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Stability analysis for linear systems with asymmetric input backlash and dead-zone through LMI conditions

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***Abstract*— This paper deals with the stability analysis of the interconnection between a linear system and a nonlinear operator. This operator is the combination between an asymmetric backlash and a dead-zone. It is a memory-based operator which is only piecewise differentiable. To handle such a nonlinearity, which is more complex than the standard ones, we aim to capture the closed-loop trajectories in some suitable regions of the state space. Uniform Ultimate Boundednes (UUB) property is addressed, using the intrinsic behavior of the operator and a Lyapunov-based approach. Sufficient conditions are provided in terms of Linear Matrix Inequalities (LMIs). A numerical illustration is presented to assess our contribution.**

I. Introduction

This paper is focused on the stability analysis for systems with asymmetric input backlash and dead-zone. The mechanical conception of many systems such as hydraulic cylinder induces backlash effects [1]. This memory-based phenomena is roughly speaking a lost motion caused by gaps between several parts of the mechanism. In addition, these actuators are used to set in motion a mechanical system [2]. Therefore, for low speed, dry friction can be significant and they can be modeled by the dead-zone. Finally the notion of asymmetry comes from the fact that some mechanical systems may have different behavior depending on the direction of motion (external forces, simple effect cylinder, etc). For practical applications such as in [3] or [4], these nonlinearities must be considered to avoid degradation of the closed-loop performance or to avoid loosing stability. The main difficulty of such a nonlinear operator is the fact that it is not differentiable and is only piecewise differentiable. Unfortunately, in the literature, there is only a few contributions dealing with asymmetric backlash: we can cite papers focused on their identification [5], [6] or their compensation [7]. For its control, we can cite [4] using a fuzzy approach or [2] using a Linear Quadratic Regulator (LQR).

We consider here a linear dynamical system with an actuator which behaves accordingly with the nonlinear operator having aforementioned characteristics. Among the many approaches in the literature analyzing the stability, we proposed here, inspired by [8], a Lyapunov analysis in order to provide sufficient conditions to characterize a finite-time attractor for the state trajectories, for all initial conditions. Our approach considers also an asymmetric partition of the state-space, guided by [9] for instance, allowing to consider various and asymmetric sector conditions in each area of the state-space partition. Once the state is inside this attracting set, we can show that the state trajectories are uniformly ultimately bounded. More specifically, we propose sector conditions tailored to the nonlinear operator and the Lyapunov analysis gives rise to Linear Matrix Inequalities (LMI) conditions to prove the finite-time attractivity of a compact set for all initial conditions.

In comparison to existing results such as in [8], [10], [11], the nonlinear operator considered here is more general than the classical backlash, due in particular to the asymmetry of the backlash and also the presence of the dead-zone.

The paper is organized as follows. A formulation of the problem is provided in Section II where the system and the operator are described. After some preliminary results, Section III presents the main result of the paper, which consists in providing sufficient conditions for the characterization of an attractor in terms of LMIs. A numerical illustration and a related discussion are presented in Section IV. Section V gathers concluding remarks and few perspectives.

Notation: We denote by $\mathcal{C}^1_{\text{pw}}([0,+\infty);\mathbb{R}^n)$ the set of continuous piece-wise differentiable functions $f:[0,+\infty)\to\mathbb{R}^n$. In other words, this is the set of continuous functions f being, for some unbounded sequence $(t_j)_{j=0}^{\infty}$ in $[0,+\infty)$ with $t_0=0$, continuously differentiable on (t_{j-1},t_j) for all $j\in\mathbb{N}$. For a symmetric matrix M, $M\succ 0$ ($M\prec 0$, *resp.*) means that M is positive definite (negative definite, *resp.*). In matrices, the symbol $\star$ represents a symmetric block. For a matrix M, $M^\top$ denotes its transpose. For a square matrix M, we define the operator $\text{He}(M)=M+M^\top$. For two square matrices M_1 and M_2, $\text{diag}\{M_1,M_2\}$ means the diagonal block matrix with M_1 and M_2 as blocks. I_n is the Identity matrix of dimension n and $0_{n\times m}$ is the null matrix of dimensions $n\times m$.

