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Abstract

We consider the time optimal control problem, with a point target, for an infinite dimensional system described by the Kirchhoff plate equation with distributed control. The main contribution we bring in consists in solving a problem left open by previous work on the subject: we prove that time optimal controls have a bang-bang-property and, consequently, that they are unique. The main ingredients used to achieve this goal is a new approximate observability property from measurable sets for the system described by the Kirchhoff equation and an abstract result for systems with skew-adjoint generator.

Keywords: Time optimal control, Bang-Bang property, Schrödinger equation, Kirchhoff equation
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1. Introduction

Time optimal control, with a point target, is a classical problem for time invariant linear systems. In the finite dimensional case it is known that time optimal controls satisfy Pontryagyn's maximum principle and that they are *bang-bang* (see Bellman, Glicksberg and Gross [2]). The first extensions of these results to infinite dimensional linear systems have been given in Fattorini's paper [5]. The progress made in this field has been successively reported in the books of Lions [10] and of Fattorini [6]. As far as we know, in the infinite dimensional case the bang-bang property of time optimal controls is established in a context which includes one of the following assumptions:

1. The considered system is null controllable with inputs active for times in an arbitrary set of positive measure. This idea, going back to Mizel and Seidman [13], has been largely exploited for systems described by parabolic equations (see, for instance, Wang [17], Phung and Wang [14], Kunisch and Wang [9], Micu, Roventa and Tucsnak [12]).

2. The considered system is time reversible, it is exactly controllable in any time by means of inputs which are L^∞ functions of time and it is approximately controllable with inputs active for times in an arbitrary set of positive measure. This situation occurs, in particular, for invertible input operators. In the case of systems governed by PDE's, this means, roughly speaking, that the control is active in the entire spatial domain where the PDE is considered. This situation has been already tackled in [10] and [6]. The case

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when control is active only in a part of the considered domain (or of its boundary) has been considered only relatively recently (see Lohéac and Tucsnak [11]), with applications to systems described by the Schrödinger equation.

In this work we consider a control system described by the Kirchhoff plate equation, which clearly does not fall in the first category described above. The system we consider partially falls in the second class of systems described above. Indeed, as shown below, it is time reversible and exactly controllable in any time. One of the main novelties we bring in this work is that we establish the approximate controllability from measurable sets property, which allows us to show that it fits the framework introduced at the second point above.

More precisely, using the standard notation Δ for the Laplace operator and $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ for Sobolev spaces, we consider the control system

$$\ddot{w}(t,x) = -\Delta^2 w(t,x) + \chi_{\mathcal{O}}(x)u(t,x) \qquad (t \geqslant 0,\ x \in \Omega)\,, \quad (1.1)$$

$$w(t,x) = \Delta w(t,x) = 0 \quad (t \geqslant 0,\ x \in \partial\Omega)\,, \quad (1.2)$$

$$w(0,x) = f_0(x), \quad \dot{w}(0,x) = g_0(x) \quad (x \in \Omega), \quad (1.3)$$

where $\Omega \subset \mathbb{R}^n$, with $n \in \mathbb{N}^*$, is an open bounded set, $\mathcal{O}$ is an open subset of Ω, $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$, $f_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $g_0 \in L^2(\Omega)$ and the control u is such that $\|u(t,\cdot)\|_{L^2(\mathcal{O})} \leqslant 1$ for almost every $t \geqslant 0$. This system describes the transverse vibrations of a Kirchhoff plate occupying the domain $\Omega \subset \mathbb{R}^n$, hinged at the boundary and with the distributed control u acting in the open subset $\mathcal{O}$ of Ω.

