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OBSERVED-BASED EXPONENTIAL STABILITY OF THE FRACTIONAL HEAT EQUATION

HUGO PARADA

ABSTRACT. In this work, the exponential stability of the nonlocal fractional heat equation is studied. The fractional Laplacian is defined via a singular integral. Using the spectral properties of the fractional Laplacian and a state-decomposition. The feedback control is build taking account the first N modes and an observer defined via a bounded operator. Different kind of configurations are studied to mention, interior controller and interior observation, interior controller and exterior observation. Using the recent result about simplicity of the eigenvalues [11], some of our stabilization results are valid for $s \in (0,1)$, in particular for $s \in (0,1/2)$ in which case the fractional heat equation is not null controllable.

1. INTRODUCTION

We consider the following fractional heat equation described for $t > 0$

$$\begin{cases} \partial_t w + (-d_x^2)^s w - qw = \mathcal{F}(w), & t > 0, x \in \Omega, \\ w(t, x) = 0, & t > 0, x \in \mathbb{R} \setminus \Omega, \\ w(0, x) = w_0(x), & x \in \Omega, \\ z(t) = \mathcal{C}w, & t > 0, \end{cases} \quad (1.1)$$

where q is positive constant, $\Omega = (-1,1)$, $w_0 \in L^2(\Omega)$ is a given initial datum, \mathcal{F} is a feedback controller and \mathcal{C} is an observation operator. Here $(-d_x^2)^s$ denotes the one-dimensional, nonlocal fractional Laplace operator, for all $s \in (0,1)$, which is defined as the following singular integral (see [1]):

$$(-d_x^2)^s w(x) = c_s \text{pv} \int_{\mathbb{R}} \frac{w(x) - w(y)}{|x - y|^{1+2s}} dy, \quad (1.2)$$

with normalization constant

$$c_s := \frac{s2^s \Gamma\left(\frac{1+2s}{2}\right)}{\sqrt{\pi} \Gamma(1-s)}, \quad (1.3)$$

where Γ denotes the Euler Gamma function. Due to the presence of $q > 0$ in (1.1), this system could be unstable, the idea of this work is to build a finite dimensional feedback controller \mathcal{F} using an observation operator \mathcal{C} to achieve the exponential stability of (1.1). This is not a straightforward task, in particular because for the fractional Laplacian the eigenvalues and eigenfunctions are not known explicitly. The reader should note that the fractional Laplacian $(-d_x^2)^s$ considered in this work is a different operator that the *spectral* fractional Laplacian $(-\Delta)^s$. The first one is defined by the singular integral (1.2) for functions satisfying the exterior zero condition $f = 0$ in $\mathbb{R} \setminus \Omega$, while the second one can be defined as the fractional powers of the positive self-adjoint operator $(-\Delta)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Those two operators are different, in fact their eigenvalues and eigenfunctions are not the same, see Section 2.1. Even if this two operators are different, similar results to those presented in this work can be obtain for the spectral fractional Laplacian. In fact, that case is easier because the eigenvalues and eigenfunctions are explicitly known. Other approach, that will not treated in this work but in a future is to consider fractional in time derivative. For instance, we can replace in (1.1) ∂_t by the Caputo fractional derivative [29], defined for $\alpha \in (0,1)$ as

$${}^c D_t^\alpha v(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \partial_\tau v(\tau) d\tau. \quad (1.4)$$

Key words and phrases. fractional Laplacian; output feedback; exponential stability;

The introduction of this fractional time derivative requires the use of fractional order calculus and the Mittag-Leffler matrix function. Other definitions can be used to model fractional in time derivatives, for instance, the Riemann-Liouville derivative, the Riesz derivative, the Weyl derivative, etc. In the last years, nonlocal PDEs and fractional order operators have become very popular in modeling various subject of science for instance, turbulence, image processing, porous media [4, 6, 26]. Nonlocal PDEs (operators) are a type of partial differential equations that differ from the typical differential operators because the nonlocal operators take in account the interactions between distant points in a system, and they are relevant to numerous scientific and engineering. They are particularly useful in the develop of stochastic models for anomalous diffusion problems. Several stochastic models have been proposed, we mention fractional Brownian motion, the continuous-time random-walk, the Lévy flights, etc [10, 12, 24]. These models are used because offer a more flexible and accurate representation of complex physical system involving nonlocal interactions.

With respect to controllability of fractional heat equations in open subsets of \mathbb{R}^N , the question is not fully answered. A powerful tool, used typically to prove controllability properties for parabolic equations are Carleman inequalities which are not available yet in the fractional case. In the case multidimensional case $N \geq 2$ the best controllability result know for the nonlocal fractional heat equation in a open subset is the approximate controllability with interior or exterior controls [17, 31]. However, in the one dimensional setting, better results are know in the case $s \in (1/2, 1)$, i.e the fractional heat equation is null controllable with interior or exterior controls [5, 33]. Recently in the case of the spectral fractional Laplacian, the null controllability was proved in the case of $s \in (0, 1/2)$ in [27] using a moving control strategy. It is expected (as is mention in [27]) that the strategy of moving control also helps to achieve the null controllability in the case $s \in (0, 1/2)$ for the nonlocal fractional Laplacian, but until now it is an open problem. Moving to the stabilization problem, up to our knowledge this is not as well researched than the controllability, we can mention [15, 14] where the boundary stabilization of a time-space fractional heat and wave equation were investigated, it is worth to mention that in those works the spectral fractional Laplacian is used. In [2] the rapid stabilization of a fractional diffusive equation on the whole space was derived. We also mention the works, [23] where the lack of controllability for fractional ODE and PDE was studied, [25] where controllability and observability of finite dimensional fractional systems was investigated, [16] where the controllability problem for fractional order system in Banach spaces was dealt, [22] where the Bang bang principle and null controllability was studied for fractional order parabolic systems.

The goal of this work is to study the exponential stability of the fractional heat equation (1.1) by acting with an observer-based finite dimensional controller. Using the spectral properties of the fractional Laplacian, we decompose the state into its stable and unstable part, then we manage to build a controller using the first N modes. In the case of the *classical* one dimensional Laplacian with Dirichlet conditions, we can use as observation (among others) an internal observation

$$\mathcal{C}w = \int_{\Omega} c(x)w(t, x)dx, \quad (1.5)$$

which is a bounded operator in $L^2(\Omega)$ and the Neumann trace of the solution

$$\mathcal{C}w = \partial_x w(t, 0). \quad (1.6)$$

In that case the associate observation operator is unbounded in $L^2(\Omega)$. Here for the nonlocal fractional Laplacian, we deal with two kinds of observations, the first one corresponds to an internal observation as (1.5) and is analyzed in Section 3. In the case of (1.1) the analogous of the Neumann trace is the *nonlocal normal derivative*

$$\mathcal{N}_s w(x) = c_s \int_{\Omega} \frac{w(x) - w(y)}{|x - y|^{1+2s}} dy, \quad x \in \mathbb{R} \setminus \bar{\Omega}.$$

Contrary to the Neumann case for the classical Laplacian, the nonlocal normal derivative define a bounded observation operator (see Section 4). Furthermore, observe that the nonlocal normal derivative is defined for points x outside of Ω . This allows us to design the control using exterior

measures.

The paper is organized as follows in Section 2 we define the fractional Laplacian and its spectral properties. In Section 3 we study the exponential stabilization using an internal observation via a bounded operator. Then, in Section 4 we study the exponential stabilization using an exterior observation. In Section 5 we deal with a more general form of the controller, in that case if the control is localized we a weak spectral inequality in order to show the exponential stability via a Lyapunov methods. We conclude the discussion with some future research lines to be investigated.

2. PRELIMINARIES

In this section we define the fractional Laplacian and we describe some properties about it that will be useful along this work. We consider the space

$$\mathcal{L}_s^1(\mathbb{R}) := \left\{ w : \mathbb{R} \rightarrow \mathbb{R} : w \text{ measurable, } \int_{\mathbb{R}} \frac{|w(x)|}{(1+|x|)^{1+2s}} dx < \infty \right\}, \quad (2.1)$$

and, for any $w \in \mathcal{L}_s^1$ and $\varepsilon > 0$, we set

$$(-d_x^2)_\varepsilon^s w(x) = c_s \int_{|x-y|>\varepsilon} \frac{w(x) - w(y)}{|x-y|^{1+2s}} dy, \quad x \in \mathbb{R}. \quad (2.2)$$

The *fractional Laplacian* is then defined by the singular integral,

$$(-d_x^2)^s w(x) = c_s \text{pv} \int_{\mathbb{R}} \frac{w(x) - w(y)}{|x-y|^{1+2s}} dy = \lim_{\varepsilon \rightarrow 0^+} (-d_x^2)_\varepsilon^s w(x), \quad x \in \mathbb{R}, \quad (2.3)$$

provided that the limit exists. Given $s \in (0, 1)$ and $\Omega \subset \mathbb{R}$, the fractional Sobolev space $H^s(\Omega)$ is defined as

$$H^s(\Omega) := \left\{ w \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^{1+2s}} dx dy < \infty \right\}. \quad (2.4)$$

This is a Hilbert space, endowed with the norm (derived from the scalar product)

$$\|w\|_{H^s(\Omega)} := \left[\|w\|_{L^2(\Omega)}^2 + |w|_{H^s(\Omega)}^2 \right]^{\frac{1}{2}}, \quad (2.5)$$

where the term

$$|w|_{H^s(\Omega)} := \left(\int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^{1+2s}} dx dy \right)^{\frac{1}{2}}, \quad (2.6)$$

is the so-called Gagliardo seminorm of w . We set

$$H_0^s(\overline{\Omega}) := \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}, \quad (2.7)$$

the closure of the continuous infinitely differentiable functions with compact support in Ω with respect to the $H^s(\Omega)$ -norm. Then, for $0 < s \leq \frac{1}{2}$, the identity $H_0^s(\overline{\Omega}) = H^s(\Omega)$ holds. This is because, in this case, the $C_0^\infty(\Omega)$ functions are dense in $H^s(\Omega)$; on the other hand, for $\frac{1}{2} < s < 1$, we have $H_0^s(\overline{\Omega}) = \{w \in H^s(\mathbb{R}) : w = 0 \text{ in } \mathbb{R} \setminus \Omega\}$.