II. Problem formulation

Consider the following continuous-time system, consisting in the interconnection between a linear system and an isolated nonlinearity:

$$\begin{array}{rcl}\dot{x}(t) &=& Ax(t)+B\Phi[u](t),\\ u(t) &=& Kx(t),\end{array} \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}$ is the single input of the nonlinearity Φ; $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times 1}$ are given matrices. The nonlinear operator Φ is composed of an asymmetric backslash (described in [8]) and an asymmetric

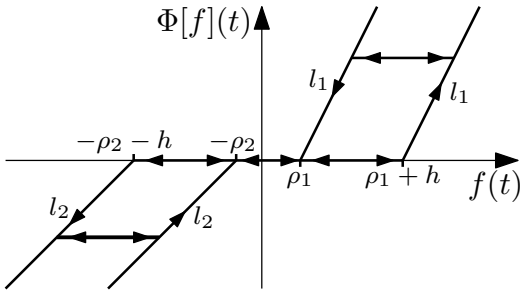

Fig. 1. Depiction of the nonlinear operator.

dead-zone. The operator Φ is defined as follows, for all $f \in \mathcal{C}^1_{\text{pw}}([0,+\infty);\mathbb{R})$, for all $j \in \mathbb{N}$, for all (t_{j-1}, t_j):

$$\dot{\overbrace{\Phi[f](t)}} = \begin{cases} l_1\dot{f}(t) \quad \text{if } \Phi[f](t) \geq 0 \text{ and } \Big(\big(\dot{f} \geq 0 \text{ and } \\ \quad \Phi[f](t) = l_1(f(t)-\rho_1-h)\big) \text{ or } \\ \quad \big(\dot{f}(t) \leq 0 \text{ and } \Phi[f](t) = l_1(f(t)-\rho_1)\big)\Big) \\ l_2\dot{f}(t) \quad \text{if } \Phi[f](t) \leq 0 \text{ and } \Big(\big(\dot{f} \leq 0 \text{ and } \\ \quad \Phi[f](t) = l_2(f(t)+\rho_2+h)\big) \text{ or } \\ \quad \big(\dot{f}(t) \geq 0 \text{ and } \Phi[f](t) = l_2(f(t)+\rho_2)\big)\Big) \\ 0 \quad \text{otherwise} \end{cases}$$

where $0 = t_0 < t_1 < \dots$ is a partition of $[0;+\infty)$ such that f is continuously differentiable on each of the intervals (t_{j-1}, t_j), $j \in \mathbb{N}$. The parameters $h, l_1, l_2, \rho_1, \rho_2 \in \mathbb{R}_{>0}$ correspond to the backslash width, inclination and threshold when f is positive and negative respectively. This operator is represented on Figure 1.

Let us show that the nonlinearity is uniquely defined by following [12]. To this end, we introduce the sets related to the nonlinearity:

$$\Sigma_1 = \{(f,\Phi) \in \mathbb{R}^2, \Phi \geq 0, \Phi + l_1\rho_1 \leq l_1 f \leq \Phi + l_1(\rho_1+h)\},$$
$$\Sigma_2 = \{(f,\Phi) \in \mathbb{R}^2, 0 \leq f \leq \rho_1,\ \Phi = 0\},$$
$$\Sigma_3 = \{(f,\Phi) \in \mathbb{R}^2, -\rho_2 \leq f \leq 0,\ \Phi = 0\},$$
$$\Sigma_4 = \{(f,\Phi) \in \mathbb{R}^2, \Phi \leq 0, \Phi - l_2(\rho_2+h) \leq l_2 f \leq \Phi - l_2\rho_2\},$$
$$\Sigma = \cup_{i\in\{1,2,3,4\}}\Sigma_i.$$

For any $(f_k, \Phi_k) \in \Sigma$, we define two functions, $F_i(\cdot, f_k, \Phi_k) : [f_k, +\infty) \to [\Phi_k, +\infty)$ and $F_d(\cdot, f_k, \Phi_k) : (-\infty, f_k] \to (-\infty, \Phi_k]$:

$$F_i(f, f_k, \Phi_k) = \begin{cases} \max(\Phi_k, l_1(f-\rho_1-h)), \text{ if } (f_k,\Phi_k) \in \Sigma_1, \\ \max(l_1(f-\rho_1-h), \min(0, \max(\Phi_k, l_2(f+\rho_2)))), \\ \quad \text{if } (f_k,\Phi_k) \in \cup_{i=2,3,4}\Sigma_i. \end{cases}$$

$$F_d(f, f_k, \Phi_k) = \begin{cases} \min(\Phi_k, l_2(f+\rho_2+h)), \text{ if } (f_k,\Phi_k) \in \Sigma_4, \\ \min(l_2(f+\rho_2+h), \max(0, \min(\Phi_k, l_1(f-\rho_1)))), \\ \quad \text{if } (f_k,\Phi_k) \in \cup_{i=1,2,3}\Sigma_i, \end{cases}$$

