

# QUANTIZATION ON GROUPS AND GÅRDING INEQUALITY

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ABSTRACT. In this paper, we introduce Wick’s quantization on groups and discuss its links with Kohn-Nirenberg’s one. By quantization, we mean an operation that associates an operator to a symbol. The notion of symbols for both quantizations is based on representation theory via the group Fourier transform and the Plancherel theorem. As an application, we give a simple proof of Gårding inequalities for three globally symbolic pseudo-differential calculi on groups.

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## 1. INTRODUCTION

On  $\mathbb{R}^n$ , the links between the Kohn-Nirenberg and Wick quantizations provide a simple proof of Gårding inequalities; this is briefly sketched in Appendix A for the Hörmander calculus on  $\mathbb{R}^n$  while a reference for the semi-classical is for instance Jean-Marc Bouclet’s lecture notes [2], see also [26, 20]. In this paper, we show how these ideas extend readily to groups.

We start by explaining that, as for the Kohn-Nirenberg quantization, the definition of the Wick quantization extend naturally to groups that satisfy some hypotheses allowing for the definition of the group Fourier transform (based on representation theory) and the associated Plancherel

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theorem. Using this observation, we then prove Gårding inequalities for three globally symbolic pseudo-differential calculi on compact and graded nilpotent Lie groups. This topic, that is, Gårding inequalities for global pseudo-differential calculi on groups, has been the subject of many papers in recent years, see e.g. [14, 22, 3, 4].

Our results regarding Gårding inequalities are summarised in the three following theorems, although their statement will use the notation for the settings and the calculi recalled later on in the paper. Our approach provides a simple proof of strong Gårding inequalities, that is, with gain of one derivative, in a context where the symbols are not scalar-valued but operator-valued. Moreover, these Gårding inequalities are sharp in the sense that the lower bound involves only one term (here, given with gain of one derivative).

The first inequality is set on compact Lie groups and considers the symbolic pseudo-differential calculus proposed in [21, 23], studied in [10] and briefly recalled in Section 3.1. We obtain the following Gårding inequality.

**Theorem 1.1.** *Let  $G$  be a connected compact Lie group. Let  $m \in \mathbb{R}$ . Assume that the symbol  $\sigma \in S^{2m+1}(G)$  satisfies the elliptic condition  $\sigma \geq c(\text{id} + \widehat{\mathcal{L}})^{\frac{2m+1}{2}}$  for some constant  $c > 0$ . Then there exists a constant  $C > 0$  such that*

$$\forall f \in C^\infty(G), \quad \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -C\|f\|_{H^m(G)}^2.$$

Above, the spaces  $H^m(G)$  denote the usual Sobolev spaces defined on any compact manifold, here  $G$ , while the definitions of the symbol classes  $S^m(G)$  and the Laplace-Beltrami operator  $\mathcal{L}$  as well as its Fourier transform are recalled in Section 3.1. We also discuss below what the method of proof gives regarding  $(\rho, \delta)$ -generalisation of the latter Gårding inequality (see Theorem 3.6).

The next result concerns the symbolic pseudo-differential calculus on a graded Lie group  $G$  [12, 7], briefly recalled in Section 4.2. The Sobolev space  $L_s^2(G)$  will be the ones adapted to this setting [12, 13]. We obtain the following Gårding inequality.

**Theorem 1.2.** *Let  $G$  be a graded nilpotent Lie group. Let  $m \in \mathbb{R}$ . Assume that  $\sigma \in S^{2m+1}(G)$  satisfies the elliptic condition  $\sigma \geq c(\text{id} + \widehat{\mathcal{R}})^{\frac{2m+1}{\nu}}$  for some constant  $c > 0$  where  $\mathcal{R}$  is a positive Rockland operator of homogeneous degree  $\nu$ . Then there exists a constant  $C > 0$  such that*

$$\forall f \in C_c^\infty(G), \quad \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -C\|f\|_{L_m^2(G)}^2.$$

We also discuss below what the method of proof gives regarding  $(\rho, \delta)$ -generalisation of the latter Gårding inequality (see Theorem 4.3).

Still in the context of graded Lie groups, our method is particularly adapted to the semi-classical counter-part of Theorem 1.2:

**Theorem 1.3.** *Let  $\sigma \in \mathcal{A}_0$ , that is, the symbol  $\sigma$  is a smoothing symbol with  $x$ -compact support. If  $\sigma$  is non-negative, then there exists a constant  $C > 0$  such that*

$$\forall f \in L^2(G), \quad \forall \varepsilon \in (0, 1], \quad \Re(\text{Op}_\varepsilon^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -C\varepsilon\|f\|_{L^2(G)}^2.$$

The semi-classical calculus in this setting [8, 9] is recalled in Section 5.1, as well as the definitions for  $\mathcal{A}_0$  and  $\text{Op}_\varepsilon^{\text{KN}}$ .

On a compact Lie group  $G$ , the pseudo-differential calculus mentioned above with  $\rho > \delta$  and  $\rho \geq 1 - \delta$  coincides with Hörmander's pseudo-differential calculus defined via charts on the compact Lie group  $G$  viewed as a compact manifold [10, 23]. However, the notion of symbols are not the same in these two calculi: the one presented here or in [10, 21, 23] is global and based on the representations of the group. For a graded nilpotent Lie group  $G$ , the pseudo-differential calculus mentioned above coincides with the (global) Hörmander calculus only when  $G$  is abelian, that is,

only when  $G$  is the abelian group  $(\mathbb{R}^n, +)$  with  $n = \dim G$ . Otherwise, although a graded nilpotent Lie group is globally diffeomorphic to  $\mathbb{R}^n$  as a manifold, the calculi will not be comparable. At this point, we ought to clarify what we mean by pseudo-differential calculus on a smooth manifold  $M$  in this paper:

**Definition 1.4.** For each  $m \in \mathbb{R}$ , let  $\Psi^m(M)$  be a given Fréchet space of continuous operators  $\mathcal{D}(M) \rightarrow \mathcal{D}(M)$ . We say that the space  $\Psi^\infty(M) := \cup_m \Psi^m(M)$  form a *pseudo-differential calculus* when it is an algebra of operators satisfying:

- (1) The continuous inclusions  $\Psi^m(M) \subset \Psi^{m'}(M)$  hold for any  $m \leq m'$ .
- (2)  $\Psi^\infty(M)$  is an algebra of operators. Furthermore if  $T_1 \in \Psi^{m_1}(M)$ ,  $T_2 \in \Psi^{m_2}(M)$ , then  $T_1 T_2 \in \Psi^{m_1+m_2}(M)$ , and the composition is continuous as a map  $\Psi^{m_1}(M) \times \Psi^{m_2}(M) \rightarrow \Psi^{m_1+m_2}(M)$ .
- (3)  $\Psi^\infty(M)$  is stable under taking the adjoint. Furthermore if  $T \in \Psi^m(M)$  then  $T^* \in \Psi^m(M)$ , and taking the adjoint is continuous as a map  $\Psi^m(M) \rightarrow \Psi^m(M)$ .

The paper is organised as follows. We start with recalling the definition of the Kohn-Nirenberg quantization on groups and introducing Wick's (Section 2). Then we show that a link between these two quantizations in the symbolic calculi provides a proof of Gårding inequalities in the cases of compact Lie groups  $G$  (Section 3), and of graded nilpotent Lie groups  $G$  (Section 4). In the latter case, we also study the semi-classical analogue in Section 5. In Appendix A, we develop the same strategy of proof in the Euclidean case; to our knowledge, the proof which is the closest to ours is Folland's one [17, Chapter 2, Section 6].

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## 2. QUANTIZATIONS ON GROUPS

In this section, we discuss two quantizations of operators on groups that are based on the group Fourier transform and the associated Plancherel theorem. These latter notions require some hypotheses on the group we now list. The group  $G$  is a separable locally compact group. We assume that it is unimodular, that is, its left (resp. right) Haar measures are also right (resp. left) invariant. We also assume that it is of type I. The paper may be read without understanding these technical hypotheses. It suffices to know that they ensure that the Plancherel theorem holds, and that they are naturally satisfied on Lie groups that are compact or nilpotent.

### 2.1. Fourier analysis.

2.1.1. *The dual set.* Recall that a (unitary) *representation*  $(\mathcal{H}_\pi, \pi)$  of  $G$  is a pair consisting in a Hilbert space  $\mathcal{H}_\pi$  and a group morphism  $\pi$  from  $G$  to the set of unitary transforms on  $L^2(\mathcal{H}_\pi)$ . In this paper, the representations will always be assumed (unitary) strongly continuous, and their Hilbert spaces separable. A representation is said to be *irreducible* if the only closed subspaces of  $\mathcal{H}_\pi$  that are stable under  $\pi$  are  $\{0\}$  and  $\mathcal{H}_\pi$  itself. Two representations  $\pi_1$  and  $\pi_2$  are equivalent if there exists a unitary transform  $\mathbb{U}$  called an *intertwining map* that sends  $\mathcal{H}_{\pi_1}$  on  $\mathcal{H}_{\pi_2}$  with

$$\pi_1 = \mathbb{U}^{-1} \circ \pi_2 \circ \mathbb{U}.$$

The *dual set*  $\widehat{G}$  is obtained by taking the quotient of the set of irreducible representations by this equivalence relation. We may still denote by  $\pi$  the elements of  $\widehat{G}$  and we keep in mind that different representations of the class are equivalent through intertwining operators.

2.1.2. *Fixing a Haar measure.* We fix a Haar measure that we denote  $dx$  when the variable of integration is  $x \in G$ , or  $dy$  if the variable is  $y$ .

The non-commutative convolution is given via

$$(f_1 * f_2)(x) := \int_G f_1(y) f_2(y^{-1}x) dy, \quad x \in G.$$

for  $f_1, f_2 \in \mathcal{C}_c(G)$ ; here  $\mathcal{C}_c(G)$  denotes the space of continuous complex-valued functions on  $G$  with compact support.

2.1.3. *The Fourier transform.* The *Fourier transform* of an integrable function  $f \in L^1(G)$  at a representation  $\pi$  of  $G$  is the operator acting on  $\mathcal{H}_\pi$  via

$$\widehat{f}(\pi) := \mathcal{F}(f)(\pi) := \int_G f(z) (\pi(z))^* dz.$$

Note that if  $f_1, f_2 \in \mathcal{C}_c(G)$  then

$$(2.1) \quad \widehat{f_1 * f_2} = \widehat{f_2} \widehat{f_1}.$$

If  $\pi_1, \pi_2$  are two equivalent representations of  $G$  with  $\pi_1 = \mathbb{U}^{-1} \circ \pi_2 \circ \mathbb{U}$  for some intertwining operator  $\mathbb{U}$ , then

$$\mathcal{F}(f)(\pi_1) = \mathbb{U}^{-1} \circ \mathcal{F}(f)(\pi_2) \circ \mathbb{U}.$$

Hence, this defines the measurable field of operators  $\{\mathcal{F}(f)(\pi), \pi \in \widehat{G}\}$  modulo equivalence. Here, the unitary dual  $\widehat{G}$  is equipped with its natural Borel structure, and the equivalence comes from quotienting the set of irreducible representations of  $G$  together with understanding the resulting fields of operators modulo intertwiners.

2.1.4. *The Plancherel Theorem.* Here, we recall the Plancherel Theorem due to Dixmier [6, Ch. 18]. Among other results, it states the existence and uniqueness of the *Plancherel measure*, that is, the positive Borel measure  $\mu$  on  $\widehat{G}$  such that the Plancherel formula

$$(2.2) \quad \int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\widehat{f}(\pi)\|_{HS(\mathcal{H}_\pi)}^2 d\mu(\pi),$$

holds for any  $f \in \mathcal{C}_c(G)$ . Here  $\|\cdot\|_{HS(\mathcal{H}_\pi)}$  denotes the Hilbert-Schmidt norm on  $\mathcal{H}_\pi$ . This implies that the group Fourier transform is a unitary map from  $L^1(G) \cap L^2(G)$  equipped with the norm of  $L^2(G)$  to the Hilbert space

$$L^2(\widehat{G}) := \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{H}_\pi^* d\mu(\pi).$$

We identify  $L^2(\widehat{G})$  with the space of  $\mu$ -square integrable Hilbert-Schmidt fields on  $\widehat{G}$ ; its Hilbert norm and scalar products are then given by

$$\begin{aligned} \|\tau\|_{L^2(\widehat{G})}^2 &= \int_{\widehat{G}} \|\tau(\pi)\|_{HS(\mathcal{H}_\pi)}^2 d\mu(\pi), \quad \tau \in L^2(\widehat{G}), \\ (\tau_1, \tau_2)_{L^2(\widehat{G})} &= \int_{\widehat{G}} \text{tr}_{\mathcal{H}_\pi}(\tau_1(\pi) \tau_2(\pi)^*) d\mu(\pi), \quad \tau_1, \tau_2 \in L^2(\widehat{G}). \end{aligned}$$

Here  $\text{tr}_{\mathcal{H}_\pi}$  denotes the trace of operators on the Hilbert space  $\mathcal{H}_\pi$ . The group Fourier transform  $\mathcal{F}$  extends unitarily from  $L^2(G)$  onto  $L^2(\widehat{G})$ .