Let $(-d_x^2)_D^s$ be the self-adjoint operator on $L^2(\Omega)$ associated with the closed and bilinear form

$$\mathcal{E} = \frac{c_s}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(w(x) - w(y))(v(x) - v(y))}{|x-y|^{1+2s}} dx dy, \quad w, v \in H_0^s(\overline{\Omega}). \quad (2.8)$$

That is,

$$D \left((-d_x^2)_D^s \right) = \left\{ w \in H_0^s(\overline{\Omega}) : (-d_x^2)^s w \in L^2(\Omega) \right\}, \quad (-d_x^2)_D^s w = (-d_x^2)^s w. \quad (2.9)$$

Then $(-d_x^2)_D^s$ is the realization in $L^2(\Omega)$ of the fractional Laplace operator $(-d_x^2)^s$ with zero exterior condition $w = 0$ on $\mathbb{R} \setminus \Omega$. The operator $(-d_x^2)_D^s$ has a compact resolvent and its eigenvalues from a non-decreasing sequence of real numbers $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$, satisfying

$\lim_{n \rightarrow \infty} \lambda_n = \infty$. Let $(\phi_n)_{n \in \mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with $(\lambda_n)_{n \in \mathbb{N}}$, i.e.

$$\begin{cases} (-d_x^2)^s \phi_n = \lambda_n \phi_n & \text{in } \Omega, \\ \phi_n = 0 & \text{in } \mathbb{R} \setminus \Omega. \end{cases} \quad (2.10)$$

Moreover, the following asymptotic behaviour is known [18, Theorem 1]

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-2s)\pi}{8} \right)^{2s} + O\left(\frac{1}{n}\right). \quad (2.11)$$

For any real $\sigma > 0$, we define the space $\mathbb{H}_s^\sigma(\Omega)$ as the domain of the σ -power of $(-d_x^2)_D^s$. More precisely,

$$\mathbb{H}_s^\sigma(\Omega) := \left\{ w \in L^2(\Omega) : \sum_{n \in \mathbb{N}} |\lambda_n^\sigma(w, \phi_n)_{L^2(\Omega)}|^2 < \infty \right\}, \quad (2.12)$$

and

$$\|w\|_{\mathbb{H}_s^\sigma(\Omega)} := \left(\sum_{n \in \mathbb{N}} |\lambda_n^\sigma(w, \phi_n)_{L^2(\Omega)}|^2 \right)^{\frac{1}{2}}. \quad (2.13)$$

Note that clearly, $\mathbb{H}_s^{\frac{1}{2}}(\Omega) = H_0^s(\overline{\Omega})$, with equivalent norms. Finally, in what follows, we will indicate with $H^{-s}(\Omega) = (H^s(\Omega))'$ (resp. $\mathbb{H}_s^{-\sigma}(\Omega) = (\mathbb{H}_s^\sigma(\Omega))'$) the dual space of $H^s(\Omega)$ (resp. $\mathbb{H}_s^\sigma(\Omega)$) with respect to the pivot space $L^2(\Omega)$. In particular, we consider $\mathbb{H}_s^{-\sigma}(\Omega)$ endowed with the norm

$$\|w\|_{\mathbb{H}_s^{-\sigma}(\Omega)} := \left(\sum_{n \in \mathbb{N}} |\rho_n^{-\sigma}(w, \phi_n)_{L^2(\Omega)}|^2 \right)^{\frac{1}{2}}. \quad (2.14)$$

Then we have the following embeddings

$$\mathbb{H}_s^\sigma(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{H}_s^{-\sigma}(\Omega). \quad (2.15)$$

We recall the following integration by parts formula in the case of fractional Laplacian given in [9, Lemma 3.3]. For $w \in H^s(\mathbb{R})$, we introduce the *nonlocal normal derivative* \mathcal{N}_s given by

$$\mathcal{N}_s w(x) := c_s \int_{\Omega} \frac{w(x) - w(y)}{|x - y|^{1+2s}} dy, \quad x \in \mathbb{R} \setminus \overline{\Omega}. \quad (2.16)$$

Lemma 2.1. *Let $w \in H_0^s(\overline{\Omega})$, be such that $(-d_x^2)^s w \in L^2(\Omega)$ and $\mathcal{N}_s w \in L^2(\mathbb{R} \setminus \Omega)$. Then for every $v \in H^2(\mathbb{R})$, we have*

$$\frac{c_s}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(w(x) - w(y))(v(x) - v(y))}{|x - y|^{1+2s}} dx dy = \int_{\Omega} v (-d_x^2)^s w dx + \int_{\mathbb{R} \setminus \Omega} v \mathcal{N}_s w dx. \quad (2.17)$$

Finally we mention the following *unique continuation properties* which proofs can be founded in [5, Theorem 1.2] and [32, Theorem 16] respectively. The first one with internal observation

Let $\lambda > 0$ be a real number and $\omega \subset \Omega$ an arbitrary nonempty open set.
If $\varphi \in (-d_x^2)_D^s$ satisfies $(-d_x^2)_D^s \varphi = \lambda \varphi$ in Ω and $\varphi = 0$ in ω , then $\varphi = 0$ in \mathbb{R} . (IntUC)

The second one with exterior *Neumann* observation

Let $\lambda > 0$ be a real number and $\mathcal{O} \subset \mathbb{R} \setminus \overline{\Omega}$ an arbitrary nonempty open set.
If $\varphi \in (-d_x^2)_D^s$ satisfies $(-d_x^2)_D^s \varphi = \lambda \varphi$ in Ω and $\mathcal{N}_s \varphi = 0$ in \mathcal{O} , then $\varphi = 0$ in \mathbb{R} . (ExtUC)

2.1. Two different operators. In order to make more clear the difference of the operator $((-d_x^2)^s)$ and the spectral fractional Laplacian $(-\Delta)^s$, in this part we give the definition of $(-\Delta)^s$ emphasizing the main differences.

It is well known that the Laplacian operator $((-\Delta), D(-\Delta))$, with $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ is a positive self-adjoint operator, in particular its eigenvalues and eigenfunctions $(\mu_n, \varphi_n)_{n \in \mathbb{N}}$ satisfy the equation

$$\begin{cases} -\varphi_n'' = \mu_n \varphi, & \text{in } \Omega, \\ \varphi_n(-1) = \varphi_n(1) = 0, \end{cases} \quad (2.18)$$

and are given by

$$\mu_n = \left(\frac{n\pi}{2}\right)^2, \quad \varphi_n(x) = \begin{cases} \sin(k\pi x), & n = 2k, \\ \cos(k\pi x), & n = 2k + 1. \end{cases} \quad (2.19)$$

Moreover $(\varphi_n)_{n \in \mathbb{N}}$ forms an orthonormal basis of $L^2(\Omega)$ in particular, for $f \in L^2(\Omega)$

$$f = \sum_{n \in \mathbb{N}} (f, \varphi_n)_{L^2(\Omega)} \varphi_n, \quad \|f\|_{L^2(\Omega)}^2 = \sum_{n \in \mathbb{N}} (f, \varphi_n)_{L^2(\Omega)}^2. \quad (2.20)$$

If $f \in D(-\Delta)$

$$(-\Delta)f = \sum_{n \in \mathbb{N}} \mu_n (f, \varphi_n)_{L^2(\Omega)} \varphi_n. \quad (2.21)$$

Let $s > 0$, this motivates to define the spectral fractional Laplacian $(-\Delta)^s$ (similar to (2.12),(2.13)) as

$$(-\Delta)^s f = \sum_{n \in \mathbb{N}} \mu_n^s (f, \varphi_n)_{L^2(\Omega)} \varphi_n, \quad D[(-\Delta)^s] = \{f \in L^2(\Omega), \sum_{n \in \mathbb{N}} \mu_n^{2s} (f, \varphi_n)_{L^2(\Omega)}^2 < \infty\}. \quad (2.22)$$

From the above definition, we remark the following:

- The eigenvalues of $(-\Delta)^s$ are $\mu_n^{2s} = \left(\frac{n\pi}{2}\right)^{2s}$, which are different to λ_n (2.11) but asymptotically close.
- The eigenfunctions of $(-\Delta)^s$ are φ_n and they are smooth $\varphi_n \in C^\infty(\Omega)$, contrary to those of $(-d_x^2)^s$ which are only Hölder continuous up to the boundary [30, Theorem 1.1].