We are interested in the following time optimal control problem:

[OT] *Given $f_1 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $g_1 \in L^2(\Omega)$, determine $\tau^* > 0$ and $u^* \in L^\infty([0,\tau^*]; L^2(\mathcal{O}))$, with $\|u^*\|_{L^\infty([0,\tau^*];L^2(\mathcal{O}))} \leqslant 1$, such that*

- *The solution w^* of (1.1)-(1.3) with $u = u^*$ satisfies $w^*(\tau^*,\cdot) = f_1$ and $\dot{w}^*(\tau^*,\cdot) = g_1$.*
- *If $\tau > 0$ and $u \in L^\infty([0,\tau]; L^2(\mathcal{O}))$, with $\|u\|_{L^\infty([0,\tau];L^2(\mathcal{O}))} \leqslant 1$, are such that the solution z of (1.1)-(1.3) satisfies $w(\tau,\cdot) = f_1$ and $\dot{w}(\tau,\cdot) = g_1$, then $\tau \geqslant \tau^*$.*

The following result has been proved in Lohéac and Tucsnak [11].

Proposition 1.1. *Assume that one of the following sets of assumptions is verified:*

1. *The open set Ω is bounded, $\partial\Omega$ is of class C^2 and $\mathcal{O}$ satisfies the geometric optics condition;*
2. *The open set Ω is a rectangular domain and $\mathcal{O}$ is an arbitrary nonempty open subset of Ω.*

Then for every $f_0, f_1 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $g_0, g_1 \in L^2(\Omega)$, with $(f_0, g_0) \neq (f_1, g_1)$, there exists a solution (τ^*, u^*) of the time optimal control problem **[OT]**.

Moreover, in [11] it has been shown that any solution of the time optimal control problem satisfies a Pontryagyn type maximum principle, which we do not directly use in this work, so we do not give its precise statement.

By analogy with finite or some other infinite dimensional time optimal control problems (see, for instance, Apraiz et al. [1] for sytems described by te heat equations or [11] for systems described by the Schrödinger equation) one could expect that the optimal control u^* in Proposition 1.1 satisfies the bang-bang property

$$\|u^*(\cdot,t)\|_{L^2(\mathcal{O})} = 1 \qquad (t \in [0,\tau^*] \quad \text{a.e.})\,. \quad (1.4)$$

However, this property is stated only as an open question in [11], where it is pointed out that an important point missing in solving this problem was the establishment of a nonstandard approximate observability property. In this paper we prove that this property holds and, by using an abstract result established in [11], we obtain the main result in this work:

Theorem 1.2. *With the notation and assumptions in Proposition 1.1, the solution (τ^*, u^*) of the time optimal control problem* **[OT]** *satisfies the bang-bang property (1.4). Moreover (τ^*, u^*) is the unique solution of* **[OT]**.

The remaining part of this work is organized as follows. In Section 2 we recall some known facts on observability, controllability and time optimal control problems for linear time invariant infinite dimensional systems. Section 3 is devoted to some background on the Kirchhoff plate equation. Our main result is proved in Section 4. Finally, Section 5 describes some extensions, comments and open problems.

2. Some background on observability, controllability and time optimal control problems

In this section we gather, for easy reference, some known facts from the observability and controllability theory for time invariant infinite dimensional linear systems. Moreover, we recall some basic facts on time optimal control problems, with emphasis on systems with skew-adjoint generators.

Within this section, X, U and Y are abstract Hilbert spaces, $A : \mathcal{D}(A) \to X$ is the generator of a C^0 semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U, X)$ is a control operator and $C \in \mathcal{L}(X, Y)$ is an observation operator. For $\tau > 0$ we consider the *output map* Ψ_τ defined by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \qquad (z_0 \in X,\ t \in [0, \tau])\,.$$

If $e \subset [0, \tau]$ is a set of positive measure, we consider the restriction of the above defined initial state to output map to a set of positive measure $e \subset [0, \tau]$, which is defined by

$$\Psi_{\tau,e} \in \mathcal{L}(X, L^2([0, \tau], Y)), \qquad \Psi_{\tau,e} = \chi_e \Psi_\tau\,,$$

where χ_e is the characteristic function of e.

There are several generalizations of the concept of observability to infinite dimensional linear systems. In this paper we need only the following ones.

Definition 2.1. *Let $\tau > 0$ and $e \subset [0, \tau]$ be a set of positive measure. The pair (A, C) is said*

- *approximately observable in time τ if* Ker $\Psi_\tau = \{0\}$.
- *approximately observable from e if* Ker $\Psi_{\tau,e} = \{0\}$.

