



HAL
open science

Boundary null controllability of KdV equation in a star-shaped network

Hugo Parada

► **To cite this version:**

Hugo Parada. Boundary null controllability of KdV equation in a star-shaped network. 2023. hal-04169908

HAL Id: hal-04169908

<https://hal.science/hal-04169908v1>

Preprint submitted on 24 Jul 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BOUNDARY NULL CONTROLLABILITY OF KDV EQUATION IN A STAR-SHAPED NETWORK

HUGO PARADA*

Abstract. The boundary null controllability of a system of N linear KdV equations posed on a star-shaped network is studied. The controls are located on the right Dirichlet and Neumann conditions of $N - 1$ edges. First, we study the well-posedness of the system considered by using the notion of solution by transposition and interpolation arguments. Then, a Carleman inequality is shown for the adjoint system. Finally, using a dissipation estimate and the Carleman estimate, we show an observability inequality that is equivalent to null controllability.

Key words. Korteweg-de Vries equation, star-shaped network, controllability, carleman estimates

1. Introduction and presentation of our results. In this paper, we study the null controllability problem of the linear Korteweg-de Vries equation posed in a star-shaped network. The nonlinear Korteweg-de Vries equation is given by $\partial_t u + \partial_x u + \partial_x^3 u + u \partial_x u = 0$ and was introduced in [16] to model the propagation of long water waves in a channel. In the network case, this third-order dispersive equation, it was proposed to model the pressure on the arterial tree [1]. The controllability of the KdV equation has been very well addressed in recent years. We mention, for instance, the pioneer work [23] where the exact boundary controllability of the linear and nonlinear KdV equation was showed. Regarding null controllability, there is the work [24], where null controllability was achieved by using a Dirichlet control on the left, [13] where a uniform null controllability result was obtained and [14] where null controllability was derived in the case of special boundary conditions. For a good introduction to the controllability and stabilization problems for the KdV equation, we recommend the surveys [9] and [25]. When it is posed in a network, the KdV equation has been studied in several works, for instance [7, 21, 26]. More specifically, the exact controllability was studied in [1, 10] and the stabilization in [19, 20, 18]. Regarding controllability problems in networks, studies such as [12] and [17] have addressed control of multi-structures, [8] established the null controllability of the Kuramoto Sivashinsky equation on a star-shaped network. In [4] and [15] Carleman inequalities were obtained for wave, heat and Schrödinger equations in networks. Finally, we want to mention the recent work [2] where null controllability properties of parabolic and hyperbolic equations in trees were studied. Although we do not study in this work the null controllability of the nonlinear KdV equation posed in a star-shaped network, we expect that our result can be extended to the nonlinear case.

Let $K = \{k_j : 1 \leq n \leq N\}$ be the set of N edges of a network \mathcal{T} described as the intervals $I_j = (0, \ell_j)$ for $j = 1, \dots, N$, where $\ell_j > 0$ for $j = 1, \dots, N$. The network \mathcal{T} is defined by $\mathcal{T} = \bigcup_{j=1}^N k_j$. In this work, we consider the next evolution problem for the linear KdV equation. Let $T > 0$ and define $Q_j = (0, T) \times I_j$

$$\left\{ \begin{array}{ll} \partial_t u_j(t, x) + \partial_x u_j(t, x) + \partial_x^3 u_j(t, x) = 0, & (t, x) \in Q_j, \quad j = 1, \dots, N, \\ u_j(t, 0) = u_1(t, 0)(t), & t \in (0, T), \quad \forall j = 2, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\lambda u_1(t, 0), & t \in (0, T), \\ u_j(t, \ell_j) = g_j(t), \quad \partial_x u_j(t, \ell_j) = p_j(t), & t \in (0, T), \quad j = 1, \dots, N, \\ u_j(0, x) = u_j^0(x), & x \in I_j, \end{array} \right. \quad (\text{LKdV})$$

where $\lambda > \frac{N}{2}$. The system under consideration has control inputs $\underline{g} = (g_1, \dots, g_N)^T$ and $\underline{p} = (p_1, \dots, p_N)^T \in (L^2(0, T))^N$, which act on the exterior Dirichlet and Neumann conditions respectively. The conditions of the central node are inspired by the previous works [1, 19, 20]. According to [20], we can use u_j to represent the dimensionless and scaled variable standing for the deflection from rest position. Take v_j to represent the

*Université Grenoble Alpes, CNRS, Laboratoire Jean Kuntzmann, F-38000, Grenoble, France (e-mail: hugo.parada@univ-grenoble-alpes.fr). This work was supported by the French National Research Agency in the framework of the "Investissements d'avenir" program (ANR-15-IDEX-02)

velocity on the branch j of long water waves. With this in mind we have

$$\begin{cases} \partial_t u_j + \partial_x u_j + \partial_x^3 u_j + u_j \partial_x u_j = 0, & x \in (0, \ell_j), t \in (0, T), j = 1, \dots, N, \\ v_j = u_j - \frac{1}{6} u_j^2 + 2 \partial_x^2 u_j, & x \in (0, \ell_j), t \in (0, T), j = 1, \dots, N. \end{cases} \quad (1.1)$$

In central node, if we assume water level to be the same and that the sum of the flux is null, this implies:

$$\begin{cases} u_j(t, 0) = u_1(t, 0), & t \in (0, T), j = 2, \dots, N, \\ \sum_{j=1}^N u_j(t, 0) v_j(t, 0) = 0, & t \in (0, T). \end{cases}$$

Then, if we linearize (1.1) around the 0, we obtain the following boundary conditions

$$\begin{cases} u_j(t, 0) = u_1(t, 0), & t \in (0, T), j = 2, \dots, N, \\ \sum_{j=1}^N \partial_x^2 u_j(t, 0) = -\frac{N}{2} u_1(t, 0), & t \in (0, T). \end{cases}$$

In this sense, we can view the controls g_j , $j = 1, \dots, N$ as our possibility to move the deflection u_j with respect to the rest position of the water wave, but only at the external vertices.

Following [1], we consider the following functional setting. Let $\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^N L^2(0, \ell_j)$, endowed with the inner product

$$(\underline{u}, \underline{z})_{\mathbb{L}^2(\mathcal{T})} = \sum_{j=1}^N \int_0^{\ell_j} u_j z_j dx, \quad \forall \underline{u}, \underline{z} \in \mathbb{L}^2(\mathcal{T}).$$

Similarly, $\mathbb{H}^k(\mathcal{T})$ be the cartesian product of $H^k(0, \ell_j)$, for $k = -2, -1, 2, 3, 4$ and $\mathbb{H}_0^2(\mathcal{T})$ be the cartesian product of $H_0^2(0, \ell_j)$. For $k = 1, 2$ define

$$H_r^k(0, \ell_j) := \left\{ v \in H^k(0, \ell_j), \left(\frac{d}{dx} \right)^{i-1} v(\ell_j) = 0, 1 \leq i \leq k \right\},$$

where the index r is related to the null right boundary conditions, the space $\mathbb{H}_r^k(\mathcal{T}) := \prod_{j=1}^N H_r^k(0, \ell_j)$, with the norm

$$\|\underline{u}\|_{\mathbb{H}_r^k(\mathcal{T})}^2 = \sum_{j=1}^N \|u_j\|_{H^k(0, \ell_j)}^2, \quad k = 1, 2.$$

Finally, define $H_r^{-1}(0, \ell_j) := (H_r^1(0, \ell_j))'$ the dual space of $H_r^1(0, \ell_j)$ with respect to the pivot space $L^2(0, \ell_j)$ and $\mathbb{H}_r^{-1}(\mathcal{T})$ be the cartesian product of $H_r^{-1}(0, \ell_j)$.

The focus of this work is to study the well-posedness and null controllability of the linear KdV equation posed in a star-shaped network without acting on all the branches. In particular, in this work we consider the case of $N - 1$ acted edges, i.e. $2N - 2$ boundary controls. It is worth mentioning that in the case of null controllability of a single KdV equation by the right via Carleman estimates two controls are needed. In this sense in the network case we are able to reduce the number of controls. To prove our null controllability result, we use the classical dual characterization given by the HUM. Thus, we have to prove an appropriated observability inequality for the adjoint system. Without loss of generality, we will consider that the uncontrolled edge is the first one, that is $p_1 = g_1 = 0$.

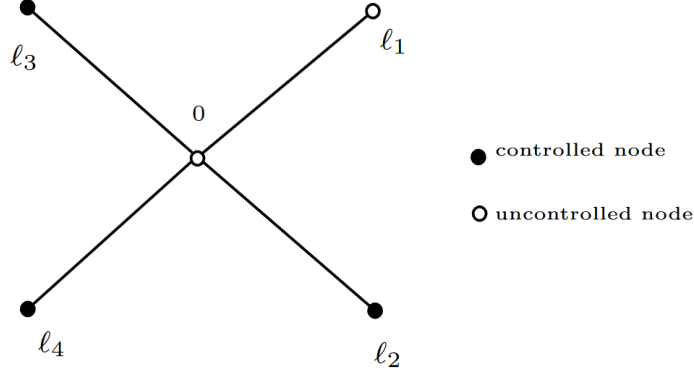


FIG. 1.1. Star-shaped network of $N = 4$ branches

It is not difficult to see that the adjoint system is given by

$$\begin{cases} -\partial_t \varphi_j(t, x) - \partial_x \varphi_j(t, x) - \partial_x^3 \varphi_j(t, x) = 0, & (t, x) \in Q_j, j = 1, \dots, N, \\ \varphi_j(t, 0) = \varphi_1(t, 0), & t \in (0, T), \forall j = 2, \dots, N, \\ \sum_{j=1}^N \partial_x^2 \varphi_j(t, 0) = (\lambda - N)\varphi_1(t, 0), & t \in (0, T), \\ \varphi_j(t, \ell_j) = \partial_x \varphi_j(t, 0) = 0, & t \in (0, T), j = 1, \dots, N, \\ \varphi_j(T, x) = \varphi_j^T(x), & x \in I_j, \end{cases} \quad (\text{Adj})$$

where $\varphi^T \in \mathbb{L}^2(\mathcal{T})$. Note that if the controls g_2, \dots, g_N and p_2, \dots, p_N steer the system (LKdV) from \underline{u}^0 to $\underline{0}$ in $\mathbb{L}^2(\mathcal{T})$, then, multiplying (LKdV) by φ solution of (Adj) and integrating with respect to (t, x) on Q_j , we get the next after some integrations by parts

$$\sum_{j=1}^N \int_0^{\ell_j} u_j^0 \varphi_j(0, x) dx = \sum_{j=2}^N \int_0^T g_j(t) \partial_x^2 \varphi_j(t, \ell_j) dt - \sum_{j=2}^N \int_0^T p_j(t) \partial_x \varphi_j(t, \ell_j) dt,$$

which leads to the following observability inequality;

$$\|\underline{\varphi}(0, x)\|_{\mathbb{L}^2(\mathcal{T})}^2 \leq C \sum_{j=2}^N \left(\|\partial_x^2 \varphi_j(t, \ell_j)\|_{L^2(0, T)}^2 + \|\partial_x \varphi_j(t, \ell_j)\|_{L^2(0, T)}^2 \right). \quad (\text{Obs})$$

In this sense, the subsequent is the main result of this work.

THEOREM 1.1. *Let $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, then, for every $T > 0$, there exist controls $\underline{g} = (0, g_2, \dots, g_N)^T$, $\underline{p} = (0, p_2, \dots, p_N)^T \in (L^2(0, T))^N$, such that the unique solution \underline{u} of (LKdV) satisfies $u_j(T, \cdot) = 0$, for all $j = 1, \dots, N$.*

Regarding the well-posedness, the classical approach based on semigroup theory for the KdV equation on networks, used in several works [1, 10, 19, 20] does not fit at all when using Dirichlet controls. For example, in [13, 5, 14], the main strategy to prove well-posedness is to use the notion of “solutions by transposition” and interpolation arguments. However, these ideas were not yet available in the literature for the KdV equation on networks, particularly due to the network structure the computations generates some traces terms which are not too easy to deal with them.

This article is organized as follows: In Section 2 we give the definition of transposition solutions and we prove the well-posedness of (LKdV). Section 3 is devoted to the Carleman estimates. In Section 4 we prove our null controllability result by using the Carleman inequality and a dissipation estimates. Finally, in Section 5 we give collect some open question related with this work.

2. Well-posedness results. This section is devoted to prove the well-posedness of (LKdV). First, we explain what we mean by a solution of (LKdV). This section is mainly based on the ideas presented in [13, 14].

DEFINITION 2.1. *Given $T > 0$, $\underline{u}^0 \in \mathbb{H}^{-1}(\mathcal{T})$ and $(\underline{g}, \underline{p}) \in (L^2(0, T))^N \times (H^{-1/3}(0, T))^N$, a solution by transposition of (LKdV) is a function $\underline{u} \in L^2(0, T; \mathbb{L}^2(\mathcal{T}))$ satisfying*

$$\begin{aligned} \sum_{j=1}^N \int_0^T \int_0^{\ell_j} u_j f_j dx dt &= \sum_{j=1}^N \langle u_j^0, \varphi_j(0, \cdot) \rangle_{H_r^{-1}(0, \ell_j) \times H_r^1(0, \ell_j)} - \sum_{j=1}^N \int_0^T g(t) \partial_x^2 \varphi(t, \ell_j) dt \\ &\quad - \sum_{j=1}^N \langle p_j, \partial_x \varphi_j(\cdot, \cdot) \rangle_{H^{-1/3}(0, T) \times H^{1/3}(0, T)}, \quad \forall \underline{f} \in L^2(0, T; \mathbb{L}^2(\mathcal{T})), \end{aligned} \quad (2.1)$$

where $\underline{\varphi}$ is the solution of

$$\begin{cases} -\partial_t \varphi_j(t, x) - \partial_x \varphi_j(t, x) - \partial_x^3 \varphi_j(t, x) = f_j(t, x), & (t, x) \in Q_j, \quad j = 1, \dots, N, \\ \varphi_j(t, 0) = \varphi_1(t, 0), & t \in (0, T), \quad \forall j = 2, \dots, N, \\ \sum_{j=1}^N \partial_x^2 \varphi_j(t, 0) = (\lambda - N) \varphi_1(t, 0), & t \in (0, T), \\ \varphi_j(t, \ell_j) = \partial_x \varphi_j(t, 0) = 0, & t \in (0, T), \quad j = 1, \dots, N, \\ \varphi_j(T, x) = 0, & x \in I_j. \end{cases} \quad (2.2)$$

A first remark to be noted is that in order to be consistence, we need the solution $\underline{\varphi}$ of (2.2) to satisfy

$$\varphi_j(0, \cdot) \in H_r^1(0, \ell_j), \quad \partial_x^2 \varphi(\cdot, \ell_j) \in L^2(0, T) \quad \text{and} \quad \partial_x \varphi(\cdot, \ell_j) \in H^{1/3}(0, T), \quad \forall j = 1, \dots, N.$$

The following is the main result of this section, concerning the existence and uniqueness of the Cauchy problem (LKdV).

