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ORIENTED EMBEDDING FUNCTORS OF TORI AS HOMOGENEOUS SPACES

PHILIPPE GILLE AND TING-YU LEE

ABSTRACT. We provide a characterization of homogeneous spaces under a reductive group scheme such that the geometric stabilizers are maximal tori. The quasi-split case over a semilocal base (and more generally over a LG-ring) is of special interest and permits to answer a question raised by Marc Levine on SL_n -homogeneous spaces. At the end, we provide an application to the local-global principles for embeddings of étale algebras with involution into central simple algebras with involution.

Keywords: Reductive group schemes, tori, torsors, homogeneous spaces, embeddings, quasi-split groups, local-global principles, étale algebras, central simple algebras.

MSC 2000: 14L15, 20G35

1. INTRODUCTION

Let k be a field and let k_s be a separable closure of k ; we denote by $\Gamma_k = \mathrm{Gal}(k_s/k)$ the absolute Galois group of k . Let X be an affine k -variety that is a SL_n -homogeneous space such that the stabilizer of a geometric point is a maximal torus. Marc Levine asked whether X is G -isomorphic to SL_n/T where T is a maximal torus of SL_n .

Our main classification result (Theorem 3.4) tells us that X is the variety¹ of embeddings of a suitable oriented torus T in SL_n and we explain now the meaning of $X(k)$. Denoting by $\mathbb{G}_m^{n,1}$ the standard split torus of SL_n , we remind the reader that the automorphism group of the root system $A_{n-1} = \Phi(\mathrm{SL}_n, \mathbb{G}_m^{n,1})$ is $S_n \times \mathbb{Z}/2\mathbb{Z}$ if $n \geq 3$ and $\mathbb{Z}/2\mathbb{Z}$ for $n = 2$. We have that $T = R_{A/k}^1(\mathbb{G}_m)$ for an étale k -algebra A (see [18, §7.5]) and that $X(k)$ corresponds to the set of embeddings $f : T \rightarrow \mathrm{SL}_n$ such that the class of the $S_n \times \mathbb{Z}/2\mathbb{Z}$ -torsor $\mathrm{Isom}\left(\Phi(\mathrm{SL}_{n,k_s}, \mathbb{G}_{m,k_s}^{n,1}), \Phi(\mathrm{SL}_{n,k_s}, f(T)_{k_s})\right)$ is $[A] \times 1$ if $n \geq 3$ and simply $[A]$ if $n = 2$. We used implicitly there that the Galois cohomology set $H^1(k, S_n)$ classifies isomorphism classes of étale algebras of degree n and that the oriented type of T is the class of the above torsor.

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¹Those varieties (and their generalizations over a base) have been defined by the second author [20].

On the other hand, if we see A as a k -vector space of dimension n , we have an isomorphism $\mathrm{SL}(A) \xrightarrow{\sim} \mathrm{SL}_n$ and the natural embedding $T = R_{A/k}^1(\mathbb{G}_m) \hookrightarrow \mathrm{SL}_1(A)$ defines then an embedding $T \rightarrow \mathrm{SL}_n$ of oriented type $[A] \times 1$. In other words, the SL_n -homogeneous space X carries a k -point whose stabilizer is $T = R_{A/k}^1(\mathbb{G}_m)$. Thus X is SL_n -isomorphic to SL_n/T as desired.

This statement can be strengthened in two ways. First we can replace SL_n by any quasi-split reductive k -group, the presence of k -points on a such homogeneous space being the Gille-Raghunathan's theorem [14, 26].

With more effort, we can replace further the base field k by an arbitrary LG ring (with infinite residue fields). This is Corollary 4.8 which involves a generalization of the existence of maximal tori of any orientation in such group schemes (see Theorem 4.6). As in Steinberg's section theorem, the case of type A_{2n} requires an additional argument. Also this generalization needs the notion of versal torsor in that setting (Reichstein-Tossici [28]); we introduced a variation of this technique involving algebraic spaces (see §4.3).

Finally we provide an arithmetic application to the embeddings of étale algebras with involution into central simple algebras with involution. Let (E, σ) be an étale algebra E over k with involution σ and (A, τ) be a central simple algebra A over k with involution τ . We want to know when (E, σ) can be embedded into (A, τ) under some constraints on the dimensions of (E, σ) and (A, τ) (see [25], [2], [3]). In Theorem 4.5.1, we show that when k is a number field and the unitary group associated to (A, τ) is quasi-split, the local-global principle always holds for the embeddings of (E, σ) into (A, τ) .

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Notation. We use mainly the terminology and notations of Grothendieck-Dieudonné [12, §9.4 and 9.6] which agree with that of Demazure-Grothendieck used in [31, Exp. I.4]

(a) Let S be a scheme and let \mathcal{E} be a quasi-coherent sheaf over S . For each morphism $f : T \rightarrow S$, we denote by $\mathcal{E}_T = f^*(\mathcal{E})$ the inverse image of \mathcal{E} by the morphism f . We denote by $\mathbf{V}(\mathcal{E})$ the affine S -scheme defined by $\mathbf{V}(\mathcal{E}) = \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathcal{E}))$; it is affine over S and represents the S -functor $Y \mapsto \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{E}_Y, \mathcal{O}_Y)$ [12, 9.4.9].

(b) We assume now that \mathcal{E} is locally free of finite rank and denote by \mathcal{E}^\vee its dual. In this case the affine S -scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S -functor $Y \mapsto H^0(Y, \mathcal{E}_Y) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{E}_Y)$ is representable by the affine S -scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [31, I.4.6].

It applies to the locally free coherent sheaf $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}$ over S so that we can consider the affine S -scheme $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ that is an S -functor in associative

commutative and unital algebras [12, 9.6.2]. Now we consider the S -functor $Y \mapsto \text{Aut}_{\mathcal{O}_Y}(\mathcal{E}_Y)$. It is representable by an open S -subscheme of $\mathbf{V}(\mathcal{E}nd(\mathcal{E}))$ which is denoted by $\text{GL}(\mathcal{E})$ (*loc. cit.*, 9.6.4).

2. GENERALITIES ON HOMOGENEOUS SPACES

Let S be a scheme and let \bullet be the final object of the category of fppf S -sheaves of groups, that is $\bullet(T) = \text{Hom}_{S\text{-sch}}(T, S) = h_S(T)$ for each S -scheme T .

Let \underline{G} be an fppf S -sheaf of groups. Let \underline{X} be an fppf S -sheaf equipped with a left action of \underline{G} . We say that \underline{X} is homogeneous (resp. principal homogeneous) under \underline{G} if the map $\underline{X} \rightarrow \bullet$ is an epimorphism of fppf S -sheaves and if the action map $\underline{G} \times_S \underline{X} \rightarrow \underline{X} \times_S \underline{X}$, $(g, x) \mapsto (x, g.x)$ is an epimorphism (resp. an isomorphism) of fppf S -sheaves. Similarly we have the notion of right homogeneous (resp. principal homogeneous) spaces.

Let \underline{E} be a right principal homogeneous \underline{G} -space and denote by \underline{G}' the twist of \underline{G} by \underline{E} through inner automorphisms. Then twisting by \underline{E} gives rise to an equivalence of categories between the category of left homogeneous \underline{G} -spaces and that of left homogeneous \underline{G}' -spaces.

Remark 2.1. For the discussion of the same material in the representable case, see [27, VI.1].

Lemma 2.2. *Let \underline{H} be an fppf group subsheaf of \underline{G} and \underline{X} be the \underline{G} -homogeneous space $\underline{G}/\underline{H}$. Consider the S -sheaf of groups $\underline{A} = \underline{\text{Aut}}_{\underline{G}}(\underline{X})$ that acts on the left of \underline{X} .*

(1) *Let $\underline{N} = \underline{N}_{\underline{G}}(\underline{H})$ be the normalizer of \underline{H} in \underline{G} (as in [15, §2.3]). Then the map $\underline{G} \times_S \underline{N} \rightarrow \underline{G}$, $(g, n) \mapsto gn$ induces an isomorphism $\underline{N}/\underline{H} \xrightarrow{\sim} \underline{A}^{op}$.*

(2) *The action of \underline{A} on \underline{X} is free and the map $\underline{X} \rightarrow \underline{A} \backslash \underline{X}$ is a (left) principal \underline{A} -homogeneous space.*

(3) *Let \underline{F} be a right \underline{A} -torsor and let \underline{X}' be the twist of \underline{X} by \underline{F} . Then \underline{X}' is a \underline{G} -homogeneous space and there is a canonical isomorphism $\underline{A}' \backslash \underline{X}' \xrightarrow{\sim} \underline{A} \backslash \underline{X}$, where $\underline{A}' = \underline{\text{Aut}}_{\underline{G}}(\underline{X}')$.*

(4) *The S -forms of \underline{X} (as \underline{G} -spaces) are classified by $H_{right}^1(S, \underline{A})$.*

Proof. We put $\underline{W} = \underline{N}/\underline{H}$.

(1) The map $\underline{G} \times_S \underline{N} \rightarrow \underline{G}$ induces a map $\underline{X} \times \underline{W} \rightarrow \underline{X}$ hence a group homomorphism $\phi : \underline{W} \rightarrow \underline{A}^{op}$ of S -sheaves of groups.

ϕ is a monomorphism. Let T be a S -scheme and let $w \in \underline{W}(T)$ be an element such that $\phi(w) = 1$. Up to localization we can assume that w arises from an element $n \in \underline{N}(T)$ which satisfies $1 \times n \in \underline{H}(T)$. Thus $w = 1$.

ϕ is an epimorphism. Let T be an S -scheme and let $f \in \underline{A}(T)$. Denoting by [1] the S -point $S \xrightarrow{e_G} \underline{G} \xrightarrow{q} \underline{X} = \underline{G}/\underline{H}$ we put $x = f([1]) \in \underline{X}(T)$ and up to localization we can assume that x arises from some $g \in \underline{G}(T)$. We claim that $g \in \underline{N}(T)$. For each

T -scheme T' and each $g' \in \underline{G}(T)$ we have $f(g' \cdot [1]) = g' \cdot g_{T'} \cdot [1]$. In particular for each $h' \in \underline{H}(T')$, we have $h' g_{T'} \in g_{T'} \underline{H}(T')$ so that $g_{T'}$ normalizes $\underline{H}(T')$. The claim is proven and it follows that $f = \phi(w)$ where w is the image of $g \in \underline{N}(T)$ in $\underline{W}(T)$.