The result below, first stated and proved in Tucsnak and Weiss [16, Proposition 6.1.9], asserts that, in some sense, to prove approximate observability it suffices to check that the output map is one to one on a potentially much smaller space than X.

Proposition 2.2. *Suppose that for some $\tau > 0$,*

$$\text{Ker } \Psi_\tau \cap \mathcal{D}(A^\infty) = \{0\},$$

where $\mathcal{D}(A^\infty) = \bigcap_{k=1}^\infty \mathcal{D}(A^k)$. Then (A, C) is approximately observable in time $\tau + \varepsilon$ for any $\varepsilon > 0$.

It is not all clear whether the above result holds if we replace Ψ_τ by $\Psi_{\tau,e}$ in its statement. In any case, the proof provided in [16, Proposition 6.1.9] obviously cannot be adapted to this case.

We consider the infinite dimensional control system described by the equation

$$\dot z(t) = Az(t) + Bu(t), \qquad z(0) = z_0\,. \tag{2.5}$$

With the above notation, the solution z of (2.5) is defined by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \qquad \big(t \geqslant 0\big)\,, \tag{2.6}$$

where $\Phi_t \in \mathcal{L}(L^2([0, t], U); X)$ is given by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma) \mathrm{d}\sigma \qquad (u \in L^2([0, \infty), U))\,. \tag{2.7}$$

The maps (Φ_t) are called *input to state maps*.

Recall the following definition (see, for instance, [16, Sections 4.2 and 11.1] and [11]):

Definition 2.3.

- *Given $\tau > 0$, the pair (A, B) is said exactly controllable in time τ if* Ran $\Phi_\tau = X$.
- *Let $e \subset [0, \tau]$ be a set of positive Lebesgue measure. The pair (A, B) is said approximatively controllable in time τ from e if the range of the map $\Phi_{\tau,e} \in \mathcal{L}(L^2([0, \tau], U), X)$ defined by*

  $$\Phi_{\tau,e} u = \int_e \mathbb{T}_{\tau-\sigma} Bu(\sigma)\, \mathrm{d}\sigma \qquad (u \in L^2([0, \tau], U)),$$

  is dense in X.

Let us also note the following duality result, which can be proved by an obvious adaptation of the proof of the corresponding result for $e = [0, \tau]$ (see, for instance [12, Proposition 2.4]).

Proposition 2.4. *Let $\tau > 0$, $e \subset [0, \tau]$ a set of positive measure and let*

$$e' = \{\tau - t \quad | \quad t \in e\}.$$

Then the pair (A, B) is approximatively controllable in time τ from the set e if and only if (A^*, B^*) is approximatively observable from e'.

We are now in a position to introduce some basic definitions and results on time optimal control problems for systems described by (2.5).

Definition 2.5. *Let $z_0,\ z_1 \in X$ with $z_0 \neq z_1$. A function $u^* \in L^\infty([0, \infty), U)$ is said a time optimal control for the pair (A, B), associated to the initial state z_0 and the final state z_1, if there exists $\tau^* > 0$ such that*

1. *$z_1 = \mathbb{T}_{\tau^*} z_0 + \Phi_{\tau^*} u^*$ and $\|u^*\|_{L^\infty([0,\tau^*],U)} \leqslant 1$;*
2. *If $\tau > 0$ is such that there exists $u \in L^\infty([0, \tau], U)$ with*

$$z_1 = \mathbb{T}_\tau z_0 + \Phi_\tau u, \qquad \|u\|_{L^\infty([0,\tau],U)} \leqslant 1, \tag{2.8}$$

then $\tau \geqslant \tau^*$.

We will use in an essential manner the result below, which is just a juxtaposition of Theorem 1.4 and Corollary 1.5 from [11].

Theorem 2.6. *Suppose that A is skew-adjoint and that (A, B) is exactly controllable in any time $\tau > 0$. Then, for every $z_0,\ z_1 \in X,\ z_0 \neq z_1$, there exists a time optimal control u^* steering z_0 to z_1 in time $\tau^* = \tau^*(z_0, z_1)$. Moreover, assume that the pair (A^*, B^*) is approximatively observable in time τ^* from any $e \subset [0, \tau^*]$ of positive measure. Then the time optimal control u^* is bang-bang, in the sense that*

$$\|u^*(t)\|_U = 1 \qquad (t \in [0, \tau^*] \quad \text{a.e.})\,. \tag{2.9}$$

Moreover, this time optimal control is unique.