THEOREM 2.2. *Let $\underline{u}^0 \in \mathbb{H}^{-1}(\mathcal{T})$ and $(\underline{g}, \underline{p}) \in (L^2(0, T))^N \times (H^{-1/3}(0, T))^N$. Then, there exists a unique solution by transposition of (LKdV).*

Let $\underline{u}^0 \in \mathbb{H}^{-1}(\mathcal{T})$ and $(\underline{g}, \underline{p}) \in (L^2(0, T))^N \times (H^{-1/3}(0, T))^N$. Consider the map $\Lambda : L^2(0, T; \mathbb{L}^2(\mathcal{T})) \rightarrow \mathbb{R}$, defined by

$$\Lambda(\underline{f}) = \sum_{j=1}^N \langle u_j^0, \varphi_j(0, \cdot) \rangle_{H_r^{-1}(0, \ell_j) \times H_r^1(0, \ell_j)} - \sum_{j=1}^N \int_0^T g(t) \partial_x^2 \varphi(t, \ell_j) dt - \sum_{j=1}^N \langle p_j, \partial_x \varphi_j(\cdot, \cdot) \rangle_{H^{-1/3}(0, T) \times H^{1/3}(0, T)},$$

where $\underline{\varphi}$ is the associate solution to (2.2). In virtue of Definition 2.1, it suffices to show that Λ is linear and continuous. The map Λ is clearly linear, thus we focus on the continuity. To prove it, we are going to use the following result.

PROPOSITION 2.3. *If $\underline{f} \in L^2(0, T; \mathbb{L}^2(\mathcal{T}))$, then there exists a unique solution $\underline{\varphi} \in C([0, T]; \mathbb{H}_r^1(\mathcal{T})) \cap L^2(0, T; \mathbb{H}^2(\mathcal{T}))$ of (2.2). Moreover, there exists $C > 0$ such that*

$$\|\underline{\varphi}\|_{C([0, T]; \mathbb{H}_r^1(\mathcal{T})) \cap L^2(0, T; \mathbb{H}^2(\mathcal{T}))} + \sum_{j=1}^N \|\partial_x^2 \varphi_j(\cdot, \ell_j)\|_{L^2(0, T)} + \sum_{j=1}^N \|\partial_x \varphi_j(\cdot, \ell_j)\|_{H^{1/3}(0, T)} \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}. \quad (2.3)$$

Once Proposition 2.3 is proved, Theorem 2.2 follows directly. We will now proceed with the proof of Proposition 2.3, which is split into five steps. First, we establish the existence and uniqueness of solutions for (2.2). Second, we consider $\underline{f} \in L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))$ and prove the appropriated energy estimates. Third, we consider a more regular source term $\underline{f} \in L^2(0, T; \mathbb{H}^2(\mathcal{T}))$ and prove appropriated energy estimates. Fourth, we use

interpolation arguments to obtain the appropriated regularity framework. Finally, we estimated the remaining terms.

Proof:

Step 1: Existence and uniqueness of (2.2).

Inspired by [1], define the operator $\mathcal{A}\underline{v} = -\partial_x \underline{v} - \partial_x^3 \underline{v}$, with domain

$$D(\mathcal{A}) = \left\{ \underline{v} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_r^2(\mathcal{T}), v_1(0) = v_j(0), j = 1, \dots, N, \sum_{j=1}^N \frac{d^2 v_j}{dx^2}(0) = -\lambda v_1(0) \right\}.$$

The adjoint operator $\mathcal{A}^* \underline{v} = \partial_x \underline{v} \partial_x^3 \underline{v}$, with domain

$$D(\mathcal{A}^*) = \left\{ \underline{v} \in \mathbb{H}^3(\mathcal{T}) \cap \mathbb{H}_r^1(\mathcal{T}), v_1(0) = v_j(0), \frac{dv_j}{dx}(0) = 0, j = 1, \dots, N, \sum_{j=1}^N \frac{d^2 v_j}{dx^2}(0) = (\lambda - N)v_1(0) \right\}.$$

From [1, Proposition 2.3], we know that $\mathcal{A}, \mathcal{A}^*$ are closed and dissipative operators. Using classical results about inhomogeneous value problems, (see for instance [22]), we can obtain that for all $\underline{f} \in C([0, T]; D(\mathcal{A}^*))$, there exists a unique solution $\underline{\varphi} \in C^1([0, T]; D(\mathcal{A}^*))$ of (2.2). All the integration by parts in what follows can be justified using this existence result and a density argument.

Step 2: Energy estimates for $\underline{f} \in L^2(0, T; \mathbb{H}^{-1}(\mathcal{T}))$.

We start considering $\underline{f} \in L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))$. Multiplying the first line of (2.2) by φ_j and integrate by parts with respect to x . Using the boundary conditions of (2.2) we get

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 dx + \left(\lambda - \frac{N}{2} \right) (\varphi_1(t, 0))^2 + \frac{1}{2} \sum_{j=1}^N (\partial_x \varphi_j(t, \ell_j))^2 \\ & = \sum_{j=1}^N \langle f_j(t, \cdot), \varphi_j(t, \cdot) \rangle_{H_r^{-1}(0, \ell_j) \times H_r^{-1}(0, \ell_j)}, \end{aligned}$$

integrating the last expression with respect to t on $[s, T]$ for $0 \leq s \leq T$ we obtain that

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(s, x))^2 dx + \left(\lambda - \frac{N}{2} \right) \int_s^T (\varphi_1(t, 0))^2 dt + \frac{1}{2} \sum_{j=1}^N \int_s^T (\partial_x \varphi_j(t, \ell_j))^2 dt \\ & = \sum_{j=1}^N \int_s^T \langle f_j(t, \cdot), \varphi_j(t, \cdot) \rangle_{H_r^{-1}(0, \ell_j) \times H_r^{-1}(0, \ell_j)} dt. \end{aligned} \tag{2.4}$$

Now, let us multiply (2.2) by $\varphi_j e^{-x}$ obtaining after integration by parts

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx + \frac{3}{2} \sum_{j=1}^N \int_0^{\ell_j} (\partial_x \varphi_j(t, x))^2 e^{-x} dx + 2 \left(\lambda - \frac{N}{2} \right) (\varphi_1(t, 0))^2 \\ & = \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx + \sum_{j=1}^N \langle f_j(t, \cdot), \varphi_j(t, \cdot) e^{-x} \rangle_{H_r^{-1}(0, \ell_j) \times H_r^{-1}(0, \ell_j)}, \end{aligned}$$

The right-hand side term can be estimated as follows, for all $\varepsilon > 0$ we get

$$\begin{aligned} & \sum_{j=1}^N \langle f_j(t, \cdot), \varphi_j(t, \cdot) \rangle_{H_r^{-1}(0, \ell_j) \times H_r^{-1}(0, \ell_j)} dt \leq \sum_{j=1}^N \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)} \|\varphi_j(t, \cdot)\|_{H^1(0, \ell_j)} dt \\ & \leq C_\varepsilon \sum_{j=1}^N \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)}^2 dt + \varepsilon \int_0^T \sum_{j=1}^N \|\varphi_j(t, \cdot)\|_{H^1(0, \ell_j)}^2 dt. \end{aligned} \tag{2.5}$$

Then for $\varepsilon > 0$ small enough we get

$$-\frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx + \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\partial_x \varphi_j(t, x))^2 e^{-x} dx + 2 \left(\lambda - \frac{N}{2} \right) (\varphi_1(t, 0))^2 \quad (2.6)$$

$$\leq \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx + C_\varepsilon \sum_{j=1}^N \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)}^2. \quad (2.7)$$

In particular we have,

$$-\frac{1}{2} \frac{d}{dt} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx \leq \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx + C_\varepsilon \int_0^T \sum_{j=1}^N \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)}^2,$$

by Gronwall's lemma we deduce for $0 < t_1 < t_2 < T$,

$$\sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t_1, x))^2 e^{-x} dx \leq e^{C(t_2 - t_1)} \left(\sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t_2, x))^2 e^{-x} dx + \int_{t_1}^{t_2} \sum_{j=1}^N \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)}^2 \right). \quad (2.8)$$

Taking $t_2 = T$ and using that $1 \geq e^{-x} \geq e^{-\ell_j}$ for $x \in (0, \ell_j)$ we get

$$C_{min} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t_1, x))^2 dx \leq \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t_1, x))^2 e^{-x} dx \leq e^{CT} \int_0^T \sum_{j=1}^N \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)}^2,$$

where $C_{min} = \min_{j=1, \dots, N} (e^{-\ell_j})$. Thus, taking the supremum for $0 < t_1 < T$ of the last expression we get:

$$\|\underline{\varphi}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T}))} \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))}. \quad (2.9)$$

Now integrating (2.6) between 0 and T we observe

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(0, x))^2 e^{-x} dx + \frac{1}{2} \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x \varphi_j(t, x))^2 e^{-x} dx dt + 2 \left(\lambda - \frac{N}{2} \right) \int_0^T (\varphi_1(t, 0))^2 dt \\ & \leq \frac{1}{2} \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\varphi_j(t, x))^2 e^{-x} dx dt + C \sum_{j=1}^N \int_0^T \|f_j(t, \cdot)\|_{H_r^{-1}(0, \ell_j)}^2 dt. \end{aligned}$$

Using (2.9) we get

$$\|\underline{\varphi}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T})) \cap L^2(0, T; \mathbb{H}_r^1(\mathcal{T}))} + \|\varphi_1(\cdot, 0)\|_{L^2(0, T)} \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))}. \quad (2.10)$$

Finally, taking $s = 0$ in (2.4) and using (2.10)-(2.5) we conclude

$$\sum_{j=1}^N \int_0^T (\partial_x \varphi(\cdot, \ell_j))^2 dt \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))}. \quad (2.11)$$

We recall the classical definition of $H^{-1}(0, \ell_j) = (H_0^1(0, \ell_j))'$, with respect to the pivot space $L^2(0, \ell_j)$ and that for $v \in H_0^1(0, \ell_j) \subset H_r^1(0, \ell_j)$, $\|v\|_{H_0^1(0, \ell_j)} = \|v\|_{H_r^1(0, \ell_j)}$. Now let $f \in H_r^{-1}(0, \ell_j)$, then f is a linear and bounded function from $H_r^1(0, \ell_j)$ to \mathbb{R} , in particular we can consider its restriction to $H_0^1(0, \ell_j)$ (still called by f) which is linear and bounded because for $v \in H_0^1(0, \ell_j)$, $\|v\|_{H_0^1(0, \ell_j)} = \|v\|_{H_r^1(0, \ell_j)}$ thus $f \in H^{-1}(0, \ell_j)$. Moreover, by as and $H_0^1(0, \ell_j)$ is dense in $H_r^1(0, \ell_j)$ the norm $\|f\|_{H_r^{-1}(0, \ell_j)} = \|f_j\|_{H^{-1}(0, \ell_j)}$. Using this fact and a density argument we get from (2.10), (2.11)

$$\|\underline{\varphi}\|_{C([0, T]; \mathbb{L}^2(\mathcal{T})) \cap L^2(0, T; \mathbb{H}_r^1(\mathcal{T}))} + \|\varphi_1(\cdot, 0)\|_{L^2(0, T)} + \sum_{j=1}^N \int_0^T (\partial_x \varphi(\cdot, \ell_j))^2 dt \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}^{-1}(\mathcal{T}))}. \quad (2.12)$$

Step 3: Energy estimates for $\underline{f} \in L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))$.

We multiply the first line of (Adj) by $\partial_x^3 \partial_t \varphi_j e^{-x}$ and integrate with respect to x . After some integration by parts we get

$$\begin{aligned}
& -\frac{1}{2} \frac{d}{dt} \int_0^{\ell_j} (\partial_x^3 \varphi_j(t, x))^2 e^{-x} dx + \frac{3}{2} \int_0^{\ell_j} (\partial_x \partial_t \varphi_j(t, x))^2 e^{-x} dx + \frac{1}{2} (\partial_x \partial_t \varphi_j(t, \ell_j))^2 e^{-\ell_j} + \frac{1}{2} (\partial_t \varphi_1(t, 0))^2 \\
& + \partial_t \varphi_1(t, 0) \partial_x^2 \partial_t \varphi_j(t, 0) - \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} + \frac{1}{2} \frac{d}{dt} \int_0^{\ell_j} (\partial_x^2 \varphi_j(t, x))^2 e^{-x} dx \\
& - \frac{1}{2} \int_0^{\ell_j} (\partial_t \varphi_j(t, x))^2 e^{-x} dx + \int_0^{\ell_j} \partial_x^2 \varphi_j(t, x) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx - \frac{1}{2} \frac{d}{dt} \int_0^{\ell_j} (\partial_x \varphi_j(t, x))^2 e^{-x} dx \\
& - \partial_x \varphi_j(t, \ell_j) \partial_x \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} = \int_0^{\ell_j} f_j(t, x) \partial_x^3 \partial_t \varphi_j(t, x) e^{-x} dx.
\end{aligned}$$

We take now, the sum from $j = 1 \dots N$ and we integrate from $T \leq t \leq s$ obtaining

$$\begin{aligned}
& \frac{C_{\min}}{2} \sum_{j=1}^N \int_0^{\ell_j} (\partial_x^3 \varphi_j(s, x))^2 dx + \frac{3C_{\min}}{2} \sum_{j=1}^N \int_s^T \int_0^{\ell_j} (\partial_x \partial_t \varphi_j(t, x))^2 dx dt \\
& + \frac{C_{\min}}{2} \sum_{j=1}^N \int_s^T (\partial_x \partial_t \varphi_j(t, \ell_j))^2 dt + \left(\lambda - \frac{N}{2} \right) \int_s^T (\partial_t \varphi_1(t, 0))^2 dt \\
& \leq \underbrace{\sum_{j=1}^N \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} dt}_{T_1} - \underbrace{\frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} [(\partial_x^2 \varphi_j(s, x))^2 - (\partial_x \varphi_j(s, x))^2] e^{-x} dx}_{T_2} \\
& + \underbrace{\frac{1}{2} \sum_{j=1}^N \int_s^T \int_0^{\ell_j} (\partial_t \varphi_j(t, x))^2 e^{-x} dx}_{T_3} - \underbrace{\sum_{j=1}^N \int_s^T \int_0^{\ell_j} \partial_x^2 \varphi_j(t, x) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx}_{T_4} \\
& + \underbrace{\sum_{j=1}^N \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x \partial_t \varphi_j(t, \ell_j) e^{-\ell_j}}_{T_5} + \underbrace{\sum_{j=1}^N \int_s^T \int_0^{\ell_j} f_j(t, x) \partial_x^3 \partial_t \varphi_j(t, x) e^{-x} dx}_{T_6}.
\end{aligned} \tag{2.13}$$