By decomposing the time-domain in intervals $[t_k, t_{k+1})$ where $f(t)$ is monotone, it is possible to define the value of the nonlinear operator by $\Phi[f](t) = F_i(f(t), f_k, \Phi_k)$ when f is increasing (or $\Phi[f](t) = F_d(f(t), f_k, \Phi_k)$ when f is decreasing), with $f_k = f(t_k)$ and $\Phi_k = \Phi[f](t_k)$. As a consequence, Σ is invariant by applying the nonlinear operator. In other words, it means that if at $t_0 = 0$, the nonlinearity is active $(u_0, \Phi[u](0)) \in \Sigma$, then it remains active $(u(t), \Phi[u](t)) \in \Sigma$, $\forall t \in \mathbb{R}_{\geq 0}$. Throughout the paper, we consider that the nonlinearity is active.

Hereafter we assume the following:

Assumption 1: The triple (A, B, K) is supposed to be such that there exists a common positive definite matrix $P = P^\top \in \mathbb{R}^{n\times n}$ verifying $(A - Bl_iK)^\top P + P(A - Bl_iK) \prec 0$, $i \in \{1, 2\}$. □

Around the origin, the presence of the dead-zone makes the system (1) locally uncontrollable. In this case, it behaves as the autonomous system $\dot{x} = Ax$. Since A is not necessarily Hurwitz, the origin may be unstable. Nevertheless, far from the origin, the system is controlled by the state feedback Kx and it behaves as the linear system $\dot{x} = (A - Bl_iK)x$, $i \in \{1, 2\}$ for which the origin is stable thanks to Assumption 1. Let us stress that Assumption 1 deals with the closed-loop linear part and not with the whole closed-loop dynamics. Here the stability analysis will rely on the notion of Uniform Ultimate Boundedness (UUB). Let us first recall its definition [13].

Definition 1 (UUB): The trajectory of system (1) is uniformly ultimately bounded with ultimate bound b if there exist positive constants b and c, independent of $t_0 > 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that $\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b,\ \forall t \geq t_0 + T$. □

The UUB problem is stated as follows.

Problem 1 (Uniform Ultimate Boundedness analysis): Given a gain K such that Assumption 1 holds, determine the region of attraction of the system (1), denoted by $\mathcal{A}$, as small as possible and associated with the UUB property.

III. Uniform Ultimate Boundedness analysis

A. Preliminary results

For readibility reasons, we will use the shortcuts Φ for $\Phi[u](t)$ and $\dot{\Phi}$ for $\dot{\overbrace{\Phi[u](t)}}$ and we will avoid the time when there is not possible confusion.

The presence of the dead-zone and the fact that the two blacklashes are asymmetric, guide us to consider a decomposition in four zones of the nonlinearity as depicted on Figure 2 (where the number of the zone is on the bottom of the scheme) and defined as follows:

- zone 1 : $f(t) \geq \rho_1$; $\Phi(t) \geq 0$;
- zone 2 : $0 \leq f(t) \leq \rho_1$; $\Phi(t) = 0$;
- zone 3 : $-\rho_2 \leq f(t) \leq 0$; $\Phi(t) = 0$;
- zone 4 : $f(t) \leq -\rho_2$; $\Phi(t) \leq 0$.

It is worthy noticing that the decomposition between zones 2 and 3 will allow to use a suitable change of variable in the sequel of the paper. Furthermore, the sets Σ_i belong respectively to the zone i, for any $i \in \{1, \cdots, 4\}$.

