega^4 - 2339301136\omega^2 + 21290311744} - 8712}{105(\omega^4 - 7\omega^2 + 64)}$	$\approx 0.05$	$\approx 58.54$
9	$\frac{7129\omega^2 + \sqrt{110781799921\omega^4 - 775930003216\omega^2 + 7090035194944} - 57032}{630(\omega^4 - 7\omega^2 + 64)}$	$\approx 0.01$	$\approx 170$

These results can be further validated through numerical computation of the system's roots with respect to  $\tau$ . Indeed, Figure 9 depicts the stability loss for the cases of the three- and five-delay approximations. It is worth noticing that, as the value of  $\ell$  increases, the delay margin  $\tau_{\max}$  decreases, suggesting that utilizing a larger number of delays makes the system more sensitive to small changes on the parameter  $\tau$ .

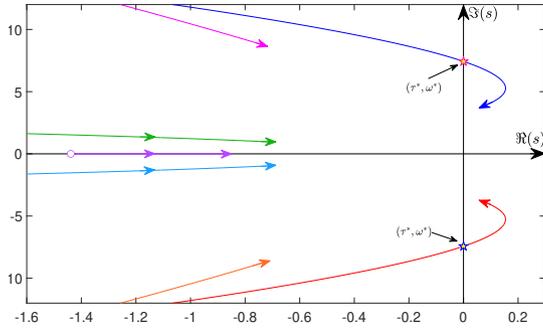
## 5 | EXPERIMENTAL EXAMPLE

In this section, we present experimental results using the QUBE-Servo 2 servo motor. The system can be described by the following transfer function that relates the output angular position ( $\Theta[\text{rad}]$ ) to the input voltage ( $V[V]$ ):

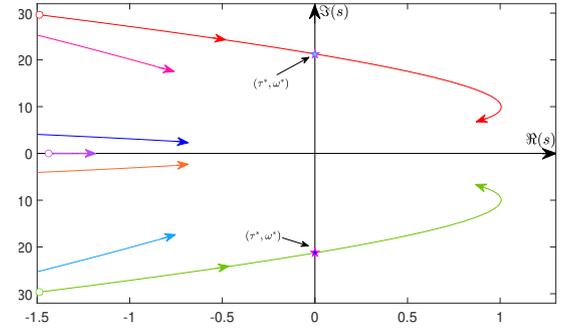
$$H_{yu}(s) = \frac{\Theta(s)}{V(s)} = \frac{k}{s(Ts + 1)}, \quad (44)$$

Here,  $k \in \mathbb{R}_+^*$  represents the open-loop gain of the plant, and  $T \in \mathbb{R}_+^*$  its time constant. Notably, this system has a relative degree of two, making issues related to improperly-posedness inconsequential. Furthermore, since this plant corresponds to an "integral" plant, opting for a PD control scheme is both intuitive and ideal.

The plant parameters, specifically  $k = \frac{473}{20}$  and  $T = \frac{69}{200}$ , were determined through identification. As a preliminary step, we consider the closed-loop system with the controller  $C_0(s)$  defined by (5). This controller leads to the closed-loop characteristic equation  $\Delta(s) : \mathbb{C} \rightarrow \mathbb{C}$ :

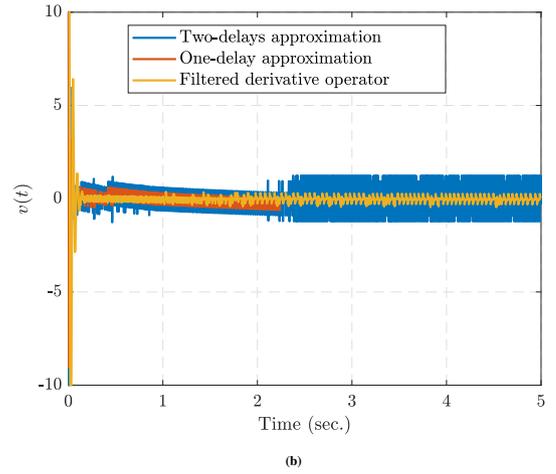
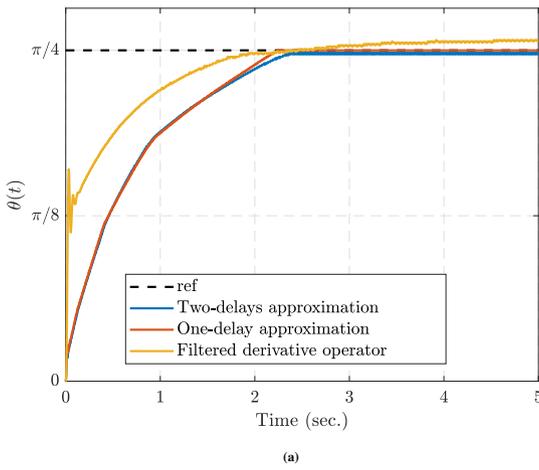