(2) We start by establishing that \underline{A} acts freely on \underline{X} . By (1), it is equivalent to prove that \underline{W} acts freely on \underline{X} . Let T be a scheme and $x \in \underline{X}(T)$. After localization we can assume that x arises from an element $g \in \underline{G}(T)$. Consider the T -sheaf of groups $\underline{\text{Stab}}_{\underline{W}}(x)$. For each T -scheme T' and $w \in \underline{\text{Stab}}_{\underline{W}}(x)(T')$, up to localization we can assume that w comes from an element $n \in \underline{N}(T')$. Then $[g] = [g] \cdot w = [gn]$, that implies $n \in \underline{H}(T')$ and w is the identity in $\underline{W}(T')$. Hence \underline{W} acts freely on \underline{X} . We put $\underline{Y} = \underline{A} \backslash \underline{X}$. In view of [17, III.3.1.2], the map $\underline{X} \rightarrow \underline{Y}$ satisfies then the first item of the definition of \underline{A} -torsor, that is, the map $\underline{A}_{\underline{Y}} \times_{\underline{Y}} \underline{X} \rightarrow \underline{X} \times_{\underline{Y}} \underline{X}$, $(a, x) \mapsto (x, ax)$, is an isomorphism. Since $\underline{X} \rightarrow \underline{Y}$ admits locally sections for the fppf topology, the second requirement is satisfied as well [17, definition III.4.1]. Thus $\underline{X} \rightarrow \underline{Y}$ is a left \underline{A} -torsor.

(3) Let p be the projection from \underline{X} to the quotient sheaf $\underline{A} \backslash \underline{X}$. Consider the map $\pi : \underline{F} \times \underline{X} \rightarrow \underline{A} \backslash \underline{X}$, which projects $\underline{F} \times \underline{X}$ to \underline{X} and then to the quotient sheaf $\underline{A} \backslash \underline{X}$. Then clearly π induces a map from \underline{X}' to $\underline{A} \backslash \underline{X}$, which we still denote by π .

Note that \underline{G} acts on the \underline{X} -factor of $\underline{F} \times \underline{X}$. As the G -action and the A -action commutes on \underline{X} , this defines an \underline{G} -action on \underline{X}' . Choose a fppf-cover $\{S_i\}$ of S that trivializes the \underline{A} -torsor \underline{F} . For each S_i , there is an isomorphism of right \underline{A} -torsors between \underline{F}_{S_i} and \underline{A}_{S_i} . Hence $\underline{X}'_{S_i} \simeq \underline{X}_{S_i}$ and $\underline{A}'_{S_i} \simeq \underline{A}_{S_i}$.

Fix an isomorphism of \underline{A}_{S_i} -torsors $\iota_i : \underline{F}_{S_i} \rightarrow \underline{A}_{S_i}$. The map ι_i induces an isomorphism between \underline{X}'_{S_i} and \underline{X}_{S_i} that sends $[f, x]$ to $\iota_i(f)(x)$ for $x \in \underline{X}(T_i)$, $f \in \underline{F}(T_i)$ and for arbitrary S_i -scheme T_i . We denote this induced isomorphism still by ι_i . The map ι_i induces an isomorphism between \underline{A}'_{S_i} and \underline{A}_{S_i} by sending a' to $\iota_i \circ a' \circ \iota_i^{-1}$ for $a' \in \underline{A}'(T_i)$ and T_i an S_i -scheme. Then clearly ι_i induces an isomorphism $\bar{\iota}_i : (\underline{A}' \backslash \underline{X}')_{S_i} \rightarrow (\underline{A} \backslash \underline{X})_{S_i}$.

One checks that $p \circ \iota_i = \pi_{S_i}$. Hence $\bar{\iota}_i \circ p' = \pi_{S_i}$ where p' is the projection from \underline{X}' to $\underline{A}' \backslash \underline{X}'$. This implies that $\pi(x') = \pi(a' \cdot x')$ for $x' \in \underline{X}'(T)$, $a' \in \underline{A}'(T)$ and for arbitrary S -scheme T . Hence π induces a map $\bar{\pi}$ from $\underline{A}' \backslash \underline{X}'$ to $\underline{A} \backslash \underline{X}$. On each S_i , we have $\bar{\pi}_{S_i} = \bar{\iota}_i$. As $\bar{\iota}_i$ is an isomorphism, $\bar{\pi}$ is an isomorphism between $\underline{A}' \backslash \underline{X}'$ and $\underline{A} \backslash \underline{X}$.

(4) Let X' be an S -form of X as \underline{G} -spaces. Then the sheaf of isomorphisms $\underline{\text{Isom}}_{\underline{G}}(X, X')$ is a right \underline{A} -torsor and the contracted product $\underline{\text{Isom}}_{\underline{G}}(\underline{X}, \underline{X}') \wedge^{\underline{A}} \underline{X}$ (as defined in [17, §III.1.3]) is canonically isomorphic to \underline{X}' .

Conversely given a right \underline{A} -torsor \underline{F} , let \underline{X}' be the twist of \underline{X} by \underline{F} . According to the proof of (3), the twist \underline{X}' is a G -homogeneous space and is isomorphic to \underline{X} fppf-locally. Hence \underline{X}' is an S -form of \underline{X} . We claim that the sheaf of isomorphisms $\underline{\text{Isom}}(X, X')$ is isomorphic to \underline{F} .

Note that $\underline{A} \wedge^{\underline{A}} \underline{X}$ is canonically isomorphic to \underline{X} and the \underline{A} -action on \underline{X} corresponds to the \underline{A} -action on the \underline{A} -factor of $\underline{A} \wedge^{\underline{A}} \underline{X}$. Regard \underline{A} as a right \underline{A} -torsor. Then there is

a natural map ι between $\underline{\text{Isom}}_{\underline{A}}(\underline{A}, \underline{F})$ and $\underline{\text{Isom}}_{\underline{G}}(\underline{X}, \underline{X}') = \underline{\text{Isom}}_{\underline{G}}(\underline{A} \wedge^{\underline{A}} \underline{X}, \underline{F} \wedge^{\underline{A}} \underline{X}')$. To be precise, for an S -scheme S' and $\phi \in \underline{\text{Isom}}_{\underline{A}}(\underline{A}, \underline{F})(S')$, $\iota(\phi)([(a, x)]) = [(\phi(a), x)]$ for any S' -scheme S'' and $[(a, x)] \in (\underline{A} \wedge^{\underline{A}} \underline{X})(S'')$.

The sheaf of automorphisms of \underline{A} (as a right \underline{A} -torsor) is \underline{A} itself which acts on the left of \underline{A} . As $\underline{\text{Isom}}_{\underline{A}}(\underline{A}, \underline{F})$ is a right $\underline{\text{Aut}}_{\underline{A}}(\underline{A})$ -torsor, it is a right \underline{A} -torsor. It is easy to check that ι is compatible with the \underline{A} -action. Since $\underline{\text{Isom}}_{\underline{A}}(\underline{A}, \underline{F})$ and $\underline{\text{Isom}}_{\underline{G}}(\underline{X}, \underline{X}')$ are both right \underline{A} -torsors, and ι is compatible with the \underline{A} -action, ι is an isomorphism.

There is a natural map i from $\underline{\text{Isom}}_{\underline{A}}(\underline{A}, \underline{F})$ to \underline{F} that sends $\phi \in \underline{\text{Isom}}_{\underline{A}}(\underline{A}, \underline{F})(S')$ to $\phi(1_{\underline{A}}) \in \underline{F}(S')$ for all S -schemes S' . Since i is compatible the right \underline{A} -action, i is an isomorphism between right \underline{A} -torsors, and $i \circ \iota^{-1}$ gives the desired isomorphism between $\underline{\text{Isom}}_{\underline{G}}(\underline{X}, \underline{X}')$ and \underline{F} . Therefore the map that sends an S -form \underline{X}' to the right \underline{A} -torsor $\underline{\text{Isom}}_{\underline{G}}(\underline{X}, \underline{X}')$ is a bijection. \square

3. EMBEDDING FUNCTORS

Let S be a scheme. For a point $s \in S$, let $\kappa(s)$ be the residue field of s and $\overline{\kappa(s)}$ be the algebraic closure of $\kappa(s)$. Let \bar{s} be the scheme $\text{Spec}(\overline{\kappa(s)})$.

3.1. Twisted root data and Weyl groups. Let M be a \mathbf{Z} -lattice and M^\vee be its dual lattice. Let R and R^\vee be finite subsets of M and M^\vee respectively. Suppose that (M, M^\vee, R, R^\vee) satisfy the root data axioms.

Denote by \underline{M}_S (resp. \underline{R}_S) the constant sheaf on S associated to M (resp. R). Let T be an S -torus and denote by \mathcal{M} (resp. \mathcal{M}^\vee) the fppf sheaf of characters (resp. cocharacters) of T . Let $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ be a twisted root datum. (c.f. [31] Exp. XXII, 1.9). We say that the twisted root datum Ψ is of *type* (M, M^\vee, R, R^\vee) at $s \in S$ if $\Psi_{\bar{s}} \simeq (\underline{M}_{\bar{s}}, \underline{M}_{\bar{s}}^\vee, \underline{R}_{\bar{s}}, \underline{R}_{\bar{s}}^\vee)$.

We denote by $W(\Psi)$ the Weyl S -group scheme of the twisted root datum Ψ . If Ψ is split of constant type, Ψ can be written as $(\underline{M}_S, \underline{M}_S^\vee, \underline{R}_S, \underline{R}_S^\vee)$ for some root datum (M, M^\vee, R, R^\vee) . In this case, we have $W(\Psi) = W_S$ where W is the (abstract) Weyl group of the root datum (M, M^\vee, R, R^\vee) ; we recall that W is a finite group generated by the reflections s_α defined by

$$s_\alpha(x) = x - (\alpha^\vee, x)\alpha, \text{ for } \alpha \in R \text{ and } x \in M.$$

For $\alpha \in R$, the reflection s_α induces an automorphism \tilde{s}_α on T by

$$\tilde{s}_\alpha(t) = t(\alpha^\vee(\alpha(t)))^{-1} \text{ for any } S\text{-scheme } S', \alpha \in R \text{ and } t \in T(S').$$

(See [31] Exp. XXII, 3.3.)

The map sending s_α to \tilde{s}_α defines an action of $W(\Psi)^{op}$ on T . For $w \in W(\Psi)(S)$, we denote by \tilde{w} its image in $\underline{\text{Aut}}(T)(S)$. In general, $W(\Psi)$ is a finite étale group scheme over S , which is regarded as a subgroup scheme of $\underline{\text{Aut}}(\mathcal{M})$.