In order to take the most advantage of Assumption 1, we propose the change of variable concerning the nonlinearity Φ,

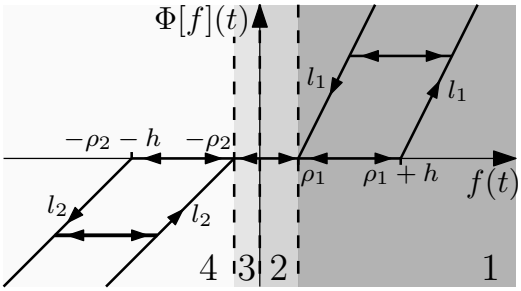

Fig. 2. Depiction of the nonlinear operator Φ with its related zones.

by introducing an extension of the classical dual nonlinearity Ψ (see [8]) in adequacy with the asymmetric property:

$$\Psi(t) = \Psi[f](t) = \begin{cases} \Phi(t) - l_1 f(t) \text{ if } f(t) \geq 0, \\ \Phi(t) - l_2 f(t) \text{ elsewhere.} \end{cases} \quad (2)$$

The characteristic of the dual nonlinearity Φ is depicted on Figure 3 where the four zones are differently colored and with the number of the zone being at the bottom of the picture.

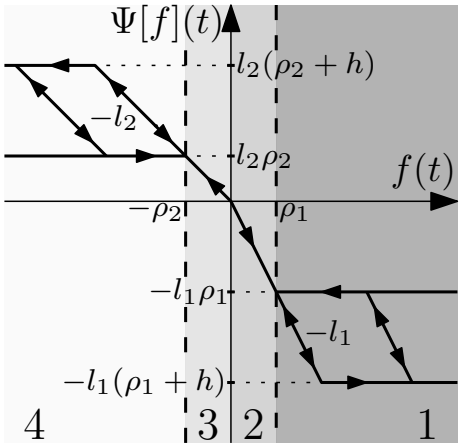

Fig. 3. Characteristic of the dual nonlinearity Ψ.

In order to simplify the expressions mentioned below, we will consider the augmented vectors

$$\begin{aligned} z &= \begin{pmatrix} x^\top & \Psi & \dot{\Psi} & 1 \end{pmatrix}^\top \in \mathbb{R}^{n+3}, \\ y &= \begin{pmatrix} x^\top & \Phi & \dot{\Phi} & 1 \end{pmatrix}^\top \in \mathbb{R}^{n+3}, \end{aligned}$$

that are associated with the change of variables $y = N_i z$, where N_i depends on the zones ($N_i = N_1$ on the zones 1 and 2; $N_i = N_2$ on the zones 3 and 4). N_i is given by

$$N_i = \begin{pmatrix} I_n & 0 & 0 & 0 \\ l_i K & 1 & 0 & 0 \\ l_i K\,(A + B l_i K) & l_i K B & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By replacing the nonlinearity Φ by its dual Ψ in the dynamics (1), we have

$$\dot{x}(t) = \begin{cases} (A + Bl_1K)x(t) + B\Psi[Kx](t) = \Gamma_1 z(t), \\ \qquad \text{if } Kx(t) \geq 0, \\ (A + Bl_2K)x(t) + B\Psi[Kx](t) = \Gamma_2 z(t), \\ \qquad \text{elsewhere,} \end{cases}$$

where matrices $\Gamma_i \in \mathbb{R}^{n\times(n+3)}$, $i \in \{1, 2\}$, are defined by

$$\Gamma_i = \begin{pmatrix} A + Bl_iK & B & 0_{n\times 1} & 0_{n\times 1} \end{pmatrix}.$$

Let us now determine several quadratic forms in z (the weighting matrices $M_{i,j}$ are defined in relations (6)–(7)) that are nonpositive in given zones:

- In zone 1, $f \geq \rho_1 \Rightarrow z^\top M_{1,1} z \leq 0$ and $\Phi \geq 0 \Rightarrow z^\top M_{2,1} z \leq 0$;
- In zone 2, $0 \leq f \leq \rho_1 \Rightarrow z^\top M_{1,2} z \leq 0$ and $\Phi \geq 0 \Rightarrow z^\top M_{2,2} z \leq 0$;
- In zone 3, $-\rho_2 \leq f \leq 0 \Rightarrow z^\top M_{1,3} z \leq 0$ and $\Phi \leq 0 \Rightarrow z^\top M_{2,3} z \leq 0$;
- In zone 4, $f \leq -\rho_2 \Rightarrow z^\top M_{1,4} z \leq 0$ and finally $\Phi \leq 0 \Rightarrow z^\top M_{2,4} z \leq 0$.

We are now armed to deal with sector conditions for the nonlinearity Φ or Ψ, depending on the zones. The results are gathered into the following lemmas. The weighting matrices in Lemmas 1 to 4 are given in Equations (7)–(11).