Claim 1. For $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\sum_{i=1}^6 T_i \leq \varepsilon \left(\sum_{j=1}^N \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + \|\underline{\varphi}\|_{\mathcal{C}([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + \|\underline{\varphi}\|_{H^1(0, T; \mathbb{H}^1(\mathcal{T}))}^2 \right) + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}^2. \tag{2.14}$$

We prove this claim in Appendix A. Using this claim in (2.13) with $\varepsilon > 0$ small enough we get

$$\begin{aligned}
& \|\underline{\varphi}\|_{\mathcal{C}([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + \|\underline{\varphi}\|_{H^1(0, T; \mathbb{H}^1(\mathcal{T}))}^2 + \sum_{j=1}^N \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt \\
& + \left(\lambda - \frac{N}{2} \right) \int_0^T (\partial_t \varphi_1(t, 0))^2 dt \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}^2.
\end{aligned}$$

Now as $\partial_x^4 \varphi_j = \partial_x f_j - \partial_x^2 \varphi_j - \partial_x \partial_t \varphi_j$, we deduce $\partial_x^4 \varphi_j \in L^2(0, T; L^2(0, \ell_j))$. Moreover

$$\|\underline{\varphi}\|_{L^2(0, T; \mathbb{H}^4(\mathcal{T}))} \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}.$$

Step 4: Interpolation arguments. Consider the linear operators Φ and Ψ , defined by $\Phi(\underline{f}) = \underline{\varphi}$ and $\Psi(\underline{f}) = (\partial_x \varphi_1(\cdot, \ell_1) \dots \partial_x \varphi_N(\cdot, \ell_N))$, where $\underline{\varphi}$ is the associate solution of (2.2) with right-hand side \underline{f} . From

the Steps 2 and 3 we get that

$$\begin{aligned}
\Phi &: L^2(0, T; \mathbb{H}^{-1}(\mathcal{T})) \rightarrow C([0, T]; \mathbb{L}^2(\mathcal{T})) \cap L^2(0, T; \mathbb{H}^1(\mathcal{T})) \text{ is continuous,} \\
\Phi &: L^2(0, T; \mathbb{H}_0^2(\mathcal{T})) \rightarrow C([0, T]; \mathbb{H}^3(\mathcal{T})) \cap L^2(0, T; \mathbb{H}^4(\mathcal{T})) \text{ is continuous,} \\
\Psi &: L^2(0, T; \mathbb{H}^{-1}(\mathcal{T})) \rightarrow (L^2(0, T))^N \text{ is continuous,} \\
\Psi &: L^2(0, T; \mathbb{H}_0^2(\mathcal{T})) \rightarrow (H^1(0, T))^N \text{ is continuous.}
\end{aligned}$$

By interpolation arguments as in [13, Section 2.3.2] we obtain the following estimates

$$\|\underline{v}\|_{C([0, T]; \mathbb{H}_r^1(\mathcal{T})) \cap L^2(0, T; \mathbb{H}^2(\mathcal{T}))} + \sum_{j=1}^N \|\partial_x \varphi_j(\cdot, \ell_j)\|_{H^{1/3}(0, T)} \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}. \quad (2.15)$$

Remark 1. In [14], the interpolation arguments are done dealing with the spaces $H_\ell^1(0, L) =: \{v \in H^1(0, L), v(0) = 0\}$ and its dual $H_\ell^{-1}(0, L) = (H_\ell^1(0, L))'$ with respect to the pivot space $L^2(0, L)$. With this choice is not too clear for us how adapt [13, Section 2.3.2], this is why we prove (2.12) involving the $H^{-1}(0, \ell_j)$ -norm and not $H_r^{-1}(0, \ell_j)$. \circ

Step 5: Estimate the $L^2(0, T)$ -norm of $\partial_x^2 \varphi_j(\cdot, \ell_j)$. With the estimate (2.15) we are almost ready with the proof of (2.3). Only left to estimate the $L^2(0, T)$ -norm of $\partial_x^2 \varphi_j(\cdot, \ell_j)$. We multiply (2.2) by $x \partial_x^2 \varphi_j$ and integrate x on $(0, \ell_j)$. We obtain

$$-\int_0^{\ell_j} \partial_t \varphi_j x \partial_x^2 \varphi_j dx - \int_0^{\ell_j} \partial_x \varphi_j x \partial_x^2 \varphi_j dx - \int_0^{\ell_j} \partial_x^3 \varphi_j x \partial_x^2 \varphi_j dx = \int_0^{\ell_j} f_j x \partial_x^2 \varphi_j dx.$$

By integrating by parts

$$\begin{aligned}
-\int_0^{\ell_j} \partial_t \varphi_j x \partial_x^2 \varphi_j dx &= \int_0^{\ell_j} \partial_t \partial_x \varphi_j x \partial_x \varphi_j dx + \int_0^{\ell_j} \partial_t \varphi_j \partial_x \varphi_j dx \\
&= \frac{1}{2} \int_0^{\ell_j} x \partial_t (\partial_x \varphi_j)^2 + \int_0^{\ell_j} (-\partial_x \varphi_j - \partial_x^3 \varphi_j - f_j) \partial_x \varphi_j dx \\
&= \frac{1}{2} \int_0^{\ell_j} x \partial_t (\partial_x \varphi_j)^2 - \int_0^{\ell_j} (\partial_x \varphi_j)^2 dx + \int_0^{\ell_j} (\partial_x^2 \varphi_j)^2 dx - \int_0^{\ell_j} f_j \partial_x \varphi_j dx \\
&\quad - \partial_x \varphi_j(\cdot, \ell_j) \partial_x^2 \varphi_j(\cdot, \ell_j), \\
-\int_0^{\ell_j} \partial_x \varphi_j x \partial_x^2 \varphi_j dx &= -\frac{1}{2} \int_0^{\ell_j} x \partial_x (\partial_x \varphi_j)^2 dx = \frac{1}{2} \int_0^{\ell_j} (\partial_x \varphi_j)^2 dx - \frac{\ell_j}{2} (\partial_x \varphi_j(\cdot, \ell_j))^2, \\
-\int_0^{\ell_j} \partial_x^3 \varphi_j x \partial_x^2 \varphi_j dx &= -\frac{1}{2} \int_0^{\ell_j} x \partial_x (\partial_x^2 \varphi_j)^2 dx = \frac{1}{2} \int_0^{\ell_j} (\partial_x^2 \varphi_j)^2 dx - \frac{\ell_j}{2} (\partial_x^2 \varphi_j(\cdot, \ell_j))^2.
\end{aligned}$$

Joining these computations

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} x \partial_t (\partial_x \varphi_j)^2 dx - \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} (\partial_x \varphi_j)^2 dx dt + \frac{3}{2} \sum_{j=1}^N \int_0^{\ell_j} (\partial_x^2 \varphi_j)^2 dx dt - \sum_{j=1}^N \partial_x \varphi_j(\cdot, \ell_j) \partial_x^2 \varphi_j(\cdot, \ell_j) \\
- \sum_{j=1}^N \frac{\ell_j}{2} (\partial_x \varphi_j(\cdot, \ell_j))^2 - \sum_{j=1}^N \frac{\ell_j}{2} (\partial_x^2 \varphi_j(\cdot, \ell_j))^2 = \sum_{j=1}^N \int_0^{\ell_j} (x \partial_x^2 \varphi_j + \partial_x \varphi_j) f_j dx.
\end{aligned}$$

Integrating t over $(0, T)$

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^N \int_0^{\ell_j} x (\partial_x \varphi_j(0, x))^2 dx dt + \frac{1}{2} \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x \varphi_j)^2 dx dt + \sum_{j=1}^N \int_0^T \frac{\ell_j}{2} (\partial_x \varphi_j(t, \ell_j))^2 dt \\ & + \sum_{j=1}^N \frac{\ell_j}{2} \int_0^T (\partial_x^2 \varphi_j(t, \ell_j))^2 dt = \frac{3}{2} \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (\partial_x^2 \varphi_j)^2 dx dt - \sum_{j=1}^N \int_0^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \varphi_j(\cdot, \ell_j) dt \\ & \quad - \sum_{j=1}^N \int_0^T \int_0^{\ell_j} (x \partial_x^2 \varphi_j + \partial_x \varphi_j) f_j dx, \end{aligned}$$

using Young's inequality, there exists $C > 0$ such that

$$\sum_{j=1}^N \int_0^T (\partial_x^2 \varphi_j(t, \ell_j))^2 dt \leq C \left(\|f\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2 + \|\underline{\varphi}\|_{L^2(0, T; \mathbb{H}^2(\mathcal{T}))}^2 + \sum_{j=1}^N \|\partial_x \varphi_j(\cdot, \ell_j)\|_{L^2(0, T)}^2 \right).$$

Finally by (2.15) we conclude

$$\sum_{j=1}^N \int_0^T (\partial_x^2 \varphi_j(t, \ell_j))^2 dt \leq C \|f\|_{L^2(0, T; \mathbb{L}^2(\mathcal{T}))}^2,$$

and thus we have proved Proposition 2.3 □

3. Carleman estimate. In this section, we prove a Carleman estimate for the system (Adj) in order to deduce our observability inequality. Following [5, 13], let us introduce the following weights functions.

$$\begin{aligned} \alpha_j(t, x) &= \frac{\beta_j(x)}{t^{1/2}(T-t)^{1/2}} = \beta_j(x) \rho(t), \quad (t, x) \in Q_j, \\ \beta_j(x) &= \begin{cases} -a_1 x^2 + b_1 \ell_1 x + c_1 \ell_1^2, & x \in (0, \ell_1), j = 1, \\ -a_j x^2 - b_j \ell_j x + c_j \ell_j^2, & x \in (0, \ell_j), j = 2, \dots, N. \end{cases} \end{aligned} \quad (3.1)$$

where $a_j, b_j, c_j > 0$, are taken such that β_j be strictly positive, and concave for $j = 1, \dots, N$. Moreover, β_1 is strictly decreasing, and β_j is strictly increasing for $j = 2, \dots, N$. The objective of this section is to prove the following Carleman estimate.

Remark 2. We use a temporal weight with power $1/2$ ($t^{1/2}(T-t)^{1/2}$), which is optimal for the KdV equation (see [13]). ◦

PROPOSITION 3.1. Let $T > 0$, there exists $C > 0$ and $s^* > 0$, such that for any $\underline{\varphi}^T \in \mathbb{L}^2(\mathcal{T})$ and $s \geq s^*$, the solution $\underline{\varphi}$ of (Adj) satisfies

$$\sum_{j=1}^N \iint_{Q_j} e^{-2\alpha_j} (s^5 \rho^5 (\varphi_j)^2 + s^3 \rho^3 (\partial_x \varphi_j)^2) dx dt \leq C \sum_{j=2}^N \int_0^T e^{-2s\alpha_j(t, \ell_j)} (s^3 \rho^3 (\varphi_j(t, \ell_j))^2 + s \rho (\partial_x^2 \varphi_j(t, \ell_j))^2) dt. \quad (3.2)$$

Proof:

Let $s > 0$ and define $\psi_j = e^{-s\alpha_j} \varphi_j$. Consider the operator $\mathcal{L}\varphi_j = \partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j$ and the conjugated operator $\mathcal{P}\varphi_j = e^{-s\alpha_j} \mathcal{L}(e^{s\alpha_j} \varphi_j)$. After some computations, we can decompose $\mathcal{P}\psi_j$ as $\mathcal{P}\psi_j = \mathcal{P}_1\psi_j + \mathcal{P}_2\psi_j + \mathcal{R}\psi_j$, where

$$\begin{aligned} \mathcal{P}_1\psi_j &= \partial_x^3 \psi_j + \partial_t \psi_j + 3s^2 (\partial_x \alpha_j)^2 \partial_x \psi_j + \partial_x \psi_j, \\ \mathcal{P}_2\psi_j &= s^3 (\partial_x \alpha_j)^3 \psi_j + 3s (\partial_x \alpha_j) \partial_x^2 \psi_j + s (\partial_t \alpha_j) \psi_j + 3s (\partial_x^2 \alpha_j) \partial_x \psi_j + s (\partial_x \alpha_j) \psi, \\ \mathcal{R}\psi_j &= 3s^2 (\partial_x \alpha_j) (\partial_x^2 \alpha_j) \psi_j + s (\partial_x^3 \alpha_j) \psi_j. \end{aligned}$$

Now, taking L^2 -norm in Q_j of $\mathcal{P}_1\psi_j + \mathcal{P}_2\psi_j = \mathcal{P}\psi_j - \mathcal{R}\psi_j$ we get

$$\iint_{Q_j} \mathcal{P}_1\psi_j\mathcal{P}_2\psi_j dxdt \leq \|\mathcal{P}\psi_j\|_{L^2(Q_j)}^2 + \|\mathcal{R}\psi_j\|_{L^2(Q_j)}^2$$

From now on, we will use \iint to denote \iint_{Q_j} . Consider also $I_{i,k}$ the L^2 -product in Q_j of the i -th term of $\mathcal{P}_1\psi_j$ with the k -th term of $\mathcal{P}_2\psi_j$. Before starting to estimate the terms, $I_{i,k}$ we recall some properties of ψ_j . Note first that as $\psi_j = e^{-s\alpha_j}\varphi_j$, using the boundary conditions of (Adj) $\psi_j(t, \ell_j) = 0$ and $\psi_j(t, 0) = e^{-s\alpha_j(t,0)}\varphi_1(t, 0)$. Thus asking to α_j satisfy $\alpha_j(t, 0) = \alpha_1(t, 0)$ for all $j = 1, \dots, N$, which is equivalent to take, $c_j\ell_j^2 = c_1\ell_1^2$ we deduce $\psi_j(t, 0) = \psi_1(t, 0)$.