The proof of Theorem 2.6 uses in an essential manner the fact that standard exact controllability (using signals which are L^2 functions of time) implies the exact controllability by signals which are L^∞ functions of time. This is used in [11] to derive a maximum principle which is then shown to imply the assessed bang-bang property.

3. Some background on the Kirchhoff system

Let $n \in \mathbb{N}$ and let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary or Ω is a rectangular subset of $\mathbb{R}^n$. Let $H := L^2(\Omega)$. The inner product and the norm on H will be simply denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We denote by $-A_0$ the Dirichlet Laplacian on Ω. More precisely $A_0 : \mathcal{D}(A_0) \to H$ is defined by

$$\mathcal{D}(A_0) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega),$$

$$A_0 \varphi = -\Delta \varphi \qquad\qquad (\varphi \in \mathcal{D}(A_0)).$$

It is known, see for instance [16, Section 3.6] that A_0 is a strictly positive operator and that A_0 is diagonalizable with an orthonormal basis of eigenvectors $\{\varphi_k\}_{k \geqslant 0}$ and corresponding family of eigenvalues $\{\lambda_k\}_{k \geqslant 0}$, where the sequence (λ_k) is positive, non decreasing and satisfies $\lambda_k \to \infty$. With the above notation, we have

$$A_0 z = \sum_{k \geqslant 0} \lambda_k \langle z, \varphi_k \rangle \varphi_k \qquad (z \in \mathcal{D}(A_0)), \tag{3.10}$$

Moreover, for any $\beta > 0$, the sets

$$H_\beta := \left\{ z \in H : \textstyle\sum_{k \geqslant 0} \left(1 + \lambda_k^2\right)^\beta |\langle z, \varphi_k \rangle|^2 < \infty \right\}$$

endowed with the inner product

$$\langle y, z \rangle_\beta = \sum_{k \geqslant 0} \left(1 + \lambda_k^2\right)^\beta \langle y, \varphi_k \rangle \ \overline{\langle z, \varphi_k \rangle} \quad (z,\, y \in X_\beta)$$

are Hilbert spaces. The scale $\{H_\beta\}_{\beta \geqslant 0}$ of Hilbert spaces can be extended to a scale $\{H_\beta\}_{\beta \in \mathbb{R}}$ by defining $H_0 = H$ and, for every $\beta < 0$, H_β as the completion of H with respect to the norm associated to the inner product$\langle\cdot,\cdot\rangle_\beta$. Alternatively, $H_{-\beta}$ may be defined, for $\beta>0$, as the dual of H_β with respect to the pivot space H.

For every $\beta>0$, A_0 can be canonically extended to a strictly positive operator on $H_{-\beta}$ by setting

$$A_0 z=\sum_{k\geqslant 0}\lambda_k\langle z,\varphi_k\rangle_{-\beta,\beta}\,\varphi_k \qquad (z\in H_{-\beta+1}),$$

where the notation $\langle\cdot,\cdot\rangle_{-\beta,\beta}$ stands for the duality between $H_{-\beta}$ and H_β. These extensions will be still denoted by A_0. Note that, for every $\beta\in\mathbb{R}$, the family $\{(1+\lambda_k^2)^{\beta/2}\varphi_k\}_{k\geqslant 0}$ is an orthonormal basis in H_β.