Lemma 1: In zone 1, the nonlinearities Φ and Ψ satisfy the following inequalities, for any $\alpha_1 > 1$, for any time t, except the times t_k of the partition presented above.

$$\dot{\Phi}\left(\Psi + l_1\left(\rho_1 + h/2\right)\right) \leq 0 \Rightarrow z^\top M_{3,1} z \leq 0; \quad (3)$$

$$\dot{\Phi}\left(\dot{\Phi} - \alpha_1 l_1 K \dot{x}\right) \leq 0 \Rightarrow z^\top M_{4,1} z \leq 0; \quad (4)$$

$$-l_1(\rho_1 + h) \leq \Psi \leq -l_1\rho_1 \Rightarrow z^\top M_{5,1} z \leq 0. \quad (5)$$

□

Proof: In zone 1, let us consider several cases to prove (3):

- If $\dot{f} \geq 0$ and $\Phi = l_1(f - \rho_1 - h)$, then $\dot{\Phi} = l_1\dot{f} \geq 0$ and $\Psi = -l_1(\rho_1 + h)$ or $\Psi + l_1(\rho_1 + h/2) = -l_1 h/2 \leq 0$. We have $\dot{\Phi}\left(\Psi + l_1(\rho_1 + h/2)\right) = -l_1^2 h/2\dot{f} \leq 0$.
- If $\dot{f} \leq 0$ and $\Phi = l_1(f - \rho_1)$, then $\dot{\Phi} = l_1\dot{f} \leq 0$ and $\Psi + l_1 f = l_1(f - \rho_1)$ or $\Psi + l_1(\rho_1 + h/2) = l_1 h/2 \geq 0$. Finally, we have $\dot{\Phi}\left(\Psi + l_1(\rho_1 + h/2)\right) = l_1^2 h/2\dot{f} \leq 0$.
- In the other cases, we have $\dot{\Phi} = \dot{\Phi}\left(\Psi + l_1(\rho_1 + h/2)\right) = 0$.

In zone 1, let us consider several cases to prove (4):

- If $\dot{f} \geq 0$ and $\Phi = l_1(f - \rho_1 - h)$, then $\dot{\Phi} = l_1\dot{f} = l_1 K\dot{x} \geq 0$. We deduce that $\dot{\Phi}\left(\dot{\Phi} - \alpha_1 l_1 K\dot{x}\right) = l_1^2(K\dot{x})^2(1 - \alpha_1) \leq 0$.
- If $\dot{f} \geq 0$ and $\Phi = l_1(f - \rho_1)$, then $\dot{\Phi} = l_1 K\dot{x} \leq 0$. We deduce $\dot{\Phi}\left(\dot{\Phi} - \alpha_1 l_1 K\dot{x}\right) = l_1^2(K\dot{x})^2(1 - \alpha_1) \leq 0$.
- In the other cases, we have $\dot{\Phi} = 0$. That leads to $\dot{\Phi}\left(\dot{\Phi} - \alpha_1 l_1 K\dot{x}\right) = 0$.

To prove the last term (5), in zone 1, we have $f \geq \rho_1$ and $l_1(f - \rho_1 - h) \leq \Phi \leq l_1(f - \rho_1)$. Thanks to the change of

$$M_{1,1}=-M_{1,2}=\begin{pmatrix}0&0&0&-\rho_1K^\top\\ \star&0&0&0\\ \star&\star&0&0\\ \star&\star&\star&2\rho_1^2\end{pmatrix},\ M_{1,3}=-M_{1,4}=\begin{pmatrix}0&0&0&-K^\top\rho_2\\ \star&0&0&0\\ \star&\star&0&0\\ \star&\star&\star&-2\rho_2^2\end{pmatrix},\ M_{2,1}=M_{2,2}=\begin{pmatrix}0&0&0&-l_1K^\top\\ \star&0&0&-1\\ \star&\star&0&0\\ \star&\star&\star&0\end{pmatrix}, \tag{6}$$

$$M_{2,3}=M_{2,4}=\begin{pmatrix}0&0&0&l_2K^\top\\ \star&0&0&1\\ \star&\star&0&0\\ \star&\star&\star&0\end{pmatrix},\ M_{5,2}=\begin{pmatrix}0&0&0&0\\ \star&1&0&l_1\rho_1/2\\ \star&\star&0&0\\ \star&\star&\star&0\end{pmatrix},\ M_{5,3}=\begin{pmatrix}0&0&0&0\\ \star&1&0&-l_2\rho_2/2\\ \star&\star&0&0\\ \star&\star&\star&0\end{pmatrix}, \tag{7}$$