3.1. Computing the terms $I_{i,k}$. We compute first the terms $I_{1,k}$,

- $I_{1,1} = s^3 \iint (\partial_x \alpha_j)^3 \psi_j \partial_x^3 \psi_j dxdt$. Performing integration by parts and using $\psi_j(t, \ell_j) = 0$, $\psi_j(t, 0) = \psi_1(t, 0)$ and $\partial_x^3 \alpha_j = 0$, we get

$$\begin{aligned} I_{1,1} = & -3s^3 \iint (\psi_j)^2 (\partial_x^2 \alpha_j)^3 dxdt + \frac{9}{2}s^3 \iint (\partial_x \psi_j)^2 (\partial_x \alpha_j)^2 \partial_x^2 \alpha_j dxdt - \frac{s^3}{2} \int_0^T (\partial_x \psi_j(t, \ell_j))^2 (\partial_x \alpha_j(t, \ell_j))^3 dt \\ & + \frac{s^3}{2} \int_0^T (\partial_x \psi_j(t, 0))^2 (\partial_x \alpha_j(t, 0))^3 dt + 3s^3 \int_0^T \partial_x \psi_j(t, 0) \psi_1(t, 0) \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^2 dt \\ & - 3s^3 \int_0^T (\psi_1(t, 0))^2 \partial_x \alpha_j(t, 0) (\partial_x^2 \alpha_j(t, 0))^2 dt - s^3 \int_0^T \partial_x^2 \psi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt. \end{aligned}$$

- $I_{1,2} = 3s \iint (\partial_x \alpha_j) \partial_x^2 \psi_j \partial_x^3 \psi_j dxdt$. Integrating by parts, we get

$$I_{1,2} = -\frac{3}{2}s \iint (\partial_x^2 \psi_j)^2 \partial_x^2 \alpha_j dxdt + \frac{3}{2}s \int_0^T (\partial_x^2 \psi_j(t, \ell_j))^2 \partial_x \alpha_j(t, \ell_j) dt - \frac{3}{2}s \int_0^T (\partial_x^2 \psi_j(t, 0))^2 \partial_x \alpha_j(t, 0) dt.$$

- $I_{1,3} = s \iint (\partial_t \alpha_j) \partial_x^3 \psi_j \psi_j dxdt$. Again, using $\psi_j(t, \ell_j) = 0$, $\psi_j(t, 0) = \psi_1(t, 0)$ and $\partial_x^3 \alpha_j = 0$, we get

$$\begin{aligned} I_{1,3} = & \frac{3}{2}s \iint (\partial_x \psi_j)^2 \partial_{xt} \alpha_j dxdt - \frac{s}{2} \int_0^T (\partial_x \psi_j(t, \ell_j))^2 \partial_t \alpha_j(t, \ell_j) dt + \frac{s}{2} \int_0^T (\partial_x \psi_j(t, 0))^2 \partial_t \alpha_j(t, 0) dt \\ & + s \int_0^T \partial_x \psi_j(t, 0) \psi_1(t, 0) \partial_{xt} \alpha_j(t, 0) dt - \frac{s}{2} \int_0^T (\psi_1(t, 0))^2 \partial_{xxt} \alpha_j(t, 0) dt \\ & - s \int_0^T \partial_x^2 \psi_j(t, 0) \psi_1(t, 0) \partial_t \alpha_j(t, 0) dt. \end{aligned}$$

- $I_{1,4} = 3s \iint (\partial_x^2 \alpha_j) \partial_x^3 \psi_j \partial_x \psi_j dxdt$. Integrating by parts and by $\partial_x^3 \alpha_j = 0$,

$$\begin{aligned} I_{1,4} = & -3s \iint (\partial_x^2 \psi_j)^2 \partial_x^2 \alpha_j dxdt + 3s \int_0^T \partial_x^2 \psi_j(t, \ell_j) \partial_x \psi_j(t, \ell_j) \partial_x^2 \alpha_j(t, \ell_j) dt \\ & - 3s \int_0^T \partial_x^2 \psi_j(t, 0) \partial_x \psi_j(t, 0) \partial_x^2 \alpha_j(t, 0) dt. \end{aligned}$$

- $I_{1,5} = s \iint (\partial_x \alpha_j) \partial_x^3 \psi_j \psi_j dxdt$. Using $\psi_j(t, \ell_j) = 0$, $\psi_j(t, 0) = \psi_1(t, 0)$ and $\partial_x^3 \alpha_j = 0$, we get

$$\begin{aligned} I_{1,5} = & \frac{3}{2}s \iint (\partial_x^2 \psi_j)^2 \partial_x^2 \alpha_j dxdt - \frac{s}{2} \int_0^T (\partial_x \psi_j(t, \ell_j))^2 \partial_x \alpha_j(t, \ell_j) dt + \frac{s}{2} \int_0^T (\partial_x \psi_j(t, 0))^2 \partial_x \alpha_j(t, 0) dt \\ & + s \int_0^T \partial_x \psi_j(t, 0) \psi_1(t, 0) \partial_x^2 \alpha_j(t, 0) dt - s \int_0^T \partial_x^2 \psi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) dt. \end{aligned}$$

Now, we follow computing the terms $I_{2,k}$,

- $I_{2,1} = s^3 \iint (\partial_x \alpha_j)^3 \partial_t \psi_j \psi_j dx dt$. By performing integration by parts and using $\psi_j(0, \cdot) = \psi_j(T, \cdot) = 0$ we get

$$I_{2,1} = -\frac{3}{2} s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 \partial_{xt} \alpha_j dx dt.$$

- $I_{2,2} = 3s \iint (\partial_x \alpha_j) \partial_t \psi_j \partial_x^2 \psi_j dx dt$. Using $\partial_x \psi_j(0, \cdot) = \partial_x \psi_j(T, \cdot) = 0$ and the boundary conditions, we get the following.

$$I_{2,2} = \frac{3}{2} s \iint (\partial_x \psi_j)^2 \partial_{xt} \alpha_j dx dt - 3s \iint (\partial_x^2 \alpha_j) \partial_t \psi_j \partial_x \psi_j dx dt - 3s \int_0^T \partial_t \psi_j(t, 0) \partial_x \psi_j(t, 0) \partial_x \alpha_j(t, 0) dt.$$

- $I_{2,3} = s \iint (\partial_t \alpha_j) \partial_t \psi_j \psi_j dx dt$. Using $\psi_j(0, \cdot) = \psi_j(T, \cdot) = 0$

$$I_{2,3} = -\frac{s}{2} \iint (\psi_j)^2 \partial_t^2 \alpha_j dx dt$$

- $I_{2,4} = 3s \iint (\partial_x^2 \alpha_j) \partial_t \psi_j \partial_x \psi_j dx dt$. Observe, that this term eliminates the second term of $I_{2,2}$.
- $I_{2,5} = s \iint (\partial_x \alpha_j) \partial_t \psi_j \psi_j dx dt$. Again by $\psi_j(0, \cdot) = \psi_j(T, \cdot) = 0$.

$$I_{2,5} = -\frac{s}{2} \iint (\psi_j)^2 \partial_{xt} \alpha_j dx dt.$$

We study now the terms $I_{3,k}$.

- $I_{3,1} = 3s^5 \iint (\partial_x \alpha_j)^5 \partial_x \psi_j \psi_j dx dt$. Thus, by the boundary conditions.

$$I_{3,1} = -\frac{15}{2} s^5 \iint (\psi_j)^2 (\partial_x \alpha_j)^4 \partial_x^2 \alpha_j dx dt - \frac{3}{2} s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt.$$

- $I_{3,2} = 9s^3 \iint (\partial_x \alpha_j)^3 \partial_x^2 \psi_j \partial_x \psi_j dx dt$. Then, after some integration by parts

$$\begin{aligned} I_{3,2} = & -\frac{27}{2} s^3 \iint (\partial_x \psi_j)^2 (\partial_x \alpha_j)^2 \partial_x^2 \alpha_j dx dt + \frac{9}{2} s^3 \int_0^T (\partial_x \psi_j(t, \ell_j))^2 (\partial_x \alpha_j(t, \ell_j))^3 dt \\ & - \frac{9}{2} s^3 \int_0^T (\partial_x \psi_j(t, 0))^2 (\partial_x \alpha_j(t, 0))^3 dt. \end{aligned}$$

- $I_{3,3} = 3s^3 \iint (\partial_x \alpha_j)^2 \partial_t \alpha_j \partial_x \psi_j \psi_j dx dt$. Using the boundary conditions

$$\begin{aligned} I_{3,3} = & -3s^3 \iint (\psi_j)^2 \partial_x \alpha_j \partial_x^2 \alpha_j \partial_t \alpha_j dx dt - \frac{3}{2} s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 \partial_{xt} \alpha_j dx dt \\ & - \frac{3}{2} s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 \partial_t \alpha_j(t, 0) dt. \end{aligned}$$

- $I_{3,4} = 9s^3 \iint (\partial_x \psi_j)^2 (\partial_x \alpha_j)^2 (\partial_x^2 \alpha_j) dx dt$.

- $I_{3,5} = 3s^3 \iint (\partial_x \alpha_j)^3 \partial_x \psi_j \psi_j dx dt$. Integrating by parts and by the boundary conditions

$$I_{3,5} = -\frac{9}{2}s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 (\partial_x^2 \alpha_j) dx dt - \frac{3}{2}s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^3 dt.$$

We finish this part by computing $I_{4,k}$.

- $I_{4,1} = s^3 \iint (\partial_x \alpha_j)^3 \partial_x \psi_j \psi_j dx dt$. Noticing that $I_{4,1} = \frac{1}{3}I_{3,5}$, we get

$$I_{4,1} = -\frac{3}{2}s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 (\partial_x^2 \alpha_j) dx dt - \frac{s^3}{2} \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^3 dt.$$

- $I_{4,2} = 3s \iint (\partial_x \alpha_j) \partial_x^2 \psi_j \partial_x \psi_j dx dt$. Integrating by parts

$$I_{4,2} = -\frac{3}{2}s \iint (\partial_x \psi_j)^2 (\partial_x^2 \alpha_j) dx dt + \frac{3}{2}s \int_0^T (\partial_x \psi_j(t, \ell_j))^2 (\partial_x \alpha_j(t, \ell_j)) dt - \frac{3}{2}s \int_0^T (\partial_x \psi_j(t, 0))^2 (\partial_x \alpha_j(t, 0)) dt.$$

- $I_{4,3} = s \iint (\partial_t \alpha_j) \partial_x \psi_j \psi_j dx dt$. Using the boundary conditions.

$$I_{4,3} = -\frac{s}{2} \iint (\psi_j)^2 \partial_{xt} \alpha_j dx dt - \frac{s}{2} \int_0^T (\psi_1(t,0))^2 \partial_t \alpha_j(t,0) dt.$$

- $I_{4,4} = 3s \iint (\partial_x \psi_j)^2 \partial_x^2 \alpha_j dx dt$.

- $I_{4,5} = s \iint (\partial_x \alpha_j) \partial_x \psi_j \psi_j dx dt$. Using the boundary conditions.

$$I_{4,5} = -\frac{s}{2} \iint (\psi_j)^2 \partial_x^2 \alpha_j dx dt - \frac{s}{2} \int_0^T (\psi_1(t,0))^2 \partial_x \alpha_j(t,0) dt.$$

Finally, we obtain

$$\iint \mathcal{P}_1 \psi_j \mathcal{P}_2 \psi_j dx dt = \sum_{i=1}^4 \sum_{k=1}^5 I_{i,k} = \mathcal{I} + \mathcal{B}_{\ell_j} + \mathcal{B}_0,$$

where

$$\begin{aligned} \mathcal{I} = & -\frac{15}{2}s^5 \iint (\psi_j)^2 (\partial_x \alpha_j)^4 \partial_x^2 \alpha_j dx dt - 3s^3 \iint (\psi_j)^2 (\partial_x^2 \alpha_j)^3 dx dt - 3s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 \partial_{xt} \alpha_j dx dt \\ & - 3s^3 \iint (\psi_j)^2 \partial_x \alpha_j \partial_x^2 \alpha_j \partial_t \alpha_j dx dt - 6s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 \partial_x^2 \alpha_j dx dt - \frac{9}{2}s \iint (\partial_x^2 \psi_j)^2 \partial_x^2 \alpha_j dx dt \\ & + 3s \iint (\partial_x \psi_j)^2 \partial_{xt} \alpha_j dx dt + 3s \iint (\partial_x \psi_j)^2 \partial_x^2 \alpha_j dx dt - s \iint (\psi_j)^2 \partial_{xt} \alpha_j dx dt \\ & - \frac{s}{2} \iint (\psi_j)^2 \partial_t^2 \alpha_j dx dt - \frac{s}{2} \iint (\psi_j)^2 \partial_x^2 \alpha_j dx dt, \end{aligned}$$

$$\begin{aligned} \mathcal{B}_{\ell_j} = & 4s^3 \int_0^T (\partial_x \psi_j(t, \ell_j))^2 (\partial_x \alpha_j(t, \ell_j))^3 dt + \frac{3}{2}s \int_0^T (\partial_x^2 \psi_j(t, \ell_j))^2 \partial_x \alpha_j(t, \ell_j) dt \\ & - \frac{s}{2} \int_0^T (\partial_x \psi_j(t, \ell_j))^2 \partial_t \alpha_j(t, \ell_j) dt + 3s \int_0^T \partial_x^2 \psi_j(t, \ell_j) \partial_x \psi_j(t, \ell_j) \partial_x^2 \alpha_j(t, \ell_j) dt \\ & + s \int_0^T (\partial_x \psi_j(t, \ell_j))^2 \partial_x \alpha_j(t, \ell_j) dt, \end{aligned}$$

$$\mathcal{B}_0 = \mathcal{B}_0^1 + \mathcal{B}_0^2 + \mathcal{B}_0^3, \quad (3.3)$$

where

$$\begin{aligned} \mathcal{B}_0^1 = & -\frac{3}{2}s^5 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^5 dt - 3s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0)) (\partial_x^2 \alpha_j(t,0))^2 dt \\ & - \frac{3}{2}s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^2 \partial_t \alpha_j(t,0) dt - 2s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^3 dt \\ & - \frac{s}{2} \int_0^T (\psi_1(t,0))^2 (\partial_{xt} \alpha_j(t,0)) dt - \frac{s}{2} \int_0^T (\psi_1(t,0))^2 \partial_t \alpha_j(t,0) dt - \frac{s}{2} \int_0^T (\psi_1(t,0))^2 \partial_x \alpha_j(t,0) dt, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \mathcal{B}_0^2 = & 3s^3 \int_0^T \partial_x \psi_j(t,0) \psi_1(t,0) \partial_x^2 \alpha_j(t,0) (\partial_x \alpha_j(t,0))^2 dt - 4s^3 \int_0^T (\partial_x \psi_j(t,0))^2 (\partial_x \alpha_j(t,0))^3 dt \\ & + \frac{s}{2} \int_0^T (\partial_x \psi_j(t,0))^2 (\partial_t \alpha_j(t,0)) dt + s \int_0^T \partial_x \psi_j(t,0) \psi_1(t,0) \partial_{xt} \alpha_j(t,0) dt \\ & - s \int_0^T (\partial_x \psi_j(t,0))^2 (\partial_x \alpha_j(t,0)) dt + s \int_0^T \partial_x \psi_j(t,0) \psi_1(t,0) (\partial_x^2 \alpha_j(t,0)) dt \\ & - 3s \int_0^T \partial_t \psi_j(t,0) \partial_x \psi_j(t,0) (\partial_x \alpha_j(t,0)) dt, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \mathcal{B}_0^3 = & -s^3 \int_0^T \partial_x^2 \psi_j(t,0) \psi_1(t,0) (\partial_x \alpha_j(t,0))^3 dt - s \int_0^T \partial_x^2 \psi_j(t,0) \psi_1(t,0) (\partial_t \alpha_j(t,0)) dt \\ & - 3s \int_0^T \partial_x^2 \psi_j(t,0) \partial_x \psi_j(t,0) (\partial_x^2 \alpha_j(t,0)) dt - s \int_0^T \partial_x^2 \psi_j(t,0) \psi_1(t,0) (\partial_x \alpha_j(t,0)) dt \\ & - \frac{3}{2}s \int_0^T (\partial_x^2 \psi_j(t,0))^2 (\partial_x \alpha_j(t,0)) dt. \end{aligned} \quad (3.6)$$