3. EXPONENTIAL STABILITY WITH INTERNAL OBSERVATION

In this section, we prove our main result related with the finite-dimensional observer-based exponential stabilization of the fractional heat equation with bounded observation. We base our ideas on [21]. We search for a control in the form $u(t)f(x)$, this choice of the feedback controller is made to simplify the computations, in a more general case, we can not directly apply the same strategy shown here, we explain more details about that in Section 5. Therefore, with this choice of the control we focus on the system

$$\begin{cases} \partial_t w + (-d_x^2)^s w - qw = u(t)f(x), & t > 0, x \in \Omega, \\ w(t, x) = 0, & t > 0, x \in \mathbb{R} \setminus \Omega, \\ w(0, x) = w_0(x), & x \in \Omega, \\ z(t) = \int_{\Omega} c(x)w(t, x) dx, & t > 0. \end{cases} \quad (3.1)$$

Denoting by $\mathcal{A} = (-d_x^2)_D^s$ the fractional Laplacian operator defined on Section 2 we can write system (3.1) as

$$\partial_t w(t, \cdot) = (-\mathcal{A} + \mathcal{I}q)w(t, \cdot) + fu(t), \quad t > 0, \quad (3.2)$$

where \mathcal{I} , denotes the identity map. Now, we observe that for any $w(t, \cdot) \in D(\mathcal{A})$, we can write $w(t, x) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$, where $(\phi_n)_{n \in \mathbb{N}}$ are the eigenfunctions of \mathcal{A} and $w_n(t)$ is the coefficient projection of w onto the subspace generated by ϕ_n , i.e $w_n(t) = \langle w(t, \cdot), \phi_n \rangle_{L^2(\Omega)}$. Similarly we consider, $f_n = \langle f, \phi_n \rangle_{L^2(\Omega)}$ and $c_n = \langle c, \phi_n \rangle_{L^2(\Omega)}$. Therefore, we obtain the following infinite dimensional system

$$\begin{cases} \partial_t w_n(t) = (-\lambda_n + q)w_n(t) + f_n u(t), & t > 0, n \in \mathbb{N}, \\ z(t) = \sum_{i=1}^{\infty} c_i w_i(t), & t > 0, \end{cases} \quad (3.3)$$

where $(\lambda_n)_{n \in \mathbb{N}}$ are the eigenvalues of \mathcal{A} . Our idea now is to split the above system in its unstable and stable part. Let $N_0 \in \mathbb{N}$ and $\delta > 0$ be given such that $-\lambda_n + q < -\delta < 0$ for all $n \geq N_0 + 1$. Let $N \geq N_0 + 1$, we design an observer with the task of estimate the first N eigenmodes of the system. We introduce

$$W_{N_0} = \begin{bmatrix} w_1 \\ \vdots \\ w_{N_0} \end{bmatrix}, \quad A_{N_0} = \begin{bmatrix} -\lambda_1 + q & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\lambda_{N_0} + q \end{bmatrix}, \quad F_{N_0} = \begin{bmatrix} f_1 \\ \vdots \\ f_{N_0} \end{bmatrix}. \quad (3.4)$$

Thus,

$$\dot{W}_{N_0}(t) = A_{N_0} W_{N_0}(t) + F_{N_0} u(t). \quad (3.5)$$

We write now the observer dynamics. Consider observer gains $\ell_n \in \mathbb{R}$ for $1 \leq n \leq N$, such that $\ell_n = 0$ for $N_0 + 1 \leq n \leq N$

$$\partial_t \hat{w}_n(t) = (-\lambda_n + q)\hat{w}_n(t) + f_n u(t) - \ell_n \left(\int_{\Omega} c(x) \sum_{i=1}^N \hat{w}_i(t) \phi_i(x) dx - z(t) \right), \quad (3.6)$$

where we have taken null initial condition for the observer dynamics ($\hat{w}_n(0) = 0$, for $1 \leq n \leq N$). We define for $1 \leq n \leq N$, the observation error e_n as

$$e_n(t) = w_n(t) - \hat{w}_n(t). \quad (3.7)$$

Observe that

$$\begin{aligned} \int_{\Omega} c(x) \sum_{i=1}^N \hat{w}_i(t) \phi_i(x) dx - z(t) &= \int_{\Omega} c(x) \left(\sum_{i=1}^N \hat{w}_i(t) \phi_i(x) - \sum_{i=1}^{\infty} w_i(t) \phi_i(x) \right) dx \\ &= \int_{\Omega} c(x) \left(\sum_{i=1}^N (\hat{w}_i(t) - w_i(t)) \phi_i(x) - \sum_{i=N+1}^{\infty} w_i(t) \phi_i(x) \right) dx \\ &= - \sum_{i=1}^N e_i(t) \int_{\Omega} c(x) \phi_i(x) dx - \sum_{i=N+1}^{\infty} w_i(t) \int_{\Omega} c(x) \phi_i(x) dx \\ &= - \sum_{i=1}^N e_i(t) c_i - \sum_{i=N+1}^{\infty} w_i(t) c_i. \end{aligned}$$

Defining $\zeta(t) = \sum_{i=N+1}^{\infty} w_i(t) c_i$, we can write the observer dynamics as follows

$$\partial_t \hat{w}_n(t) = (-\lambda_n + q)\hat{w}_n(t) + f_n u(t) + \ell_n \sum_{i=1}^N e_i(t) c_i + \ell_n \zeta(t). \quad (3.8)$$

We introduce now,

$$\begin{aligned} \widehat{W}_{N_0} &= \begin{bmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_{N_0} \end{bmatrix}, \quad E_{N_0} = \begin{bmatrix} e_1 \\ \vdots \\ e_{N_0} \end{bmatrix}, \quad E_{N-N_0} = \begin{bmatrix} e_{N_0+1} \\ \vdots \\ e_N \end{bmatrix}, \quad L_{N_0} = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_{N_0} \end{bmatrix}, \\ C_{N_0} &= [c_1 \dots c_{N_0}], \quad C_{N-N_0} = [c_{N_0+1} \dots c_N]. \end{aligned} \quad (3.9)$$

Thus, we have

$$\dot{\widehat{W}}_{N_0}(t) = A_{N_0} \widehat{W}_{N_0}(t) + u(t) F_{N_0} + L_{N_0} C_{N_0} E_{N_0}(t) + L_{N_0} C_{N-N_0} E_{N-N_0}(t) + L_{N_0} \zeta(t). \quad (3.10)$$

Suppose now that for all $1 \leq n \leq N_0$, $f_n \neq 0$. In that case, we have that the pair (A_{N_0}, F_{N_0}) is controllable. Indeed, the pair (A_{N_0}, F_{N_0}) is controllable if and only if the Kalman matrix $\mathcal{K}_{A_{N_0}, F_{N_0}}$ has full rank. We can easily check that

$$\det(\mathcal{K}_{A_{N_0}, F_{N_0}}) = \left(\prod_{n=1}^{N_0} f_n \right) \text{Vdm}_{-\lambda_1+q, \dots, -\lambda_{N_0}+q}. \quad (3.11)$$

The determinant $\text{Vdm}_{-\lambda_1+q, \dots, -\lambda_{N_0}+q}$ the Vandermonde determinant and is never zero because for all $s \in (0, 1)$, the eigenvalues of the fractional Laplacian are simple. The simplicity in the case $s \in (1/2, 1)$ is known from 2012 [18, Proposition 3], the proof is based in the explicit formula for the eigenvalue problem for the fractional Laplacian in the half-line $(0, \infty)$. From the other side, in a very recent work [11] the simplicity was extended to $s \in (0, 1)$, in this case the proof use a splitting condition and a domain perturbation argument.

Remark 3.1. *It is not difficult to build a function $f(x)$ satisfying that for all $1 \leq n \leq N_0$, $f_n \neq 0$.*

We can just take $a_1, \dots, a_{N_0} \neq 0$, $f(x) = \sum_{n=1}^{N_0} a_n \phi_n(x)$ and recall that $(\phi_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$.