We denote by  $L^\infty(\widehat{G})$  the space of measurable fields (modulo equivalence) of bounded operators  $\sigma = \{\sigma(\pi) \in \mathcal{L}(\mathcal{H}_\pi) : \pi \in \widehat{G}\}$  on  $\widehat{G}$  such that

$$\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma(\pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}$$

is finite; here the supremum refers to the essential supremum with respect to the Plancherel measure  $\mu$  of  $\widehat{G}$ . In fact,  $L^\infty(\widehat{G})$  is naturally a Banach space and moreover a von Neumann algebra, sometimes called the von Neumann algebra of the group  $G$ . It acts naturally on  $L^2(\widehat{G})$  by composition on the left:

$$(\sigma\tau)(\pi) = \sigma(\pi) \tau(\pi), \quad \pi \in \widehat{G}, \quad \sigma \in L^\infty(\widehat{G}) \text{ and } \tau \in L^2(\widehat{G}),$$

(it also acts on the right) and this action is continuous

$$\|\sigma\tau\|_{L^2(\widehat{G})} \leq \|\sigma\|_{L^\infty(\widehat{G})} \|\tau\|_{L^2(\widehat{G})}.$$

Dixmier's Plancherel theorem implies that  $L^\infty(\widehat{G})$  is isomorphic to the von Neumann algebra  $\mathcal{L}(L^2(G))^G$  of linear bounded operators on  $G$  that are invariant under left translations. The isomorphism is given by the fact that the the Fourier multiplier with symbol  $\sigma$ , i.e. the operator  $f \mapsto \mathcal{F}^{-1}(\sigma\widehat{f})$ , is an operator in  $\mathcal{L}(L^2(G))^G$ .

Note that  $\mathcal{FL}^1(G) \subseteq L^\infty(\widehat{G})$  with

$$\forall f \in L^1(G), \quad \|\widehat{f}\|_{L^\infty(\widehat{G})} \leq \|f\|_{L^1(G)}.$$

**2.2. The Kohn-Nirenberg quantization.** In this section, we recall some results related to the symbolic quantization on groups introduced by Michael Taylor [24]. When  $G$  is the abelian group  $\mathbb{R}^n$ , this is the quantization often used in the field of Partial Differential Equations and called the Kohn-Nirenberg quantization or classical quantization [19, 1]. We keep this vocabulary in the group case.

**2.2.1. The space  $L^2(G \times \widehat{G})$ .** We may identify the tensor product  $L^2(G) \otimes L^2(\widehat{G})$  with the space denoted by  $L^2(G \times \widehat{G})$  of measurable fields  $\tau = \{\tau(x, \pi) \in HS(\mathcal{H}_\pi) : (x, \pi) \in G \times \widehat{G}\}$  of Hilbert-Schmidt operators (up to equivalence) such that the quantities

$$\|\tau\|_{L^2(G \times \widehat{G})}^2 := \int_{G \times \widehat{G}} \|\tau(x, \pi)\|_{HS(\mathcal{H}_\pi)}^2 dx d\mu(\pi)$$

are finite. It is naturally a separable Hilbert space with norm  $\|\cdot\|_{L^2(G \times \widehat{G})}$  and scalar product given by

$$(\tau_1, \tau_2)_{L^2(G \times \widehat{G})} = \int_{G \times \widehat{G}} \text{tr}_{\mathcal{H}_\pi} (\tau_1(x, \pi) \tau_2(x, \pi)^*) dx d\mu(\pi), \quad \sigma_1, \sigma_2 \in L^2(G \times \widehat{G}).$$

By the Plancherel theorem, the Hilbert space  $L^2(G \times \widehat{G})$  and  $L^2(G \times G)$  are isomorphic via the Fourier transform in the second variable:

$$L^2(G \times G) \longrightarrow L^2(G \times \widehat{G}), \quad \kappa \longmapsto (\text{id} \otimes \mathcal{F})\kappa.$$

In other words, any  $\tau \in L^2(G \times \widehat{G})$  may be written as

$$\tau(x, \pi) = \widehat{\kappa}_{\tau, x}(\pi)$$

for a unique function  $\kappa_\tau : (x, y) \mapsto \kappa_{\tau, x}(y) = \kappa(x, y)$  in  $L^2(G \times G)$ .

**2.2.2. The quantization  $\text{Op}^{\text{KN}}$  on  $L^2(G \times \widehat{G})$ .** For any  $f \in \mathcal{C}_c(G)$ , the symbol

$$\tau\widehat{f} := \{\tau(x, \pi)\pi(f) : (x, \pi) \in G \times \widehat{G}\}$$

is measurable on  $G \times \widehat{G}$  and satisfies

$$\|\tau\widehat{f}\|_{L^2(G \times \widehat{G})} \leq \|\tau\|_{L^2(G \times \widehat{G})} \|\widehat{f}\|_{L^\infty(\widehat{G})}.$$

Hence  $\tau\widehat{f} \in L^2(G \times \widehat{G})$  and we can define  $(\text{id} \otimes \mathcal{F}^{-1})(\tau\widehat{f}) \in L^2(G \times G)$ . By (2.1), we have:

$$(\text{id} \otimes \mathcal{F}^{-1})(\tau\widehat{f})(x, z) = \int_G f(y) \kappa_{\tau, x}(y^{-1}z) dy = \int_G f(zy^{-1}) \kappa_{\tau, x}(y) dy = f * \kappa_x(z).$$

As  $f \in \mathcal{C}_c(G)$ ,  $(\text{id} \otimes \mathcal{F}^{-1})(\tau \widehat{f})(x, z)$  is in fact continuous in  $z$  and it makes sense to define:

$$(2.3) \quad \text{Op}^{\text{KN}}(\tau)f(x) := (\text{id} \otimes \mathcal{F}^{-1})(\tau \widehat{f})(x, x) = f * \kappa_{\tau, x}(x), \quad f \in \mathcal{C}_c(G), x \in G.$$

It follows from the formula above that the integral kernel of  $\text{Op}^{\text{KN}}(\tau)$  is given by

$$G \times G \ni (x, y) \longmapsto \kappa_{\tau, x}(y^{-1}x).$$

Hence, the operator  $\text{Op}^{\text{KN}}(\tau)$  extends uniquely into a Hilbert-Schmidt operator on  $L^2(G)$  with norm

$$\|\text{Op}^{\text{KN}}(\tau)\|_{HS(L^2(G))} = \|\kappa_{\tau}\|_{L^2(G \times G)} = \|\tau\|_{L^2(G \times \widehat{G})}$$

Consequently,  $\text{Op}^{\text{KN}}$  is an isometry from  $L^2(G \times \widehat{G})$  onto  $HS(L^2(G))$ .

**2.2.3. Extension of  $\text{Op}^{\text{KN}}$  to  $\mathcal{C}(G, \mathcal{FL}^1(G))$ .** Clearly, we can extend naturally  $\text{Op}^{\text{KN}}$  via (2.3) to  $\mathcal{C}(G, \mathcal{FL}^1(G))$ , that is, to the symbols  $\sigma$  of the form  $\sigma(x, \pi) = \widehat{\kappa}_x(\pi)$  with convolution kernel  $\kappa \in \mathcal{C}(G, L^1(G))$ . By injectivity of the Fourier transform, the two possible definitions of  $\text{Op}^{\text{KN}}$  on symbols in  $L^2(G \times \widehat{G})$  and  $\mathcal{C}(G, \mathcal{FL}^1(G))$  coincide. Note that  $\text{Op}^{\text{KN}}(\sigma)$  with  $\sigma \in \mathcal{C}(G, \mathcal{FL}^1(G))$  will act on  $\mathcal{C}_c(G)$  and the Young convolution inequality implies the following estimate for the operator norm as operators on  $L^2(G)$ .

**Lemma 2.1.** *If  $\sigma \in \mathcal{C}(G, \mathcal{FL}^1(G))$  then*

$$\|\text{Op}^{\text{KN}}(\sigma)\|_{\mathcal{L}(L^2)} \leq \int_G \sup_{x \in G} |\kappa_x(y)| dy$$

*Proof.* Let  $\kappa \in \mathcal{C}(G, L^1(G))$  and  $f \in \mathcal{C}(G)$ . We have

$$|f * \kappa_x(x)| \leq |f| * \sup_{x' \in G} |\kappa_{x'}|(x),$$

so the Young convolution inequality yields

$$\sqrt{\int_G |f * \kappa_x(x)|^2 dx} \leq \| |f| * \sup_{x' \in G} |\kappa_{x'}| \|_{L^2(G)} \leq \|f\|_{L^2(G)} \|\sup_{x' \in G} |\kappa_{x'}|\|_{L^1(G)}.$$

□

If  $\sigma \in \mathcal{C}(G, L^1(G))$ , we define

$$(2.4) \quad \|\sigma\|_{\mathcal{A}_0} := \int_G \sup_{x \in G} |\kappa_x(y)| dy, \quad \sigma(x, \pi) = \mathcal{F}\kappa_x(\pi),$$

and we denote by  $\mathcal{C}_b(G, \mathcal{FL}^1(G))$  the subspace of  $\sigma \in \mathcal{C}(G, \mathcal{FL}^1(G))$  such that  $\|\sigma\|_{\mathcal{A}_0}$  is finite. We also denote by  $\mathcal{C}_b(G, L^1(G))$  the space of  $\kappa \in \mathcal{C}(G, L^1(G))$  such that  $\int_G \sup_{x \in G} |\kappa_x(y)| dy$  is finite.

**2.2.4. Extension of  $\text{Op}^{\text{KN}}$  via the inversion formula.** We can extend  $\text{Op}^{\text{KN}}$  to a larger space of symbols than  $\mathcal{C}(G, \mathcal{FL}^1(G))$  but acting on a smaller space of functions than  $\mathcal{C}_c(G)$  under some further technical assumptions. Indeed, let us consider the space  $\mathcal{C}(G, L^\infty(\widehat{G}))$  of symbols  $\sigma$  that are continuous maps from  $G$  to  $L^\infty(\widehat{G})$ . A symbol  $\sigma$  in  $\mathcal{C}(G, L^\infty(\widehat{G}))$  is naturally identified with a measurable field (up to equivalence) of operators  $\sigma = \{\sigma(x, \pi) \in \mathcal{L}(\mathcal{H}_\pi) : (x, \pi) \in G \times \widehat{G}\}$  satisfying conditions of continuity in  $x$  and boundedness in  $\pi$ . We also consider, when it exists, a space  $S$  of bounded, continuous and integrable functions satisfying:

- the space  $S \cap \mathcal{C}_c(G)$  is dense in  $L^2(G)$ , and
- for any  $f \in S$ , the operators  $\widehat{f}(\pi)$ ,  $\pi \in \widehat{G}$ , are trace-class and the following quantity is finite:

$$\int_{\widehat{G}} \text{tr}_{\mathcal{H}_\pi} |\widehat{f}(\pi)| d\mu(\pi) < \infty.$$

As a consequence of the Plancherel formula, the following inversion formula holds:

$$f(x) = \int_{\widehat{G}} \text{tr}_{\mathcal{H}_\pi} \left( \pi(x) \widehat{f}(\pi) \right) d\mu(\pi), \quad f \in S, \quad x \in G,$$

provided that  $G$  is amenable. We will not discuss here these technical assumptions (existence of  $S$  and amenability of  $G$ ), but just comment on the fact that they are naturally satisfied for compact or nilpotent Lie groups with  $S$  being the space of smooth functions with compact support; in the nilpotent case, we can take  $S$  to be the space of Schwartz functions. With the inversion formula,  $\text{Op}^{\text{KN}}$  extends to the quantization given for symbols  $\sigma$  in  $\mathcal{C}(G, L^\infty(\widehat{G}))$  by:

$$\text{Op}^{\text{KN}}(\sigma)f(x) = \int_{\widehat{G}} \text{tr}_{\mathcal{H}_\pi} \left( \pi(x) \sigma(x, \pi) \widehat{f}(\pi) \right) d\mu(\pi), \quad f \in S, \quad x \in G.$$

Naturally, this coincides with the quantization defined above for  $\sigma \in L^2(G \times \widehat{G})$  and for  $\sigma \in \mathcal{C}(G, \mathcal{FL}^1(G))$ .

**2.3. The Wick quantization.** Another natural symbolic quantization appears on the (locally compact, unimodular, type I) group  $G$ , in the same flavour as Wick's quantization (see [20]). For this, we start by defining the transformation  $\mathcal{B} = \mathcal{B}_a$  associated with a continuous, square-integrable and bounded function  $a$  satisfying  $\|a\|_{L^2} = 1$ .