Let $\beta\in\mathbb{R}$. We denote $\mathcal{X}_\beta=H_{\beta+1}\times H_\beta$, which is a Hilbert space with the inner product

$$\left\langle \begin{bmatrix} f_1\\ g_1\end{bmatrix}, \begin{bmatrix} f_2\\ g_2\end{bmatrix}\right\rangle_{\mathcal{X}_\beta}=\langle f_1,f_2\rangle_{\beta+1}+\langle g_1,g_2\rangle_\beta.$$

We define $\mathcal{A}_\beta:\mathcal{D}(\mathcal{A}_\beta)\to\mathcal{X}_\beta$ by $\mathcal{D}(\mathcal{A}_\beta)=H_{\beta+2}\times H_{\beta+1}$ and

$$\mathcal{A}_\beta\begin{bmatrix} f\\ g\end{bmatrix}=\begin{bmatrix} g\\ -A_0^2 f\end{bmatrix}. \tag{3.11}$$

In the above formula we made the convention to still denote by A_0 the restriction or extension of the initial operator A_0 to a positive operator on H_β, with domain $H_{\beta+1}$. We note that, with the above notation, equations (1.1)-(1.3) define the control system, with state space $\mathcal{X}_0$ and control space H

$$\dot z(t)=\mathcal{A}_0 z(t)+\mathcal{B}u(t) \qquad (t\geqslant 0),$$

$$z(0)=\begin{bmatrix} f_0\\ g_0\end{bmatrix},$$

where the control operator $\mathcal{B}\in\mathcal{L}(H,\mathcal{X}_0)$ is defined by

$$\mathcal{B}\mathrm{u}=\begin{bmatrix} 0\\ \mathrm{u}\chi_{\mathcal{O}}\end{bmatrix} \qquad (\mathrm{u}\in L^2(\mathcal{O})).$$

Since $A_0^2>0$, according to a standard result (see, for instance Proposition 3.7.6 in [16]), $\mathcal{A}_\beta$ is skew-adjoint and $0\in\rho(\mathcal{A}_\beta)$. By the theorem of Stone, $\mathcal{A}_\beta$ generates a unitary group $\mathbb{T}$ on X_β. Consequently, we have:

Proposition 3.1. *For every $\beta\in\mathbb{R}$, $w_0\in H_{\beta+1}$, $w_1\in H_\beta$ there exists a unique function*

$$w\in\bigcap_{k=0}^{\infty}C^k([0,\infty);H_{\beta+1-k}), \tag{3.12}$$

satisfying

$$\ddot w(t)+A_0^2 w(t)=0 \qquad (t\geqslant 0), \tag{3.13}$$

$$w(0)=w_0, \qquad \dot w(0)=w_1. \tag{3.14}$$

This solutions is given by

$$\begin{bmatrix} w(t)\\ \dot w(t)\end{bmatrix}=\mathbb{T}_t\begin{bmatrix} w_0\\ w_1\end{bmatrix} \qquad (t\in\mathbb{R}),$$

where $\mathbb{T}$ is the unitary group in X_β generated by $\mathcal{A}_\beta$ defined in (3.11).

The result below, easy to check by a direct calculation, can be seen as a consequence of the decomposition of the abstract Kirchhoff operator $\frac{\partial^2}{\partial t^2}+A_0^2$ as

$$\frac{\partial^2}{\partial t^2}+A_0^2=\left(\frac{\partial}{\partial t}-iA_0\right)\left(\frac{\partial}{\partial t}+iA_0\right).$$

Proposition 3.2. *Let w be the solution of* (3.13), (3.14) *for $\beta=0$ and denote $\psi=\dot w$. Then by setting*

$$\gamma=\dot\psi-iA_0\psi, \tag{3.15}$$

we have $\gamma\in C([0,\infty);H_{-1})\cap C^1([0,\infty);H_{-2})$ and

$$\dot\gamma(t)+iA_0\gamma(t)=0 \qquad (t\geqslant 0), \tag{3.16}$$

in H_{-2}.

Assuming that the time derivative $\dot w$ of the solution w of (3.13), (3.14) (with $\beta=0$) vanishes for t in a set of positive measure and $x\in\mathcal{O}$, we need below to have information on the set where γ defined in (3.15) vanishes (in some sense). This is the described in the result below.