As the pair (A_{N_0}, F_{N_0}) is controllable, there exists a matrix $K \in \mathbb{R}^{1 \times N_0}$ such that the matrix $A_{N_0} + F_{N_0}K$ is Hurwitz. Take the control $u(t)$ as $u(t) = K\widehat{W}_{N_0}(t)$, then

$$\dot{\widehat{W}}_{N_0}(t) = (A_{N_0}(t) + F_{N_0}K)\widehat{W}_{N_0}(t) + L_{N_0}C_{N_0}E_{N_0}(t) + L_{N_0}C_{N-N_0}E_{N-N_0}(t) + L_{N_0}\zeta(t). \quad (3.12)$$

$$\dot{W}_{N_0}(t) = A_{N_0}(t)W_{N_0}(t) + F_{N_0}K\widehat{W}_{N_0}(t). \quad (3.13)$$

$$\dot{E}_{N_0}(t) = (A_{N_0}(t) - L_{N_0}C_{N_0})E_{N_0}(t) - L_{N_0}C_{N_0}E_{N_0}(t) - L_{N_0}\zeta(t). \quad (3.14)$$

Assuming that $c_n \neq 0$ for all $1 \leq n \leq N_0$, we get that the pair (A_{N_0}, C_{N_0}) is observable. We focus now on $N_0 + 1 \leq n \leq N$. Define

$$\widehat{W}_{N-N_0} = \begin{bmatrix} \widehat{w}_{N_0+1} \\ \vdots \\ \widehat{w}_N \end{bmatrix}, \quad A_{N-N_0} = \begin{bmatrix} -\lambda_{N_0+1} + q & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\lambda_N + q \end{bmatrix}, \quad F_{N-N_0} = \begin{bmatrix} f_{N_0+1} \\ \vdots \\ f_N \end{bmatrix}. \quad (3.15)$$

Since $\ell_n = 0$ for $N_0 + 1 \leq n \leq N$, we get

$$\dot{\widehat{W}}_{N-N_0}(t) = A_{N-N_0}(t)\widehat{W}_{N-N_0}(t) + F_{N-N_0}K\widehat{W}_{N_0}(t), \quad (3.16)$$

$$\dot{W}_{N-N_0}(t) = A_{N-N_0}(t)W_{N_0}(t) + F_{N-N_0}K\widehat{W}_{N_0}(t), \quad (3.17)$$

$$\dot{E}_{N-N_0}(t) = A_{N-N_0}(t)E_{N-N_0}. \quad (3.18)$$

Remark 3.2. *If we take $f(x) = \sum_{n=1}^{N_0} a_n \phi_n$, we get $F_{N-N_0} = 0$.*

Define

$$\mathbf{X} = \begin{bmatrix} \widehat{W}_{N_0} \\ E_{N_0} \\ \widehat{W}_{N-N_0} \\ E_{N-N_0} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_{N_0} + F_{N_0}K & L_{N_0}C_{N_0} & 0 & L_{N_0}C_{N-N_0} \\ 0 & A_{N_0} - L_{N_0}C_{N_0} & 0 & -L_{N_0}C_{N-N_0} \\ F_{N-N_0}K & 0 & A_{N-N_0} & 0 \\ 0 & 0 & 0 & A_{N-N_0} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} L_{N_0} \\ -L_{N_0} \\ 0 \\ 0 \end{bmatrix}, \quad (3.19)$$

$$\mathbf{K} = [K \ 0 \ 0 \ 0]. \quad (3.20)$$

We get thus,

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X} + \mathbf{L}\zeta(t), \quad (3.21)$$

and that the control $u(t)$ can be written $u(t) = \mathbf{K}\mathbf{X}(t)$. Finally, we can state our stabilization result

Theorem 3.1. *Let $N_0 \geq 1$ and $\delta > 0$ given such that $-\lambda_n + q < -\delta < 0$ for all $n \geq N_0 + 1$. Assume that $f_n \neq 0$ and $c_n \neq 0$ for all $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1 \times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that the matrix $A_{N_0} + F_{N_0}K$ and $A_{N_0} - L_{N_0}C_{N_0}$ are Hurwitz with eigenvalues that have real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist $\mathbf{P} \succ 0$, $\alpha > 1$ and $\beta, \gamma > 0$ such that*

$$\Theta_1 = \begin{bmatrix} \mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} + 2\delta \mathbf{P} + \alpha\gamma \|f\|_{L^2(\Omega)} \mathbf{K}^\top \mathbf{K} & \mathbf{P}\mathbf{L} \\ \mathbf{L}^\top \mathbf{P} & -\beta \end{bmatrix} \preceq 0, \quad (3.22)$$

$$\Theta_2 = 2\gamma \left[-\left(1 - \frac{1}{2\alpha}\right) \lambda_{N+1} + q + \delta \right] + \frac{\beta \|c\|_{L^2(\Omega)}}{\lambda_{N+1}} \leq 0. \quad (3.23)$$

Then, for the closed loop system composed of (3.1), observer (3.6) with null initial condition and controller $u(t) = K\widehat{W}_{N_0}(t)$, there exists $M > 0$ such that for any $w_0 \in D(\mathcal{A})$, the classical solution $w(t, \cdot) \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, L^2(\Omega))$ satisfies

$$u(t)^2 + \sum_{n=1}^N \widehat{w}_n(t)^2 + \|w(t, \cdot)\|_{H_0^s(\overline{\Omega})}^2 \leq M e^{-2\delta t} \|w_0\|_{H_0^s(\overline{\Omega})}^2 \quad (3.24)$$

Proof. This proof is inspired on [21]. For the well-posedness for classical solutions, first note that the operator \mathcal{A} generates a strongly continuous submarkovian semigroup on $L^2(\Omega)$ [7] and as we are dealing with bounded observation, the well-posedness follows directly from [28, Chapter 6]. For a classical solution $w(t, \cdot) \in D(\mathcal{A})$, for all $t \geq 0$, consider the following Lyapunov candidate

$$V(\mathbf{X}, w) = \mathbf{X}^\top \mathbf{P}\mathbf{X} + \gamma \sum_{n=N+1}^{\infty} \lambda_n w_n^2. \quad (3.25)$$

Taking the time derivative of V we get

$$\begin{aligned} \dot{V} + 2\delta V &= \mathbf{X}^\top (\mathbf{A}^\top \mathbf{P} + \mathbf{P}\mathbf{A} + 2\delta \mathbf{P})\mathbf{X} + 2\mathbf{X}^\top \mathbf{P}\mathbf{L}\zeta + 2\gamma \sum_{n=N+1}^{\infty} \lambda_n (-\lambda_n + q + \delta) w_n^2 \\ &\quad + 2\gamma \sum_{n=N+1}^{\infty} \lambda_n f_n w_n u. \end{aligned} \quad (3.26)$$

Note that for $\alpha > 0$ and using that $u(t) = \mathbf{K}\mathbf{X}(t)$

$$\begin{aligned} 2\gamma \sum_{n=N+1}^{\infty} \lambda_n f_n w_n u &\leq \frac{\gamma}{\alpha} \sum_{n=N+1}^{\infty} \lambda_n^2 w_n^2 + \gamma\alpha \sum_{n=N+1}^{\infty} f_n^2 u^2 \\ &\leq \frac{\gamma}{\alpha} \sum_{n=N+1}^{\infty} \lambda_n^2 w_n^2 + \gamma\alpha \|f\|_{L^2(\Omega)} \mathbf{X}^\top \mathbf{K}^\top \mathbf{K}\mathbf{X}. \end{aligned}$$

Recalling $\zeta = \sum_{n=N+1}^{\infty} c_n w_n$, we easily get $\zeta^2 \leq \|c\|_{L^2(\Omega)}^2 \sum_{n=N+1}^{\infty} w_n^2$. Thus for all $\beta > 0$

$$\begin{aligned} 0 &\leq \beta \|c\|_{L^2(\Omega)}^2 \sum_{n=N+1}^{\infty} w_n^2 - \beta \zeta^2 \\ &= \beta \|c\|_{L^2(\Omega)}^2 \sum_{n=N+1}^{\infty} \frac{\lambda_n}{\lambda_n} w_n^2 - \beta \zeta^2 \\ &\leq \beta \frac{\|c\|_{L^2(\Omega)}^2}{\lambda_{N+1}} \sum_{n=N+1}^{\infty} \lambda_n w_n^2 - \beta \zeta^2. \end{aligned}$$

Thus,

$$\dot{V} + 2\delta V \leq \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix}^\top \Theta_1 \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix} + \sum_{n=N+1}^{\infty} \left[2\gamma \left(-\lambda_n \left(1 - \frac{1}{2\alpha}\right) + q + \delta \right) + \beta \frac{\|c\|_{L^2(\Omega)}^2}{\lambda_{N+1}} \right] \lambda_n w_n^2. \quad (3.27)$$

Then for $\alpha > \frac{1}{2}$

$$\begin{aligned} \dot{V} + 2\delta V &\leq \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix}^\top \Theta_1 \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix} + \sum_{n=N+1}^{\infty} \left[2\gamma \left(-\lambda_{N+1} \left(1 - \frac{1}{2\alpha} \right) + q + \delta \right) + \beta \frac{\|c\|_{L^2(\Omega)}^2}{\lambda_{N+1}} \right] \lambda_n w_n^2 \\ &= \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix}^\top \Theta_1 \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix} + \Theta_2 \sum_{n=N+1}^{\infty} \lambda_n w_n^2 \\ &\leq 0. \end{aligned} \quad (3.28)$$

□

Remark 3.3. *It is important to remark, that the inequalities (3.22) and (3.23) are always feasible for N big enough, we refer to [21] for proof of this feasibility.*

In this part we describe some extensions of Theorem 3.1 in different configurations.