2.3.1. *The transformation  $\mathcal{B}$ .* First, for each  $x \in G$ , we define the symbol  $\mathcal{F}_x$  on  $G \times \widehat{G}$  via

$$\mathcal{F}_x := \{ \mathcal{F}_{x,\pi}(y) : (y, \pi) \in G \times \widehat{G} \}, \quad \mathcal{F}_{x,\pi}(y) = a(yx^{-1})\pi(y)^*.$$

We check readily that  $\mathcal{F}_x \in \mathcal{C}(G, L^\infty(\widehat{G}))$  with

$$\sup_{y \in G} \|\mathcal{F}_x(y, \cdot)\|_{L^\infty(\widehat{G})} \leq \|a\|_{L^\infty(G)}.$$

We can now define the operator  $\mathcal{B} = \mathcal{B}_a$  on  $\mathcal{C}_c(G)$  via

$$\mathcal{B}[f](x, \pi) = \int_G f(y) \mathcal{F}_{x,\pi}(y) dy, \quad f \in \mathcal{C}_c(G), \quad (x, \pi) \in G \times \widehat{G}.$$

We observe that  $\mathcal{B}[f]$  is the field of operators on  $G \times \widehat{G}$  given by

$$(2.5) \quad \mathcal{B}[f](x, \pi) = \mathcal{F}(f a(\cdot x^{-1}))(\pi), \quad (x, \pi) \in G \times \widehat{G}.$$

This map has frame's properties:

**Proposition 2.2.** (1) *For any  $f \in \mathcal{C}_c(G)$ ,  $\mathcal{B}[f]$  defines an element of  $L^2(G \times \widehat{G})$  with norm*

$$\|\mathcal{B}[f]\|_{L^2(G \times \widehat{G})} = \|f\|_{L^2(G)}.$$

(2) *The map  $\mathcal{B}$  extends uniquely to an isometry from  $L^2(G)$  to  $L^2(G \times \widehat{G})$  for which we keep the same notation. Its adjoint map  $\mathcal{B}^* : L^2(G \times \widehat{G}) \rightarrow L^2(G)$  is given by*

$$\mathcal{B}^*[\tau](y) = \int_{G \times \widehat{G}} \text{tr}_{\mathcal{H}_\pi} (\tau(x, \pi) (\mathcal{F}_{x,\pi}(y))^*) dx d\mu(\pi), \quad \tau \in L^2(G \times \widehat{G}), \quad y \in G,$$

*in the sense that for any  $f \in L^2(G)$ ,*

$$(\mathcal{B}^*[\tau], f)_{L^2(G)} = \int_{G \times \widehat{G}} \text{tr}_{\mathcal{H}_\pi} \left( \tau(x, \pi) (\mathcal{F}(f a(\cdot x^{-1}))(\pi))^* \right) dx d\mu(\pi).$$

*If  $\tau = (\text{id} \otimes \mathcal{F})\kappa$ ,  $\kappa \in L^2(G \times G)$ , then*

$$\mathcal{B}^*[\tau](y) = \int_G \kappa_x(y) \bar{a}(yx^{-1}) dx = (\bar{a} * \kappa(\cdot))(y).$$

(3) We have  $\mathcal{B}^*\mathcal{B} = \text{id}_{L^2(G)}$  while  $\mathcal{B}\mathcal{B}^*$  is a projection on a closed subspace of  $L^2(G \times \widehat{G})$ .

*Proof.* From (2.5) and the Plancherel formula (2.2), we obtain

$$\int_{\widehat{G}} \|\mathcal{B}[f](x, \pi)\|_{HS(\mathcal{H}_\pi)}^2 d\mu(\pi) = \|f a(\cdot x^{-1})\|_{L^2(G)}^2, \quad x \in G.$$

Integrating against  $dx$  yields Part (1). Parts (2) and (3) follow readily.  $\square$

2.3.2. *The quantization*  $\text{Op}^{\text{Wick}}$ . We can now define the quantization

$$\text{Op}^{\text{Wick}} = \text{Op}^{\text{Wick}, a}, \quad \text{with} \quad \text{Op}^{\text{Wick}}(\sigma)f = \mathcal{B}^*\sigma\mathcal{B}[f], \quad f \in L^2(G), \sigma \in L^\infty(G \times \widehat{G}).$$

Here,  $L^\infty(G \times \widehat{G})$  denotes the space of symbols  $\sigma = \{\sigma(x, \pi) : (x, \pi) \in G \times \widehat{G}\}$  which are bounded in  $(x, \pi) \in G \times \widehat{G}$ , i.e. a measurable field of operators in  $(x, \pi) \in G \times \widehat{G}$  such that

$$\|\sigma\|_{L^\infty(G \times \widehat{G})} := \sup_{(x, \pi) \in G \times \widehat{G}} \|\sigma(x, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}$$

is finite, the supremum referring to the essential supremum for the measure  $dx d\mu$  on  $G \times \widehat{G}$ . This is naturally a Banach space (even a von Neumann algebra). Moreover, it acts naturally continuously on  $L^2(G \times \widehat{G})$  by left composition (and even right composition) with

$$\|\sigma\tau\|_{L^2(G \times \widehat{G})} \leq \|\sigma\|_{L^\infty(G \times \widehat{G})} \|\tau\|_{L^2(G \times \widehat{G})}, \quad \sigma \in L^\infty(G \times \widehat{G}), \tau \in L^2(G \times \widehat{G}).$$

This implies that the quantization  $\text{Op}^{\text{Wick}}$  is well defined:

**Proposition 2.3.** *The symbolic quantization*  $\text{Op}^{\text{Wick}}$  *is well defined on*  $L^\infty(G \times \widehat{G})$  *and satisfies*

$$\forall \sigma \in L^\infty(G \times \widehat{G}), \quad \|\text{Op}^{\text{Wick}}(\sigma)\|_{\mathcal{L}(L^2(G))} \leq \|\sigma\|_{L^\infty(G \times \widehat{G})}.$$

*Proof.* We have for any  $f \in L^2(G)$ :

$$\begin{aligned} \|\text{Op}^{\text{Wick}}(\sigma)f\|_{L^2(G)} &= \|\mathcal{B}^*\sigma\mathcal{B}[f]\|_{L^2(G)} \\ &\leq \|\mathcal{B}^*\|_{\mathcal{L}(L^2(G \times \widehat{G}), L^2(G))} \|\sigma\|_{L^\infty(G \times \widehat{G})} \|\mathcal{B}\|_{\mathcal{L}(L^2(G), L^2(G \times \widehat{G}))} \|f\|_{L^2(G)}. \end{aligned}$$

Since  $\mathcal{B}$  is an isometry, the operator norms of  $\mathcal{B}$  and  $\mathcal{B}^*$  are equal to 1.  $\square$

In the case of  $G = \mathbb{R}^n$  and  $a$  being a Gaussian function, we recognize  $\mathcal{B}$  as the Bargmann transform and  $\text{Op}^{\text{Wick}}$  as the Wick quantization [17, 20, 5]. This explains the notation above. We extend this vocabulary to the group case.

As an example, we check readily that  $\text{Op}^{\text{Wick}}(\text{id}) = \text{id}_{L^2(G)}$  where  $\text{id}$  is the symbol  $\text{id} = \{\text{id}_{\mathcal{H}_\pi}, (x, \pi) \in G \times \widehat{G}\}$ .

The following computation will allow for the comparison between the Wick and Kohn-Nirenberg quantizations on  $\mathcal{C}_b(G, \mathcal{FL}^1(G))$ ; note that a symbol in  $\mathcal{C}_b(G, \mathcal{FL}^1(G))$  is in  $L^\infty(G \times \widehat{G})$ .

**Lemma 2.4.** *If a symbol*  $\sigma$  *is in*  $\mathcal{C}_b(G, \mathcal{FL}^1(G))$ , *then*

$$\text{Op}^{\text{Wick}}(\sigma)f(x) = f * \kappa_x^{\text{Wick}}(x), \quad f \in \mathcal{C}_c(G), x \in G,$$

where  $\kappa_x^{\text{Wick}} \in \mathcal{C}_b(G, L^1(G))$  is given by:

$$\begin{aligned} \kappa_x^{\text{Wick}}(w) &= \int_G a(w^{-1}xz^{-1})\bar{a}(xz^{-1})\kappa_z(w)dz \\ &= \int_G a(w^{-1}z')\bar{a}(z')\kappa_{z'^{-1}x}(w)dz', \end{aligned}$$

and  $\kappa \in \mathcal{C}_b(G, L^1(G))$  is the convolution kernel of  $\sigma$  in the sense that  $\sigma(x, \pi) = \mathcal{F}\kappa_x(\pi)$ .

We will call  $\kappa^{\text{Wick}}$  the Wick convolution kernel of  $\sigma$ .



*Proof.* We check readily that  $\kappa^{\text{Wick}} \in \mathcal{C}_b(G, L^1(G))$  with

$$\int_G \sup_{x \in G} |\kappa_x^{\text{Wick}}(w)| dw \leq \int_G \sup_{x' \in G} |\kappa_{x'}(w)| dw \int_G |a|(w^{-1}z')|a|(z') dz' \leq \int_G \sup_{x' \in G} |\kappa_{x'}(w)| dw,$$

by the Cauchy-Schwartz inequality since  $\|a\|_{L^2(G)} = 1$ .

Let  $f \in \mathcal{C}_c(G)$  and  $x \in G$ . Property (2.1) yields

$$\sigma \mathcal{B}[f](x, \pi) = (\widehat{\kappa}_x \mathcal{F}(f a(\cdot x^{-1}))) (\pi) = \mathcal{F}((f a(x^{-1}\cdot)) * \kappa_x) (\pi)$$

so with Part 2 of Proposition 2.2, we obtain

$$\begin{aligned} \text{Op}^{\text{Wick}}(\sigma) f(x) &= \int_G (f a(\cdot z^{-1})) * \kappa_z(x) \bar{a}(xz^{-1}) dz \\ &= \int_{G \times G} f(y) a(yz^{-1}) \kappa_z(y^{-1}x) \bar{a}(xz^{-1}) dy dz, \end{aligned}$$

and we recognise  $f * \kappa_x^{\text{Wick}}(x)$ . □

**2.3.3. Some properties of  $\text{Op}^{\text{KN}}$  and  $\text{Op}^{\text{Wick}}$ .** In our definitions of the quantizations, we choose to act on the left by  $\tau$  in (2.3) or equivalently to place  $\kappa_x$  on the right of the convolution product in (2.3) while we made choices in the writing of  $\mathcal{F}_{x,\pi}$ . These choices imply that our quantization interact well with the left translations  $L_{x_0}$  by  $x_0$  on functions, i.e.  $L_{x_0} f(x) = f(x_0 x)$  for any function  $f$  defined on  $G$ , and also on symbols:  $L_{x_0} \sigma(x, \pi) = \sigma(x_0 x, \pi)$ . Indeed, we check readily that

$$(2.6) \quad \text{Op}^{\text{Wick},a}(L_{x_0} \sigma) = \text{Op}^{\text{Wick},L_{x_0}a}(\sigma) \quad \text{while} \quad \text{Op}^{\text{KN}}(L_{x_0} \sigma) = L_{x_0} \text{Op}^{\text{KN}}(\sigma) L_{x_0}^{-1}.$$

The Wick quantization  $\text{Op}^{\text{Wick}}$  has the advantage of preserving self-adjointness and of being naturally positive. Indeed, for any  $\sigma \in L^\infty(G \times \widehat{G})$ , we have

$$(\text{Op}^{\text{Wick}}(\sigma))^* = \mathcal{B}^* \sigma^* \mathcal{B} = \text{Op}^{\text{Wick}}(\sigma^*),$$

so if  $\sigma$  is self-adjoint in the sense that  $\sigma(x, \pi) = \sigma(x, \pi)^*$  for almost all  $(x, \pi) \in G \times \widehat{G}$ , then  $\text{Op}^{\text{Wick}}(\sigma)$  is self-adjoint. Moreover, if  $\sigma$  is a non-negative symbol in the sense that the operator  $\sigma(x, \pi)$  is bounded below by 0 for almost every  $(x, \pi) \in G \times \widehat{G}$ , then the corresponding operator acting on  $L^2(G \times \widehat{G})$  is also non-negative so

$$(2.7) \quad (\text{Op}^{\text{Wick}}(\sigma) f, f)_{L^2(G)} = (\sigma \mathcal{B}[f], \mathcal{B}[f])_{L^2(G \times \widehat{G})} \geq 0 \quad \sigma \in L^\infty(G \times \widehat{G}), \quad f \in S,$$

and  $\text{Op}^{\text{Wick}}(\sigma)$  is a non-negative operator on  $L^2(G)$ .

In general, the Kohn-Nirenberg quantization  $\text{Op}^{\text{KN}}$  will not be positive. However, weaker properties of positivity may be recovered in certain cases via Gårding inequalities in pseudo-differential calculi. The rest of this paper is devoted to showing Gårding inequalities in the case of graded nilpotent Lie groups and compact Lie groups.

### 3. GÅRDING INEQUALITY ON COMPACT LIE GROUPS

Here,  $G$  is a connected compact Lie group. Automatically, all the technical assumptions mentioned in Section 2 (locally compact, unimodular, type I, amenable) are satisfied. In this case, every irreducible representation is finite dimensional, the dual set  $\widehat{G}$  is discrete and the Plancherel measure is known explicitly:  $\mu(\{\pi\}) = d_\pi$  is the dimension of  $\pi \in \widehat{G}$ , so that we have the Plancherel formula:

$$\|f\|_{L^2(G)}^2 = \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{HS(\mathcal{H}_\pi)}^2.$$

A symbol is a family  $\sigma = \{\sigma(x, \pi) \in \mathcal{L}(\mathcal{H}_\pi) : (x, \pi) \in G \times \widehat{G}\}$  of finite dimensional linear maps parametrised by  $(x, \pi)$ , each acting on the (finite dimensional) space of the representation. We can define the Fourier transform not only of integrable functions, but also of any distributions.

**3.1. The pseudo-differential calculus.** In this section, we set some notation and recall briefly the global symbol classes defined on  $G$  together with some properties of the pseudo-differential calculus. We refer to [10] for more details.

We fix a basis  $X_1, \dots, X_n$  for the Lie algebra  $\mathfrak{g}$  of the group  $G$ . We keep the same notation for the associated left-invariant vector fields on  $G$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ , we set  $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ .

We fix a scalar product on  $\mathfrak{g}$  that is invariant under the adjoint action. The Laplace-Beltrami operator is the differential operator  $\mathcal{L} = -X_1^2 - \dots - X_n^2$  for any orthonormal basis  $X_1, \dots, X_n$  of  $\mathfrak{g}$ . Identified with an element of the universal enveloping algebra and keeping the same notation for a representation  $\pi$  of  $G$  and its infinitesimal counterpart for  $\mathfrak{g}$ ,  $\pi(\mathcal{L})$  is scalar when  $\pi$  is irreducible.

$$\widehat{\mathcal{L}}(\pi) := \pi(\mathcal{L}) = \lambda_\pi \text{id}_{\mathcal{H}_\pi},$$

with  $\lambda_\pi \geq 0$ . In fact,  $\lambda_1 = 0$  when  $\pi$  is the trivial representation 1, while  $\lambda_\pi > 0$  when  $\pi \neq 1$ .

Let  $m \in \mathbb{R}$  and  $1 \geq \rho \geq \delta \geq 0$ . A symbol  $\sigma$  is in  $S_{\rho, \delta}^m(G)$  when for any multi-indices  $\alpha, \beta$ , there exists  $C = C(\alpha, \beta) \geq 0$  such that

$$\|X^\beta \Delta^\alpha \sigma(x, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \leq C(1 + \lambda_\pi)^{\frac{m - \rho|\alpha| + \delta|\beta|}{2}};$$

above,  $\Delta^\alpha$  denotes the intrinsic difference operators (see [10, 11] for more details) or the RT-difference operators (see (3.1) below). This yields the following semi-norm

$$\|\sigma\|_{S_{\rho, \delta}^{m, a, b}} := \max_{|\alpha| \leq a, |\beta| \leq b} \sup_{(x, \pi) \in G \times \widehat{G}} (1 + \lambda_\pi)^{-\frac{m - \rho|\alpha| + \delta|\beta|}{2}} \|X^\beta \Delta^\alpha \sigma(x, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)}.$$

If  $(\rho, \delta) = (1, 0)$ , we simply write  $S^m(G) = S_{1, 0}^m(G)$ .

The following theorem summarises the main property of the classes of operators obtained by the  $\text{Op}^{\text{KN}}$ -quantization of the classes  $S_{\rho, \delta}^m(G)$ :

**Theorem 3.1.** *For each  $m \in \mathbb{R}$ , and  $1 \geq \rho \geq \delta \geq 0$ , equipped with the semi-norm  $\|\cdot\|_{S_{\rho, \delta}^{m, a, b}}$ ,  $S_{\rho, \delta}^m(G)$  becomes a Fréchet space. The space of operators  $\Psi_{\rho, \delta}^m(G) := \text{Op}^{\text{KN}}(S_{\rho, \delta}^m(G))$  inherits this structure of Fréchet space. The classes of operators  $\Psi_{\rho, \delta}^\infty(G) = \cup_{m \in \mathbb{R}} \Psi_{\rho, \delta}^m(G)$  is a pseudo-differential calculus in the sense of Definition 1.4. Moreover, we have the following properties:*

- (1) *The calculus  $\Psi_{\rho, \delta}^\infty(G)$  acts continuously on the Sobolev spaces  $H^s(G)$  in the following sense: if  $\sigma \in S_{\rho, \delta}^m(G)$  then  $\text{Op}^{\text{KN}}(\sigma)$  maps  $H^s(G)$  to  $H^{s-m}(G)$  for any  $s \in \mathbb{R}$ . Furthermore, the map  $\sigma \mapsto \text{Op}^{\text{KN}}(\sigma)$  is continuous  $S^m(G) \rightarrow \mathcal{L}(H^s(G), H^{s-m}(G))$ .*
- (2) *For any  $\sigma_1 \in S_{\rho, \delta}^{m_1}$  and  $\sigma_2 \in S_{\rho, \delta}^{m_2}$ , we have*

$$\text{Op}^{\text{KN}}(\sigma_1)\text{Op}^{\text{KN}}(\sigma_2) - \text{Op}^{\text{KN}}(\sigma_1\sigma_2) \in \Psi_{\rho, \delta}^{m_1+m_2-(\rho-\delta)}(G).$$

*Furthermore, the map  $(\sigma_1, \sigma_2) \mapsto \text{Op}^{\text{KN}}(\sigma_1)\text{Op}^{\text{KN}}(\sigma_2) - \text{Op}^{\text{KN}}(\sigma_1\sigma_2)$  is continuous  $S_{\rho, \delta}^{m_1} \times S_{\rho, \delta}^{m_2} \rightarrow \Psi_{\rho, \delta}^{m_1+m_2-(\rho-\delta)}(G)$ .*

- (3) *For any  $\sigma \in S_{\rho, \delta}^m$ , we have*

$$\text{Op}^{\text{KN}}(\sigma)^* - \text{Op}^{\text{KN}}(\sigma^*) \in \Psi_{\rho, \delta}^{m-(\rho-\delta)}(G).$$

*Furthermore, the map  $\sigma \mapsto \text{Op}^{\text{KN}}(\sigma)^* - \text{Op}^{\text{KN}}(\sigma^*)$  is continuous  $S_{\rho, \delta}^m \rightarrow \Psi_{\rho, \delta}^{m-(\rho-\delta)}(G)$ .*

Whenever it makes sense, this calculus coincides with Hörmander's:

**Theorem 3.2.** *When  $\rho > \delta$  and  $\rho \geq 1 - \delta$ ,  $\Psi_{\rho,\delta}^\infty(G)$  coincides with the Hörmander pseudo-differential calculus defined locally via charts.*

Any  $\sigma \in S_{\rho,\delta}^m(G)$  admits a distributional convolution kernel  $\kappa : z \mapsto \kappa_x(z) \in \mathcal{C}^\infty(G, \mathcal{D}'(G))$ , i.e.  $\sigma(x, \pi) = \widehat{\kappa}_x(\pi)$  and  $\text{Op}^{\text{KN}}(\sigma)f(x) = f * \kappa_x(x)$ ,  $f \in \mathcal{D}(G)$ . This allows for an application of the RT-difference operators. Let us recall that the RT-difference operator  $\Delta_q$  associated to  $q \in \mathcal{C}^\infty(G)$  is defined via:

$$(3.1) \quad \Delta_q \widehat{\kappa} = \mathcal{F}(q\kappa), \quad \kappa \in \mathcal{D}'(G).$$

The following property of RT-difference operators follows readily from [10, Section 5]:

**Lemma 3.3.** *Let  $m \in \mathbb{R}$  and  $1 \geq \rho \geq \delta \geq 0$ .*

- (1) *If  $q \in \mathcal{D}(G)$ , then the map  $\sigma \mapsto \Delta_q \sigma$  is continuous  $S_{\rho,\delta}^m(G) \rightarrow S_{\rho,\delta}^m(G)$ . Moreover,  $q \mapsto \Delta_q$  is continuous  $\mathcal{D}(G) \rightarrow \mathcal{L}(S_{\rho,\delta}^m(G))$ .*
- (2) *The map  $\sigma \mapsto \Delta_{q-q(e_G)} \sigma$  is continuous  $S_{\rho,\delta}^m(G) \rightarrow S_{\rho,\delta}^{m-(\rho-\delta)}(G)$  for any  $m \in \mathbb{R}$ . Moreover,  $q \mapsto \Delta_q$  is continuous  $\mathcal{D}(G) \rightarrow \mathcal{L}(S_{\rho,\delta}^m(G), S_{\rho,\delta}^{m-(\rho-\delta)}(G))$ .*

We observe that translating a symbol will not affect its membership to a  $S_{\rho,\delta}^m(G)$ -class or the action of a difference operator:

**Lemma 3.4.** *Let  $m \in \mathbb{R}$  and  $1 \geq \rho \geq \delta \geq 0$ . If  $x_0 \in G$ , then for any  $\sigma \in S_{\rho,\delta}^m(G)$ , the symbol  $L_{x_0} \sigma = \{\sigma(x_0 x, \pi) : (x, \pi) \in G \times \widehat{G}\}$  is in  $S_{\rho,\delta}^m(G)$  with*

$$\|L_{x_0} \sigma\|_{S_{\rho,\delta}^{m,a,b}} = \|\sigma\|_{S_{\rho,\delta}^{m,a,b}}.$$

Moreover, the map  $(x_0, \sigma) \mapsto L_{x_0} \sigma$  is continuous  $G \times S_{\rho,\delta}^m(G) \rightarrow S_{\rho,\delta}^{m+\delta}(G)$ .

*Proof.* Taylor estimate yields for any  $(x, \pi) \in G \times \widehat{G}$ ,

$$\|\sigma(x x_0, \pi) - \sigma(x, \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)} \lesssim_G |x_0| \sup_{x' \in G} \|X_{x'} \sigma(x', \pi)\|_{\mathcal{L}(\mathcal{H}_\pi)},$$

where  $|x_0|$  denotes the Riemann distance of  $x_0 \in G$  to the neutral element  $e_G$ . More generally for any semi-norm  $\|\cdot\|_{S_{\rho,\delta}^{m,a_0,b_0}}$ , we have

$$\|L_{x_0} \sigma - \sigma\|_{S_{\rho,\delta}^{m+\delta,a_0,b_0}} \lesssim_G |x_0| \|\sigma\|_{S_{\rho,\delta}^{m,a_0,b_0+1}}.$$

□

This implies readily the following property of convolution in the  $x$ -variable of a symbol:

**Corollary 3.5.** *If  $\sigma \in S_{\rho,\delta}^m(G)$  and  $\varphi \in \mathcal{C}^\infty(G)$ , then denoting by  $\varphi * \sigma$  the symbol*

$$\varphi * \sigma = \{\varphi * \sigma(x, \pi) : (x, \pi) \in G \times \widehat{G}\}, \quad \varphi * \sigma(x, \pi) = \int_G \varphi(z) \sigma(z^{-1} x, \pi) dz,$$

we have  $\varphi * \sigma \in S_{\rho,\delta}^m(G)$  with

$$\|\varphi * \sigma\|_{S_{\rho,\delta}^{m,a,b}} \leq \|\varphi\|_{L^1(G)} \|\sigma\|_{S_{\rho,\delta}^{m,a,b}}.$$

**3.2. Proof of the Gårding inequality.** Here, we prove the following  $(\rho, \delta)$ -generalisation of Theorem 1.1:

**Theorem 3.6.** *Let  $G$  be a connected compact Lie group. Let  $m \in \mathbb{R}$  and  $1 \geq \rho > \delta \geq 0$ . Assume that the symbol  $\sigma \in S_{\rho, \delta}^m(G)$  satisfies the positivity condition  $\sigma \geq 0$ . Then, for all  $\eta > 0$ , there exists a constant  $C_\eta > 0$  such that*

$$(3.2) \quad \forall f \in C^\infty(G), \quad \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -\eta \|f\|_{H^{\frac{m+\delta}{2}}(G)}^2 - C_\eta \|f\|_{H^{\frac{m-(\rho-\delta)}{2}}(G)}^2.$$

Moreover, if  $\delta = 0$  and  $\sigma \in S_{\rho, \delta}^m(G)$  satisfies the ellipticity condition  $\sigma \geq c(\text{id} + \widehat{\mathcal{L}})^{m/2}$ , that is,

$$\sigma(x, \pi) \geq c(1 + \lambda_\pi)^{\frac{m}{2}} \text{id}_{\mathcal{H}_\pi}, \quad (x, \pi) \in G \times \widehat{G},$$

for some constant  $c > 0$ . Then, there exists a constant  $C > 0$  such that

$$\forall f \in C^\infty(G), \quad \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -C \|f\|_{H^{\frac{m-(\rho-\delta)}{2}}(G)}^2.$$

When  $\delta = 0$  and  $\rho = 1$ , the second part of Theorem 3.6 is Theorem 1.1. Note that when  $\delta = 0$  and  $\rho \in (0, 1)$  the term involving  $\|f\|_{H^{\frac{m-\delta}{2}}(G)}^2$  in the right hand-side of the weak Gårding inequality (3.2) disappears.

The main step in our proof is the following analysis of the Wick quantization in the  $\Psi_{\rho, \delta}^\infty$ -calculus:

**Lemma 3.7.** *Here, we consider the Wick quantization  $\text{Op}^{\text{Wick}, a}$  with a smooth function  $a : G \rightarrow \mathbb{C}$ . Let  $m \in \mathbb{R}$  and  $1 \geq \rho \geq \delta \geq 0$ .*

- (1) *If  $\sigma \in S_{\rho, \delta}^m(G)$  with  $m \leq 0$ , then  $\sigma \in L^\infty(G \times \widehat{G})$  and  $\text{Op}^{\text{Wick}}(\sigma) \in \Psi_{\rho, \delta}^m(G)$ . Moreover, the map  $\sigma \mapsto \text{Op}^{\text{Wick}}(\sigma)$  is continuous  $S_{\rho, \delta}^m(G) \rightarrow \Psi_{\rho, \delta}^m(G)$ .*
- (2) *If  $\sigma \in S_{\rho, \delta}^m(G)$  with  $m \leq 0$ , then we have*

$$\text{Op}^{\text{Wick}}(\sigma) - \text{Op}^{\text{KN}}(|a|^2 * \sigma) \in \Psi_{\rho, \delta}^{m-(\rho-\delta)}(G).$$

Moreover, the map  $\sigma \mapsto \text{Op}^{\text{Wick}}(\sigma) - \text{Op}^{\text{KN}}(|a|^2 * \sigma)$  is continuous  $S_{\rho, \delta}^m(G) \rightarrow \Psi_{\rho, \delta}^{m-(\rho-\delta)}(G)$ .