3.2. Estimation of internal terms. We begin this section by recalling some properties of the weight functions,

$$\begin{aligned} \ell_1^2 c_1 \rho \leq \alpha_1 \leq \ell_1^2 (-a_1 + b_1 + c_1) \rho, \quad \ell_1 (b_1 - 2a_1) \rho \leq \partial_x \alpha_1 \leq \ell_1 b_1 \rho, \quad \partial_x^2 \alpha_1 = -2a_1 \rho, \\ \ell_j^2 (-a_j - b_j + c_j) \rho \leq \alpha_j \leq \ell_j^2 c_j \rho, \quad \ell_j (-b_j - 2a_j) \rho \leq \partial_x \alpha_j \leq -\ell_j b_j \rho, \quad \partial_x^2 \alpha_j = -2a_j \rho, \quad j = 2, \dots, N, \end{aligned} \quad (3.7)$$

where

$$b_1 > 2a_1, \text{ and } c_j \ell_j^2 = c_1 \ell_1^2, \quad c_j > a_j + b_j, \quad \forall j = 2, \dots, N. \quad (3.8)$$

We now focus our attention on the dominant terms of \mathcal{I} . From now on, $C > 0$ will be a generic positive constant

$$-\frac{15}{2}s^5 \iint (\psi_j)^2 (\partial_x \alpha_j)^4 \partial_x^2 \alpha_j dx dt - \frac{9}{2}s \iint (\partial_x^2 \psi_j)^2 \partial_x^2 \alpha_j dx dt.$$

Using (3.7) we can observe that there exists $C > 0$

$$\begin{aligned} & -\frac{15}{2}s^5 \iint (\psi_1)^2 (\partial_x \alpha_1)^4 \partial_x^2 \alpha_1 dx dt - \frac{9}{2}s \iint (\partial_x^2 \psi_1)^2 \partial_x^2 \alpha_1 dx dt \\ & \geq C \left(\iint s^5 \rho^5 (\psi_j)^2 dx dt + \iint s \rho (\partial_x^2 \psi_j)^2 dx dt. \right) \end{aligned}$$

To deal with the terms involving $\partial_x \psi_j$ in \mathcal{I} , note that by integrating by parts, we can write

$$\iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt = - \iint s^3 \rho^3 \partial_x^2 \psi_j \psi_j dx dt - \int_0^T \ell_j^2 s^3 \rho^3 \partial_x \psi_j(t, 0) \psi_1(t, 0) dt.$$

Moreover, by Young's inequality,

$$\iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt \leq C \left(\iint s^5 \rho^5 (\psi_j)^2 dx dt + \iint s \rho (\partial_x^2 \psi_j)^2 dx dt - \mathcal{B}_0^* \right),$$

where

$$\mathcal{B}_0^* = \int_0^T s^3 \rho^3 \partial_x \psi_j(t, 0) \psi_1(t, 0) dt.$$

Thus, the dominating terms satisfy

$$\begin{aligned} & -\frac{15}{2} s^5 \iint (\psi_j)^2 (\partial_x \alpha_j)^4 \partial_x^2 \alpha_j dx dt - \frac{9}{2} s \iint (\partial_x^2 \psi_j)^2 \partial_x^2 \alpha_j dx dt \\ & \geq C \left(\iint s^5 \rho^5 (\psi_j)^2 dx dt + \iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt + \mathcal{B}_0^* \right). \end{aligned}$$

We can prove that the weight functions satisfy:

$$|\partial_t \alpha_j| + |\partial_{xt} \alpha_j| + |\partial_{xxt} \alpha_j| \leq CT \rho^3, \quad |\partial_t^2 \alpha_j| \leq CT^2 \rho^5. \quad (3.9)$$

Using $\rho^{-1} \leq \frac{T}{2}$, we obtain for the remainder internal terms

$$-3s^3 \iint (\psi_j)^2 (\partial_x^2 \alpha_j)^3 dx dt = 24 \iint a_j^3 s^3 \rho^3 (\psi_j)^2 dx dt \geq 0,$$

$$\left| 3s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 \partial_{xt} \alpha_j dx dt \right| \leq \frac{C_1 T}{s^2} \iint s^5 \rho^5 (\psi_j)^2 dx dt,$$

$$\left| 3s^3 \iint (\psi_j)^2 \partial_x \alpha_j \partial_x^2 \alpha_j \partial_t \alpha_j dx dt \right| \leq \frac{C_2}{s^2} \iint s^5 \rho^5 (\psi_j)^2 dx dt,$$

$$\left| 6s^3 \iint (\psi_j)^2 (\partial_x \alpha_j)^2 \partial_{xt} \alpha_j dx dt \right| \leq \frac{C_3 T^2}{s^2} \iint s^5 \rho^5 (\psi_j)^2 dx dt,$$

$$\left| 3s \iint (\partial_x \psi_j)^2 \partial_{xt} \alpha_j dx dt \right| \leq \frac{C_4 T}{s^2} \iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt,$$

$$\left| 3s \iint (\partial_x \psi_j)^2 \partial_x^2 \alpha_j dx dt \right| \leq \frac{C_5 T^2}{s^2} \iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt,$$

$$\left| s \iint (\psi_j)^2 \partial_{xt} \alpha_j dx dt \right| \leq \frac{C_6 T^3}{s^4} \iint s^5 \rho^5 (\psi_j)^2 dx dt,$$

$$\left| \frac{s}{2} \iint (\psi_j)^2 \partial_t^2 \alpha_j dx dt \right| \leq \frac{C_7 T^2}{s^4} \iint s^5 \rho^5 (\psi_j)^2 dx dt,$$

$$-\frac{s}{2} \iint (\psi_j)^2 \partial_x^2 \alpha_j dx dt = s \iint a_j \rho (\psi_j)^2 dx dt \geq 0.$$

for some $C_1, C_2, C_3, C_4, C_5, C_6, C_7, > 0$. Joining all these estimates, we derive,

$$\begin{aligned} \mathcal{I} \geq & \left(C - \frac{C_1 T}{s^2} - \frac{C_2}{s^2} - \frac{C_3 T^2}{s^2} - \frac{C_6 T^3}{s^4} - \frac{C_7 T^2}{s^4} \right) \iint s^5 \rho^5 (\psi_j)^2 dx dt \\ & + \left(C - \frac{C_4 T}{s^2} - \frac{C_5 T^2}{s^2} \right) \iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt + C \mathcal{B}_{0,j}^*. \end{aligned}$$

We deduce the existence of $s_0 > 0$, such that, taken $s \geq s_0$ we get

$$\mathcal{I} \geq C \left(\iint \ell_j^4 s^5 \rho^5 (\psi_j)^2 dx dt + \iint \ell_j^2 s^3 \rho^3 (\partial_x \psi_j)^2 dx dt + \mathcal{B}_0^* \right), \quad (3.10)$$

for some $C > 0$.

3.3. Estimation of exterior boundary terms. For the external boundary terms, notice that for $j = 2, \dots, N$ these terms are our observations. Moreover, we can show that

$$\mathcal{B}_{\ell_j} \leq C \left(\iint s^3 \rho^3 (\partial_x \psi_j(t, \ell_j))^2 dt + \iint s \rho (\partial_x^2 \psi_j(t, \ell_j))^2 dt \right), \quad \forall j = 2, \dots, N. \quad (3.11)$$

For $j = 1$, observe that we have the following dominating terms

$$4s^3 \int_0^T (\partial_x \psi_1(t, \ell_1))^2 (\partial_x \alpha_1(t, \ell_1))^3 dt \geq C \int_0^T s^3 \rho^3 (\partial_x \psi_1(t, \ell_1))^2 dt,$$

$$\frac{3}{2}s \int_0^T (\partial_x^2 \psi_1(t, \ell_1))^2 \partial_x \alpha_1(t, \ell_1) dt.$$

For the remaining terms, we apply the same strategy as in the case of internal terms.

$$\left| \frac{s}{2} \int_0^T (\partial_x \psi_1(t, \ell_1))^2 \partial_t \alpha_1(t, \ell_1) dt \right| \leq \frac{CT}{s^2} \int_0^T s^3 \rho^3 (\partial_x \psi_1(t, \ell_1))^2 dt,$$

Using $ab \leq (a^2 + b^2)/2$,

$$\begin{aligned} \left| 3s \int_0^T \partial_x^2 \psi_1(t, \ell_1) \partial_x \psi_1(t, \ell_1) \partial_x^2 \alpha_1(t, \ell_1) dt \right| & \leq \frac{3}{2}s \int_0^T \frac{(\partial_x^2 \alpha_1(t, \ell_1))^2}{\partial_x \alpha_1(t, \ell_1)} (\partial_x \psi_1(t, \ell_1))^2 dt \\ & + \frac{3}{2}s \int_0^T (\partial_x^2 \psi_1(t, \ell_1))^2 \partial_x \alpha_1(t, \ell_1) dt \\ & \leq \frac{CT^2}{s^2} \int_0^T s^3 \rho^3 (\partial_x \psi_1(t, \ell_1))^2 dt \\ & + \frac{3}{2}s \int_0^T (\partial_x^2 \psi_1(t, \ell_1))^2 \partial_x \alpha_1(t, \ell_1) dt. \end{aligned}$$

Note that the last term can be eliminated by the second dominating term.

$$\left| s \int_0^T (\partial_x \psi_j(t, \ell_j))^2 \partial_x \alpha_j(t, \ell_j) dt \right| \leq \frac{CT^2}{s^2} \int_0^T s^3 \rho^3 (\partial_x \psi_1(t, \ell_1))^2 dt.$$

Gathering these estimates and similar to the internal estimates, we obtain for all $s \geq s_1 > 0$

$$\mathcal{B}_{\ell_1} \geq C \int_0^T s^3 \rho^3 (\partial_x \psi_1(t, \ell_1))^2 dt \geq 0. \quad (3.12)$$

3.4. Estimation of the central node terms. Let $\tilde{\mathcal{B}}_0 = \mathcal{B}_0 + C\mathcal{B}_0^*$, where, \mathcal{B}_0 comes from Section 3.1 and \mathcal{B}_0^* from Section 3.2. In this section, we show that we can choose the weights α_j appropriately, such that $\tilde{\mathcal{B}}_0 \geq 0$. The main idea is re-write $\tilde{\mathcal{B}}_0$ in terms only of $\psi_1(t, 0)$ and $\partial_x^2 \varphi_j(t, 0)$. The term \mathcal{B}_0^1 in (3.4) is already in the desired form. For the term $\mathcal{B}_0^2 + C\mathcal{B}_0^*$, we notice that $\partial_x \psi_j(t, 0) = -s \partial_x \alpha_j(t, 0) \psi_1(t, 0)$ and

$$3s \int_0^T \partial_t \psi_j(t, 0) \partial_x \psi_j(t, 0) (\partial_x \alpha_j(t, 0)) dt = 3s^2 \int_0^T \frac{d}{dt} (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 dt.$$

After an integration by parts and using, $\psi_j(0, \cdot) = \psi_j(T, \cdot) = 0$ we get

$$3s \int_0^T \partial_t \psi_j(t, 0) \partial_x \psi_j(t, 0) (\partial_x \alpha_j(t, 0)) dt = -6s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x \alpha_j(t, 0) \partial_{xt} \alpha_j(t, 0) dt.$$

Therefore

$$\begin{aligned} \mathcal{B}_0^2 + C\mathcal{B}_0^* &= -3s^4 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^3 dt - 4s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt \\ &\quad + \frac{s^3}{2} \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 (\partial_t \alpha_j(t, 0)) dt + 5s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x \alpha_j(t, 0) \partial_{xt} \alpha_j(t, 0) dt \\ &\quad - s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^3 dt - s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x \alpha_j(t, 0) (\partial_x^2 \alpha_j(t, 0)) dt \\ &\quad - Cs^4 \int_0^T (\psi_1(t, 0))^2 \rho^3(t) \partial_x \alpha_j(t, 0) dt. \end{aligned}$$