3.1. $L^2(\Omega)$ -stability. As we see in Theorem 3.1. Our stabilization results ask to consider initial data in $H_0^s(\bar{\Omega})$, but under some minor modification we can prove the following exponential stability estimate on $L^2(\Omega)$:

$$u(t)^2 + \sum_{n=1}^N \hat{w}_n(t)^2 + \|w(t, \cdot)\|_{L^2(\Omega)}^2 \leq M e^{-2\delta t} \|w_0\|_{L^2(\Omega)}^2. \quad (3.29)$$

The main difference is that we have to consider the following Lyapunov candidate

$$V(\mathbf{X}, w) = \mathbf{X}^\top \mathbf{P} \mathbf{X} + \gamma \sum_{n=N+1}^{\infty} w_n^2. \quad (3.30)$$

Following the same steps as before we can prove

$$\dot{V} + 2\delta V \leq \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix}^\top \Theta_1 \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix} + \left(2\gamma \left[-\lambda_{N+1} + \frac{1}{2\alpha} + q + \delta \right] + \beta \|c\|_{L^2(\Omega)} \right) \sum_{n=N+1}^{\infty} w_n^2 \leq 0, \quad (3.31)$$

under the assumptions (3.22) and

$$\tilde{\Theta}_2 = 2\gamma \left[-\lambda_{N+1} + \frac{1}{2\alpha} + q + \delta \right] + \beta \|c\|_{L^2(\Omega)} \leq 0. \quad (3.32)$$

Then, it is enough to observe that $\|w(t, \cdot)\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} w_n^2$. Moreover, regarding the structure of the Lyapunov functions (3.25)-(3.30) we can go a little further and prove the exponential stability for the intermediate spaces $\mathbb{H}_g^\sigma(\Omega)$ ¹ for $0 \leq \sigma \leq \frac{1}{2}$. In fact, consider the Lyapunov candidate

$$V(\mathbf{X}, w) = \mathbf{X}^\top \mathbf{P} \mathbf{X} + \gamma \sum_{j=N+1}^{\infty} \lambda_n^{2\sigma} w_n^2. \quad (3.33)$$

Then, we can obtain

$$\dot{V} + 2\delta V \leq \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix}^\top \Theta_1 \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix} + \sum_{n=N+1}^{\infty} \left(2\gamma \left[-\lambda_n + \frac{\lambda_n^{2\sigma}}{2\alpha} + q + \delta \right] + \beta \frac{\|c\|_{L^2(\Omega)}}{\lambda_{N+1}^{2\sigma}} \right) \lambda_n^{2\sigma} w_n^2. \quad (3.34)$$

Recalling the following estimate² from [18] (or [8]) we know that

$$\frac{1}{2} \left(\frac{n\pi}{2} \right)^{2s} \leq \lambda_n \leq \left(\frac{n\pi}{2} \right)^{2s}, \quad \text{for all } n \in \mathbb{N}. \quad (3.35)$$

¹We call $\mathbb{H}_g^\sigma(\Omega)$ an intermediate space, because by [32] for $0 < \sigma < 1$, $\mathbb{H}_g^\sigma(\Omega) = \left[D \left((-d_x^2)_D \right)^s, L^2(\Omega) \right]_{1-\sigma}$, the complex interpolation spaces.

²This estimate was proved in [8], using the theory of Cauchy process and mini-max theorem

Our idea is to use (3.35) to ensure that $\lambda_n \geq 1$ for all $n > 1$. Note that by (3.35) $\lambda_1 \geq \frac{1}{2} \left(\frac{\pi}{2}\right)^{2s}$, in particular $\lambda_1 > 1$ if

$$\begin{aligned} \frac{1}{2} \left(\frac{\pi}{2}\right)^{2s} > 1 &\iff 2s \ln \left(\frac{\pi}{2}\right) > \ln(2), \\ s > s^* &= \frac{\ln(2)}{2 \ln \left(\frac{\pi}{2}\right)} \approx 0,77. \end{aligned} \quad (3.36)$$

Here we have to cases

- For $s \in (s^*, 1)$ we have that $\lambda_n \geq 1$ for all $n > 1^3$. Thus, we get $\lambda_n^{2\sigma} \leq \lambda_n$ for $n > 1$.
- For $s \in (0, s^*]$, by (3.35) there exists a $N^* \in \mathbb{N}$ such that for $n > N^*$, $\lambda_n \geq 1$. Thus, in this case we redefine N (taking more modes) such that $\lambda_n \geq 1$ for all $n > N + 1$.

We get, from (3.34)

$$\dot{V} + 2\delta V \leq \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix}^\top \Theta_1 \begin{bmatrix} \mathbf{X} \\ \zeta \end{bmatrix} + \sum_{n=N+1}^{\infty} \left(2\gamma \left[-\lambda_n \left(1 - \frac{1}{2\alpha} \right) + q + \delta \right] + \beta \frac{\|c\|_{L^2(\Omega)}}{\lambda_{N+1}^{2\sigma}} \right) \lambda_n^{2\sigma} w_n^2. \quad (3.37)$$

Thus, under the assumptions (3.22) and

$$\bar{\Theta}_2 = 2\gamma \left[-\lambda_{N+1} \left(1 - \frac{1}{2\alpha} \right) + q + \delta \right] + \beta \frac{\|c\|_{L^2(\Omega)}}{\lambda_{N+1}^{2\sigma}} \leq 0, \quad (3.38)$$

we deduce $V(t) \leq e^{-2\delta t} V(0)$, finally by the definition of the norm of $\mathbb{H}_s^\sigma(\Omega)$ (2.13) we conclude the exponential stability in the space $\mathbb{H}_s^\sigma(\Omega)$.

3.2. Localized actuator. In Theorem 3.1 a very important assumption is that the pair (A_{N_0}, F_{N_0}) is controllable, which is equivalent to $f_n = \int_{\Omega} f \phi_n dx \neq 0$ for all $1 \leq n \leq N_0$. By Remark 3.1 we build a function satisfying this condition. In the context of controllability, a very common hypothesis is that the control function is supported in a non-empty open subset $\omega \subset \Omega$. Note that the function defined in Remark 3.1 does not satisfy this condition because is a linear combination of the eigenfunctions. In this subsection, we state the existence of a function $f = \mathbf{1}_\omega f$ such that the pair (A_{N_0}, F_{N_0}) is controllable i.e. Given $\omega \subset \Omega$ we are looking for a function $f \in L^2(\omega)$ such that

$$\int_{\omega} f \phi_n dx \neq 0, \quad 1 \leq n \leq N_0. \quad (3.39)$$

By contradiction, suppose that for all $f \in L^2(\omega)$ there exists $1 \leq n \leq N_0$ such that

$$\int_{\omega} f \phi_n dx = 0. \quad (3.40)$$

In particular, as $L^2(\omega)$ is infinite dimensional, there exists an $K \subset L^2(\omega)$, with $\text{card}(K) = \infty$ and $1 \leq n_K \leq N_0$ such that

$$\int_{\omega} f \phi_{n_K} dx = 0, \quad \forall f \in K, \quad (3.41)$$

from where we can deduce that $\phi_{n_K} \equiv 0$ in ω . Now as $(-d_x^2)^s \phi_{n_K} = \lambda_{n_K} \phi_{n_K}$ and $\phi_{n_K} \equiv 0$ in ω , by the unique continuation property (IntUC), we get $\phi_{n_K} \equiv 0$ in Ω , which is a contradiction. Finally, we conclude the existence of $f = \mathbf{1}_\omega f \in L^2(\omega)$ such that (3.39) holds.

³For $s \in (0, 1)$ we get that for $N \in \mathbb{N}$ big enough $\lambda_n > 1$ for all $n \geq N$. In [18] it is shown numerically that even for $s = 0.005$, $\lambda_2 \approx 1.0086 > 1$.

4. EXPONENTIAL STABILITY WITH EXTERIOR OBSERVATION

The analysis developed in the past section is also true for the observation function $c \in L^2(\Omega)$. That means for instance that we can act with our control in an open set $\omega_1 \subset \Omega$ and measure of solution in other open set $\omega_2 \subset \Omega$. Due to the nonlocal nature of the fractional Laplacian, a natural question arises: Can we build the controller using an exterior observation? Let $\mathcal{O} \subset \mathbb{R} \setminus \Omega$ a nonempty open set and consider the following system with exterior observation

$$\begin{cases} \partial_t w + (-d_x^2)^s w - qw = u(t)f(x), & t > 0, x \in \Omega, \\ w(t, x) = 0, & t > 0, x \in \mathbb{R} \setminus \Omega, \\ w(0, x) = w_0(x), & x \in \Omega, \\ \tilde{z}(t) = \int_{\mathcal{O}} \mathcal{N}_s w(t, x) dx, & t > 0, \end{cases} \quad (4.1)$$

where \mathcal{N}_s is the nonlocal normal derivative operator defined on (2.16), this case is in some sense the analogue of the Neumann trace observer for the classical heat equation. First, recalling that we can write $w(t, x) = \sum_{i=1}^{\infty} w_i(t)\phi_i(x)$, we write the observation operator \tilde{z} as

$$\tilde{z}(t) = \sum_{i=1}^{\infty} w_i(t) \int_{\mathcal{O}} \mathcal{N}_s \phi_i(x) dx. \quad (4.2)$$

The observer dynamics writes as

$$\partial_t \hat{w}_n(t) = (-\lambda_n + q)\hat{w}_n(t) + f_n u(t) - \ell_n \left(\sum_{i=1}^N \hat{w}_i(t) \int_{\mathcal{O}} \mathcal{N}_s \phi_i(x) dx - \tilde{z}(t) \right), \quad (4.3)$$

where we have taken a null initial condition for the observer dynamics and observer gains $\ell_n \in \mathbb{R}$ for $1 \leq n \leq N$, such that $\ell_n = 0$ for $N_0 + 1 \leq n \leq N$. Recalling for $1 \leq n \leq N$, the observation error e_n and defining $\tilde{\zeta}(t) = \sum_{i=N+1}^{\infty} w_i(t) \int_{\mathcal{O}} \mathcal{N}_s \phi_i(x) dx$, we can write the observer dynamics as follows

$$\partial_t \hat{w}_n(t) = (-\lambda_n + q)\hat{w}_n(t) + f_n u(t) + \ell_n \sum_{i=1}^N e_i(t) \int_{\mathcal{O}} \mathcal{N}_s \phi_i(x) dx + \ell_n \tilde{\zeta}(t). \quad (4.4)$$