*Proof of Lemma 3.7.* We may rephrase Lemma 2.4 as  $\text{Op}^{\text{Wick}}(\sigma) = \text{Op}^{\text{KN}}(\sigma^{\text{Wick}})$  with

$$\sigma^{\text{Wick}}(x, \pi) = \int_G \Delta_{q_z} \sigma(z^{-1}x, \pi) dz$$

where  $q_z(w) = a(w^{-1}z)\bar{a}(z)$ . By Lemma 3.4, Corollary 3.5 and Lemma 3.3 (1), this implies Part 1.

We observe that

$$\int_G \Delta_{q_z(0)} \sigma(z^{-1}x, \pi) dz = \int_G |a|^2(z) \sigma(z^{-1}x, \pi) dz = |a|^2 * \sigma(x, \pi).$$

Hence, Lemma 3.3 (2) implies Part 2. □

The choice of the functions  $a$  for the Wick quantization is at the core of our proof of the Gårding inequality. We do it in relation to an approximation of the identity. By an approximation of the identity on a compact Lie group  $G$ , we mean a family of functions  $\varphi_t \in \mathcal{D}(G)$ ,  $t > 0$ , satisfying  $\varphi_t(z) \geq 0$ ,  $\int_G \varphi_t(z) dz = 1$  for any  $t > 0$  and for any neighbourhood  $V$  of the neutral element  $e_G$ ,  $\lim_{t \rightarrow 0} \int_{z \in V} \varphi_t(z) dz = 1$ . We then have

$$\lim_{t \rightarrow 0} \max_{x \in G} |\psi(x) - \varphi_t * \psi(x)| = 0,$$

for any continuous function  $\psi : G \rightarrow \mathbb{C}$ ; this extends to continuous functions  $\psi$  that are Banach valued. From this, we obtain the properties regarding the approximation of the identity that we will use in our proof of Gårding inequalities below and summarised in the following lemma:

**Lemma 3.8.** *Let  $\varphi_t$ ,  $t > 0$ , be an approximation of the identity on  $G$ . There exists  $C > 0$  such that for all  $t > 0$ ,*

$$\forall \sigma \in S_{\rho,\delta}^m(G), \quad \|\sigma - \varphi_t * \sigma\|_{S_{\rho,\delta}^{m+\delta}(G), a_0, b_0} \leq Ct \|\sigma\|_{S_{\rho,\delta}^m, a_0, b_0+1},$$

for any  $a_0, b_0 > 0$ ,  $m \in \mathbb{R}$ ,  $1 \geq \rho \geq \delta \geq 0$ .

*Proof of Lemma 3.8.* We observe that for any  $\varphi \in C^\infty(G)$ , we have

$$\|\varphi * \sigma - \sigma\|_{S_{\rho,\delta}^{m+\delta}, a_0, b_0} \leq \int_G \|L_{y^{-1}}\sigma - \sigma\|_{S_{\rho,\delta}^{m+\delta}, a_0, b_0} |\varphi(y)| dy \lesssim_G \int_G |y| |\varphi(y)| dy \|\sigma\|_{S_{\rho,\delta}^m, a_0, b_0+1},$$

by Lemma 3.4 and its proof. Hence, we have

$$\|\varphi_t * \sigma - \sigma\|_{S_{\rho,\delta}^m, a_0, b_0} \lesssim_G \int_G |y| |\varphi_t(y)| dy \|\sigma\|_{S_{\rho,\delta}^m, a_0, b_0+1},$$

for any  $\sigma \in S_{\rho,\delta}^m$ . □

In the proof of the Gårding inequality for symbol of order 0 below, we will choose an approximation of the identity  $\varphi_t$ ,  $t > 0$ , that never vanishes, i.e.  $\varphi_t(x) > 0$  for any  $x \in G$  and  $t > 0$ , and then take  $a := \sqrt{\varphi_t}$ . Such an approximation of the identity  $\varphi_t$  is obtained by considering the heat kernel  $p_t$  [25, 10], that is, the convolution kernel of  $e^{-t\mathcal{L}}$ .

*Proof of Theorem 3.6 for  $m = 0$ .* Let  $\sigma \in S_{\rho,\delta}^0(G)$  satisfying  $\sigma(x, \pi) = \sigma(x, \pi)^*$  for any  $(x, \pi) \in G \times \widehat{G}$ . The link between the Wick and Kohn-Nirenberg quantizations (Lemma 3.7) and the properties of the pseudo-differential calculus (Theorem 3.1) imply

$$\frac{1}{2}(\text{Op}^{\text{KN}}(\sigma) + \text{Op}^{\text{KN}}(\sigma)^*) = \text{Op}^{\text{Wick}}(\sigma) + \text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma) + \text{Op}^{\text{KN}}(\rho),$$

with  $\rho \in S_{\rho,\delta}^{-(\rho-\delta)}(G)$ . Hence, we obtain:

$$\begin{aligned} \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} &\geq (\text{Op}^{\text{Wick}}(\sigma)f, f)_{L^2(G)} - \left| (\text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma)f, f)_{L^2(G)} \right| \\ &\quad - \|\text{Op}^{\text{KN}}(\rho)\|_{\mathcal{L}(H^{-\frac{\rho-\delta}{2}}(G), H^{\frac{\rho-\delta}{2}}(G))} \|f\|_{H^{-\frac{\rho-\delta}{2}}(G)}^2. \end{aligned}$$

The property (2.7) of the Wick quantization and the hypothesis  $\sigma(x, \pi) \geq 0$  for any  $(x, \pi) \in G \times \widehat{G}$  yield

$$(\text{Op}^{\text{Wick}}(\sigma)f, f)_{L^2(G)} \geq 0$$

while the properties of the pseudo-differential calculus (Theorem 3.1) imply that the operator norm  $\|\text{Op}^{\text{KN}}(\rho)\|_{\mathcal{L}(H^{-\frac{\rho-\delta}{2}}(G), H^{\frac{\rho-\delta}{2}}(G))}$  is finite. We then choose  $a := \sqrt{\varphi_t}$  with  $\varphi_t(x) > 0$  an approximation of the identity. By Lemma 3.8, one can find  $t = t(\eta)$  such that

$$\left| (\text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma)\|_{\mathcal{L}(L^2(G))} f, f)_{L^2(G)} \right| \leq \eta \|f\|_{H^{\frac{\delta}{2}}(G)}.$$

This concludes the first part of the Theorem, in the case  $\sigma \in S_{\rho,\delta}^0(G)$ . The second part of the Theorem, with  $\delta = 0$ , is obtained in a similar manner by observing that the property (2.7) of the Wick quantization and the hypothesis  $\sigma(x, \pi) \geq c$  for any  $(x, \pi) \in G \times \widehat{G}$  yield

$$(\text{Op}^{\text{Wick}}(\sigma)f, f)_{L^2(G)} \geq c \|f\|_{L^2(G)}^2.$$

One then choose  $\eta \leq c$  so that the term  $\eta \|f\|_{H^{\frac{\delta}{2}}(G)} = \eta \|f\|_{L^2(G)}$  is absorbed by  $c \|f\|_{L^2(G)}^2$ . □

In the proof above, we observe that  $\|\text{Op}^{\text{KN}}(\rho)\|_{\mathcal{L}(H^{-\frac{\rho-\delta}{2}}(G), H^{\frac{\rho-\delta}{2}}(G))}$  depends on our choice of  $a := \sqrt{\varphi_t}$ , and thus of  $\eta$ .

*Proof of Theorem 3.6 for any  $m \in \mathbb{R}$ .* Let  $\sigma \in S_{\rho, \delta}^m(G)$  be such that  $\sigma \geq 0$ . By the properties of the pseudo-differential calculus, we may write

$$(\text{id} + \mathcal{L})^{-m/4} \text{Op}^{\text{KN}}(\sigma) (\text{id} + \mathcal{L})^{-m/4} = \text{Op}^{\text{KN}}(\sigma_1) + \text{Op}^{\text{KN}}(\rho_1),$$

with  $\sigma_1 \in S_{\rho, \delta}^0(G)$  given via  $\sigma_1(x, \pi) := (1 + \lambda_\pi)^{-m/2} \sigma(x, \pi)$ , and  $\rho_1 \in S_{\rho, \delta}^{-(\rho - \delta)}(G)$ . We observe that  $\sigma_1$  satisfies the hypothesis of Theorem 3.6 with  $m = 0$ .

For any  $f \in \mathcal{C}^\infty(G)$ , setting  $f_1 = (\text{id} + \mathcal{L})^{m/4} f$ , we have

$$\begin{aligned} \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} &= \Re((\text{id} + \mathcal{L})^{-\frac{m}{4}} \text{Op}^{\text{KN}}(\sigma) (\text{id} + \mathcal{L})^{-\frac{m}{4}} f_1, f_1)_{L^2(G)} \\ &= \Re(\text{Op}^{\text{KN}}(\sigma_1)f_1, f_1)_{L^2(G)} + \Re(\text{Op}^{\text{KN}}(\rho_1)f_1, f_1)_{L^2(G)} \\ &\geq -\eta \|f_1\|_{H^{\frac{\delta}{2}}(G)} - C_\eta \|f_1\|_{H^{-\frac{\rho - \delta}{2}}(G)}, \end{aligned}$$

by Theorem 3.6 with  $m = 0$  applied to  $\sigma_1$  and the properties of the pseudo-differential calculus applied to  $\rho_1$ . The conclusion follows from  $\|f_1\|_{H^{-\frac{\rho - \delta}{2}}(G)} \asymp \|f\|_{H^{\frac{m - (\rho - \delta)}{2}}(G)}$ .  $\square$

#### 4. GÅRDING INEQUALITY ON GRADED NILPOTENT LIE GROUPS

Here, we prove the Gårding inequality on a graded nilpotent Lie group  $G$ . Before this, we recall some definitions and notations about this class of groups and the associated pseudo-differential calculus.

**4.1. Preliminaries on graded nilpotent groups.** A graded group  $G$  is a connected simply connected nilpotent Lie group whose (finite dimensional, real) Lie algebra  $\mathfrak{g}$  admits an  $\mathbb{N}$ -gradation into linear subspaces,

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} \mathfrak{g}_j \quad \text{with} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}, \quad 1 \leq i \leq j,$$

where all but a finite number of subspaces  $\mathfrak{g}_j$  are trivial. We denote by  $j = n_G$  the smallest integer such that all the subspaces  $\mathfrak{g}_j$ ,  $j > n_G$ , are trivial. If the first stratum  $\mathfrak{g}_1$  generates the whole Lie algebra, then  $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$  for all  $j \in \mathbb{N}_0$  and  $n_G$  is the step of the group; the group  $G$  is then said to be stratified, and also (after a choice of basis or inner product for  $\mathfrak{g}_1$ ) Carnot.

The product law on  $G$  is derived from the exponential map  $\exp_G : \mathfrak{g} \rightarrow G$  which is a global diffeomorphism from  $\mathfrak{g}$  onto  $G$ . Once a basis  $X_1, \dots, X_n$  for  $\mathfrak{g}$  has been chosen, we may identify the points  $(x_1, \dots, x_n) \in \mathbb{R}^n$  with the points  $x = \exp(x_1 X_1 + \dots + x_n X_n)$  in  $G$ . It allows us to define the (topological vector) spaces  $\mathcal{C}^\infty(G)$  and  $\mathcal{S}(G)$  of smooth and Schwartz functions on  $G$  identified with  $\mathbb{R}^n$ ; note that the resulting spaces are intrinsically defined as spaces of functions on  $G$  and do not depend on a choice of basis.

The exponential map induces a Haar measure  $dx$  on  $G$  which is invariant under left and right translations and defines Lebesgue spaces on  $G$ .

We now construct a basis adapted to the gradation. Set  $d_j = \dim \mathfrak{g}_j$  for  $1 \leq j \leq n_G$ . We choose a basis  $\{X_1, \dots, X_{n_1}\}$  of  $\mathfrak{g}_1$  (this basis is possibly reduced to  $\emptyset$ ), then  $\{X_{n_1+1}, \dots, X_{n_1+n_2}\}$  a basis of  $\mathfrak{g}_2$  (possibly  $\{0\}$ ) and so on. Such a basis  $\mathcal{B} = (X_1, \dots, X_{d_1+\dots+d_{n_G}})$  of  $\mathfrak{g}$  is said to be adapted to the gradation.

The Lie algebra  $\mathfrak{g}$  is a homogeneous Lie algebra equipped with the family of dilations  $\{\delta_r, r > 0\}$ ,  $\delta_r : \mathfrak{g} \rightarrow \mathfrak{g}$ , defined by  $\delta_r X = r^\ell X$  for every  $X \in \mathfrak{g}_\ell$ ,  $\ell \in \mathbb{N}$  [16, 12]. We re-write the set of integers  $\ell \in \mathbb{N}$  such that  $\mathfrak{g}_\ell \neq \{0\}$  into the increasing sequence of positive integers  $v_1, \dots, v_n$  counted with multiplicity, the multiplicity of  $\mathfrak{g}_\ell$  being its dimension. In this way, the integers  $v_1, \dots, v_n$  become the weights of the dilations and we have  $\delta_r X_j = r^{v_j} X_j$ ,  $j = 1, \dots, n$ , on the chosen basis of  $\mathfrak{g}$ . The associated group dilations are defined by

$$\delta_r(x) = rx := (r^{v_1} x_1, r^{v_2} x_2, \dots, r^{v_n} x_n), \quad x = (x_1, \dots, x_n) \in G, \quad r > 0.$$

In a canonical way, this leads to the notions of homogeneity for functions and operators. For instance, the Haar measure is homogeneous of degree

$$Q := \sum_{\ell} \nu_{\ell} \dim \mathfrak{g}_{\ell},$$

which is called the homogeneous dimension of the group. Another example is the vector field corresponding to an element  $X \in \mathfrak{g}_{\ell}$ : it is  $\ell$ -homogeneous.