Proposition 3.3. *With the notation and assumption in Proposition 3.2, assume that there exists a set of positive measure $e\subset[0,\infty)$ and an open set $\mathcal{O}\subset\Omega$ such that*

$$\dot w(t,x)=0 \qquad (t\in e,\ x\in\mathcal{O}). \tag{3.17}$$

Then there exists a set of positive measure $\widetilde{e} \subset e$ such that, for each $\varphi \in W_0^{2,2}(\mathcal{O})$, we have

$$\langle \gamma(t,\cdot), \varphi\rangle_{-1,1} = 0 \qquad (t \in \widetilde{e}). \tag{3.18}$$

Proof. We know from Proposition 3.2 that $\psi = \dot{w}$ satisfies

$$\psi \in C([0,\infty); H) \cap C^1([0,\infty); H_{-1}), \tag{3.19}$$

and from (3.17) that

$$\psi(t,x) = 0 \qquad (t \in e,\ x \in \mathcal{O}). \tag{3.20}$$

Moreover, for any $\varphi \in H_1$, we have that

$$\langle A_0\psi(t,\cdot), \varphi\rangle_{-1,1} = \langle \psi(t,\cdot), -\Delta\varphi\rangle \quad (t \in e).$$

Taking $\varphi \in \mathcal{D}(\mathcal{O})$ in the above formula and using (3.20), it follows that

$$\langle A_0\psi(t,\cdot), \varphi\rangle_{-1,1} = 0 \qquad (t \in e). \tag{3.21}$$

On the other hand, the set of isolated points in e is countable, thus there exists a positive measure subset $\widetilde{e}$ of e such that for every $t \in \widetilde{e}$ there exists a sequence $(t_k) \subset e$ such that $t_k \neq t$ for all k and $t_k \to t$. This fact, together with (3.20) and (3.19) implies that

$$\langle \dot{\psi}(t,\cdot), \varphi\rangle_{-1,1} = 0 \qquad (t \in \widetilde{e},\ \varphi \in \mathcal{D}(\mathcal{O})).$$

The above formula, combined with (3.21) implies that the conclusion (3.18) holds for $\varphi \in \mathcal{D}(\mathcal{O})$. Since the left-hand side of (3.18) defines a bounded linear form on $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ it follows that (3.18) holds for φ in the closure of $\mathcal{D}(O)$ with respect to the $W^{2,2}(\Omega)$ topology, that is for $\varphi \in W_0^{2,2}(\mathcal{O})$, which ends the proof. □

4. Proof of the main result

In this section we continue to use the notation introduced in Section 3. In particular, we design by H the space $L^2(\Omega)$, with the inner product in H simply denoted by $\langle\cdot,\cdot\rangle$ and we use the scale of Hilbert spaces H_β from Section 3.

The main ingredient in the proof of Theorem 1.2 is the following unique continuation result:

Theorem 4.1. *Under the assumptions of Proposition 1.1, let w be a solution of (3.13), (3.14) satisfying (3.12) for $\beta = 0$, together with (3.17) for some set of positive measure $e \subset [0,\infty)$. Then $w(t,x)$ vanishes for every $t \geqslant 0$ and almost every $x \in \Omega$.*

Proof. Using the notation in Proposition 3.2 and the conclusion of Propositions 3.2 and 3.3, we know that $\gamma = \ddot{w} - iA_0\dot{w}$ satisfies (3.16) and (3.18). The first of the two facts mentioned above implies, denoting $a_k = \langle\gamma(0), \varphi_k\rangle_{-1,1}$, that

$$\gamma(t) = \sum_{k\geqslant 1} a_k \mathrm{e}^{-i\lambda_k t}\varphi_k \quad \text{in } H_{-1} \qquad (t \geqslant 0), \tag{4.22}$$

with $\left(\dfrac{a_k}{\lambda_k}\right) \in l^2(\mathbb{C})$. By combining (4.22) and (3.18) it follows that there exists a set of positive measure $\widetilde{e} \subset e$ such that

$$\sum_{k\geqslant 1} a_k \mathrm{e}^{-i\lambda_k t}\langle v, \varphi_k\rangle = 0 \qquad (t \in \widetilde{e},\ v \in \mathcal{D}(\mathcal{O})).$$

Since $v \in \mathcal{D}(A_0^\infty)$ we have that the sequence $(a_k\langle v, \varphi_k\rangle)$ is in $l^1(\mathbb{C})$. By a consequence of Privalov's uniqueness theorem (see, for instance, [11, Lemma 4.1]) it follows that

$$\sum_{k\geqslant 1} a_k \mathrm{e}^{-i\lambda_k t}\langle v, \varphi_k\rangle = 0 \qquad (t \in \mathbb{R},\ v \in \mathcal{D}(\mathcal{O})).$$