We introduce now,

$$\tilde{C}_{N_0} = \left[\int_{\mathcal{O}} \mathcal{N}_s \phi_1(x) dx \dots \int_{\mathcal{O}} \mathcal{N}_s \phi_{N_0}(x) dx \right], \quad \tilde{C}_{N-N_0} = \left[\int_{\mathcal{O}} \mathcal{N}_s \phi_{N_0+1}(x) dx \dots \int_{\mathcal{O}} \mathcal{N}_s \phi_N(x) dx \right]. \quad (4.5)$$

Thus, we have

$$\dot{\hat{W}}_{N_0}(t) = A_{N_0} \hat{W}_{N_0}(t) + u(t)F_{N_0} + L_{N_0} \tilde{C}_{N_0} E_{N_0}(t) + L_{N_0} \tilde{C}_{N-N_0} E_{N-N_0}(t) + L_{N_0} \tilde{\zeta}(t). \quad (4.6)$$

Suppose now that the pair (A_{N_0}, F_{N_0}) is controllable, thus there exists a matrix $K \in \mathbb{R}^{1 \times N_0}$ such that the matrix $A_{N_0} + F_{N_0}K$ is Hurwitz. Take the control $u(t)$ as $u(t) = K \hat{W}_{N_0}(t)$. We need now that the pair $(A_{N_0}, \tilde{C}_{N_0})$ be observable, in particular we prove that there exists an exterior observation set \mathcal{O} , such that the pair $(A_{N_0}, \tilde{C}_{N_0})$ is observable. Note first that the pair $(A_{N_0}, \tilde{C}_{N_0})$ is observable if and only if,

$$\int_{\mathcal{O}} \mathcal{N}_s \phi_i(x) dx \neq 0, \quad \forall 1 \leq i \leq N_0. \quad (4.7)$$

Suppose that it is not possible, then similar as the past section we get the existence of $1 \leq \tilde{n} \leq N_0$ and a family $\mathcal{M} \subset \mathcal{P}(\mathbb{R} \setminus \Omega)$ with $\text{card}(\mathcal{M}) = \infty$ such that

$$\int_{\mathcal{O}} \mathcal{N}_s \phi_{\tilde{n}}(x) dx = 0, \quad \forall \mathcal{O} \in \mathcal{M}. \quad (4.8)$$

Regarding the definition of \mathcal{N}_s and that $\phi_{\tilde{n}} \equiv 0$ on $\mathbb{R} \setminus \Omega$ we get

$$\mathcal{N}_s \phi_{\tilde{n}}(x) = \int_{\Omega} \frac{\phi_{\tilde{n}}(y)}{|x-y|^{1+2s}} dy, \quad x \in \mathcal{O}. \quad (4.9)$$

Now, by the continuity of eigenfunctions, we get that $\mathcal{N}_s \phi_{\tilde{n}} \equiv 0$ on \mathcal{O} , now by the unique continuation property (ExtUC) we get that $\phi_{\tilde{n}} \equiv 0$ on Ω that it is a contradiction and thus $(A_{N_0}, \tilde{C}_{N_0})$ is observable. Define

$$\mathbf{X} = \begin{bmatrix} \widehat{W}_{N_0} \\ E_{N_0} \\ \widehat{W}_{N-N_0} \\ E_{N-N_0} \end{bmatrix}, \quad \tilde{\mathbf{A}} = \begin{bmatrix} A_{N_0} + F_{N_0}K & L_{N_0}\tilde{C}_{N_0} & 0 & L_{N_0}\tilde{C}_{N-N_0} \\ 0 & A_{N_0} - L_{N_0}\tilde{C}_{N_0} & 0 & -L_{N_0}\tilde{C}_{N-N_0} \\ F_{N-N_0}K & 0 & A_{N-N_0} & 0 \\ 0 & 0 & 0 & A_{N-N_0} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} L_{N_0} \\ -L_{N_0} \\ 0 \\ 0 \end{bmatrix}. \quad (4.10)$$

$$\mathbf{K} = [K \quad 0 \quad 0 \quad 0]. \quad (4.11)$$

We get thus,

$$\dot{\mathbf{X}}(t) = \tilde{\mathbf{A}}\mathbf{X} + \mathbf{L}\tilde{\zeta}(t), \quad (4.12)$$

and that the control $u(t)$ can be written $u(t) = \mathbf{K}\mathbf{X}(t)$. Finally, we can state our stabilization result

Theorem 4.1. *Let $s \in (1/2, 1)$, $N_0 \geq 1$ and $\delta > 0$ given such that $-\lambda_n + q < -\delta < 0$ for all $n \geq N_0 + 1$. Assume that $f_n \neq 0$ and $\int_{\mathcal{O}} \mathcal{N}_s \phi_n dx \neq 0$ for all $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1 \times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that the matrix $A_{N_0} + F_{N_0}K$ and $A_{N_0} - L_{N_0}\tilde{C}_{N_0}$ are Hurwitz with eigenvalues that have real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist $\mathbf{P} \succ 0$, $\alpha > 1$ and $\beta, \gamma > 0$ such that*

$$\tilde{\Theta}_1 = \begin{bmatrix} \tilde{\mathbf{A}}^\top \mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} + 2\delta\mathbf{P} + \alpha\gamma\|f\|_{L^2(\Omega)}\mathbf{K}^\top \mathbf{K} & \mathbf{P}\mathbf{L} \\ \mathbf{L}^\top \mathbf{P} & 0 \end{bmatrix} \preceq 0, \quad (4.13)$$

$$\tilde{\Theta}_2 = \lambda_N \left(-2\gamma + \frac{\gamma}{\alpha} + \beta S \lambda_{N+1}^{2\kappa-2} \right) + 2\gamma(q + \delta) \leq 0. \quad (4.14)$$

Then, for the closed loop system composed of (4.1), observer (4.3) with null initial condition and controller $u(t) = K\widehat{W}_{N_0}(t)$, there exists $M > 0$ such that for any $w_0 \in D(\mathcal{A})$, the classical solution $w(t, \cdot) \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, L^2(\Omega))$ satisfies

$$u(t)^2 + \sum_{n=1}^N \widehat{w}_n(t)^2 + \|w(t, \cdot)\|_{H_0^s(\bar{\Omega})}^2 \leq M e^{-2\delta t} \|w_0\|_{H_0^s(\bar{\Omega})}^2. \quad (4.15)$$

Proof. The proof is quite similar to Theorem 3.1, thus we emphasize on the main differences. For a classical solution $w(t, \cdot) \in D(\mathcal{A})$, for all $t \geq 0$, consider the following Lyapunov candidate V defined on (3.25)

$$V(\mathbf{X}, w) = \mathbf{X}^\top \mathbf{P}\mathbf{X} + \gamma \sum_{n=N+1}^{\infty} \lambda_n w_n^2. \quad (4.16)$$

Taking the time derivative of V we get

$$\begin{aligned} \dot{V} + 2\delta V &= \mathbf{X}^\top (\tilde{\mathbf{A}}^\top \mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} + 2\delta\mathbf{P})\mathbf{X} + 2\mathbf{X}^\top \mathbf{P}\mathbf{L}\tilde{\zeta} + 2\gamma \sum_{n=N+1}^{\infty} \lambda_n (-\lambda_n + q + \delta) w_n^2 \\ &\quad + 2\gamma \sum_{n=N+1}^{\infty} \lambda_n f_n w_n u. \end{aligned} \quad (4.17)$$

Note that for $\alpha > 0$ and using that $u(t) = \mathbf{K}\mathbf{X}(t)$

$$2\gamma \sum_{n=N+1}^{\infty} \lambda_n f_n w_n u \leq \frac{\gamma}{\alpha} \sum_{n=N+1}^{\infty} \lambda_n^2 w_n^2 + \gamma\alpha\|f\|_{L^2(\Omega)}\mathbf{X}^\top \mathbf{K}^\top \mathbf{K}\mathbf{X}.$$

Recalling $\tilde{\zeta} = \sum_{n=N+1}^{\infty} w_n \int_{\mathcal{O}} \mathcal{N}_s \phi_n \, dx$, we easily get for $\kappa \in \mathbb{R}$

$$\begin{aligned} \tilde{\zeta}^2 &\leq \sum_{n=N+1}^{\infty} \lambda_n^{2\kappa} w_n^2 \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n^{2\kappa}} \left(\int_{\mathcal{O}} \mathcal{N}_s \phi_n \, dx \right)^2 \\ &\leq \sum_{n=N+1}^{\infty} \lambda_n^{2\kappa} w_n^2 \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n^{2\kappa}} \int_{\mathcal{O}} (\mathcal{N}_s \phi_n)^2 \, dx. \end{aligned}$$

Now, we pass to estimate $\int_{\mathcal{O}} (\mathcal{N}_s \phi_n)^2 \, dx$. By [32, Lemma 12] we have

$$\int_{\mathcal{O}} (\mathcal{N}_s \phi_n)^2 \, dx \leq \|\mathcal{N}_s \phi_n\|_{L^2(\mathbb{R} \setminus \Omega)}^2 \leq C \|\phi_n\|_{H^s(\mathbb{R})}^2. \quad (4.18)$$