In the rest of the paper, we assume that we have fixed a basis  $X_1, \dots, X_n$  of  $\mathfrak{g}$  adapted to the gradation. We keep the same notation for the associated left-invariant vector fields on  $G$ , and we denote the corresponding right invariant vector fields by  $\tilde{X}_1, \dots, \tilde{X}_n$ . For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ , we set  $X^{\alpha} = X_1^{\alpha_1} \dots X_n^{\alpha_n}$  and  $\tilde{X}^{\alpha} = \tilde{X}_1^{\alpha_1} \dots \tilde{X}_n^{\alpha_n}$ . The differential operators  $X^{\alpha}$  and  $\tilde{X}^{\alpha}$  are homogeneous of degree

$$[\alpha] = \nu_1 \alpha_1 + \dots + \nu_n \alpha_n.$$

**4.2. The pseudo-differential calculus.** In this section, we set some notation and recall briefly the global symbol classes defined on  $G$  together with some properties of the pseudo-differential calculus. We refer to [12] for more details.

A symbol  $\sigma$  is in  $S_{\rho, \delta}^m(G)$  when for any multi-indices  $\alpha, \beta \in \mathbb{N}_0^n$  and  $\gamma \in \mathbb{R}$ , there exists  $C = C(\alpha, \beta)$  such that we have for almost  $(x, \pi) \in G \times \hat{G}$ ,

$$(4.1) \quad \|\pi(\text{id} + \mathcal{R})^{-\frac{m-\rho[\alpha]+\delta[\beta]+\gamma}{\nu}} X^{\beta} \Delta^{\alpha} \sigma(x, \pi) \pi(\text{id} + \mathcal{R})^{\frac{\gamma}{\nu}}\|_{\mathcal{L}(\mathcal{H}_{\pi})} \leq C_{\alpha, \beta, \gamma},$$

where  $\mathcal{R}$  is a (and then any) positive Rockland operator of homogeneous degree  $\nu$ ; we may assume  $\gamma \in \mathbb{Z}$ . The difference operator  $\Delta^{\alpha}$  is the difference operator  $\Delta_{x^{\alpha}}$  for the monomial  $x^{\alpha}$  in the coordinates  $x_j$ . Generalising the definition in the compact setting, the difference operator  $\Delta_q$  associated to  $q \in \mathcal{C}^{\infty}(G)$  is defined as in (3.1) via  $\Delta_q \hat{\kappa} = \mathcal{F}(q\kappa)$  for any  $\kappa \in \mathcal{S}'(G)$  for which  $\kappa$  and  $q\kappa$  admits a Fourier transform (see [12]). We set  $\|\sigma\|_{S^{m, a, b, c}} := \max_{|\alpha| \leq a, |\beta| \leq b, |\gamma| \leq c} C_{\alpha, \beta, \gamma}$  for the best constants  $C_{\alpha, \beta, \gamma}$  in (4.1) and  $a, b, c \in \mathbb{N}_0$ . If  $(\rho, \delta) = (1, 0)$ , we simply write  $S^m(G) = S_{1, 0}^m(G)$ .

The following theorem summarises the main property of the classes of operators obtained by the  $\text{Op}^{\text{KN}}$ -quantization of the classes  $S^m(G)$ ; the Sobolev spaces  $L_s^2(G)$  adapted to the graded nilpotent Lie group  $G$  were studied in [12, 13] generalising slightly the stratified case [15].

**Theorem 4.1.** *Theorem 3.1 holds for  $G$  a graded nilpotent Lie group when replacing the symbol classes with the ones defined above and the Sobolev spaces with  $L_s^2(G)$ .*

Any  $\sigma \in S^m(G)$  admits a distributional convolution kernel  $\kappa : z \mapsto \kappa_x(z) \in \mathcal{C}^{\infty}(G, \mathcal{S}'(G))$ , i.e.  $\sigma(x, \pi) = \hat{\kappa}_x(\pi)$  and  $\text{Op}^{\text{KN}}(\sigma)f(x) = f * \kappa_x(x)$ ,  $f \in \mathcal{S}(G)$ . The statement of Lemma 3.3 for difference operators also holds in the context of graded Lie groups if  $a \in \mathcal{D}(G)$  is replaced with  $a \in \mathcal{S}(G)$ .

*Sketch of the proof for Lemma 3.3 for graded  $G$ .* The properties of the convolution kernels of symbols in  $S^m(G)$ , for instance being Schwartz away from the origin, implies that we may assume  $q \in \mathcal{D}(G)$  with compact support near the origin. Let  $\chi \in \mathcal{D}(G)$  be such that  $\chi \equiv 1$  on the support of  $q$ . Let  $P_N^q(z)$  be the Taylor polynomial at order  $N$  in the sense of Folland-Stein [16, 12] for  $q$ . The estimates of the convolution kernel  $\kappa$  of  $\sigma$  imply that

$$\sup_{x, y \in G} |\chi(q - P_N^q) \kappa_x(y)|$$

will be finite for  $N$  large enough, with a constant given by semi-norms in  $\sigma$ , and similarly for with  $X_y^{\beta_2} \tilde{X}_y^{\beta_2} y^{\alpha} \chi(q - P_N^q) X_x^{\beta} \kappa_x$ . This implies the statement.  $\square$

*Remark 4.2.* It is straightforward to check that Lemma 3.4 on the left-translations of a symbol, Corollary 3.5 on the  $x$ -convolution with a symbol and then Lemma 3.7 on the Wick quantization in the calculus  $\Psi_{\rho,\delta}^\infty(G)$  with  $a \in \mathcal{S}(G)$  hold on a graded group  $G$ .

**4.3. Proof of the Gårding inequality.** Here, we prove the following  $(\rho, \delta)$ -generalisation of Theorem 1.2:

**Theorem 4.3.** *Let  $G$  be a graded nilpotent Lie group. Let  $m \in \mathbb{R}$  and  $1 \geq \rho > \delta \geq 0$ . Assume that the symbol  $\sigma \in S_{\rho,\delta}^m(G)$  satisfies the positivity condition  $\sigma \geq 0$ . Then, for all  $\eta > 0$ , there exists a constant  $C_\eta > 0$  such that*

$$\forall f \in \mathcal{S}(G), \quad \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -\eta \|f\|_{L^2_{\frac{m+\delta}{2}}(G)}^2 - C_\eta \|f\|_{L^2_{\frac{m-(\rho-\delta)}{2}}(G)}^2.$$

Moreover, if  $\delta = 0$  and  $\sigma \in S_{\rho,\delta}^m(G)$  satisfies the elliptic condition  $\sigma \geq c(\text{id} + \widehat{\mathcal{R}})^\frac{m}{\nu}$ , for some constant  $c > 0$  where  $\mathcal{R}$  is a positive Rockland operator of homogeneous degree  $\nu$ . Then there exists a constant  $C > 0$  such that

$$\forall f \in \mathcal{S}(G), \quad \Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq -C \|f\|_{L^2_{\frac{m-(\rho-\delta)}{2}}(G)}^2.$$

The proof of Theorem 4.3 is a simple modification of the case of compact groups given in Section 3.2. We will need to replace the Sobolev spaces  $H^s(G)$  with the Sobolev space  $L_s^2(G)$  adapted to the graded nilpotent case. Moreover, in the final argument showing that the case of a symbol  $\sigma \in S_{\rho,\delta}^m(G)$  follows from the case of a symbol of order 0, we need to replace  $\sigma_1$  with

$$\sigma_1 = (\text{id} + \widehat{\mathcal{R}})^{-\frac{m}{2\nu}} \sigma (\text{id} + \widehat{\mathcal{R}})^{-\frac{m}{2\nu}}.$$

Below, we will detail the proof for a symbol  $\sigma \in S_{\rho,\delta}^0(G)$  of order 0. The main difference with the compact case will come from the construction of a suitable approximation of the identity. Before starting the proof, let us recall some properties of the approximations of the identity constructed using dilations on nilpotent Lie groups. In the proof of Gårding inequalities below, we will crucially use Part (3):

**Lemma 4.4.** *Let  $\varphi_1 \in \mathcal{S}(G)$  with  $\int_G \varphi_1 = 1$ . Consider the family of integrable functions  $\varphi_t = t^{-Q} \varphi_1 \circ \delta_{t^{-1}}$ ,  $t > 0$ .*

- (1) *The family of functions  $\varphi_t$ ,  $t > 0$ , form an approximation of identity on  $L^p(G)$ ,  $p \in [1, \infty)$  and on the space  $\mathcal{C}_0(G)$  of continuous functions vanishing at infinity, that is,*

$$\lim_{t \rightarrow 0} \|\psi - \varphi_t * \psi\|_{L^p(G)} = 0,$$

*for  $\psi \in L^p(G)$  (resp.  $\mathcal{C}_0(G)$ ) for  $p \in [1, \infty)$  (resp.  $p = \infty$ ).*

- (2) *There exist constants  $C, C' > 0$ , depending on the group and on  $\varphi_1$ , such that for any  $\psi \in \mathcal{C}^\infty(G)$ , we have*

$$\forall t \in (0, 1], \quad x \in G, \quad |\psi(x) - \varphi_t * \psi(x)| \leq Ct \max_{j=1, \dots, n} \sup_{x' \in G} |\tilde{X}_j \psi(x')|,$$

*and*

$$\forall t \in (0, 1], \quad x \in G, \quad |\psi(x) - \varphi_t * \psi(x)| \leq C't \max_{j=1, \dots, n} \sup_{x' \in G} |X_j \psi(x')|.$$

- (3) *There exists  $C > 0$  such that for all  $T > 0$*

$$\forall \sigma \in S_{\rho,\delta}^m(G), \quad \|\sigma - \varphi_t * \sigma\|_{S_{\rho,\delta}^{m+\delta}(G), a_0, b_0} \leq Ct \|\sigma\|_{S_{\rho,\delta}^m(G), a_0, b_0+1},$$

*for any  $a_0, b_0 > 0$ ,  $m \in \mathbb{R}$ ,  $1 \geq \rho > \delta \geq 0$ .*



The heat kernels  $p_t$  of a positive Rockland operators  $\mathcal{R}$ , i.e. the convolution kernels of  $e^{-t\mathcal{R}}$ ,  $t > 0$ , provides such an approximation of the identity [12, 16]. When  $G$  is stratified and  $\mathcal{R}$  is a sub-Laplacian, the heat kernels will be non-negative and never vanishing. We observe that the heat kernel being positive and never vanishing was used in the proof of the compact case in Section 3.2. These properties of the heat kernel for a general positive Rockland operator are not guaranteed in the graded case. However, in the graded case, Lemma 4.4 does not require an approximation of the identity that is non-negative and never vanishing, and this lemma will be sufficient in our proof of Theorem 4.3.

*Proof of Lemma 4.4.* Part 1 is well known (see [16, 12]) but will not be used in this paper. In Part 2, the first estimate follows by the Taylor estimates due to Folland and Stein [16, 12]. As  $X_j$  may be expressed as a sum of  $\tilde{X}_k$  with polynomial coefficients, we obtain the second inequality only for  $\psi$  with compact support, and a constant depending on this support. We now extend this to any function  $\psi$  with an argument of partition of unity.

From the partition of unity due to Folland and Stein [16], there exist a function  $\chi \in \mathcal{D}(G)$  and a sequence  $(x_j)_{j \in \mathbb{N}}$  of points in  $G$  such that  $\sum_{j \in \mathbb{N}} \chi_j(x) = 1$  for all  $x \in G$ , where  $\chi_j(x) = \chi(xx_j)$ ; moreover, this sum is finite for every  $x \in G$ , and in fact uniformly finite on any compact set of  $G$ . We now observe setting  $\psi_j(x) := \psi(xx_j^{-1})$

$$\psi(x) - \varphi_t * \psi(x) = \sum_j (\psi \chi_j)(x) - \varphi_t * (\psi \chi_j)(x) = \sum_j ((\psi_j \chi) - \varphi_t * (\psi_j \chi))(xx_j).$$

We obtain the second estimates of Part (2) with the estimate for functions with a fixed compact support and the sum being uniformly finite.