The above equality, combined with (4.22), implies that the following relation holds for each $v \in W_0^{2,2}(\mathcal{O})$:

$$\langle \gamma(t), v\rangle_{-1,1} = 0 \qquad (t \in [0,\infty)). \tag{4.23}$$

Let $Y = W^{-2,2}(\mathcal{O})$ and the operator $C \in \mathcal{L}(H_{-1}, Y)$ be defined by

$$\langle C\eta, \varphi\rangle_{W^{-2,2}(\mathcal{O}), W_0^{2,2}(\mathcal{O})} = \langle \eta, \varphi\rangle_{-1,1}, \tag{4.24}$$

for any $\varphi \in W_0^{2,2}(\mathcal{O})$. We clearly have $C \in \mathcal{L}(H_{-1}, Y)$ and, from (4.23), we have that $C\gamma(t) = 0$ for every $t \in [0,\infty)$. Our next aim is to show that $\gamma = 0$. To this purpose it suffices to show that the pair (iA_0, C), with state space H_{-1} and output space Y is approximately observable in any time. To this end we first note that (iA_0, C), with state space H, is exactly observable in any time. Indeed, if $\partial\Omega$ is smooth and $\mathcal{O}$ satisfies the geometric optics condition this follows, for instance by combining Remark 7.4.4 and Theorem 6.7.2 from [16], whereas for Ω rectangular and $\mathcal{O}$ an arbitrary open subset of Ω, this fact has been proved in Jaffard [7] and Komornik [8]. Using this fact, the approximate observability of (iA_0, C), with state space H_{-1}, follows from Proposition 2.2. We have thus shown that $\gamma(t)$ vanishes for every $t \geqslant 0$. This means, according to (3.15), that

$$\dot{\psi}(t) - iA_0\psi(t) = 0 \qquad (t \in [0,\infty)).$$

Using again the fact that (iA_0, C) (with state space H) is exactly observable in any positive time, combined with (3.20) and [11, Lemma 4.2], we obtain that that ψ vanishes on $[0,\infty)\times\Omega$, which clearly implies that w is identically zero. □

Theorem 4.1 can be rephrased as an approximate observability result as follows:

Corollary 4.2. *With the notation in Section 3 and the assumptions of Proposition 1.1, the pair $(\mathcal{A}_0, \mathcal{C})$, with*

$$\mathcal{C}\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \psi\chi_{\mathcal{O}} \qquad \left(\begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in \mathcal{X}_0\right), \qquad (4.25)$$

is approximately observable from any set of positive measure $e \subset [0,\infty)$.

We are now in a position to prove the main result of this paper.

Proof of Theorem 1.2. The idea is of to apply Theorem 2.6, with an appropriate choice of spaces and operators. To this aim, we take, using the notation in Section 3 and (4.25),

$$X = \mathcal{X}_0, \quad U = H, \quad A = \mathcal{A}_0,$$

$$Bu = \begin{bmatrix}0\\ u\chi_{\mathcal{O}}\end{bmatrix} \qquad (u \in H).$$

With the above choice of spaces and operators we have

- The pair (A, B) is exactly controllable in any positive time. Indeed, this follows from the facts that $B^* = \mathcal{C}$, where $\mathcal{C}$ has been defined in (4.25) from Corollary 4.2, and the exact observability in any time of the pair $(A, \mathcal{C})$ (we refer again to Remark 7.4.4, Theorem 6.7.2 from [16], [7] and [8] for the proof of this assertion).

- The pair (A, B) is approximately controllable from any set of positive measure $e \subset [0,\infty)$. Indeed, this follows from Proposition 2.4, Corollary 4.2 and the fact that $B^* = \mathcal{C}$.