By definition

$$\|\phi_n\|_{H^s(\mathbb{R})}^2 = \int_{\mathbb{R}} (\phi_n)^2 \, dx + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\phi_n(x) - \phi_n(y))^2}{|x - y|^{1+2s}} \, dx \, dy. \quad (4.19)$$

Using that $\phi_n \in H_0^s(\bar{\Omega})$ and the integration by parts formula Lemma 2.1, we obtain

$$\|\phi_n\|_{H^s(\mathbb{R})}^2 = \int_{\Omega} (\phi_n)^2 \, dx + \frac{2}{c_s} \left(\int_{\mathbb{R} \setminus \Omega} \phi_n \mathcal{N}_s \phi_n \, dx + \int_{\Omega} \phi_n (-d_x^2)^s \phi_n \, dx \right) = 1 + \frac{2\lambda_n}{c_s}. \quad (4.20)$$

Therefore, we get

$$\tilde{\zeta}^2 \leq \left(\sum_{n=N+1}^{\infty} \lambda_n^{2\kappa} w_n^2 \right) \underbrace{\left(C \sum_{n=N+1}^{\infty} \left(\frac{1}{\lambda_n^{2\kappa}} + \frac{2}{c_s \lambda_n^{2\kappa-1}} \right) \right)}_S.$$

Taking $\kappa > \frac{1}{4s} + \frac{1}{2}$, we get that the sum S is bounded. Thus for all $\beta > 0$

$$0 \leq \beta S \sum_{n=N+1}^{\infty} \lambda_n^{2\kappa} w_n^2 - \beta \tilde{\zeta}^2.$$

Finally

$$\dot{V} + 2\delta V \leq \begin{bmatrix} \mathbf{X} \\ \tilde{\zeta} \end{bmatrix}^\top \tilde{\Theta}_1 \begin{bmatrix} \mathbf{X} \\ \tilde{\zeta} \end{bmatrix} + \sum_{n=N+1}^{\infty} \left[\lambda_n \left(-2\gamma + \frac{\gamma}{\alpha} + \beta S \lambda_n^{2\kappa-2} \right) + 2\gamma(q + \delta) \right] \lambda_n w_n^2. \quad (4.21)$$

Then, for $\kappa < 1$, which is always possible because $s \in (1/2, 1)$ we observe that

$$\lambda_n \left(-2\gamma + \frac{\gamma}{\alpha} + \beta S \lambda_n^{2\kappa-2} \right) \leq \lambda_n \left(-2\gamma + \frac{\gamma}{\alpha} + \beta S \lambda_{N+1}^{2\kappa-2} \right).$$

Now, as $\kappa < 1$, $\lim_{N \rightarrow \infty} \lambda_{N+1}^{2\kappa-2} = 0$, we get that for all $n \geq N$, with N big enough

$$\lambda_n \left(-2\gamma + \frac{\gamma}{\alpha} + \beta S \lambda_{N+1}^{2\kappa-2} \right) + 2\gamma(q + \delta) \leq \underbrace{\lambda_N \left(-2\gamma + \frac{\gamma}{\alpha} + \beta S \lambda_{N+1}^{2\kappa-2} \right) + 2\gamma(q + \delta)}_{\hat{\Theta}_2},$$

and in conclusion if (4.13) and (4.14) hold, then we deduce the exponential stability. \square

5. MORE GENERAL STRUCTURE OF THE CONTROLLER.

In the past result, we proved our stabilization results by considering a controller in the form

$$\mathcal{F} = f(x)u(t) = u(t) \sum_{n=1}^N f_n \phi_n. \quad (5.1)$$

The objective of this section is to go a little further and consider a localized control in the form

$$\mathcal{F} = \mathbb{1}_\omega \left(\sum_{n=1}^N \phi_n(x) u_n(t) \right), \quad (5.2)$$

and build a full state finite dimensional feedback, in this case we are able to prove the stabilization for all $s \in (0, 1)$ for a localized controller, the main idea is to prove a weak spectral inequality and Lyapunov methods. We based some of our ideas on [3] where the internal stabilization with finite-dimensional controllers of the Navier-Stokes equation was shown using state-decomposition and Riccati methods, see also [34] where a finite-time stabilization result was proved for the heat equation.

5.1. Full action controller. In this first part we consider a simpler case where the controller acts on the whole domain Ω and our aim is to build an feedback controller (without observation). In particular we take the feedback operator \mathcal{F} in the following form

$$\mathcal{F} = \sum_{n=1}^N \phi_n(x) u_n(t), \quad (5.3)$$

where $N \in \mathbb{N}$ is such that $-\lambda_n + q < -\delta < 0$ for all $n \geq N + 1$ for some $\delta > 0$. Thus the system considered reads as

$$\begin{cases} \partial_t w + (-d_x^2)^s w - qw = \sum_{n=1}^N \phi_n(x) u_n(t), & x \in \Omega, \\ w(t, x) = 0, & x \in \mathbb{R} \setminus \Omega, \\ w(0, x) = w_0(x), & x \in \Omega. \end{cases} \quad (5.4)$$

Therefore we get the following infinite dimensional system

$$\begin{cases} \partial_t w_n(t) = (-\lambda_n + q)w_n(t) + u_n(t), & 1 \leq n \leq N, \\ \partial_t w_n(t) = (-\lambda_n + q)w_n(t), & N + 1 \leq n < \infty, \end{cases} \quad (5.5)$$

Define

$$U_N = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}. \quad (5.6)$$

Thus,

$$\dot{W}_N(t) = A_N W_N(t) + U_N(t). \quad (5.7)$$

Now we take $u_n = -\gamma w_n$, for some $\gamma > 0$, then we have

$$\dot{W}_N(t) = (A_N - \gamma I) W_N(t). \quad (5.8)$$

Consider the following Lyapunov candidate

$$V(w) = \frac{\mu}{2} \|W_N\|^2 + \frac{1}{2} \sum_{n=N+1}^{\infty} w_n^2, \quad (5.9)$$

for $\mu > 0$, it is not difficult to see that V is equivalent to the $L^2(\Omega)$ -norm. Moreover

$$\dot{V} = \mu \sum_{n=1}^N (-\lambda_n + q - \gamma) w_n^2 + \sum_{n=N+1}^{\infty} (-\lambda_n + q) w_n^2, \quad (5.10)$$

taking $\mu = 1$ and $\gamma = q$ we get

$$\dot{V} = - \sum_{n=1}^N \lambda_n w_n^2 + \sum_{n=N+1}^{\infty} (-\lambda_n + q) w_n^2 \leq \max(-\lambda_1, -\lambda_{N+1} + q) \|w(t, \cdot)\|_{L^2(\Omega)}^2, \quad (5.11)$$

from where we deduce the exponential stability in $L^2(\Omega)$.

5.2. Localized controller. Now, following the steps of the past section we prove the exponential stability in the case where the feedback controller is localized in an open subset of Ω . Let $\omega \subset \Omega$ and open set and consider the following system

$$\begin{cases} \partial_t w + (-d_x^2)^s w - qw = \mathbb{1}_\omega \left(\sum_{n=1}^N \phi_n(x) u_n(t) \right), & x \in \Omega, \\ w(t, x) = 0, & x \in \mathbb{R} \setminus \Omega, \\ w(0, x) = w_0(x), & x \in \Omega. \end{cases} \quad (5.12)$$

Recall that $-\lambda_n + q \geq 0$ for all $n \leq N$, thus the uncontrolled system is unstable. We prove that we can chose u_n in suitable way such that (5.12) be exponentially stable. With the same strategy as before, we have the following infinite dimensional system

$$\partial_t w_n(t) = (-\lambda_n + q)w_n(t) + \sum_{j=1}^N \langle \phi_n, \phi_j \rangle_{L^2(\omega)} u_j(t), \quad n \in \mathbb{N}, \quad (5.13)$$

where $\langle \phi_n, \phi_j \rangle_{L^2(\omega)} = \langle \mathbb{1}_\omega \phi_n, \phi_j \rangle_{L^2(\Omega)}$. Define J_N as the symmetric matrix

$$J_N = \begin{bmatrix} \langle \phi_1, \phi_1 \rangle_{L^2(\omega)} & \langle \phi_2, \phi_1 \rangle_{L^2(\omega)} & \cdots & \langle \phi_N, \phi_1 \rangle_{L^2(\omega)} \\ * & \langle \phi_2, \phi_2 \rangle_{L^2(\omega)} & \cdots & \vdots \\ * & * & \ddots & \vdots \\ * & * & * & \langle \phi_N, \phi_N \rangle_{L^2(\omega)} \end{bmatrix}. \quad (5.14)$$

Therefore,

$$\dot{W}_N(t) = A_N W_N(t) + J_N U_N(t). \quad (5.15)$$

We consider $U_N = -\gamma W_N$, thus we obtain the following closed-loop system:

$$\dot{W}_N(t) = (A_N - \gamma J_N) W_N(t). \quad (5.16)$$

Consider the following Lyapunov candidate

$$V(w) = \frac{\mu}{2} \|W_N\|^2 + \frac{1}{2} \sum_{n=N+1}^{\infty} w_n^2. \quad (5.17)$$

Then

$$\dot{V} = \mu W_N^\top (A_N - \gamma J_N) W_N + \sum_{n=N+1}^{\infty} (-\lambda_n + q) w_n^2 + \sum_{n=N+1}^{\infty} \sum_{j=1}^N w_n \langle \phi_n, \phi_j \rangle_{L^2(\omega)} u_j. \quad (5.18)$$

Now, we will estimate the above terms. It is not difficult to see that

$$W_N^\top J_N W_N = \left\| \sum_{n=1}^N w_n \phi_n \right\|_{L^2(\omega)}^2, \quad (5.19)$$

then

$$W_N^\top (A_N - \gamma J_N) W_N = \sum_{n=1}^N (-\lambda_n + q) w_n^2 - \gamma \left\| \sum_{n=1}^N w_n \phi_n \right\|_{L^2(\omega)}^2. \quad (5.20)$$

Now we use the following *weak spectral inequality* which proof is on Appendix A.