Let G be a reductive group scheme over S . Suppose that G has a maximal torus T over S . We denote by $\Phi(G, T)$ the twisted root datum of G with respect to T . Let

$\underline{N}_G(T)$ the normalizer of T in G . The conjugation action of $\underline{N}_G(T)/T$ on T gives an isomorphism from $\underline{N}_G(T)/T$ to $W(\Phi(G, T))^{op}$. (cf. [31] Exp. XXII, 3.1-3.4)

3.2. The Orientation. Let Ψ_1, Ψ_2 be two twisted root data. Suppose that Ψ_1 and Ψ_2 are of the same type at each $s \in S$. Let $\underline{\text{Isom}}(\Psi_1, \Psi_2)$ be the sheaf of isomorphisms between Ψ_1 and Ψ_2 . Then $\underline{\text{Isom}}(\Psi_1, \Psi_2)$ is a right principal homogeneous space of $\underline{\text{Aut}}(\Psi_1)$ and a left principal homogeneous of $\underline{\text{Aut}}(\Psi_2)$. Define

$$\underline{\text{Isomext}}(\Psi_1, \Psi_2) = W(\Psi_2) \backslash \underline{\text{Isom}}(\Psi_1, \Psi_2).$$

Write the twisted root datum Ψ_i as $(\mathcal{M}_i, \mathcal{M}_i^\vee, \mathcal{R}_i, \mathcal{R}_i^\vee)$. For an S -scheme S' and $f \in \underline{\text{Isom}}(\Psi_1, \Psi_2)(S')$, f induces an isomorphism between the Weyl groups $W(\Psi_{1, S'})$ and $W(\Psi_{2, S'})$. Namely for any S' -scheme S'' and any root α of $\mathcal{R}_1(S'')$, f sends the reflection s_α to $s_{f(\alpha)}$ and $f \circ s_\alpha = s_{f(\alpha)} \circ f$. Hence $\underline{\text{Isomext}}(\Psi_1, \Psi_2)$ is canonically isomorphic to $\underline{\text{Isom}}(\Psi_1, \Psi_2)/W(\Psi_1)$. As a consequence, the natural isomorphism between sheaves $\underline{\text{Isom}}(\Psi_1, \Psi_2)$ and $\underline{\text{Isom}}(\Psi_2, \Psi_1)$, which sends $f \in \underline{\text{Isom}}(\Psi_1, \Psi_2)(S')$ to f^{-1} , gives an isomorphism between $\underline{\text{Isomext}}(\Psi_1, \Psi_2)$ and $\underline{\text{Isomext}}(\Psi_2, \Psi_1)$.

Suppose that $\underline{\text{Isomext}}(\Psi_1, \Psi_2)(S)$ is nonempty. For $u \in \underline{\text{Isomext}}(S)$, we define $\underline{\text{Isomint}}_u(\Psi_1, \Psi_2)$ to be the fiber of $\underline{\text{Isom}}(\Psi_1, \Psi_2) \rightarrow \underline{\text{Isomext}}(\Psi_1, \Psi_2)$ at u .

Let G be a reductive group scheme over S . The *type* of G at s is the isomorphism class of the root datum of $G_{\bar{s}}$ with respect to its maximal torus. (ref. [31], Exp. XXII, Def. 2.6.1, 2.7)

Suppose that G has a maximal torus T . We define $\underline{\text{Isomext}}(G, \Psi)$ to be $\underline{\text{Isomext}}(\Phi(G, T), \Psi)$. Let T^\sharp be another maximal torus of G over S . For an S -scheme S' , every element of the transporter $\underline{\text{Transp}}_G(T^\sharp, T)(S')$ gives an isomorphism from $T_{S'}^\sharp$ to T_S via conjugation, which in turns gives an isomorphism from $\Phi(G, T)$ to $\Phi(G, T^\sharp)$. As $\underline{\text{Transp}}_G(T^\sharp, T)$ is a right homogeneous space under $\underline{N}_G(T^\sharp)$, through the identification of $\underline{N}_G(T^\sharp)/T^\sharp$ with $W(\Phi(G, T^\sharp))^{op}$, we see that there is a natural morphism between

$$\underline{\text{Isom}}(\Phi(G, T^\sharp), \Psi) \wedge^{\underline{N}(T^\sharp)^{op}} \underline{\text{Transp}}_G(T^\sharp, T) \rightarrow \underline{\text{Isom}}(\Phi(G, T), \Psi).$$

This gives a canonical isomorphism from $\underline{\text{Isomext}}(\Phi(G, T^\sharp), \Psi)$ to $\underline{\text{Isomext}}(\Phi(G, T), \Psi)$. Hence $\underline{\text{Isomext}}(G, \Psi)$ is well-defined. For those G without maximal torus over S , we define $\underline{\text{Isomext}}(G, \Psi)$ be descent (ref. [20, §1.2.1]).

We define $\underline{\text{Isomext}}(\Psi, G)$ in a similar way. As there is a canonical isomorphism between $\underline{\text{Isomext}}(\Phi(G, T), \Psi)$ and $\underline{\text{Isomext}}(\Psi, \Phi(G, T))$, the two functors $\underline{\text{Isomext}}(G, \Psi)$ and $\underline{\text{Isomext}}(\Psi, G)$ are canonically isomorphic.

An *orientation* of G with respect to Ψ is an element of $\underline{\text{Isomext}}(G, \Psi)(S)$.

A twisted root datum Ψ is *admissible* for G if at each $s \in S$, the type of $G_{\bar{s}}$ is the same as the type of $\Psi_{\bar{s}}$, and $\underline{\text{Isomext}}(G, \Psi)(S) \neq \emptyset$. The *admissibility* condition is a necessary condition for the existence of a maximal torus T of G such that $\Phi(G, T)$ is isomorphic to Ψ .

Let G' be an S -form of G . Let $\underline{\text{Isom}}(G, G')$ be the sheaf of group isomorphisms between G and G' . Note that G acts on itself by conjugation. Thus we can define the right quotient of $\underline{\text{Isom}}(G, G')$ by the adjoint quotient G_{ad} of G and denote $\underline{\text{Isom}}(G, G')/G_{ad}$ by $\underline{\text{Isomext}}(G, G')$.

Let Ψ_1, Ψ_2 and Ψ_3 be twisted root data over S . Suppose that all of them are of the same type at each geometric fibre of S . Then we have the following morphism :

$$(3.1) \quad \underline{\text{Isomext}}(\Psi_1, \Psi_2) \times \underline{\text{Isomext}}(\Psi_2, \Psi_3) \rightarrow \underline{\text{Isomext}}(\Psi_1, \Psi_3),$$

that comes from the composition of isomorphisms

$$\underline{\text{Isom}}(\Psi_1, \Psi_2) \times \underline{\text{Isom}}(\Psi_2, \Psi_3) \rightarrow \underline{\text{Isom}}(\Psi_1, \Psi_3).$$

Suppose G and Ψ_1 are of the same type at each geometric fiber of S . Similarly we have the following pairings coming from the composition of isomorphisms between root data.

Proposition 3.2.1. *We have the following pairings*

- (1) $\underline{\text{Isomext}}(G, \Psi_1) \times \underline{\text{Isomext}}(\Psi_1, \Psi_2) \rightarrow \underline{\text{Isomext}}(G, \Psi_2)$.
- (2) $\underline{\text{Isomext}}(G, \Psi_1) \times \underline{\text{Isomext}}(G, \Psi_2) \rightarrow \underline{\text{Isomext}}(\Psi_1, \Psi_2)$.
- (3) $\underline{\text{Isomext}}(G, \Psi_1) \times \underline{\text{Isomext}}(G', \Psi_1) \rightarrow \underline{\text{Isomext}}(G, G')$.
- (4) $\underline{\text{Isomext}}(G', G) \times \underline{\text{Isomext}}(G, \Psi) \rightarrow \underline{\text{Isomext}}(G', \Psi)$.

Proof. The first two assertions come from the definition of $\underline{\text{Isomext}}(G, \Psi)$ and the composition of isomorphisms between root data. For (3) and (4), we notice that if G and G' have maximal tori T and T' respectively, then $\underline{\text{Isomext}}(\Phi(G, T), \Phi(G', T'))$ is representable and isomorphic to $\underline{\text{Isomext}}(G, G')$. As reductive groups have maximal tori étale locally, assertions (3) and (4) can be deduced from descent. We refer to [20, Prop. 1.4] for more details. \square

Another complement is the following criterion of reduction of torsors. Let G be a reductive group scheme over S with a maximal torus T . Let E be right $fppf$ G -torsor over S . Consider G acts on itself by conjugation and set $G^\sharp = E \wedge^G G$. As G^\sharp is an inner twisted form of G , there is a canonical element $c \in \underline{\text{Isomext}}(G, G^\sharp)(S)$. As $\underline{\text{Isomext}}(G, G^\sharp)$ is canonically isomorphic to $\underline{\text{Isomext}}(\Psi, \Psi^\sharp)$, we can regard c as an element in $\underline{\text{Isomext}}(\Psi, \Psi^\sharp)(S)$ for any pair (Ψ, Ψ^\sharp) of twisted root data where Ψ (resp. Ψ^\sharp) is admissible for G (resp. G^\sharp).

Proposition 3.2.2. *Let E, T, G and G^\sharp be as above and consider the twisted root datum $\Psi = \Phi(G, T)$. Then the following assertions are equivalent:*

- (i) E admits a reduction to T , i.e. there is a right T -torsor F such that $E \simeq F \wedge^T G$ as G -torsors;
- (ii) G^\sharp admits a maximal S -torus T^\sharp satisfying the following property:
the set $\underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S)$ is nonempty, with the notation $\Psi = \Phi(G^\sharp, T^\sharp)$.

Proof. Let $\underline{\text{Isom}}_G(G, E)$ be the fppf-sheaf of isomorphisms of right G -torsors. Denote by 1_G the neutral element of $G(S)$. There is an isomorphism ι from $\underline{\text{Isom}}_G(G, E)$ to E which sends $f \in \underline{\text{Isom}}_G(G, E)(S')$ to $f(1_G)$ for any S -scheme S' . Regard G as a right G -torsor, the left multiplication of G on itself forms the automorphism group $\underline{\text{Aut}}_G(G)$. Thus $\underline{\text{Isom}}_G(G, E)$ is a right G -torsor and ι is an isomorphism of right G -torsors.

Denote by $\underline{\text{Isomint}}_c(G, G^\sharp)$ the fiber of $\underline{\text{Isom}}(G, G^\sharp) \rightarrow \underline{\text{Isomext}}(G, G^\sharp)$ at c . Clearly an isomorphism between the G -torsors G and E gives an isomorphism between reductive groups G and G^\sharp compatible with the orientation c . Hence there is a natural surjective morphism of sheaves from $\underline{\text{Isom}}_G(G, E)$ to $\underline{\text{Isomint}}_c(G, G^\sharp)$.