(a) Closed-loop characteristic roots behavior with respect to  $\tau$  when the system described by the transfer function in (42) is in closed-loop with the PD controller considering a three-delays approximation of the derivative action.



(b) Closed-loop characteristic roots behavior with respect to  $\tau$  when the system described by the transfer function in (42) is in closed-loop with the PD controller considering a five-delays approximation of the derivative action.

**Figure 9** Behavior of the closed-loop system characteristic roots under different configurations of the derivative approximation.



**Figure 10** System output considering the different proposed controllers and  $r(t) = \frac{\pi}{4}$ . (a) Plant output  $\theta(t)$  when in closed-loop with the chosen controllers. (b) Control output  $v(t)$  of the filtered PD controller and both approximations.

$$\Delta(s) = s^2 + s \frac{1}{T} (1 + k k_d) + \frac{k}{T} k_p. \quad (45)$$

From Equation (45), it is evident that the closed-loop system will maintain stability if and only if  $k_p > 0$  and  $k_d > -\frac{1}{k}$ . In this scenario, we have chosen the control parameters  $(k_p, k_d) = \left(\frac{700}{473}, 1\right)$ , which positions the right-most root of the system at  $s = -\frac{100}{69}$ .

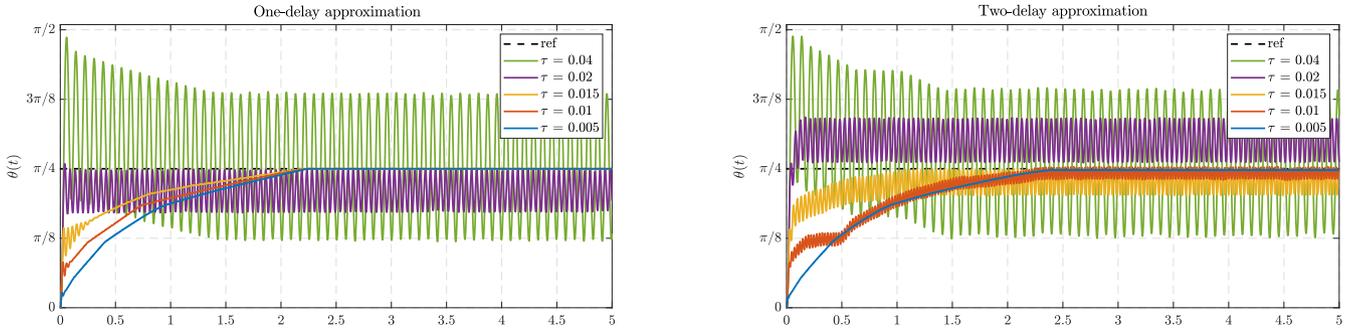
Next, we explore three distinct approaches for implementation: (i) the one-delay approximation of the derivative action, (ii) the two-delay approximation of the derivative action, and (iii) the filtered derivative action. For the latter, the controller can be expressed as:

$$C_f(s) = k_p + k_d \frac{Ns}{Ns + 1}. \quad (46)$$

From the plant documentation, we are informed that the smallest usable delay is  $\tau = 0.002$ . However, for the sake of simplicity, we have opted to use  $\tau = 0.005$ . Additionally, for the filtered implementation, we have selected a value of  $N = 100$ . The results obtained from these choices are illustrated in Figure 10. Some noteworthy observations can be made from the figure.

Firstly, it is evident that the filtered implementation tends to reach the reference signal more swiftly. Conversely, the implementations using the derivative approximation exhibit reduced oscillations and an smaller error.

Secondly, a close examination of the control output reveals certain similarities, yet, in general, it is apparent that the control effort required for the filtered derivative implementation is lower.