For the term \mathcal{B}_0^3 , we have the following (see Appendix B):

$$\begin{aligned} \mathcal{B}_0^3 &= -\frac{5}{2}s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt + 7s^4 \int_0^T (\psi_j(t, 0))^2 \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\ &\quad - \frac{9}{2}s^3 \int_0^T (\psi_j(t, 0))^2 (\partial_x^2 \alpha_j(t, 0))^2 \partial_x \alpha_j(t, 0) dt - s^3 \int_0^T (\psi_1(t, 0))^2 (\alpha_j(t, 0))^3 dt \\ &\quad - s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 \partial_t \alpha_j(t, 0) dt - 4s^3 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\ &\quad + s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) \partial_t \alpha_j(t, 0) dt + 6s^2 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) \partial_x^2 \alpha_j(t, 0) dt \\ &\quad - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_t \alpha_j(t, 0) dt - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) dt \\ &\quad - \frac{3}{2}s \int_0^T e^{-2s\alpha_1(t, 0)} \partial_x \alpha_j(t, 0) (\partial_x^2 \varphi_j(t, 0))^2 dt. \end{aligned}$$

Now, we can obtain

$$\mathcal{B}_0 = A_1 + A_2 + A_3,$$

where

$$\begin{aligned} A_1 = & -8s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt + 4s^4 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\ & - \frac{15}{2} s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0)) (\partial_x^2 \alpha_j(t, 0))^2 dt - 2s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 \partial_t \alpha_j(t, 0) dt \\ & - 4s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^3 dt + 5s^2 \int_0^T (\psi_j(t, 0))^2 \partial_x \alpha_j(t, 0) \partial_{xt} \alpha_j(t, 0) dt \\ & + s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) \partial_t \alpha_j(t, 0) dt - \frac{s}{2} \int_0^T (\psi_1(t, 0))^2 (\partial_{xxt} \alpha_j(t, 0)) dt - \frac{s}{2} \int_0^T (\psi_1(t, 0))^2 \partial_t \alpha_j(t, 0) dt \\ & - \frac{s}{2} \int_0^T (\psi_1(t, 0))^2 \partial_x \alpha_j(t, 0) dt, \end{aligned}$$

$$\begin{aligned} A_2 = & -4s^3 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt + 6s^2 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) \partial_x^2 \alpha_j(t, 0) dt \\ & - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_t \alpha_j(t, 0) dt - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) dt, \end{aligned}$$

$$A_3 = -\frac{3}{2} s \int_0^T e^{-2s\alpha_1(t, 0)} \partial_x \alpha_j(t, 0) (\partial_x^2 \varphi_j(t, 0))^2 dt.$$

The purpose of the following is to show that under suitable choice of the weights the sum for $j = 1, \dots, N$ of A_1 , A_2 and A_3 are positive. In order to simplify the computations, we take the weights α_j such that

$$\partial_x \alpha_j(t, 0) = \partial_x \alpha_k(t, 0), \quad \forall j, k = 2, \dots, N \quad (3.13)$$

$$\partial_x^2 \alpha_1(t, 0) = \partial_x^2 \alpha_j(t, 0), \quad \forall j = 2, \dots, N. \quad (3.14)$$

Note that these conditions are equivalent to take

$$\begin{aligned} b_j \ell_j &= b_k \ell_k, \quad \forall j, k = 2, \dots, N \\ a_1 &= a_j, \quad \forall j = 2, \dots, N. \end{aligned}$$

We start by studying A_1 , in particular, the term with the greatest power, i.e. $-8s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt$. We take now, the sum for $j = 1, \dots, N$ of this term, using (3.13) and the explicit expression for the weights given in (3.1)

$$-\sum_{j=1}^N 8s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt = [(N-1)(b_2 \ell_2)^5 - (b_1 \ell_1)^5] \int_0^T \rho^5 (\psi_1(t, 0))^2 dt,$$

by choosing $b_1 \ell_1 = \frac{1}{K} b_2 \ell_2$, for $K > 0$ big enough we get

$$\begin{aligned}
-\sum_{j=1}^N 8s^5 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^5 dt &= [(N-1)(b_2 \ell_2)^5 - (b_1 \ell_1)^5] \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt \\
&= \left[(N-1) - \frac{1}{K^5} \right] (b_2 \ell_2)^5 \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt > 0,
\end{aligned}$$

For the other terms of A_1 we can observe that

$$\left| 4s^4 \int_0^T (\psi_j(t,0))^2 \partial_x^2 \alpha_j(t,0) (\partial_x \alpha_j(t,0))^3 dt \right| \leq \frac{CT}{s} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| \frac{15}{2} s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0)) (\partial_x^2 \alpha_j(t,0))^2 dt \right| \leq \frac{CT^2}{s^2} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| 2s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^2 \partial_t \alpha_j(t,0) dt \right| \leq \frac{CT}{s^2} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| 4s^3 \int_0^T (\psi_1(t,0))^2 (\partial_x \alpha_j(t,0))^3 dt \right| \leq \frac{CT^2}{s^2} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| 5s^2 \int_0^T (\psi_j(t,0))^2 \partial_x \alpha_j(t,0) \partial_{xt} \alpha_j(t,0) dt \right| \leq \frac{CT^2}{s^3} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| s^2 \int_0^T (\psi_1(t,0))^2 \partial_x^2 \alpha_j(t,0) \partial_t \alpha_j(t,0) dt \right| \leq \frac{CT^2}{s^3} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| \frac{s}{2} \int_0^T (\psi_1(t,0))^2 (\partial_{xxt} \alpha_j(t,0)) dt \right| \leq \frac{CT^3}{s^4} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| \frac{s}{2} \int_0^T (\psi_1(t,0))^2 \partial_t \alpha_j(t,0) dt \right| \leq \frac{CT^3}{s^4} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

$$\left| \frac{s}{2} \int_0^T (\psi_1(t,0))^2 \partial_x \alpha_j(t,0) dt \right| \leq \frac{CT^4}{s^4} \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt$$

Thus, similar as in the past terms, for $s \geq s_2 > 0$ we deduce

$$\sum_{j=1}^N A_1 > C \int_0^T s^5 \rho^5 (\psi_1(t,0))^2 dt.$$

Now we take the sum of $j = 1, \dots, N$ of A_3 , to obtain

$$\begin{aligned} \sum_{j=1}^N A_3 &= -\frac{3}{2} \sum_{j=1}^N s \int_0^T e^{-2s\alpha_1(t,0)} \partial_x \alpha_j(t,0) (\partial_x^2 \varphi_j(t,0))^2 dt \\ &= \frac{3}{2} s b_2 \ell_2 \int_0^T \rho e^{-2s\alpha_1(t,0)} \left[\sum_{j=2}^N (\partial_x^2 \varphi_j(t,0))^2 - \frac{1}{K} (\partial_x^2 \varphi_1(t,0))^2 \right] dt \end{aligned}$$

By the junction conditions of the adjoint equation (Adj) we observe that $\partial_x^2 \varphi_1(t,0) = (\lambda - N)\varphi_1(t,0) - \sum_{j=2}^N \partial_x^2 \varphi_j(t,0)$. Thus,

$$(\partial_x^2 \varphi_1(t,0))^2 \leq 2(\lambda - N)^2 (\varphi_1(t,0))^2 + 2 \sum_{j=2}^N (\partial_x^2 \varphi_j(t,0))^2, \quad (3.15)$$

from where we can derive

$$\sum_{j=1}^N A_3 \geq \frac{3}{2} s b_2 \ell_2 \int_0^T \rho e^{-2s\alpha_1(t,0)} \left[\left(1 - \frac{2}{K}\right) \sum_{j=2}^N (\partial_x^2 \varphi_j(t,0))^2 - \frac{2(\lambda - N)^2}{K} (\varphi_1(t,0))^2 \right] dt.$$

Therefore, for $K > 0$ big enough,

$$\sum_{j=1}^N A_3 \geq C s \int_0^T \rho e^{-2s\alpha_1(t,0)} \sum_{j=2}^N (\partial_x^2 \varphi_j(t,0))^2 dt - C \int_0^T s \rho (\psi_1(t,0))^2 dt.$$

Note that the second term can be absorbed by $\sum_{j=1}^N A_1$. We take now the sum of $j = 1, \dots, N$ of A_2 , thus

$$\begin{aligned} \sum_{j=1}^N A_2 &= -4s^3 \int_0^T (\partial_x \alpha_2(t,0))^3 \psi_1(t,0) \sum_{j=2}^N e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t,0) dt - 4s^3 \int_0^T (\partial_x \alpha_1(t,0))^3 \psi_1(t,0) e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_1(t,0) dt \\ &\quad + 6s^2 \int_0^T \partial_x \alpha_2(t,0) \partial_x^2 \alpha_1(t,0) \psi_1(t,0) \sum_{j=2}^N e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t,0) dt \\ &\quad + 6s^2 \int_0^T \partial_x \alpha_1(t,0) \partial_x^2 \alpha_1(t,0) \psi_1(t,0) e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_1(t,0) dt \\ &\quad - s \int_0^T \partial_t \alpha_1(t,0) \psi_1(t,0) e^{-s\alpha_1(t,0)} \sum_{j=1}^N \partial_x^2 \varphi_j(t,0) dt - s \int_0^T \partial_x \alpha_2(t,0) \psi_1(t,0) \sum_{j=2}^N e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t,0) dt \\ &\quad - s \int_0^T \partial_x \alpha_1(t,0) \psi_1(t,0) e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t,0) dt. \end{aligned}$$

We show, that all these terms can be absorbed by the dominating terms of A_1 and A_3 . In fact, using Young's inequality we observe that

$$\begin{aligned} \left| -4s^3 \int_0^T (\partial_x \alpha_2(t,0))^3 \psi_1(t,0) \sum_{j=2}^N e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t,0) dt \right| &\leq C \int_0^T \rho \left(\rho^2 s^{2.1} |\psi_1(t,0)| s^{0.9} \left| \sum_{j=2}^N e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t,0) \right| \right) dt \\ &\leq C \int_0^T \rho \left(\rho^4 s^{4.41} (\psi_1(t,0))^2 + s^{0.81} \sum_{j=2}^N e^{-2s\alpha_1(t,0)} (\partial_x^2 \varphi_j(t,0))^2 \right) dt \\ &= \frac{C}{s^{0.59}} \int_0^T \rho^5 s^5 (\psi_1(t,0))^2 dt \\ &\quad + \frac{C}{s^{0.19}} \int_0^T \rho s \sum_{j=2}^N e^{-2s\alpha_1(t,0)} (\partial_x^2 \varphi_j(t,0))^2 dt, \end{aligned}$$

with the same strategy as before we derive

$$\begin{aligned} \left| -4s^3 \int_0^T (\partial_x \alpha_1(t, 0))^3 \psi_1(t, 0) e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_1(t, 0) dt \right| &\leq \frac{C}{s^{0.59}} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{C}{s^{0.19}} \int_0^T \rho s e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_1(t, 0))^2 dt, \end{aligned}$$

using (3.15) we observe

$$\begin{aligned} \left| -4s^3 \int_0^T (\partial_x \alpha_1(t, 0))^3 \psi_1(t, 0) e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_1(t, 0) dt \right| &\leq \frac{C}{s^{0.59}} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{CT^4}{s^{4.19}} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{C}{s^{0.19}} \int_0^T \rho s \sum_{j=2}^N e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_j(t, 0))^2 dt, \end{aligned}$$

similarly

$$\begin{aligned} \left| 6s^2 \int_0^T \partial_x \alpha_2(t, 0) \partial_x^2 \alpha_1(t, 0) \psi_1(t, 0) \sum_{j=2}^N e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) dt \right| &\leq \frac{CT}{s} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{CT}{s} \int_0^T \rho s \sum_{j=2}^N e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_j(t, 0))^2 dt, \end{aligned}$$

$$\begin{aligned} \left| 6s^2 \int_0^T \partial_x \alpha_1(t, 0) \partial_x^2 \alpha_1(t, 0) \psi_1(t, 0) e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_1(t, 0) dt \right| &\leq \frac{CT}{s} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt + \frac{CT^5}{s^5} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{CT}{s} \int_0^T \rho s \sum_{j=2}^N e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_j(t, 0))^2 dt, \end{aligned}$$

$$\begin{aligned} \left| -s \int_0^T \partial_t \alpha_1(t, 0) \psi_1(t, 0) e^{-s\alpha_1(t, 0)} \sum_{j=1}^N \partial_x^2 \varphi_j(t, 0) dt \right| &= \left| -s(\lambda - N) \int_0^T \partial_t \alpha_1(t, 0) (\psi_1(t, 0))^2 dt \right| \\ &\leq \frac{CT^2}{s^4} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \end{aligned}$$

$$\begin{aligned} \left| -s \int_0^T \partial_x \alpha_2(t, 0) \psi_1(t, 0) \sum_{j=2}^N e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) dt \right| &\leq \frac{CT^3}{s^3} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{CT}{s} \int_0^T \rho s \sum_{j=2}^N e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_j(t, 0))^2 dt, \end{aligned}$$

$$\begin{aligned} \left| -s \int_0^T \partial_x \alpha_1(t, 0) \psi_1(t, 0) e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_1(t, 0) dt \right| &\leq \frac{CT^3}{s^3} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt + \frac{CT^5}{s^5} \int_0^T \rho^5 s^5 (\psi_1(t, 0))^2 dt \\ &+ \frac{CT}{s} \int_0^T \rho s \sum_{j=2}^N e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_j(t, 0))^2 dt. \end{aligned}$$

We gather all these estimates for to obtain for $s \geq s_3$

$$\tilde{B}_0 \geq C \left(\int_0^T s^5 \rho^5 (\psi_1(t, 0))^2 dt + \int_0^T s \rho \sum_{j=2}^N e^{-2s\alpha_1(t, 0)} (\partial_x^2 \varphi_j(t, 0))^2 dt \right). \quad (3.16)$$

3.5. Coming back to φ_j . First, note that

$$\|\mathcal{R}\psi_j\|_{L^2(Q_j)}^2 \leq \frac{C}{s} \iint s^5 \rho^5 (\psi_j)^2 dx dt.$$

Thus, for all $s \geq s_4 > 0$, this term can be absorbed by (3.10). Therefore, joining the estimates (3.10), (3.12), and (3.16) and noticing that $\mathcal{P}\psi_j = 0$ we get that for all $s \geq s^* = \max(s_0, s_1, s_2, s_3, s_4)$

$$\iint s^5 \rho^5 (\psi_j)^2 dx dt + \iint s^3 \rho^3 (\partial_x \psi_j)^2 dx dt \leq C \sum_{j=2}^N \left(\iint s^3 \rho^3 (\partial_x \psi_j(t, \ell_j))^2 dt + \iint s \rho (\partial_x^2 \psi_j(t, \ell_j))^2 dt \right).$$