$$M_{3,1}=N_1^\top\begin{pmatrix}0&0&-K^\top l_1&0\\ \star&0&1&0\\ \star&\star&0&l_1\left(\rho_1+\frac{h}{2}\right)\\ \star&\star&\star&0\end{pmatrix}N_1,\ M_{4,1}=N_1^\top\begin{pmatrix}0&0&-(\alpha_1l_1KA)^\top&0\\ \star&0&-(\alpha_1l_1KB)^\top&0\\ \star&\star&2&0\\ \star&\star&\star&0\end{pmatrix}N_1,\ M_{5,1}=\begin{pmatrix}0&0&0&0\\ \star&1&0&l_1\left(\rho_1+\frac{h}{2}\right)\\ \star&\star&0&0\\ \star&\star&\star&l_1^2\rho_1\left(\rho_1+h\right)\end{pmatrix}, \tag{8}$$

$$M_{3,4}=N_2^\top\begin{pmatrix}0&0&-K^\top l_2&0\\ \star&0&1&0\\ \star&\star&0&-l_2\left(\rho_2+\frac{h}{2}\right)\\ \star&\star&\star&0\end{pmatrix}N_2,\ M_{4,4}=N_2^\top\begin{pmatrix}0&0&-(\alpha_2l_2KA)^\top&0\\ \star&0&-(\alpha_2l_2KB)^\top&0\\ \star&\star&2&0\\ \star&\star&\star&0\end{pmatrix}N_2,\ M_{5,4}=\begin{pmatrix}0&0&0&0\\ \star&1&0&-l_2\left(\rho_2+\frac{h}{2}\right)\\ \star&\star&0&0\\ \star&\star&\star&l_2^2\rho_2\left(\rho_2+h\right)\end{pmatrix}, \tag{9}$$

$$M_{3,2}=N_1^\top\begin{pmatrix}0&-\epsilon_{3,2}K^\top&0&0\\ \star&2&0&0\\ \star&\star&0&0\\ \star&\star&\star&0\end{pmatrix}N_1,\ M_{4,2}=N_1^\top\begin{pmatrix}0&0&-(\epsilon_{4,2}KA)^\top&0\\ \star&0&-\epsilon_{4,2}KB&0\\ \star&\star&2&0\\ \star&\star&\star&0\end{pmatrix}N_1, \tag{10}$$

$$M_{3,3}=N_2^\top\begin{pmatrix}0&-\epsilon_{6,3}K^\top&0&0\\ \star&2&0&0\\ \star&\star&0&0\\ \star&\star&\star&0\end{pmatrix}N_2,\ M_{4,3}=N_2^\top\begin{pmatrix}0&0&-\epsilon_{7,3}(KA)^\top&0\\ \star&0&-\epsilon_{7,3}KB&0\\ \star&\star&2&0\\ \star&\star&\star&0\end{pmatrix}N_2. \tag{11}$$

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coordinates (2), the latter leads to $-l_1(\rho_1+h)\leq\Psi\leq-l_1\rho_1$ or $-l_1h/2\leq\Psi+l_1(\rho_1+h/2)\leq l_1h/2$. Squaring these inequalities yields to $(\Psi+l_1(\rho_1+h/2))^2-(l_1h/2)^2\leq 0$, which corresponds to the expression $z^\top M_{5,1}z\leq 0$. ■

The proofs of the three following lemmas follow closely the one of Lemma 1 and are not reproduced here.

Lemma 2: In zone 2, the nonlinearities Φ and Ψ satisfy the following inequalities, for any (small) $\epsilon_{3,2}>0$ and $\epsilon_{4,2}>0$, for any time t, except the times t_k of the partition presented above.

$$0\leq\Phi\leq\epsilon_{3,2}Kx\Rightarrow z^\top M_{3,2}z\leq 0;$$

$$0\leq\dot\Phi\leq\epsilon_{4,2}K\dot x\Rightarrow z^\top M_{4,2}z\leq 0;$$

$$-l_1\rho_1\leq\Psi\leq 0\Rightarrow z^\top M_{5,2}z\leq 0.\qquad\square$$

Lemma 3: In zone 3, the nonlinearities Φ and Ψ satisfy the following inequalities, for any (small) $\epsilon_{3,3}>0$ and $\epsilon_{4,3}>0$, for any time t, except the times t_k of the partition presented above.