It suffices now to apply Theorem 2.6 to obtain the announced conclusions. □

5. Extensions, comments and open problems

5.1. More general equations

An analysis of the proofs in the previous sections indicates that the bang-bang property still holds if the Dirichlet Laplacian is replaced by $-A_0$, where A_0 is a strictly positive operator with compact resolvents on $H = L^2(\Omega)$, provided that we assume that

$$\mathcal{D}(\Omega) \subset \mathcal{D}(A_0), \qquad (5.26)$$

$$(A_0\psi)_{|\mathcal{O}} = 0 \qquad (\psi \in \mathcal{D}(\Omega), \psi_{|\mathcal{O}} = 0), \qquad (5.27)$$

and that (iA_0, B_0) is exactly controllable in any time (with B_0 still given by $B_0 u = \chi_{\mathcal{O}} u$ for every $u \in H$). We can think, in particular, to the case in which we replace the Dirichlet Laplacian by the operator $-A_0$ with

$$\mathcal{D}(A_0) = \left\{\varphi \in W^{2m,2}(\Omega) \,|\, \left(\Delta^{2k}\varphi\right)_{|\partial\Omega} = 0,\ k \leqslant m-1\right\}, \qquad (5.28)$$

$$A_0\varphi = (-\Delta)^m\varphi \qquad (\varphi \in \mathcal{D}(A_0)), \qquad (5.29)$$

for some $m \in \mathbb{N}$, with Ω and $\mathcal{O}$ satisfying the assumptions in Theorem 1.2. Indeed, in this case we can still apply the abstract result in Theorem 2.6, for the following reasons:

- The exact controllability in any time of (iA_0, B_0) follows easily from the corresponding result for the case in which $-A_0$ is the Dirichlet Laplacian.

- The approximate observability from measurable sets type result in Theorem 4.1 still holds. Indeed, essential properties of the Laplacian used in the proof of Theorem 4.1 are that it is diagonal and that it has negative eigenvalues. These properties, together with (5.26) and (5.27), being shared by $-A_0$, with A_0 defined in (5.28), (5.29), the proof of Theorem 4.1 fully adapts to the situation considered here.

We can also think to the case in which the Dirichlet Laplacian is replaced by the operator $-A_0$ with A_0 defined by $\mathcal{D}(A_0) = W^{2,2}(\Omega)$ and

$$A_0\varphi = -\Delta\varphi + a\varphi \qquad (\varphi \in \mathcal{D}(A_0)),$$

where $a \in L^\infty(\Omega, \mathbb{R})$ is a positive function. The idea is still to apply Theorem 2.6. To this aim, we first remark that the analogue of Theorem 4.1 can be proved following line by line the method we have used for $a = 0$. The exact controllability in any time if (iA_0, B_0), under the assumptions on Ω and $\mathcal{O}$ in Theorem 1.2, is a much more delicate issue for $a \neq 0$. However, under the assumption of Theorem 1.2, this property is known to hold, at least in two space dimensions (see Burq and Zworski [4] and Bournissou, Ervedoza and Tucsnak [3]).

5.2. Boundary control and L^∞ constraints with respect to the space variable

A natural question is whether the result in Theorem 2.6 extends to the case of controls acting on the boundary of Ω. Although the boundary controllability of Kirchhoff type systems is by now well understood (see, for instance, Tenenbaum and Tucsnak [15] and references therein), we have no answer at this stage to this question. However, we can remark that one of the main ingredients of our approach is not adaptable to the boundary control case. More precisely, as already mentioned in the comment after Theorem 2.6, we have implicitly used the fact that for a control system with bounded input operator, the exact controllability with standard L^2 in time signals implies the exact controllability with signals which are L^∞ with respect to time. Or, this property is not generally valid for systems with unbounded input operator, such as the boundary control systems.

Another natural question is to replace the constraint $\|u^*\|_{L^\infty([0,\tau^*];L^2(\mathcal{O}))} \leqslant 1$ in the formulation of problem **[OT]** by

$$|u(t,x)| \leqslant 1 \qquad \text{for almost every } t \text{ and } x. \tag{5.30}$$

This type of constraint has already been successfully tackled for systems described by the heat equation (see [12] and references therein). In the case of systems described by the Schrödinger or Kirchhoff plate equations the main difficulty is the study of the reachable space (in particular of the null controllability) under the constraint (5.30). This is, as far as we know, an open question.

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