Now, as $\psi_j = e^{-s\alpha_j} \varphi_j$, and $\partial_x \varphi = s\alpha_j e^{s\alpha_j} \psi_j + e^{s\alpha_j} \partial_x \psi_j$, we get

$$e^{-2\alpha_j} (\varphi_j)^2 = (\psi_j)^2, \quad e^{-2\alpha_j} (\partial_x \varphi_j)^2 \leq C (s^2 \rho^2 (\psi_j)^2 + (\partial_x \psi_j)^2).$$

In the same way, we have

$$\begin{aligned} \partial_x \psi_j(t, \ell_j) &= e^{-s\alpha_j(t, \ell_j)} \partial_x \varphi_j(t, \ell_j) \\ \partial_x^2 \psi_j(t, \ell_j) &= -2s \partial_x \alpha_j(t, \ell_j) e^{-s\alpha_j(t, \ell_j)} \partial_x \varphi(t, \ell_j) + e^{-s\alpha_j(t, \ell_j)} \partial_x^2 \varphi_j(t, \ell_j), \end{aligned}$$

and thus

$$\begin{aligned} (\partial_x \psi_j(t, \ell_j))^2 &= e^{-2s\alpha_j(t, \ell_j)} (\partial_x \varphi_j(t, \ell_j))^2 \\ (\partial_x^2 \psi_j(t, \ell_j))^2 &\leq C \left(s^2 \rho^2 e^{-2s\alpha_j(t, \ell_j)} (\partial_x \varphi_j(t, \ell_j))^2 + e^{-2s\alpha_j(t, \ell_j)} (\partial_x^2 \varphi_j(t, \ell_j))^2 \right). \end{aligned}$$

Gathering all these computations, we derive the following Carleman estimate.

$$\sum_{j=1}^N \iint e^{-2\alpha_j} (s^5 \rho^5 (\varphi_j)^2 + s^3 \rho^3 (\partial_x \varphi_j)^2) dx dt \leq C \sum_{j=2}^N \int_0^T e^{-2s\alpha_j(t, \ell_j)} (s^3 \rho^3 (\partial_x \varphi_j(t, \ell_j))^2 + s \rho (\partial_x^2 \varphi_j(t, \ell_j))^2) dt.$$

From where we obtain (3.2). \square

Remark 3. Note that our Carleman estimate (3.2) is optimal in the sense of s and ρ powers, $s^5 \rho^5$ for φ , $s^3 \rho^3$ for $\partial_x \varphi$ and $s \rho$ for $\partial_x^2 \varphi$. \circ

4. Proof of the null controllability result. With all the tools developed in the past sections, we are in position to prove our null controllability result. By the structure of the temporal weight (3.1) and the Carleman inequality (3.2) we get as in [14, 13] that there exists $C > 0$ such that

$$\sum_{j=1}^N \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\ell_j} (\varphi_j)^2 dx dt \leq C \sum_{j=2}^N \int_0^T e^{-2s\alpha_j(t, \ell_j)} (s^3 \rho^3 (\partial_x \varphi_j(t, \ell_j))^2 + s \rho (\partial_x^2 \varphi_j(t, \ell_j))^2) dt. \quad (4.1)$$

The next step is to obtain a dissipation estimate. We use (2.8) with $\underline{f} \equiv 0$, obtaining for all $0 < t_1 < t_2 < T$,

$$\sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t_1, x))^2 e^{-x} dx \leq e^{C(t_2-t_1)} \sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(t_2, x))^2 e^{-x} dx.$$

Taking $t_1 = 0$ and $t_2 \in (\frac{T}{3}, \frac{3T}{4})$, we see that

$$\sum_{j=1}^N \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\ell_j} (\varphi_j(0, x))^2 e^{-x} dx dt \leq C \left(\sum_{j=1}^N \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^{\ell_j} (\varphi_j(t_2, x))^2 e^{-x} dx dt \right).$$

Finally, taking account that $e^{-\ell_j} \leq e^{-x} \leq 1$ for all $x \in (0, \ell_j)$, the above inequality and (4.1), we conclude the existence of $C > 0$ such that

$$\sum_{j=1}^N \int_0^{\ell_j} (\varphi_j(0, x))^2 dx dt \leq C \sum_{j=2}^N \int_0^T e^{-2s\alpha_j(t, \ell_j)} (s^3 \rho^3 (\partial_x \varphi_j(t, \ell_j))^2 + s \rho (\partial_x^2 \varphi_j(t, \ell_j))^2) dt.$$

Thus, with the observability inequality, we have proved the null controllability result stated in Theorem 1.1.

Remark 4. *At the end, with respect to the spatial part of the Carleman weight (3.1), we only ask for $a_j, b_j, c_j > 0$ satisfying for $K > 0$ big enough:*

$$\begin{aligned} b_1 &> 2a_1, \\ a_j &= a_1, \quad c_j \ell_j^2 = c_1 \ell_1^2, \quad c_j > a_j + b_j, \quad b_j \ell_j = K b_1 \ell_1 \quad j = 2, \dots, N. \end{aligned}$$

Given $\ell_j, j = 1, \dots, N$, it is not difficult to build weights satisfying these conditions. ◦

5. Conclusion and final remarks. In this work, the well-posedness and null controllability problem of the linear KdV equation in a star-shaped network was addressed using $2N - 2$ controls acting on $N - 1$ edges. For the controllability, an observability inequality equivalent to the null controllability was proved by using an appropriate Carleman estimate and a dissipation estimate. This is the first work, to the best of our knowledge, dealing with Carleman estimates for the KdV equation in networks. The main challenge was to estimate the traces terms in the central node, that were possible to manage using appropriate choice of Carleman weights. For the well-posedness, the notion of transposition solution was utilized. Energy estimates were obtained for different regularity framework, and the desired well-posedness result was derived by interpolation arguments. Some interesting research directions for the future include:

- **Reduce the number of controls:** The possibility to reduce the numbers of controls or acted edges remains an open question. In [1], the exact boundary controllability of the KdV in a star-shaped network was proved by acting with $N + 1$ controls, N controls in the Neumann right conditions and 1 control in the junction (acting on the flux condition). However, in [10], the exact controllability was proved without the control in the central node but for a large time and small lengths. In our case, due to the Carleman inequality we are not able to eliminate acted edges. An idea could be to study the null controllability of the KdV equation on a tree network as in [11] and use the sense of propagation of the KdV equation to eliminate some controls.
- **Internal controls:** For systems of KdV equations with internal couplings, the problem of null controllability was studied in [6] for a Hirota-Satsuma system and in [3] for a Gear-Grimshaw system. In these two works, the controllability was proved by using Carleman estimates, and the controls act internally and locally. Due to the internal couplings term, they were able to use fewer controls than equations. An interesting question is if the internal null controllability holds for the KdV equation in a star-shaped network and if the central node conditions transmit enough information to reduce the number of controls.
- **Dispersive limit:** In [2], the null controllability problem of a parabolic system on networks was studied with a diffusive parameter $\varepsilon \rightarrow 0$. In this setting, a possible future research line could be to consider a dispersive parameter ν in the KdV equation on networks and to study the uniform null controllability, as in [13]. However, some complications arise, such as how to adapt the junction conditions in the hyperbolic limit problem on the network.

- **Inverse Problems:** In [1], a KdV system on networks was proposed to model blood pressure. In this context, an interesting inverse problem arises concerning the determination of cardiac outflow pressure through noninvasive measurements. Also, considering different principal coefficients in each edge of the KdV equation and studying the inverse problem of recovering them through measurements at the ends of the network could be considered. The main tool proposed to solve these kinds of problems are the Carleman inequalities, as developed here.

REFERENCES

- [1] Kaïs Ammari and Emmanuelle Crépeau. Feedback stabilization and boundary controllability of the Korteweg–de Vries equation on a star-shaped network. *SIAM Journal on Control and Optimization*, 56(3):1620–1639, 2018.
- [2] Jon Asier Bárcena-Petisco, Márcio Cavalcante, Giuseppe Maria Coclite, Nicola De Nitti, and Enrique Zuazua. Control of hyperbolic and parabolic equations on networks and singular limits, 2021. cvgmt preprint.
- [3] Jon Asier Bárcena-Petisco, Sergio Guerrero, and Ademir F. Pazoto. Local null controllability of a model system for strong interaction between internal solitary waves. *Communications in Contemporary Mathematics*, 24(02):2150003, 2022.
- [4] Lucie Baudouin, Emmanuelle Crépeau, and Julie Valein. Global carleman estimate on a network for the wave equation and application to an inverse problem. *Mathematical control and related fields*, 1(3):p–307, 2011.
- [5] Nicolás Carreño and Cristóbal Loyola. An explicit time for the uniform null controllability of a linear Korteweg-de Vries equation. Submitted, 2021.
- [6] Nicolás Carreño, Eduardo Cerpa, and Emmanuelle Crépeau. Internal null controllability of the generalized Hirota-Satsuma system. *ESAIM: Control, Optimisation and Calculus of Variations*, 26:75, 2020.
- [7] Márcio Cavalcante. The Korteweg–de Vries equation on a metric star graph. *Zeitschrift für angewandte Mathematik und Physik*, 69(5):124, 2018.
- [8] Cristian M Cazacu, Liviu I Ignat, and Ademir F Pazoto. Null-controllability of the linear Kuramoto–Sivashinsky equation on star-shaped trees. *SIAM Journal on Control and Optimization*, 56(4):2921–2958, 2018.
- [9] Eduardo Cerpa. Control of a Korteweg-de Vries equation: a tutorial. *Mathematical Control & Related Fields*, 4(1):45, 2014.
- [10] Eduardo Cerpa, Emmanuelle Crépeau, and Claudia Moreno. On the boundary controllability of the Korteweg–de Vries equation on a star-shaped network. *IMA Journal of Mathematical Control and Information*, 37(1):226–240, 2020.
- [11] Eduardo Cerpa, Emmanuelle Crépeau, and Julie Valein. Boundary controllability of the Korteweg-de Vries equation on a tree-shaped network. *Evolution Equations and Control Theory*, 9(3):673–692, 2020.
- [12] René Dáger and Enrique Zuazua. *Wave propagation, observation and control in 1-d flexible multi-structures*, volume 50. Springer Science & Business Media, 2006.
- [13] Olivier Glass and Sergio Guerrero. Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. *Asymptotic Analysis*, 60(1-2):61–100, 2008.
- [14] Jean-Philippe Guilleron. Null controllability of a linear KdV equation on an interval with special boundary conditions. *Mathematics of Control, Signals, and Systems*, 26(3):375–401, 2014.
- [15] Liviu I Ignat, Ademir F Pazoto, and Lionel Rosier. Inverse problem for the heat equation and the schrödinger equation on a tree. *Inverse Problems*, 28(1):015011, 2011.
- [16] DJ Korteweg and G de Vries. On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave. *Phil. Mag*, 39:422–443, 1895.
- [17] John E Lagnese, Günter Leugering, and EJP Georg Schmidt. *Modeling, analysis and control of dynamic elastic multi-link structures*. Springer Science & Business Media, 2012.
- [18] Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Asymptotic behavior of KdV equation in a star-shaped network with bounded and unbounded lengths. Accepted in CDC 2023, Singapore, 2022.
- [19] Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Delayed stabilization of the Korteweg–de Vries equation on a star-shaped network. *Mathematics of Control, Signals, and Systems*, 34(3):559–605, 2022.
- [20] Hugo Parada, Emmanuelle Crépeau, and Christophe Prieur. Global well-posedness of the KdV equation on a star-shaped network and stabilization by saturated controllers. *SIAM Journal on Control and Optimization*, 60(4):2268–2296, 2022.
- [21] Jaime Angulo Pava and Márcio Cavalcante. Linear instability criterion for the Korteweg–de Vries equation on metric star graphs. *Nonlinearity*, 34(5):3373, 2021.
- [22] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.
- [23] L. Rosier. Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 2:33–55, 1997.
- [24] Lionel Rosier. Control of the surface of a fluid by a wavemaker. *ESAIM: Control, Optimisation and Calculus of Variations*, 10(3):346–380, 2004.
- [25] Lionel Rosier and Bing-Yu Zhang. Control and stabilization of the Korteweg-de Vries equation: recent progresses. *Journal of Systems Science and Complexity*, 22(4):647–682, 2009.
- [26] ZA Sobirov, MI Akhmedov, and H Uecker. Cauchy problem for the linearized KdV equation on general metric star graphs. *Nanosystems: Physics, Chemistry, Mathematics*, 6(2):198, 2015.