We show Part (3) by modifying the proof of Lemma 3.8. We write for any  $\varphi \in \mathcal{S}(G)$

$$\|\sigma - \varphi * \sigma\|_{S_{\rho,\delta}^{m+\delta}, a_0, b_0, c_0} = \max_{[\alpha] \leq a_0, [\beta] \leq b_0, |\gamma| \leq c_0} \max_{x \in G} \|\sigma_{\alpha,\beta,\gamma}(x, \cdot) - \varphi * \sigma_{\alpha,\beta,\gamma}(x, \cdot)\|_{L^\infty(\hat{G})},$$

where the symbol  $\sigma_{\alpha,\beta,\gamma} \in L^\infty(\hat{G})$  is defined via

$$\sigma_{\alpha,\beta,\gamma}(x, \pi) := \pi(\text{id} + \mathcal{R})^{-\frac{m+\delta-\rho[\alpha]+\delta[\beta]+\gamma}{\nu}} X^\beta \Delta^\alpha \sigma(x, \pi) \pi(\text{id} + \mathcal{R})^{\frac{\gamma}{\nu}}.$$

This and the second estimate in Part (2) extended to Banach valued functions  $\psi$  imply

$$\|\varphi_t * \sigma - \sigma\|_{S_{\rho,\delta}^{m+\delta}, a_0, b_0} \lesssim_{G, \varphi_1} t \|\sigma\|_{S_{\rho,\delta}^m, a_0, b_0+1}.$$

for any  $\sigma \in S_{\rho,\delta}^m$ . □

*Proof of Theorem 4.3.* As explained above, it suffices to show the statement for  $\sigma \in S_{\rho,\delta}^0(G)$  satisfying  $\sigma(x, \pi) \geq 0$ . We fix a function  $a_1 \in \mathcal{S}(G)$  with  $\|a_1\|_{L^2(G)} = 1$ , and set  $a_t(x) := t^{-Q/2} a_1(\delta_t^{-1}x)$ ,  $x \in G$ ,  $t > 0$ . We observe that  $|a_t|^2 = t^{-Q} |a_1|^2 \circ \delta_t^{-1}$ ,  $t > 0$ , is an approximation of the identity in the sense of Lemma 4.4. We now consider the Wick quantization  $\text{Op}^{\text{Wick}, a}$  with  $a = a_t$  to be chosen at the end of the proof. The properties of the Wick quantization (see (2.7) and Lemma 3.7 with Remark 4.2) imply

$$\Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2(G)} \geq - \left| (\text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma)f, f)_{L^2(G)} \right| - C \|f\|_{L^2_{-\frac{\rho-\delta}{2}}(G)}^2,$$

for some constant  $C > 0$ . We now choose  $a = a_t$  with  $t > 0$  small enough so that, by Lemma 4.4 (3),

$$\left| (\text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma)f, f)_{L^2(G)} \right| \geq \eta \|f\|_{L^2_{\frac{\rho}{2}}(G)}^2.$$

This shows the case of  $\sigma \in S_{\rho,\delta}^0(G)$  and one can conclude the proof of Theorem 4.3 in a similar manner as for Theorem 3.6. □

## 5. SEMI-CLASSICAL GÅRDING INEQUALITY ON GRADED NILPOTENT LIE GROUPS

In this section, we show the semi-classical inequality stated in Theorem 1.3. The proof is inspired by Lemma 1.2 in [18]. Before this, we recall the definition of the semi-classical calculus and we introduce the Wick quantization adapted to the semi-classical setting.

**5.1. Semi-classical pseudodifferential calculus.** The set  $\mathcal{A}_0$  is the space of symbols  $\sigma = \{\sigma(x, \pi) : (x, \pi) \in G \times \widehat{G}\}$  of the form

$$\sigma(x, \pi) = \mathcal{F}\kappa_x(\pi) = \int_G \kappa_x(y)(\pi(x))^* dx,$$

where  $(x, y) \mapsto \kappa_x(y)$  is a function of the set  $\mathcal{C}_c^\infty(G, \mathcal{S}(G))$  of smooth and compactly supported functions on  $U$  valued in the set of Schwartz class functions. As before,  $x \mapsto \kappa_x$  is called the convolution kernel of  $\sigma$ .

With the symbol  $\sigma \in \mathcal{A}_0$ , we associate the (family of) *semi-classical pseudodifferential operators*

$$\text{Op}_\varepsilon(\sigma) = \text{Op}^{\text{KN}}\sigma(\cdot, \delta_\varepsilon \cdot), \quad \varepsilon \in (0, 1],$$

where the Kohn-Nirenberg quantization  $\text{Op}^{\text{KN}}$  was defined in Section 2.2.2 and  $\delta_r$  denotes the action of  $\mathbb{R}^+$  on  $\widehat{G}$  given via

$$\delta_r \pi(x) = \pi(\delta_r x), \quad x \in G, \quad \pi \in \widehat{G}, \quad r > 0.$$

In other words, we have

$$\text{Op}_\varepsilon(\sigma)f(x) = \int_{\pi \in \widehat{G}} \text{Tr}_{\mathcal{H}_\pi}(\pi(x)\sigma(x, \delta_\varepsilon \pi)\mathcal{F}_G f(\pi)) d\mu(\pi), \quad f \in \mathcal{S}(G), \quad x \in G,$$

or equivalently, in terms of the convolution kernel  $\kappa_x = \mathcal{F}_G^{-1}\sigma(x, \cdot)$ ,

$$\text{Op}_\varepsilon(\sigma)f(x) = f * \kappa_x^{(\varepsilon)}(x), \quad f \in \mathcal{S}(G), \quad x \in U,$$

where  $\kappa_x^{(\varepsilon)}$  is the following rescaling of the convolution kernel:

$$\kappa_x^{(\varepsilon)}(y) := \varepsilon^{-Q} \kappa_x(\delta_\varepsilon^{-1} y).$$

**5.2. The semi-classical Wick quantization.** Let  $a \in \mathcal{S}(G)$  such that  $\|a\|_{L^2} = 1$ . We set

$$a_\varepsilon := \varepsilon^{-\frac{Q}{4}} a \circ \delta_{\varepsilon^{-\frac{1}{2}}}, \quad \varepsilon > 0,$$

so that  $a_\varepsilon \in \mathcal{S}(G)$  with  $\|a_\varepsilon\|_{L^2(G)} = 1$ . We also set

$$\mathcal{F}_x^\varepsilon := \{\mathcal{F}_{x, \pi}^\varepsilon(y) : (y, \pi) \in G \times \widehat{G}\}, \quad \mathcal{F}_{x, \pi}^\varepsilon(y) = a_\varepsilon(yx^{-1})\delta_\varepsilon^{-1}\pi(y)^*,$$

and define the operator  $\mathcal{B}^\varepsilon$  on  $\mathcal{S}(G)$  via

$$\mathcal{B}^\varepsilon[f](x, \pi) = \varepsilon^{-\frac{Q}{2}} \int_G f(y)\mathcal{F}_{x, \pi}^\varepsilon(y) dy, \quad f \in \mathcal{S}(G), \quad (x, \pi) \in G \times \widehat{G}.$$

Note that, with respect to the operator  $\mathcal{B}_a$  defined in Section 2.3.1, we have

$$\mathcal{B}^\varepsilon[f](x, \pi) = \varepsilon^{-\frac{Q}{2}} \mathcal{B}_{a_\varepsilon}[f](x, \delta_\varepsilon^{-1}\pi).$$

Hence, by Proposition 2.2, the map  $\mathcal{B}^\varepsilon$  extends uniquely to an isometry from  $L^2(G)$  to  $L^2(G \times \widehat{G})$  for which we keep the same notation. Denoting by  $\mathcal{B}^{\varepsilon, *}: L^2(G \times \widehat{G}) \rightarrow L^2(G)$  its adjoint map, we have  $\mathcal{B}^{\varepsilon, *} \mathcal{B}^\varepsilon = \text{id}_{L^2(G)}$  while  $\mathcal{B}^\varepsilon \mathcal{B}^{\varepsilon, *}$  is a projection on a closed subspace of  $L^2(G \times \widehat{G})$ . We also define the semi-classical Wick quantization for  $\sigma \in L^\infty(\widehat{G})$

$$\text{Op}_\varepsilon^{\text{Wick}}(\sigma) := \mathcal{B}^{\varepsilon, *} \sigma \mathcal{B}^\varepsilon$$

We can compute the convolution kernel of  $\text{Op}_\varepsilon^{\text{Wick}}(\sigma)$  as in Lemma 2.4:

**Lemma 5.1.** *If  $\sigma \in \mathcal{A}_0$ , then*

$$\text{Op}_\varepsilon^{\text{Wick}}(\sigma) = \text{Op}_\varepsilon^{\text{KN}}(\sigma^{\varepsilon, \text{Wick}}),$$

where  $\sigma^{\varepsilon, \text{Wick}} \in \mathcal{A}_0$  has the following convolution kernel

$$\begin{aligned} \kappa_x^{\varepsilon, \text{Wick}}(w) &= \int_G a_\varepsilon(\delta_\varepsilon w^{-1} x z^{-1}) \bar{a}_\varepsilon(x z^{-1}) \kappa_z(w) dz \\ &= \int_G a_\varepsilon(\delta_\varepsilon w^{-1} z') \bar{a}_\varepsilon(z') \kappa_{z'^{-1}x}(w) dz' \\ &= \int_G a(\delta_{\sqrt{\varepsilon}} w^{-1} y) \bar{a}(y) \kappa_{\delta_{\sqrt{\varepsilon}} y^{-1}x}(w) dy, \end{aligned}$$

**Corollary 5.2.** *We choose a function  $a \in \mathcal{D}(G)$  that is even, i.e.  $a(x^{-1}) = a(x)$ , and real valued. Then for any  $\sigma \in \mathcal{A}_0$ , there exists  $C > 0$  such that for all  $\varepsilon \in (0, 1]$ ,*

$$\|\text{Op}_\varepsilon^{\text{KN}}(\sigma) - \text{Op}_\varepsilon^{\text{Wick}}(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C\varepsilon.$$

*Proof.* By Lemma 2.1, using the  $\mathcal{A}_0$ -norm defined in (2.4), we have

$$\|\text{Op}_\varepsilon^{\text{KN}}(\sigma) - \text{Op}_\varepsilon^{\text{Wick}}(\sigma)\|_{\mathcal{L}(L^2(G))} \leq \|\sigma - \sigma^{\varepsilon, \text{Wick}}\|_{\mathcal{A}_0} \leq I_1(\varepsilon) + I_2(\varepsilon),$$

where

$$\begin{aligned} I_1(\varepsilon) &:= \int_G \sup_{x \in G} \left| \int_G |a(z)|^2 \left( \kappa_x(w) - \kappa_{\delta_{\sqrt{\varepsilon}} z^{-1}x}(w) \right) dz \right| dw \\ I_2(\varepsilon) &:= \int_G \sup_{x \in G} \left| \int_G (a(z) - a(\delta_{\sqrt{\varepsilon}} w^{-1} z)) \bar{a}(z) \kappa_{\delta_{\sqrt{\varepsilon}} z^{-1}x}(w) dz \right| dw. \end{aligned}$$

By the Taylor estimates due to Folland and Stein [16, 12], we have:

$$\begin{aligned} I_1(\varepsilon) &= \sqrt{\varepsilon} \int_G \sup_{x \in G} \left| \sum_{[\alpha]=1} \int_G |a(z)|^2 q_\alpha(z^{-1}) dz \tilde{X}_x^\alpha \kappa_x(w) \right| dw + O(\varepsilon) \\ I_2(\varepsilon) &= \sqrt{\varepsilon} \int_G \sup_{x \in G} \left| \sum_{[\alpha]=1} \int_G \tilde{X}^\alpha a(z) \bar{a}(z) q_\alpha(w) \kappa_{x \delta_{\sqrt{\varepsilon}} z^{-1}x}(w) dz \right| dw + O(\varepsilon) \\ &\leq \sqrt{\varepsilon} \int_G \sup_{x' \in G} \sum_{[\alpha]=1} \left| \int_G \tilde{X}^\alpha a(z) \bar{a}(z) dz \right| |q_\alpha(w) \kappa_{x'}(w)| dw + O(\varepsilon). \end{aligned}$$

As  $a$  is even, for any polynomial  $q$  satisfying  $q(z^{-1}) = -q(z)$  such as the coordinate polynomials  $q_\alpha$  with  $[\alpha] = 1$ , we have  $\int_G |a(z)|^2 q(z) dz = 0$ . As  $a$  is real valued, for any left or right invariant vector field  $X$ , an integration by parts shows  $\int_G X a(z) \bar{a}(z) dz = 0$ . Consequently,  $I_1(\varepsilon) = O(\varepsilon)$  and  $I_2(\varepsilon) = O(\varepsilon)$ .  $\square$

**5.3. Proof of the semi-classical Gårding inequality.** We write

$$\Re(\text{Op}_\varepsilon^{\text{KN}}(\sigma) f, f)_{L^2(G)} \geq \left( \text{Op}_\varepsilon^{\text{Wick}}(\sigma) f, f \right)_{L^2(G)} - \|\text{Op}_\varepsilon^{\text{KN}}(\sigma) - \text{Op}_\varepsilon^{\text{Wick}}(\sigma)\|_{\mathcal{L}(L^2(G))} \|f\|_{L^2(G)}.$$

By the properties of the semi-classical Wick quantisation, if  $\sigma \geq 0$ , then

$$\left( \text{Op}_\varepsilon^{\text{Wick}}(\sigma) f, f \right)_{L^2(G)} = (\sigma \mathcal{B}^\varepsilon f, \mathcal{B}^\varepsilon f)_{L^2(G \times \hat{G})} \geq 0,$$

while by Corollary 5.2,  $\|\text{Op}_\varepsilon^{\text{KN}}(\sigma) - \text{Op}_\varepsilon^{\text{Wick}}(\sigma)\|_{\mathcal{L}(L^2(G))} = O(\varepsilon)$ . This concludes the proof of Theorem 1.3.