**Figure 11** Experiment to verify the computed delay margins.

## 5.1 | Delay-Margin Computation and Experimental Verification

As previously discussed, even in properly-defined systems, there exists an upper limit for the delay parameter beyond which the system's stability cannot be guaranteed. In this section, we calculate this stability margin for system (44) in closed-loop with PD controllers with one and two-delay approximations of the derivative action. Additionally, we experimentally validate the accuracy of our computations by demonstrating that as the delay parameter ( $\tau$ ) approaches its maximum value ( $\tau_{\max}$ ), the system exhibits significant oscillations. Following the methodology in<sup>23</sup> we obtain the following expression for delay margin in the one-delay case:

$$\tau_{\max}(\omega) = \frac{2kk_d(T\omega^2 - kk_p)}{(kk_p - T\omega^2)^2 + \omega^2} \quad \omega^* = 43.7645. \quad (47)$$

Which implies that stability might be lost as  $\tau \rightarrow 0.0747$ . Similarly, for the two-delays case we obtain:

$$\tau_{\max}(\omega) = \frac{\sqrt{k^2k_d^2(25(kk_p - T\omega^2)^2 + 16\omega^2) + 3kk_d(T\omega^2 - kk_p)}}{2((kk_p - T\omega^2)^2 + \omega^2)} \quad \omega^* = 87.1097, \quad (48)$$

thus, stability is lost as  $\tau \rightarrow 0.0366$ .

To experimentally validate the calculated margins, we conducted a comparative analysis of the system's performance while incrementally varying the delay parameter  $\tau$ . The results of this experimentation are illustrated in Figure 11, which visually shows the emergence of oscillations as the value of  $\tau$  is progressively increased. This behavior is expected, as we are aware of the eventual crossing on the imaginary axis. This crossing implies that with higher values of  $\tau$  there will be roots approaching the imaginary axis, leading to the observed behavior.

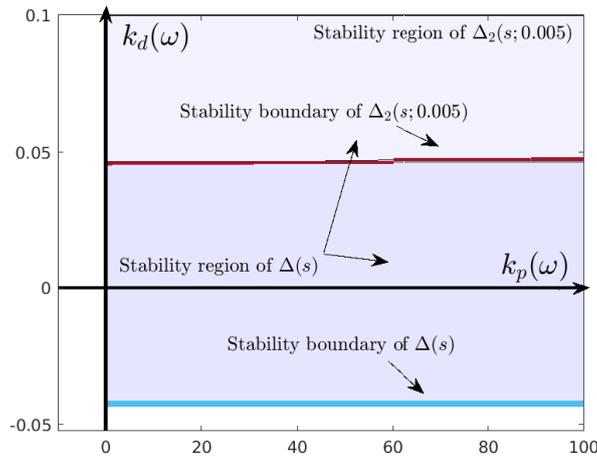
## 5.2 | Effect of the Delay-Difference Approximation on the System Stability

We have previously emphasized the critical importance of selecting an appropriate delay value. In addition to this consideration, it is essential to recognize that the stable region of the closed-loop system undergoes changes concerning the control parameters when we introduce an approximation of the derivative action.

As mentioned earlier, for the transfer function  $\Delta(s)$  defined by Equation (45), ensuring the stability of the closed-loop system is straightforward by choosing  $k_p > 0$  and  $k_d > -1/k$ . However, when we incorporate the two-delay approximation, the characteristic function  $\Delta_2 : \mathbb{C} \times \mathbb{R}_+ \rightarrow \mathbb{C}$  takes the following form:

$$\Delta_2(s; \tau) = s^2 + \frac{1}{T}s + \frac{k}{T} \left( k_p + \frac{k_d}{\tau} \left( \frac{3}{2} - 2e^{-\tau s} + \frac{1}{2}e^{-2\tau s} \right) \right). \quad (49)$$

It is natural to expect that the stability region will change as the system itself undergoes transformation due to the approximation. To explore this change, we computed the stability crossing curves, as illustrated in Figure 12. The figure reveals in particular the presence of a "gap" in which the closed-loop system with the "classic" PD controller remains stable, while the approximate value controller version leads to system instability.