Appendix A. Proof of Claim 1. In this part we derive the estimate (2.14) of Claim 1. For $\delta > 0$, C_δ denotes a generic constant (mostly big) depending on δ . Let $\varepsilon > 0$. Now, we estimate the right-hand side

terms T_i for $i = 1 \dots 6$. We start with T_1 , note that

$$\begin{aligned} \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} dt &= - \int_s^T \partial_t \partial_x \varphi_j(t, \ell_j) \partial_x^2 \varphi_j(t, \ell_j) e^{-\ell_j} dt + \partial_x \varphi_j(t, \ell_j) \partial_x^2 \varphi_j(t, \ell_j) e^{-\ell_j} \Big|_s^T \\ &\leq C \left(\int_0^T |\partial_t \partial_x \varphi_j(t, \ell_j)| |\partial_x^2 \varphi_j(t, \ell_j)| dt \right. \\ &\quad \left. + \sup_{t \in [0, T]} |\partial_x \varphi_j(t, \ell_j)| |\partial_x^2 \varphi_j(t, \ell_j) e^{-\ell_j}| \right). \end{aligned}$$

Using Young's inequality we get for $\varepsilon > 0$ and $\eta > 0$ small enough

$$\begin{aligned} \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} dt &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + C_\varepsilon \int_0^T (\partial_x^2 \varphi_j(t, \ell_j))^2 dt \\ &\quad + C \sup_{t \in [0, T]} \|\partial_x \varphi_j(t, \cdot)\|_{L^\infty(0, \ell_j)} \|\partial_x^2 \varphi_j(t, \cdot)\|_{L^\infty(0, \ell_j)} \\ &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^{5/2+\eta}(0, \ell_j))}^2 \\ &\quad + C \sup_{t \in [0, T]} \|\partial_x \varphi_j(t, \cdot)\|_{H^1(0, \ell_j)} \|\partial_x^2 \varphi_j(t, \cdot)\|_{H^1(0, \ell_j)}. \end{aligned}$$

Using interpolation between $H^3(0, \ell_j)$ and $H^1(0, \ell_j)$ we get for $\varepsilon_1 > 0$

$$\begin{aligned} \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} dt &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + \varepsilon_1 C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^3(0, \ell_j))}^2 + C_{\varepsilon_1} C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^1(0, \ell_j))}^2 \\ &\quad + C \sup_{t \in [0, T]} \|\varphi_j(t, \cdot)\|_{H^2(0, \ell_j)} \|\varphi_j(t, \cdot)\|_{H^3(0, \ell_j)} \\ &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + \varepsilon_1 C_\varepsilon \|\varphi_j\|_{C^2([0, T]; H^3(0, \ell_j))}^2 + C_{\varepsilon_1} C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^1(0, \ell_j))}^2 \\ &\quad + C \|\varphi_j\|_{C([0, T]; H^3(0, \ell_j))} \|\varphi_j\|_{L^\infty(0, T; H^2(0, \ell_j))}. \end{aligned}$$

Again by Young's inequality for $\varepsilon_2 > 0$

$$\begin{aligned} \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} dt &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + \varepsilon_1 C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^3(0, \ell_j))}^2 + C_{\varepsilon_1} C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^1(0, \ell_j))}^2 \\ &\quad + C \sup_{t \in [0, T]} \|\varphi_j(t, \cdot)\|_{H^2(0, \ell_j)} \|\varphi_j(t, \cdot)\|_{H^3(0, \ell_j)} \\ &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + (\varepsilon_1 C_\varepsilon + \varepsilon_2) \|\varphi_j\|_{C([0, T]; H^3(0, \ell_j))}^2 \\ &\quad + C_{\varepsilon_1} C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^1(0, \ell_j))}^2 + C_{\varepsilon_2} \|\varphi_j\|_{L^\infty(0, T; H^2(0, \ell_j))}, \end{aligned}$$

By interpolating between $H^3(0, \ell_j)$ and $L^2(0, \ell_j)$ we get for $\varepsilon_3 > 0$

$$C_{\varepsilon_2} \|\varphi_j\|_{L^\infty(0, T; H^2(0, \ell_j))} \leq C_{\varepsilon_2} \varepsilon_3 \|\varphi_j\|_{L^\infty(0, T; H^3(0, \ell_j))} + C_{\varepsilon_2} C_{\varepsilon_3} \|\varphi_j\|_{L^\infty(0, T; L^2(0, \ell_j))}.$$

Thus we get,

$$\begin{aligned} \int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x^2 \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} dt &\leq \varepsilon \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + (\varepsilon_1 C_\varepsilon + \varepsilon_2 \varepsilon_3 C_{\varepsilon_2}) \|\varphi_j\|_{C([0, T]; H^3(0, \ell_j))}^2 \\ &\quad + C_{\varepsilon_1} C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^1(0, \ell_j))}^2 + C_{\varepsilon_2} C_{\varepsilon_3} \|\varphi_j\|_{C([0, T]; L^2(0, \ell_j))}, \end{aligned}$$

Given $\varepsilon > 0$ we can choose $\varepsilon_1, \varepsilon_2$ and ε_3 (in that order) such that $\varepsilon_1 C_\varepsilon + \varepsilon_2 + \varepsilon_3 C_{\varepsilon_2} < \varepsilon$ using (2.10) we get

$$\begin{aligned}
T_1 &\leq \varepsilon \sum_{j=1}^N \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + \varepsilon \|\underline{\varphi}\|_{C([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))}^2 \\
&\leq \varepsilon \sum_{j=1}^N \int_0^T (\partial_t \partial_x \varphi_j(t, \ell_j))^2 dt + \varepsilon \|\underline{\varphi}\|_{C([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}^2.
\end{aligned}$$

We pass now to the term T_2 . Observe that

$$\int_0^{\ell_j} [(\partial_x^2 \varphi_j(s, x))^2 - (\partial_x \varphi_j(s, x))^2] e^{-x} dx \leq C \|\varphi_j\|_{L^\infty(0, T; H^2(0, \ell_j))}^2,$$

interpolating between $H^3(0, \ell_j)$ and $L^2(0, \ell_j)$ we get

$$\int_0^{\ell_j} [(\partial_x^2 \varphi_j(s, x))^2 - (\partial_x \varphi_j(s, x))^2] e^{-x} dx \leq \varepsilon \|\varphi_j\|_{C([0, T]; H^3(0, \ell_j))}^2 + C_\varepsilon \|\varphi_j\|_{C([0, T]; L^2(0, \ell_j))}^2,$$

using (2.10) we get

$$\begin{aligned}
T_2 &\leq \varepsilon \|\underline{\varphi}\|_{C([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))}^2 \\
&\leq \varepsilon \|\underline{\varphi}\|_{C([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}^2.
\end{aligned}$$

We study now the term T_3 .

$$\int_s^T \int_0^{\ell_j} (\partial_t \varphi_j(t, x)) e^{-x} dx \leq C \|\varphi_j\|_{H^1(0, T; L^2(0, \ell_j))}^2,$$

interpolating between $H^1(0, \ell_j)$ and $H^{-2}(0, \ell_j)$ we get

$$\int_s^T \int_0^{\ell_j} (\partial_t \varphi_j(t, x)) e^{-x} dx \leq \varepsilon \|\varphi_j\|_{H^1(0, T; H^1(0, \ell_j))}^2 + C_\varepsilon \|\varphi_j\|_{H^1(0, T; H^{-2}(0, \ell_j))}^2. \quad (\text{A.1})$$

As $\underline{f} \in L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))$, we get by (2.10) $\varphi_j \in C([0, T]; L^2(0, \ell_j)) \cap L^2(0, T; H^1(0, \ell_j))$, thus $\partial_t \varphi = -\partial_x \varphi - \partial_x^3 \varphi + f_j \in H^1(0, T; H^{-2}(0, \ell_j))$. Moreover $\|\varphi_j\|_{H^1(0, T; H^{-2}(0, \ell_j))}^2 \leq C \|\underline{f}\|_{L^2(0, T; \mathbb{H}_r^{-1}(\mathcal{T}))}^2$, using this in (A.1) we get

$$T_3 \leq \varepsilon \|\underline{\varphi}\|_{H^1(0, T; \mathbb{H}^1(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}^2.$$

We follow with T_4 , note that by for $\varepsilon > 0$ by Young's inequality we get

$$\begin{aligned}
\int_s^T \int_0^{\ell_j} \partial_x^2 \varphi_j(t, x) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx &\leq \varepsilon \|\varphi_j\|_{H^1(0, T; H^1(0, \ell_j))}^2 + C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^2(0, \ell_j))}^2 \\
&\leq \varepsilon \|\varphi_j\|_{H^1(0, T; H^1(0, \ell_j))}^2 + C_\varepsilon \|\varphi_j\|_{L^\infty(0, T; H^2(0, \ell_j))}^2,
\end{aligned}$$

the term $\|\varphi_j\|_{L^\infty(0, T; H^2(0, \ell_j))}^2$ can be estimated as in T_2 , we finally get

$$T_4 \leq \varepsilon \|\underline{\varphi}\|_{H^1(0, T; \mathbb{H}^1(\mathcal{T}))}^2 + \varepsilon \|\underline{\varphi}\|_{C([0, T]; \mathbb{H}^3(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0, T; \mathbb{H}_0^2(\mathcal{T}))}^2.$$

We continue with T_5 , by Young's inequality and $\eta > 0$ small enough

$$\begin{aligned}
\int_s^T \partial_x \varphi_j(t, \ell_j) \partial_x \partial_t \varphi_j(t, \ell_j) e^{-\ell_j} &\leq \varepsilon \int_0^T (\partial_x \partial_t \varphi_j(t, \ell_j))^2 dt + C_\varepsilon \int_0^T (\partial_x \varphi_j(t, \ell_j))^2 dt \\
&\leq \varepsilon \int_0^T (\partial_x \partial_t \varphi_j(t, \ell_j))^2 dt + C_\varepsilon \|\varphi_j\|_{L^2(0, T; H^{3/2+\eta}(0, \ell_j))}^2,
\end{aligned}$$

interpolating between $H^3(0, \ell_j)$ and $H^1(0, \ell_j)$

$$\|\varphi_j\|_{L^2(0,T;H^{3/2+\eta}(0,\ell_j))}^2 \leq \varepsilon_1 \|\varphi_j\|_{L^2(0,T;H^3(0,\ell_j))}^2 + C_{\varepsilon_1} \|\varphi_j\|_{L^2(0,T;H^1(0,\ell_j))}^2$$

taking ε_1 such that $\varepsilon_1 C_\varepsilon \leq \varepsilon$ and using (2.10) we deduce

$$T_5 \leq \varepsilon \sum_{j=1}^N \int_0^T (\partial_x \partial_t \varphi_j(t, \ell_j))^2 dt + \varepsilon \|\underline{\varphi}\|_{C([0,T];\mathbb{H}^3(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0,T;\mathbb{H}_0^2(\mathcal{T}))}^2.$$

We conclude by studying T_6 , we see that

$$\begin{aligned} \int_s^T \int_0^{\ell_j} f_j(t, x) \partial_x^3 \partial_t \varphi_j(t, x) e^{-x} dx &= - \int_s^T \int_0^{\ell_j} \partial_x f_j(t, x) \partial_x^2 \partial_t \varphi_j(t, x) e^{-x} dx + \int_s^T \int_0^{\ell_j} f_j(t, x) \partial_x^2 \partial_t \varphi_j(t, x) e^{-x} dx \\ &= \int_s^T \int_0^{\ell_j} \partial_x^2 f_j(t, x) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx - 2 \int_s^T \int_0^{\ell_j} \partial_x f_j(t, x) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx \\ &\quad + \int_s^T \int_0^{\ell_j} f_j(t, x) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx \\ &= \int_s^T \int_0^{\ell_j} (\partial_x^2 f_j(t, x) - 2 \partial_x f_j(t, x) + f_j(t, x)) \partial_x \partial_t \varphi_j(t, x) e^{-x} dx. \end{aligned}$$

By Young's inequality we get

$$T_6 \leq \varepsilon \|\underline{\varphi}\|_{H^1(0,T;\mathbb{H}^1(\mathcal{T}))}^2 + C_\varepsilon \|\underline{f}\|_{L^2(0,T;\mathbb{H}_0^2(\mathcal{T}))}^2.$$

Appendix B. Computation on \mathcal{B}_0^3 . In this part we derive (3.6). First note that using $\psi_j = e^{-s\alpha_j} \varphi_j$, we derive that

$$\partial_x^2 \psi_j(t, 0) = [-s \partial_x^2 \alpha_j(t, 0) + s^2 (\partial_x \alpha_j(t, 0))^2] \psi_1(t, 0) + e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t, 0).$$

Thus,

$$\begin{aligned} -s^3 \int_0^T \partial_x^2 \psi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt &= -s^5 \int_0^T (\psi_1(t, 0))^2 (\alpha_j(t, 0))^5 dt \\ &\quad + s^4 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\ &\quad - s^3 \int_0^T e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt, \\ -s \int_0^T \partial_x^2 \psi_j(t, 0) \psi_1(t, 0) \partial_t \alpha_j(t, 0) dt &= -s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 \partial_t \alpha_j(t, 0) dt \\ &\quad + s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) \partial_t \alpha_j(t, 0) dt \\ &\quad - s \int_0^T e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_t \alpha_j(t, 0) dt, \\ -3s \int_0^T \partial_x^2 \psi_j(t, 0) \partial_x \psi_j(t, 0) \partial_x^2 \alpha_j(t, 0) dt &= 3s^4 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^3 \partial_x^2 \alpha_j(t, 0) dt \\ &\quad - 3s^3 \int_0^T (\psi_1(t, 0))^2 \partial_x \alpha_j(t, 0) (\partial_x^2 \alpha_j(t, 0))^2 dt \\ &\quad + 3s^2 \int_0^T e^{-s\alpha_1(t,0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) \partial_x^2 \alpha_j(t, 0) dt, \end{aligned}$$

$$\begin{aligned}
-s \int_0^T \partial_x^2 \psi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) dt &= -s^3 \int_0^T (\psi_1(t, 0))^2 (\alpha_j(t, 0))^3 dt + s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) \partial_x \alpha_j(t, 0) dt \\
&\quad - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) dt,
\end{aligned}$$

$$\begin{aligned}
-\frac{3}{2} s \int_0^T (\partial_x^2 \psi_j(t, 0))^2 \partial_x \alpha_j(t, 0) dt &= -\frac{3}{2} s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt \\
&\quad + 3s^4 \int_0^T (\psi_j(t, 0))^2 \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\
&\quad - \frac{3}{2} s^3 \int_0^T (\psi_j(t, 0))^2 (\partial_x^2 \alpha_j(t, 0))^2 \partial_x \alpha_j(t, 0) dt \\
&\quad - 3s^3 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\
&\quad + 3s^2 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x^2 \alpha_j(t, 0) \partial_x \alpha_j(t, 0) dt \\
&\quad - \frac{3}{2} s \int_0^T e^{-2s\alpha_1(t, 0)} \partial_x \alpha_j(t, 0) (\partial_x^2 \varphi_j(t, 0))^2 dt.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\mathcal{B}_0^3 &= -\frac{5}{2} s^5 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^5 dt + 7s^4 \int_0^T (\psi_j(t, 0))^2 \partial_x^2 \alpha_j(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\
&\quad - \frac{9}{2} s^3 \int_0^T (\psi_j(t, 0))^2 (\partial_x^2 \alpha_j(t, 0))^2 \partial_x \alpha_j(t, 0) dt - s^3 \int_0^T (\psi_1(t, 0))^2 (\alpha_j(t, 0))^3 dt \\
&\quad - s^3 \int_0^T (\psi_1(t, 0))^2 (\partial_x \alpha_j(t, 0))^2 \partial_t \alpha_j(t, 0) dt + s^2 \int_0^T (\psi_1(t, 0))^2 \partial_x^2 \alpha_j(t, 0) \partial_t \alpha_j(t, 0) dt \\
&\quad - 4s^3 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) (\partial_x \alpha_j(t, 0))^3 dt \\
&\quad + 6s^2 \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) \partial_x^2 \alpha_j(t, 0) dt - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_t \alpha_j(t, 0) dt \\
&\quad - s \int_0^T e^{-s\alpha_1(t, 0)} \partial_x^2 \varphi_j(t, 0) \psi_1(t, 0) \partial_x \alpha_j(t, 0) dt - \frac{3}{2} s \int_0^T e^{-2s\alpha_1(t, 0)} \partial_x \alpha_j(t, 0) (\partial_x^2 \varphi_j(t, 0))^2 dt
\end{aligned}$$