(ii) \implies (i). We are given a maximal S -torus T^\sharp of G^\sharp such that $\underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S) \neq \emptyset$. Next consider the sheaf of isomorphisms $\underline{\text{Isomint}}_c((G, T), (G^\sharp, T^\sharp))$, which consists of elements in $\underline{\text{Isomint}}_c(G, G^\sharp)$ that apply T to T^\sharp . This is a subsheaf of $\underline{\text{Isomint}}_c(G, G^\sharp)$. Set

$$\mathcal{F} = \underline{\text{Isomint}}_c((G, T), (G^\sharp, T^\sharp)) \times_{\underline{\text{Isomint}}_c(G, G^\sharp)} \underline{\text{Isom}}_G(G, E).$$

We can regard \mathcal{F} as a subsheaf of $\underline{\text{Isom}}_G(G, E)$ through the second projection p_2 . Note that $\underline{N}_G(T)$ acts diagonally on the right of both factors of \mathcal{F} . Clearly p_2 is an $\underline{N}_G(T)$ -equivariant morphism. Let $\pi : \mathcal{F} \rightarrow \underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)$ be the projection from \mathcal{F} to $\underline{\text{Isomint}}_c((G, T), (G^\sharp, T^\sharp))$ composed with the natural morphism from $\underline{\text{Isomint}}_c((G, T), (G^\sharp, T^\sharp))$ to $\underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)$.

Let $f \in \underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S)$. Then the fiber of π at f is a right T -torsor and we denote it by F . Note that F embeds in E via $\iota \circ p_2$. Hence we have $E = F \wedge^T G$.

(ii) \implies (i). We suppose $E = F \wedge^T G$ for some T -torsor F . Then $G^\sharp = E \wedge^G G = F \wedge^T G$, and $T^\sharp = F \wedge^T T$ is a maximal torus of G^\sharp . It remains to show that $\underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S) \neq \emptyset$.

Let $S' \rightarrow S$ be a fppf-cover that trivializes F . We choose a trivialization $f : T_{S'} \xrightarrow{\sim} F_{S'}$. Then f induces an isomorphism between $(G_{S'}, T_{S'})$ and $(G_{S'}^\sharp, T_{S'}^\sharp)$ with respect to the orientation c , which in turn gives an isomorphism $f^\diamond \in \underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S')$.

Set $S'' = S' \times_S S'$. Let r_1 and r_2 be the projections of S'' to its first and second factors respectively. Then $(r_2^* f)^{-1} \circ (r_1^* f)$ is an automorphism of $T_{S''}$ (as a $T_{S''}$ -torsor), which corresponds to the left multiplication by some element $t \in T(S'')$. As T acts trivially on itself by conjugation, $(r_2^* f)^{-1} \circ (r_1^* f)$ induces trivial automorphism on $\Psi_{S''}$. Hence $r_2^*(f^\diamond) = r_1^*(f^\diamond)$ and f^\diamond descends to an element in $\underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S)$. Thus $\underline{\text{Isomint}}_c(\Psi, \Psi^\sharp)(S) \neq \emptyset$ as desired. \square

Denote by $G//G$ the adjoint quotient of G in the GIT framework, its formation commutes to arbitrary base change [21, §4]. Suppose that G has a maximal torus T . Let $T/W(T)$ be the quotient sheaf of T by the action of Weyl group $W(T)$. Then $T/W(T)$ is isomorphic to $G//G$ ([21] Thm. 4.1).

Consider the quotient map (of fppf-sheaves) $\pi : T \rightarrow T/W(T)$. Let $x \in (G//G)(S)$. As $T/W(T)$ and $G//G$ are isomorphic, we can regard x as an element in $(T/W(T))(S)$. Suppose that there is a semisimple regular element $t_1 \in G(S)$ whose image in $(G//G)(S)$ is x . Let T_1 be the centralizer $\underline{C}_G(t_1)$, that is also a maximal torus. Let Ψ (resp. Ψ_1) be the root datum associated to T (resp. T_1). Then the natural inclusion from T to G gives an orientation $v \in \underline{\text{Isomext}}(G, \Psi)(S)$. Similarly, we get an orientation $v_1 \in \underline{\text{Isomext}}(G, \Psi_1)(S)$ from the natural inclusion of T_1 in G . Then by 3.2.1 (2) we get an orientation $u \in \underline{\text{Isomext}}(\Psi, \Psi_1)(S)$ coming from $v \cdot v_1$. Then we have the following proposition.

Proposition 3.2.3. *The left $W(T)$ -torsor $\pi^{-1}(x)$ is isomorphic to $\underline{\text{Isomint}}_u(\Psi, \Psi_1)$.*

Proof. For an S -scheme S' and $t \in \pi^{-1}(x)(S')$, we denote by $\underline{\text{Transp}}(t, t_1)$ the transporter from t to t_1 . Both t and t_1 are mapped to the same element in $(G//G)(S)$, and hence $\underline{\text{Transp}}(t, t_1) \rightarrow \bullet$ is surjective. Since t_1 is a semisimple regular element, so is t . As t lies in T , the centralizer $\underline{C}_G(t)$ of t is the torus $T_{S'}$ and $\underline{\text{Transp}}(t, t_1)$ is a right $T_{S'}$ -torsor.

The conjugation action of T on $\Psi = \Phi(G, T)$ is trivial. Therefore the canonical morphism

$$\underline{\text{Transp}}(t, t_1) \wedge^{T_{S'}} \Psi_{S'} \rightarrow \Psi_{1, S'}$$

defines an element in $\underline{\text{Isomint}}_u(\Psi, \Psi_1)(S')$. In this way, we get a map

$$i : \pi^{-1}(x) \rightarrow \underline{\text{Isomint}}_u(\Psi, \Psi_1).$$

To be precise, write the root datum Ψ as $(\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$. For an S' -scheme S'' and $h \in \underline{\text{Transp}}(t, t_1)(S'')$, $i(t)(m) = m \circ \text{int}(h^{-1})$ for all $m \in \mathcal{M}_{S''} = \text{Hom}_{S''}(T_{S''}, \mathbb{G}_{m, S''})$. Note that $i(t)$ is independant of the choice of h since T acts trivially on \mathcal{M} .

As $\underline{\text{Isomint}}_u(\Psi, \Psi_1)$ is a right $W(\Psi)$ -torsor and $W(T) \simeq W(\Psi)^{op}$, we can regard it as left $W(T)$ -torsor. We write down explicitly this action. For $w \in W(T)_{S'}$, let n_w be a lift of w in $N(T)(S'')$ for some S' -scheme S'' . As T acts on \mathcal{M} trivially, $\text{int}(n_w^{-1})$ induces an automorphism on $\mathcal{M}_{S'}$. For $f \in \underline{\text{Isomint}}_u(\Psi, \Psi_1)(S')$, $(w \cdot f)(m) := f(m \circ \text{int}(n_w))$ for $m \in \mathcal{M}_{S'} = \text{Hom}_{S'-grp}(T_{S'}, \mathbb{G}_{m, S'})$.

Since $\pi^{-1}(x)$ and $\underline{\text{Isomint}}_u(\Psi, \Psi_1)$ are both $W(T)$ -torsors, to check i is an isomorphism, it suffices to check the map i is $W(T)$ -equivariant. Let $w \in W(T)_{S'}$ and n_w be a lift of w in $N(T)(S'')$ for some S' -scheme S'' . For $h \in \underline{\text{Transp}}(t, t_1)(S'')$, $hn_w^{-1} \in \underline{\text{Transp}}(w \cdot t, t_1)(S'')$. Thus

$$\begin{aligned} i(w \cdot t)(m) &= m \circ \text{int}((hn_w^{-1})^{-1}) \\ &= m \circ \text{int}(n_w) \circ \text{int}(h^{-1}) \\ (3.2) \quad &= i(t)(m \circ \text{int}(n_w)) \\ &= (w \cdot i(t))(m). \end{aligned}$$

Hence i is an isomorphism between $W(T)$ -torsors. \square

3.3. Embedding functors. Let G be a reductive group scheme over S and $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ be an admissible root datum for G . Let T be the S -torus with sheaf of characters \mathcal{M} .

Suppose there is an embedding f of T into G as algebraic group schemes. Then f induces an isomorphism from the character group of $f(T)$ to \mathcal{M} . We call this map f^\sharp . If f^\sharp induces an isomorphism between $\Phi(G, f(T))$ and Ψ , then we say f is an embedding with respect to the twisted root datum Ψ .

Define the *embedding functor* $\mathfrak{E}(G, \Psi)$ as follows: for each S -scheme S' ,

$$\mathfrak{E}(G, \Psi)(S') = \left\{ f : T_{S'} \hookrightarrow G_{S'} \left| \begin{array}{l} f \text{ is both a closed immersion and a group} \\ \text{homomorphism that induces an isomorphism} \\ f^\sharp : \Phi(G_{S'}, f(T_{S'})) \xrightarrow{\sim} \Psi_{S'} \end{array} \right. \right\}$$

Note that the fact that Ψ is admissible for G ensures that $\mathfrak{E}(G, \Psi)$ is not empty and allows us to fix an orientation $v \in \underline{\text{Isomext}}(G, \Psi)(S)$.

The *oriented embedding functor* $\mathfrak{E}(G, \Psi, v)$ over S is defined by

$$\mathfrak{E}(G, \Psi, v)(S') = \left\{ f : T_{S'} \hookrightarrow G_{S'} \left| \begin{array}{l} f \in \mathfrak{E}(G, \Psi)(S'), \text{ and the image of } f^\sharp \\ \text{in } \underline{\text{Isomext}}(G, \Psi)(S') \text{ is } v. \end{array} \right. \right\}$$

Since any two maximal tori of G are étale locally conjugated by G , the oriented embedding functor $\mathfrak{E}(G, \Psi, v)$ is representable by an affine S -scheme that is a left homogeneous G -space ([20, 1.6]). For an S -scheme S' and an element $f \in \mathfrak{E}(G, \Psi, v)(S')$, the stabilizer of f in $G_{S'}$ is $f(T_{S'})$, which is a maximal torus of $G_{S'}$. The S -scheme $\mathfrak{E}(G, \Psi, v)$ admits a right action of $W(T)$ and the quotient sheaf $\mathfrak{E}(G, \Psi, v)/W(T)$ identifies with the S -scheme \mathcal{T}_G of maximal tori of G . (See [20, §1.1 and §1.2] for more details.)

Remark 3.1. In [20, §1.2.2] we define the oriented embedding functor $\mathfrak{E}(G, \Psi, v')$ by orientation $v' \in \underline{\text{Isomext}}(\Psi, G)(S)$ instead of $v \in \underline{\text{Isomext}}(G, \Psi)(S)$ here. However when v' is the image of v under the canonical isomorphism $\underline{\text{Isomext}}(G, \Psi) \rightarrow \underline{\text{Isomext}}(\Psi, G)$, these two definitions are clearly equivalent.