**Figure 12** Stability regions in the control parameters plane of  $\Delta(s)$  and  $\Delta_2(s; \tau = 0.005)$

## 6 | CONCLUSIONS

Unbound solutions of the characteristic function of the closed-loop system have been characterized for systems subjected to PD controllers, where derivative action has been approximated via a delay-difference approximation including an arbitrary number of time-delays. Such characterization allowed deriving conditions on certain system parameters to identify improperly-posedness of the closed-loop system. Such results proved crucial since stability may be lost if the controller gains are selected without considering the discretization; in particular, the impact of the number of delays on the improperly-posedness criteria was explicitly addressed. Several illustrative examples showed the effectiveness of the method in detecting improperly-posedness conditions. Moreover, when the closed-loop system is properly-posed, we present examples that show the existence of a delay margin (an upper limit) of the delay parameter such that the closed-loop system preserves stability. Additionally, to exemplify the practicality of our findings, we conducted an experiment to demonstrate the effectiveness of our studied controller in regulating a real plant. This experiment also facilitated a comparative analysis between the implementation of the derivative action approximation and the filtered derivative action approach. Future research may consider non-commensurate delays in the discretization, looking to reduce the size of the improperly-posed region in the parameter-space. It is also interesting to extend the analysis to systems that naturally present a delayed input.

## ACKNOWLEDGMENTS

The work of D. TORRES GARCÍA is supported by the "ADI 2022" project funded by the IDEX Paris-Saclay (France), ANR-11-IDEX-0003-02 and by CONAHCyT-UASLP (Mexico), CVU: 929482. The authors would like to express their gratitude to M.Sc. JULIAN HERNANDEZ-GALLARDO for his valuable assistance in the development of the experiments presented in Section 5. Finally, the authors wish to thank Associate Editor and anonymous Reviewers for their useful comments and suggestions that helped the authors to improve the overall contribution and quality of the paper.

## Author contributions

This is an author contribution text. This is an author contribution text.

## Financial disclosure

None reported.

## Conflict of interest

The authors declare no potential conflict of interests.

## SUPPORTING INFORMATION

The following supporting information is available as part of the online article:

**How to cite this article:** Méndez-Barrios, C.-F., Torres-García, D., and Niculescu, S.-I., (2023), Analysis of the Improperly-Posed Case of Delay-Difference Approximations of PD-Controllers, *Int. J. Robust Nonlinear Control*, 2023;00:1–6.

## APPENDIX

### A DEGREE OF A QUASI-POLYNOMIAL

We have the following definition<sup>36,37</sup>:

**Definition 4.** For a given quasi-polynomial  $\Delta : \mathbb{C} \mapsto \mathbb{C}$  with real coefficients of the form  $\Delta(s) = \sum_{i=0}^k P_i(s)e^{-s\tau_i}$ , with  $\tau_k \in \mathbb{R}$ , such that  $\tau_0 = 0 < \tau_1 < \dots < \tau_k$ , the *degree* of  $\Delta$  is defined by  $\deg(\Delta) = k + \sum_{i=0}^k d_i$  where  $d_i$  represents the degree of the polynomial  $P_i$  for  $i = 0, 1, \dots, k$ .

### B CHEBYSHEV POLYNOMIALS

Let  $k \in \mathbb{N}$ . Then, the  $k$ -th Chebyshev polynomial of the *first kind*  $T_k : [-1, 1] \mapsto [-1, 1]$ <sup>38,39</sup>, is defined as the real part of the function  $z^k$  in the unit circle, namely

$$T_k(x) := \Re(z^k) = \frac{1}{2}(z^k + z^{-k}) = \cos(k\theta),$$

where  $x := 1/2(z + z^{-1}) = \cos(\theta)$ ,  $\theta = \cos^{-1}(x)$ . It is easy to observe that  $T_0(x) = 1$ ,  $T_1(x) = x$ , and for all  $k \geq 2$ , the polynomials  $T_k$  satisfy the following recurrent relationship:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$

For instance, for  $k = 2, 3, 4$ , one gets  $T_2(x) = 2x^2 - 1$ ,  $T_3(x) = 4x^3 - 3x$ ,  $T_4(x) = 8x^4 - 8x^2 + 1$ .

Similarly, the  $k$ -th Chebyshev polynomial of the *second kind*,

$$U_k(x) = \frac{\sin((k+1)\theta)}{\sin(\theta)},$$

satisfy the recurrent relationship:

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x),$$

but with  $U_0(x) = 1$  and  $U_1(x) = 2x$ . For instance, in this case, for  $k = 2, 3, 4$ , one gets  $U_2(x) = 4x^2 - 1$ ,  $U_3(x) = 8x^3 - 4x$ ,  $U_4(x) = 16x^4 - 12x^2 + 1$ .

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