Claim 5.1. *Exist $C > 0$ such that for all $(a_1, \dots, a_N) \in \mathbb{R}^N$, we have*

$$\left\| \sum_{n=1}^N a_n \phi_n \right\|_{L^2(\omega)}^2 \geq C \sum_{n=1}^N a_n^2. \quad (\text{Spec})$$

Using the weak spectral inequality (Spec), we get

$$W_N^\top (A_N - \gamma J_N) W_N \leq \sum_{n=1}^N (-\lambda_n + q - C\gamma) w_n^2. \quad (5.21)$$

For the other terms, we have from one-side

$$\sum_{n=N+1}^{\infty} (-\lambda_n + q)w_n^2 \leq -\delta \sum_{n=N+1}^{\infty} w_n^2 \quad (5.22)$$

and from the other side using that $U_N = -\gamma W_N$ we see that

$$\begin{aligned} \sum_{i=N+1}^{\infty} \sum_{j=1}^N w_n \langle \phi_j, \phi_n \rangle_{L^2(\omega)} u_j &\leq -\gamma \left\langle \sum_{n=N+1}^{\infty} w_n \phi_n, \sum_{j=1}^N w_j \phi_j \right\rangle_{L^2(\omega)} \\ &\leq \gamma \left\| \sum_{n=N+1}^{\infty} w_n \phi_n \right\|_{L^2(\Omega)} \left\| \sum_{j=1}^N w_j \phi_j \right\|_{L^2(\Omega)} \\ &\leq \frac{\delta}{2} \sum_{n=N+1}^{\infty} w_n^2 + \frac{\gamma^2}{2\delta} \sum_{n=1}^N w_n^2. \end{aligned} \quad (5.23)$$

Joining all these estimates, we obtain

$$\begin{aligned} \dot{V} &\leq \sum_{n=1}^N \left[\mu(-\lambda_n + q - C\gamma) + \frac{\gamma^2}{2\delta} \right] w_n^2 - \frac{\delta}{2} \sum_{n=N+1}^{\infty} w_n^2 \\ &\leq \left[\mu(-\lambda_1 + q - C\gamma) + \frac{\gamma^2}{2\delta} \right] \sum_{n=1}^N w_n^2 - \frac{\delta}{2} \sum_{n=N+1}^{\infty} w_n^2. \end{aligned}$$

For $\varepsilon > 0$, we can take $\gamma = \frac{1}{C}(-\lambda_1 + q + \varepsilon)$, then $\mu(-\lambda_1 + q - C\gamma) + \frac{\gamma^2}{2\delta} = -\mu\varepsilon + \frac{\gamma^2}{2\delta} < -\frac{\delta}{2}$, for $\mu > 0$ big enough. With this choice, we obtain

$$\begin{aligned} \dot{V} &\leq -\frac{\delta}{2} \sum_{n=1}^N w_n^2 - \frac{\delta}{2} \sum_{n=N+1}^{\infty} w_n^2 \\ &= -\frac{\delta}{2\mu} \mu \|W_N\|^2 - \frac{\delta}{2} \sum_{n=N+1}^{\infty} w_n^2 \\ &\leq \min \left(-\frac{\delta}{2\mu}, -\delta \right) V, \end{aligned} \quad (5.24)$$

from where $\dot{V} \leq -\tau V$ for some τ positive, that ensures the $L^2(\Omega)$ exponential stability, because the Lyapunov function V is equivalent to the $L^2(\Omega)$ -norm.

Remark 5.1. We call to the inequality (Spec) weak spectral inequality, motivated for the following. Let $\Omega \subset \mathbb{R}^N$ an open bounded set, and consider the Laplace operator with Dirichlet boundary condition: $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$. Let $(\lambda_n, \phi_n)_{n \in \mathbb{N}}$ its eigenvalues and eigenfunctions. In [19] the following spectral inequality was proved. Let $\delta > 0$ and N_δ such that $\lambda_{N_\delta} \leq \delta < \lambda_{N_\delta+1}$. Then for $\omega \subset \Omega$, there exists $C \geq 1$ independent of δ such that

$$\left\| \sum_{n=1}^{N_\delta} a_n \phi_n \right\|_{L^2(\omega)}^2 \geq C^{-1} e^{-C\sqrt{\delta}} \sum_{n=1}^{N_\delta} a_n^2. \quad (5.25)$$

This inequality is the key point on the proof of null controllability for the heat equation on [19] and was proved via Carleman estimates. Therefore, in the case of fractional Laplacian and $s \in (1/2, 1)$ at this time the above inequality is not known. If this spectral inequality it is true in the case $s \in (1/2, 1)$, this allow us to prove the null controllability of the semilinear fractional heat equation.

6. CONCLUSION

In this work the exponential stability via a finite-dimensional controller for the fractional heat equation was analyzed. The fractional Laplacian is taken as a non local operator, that makes the spectral analysis non explicit. First the case $s \in (1/2, 1)$ was considered, here the controller was

built using and bounded internal and exterior observation, the case where the control acts locally was also studied. Finally, the exponential stability in the case $s \in (0, 1)$ was addressed, here the main tool is an appropriate use of a weak spectral inequality. The analysis developed in this paper left some interesting open problems, we can mention for instance.

- **Nonlinearities:** As the exponential stability is mainly based on a Lyapunov function, a future research line could be the extensions of this results to the nonlinear fractional heat equation. It is important to remark that at this time, there are no results of controllability in the nonlinear case.
- **Numerical schemes:** In [21] the theoretical results obtained are complemented with some numerical simulations. When the control is given in this explicit modal form, the classical way to implement numerical simulation is to use the spectral decomposition. In our case this is quite complicated because either the eigenfunctions or the eigenvalues are not known explicitly. Even the numerical approximations for the eigenvalues are not well accurate [18].
- **Exterior controller** In the case of the classical heat equation in one dimension, an interesting problem is when the control acts thorough the boundary, this is typically manage by introducing an appropriate change of variable to pass the control from the boundary to the interior, this strategy can no be applied in the case of exterior controller. Some possible ideas could be follow the arguments of [20, 13].
- **Fractional in time-space** In this note we have considered a system modeled by a nonlocal fractional Laplacian in space variable. An interesting future research line is to consider a system with both components fractional in time and space using the nonlocal fractional Laplacian. In this direction we can mention the works [15, 14] where this problem was analyzed using the spectral fractional Laplacian.

APPENDIX A. ABOUT THE WEAK SPECTRAL INEQUALITY FOR

In this section we prove the weak spectral inequality (Spec). Suppose that it is false, then for all $m \in \mathbb{N}$, there exists $\underline{a} = (a_1^m, \dots, a_N^m)$ such that

$$\left\| \sum_{n=1}^N a_n^m \phi_n \right\|_{L^2(\omega)}^2 \leq \frac{1}{m} \sum_{n=1}^N (a_n^m)^2. \quad (\text{A.1})$$

Define $\underline{z}^m = \frac{\underline{a}^m}{\|\underline{a}^m\|_{\mathbb{R}^N}}$. Then the sequence $(\underline{z}^m)_{m \in \mathbb{N}}$ satisfies

$$\left\| \sum_{n=1}^N z_n^m \phi_n \right\|_{L^2(\omega)}^2 \leq \frac{1}{m}. \quad (\text{A.2})$$

Moreover, as $\|\underline{z}^m\|_{\mathbb{R}^N} = 1$, there exists a convergent subsequence (still denoted by \underline{z}^m), such that $\underline{z}^m \rightarrow \underline{z}$ when $m \rightarrow \infty$. Besides, \underline{z} satisfies $\|\underline{z}\|_{\mathbb{R}^N} = 1$. Taking the limit on (A.2) we deduce that $\sum_{n=1}^N z_n \phi_n = 0$ in ω . As ϕ_n is a continuous function, we can find a uncountable set $\mathcal{M} \subset \omega$ such that for all $x \in \mathcal{M}$ there exists $n = n_x$, such that $\phi_n(x) \neq 0$. In fact, if that is not the case $\phi_n \equiv 0$ in ω and by (IntUC) $\phi_n \equiv 0$ in Ω which is not possible. Therefore

$$z_1 \phi_1(x_0) + \dots + z_N \phi_N(x_0) = 0, \quad \forall x_0 \in \mathcal{M}. \quad (\text{A.3})$$

Therefore z_1, \dots, z_N solve an (infinite) overdetermined linear system. From where $z_1, \dots, z_N = 0$ which contradicts the fact that $\|\underline{z}\|_{\mathbb{R}^N} = 1$.

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