Lemma 3.2. *Keep the notation as above.*

(1) *We have an isomorphism $W(\Psi) \xrightarrow{\sim} \underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))$ as group schemes.*

(2) *Let Ψ' be another admissible root datum for G , and $v' \in \underline{\text{Isomext}}(G, \Psi')(S)$. Let $u \in \underline{\text{Isomext}}(\Psi, \Psi')(S)$ be the orientation $v' \cdot v$ by the pairing (3.2.1). Under the identification of $W(\Psi)$ and $\underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))$ in (1), there is a natural isomorphism between $\underline{\text{Isomint}}_u(\Psi, \Psi')$ and $\underline{\text{Isom}}_G(\mathfrak{E}(G, \Psi, v), \mathfrak{E}(G, \Psi', v'))$ as right $W(\Psi)$ -torsors.*

(3) *Let F be a right $W(\Psi)$ -torsor and $\Psi' = F \wedge^{W(\Psi)} \Psi$. Let c be the canonical element of $\underline{\text{Isomext}}(\Psi, \Psi')(S)$, and let $v_1 \in \underline{\text{Isomext}}(G, \Psi')(S)$ be $c \cdot v$ by the pairing (3.2.1). By identifying $W(\Psi)$ with $\underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))$, $F \wedge^{W(\Psi)} \mathfrak{E}(G, \Psi, v)$ is isomorphic to $\mathfrak{E}(G, \Psi', v_1)$.*

(4) Let E be a right G -torsor and denote by G' the twist of G by E via inner automorphisms. Let c be the canonical element of $\underline{\text{Isomext}}(G', G)(S)$. Then the homogeneous G' -space $E \wedge^G \mathfrak{E}(G, \Psi, v)$ is isomorphic to $\mathfrak{E}(G', \Psi, v')$ where $v' \in \underline{\text{Isomext}}(G', \Psi)(S)$ is $c \cdot v$ by the pairing (3.2.1).

Proof. (1) For an S -scheme S' and $w \in W(\Psi)(S')$, we denote by \tilde{w} the image of w in $W(T)(S')$. Define a morphism $\iota : W(\Psi) \rightarrow \underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))$ as follows. For an S -scheme S' and $w \in W(\Psi)(S')$,

$$\iota(w)(x) := x \circ \tilde{w} \text{ for all } S'\text{-schemes } S'', \text{ and } x \in \mathfrak{E}(G, \Psi, v)(S'').$$

For w_1 and $w_2 \in W(\Psi)(S')$,

$$\begin{aligned} \iota(w_1 \circ w_2)(x) &= x \circ \widetilde{(w_1 \circ w_2)} \\ (3.3) \quad &= x \circ \tilde{w}_2 \circ \tilde{w}_1 \\ &= (\iota(w_1) \circ \iota(w_2))(x). \end{aligned}$$

Hence ι is a group homomorphism.

Clearly ι is monomorphism. To see that ι is an isomorphism, it suffices to check ι is an isomorphism of fppf-sheaves. Let $\{S_i\}$ be a fppf-cover of S such that G_{S_i} and Ψ_{S_i} are both split. Then $\mathfrak{E}(G, \Psi, v)(S_i)$ is not empty. Choose $x_i \in \mathfrak{E}(G, \Psi, v)(S_i)$. By Lemma 2.2, $\underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))(S_i)$ is isomorphic to $(\underline{N}_G(x_i(T))/x_i(T))^{op}$, where $\underline{N}_G(x_i(T))$ is the normalizer of $x_i(T)$ in G . Since $\underline{N}_G(x_i(T))/x_i(T) \simeq W(\Phi(G_{S_i}, x_i(T_{S_i})))^{op}$, the group $\underline{N}_G(x_i(T))/x_i(T)$ is isomorphic to $W(\Psi_{S_i})^{op}$. Thus $\underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))_{S_i} \simeq W(\Psi_{S_i})$. As ι is a monomorphism and $\underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))_{S_i}$ is isomorphic to $W(\Psi_{S_i})$, ι is an isomorphism.

(2) Write $\Psi' = (\mathcal{M}', (\mathcal{M}')^\vee, \mathcal{R}', (\mathcal{R}')^\vee)$ and denote by T' the torus with sheaf of characters \mathcal{M}' . For an S -scheme S' and $f \in \underline{\text{Isomint}}_u(\Psi, \Psi')(S')$, we denote by \tilde{f} the isomorphism from $T'_{S'}$ to $T_{S'}$ induced by f . Define $\eta : \underline{\text{Isomint}}_u(\Psi, \Psi') \rightarrow \underline{\text{Isom}}_G(\mathfrak{E}(G, \Psi, v), \mathfrak{E}(G, \Psi', v'))$ by $\eta(f)(x) = x \circ \tilde{f}$, for all S' -scheme S'' , and $x \in \mathfrak{E}(G, \Psi, v)(S'')$.

Next we show that η is compatible with the $W(\Psi)$ -action under the identification of $W(\Psi)$ and $\underline{\text{Aut}}_G(\mathfrak{E}(G, \Psi, v))$. Let $w \in W(\Psi)(S')$. Then

$$\begin{aligned} \eta(f \circ w)(x) &= x \circ \widetilde{(f \circ w)} \\ (3.4) \quad &= x \circ \tilde{w} \circ \tilde{f} \\ &= \eta(f)(x \circ \tilde{w}) \\ &= \eta(f)(\iota(w)(x)) \\ &= (\eta(f) \circ \iota(w))(x). \end{aligned}$$

This shows that η is compatible with the $W(\Psi)$ -action. Since $\underline{\text{Isomint}}_u(\Psi, \Psi')$ and $\underline{\text{Isom}}_G(\mathfrak{E}(G, \Psi, v), \mathfrak{E}(G, \Psi', v'))$ are both right $W(\Psi)$ -torsors, this implies that η is an isomorphism.

(3) Write Ψ' as $(\mathcal{M}', (\mathcal{M}')^\vee, \mathcal{R}', (\mathcal{R}')^\vee)$ and let T' be the torus with character group \mathcal{M}' . For an S -scheme S' and $f \in F(S')$, we define $\varphi_f^\sharp : \mathcal{M}_{S'} \rightarrow \mathcal{M}'_{S'}$ as $\varphi_f^\sharp(m) = (f, m)$ for all S' -scheme S'' and $m \in \mathcal{M}_{S'}(S'')$. Denote by φ_f the group homomorphism from T' to T defined by φ_f^\sharp .

Define $\varsigma : F \times \mathfrak{E}(G, \Psi, v) \rightarrow \mathfrak{E}(G, \Psi', v')$ as follows. For an S -scheme S' and $(f, x) \in (F \times \mathfrak{E}(G, \Psi, v))(S')$, let $\varsigma(f, x) = x \circ \varphi_f$. Clearly $x \circ \varphi_f$ is an embedding of T' in G with orientation v' . Note that for $w \in W(\Psi)(S')$, we have

$$\begin{aligned} \varphi_{fw}^\sharp(m) &= (fw, m) \\ (3.5) \quad &= (f, w^{-1}m) \\ &= \varphi_f^\sharp(w^{-1}m). \end{aligned}$$

Therefore $\varphi_{fw}^\sharp = \varphi_f^\sharp \circ w^{-1}$ and $\varphi_{fw} = \tilde{w}^{-1} \circ \varphi_f$. It follows that

$$\begin{aligned} \varsigma(fw, x) &= x \circ \varphi_{fw} \\ (3.6) \quad &= x \circ \tilde{w}^{-1} \circ \varphi_f \\ &= (w^{-1} \cdot x) \circ \varphi_f \\ &= \varsigma(f, w \cdot x). \end{aligned}$$

Hence ς induces a morphism from $F \wedge^{W(\Psi)} \mathfrak{E}(G, \Psi, v)$ to $\mathfrak{E}(G, \Psi', v')$, that is clearly an isomorphism.

(4) For an S -scheme S' and $e \in E(S')$, we define $f_e : G_{S'} \rightarrow G'_{S'}$ by $f_e(g) = [e, g]$ for all S' -scheme S'' and $g \in G(S'')$. Then f_e is an isomorphism between $G_{S'}$ and $G'_{S'}$.

For $y = (e, x) \in (E \times_S \mathfrak{E}(G, \Psi, v))(S')$, we define

$$\iota : E \times_S \mathfrak{E}(G, \Psi, v) \rightarrow \mathfrak{E}(G', \Psi, v')$$

as $\iota(y) = f_e \circ x$. It is clear that $\iota(y) \in \mathfrak{E}(G', \Psi)(S')$. As x is an embedding of orientation v , by the definition of canonical orientation, the embedding $\iota(y)$ is of orientation v' . Since $\iota(e, x) = \iota(e \cdot g, g^{-1} \cdot x)$ for all $g \in G(S')$, ι induces an isomorphism from $E \wedge^G \mathfrak{E}(G, \Psi, v)$ to $\mathfrak{E}(G', \Psi, v')$. \square

Remark 3.3. Let G^{sc} be the simply connected cover of G and let G_{ad} be its adjoint quotient. On the other hand let Ψ^{sc} (resp. Ψ_{ad}) the simply connected (resp. adjoint) root datum associated to Ψ . Then we can attach to (Ψ, v) an oriented root datum (Ψ^{sc}, v^{sc}) (resp. (Ψ_{ad}, v_{ad})) and isomorphisms

$$\mathfrak{E}(G^{sc}, \Psi^{sc}, v^{sc}) \xrightarrow{\sim} \mathfrak{E}(G, \Psi, v) \xrightarrow{\sim} \mathfrak{E}(G_{ad}, \Psi_{ad}, v_{ad}).$$

It implies that we can deal in practice with semisimple simply connected (resp. adjoint) group schemes. Another advantage is that we have isomorphisms of finite étale group schemes

$$\underline{\text{Isomext}}(G^{sc}, \Psi^{sc}) \xrightarrow{\sim} \underline{\text{Isomext}}(G_{ad}, \Psi_{ad}) \xrightarrow{\sim} \underline{\text{Isomext}}(\underline{\text{Dyn}}(G), \underline{\text{Dyn}}(\Psi)).$$

(See [20, Cor. 1.7].)

3.4. The main result.

Theorem 3.4. *Let X be a separated S -scheme satisfying one of the following conditions:*

- (i) S is locally noetherian;
- (ii) X is locally of finite type over S .

We assume that X is a left homogeneous space under a reductive S -group scheme G and that for each point $s \in S$, the stabilizer of $X_{\bar{s}}$ is a maximal torus of $G_{\bar{s}}$.