We point out that the semi-classical case is more straightforward because we restrict ourselves to the use of  $L^2$ -norm and to a gain in the semi-classical parameter  $\varepsilon$ . The proof of Theorem 1.3 is analogous to the one of Theorem 4.3, by taking the dilation parameter  $t$  as  $\sqrt{\varepsilon}$ .

## APPENDIX A. THE EUCLIDEAN CASE

In this section, we recall the definitions and some properties of the Kohn-Nirenberg and Wick quantizations in the Euclidean case  $\mathbb{R}^n$ . We develop the same chain of arguments that show the Gårding inequality for the Hörmander calculus on  $\mathbb{R}^n$  as in the core of the paper. This leads to a proof which is close to the one of [17, Chapter 2, section 6].

**A.1. Kohn-Nirenberg and Wick quantizations.** On  $\mathbb{R}^n$ , the Kohn-Nirenberg quantization may be defined for any symbols  $\sigma \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$  via the formula

$$\text{Op}^{\text{KN}}(\sigma)f(x) = \int_{\mathbb{R}^n} e^{2i\pi x\xi} \sigma(x, \xi) \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n),$$

where  $\widehat{f} = \mathcal{F}f$  denotes the Euclidean Fourier transform of  $f$ :

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

With the *convolution kernel*  $\kappa_x := \mathcal{F}^{-1}\sigma(x, \cdot)$  of  $\sigma$ , this may be rewritten as

$$\text{Op}^{\text{KN}}(\sigma)f(x) = f * \kappa_x(x), \quad x \in \mathbb{R}^n, f \in \mathcal{S}(\mathbb{R}^n).$$

Fixing a continuous, bounded and square-integrable function  $a$  with  $\|a\|_{L^2(\mathbb{R}^n)} = 1$ , we set for any  $f \in L^2(\mathbb{R}^n)$  and  $(x, \xi) \in \mathbb{R}^n$

$$\mathcal{B}_a[f](x, \xi) := \mathcal{F}(f a(\cdot - x))(\xi) = \int_{\mathbb{R}^n} f(y) a(y - x) e^{-2i\pi y\xi} dy.$$

This defines the generalised Bargmann transform  $\mathcal{B}_a = \mathcal{B}$ . The function  $a$  is usually chosen as the Gaussian function  $a(x) = \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}$  [5]. The map  $\mathcal{B}$  is a unitary transformation  $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n \times \mathbb{R}^n)$ . Denoting by  $\mathcal{B}^*$  its adjoint, we define the Wick quantization  $\text{Op}^{\text{Wick}} = \text{Op}^{\text{Wick}, a}$ , for any symbol  $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  via:

$$\text{Op}^{\text{Wick}}(\sigma)f = \mathcal{B}^* \sigma \mathcal{B}[f], \quad f \in L^2(\mathbb{R}^n).$$

This quantization has the advantage of yielding bounded operators on  $L^2$ , of preserving self-adjointness:

$$\|\text{Op}^{\text{Wick}}(\sigma)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}, \quad \text{Op}^{\text{Wick}}(\sigma)^* = \text{Op}^{\text{Wick}}(\bar{\sigma}),$$

and of preserving positivity:

$$\sigma(x, \xi) \geq 0 \text{ for all } (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \implies (\text{Op}^{\text{Wick}}(\sigma)f, f)_{L^2(\mathbb{R}^n)} \geq 0.$$

The link between the Kohn-Nirenberg and Wick quantization for a bounded symbol is the following:

**Lemma A.1.** *Let  $a \in \mathcal{S}(\mathbb{R}^n)$  with  $\|a\|_{L^2(\mathbb{R}^n)} = 1$ , and consider the associated Wick quantization. For any symbol  $\sigma \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ , we have:*

$$\text{Op}^{\text{Wick}}(\sigma)f(x) = f * \kappa_x^{\text{Wick}}(x), \quad f \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n,$$

where  $\kappa_x^{\text{Wick}} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$  is given by

$$\kappa_x^{\text{Wick}}(y) = \int_{\mathbb{R}^n} a(z - y) \bar{a}(z) \kappa_{x-z}(y) dz,$$

where  $\kappa_x = \mathcal{F}^{-1}\sigma(x, \cdot)$  denotes the convolution kernel of  $\sigma$ . Hence,  $\text{Op}^{\text{Wick}}(\sigma) = \text{Op}^{\text{KN}}(\sigma^{\text{Wick}})$  where  $\sigma^{\text{Wick}} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$  is the symbol given by  $\sigma^{\text{Wick}}(x, \xi) = \mathcal{F}\kappa_x^{\text{Wick}}(\xi)$ .

**A.2. Gårding inequality for Hörmander symbols.** First, let us recall the definition of the Hörmander classes of symbols.

**Definition A.2.** A function  $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  is a Hörmander symbol of order  $m \in \mathbb{R}$  and index  $(\rho, \delta)$  when

$$\forall \alpha, \beta \in \mathbb{N}_0^n, \quad \exists C_{\alpha, \beta} > 0, \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \quad |\partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|^2)^{\frac{m - \rho|\alpha| + \delta|\beta|}{2}}.$$

The space of Hörmander symbols of order  $m$  and index  $(\rho, \delta)$  is denoted by  $S_{\rho, \delta}^m(\mathbb{R}^n)$ .

The space  $S_{\rho, \delta}^m(\mathbb{R}^n)$  is naturally equipped with a structure of Fréchet space, with semi-norms:  $\|\sigma\|_{S_{\rho, \delta}^{m, a, b}} = \max_{|\alpha| \leq a, |\beta| \leq b} C_{\alpha, \beta}$  with  $C_{\alpha, \beta}$  the best constant as above. The resulting class of operators  $\Psi_{\rho, \delta}^m(\mathbb{R}^n) := \text{Op}^{\text{KN}}(S_{\rho, \delta}^m(\mathbb{R}^n))$  inherits the structure of Fréchet space. Moreover, the Hörmander calculus  $\cup_{m \in \mathbb{R}} \Psi_{\rho, \delta}^m(\mathbb{R}^n)$  is a calculus in the sense of Definition 1.4. In this context, the link between the Kohn-Nirenberg and Wick quantizations is given in Part (2) of the following statement.

**Proposition A.3.** (1) Let  $m \in \mathbb{R}$  and  $1 \geq \rho \geq \delta \geq 0$ . If  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  and  $\sigma \in S_{\rho, \delta}^m(\mathbb{R}^n)$  then

$$\varphi * \sigma : (x, \xi) \mapsto \int_{\mathbb{R}^n} \varphi(z) \sigma(x - z, \xi) dz = \varphi * \sigma(\cdot, \xi)(x),$$

defines a symbol in  $S_{\rho, \delta}^m(\mathbb{R}^n)$ .

(2) We assume that  $a \in \mathcal{S}(\mathbb{R}^n)$  with  $\|a\|_{L^2(\mathbb{R}^n)} = 1$ . If  $\sigma \in S_{\rho, \delta}^0(\mathbb{R}^n)$ , then

$$\text{Op}^{\text{Wick}}(\sigma) - \text{Op}^{\text{KN}}(|a|^2 * \sigma) \in \Psi_{\rho, \delta}^{-(\rho - \delta)}(\mathbb{R}^n).$$

*Sketch of the proof of Proposition A.3.* Part (1) is easily checked. For Part (2), we may rephrase Lemma A.1 using the notion of difference operators  $\Delta_q$  defined formally via:

$$(\Delta_q \sigma)(x, \xi) = \mathcal{F}(q \mathcal{F}^{-1} \sigma(x, \cdot))(\xi) = \mathcal{F}(q \kappa_x)(\xi) = \widehat{q} * \sigma_x(\xi),$$

with  $\kappa_x = \mathcal{F}^{-1} \sigma(x, \cdot)$  the convolution kernel of  $\sigma$ . We have:

$$\text{Op}^{\text{Wick}}(\sigma) = \text{Op}^{\text{KN}}(\sigma^{\text{Wick}}) \quad \text{with} \quad \sigma^{\text{Wick}}(x, \xi) = \int_{\mathbb{R}^n} \Delta_{q_z} \sigma(x - z, \xi) dz,$$

where  $q_z(w) = a(z - w) \bar{a}(z)$ . We then conclude with the following asymptotic expansion in any symbol class  $S_{\rho, \delta}^m(\mathbb{R}^n)$  for a difference operator associated with  $q \in \mathcal{S}(\mathbb{R}^n)$

$$\Delta_q \sigma \sim q(0) \sigma + \sum_{\alpha > 0} c_\alpha \partial_x^\alpha q(0) \partial_\xi^\alpha \sigma,$$

with explicit coefficients  $c_\alpha$ . □

The properties of the Kohn-Nirenberg and Wick quantizations imply the following Gårding inequality:

**Theorem A.4.** Let  $m \in \mathbb{R}$  and  $1 \geq \rho > \delta \geq 0$ . Assume that the symbol  $\sigma \in S_{\rho, \delta}^m(\mathbb{R}^n)$  satisfies the positivity condition  $\sigma(x, \xi) \geq 0$ . Then, for all  $\eta > 0$ , there exists a constant  $C_\eta > 0$  such that

$$\forall f \in \mathcal{S}(\mathbb{R}^n), \quad \Re(\text{Op}^{\text{KN}}(\sigma) f, f)_{L^2} \geq -\eta \|f\|_{H^{\frac{m+\delta}{2}}}^2 - C_\eta \|f\|_{H^{\frac{m-(\rho-\delta)}{2}}}^2.$$

Moreover, if  $\delta = 0$  and  $\sigma \in S_{\rho, 0}^m(\mathbb{R}^n)$  satisfies the elliptic condition  $\sigma(x, \xi) \geq c$ , for some constant  $c > 0$ , then, there exists a constant  $C > 0$  such that

$$\forall f \in \mathcal{S}(\mathbb{R}^n), \quad \Re(\text{Op}^{\text{KN}}(\sigma) f, f)_{L^2} \geq -C \|f\|_{H^{\frac{m-(\rho-\delta)}{2}}}^2.$$

We will use the following property of approximation of the identity. The proof relies in crucial ways on the fact that the convolution is in  $x$  only.

**Lemma A.5.** *Let  $\varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ . Set  $\varphi_t(x) := t^{-n}\varphi_1(t^{-1}x)$ . Let  $m \in \mathbb{R}^n$  and  $1 \geq \rho > \delta \geq 0$ . For any  $\sigma \in S_{\rho,\delta}^m(\mathbb{R}^n)$ ,  $\varphi_t * \sigma$  converges to  $\sigma$  in  $S_{\rho,\delta}^m(\mathbb{R}^n)$  as  $t \rightarrow 0$ . This means that given any semi-norm  $\|\cdot\|_{S_{\rho,\delta}^{m,a,b}}$ , we have*

$$\lim_{t \rightarrow 0} \|\sigma - \varphi_t * \sigma\|_{S_{\rho,\delta}^{m,a,b}} = 0.$$

*Proof.* We observe that

$$\begin{aligned} |\sigma(x, \xi) - \varphi_t * \sigma(x, \xi)| &\leq \int_{\mathbb{R}^n} |\sigma(x, \xi) - \sigma(x - ty, \xi)| |\varphi_1|(y) dy \\ &\leq t \sup_{x' \in \mathbb{R}^n} |\nabla_x \sigma(x', \xi)| \int_{\mathbb{R}^n} |y| |\varphi_1(y)| dy. \end{aligned}$$

Since

$$\partial_x^\beta \partial_\xi^\alpha (\sigma - \varphi_t * \sigma)(x, \xi) = \partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi) - \varphi_t * \partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi),$$

we have a similar result applying  $\partial_x^\beta \partial_\xi^\alpha$ . Consequently, if  $\sigma \in S_{\rho,\delta}^m(\mathbb{R}^n)$

$$\|\sigma - \varphi_t * \sigma\|_{S_{\rho,\delta}^{m+\delta,a,b}} \leq C_{\varphi_1} t \|\sigma\|_{S_{\rho,\delta}^{m,a,b+1}}, \quad \text{with } C_{\varphi_1} := \int_{\mathbb{R}^n} |y| |\varphi_1(y)| dy.$$

□

*Sketch of the proof of Theorem A.4.* It suffices to show the case  $m = 0$ . Let  $\sigma \in S_{\rho,\delta}^0(\mathbb{R}^n)$  satisfying  $\sigma(x, \xi) \geq c$ . The properties of the Wick quantization (especially preserving positivity and Proposition A.3) imply

$$\Re(\text{Op}^{\text{KN}}(\sigma)f, f)_{L^2} \geq c \|f\|_{L^2}^2 - |(\text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma)f, f)_{L^2}| - C \|f\|_{H^{\frac{\rho-\delta}{2}}}^2,$$

for some constant  $C > 0$ . We fix a function  $a_1 \in \mathcal{S}(\mathbb{R}^n)$  with  $\|a_1\|_{L^2(\mathbb{R}^n)} = 1$  and set  $a_t(x) := t^{-n/2} a_1(t^{-1}x)$ . Applying Lemma A.5 to  $|a_t|^2 = t^{-n} |a_1(t^{-1}\cdot)|^2$ ,  $t > 0$ , we then choose  $a = a_t$  with  $t > 0$  small enough so that

$$\left| (\text{Op}^{\text{KN}}(\sigma - |a|^2 * \sigma)f, f)_{L^2(G)} \right| \leq \eta \|f\|_{H^{\frac{\delta}{2}}}.$$

The second part of the Theorem is proved as for Theorem 3.6. □

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