(1) *There exists a twisted root datum $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ that is admissible for G and an orientation $v \in \underline{\text{Isomext}}(G, \Psi)(S)$ such that X is isomorphic to $\mathfrak{E}(G, \Psi, v)$ as left G -homogeneous space.*

(2) *The oriented root datum (Ψ, v) in (1) is unique up to isomorphism.*

Proof. We shall establish first that the local stabilizers of X are maximal tori. Since G is smooth over S with connected fibers, X is smooth of finite presentation over S according to [27, prop. VI.1.2].

(1) Let $(S_i)_{i \in I}$ be an fppf cover of S such that X_{S_i} is G_{S_i} -isomorphic to G_{S_i}/H_i where H_i is a S_i -subgroup scheme of G_{S_i} (see [31, IV 6.7.3]). Since H_i is the stabilizer of a point of $X(S_i)$ (which is S_i -separated), H_i is a closed S_i -subgroup scheme of G_{S_i} and is in particular affine.

Again by [27, prop. VI.1.2], H_i is flat locally of finite presentation over S_i so is affine flat of finite presentation over S_i . As the geometric fibers of H_i are tori, it follows that H_i is a S_i -torus [31, X.4.9]. Therefore H_i is a maximal S_i -torus of G_{S_i} for each $i \in I$.

In particular, $C(G)$ acts trivially on X , so X is a G_{ad} -homogeneous space whose local stabilizers are maximal tori. By Remark 3.3 we reduce to the case that G is an *adjoint* semisimple S -group scheme. We denote by G^q a quasi-split S -form of G . Since G is an inner form of G^q , there is a canonical orientation $c \in \underline{\text{Isomext}}(G, G^q)(S)$. Let T^q be a maximal torus of G^q , Ψ^q be the root datum $\Phi(G^q, T^q)$ and $v^q \in \underline{\text{Isomext}}(G^q, \Psi^q)(S)$ be the orientation induced by the inclusion of T^q to G^q . Let $u = c \cdot v^q \in \underline{\text{Isomext}}(G, \Psi^q)(S)$ (by 3.2.1) and consider the homogeneous G -space $X^q = \mathfrak{E}(G, \Psi^q, u)$. This is a left G homogeneous space whose local stabilizers are maximal tori.

Claim. X is an S -form of X^q .

Denote by $\underline{\text{Isom}}_G(X^q, X)$ the sheaf of G -space isomorphisms over S . Since $\Phi(G_{S_i}, H_i)$ is fppf-locally isomorphic to $\Psi_{S_i}^q$ over S_i , we can find an fppf-cover S_{ij} of S_i such that there is $f \in X^q(S_{ij})$ with $f(T_{S_{ij}}^q) = H_{i, S_{ij}}$. As the stabilizer $\underline{\text{Stab}}_{G_{S_{ij}}}(f)$ is $H_{i, S_{ij}}$, we have $X_{S_{ij}}^q \simeq G_{S_{ij}}/H_{i, S_{ij}}$ as $G_{S_{ij}}$ -spaces. Hence there is an isomorphism between X^q and X over S_{ij} that preserves the G -structure. This proves that $\underline{\text{Isom}}_G(X^q, X)$ is nonempty. The canonical isomorphism $\underline{\text{Isom}}_G(X^q, X) \wedge^{\underline{\text{Aut}}_G(X^q)} X^q \rightarrow X$ then shows that X is an S -form of X^q . Our claim is established.

By identifying $W(\Psi^q)$ with $\underline{\text{Aut}}_G(X^q)$ as in Lemma 3.2 (1), we denote by Ψ the root datum $\underline{\text{Isom}}_G(X^q, X) \wedge^{\underline{\text{Aut}}_G(X^q)} \Psi^q$. Let $c' \in \underline{\text{Isomext}}(\Psi^q, \Psi)(S)$ be the canonical orientation, and v be the orientation $u \cdot c'$ by pairing (3.2.1). We conclude that X is isomorphic to the embedding functor $\mathfrak{E}(G, \Psi, v)$ by Lemma 3.2(3).

(2) It follows from Lemma 3.2(2). \square

4. THE CASE OF A LG RING

4.1. LG rings. We call R an *LG-ring* if for every $n \in \mathbb{N}$ and every $f \in R[X_1, \dots, X_n]$ the polynomial f represents a unit if and only if one of the following obviously equivalent conditions hold:

- (1) f represents a unit over every localization $R_{\mathfrak{m}}$, \mathfrak{m} a maximal ideal of R ;
- (2) f represents a unit over every field R/\mathfrak{m} , \mathfrak{m} a maximal ideal of R ;
- (3) f represents a unit over every R -field F ;
- (4) $\sum_{(r_1, \dots, r_n) \in R^n} f(r_1, \dots, r_n)R = R$,

Clearly a semilocal ring is a LG ring. An important fact is that locally free R -modules of rank n are free as reminded in §4.1. One says that a ring R *satisfies the primitive criterion* [9, 23] the following equivalent conditions hold:

- (1) for every primitive polynomial $P \in R[X]$ there exists $r \in R$ such that $P(r) \in R^\times$;
- (2) for every primitive $Q \in R[X_1, \dots, X_n]$ there exists $(r_1, \dots, r_n) \in R^n$ such that $Q(r_1, \dots, r_n) \in R^\times$;
- (3) R is LG and all residue fields are infinite.

In particular a semilocal ring with infinite residue fields is a LG ring. For more on LG rings, see [11, §11.20].

4.2. Schematically dominant morphisms. We recall that a morphism of schemes $f : Y \rightarrow X$ is schematically dominant if $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is injective; if f is an immersion we say that X is schematically dense in Y [13, 11.10.2]. If $g : X' \rightarrow X$ is flat, and f is schematically dominant and quasi-compact, then $f' : Y \times_X X' \rightarrow X'$ is schematically dominant (*loc. cit.*, 11.10.5).

If $f : X \rightarrow Y$ is an S -morphism of schemes, we say that f is schematically S -dominant (or universally schematically dominant with respect to S) if for each S -scheme S' , $f_{S'} : X_{S'} \rightarrow Y_{S'}$ is schematically dominant; similarly if f is an immersion, we say that Y is schematically S -dense in X . Under mild assumptions, this property can be checked fiberwise (*ibid*, 11.10.9). The base change property above extends mechanically: If $g : X' \rightarrow X$ is flat and $f : Y \rightarrow X$ is schematically S -dominant and quasi-compact, then $f' : Y \times_X X' \rightarrow X'$ is schematically S -dominant.

Rydh extended that to morphism of algebraic spaces [29, §7.5]. A morphism of algebraic spaces $f : X \rightarrow Y$ is schematically dominant if the morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is injective in the small étale site; if f is an immersion, we say that X is schematically dense in Y .

As in [30, §3.5] the base change property extends in that framework: given a flat morphism $g : X' \rightarrow X$ of algebraic spaces assuming that $f : Y \rightarrow X$ is schematically dominant and quasi-compact, then $f' : Y \times_X X' \rightarrow X'$ is schematically dominant.

If S is a scheme and $f : Y \rightarrow X$ is a morphism of S -algebraic spaces, we say that $f : X \rightarrow Y$ is schematically S -dominant if for each S -scheme S' , $f_{S'} : X_{S'} \rightarrow Y_{S'}$ is schematically dominant; similarly if f is an immersion, we say that Y is schematically S -dense in X . One last time, given a flat morphism $g : X' \rightarrow X$ and assuming that $f : Y \rightarrow X$ is schematically S -dominant and quasi-compact, then $f' : Y \times_X X' \rightarrow X'$ is schematically S -dominant.

4.3. (4)-Versal torsors. Our goal is to adapt the framework of versal torsors [10, §5] from fields to the LG ring setting by using algebraic spaces for which we use Olsson's book as reference [24] as well as the Stacks project [33]. It is close to Reichstein-Tossici's definition of (3)-versality for algebraic groups [28, Def. 1.3].

Let $S = \text{Spec}(R)$ be the spectrum of a LG-ring R and let G be an affine flat S -group scheme of finite presentation.

Definition 4.1. *Let $f : E \rightarrow X$ be a G -torsor where E is a quasi-compact quasi-separated S -scheme and X is a quasi-compact quasi-separated S -algebraic space. We say that $E \rightarrow X$ is (4)-versal if it satisfies the following property: for each open retrocompact subspace U schematically S -dense in X , for each R -algebra B satisfying the primitive criterion, and for each G -torsor F over B , there exists $x \in U(B)$ such that $E_x \cong F$ as G -torsors.*

Remarks 4.2. (a) We observe that if $f : E \rightarrow X$ is a (4)-versal G -torsor so is $f^{-1}(V) \rightarrow V$ for each schematically S -dense open retrocompact subspace V of X .

(b) There exists a dense open subspace X' of X that is a scheme [33, Tag 06NH]. If X' can be chosen furthermore schematically S -dense retrocompact, we see then that the open dense retrocompact subscheme $U' = U \times_X X'$ can be chosen in the definition. This is the case if S is noetherian of dimension ≤ 1 and X is reduced separated according to [33, Tag 0ADD].

Lemma 4.3. *Let $G \rightarrow \text{GL}(\mathcal{E})$ be a faithful linear representation of G where \mathcal{E} is a locally free R -module of finite rank.*

(1) *The fppf quotient $X = \text{GL}(\mathcal{E})/G$ is representable by an algebraic space and the quotient map $\text{GL}(\mathcal{E}) \rightarrow X$ is a (4)-versal G -torsor with right G -action.*

(2) *Assume that the vector S -group $\mathbf{V}(\mathcal{E})$ (defined in the notation) admits an open universally dense S -subscheme V which is G -stable and such that G acts freely on V . Let $Y = V/G$ be the quotient algebraic space. Then $V \rightarrow V/G$ is a (4)-versal G -torsor.*

Proof. (1) If R is noetherian, then the fppf quotient $X = \text{GL}(\mathcal{E})/G$ is represented by an S -algebraic space [1, th. 3.1.1]. The usual noetherian reduction trick shows that

it is the case in general. Let B be an R -ring satisfying the primitive criterion and F be a G -torsor over B .

We assume firstly that \mathcal{E} is of constant rank n so that $\mathcal{E} \cong R^n$ according to §4.1. Consider the exact sequence of pointed sets:

$$1 \rightarrow G(B) \rightarrow \mathrm{GL}_n(B) \rightarrow X(B) \xrightarrow{\delta} H_{fppf}^1(B, G) \rightarrow H_{fppf}^1(B, \mathrm{GL}_n).$$

As B is an LG ring, we have $H_{fppf}^1(B, \mathrm{GL}_n) = 1$ from the same references. Hence the torsor F is a pull-back of $\mathrm{GL}(\mathcal{E}) \rightarrow X$ at some element $x \in X(B)$.

Let U be an open retrocompact subspace of X that is schematically S -dense in X . We have the following commutative diagram

$$\begin{array}{ccc} U \times_{X, f_x} \mathrm{GL}_{n, B} & \longrightarrow & \mathrm{GL}_{n, B} \\ \downarrow & & \downarrow f_x \\ U_B & \longrightarrow & X_B \end{array}$$

The algebraic space $U \times_{X, f_x} \mathrm{GL}_{n, B}$ is representable by an open retrocompact subscheme V_x of $\mathrm{GL}_{n, B}$ which is S -dense according to base change property listed in 4.2. Since $\mathrm{GL}_{n, R}$ is S -dense in the affine space $\mathbf{V}(\mathrm{M}_n(R))$, V_x is S -dense in $\mathbf{V}(\mathrm{M}_n(R))$. According to [16, prop. 1.4], we have that $V_x(B) \neq \emptyset$. In other words, there exists $g \in \mathrm{GL}_n(B)$ such that $g \cdot x$ is in the image of $U(B)$. Since $\delta(x) = \delta(g \cdot x)$, F is a pull back of $\mathrm{GL}(\mathcal{E}) \rightarrow X$ at $g \cdot x \in U(B)$.

The reduction to the constant rank case is standard. Since the rank function is locally constant we have a decomposition $R = R_1 \times \cdots \times R_c$ such that \mathcal{E}_{R_i} has constant rank n_i for $i = 1, \dots, c$. The point is that each R_i is an LG ring and similarly each $B_i = B \otimes_R R_i$ satisfies the primitive criterion [16, Ex. 1.2.(b)]. Since $G(B) = G(B_1) \times \cdots \times G(B_c)$, the first case is enough to complete the proof.

(2) Let B be a R -ring satisfying the primitive criterion. Let U be an open retrocompact subspace of V/G which is S -schematically dense. We consider the G -torsor $P = V \times_{V/G} U$ over U and want to show that it is (4)-versal, that is, for a given G -torsor F over B to find $u \in U(B)$ such that $P_u \cong F$. The Chinese Remainder theorem implies that $V(B) \neq \emptyset$. We pick $v \in V(B)$, we can define a right G -equivariant morphism f_v from $\mathrm{GL}(\mathcal{E})$ to $\mathbf{V}(\mathcal{E})$ by

$$f_v(\sigma) = v \circ \sigma,$$

where σ is an element of $\mathrm{GL}(\mathcal{E})(C)$ and C is a B -algebra. We have then a commutative diagram of G -torsors

$$\begin{array}{ccc} \mathrm{GL}(\mathcal{E})_B & \xrightarrow{f_v} & V_B \\ q \downarrow & & q' \downarrow \\ \mathrm{GL}(\mathcal{E})_B / G_B & \xrightarrow{\bar{f}_v} & (V/G)_B. \end{array}$$

The point is that $\overline{f}_v^{-1}(U_B)$ is retrocompact and schematically B -dense in $X_B = \mathrm{GL}(\mathcal{E})_B/G_B$ according to the preliminaries of 4.2.

According to (1) applied to B , the G -torsor $\mathrm{GL}(\mathcal{E}) \rightarrow X$ is a (4)-versal torsor. There exists $x \in \overline{f}_v^{-1}(U_B)$ such that $q^{-1}(x) \cong F$. We put $u = \overline{f}_v(x)$ and conclude that $P_u = q'^{-1}(u) \cong F$. Thus $V \rightarrow V/G$ is also a (4)-versal torsor. \square

In this paper we shall use only the next special case.

Lemma 4.4. *Let G be an S -group scheme that is finite locally free. Let \mathcal{E} be a G - R -module which is locally free of finite rank as R -module. Let $U \subset \mathbf{V}(\mathcal{E})$ be a G -stable retrocompact S -dense open subscheme such that G acts freely on U . Then U/G is representable by an S -scheme X and the quotient map $U \rightarrow X$ is a versal (4)-torsor.*

Proof. According to [8, III.2.6.1, Corollaire], U/G is representable by an S -scheme X and $U \rightarrow X$ is a G -torsor. Lemma 4.3 shows that $U \rightarrow X$ is a versal (4)-torsor. \square

Remark 4.5. In this setting, it is not clear whether (4)-versal torsors always exist.

4.4. Embeddings in quasi-split groups. In this section, we assume that S is the spectrum of a ring R satisfying the primitive criterion. Let G be a quasi-split reductive S -group scheme.

Let (B, T) be a Killing couple of G . We identify the Weyl group $W(T)$ with $N_G(T)/T$ and denote by $\mathbf{Dyn}(G)$ the Dynkin scheme (which is finite étale over S). We have a canonical isomorphism $\mathbf{Dyn}(\Phi(G, T)) \xrightarrow{\sim} \mathbf{Dyn}(G)$.

According to [31, XXIV.3.13], if G is simply connected, we have an isomorphism $T \xrightarrow{\sim} R_{\mathbf{Dyn}(G)/S}(\mathbb{G}_{m, \mathbf{Dyn}(G)})$. We denote by T^{reg} the open subscheme of regular elements of T . Then $W(T)$ stabilizes T^{reg} and acts freely on it. According to [8, III.2.6.1, Corollaire], the fppf sheaf $T^{reg}/W(T)$ is representable and the map $\pi : T^{reg} \rightarrow T^{reg}/W(T)$ is a $W(T)$ -torsor.

Denote by $G//G$ the adjoint quotient. We use now Chevalley's theorem [21] in this setting $u : T/W(T) \xrightarrow{\sim} G//G$. Away of type A_{2n} , we can deal with the Steinberg cross section $c : G//G \rightarrow G$ of the quotient map $G \rightarrow G//G$ [21, th. 5.7]; in that case we can associate to an element $x \in (T^{reg}/W(T))(S)$ the semisimple regular element $g = c(u(x)) \in G(S)$ giving rise to the maximal S -subtorus $T_x = C_G(g)$ of G . The following generalizes to the LG setting a result of Gille-Raghunathan [14, 26].

Theorem 4.6. *Let S be the spectrum of a ring R satisfying the primitive criterion. Let G be a quasi-split reductive S -group scheme. Let Ψ be a root data which is admissible for G and let $v \in \mathrm{Isomext}(G, \Psi)(S)$ be an orientation. Then $\mathfrak{E}(G, \Psi, v)(S) \neq \emptyset$.*

Remark 4.7. The statement holds also in the case S is the spectrum of a finite field. Thus is obvious since all homogeneous G -spaces have a k -rational point [32, III.2.4].

Proof. Using the same kind of reduction as in the proof of Lemma 4.3.(1) we can assume without loss of generality that G is of constant type. It is harmless to assume

furthermore that G is semisimple simply connected. Furthermore using the decomposition [31, XXIV.5.5] and the usual Weil restriction argument, we can assume that the geometric fibers of G are almost simple (note that a finite étale extension of R satisfies the primitive criterion in view of [16, Ex. 1.2.(b)]). Let (B, T) be a Killing couple of G . We denote by $W(T) = N_G(T)/T$ the Weyl group and by $\mathbf{Dyn}(G)$ the Dynkin scheme (which is finite étale).

Let c be the orientation coming from the natural inclusion $T \hookrightarrow G$, and denote by $u \in \underline{\text{Isomext}}(\Phi(G, T), \Psi)(S)$ the image of (c, v) under the morphism in Prop. 3.2.1. We consider the left W -torsor $J = \underline{\text{Isomint}}_u(\Phi(G, T), \Psi)$. According to Lemma 3.2, we have $\mathfrak{E}(G, \Psi, v) = J \wedge^W \mathfrak{E}(G, \Phi(G, T), c)$.

By Lemma 4.4, $\pi : T^{\text{reg}} \rightarrow T^{\text{reg}}/W$ is a (4)-versal (left) $W(T)$ -torsor. In particular there exists $x \in (T^{\text{reg}}/W)(S)$ such that $\pi^{-1}(x) \cong J$ (as left $W(T)$ -torsors).

Case 1. G is not of A_{2n} -type. By Steinberg cross section, there is a semisimple regular element $t \in G(S)$ mapped to x . The centralizer of t is a maximal torus T' of G over S . Let $c' \in \underline{\text{Isomext}}(G, \Phi(G, T'))(S)$ be the canonical orientation induced by the natural inclusion from T' to G . Denote by $u' \in \underline{\text{Isomext}}(\Phi(G, T), \Phi(G, T'))(S)$ be the image of (c, c') under the pairing in Prop. 3.2.1. By Prop. 3.2.3 the left $W(T)$ -torsors $\pi^{-1}(x)$ and $\underline{\text{Isomint}}_{u'}(\Phi(G, T), \Phi(G, T'))$ are isomorphic. It follows that $J \wedge^W \mathfrak{E}(G, \Phi(G, T), c) = \mathfrak{E}(G, \Psi, v)$ is isomorphic to $\pi^{-1}(x) \wedge^W \mathfrak{E}(G, \Phi(G, T), c) \cong \mathfrak{E}(G, \Phi(G, T'), c')$. As $\mathfrak{E}(G, \Phi(G, T'), c')(S) \neq \emptyset$, we conclude that $\mathfrak{E}(G, \Psi, v)(S)$ is not empty.

Case 2. G is of A_{2n} -type. The key point is to reduce to the case of type A_{2n+1} . Let R_{\sharp}/R be the étale quadratic algebra corresponding to the outer type of G . Then R_{\sharp}/R splits G and we denote by σ its canonical involution.

We consider the R_{\sharp} -module $V = (R_{\sharp})^{2n+1} = V_- \oplus R_{\sharp} \oplus V_+$ with $V_- = (R_{\sharp})^n$ and $V_+ = (R_{\sharp})^n$. It is equipped with the hermitian form $h(x, y) = \sigma(x_0) y_0 + \sum_{i=1, \dots, n} \sigma(x_{-i}) y_i$

with the notation $x = (x_i)_{i=-n, \dots, n}$. Denoting by \mathbf{H}_n the standard hyperbolic space of rank $2n$, we have an orthogonal decomposition $h = \mathbf{H}_n \perp \langle 1 \rangle$.

The unitary group scheme $U(V, h)$ is reductive and its derived group $SU(V, h)$ is a semisimple simply connected R -group scheme of type A_{2n+1} [16, prop. C.6]. We consider the cocharacter $\lambda : \mathbb{G}_{m, R} \rightarrow U(V, h)$ defined by $\lambda(t) = \text{diag}(t^{-n}, t^{-n+1}, \dots, t^{-1}, 1, t, \dots, t^{2n-1}, t^{2n})$. The associated parabolic R -subgroup $P = P_{U(V, h)}(\lambda)$ carries the Levi subgroup $C_{U(V, h)}(\lambda)$ which is the R -torus $E = U(V, h) \cap R_{R_{\sharp}/R}(\mathbb{G}_m^{2n+1}) \cong \mathbb{G}_m \times_R R_{R_{\sharp}/R}(\mathbb{G}_m^n)$ so that P is an R -Borel subgroup of $U(V, h)$. In view of [16, §2.12], the R -group scheme $U(V, h)$ is then quasi-split. By considering E , we see that the outer type of $SU(V, h)$ is $[R_{\sharp}] \in H^1(R, \mathbb{Z}/2\mathbb{Z})$ so that $G \cong SU(V, h)$. The group G carries then a Cartan involution defined over R , hence if we have an embedding with respect to one orientation, we have an embedding with respect to the other orientation. Hence it suffices to show that $\mathfrak{E}(G, \Psi)(R) \neq \emptyset$.

Since $U(V, h)$ and $G = SU(V, h)$ have same automorphism group, same scheme of tori and same isomorphism classes of root data, we can work with $\tilde{G} = U(V, h)$. Let \mathfrak{S}_i be the symmetric group of the set $\{1, \dots, i\}$. We have a natural injection from \mathfrak{S}_{2n+1} to \mathfrak{S}_{2n+2} and hence an injective homomorphism ι from $\mathfrak{S}_{2n+1} \times \mathbb{Z}/2\mathbb{Z}$ to $\mathfrak{S}_{2n+1} \times \mathbb{Z}/2\mathbb{Z}$ that maps $\mathbb{Z}/2\mathbb{Z}$ isomorphically to itself. The map ι induces a map $\iota_* : H_{\acute{e}t}^1(R, \mathfrak{S}_{2n+1} \times \mathbb{Z}/2\mathbb{Z}) \rightarrow H_{\acute{e}t}^1(R, \mathfrak{S}_{2n+2} \times \mathbb{Z}/2\mathbb{Z})$. Note that $\mathfrak{S}_{2n+1} \times \mathbb{Z}/2\mathbb{Z}$ is isomorphic to the Weyl group of the split root datum of GL_{2n} . Hence $H_{\acute{e}t}^1(R, \mathfrak{S}_{2n+1} \times \mathbb{Z}/2\mathbb{Z})$ classifies those root data that are étale locally isomorphic to the split root datum of $GL_{2n+1, R}$.

Thus $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ corresponds to some $[\alpha] \in H_{\acute{e}t}^1(R, \mathfrak{S}_{2n+1} \times \mathbb{Z}/2\mathbb{Z})$. We twist the split root datum of $GL_{2n+2, R}$ by $\iota_*(\alpha)$ and denote it by $\Psi' = (\mathcal{M}', \mathcal{M}'^\vee, \mathcal{R}', \mathcal{R}'^\vee)$. Let T and T' be the tori determined by \mathcal{M} and \mathcal{M}' respectively. Note that by our construction \mathcal{M}'^\vee can be written as $\mathcal{M}^\vee \oplus \mathcal{E}^\vee$ with \mathcal{E} is a *fppf* twisted form of \mathbb{Z} in such a way that R^\vee is injectively sent to a subset of R'^\vee . Fix such an isomorphism $\mathcal{M}'^\vee \simeq \mathcal{M}^\vee \oplus \mathcal{E}^\vee$. This in turns gives an isomorphism $T' \simeq T \times \mathbf{R}_{R_\sharp/R}^{(1)}(\mathbf{G}_m)$. We denote by T_0 the subtorus $1 \times \mathbf{R}_{R_\sharp/R}^{(1)}(\mathbf{G}_m) \subseteq T'$ (under the above isomorphism).

We put $V' = V \oplus R^\sharp$ and $h'((v, r), (w, s)) = h(v, w) - \sigma(r)s$. We have an orthogonal decomposition $(V', h') = \mathbf{H}(V_-) \oplus (R_\sharp^2, h'')$ where $h''((r_1, s_1), (r_2, s_2)) = \sigma(r_1)s_1 - \sigma(r_2)s_2$. Since $(1, 1)$ is an isotropic unimodular vector of h'' , we know that h'' is hyperbolic in view of [19, prop. 3.7.1]. Since R_\sharp is a LG ring, the locally free R_\sharp -modules of constant rank are free so that $h'' \cong \mathbf{H}_1$ and $h' \cong \mathbf{H}_{n+1}$. The R -group scheme $U(V', h')$ is then a quasi-split semisimple R -group scheme of type A_{2n+1} .

By our assumption, there is an orientation between Ψ and G . Hence Ψ and G are both become inner forms after base change to R . By our construction of Ψ' and G' , they are also both of inner form after base change to R . Hence there is an orientation v' between Ψ' and G' . Since G' is quasi-split of type A_{2n+1} , by Case 1 we have $\mathfrak{E}(G', \Psi', v')(R) \neq \emptyset$.

Choose $f \in \mathfrak{E}(G', \Psi', v')(R)$. Consider the subgroup $f(T_0)$ in G' . Let C be the centralizer $\underline{\text{Centr}}_{G'}(f(T_0))$, which is a Levi subgroup of G' . Clearly $f(T') \subseteq C$. Let V_0 be the maximal R -submodule of V fixed by $f(T_0)$. Namely, V_0 is the maximal R -submodule on which $f(T_0)$ acts trivially. Since the action of $f(T_0)$ is R_\sharp -linear, V_0 is furthermore an R_\sharp -submodule of V . It can be checked étale locally that $V = V_0 \perp V_0^\perp$ and that the R_\sharp -module V_0 is locally free of rank $2n + 1$. Let h_0 be the restriction of h' on V_0 .

Claim 4.4.1. $h_0 \cong \mathbf{H}_n \perp \langle a \rangle$ for some $a \in R^\times$.

Since R_\sharp is a LG ring, the R_\sharp -module V_0 is free of rank $2n + 1$ and V_0^\sharp is free of rank 1. We have then an orthogonal decomposition $\mathbf{H}_{n+1} = h' = h_0 \perp \langle -a \rangle$ for some $a \in R^\times$. The same argument as above shows that $\langle a \rangle \perp \langle -a \rangle \cong \mathbf{H}_1$ so that $h_0 \perp \mathbf{H} \cong \mathbf{H}_{n+1} \perp \langle a \rangle$. The cancellation property [16, App. 5.5] yields that $h_0 \cong \mathbf{H}_n \perp \langle a \rangle$ as desired. The Claim is established.

The same reasoning as above shows that $U(V_0, h_0)$ is the quasi-split R -group scheme of type A_{2n} of Tits invariant $[R_{\sharp}] \in H^1(R, \mathbb{Z}/2\mathbb{Z})$. In other words we have $U(V_0, h_0) \simeq U(V, h) = \tilde{G}$.

Clearly C stabilizes V' and hence induces an action of $C/f(T_0)$ on (V', h') . This gives a group homomorphism of $C/f(T_0)$ to $U(V_0, h_0)$, which is an isomorphism at each geometric point \bar{s} of S . Therefore $C/f(T_0) \simeq U(V_0, h_0)$ and f gives an embedding of $T \simeq T'/T_0$ in \tilde{G} with respect to Ψ . \square

Corollary 4.8. *Let S be a scheme that is the spectrum of a ring R satisfying the primitive criterion. Let G be a quasi-split reductive S -group scheme. Let X be an S -scheme satisfying one of the following conditions:*

- (i) S is noetherian;
- (ii) X is locally of finite type over S .

We assume that X is a left homogeneous space under G and that for each point $s \in S$, the stabilizers of $X_{\bar{s}}$ are maximal tori of $G_{\bar{s}}$. Then $X \cong G/T$ where T is a maximal S -torus of G .

Proof. It follows immediately from Theorem 3.4 and Theorem 4.6. \square

Remark 4.9. The statement holds also in the case S is the spectrum of a finite field in view of Remark 4.7.

4.5. An application to the local-global principle for embeddings. In this section we assume that k is a global field with characteristic not 2. Let K be a field, A be a central simple algebra over K with involution τ , and E be an étale algebra over K with involution σ . Suppose that $\tau|_K = \sigma|_K$ and that k is the field of invariants K^τ .

Let $\dim_L A = n^2$. Assume that $\dim_K E = n$, and that if $K \neq k$, then $\dim_k E^\sigma = n$. If $K = k$, then we assume that $\dim_k E^\sigma = \lfloor \frac{n+1}{2} \rfloor$.

The local-global principle for embeddings of K -algebras with involution (E, σ) into (A, τ) is first considered in [25]. One can associate a root datum Ψ to (E, σ) and a reductive group $U(A, \tau)^\circ$ to (A, τ) , where $U(A, \tau)^\circ$ is the connected component of the unitary group $U(A, \tau)$. (See [20] §1.3.2.) In [20] §1.3.2 and [2] §1.2 and 1.3, it is shown that there is a bijection between the embeddings of algebras with involutions over k and the embeddings of corresponding root data into reductive groups.

Combining with Theorem 4.6, we have the following theorem.

Theorem 4.5.1. *Let (E, σ) and (A, τ) be as above. If $U(A, \tau)^\circ$ is quasi-split, then the local-global principle holds for the K -algebra embeddings of (E, σ) into (A, τ) .*

Proof. Let Ψ be the root datum associated to (E, σ) ([20] §1.3.2). Let $G = U(A, \tau)^\circ$. By Theorem 4.6, it suffices to show that $\text{Isomext}(G, \Psi)(k) \neq \emptyset$.

First we consider the case where $K \neq k$. In this case, we can always fix an orientation by [20] Theorem 1.15 (2) and Remark 1.16.

Next we consider the case where $K = k$. If τ is symplectic involution or τ is orthogonal with n odd, then G is of type $C_{n/2}$ or $B_{(n+1)/2}$. Then there is always an orientation between Ψ and G .

Consider the case that τ is orthogonal with n even. Suppose there is an embedding η_v of $(E \times_k k_v, \sigma \otimes \text{id}_{k_v})$ into $(A \times_k k_v, \sigma \otimes \text{id}_{k_v})$ for all places v of k . Then η_v gives an isomorphism between the discriminant $\Delta(E)$ of E and the center $Z(A, \tau)$ of the Clifford algebra of (A, τ) over k_v (see [3] §2.3). As they are both quadratic étale algebras over k , this means that $\Delta(E)$ is isomorphic to $Z(A, \tau)$ over k . By [2] Proposition 1.3.1, an isomorphism between $\Delta(E)$ and $Z(A, \tau)$ gives an orientation between Ψ and G .

Hence for $K = k$, we can always fix an orientation between Ψ and G . □

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