$$\epsilon_{3,3}Kx\leq\Phi\leq 0\Rightarrow z^\top M_{3,3}z\leq 0;$$

$$\epsilon_{4,3}K\dot x\leq\dot\Phi\leq 0\Rightarrow z^\top M_{4,3}z\leq 0;$$

$$0\leq\Psi\leq l_2\rho_2\Rightarrow z^\top M_{5,3}z\leq 0.\qquad\square$$

Lemma 4: In zone 4, the nonlinearities Φ and Ψ satisfy the following inequalities, for any $\alpha_2>1$, for any time t, except the times t_k of the partition presented above.

$$\dot\Phi\left(\Psi-l_2\left(\rho_2+h/2\right)\right)\leq 0\Rightarrow z^\top M_{3,4}z\leq 0;$$

$$\dot\Phi\left(\dot\Phi-\alpha_2l_2K\dot x\right)\leq 0\Rightarrow z^\top M_{4,4}z\leq 0;$$

$$l_2\rho_2\leq\Psi\leq l_2(\rho_2+h)\Rightarrow z^\top M_{5,4}z\leq 0.\qquad\square$$

B. Main result

The following theorem provides a solution to Problem 1.

Theorem 1: Let us consider system (1), with the gain $K\in\mathbb{R}^{n\times 1}$ such that Assymption 1 is verified. If there exist a symmetric positive definite matrix $P\in\mathbb{R}^{n\times n}$ and positive scalars $\tau_{j,i}$, $j\in\{0,1,2,3,4,5\}$ and $i\in\{1,2,3,4\}$ satisfying the following matrix inequalities, with $C=\begin{pmatrix}I_n&0_{n\times 3}\end{pmatrix}$, and $\overline{P}=\mathrm{diag}\left\{-P;0;0;1\right\}$,

$$\mathcal{M}_1=\mathrm{He}\left(\Gamma_1^\top PC\right)-\tau_{0,1}\overline{P}-\sum\nolimits_{j=1}^{5}\tau_{j,1}M_{j,1}\prec 0, \tag{12}$$

$$\mathcal{M}_2=\mathrm{He}\left(\Gamma_1^\top PC\right)-\tau_{0,2}\overline{P}-\sum\nolimits_{j=1}^{5}\tau_{j,2}M_{j,2}\prec 0, \tag{13}$$

$$\mathcal{M}_3=\mathrm{He}\left(\Gamma_2^\top PC\right)-\tau_{0,3}\overline{P}-\sum\nolimits_{j=1}^{5}\tau_{j,3}M_{j,3}\prec 0, \tag{14}$$

$$\mathcal{M}_4=\mathrm{He}\left(\Gamma_2^\top PC\right)-\tau_{0,4}\overline{P}-\sum\nolimits_{j=1}^{5}\tau_{j,4}M_{j,4}\prec 0, \tag{15}$$

then, for any initial admissible solution $(x(0),\Phi(0))$, with $(Kx(0),\Phi(0))\in\Sigma$, the system is uniformly ultimately bounded with respect to the set

$$\mathcal{A}=\{x\in\mathbb{R}^n,\ x^\top Px\leq 1\}. \tag{16}$$

$\square$

Proof: Let us consider the quadratic function $V(x)=x^\top Px$, as a Lyapunov function candidate for UUB property.

We would like to prove that the time derivative of this function along the trajectory $t \mapsto x(t)$ is negative definite outside the ellipsoid $\mathcal{A}$, defined by (16), which is the unitary level set of V. See for further details [13].

The idea of the proof is to ensure in each zone, and for the associated sector conditions, that $\dot{V}(x(t))$ is negative definite outside the ellipsoid $\mathcal{A}$. The preliminary results will be useful to apply a technique based on the S-procedure [14].

The time derivative of V is given by $\dot{V}(x(t)) = 2\dot{x}^\top Px = z^\top \text{He}\left(\Gamma_i^\top PC\right) z$, with $i = 1$ in zones 1 and 2 and $i = 2$ in zones 3 and 4. In addition, due to the definition of $\overline{P}$, we have that $1 - V(x) = z^\top \overline{P} z$.

For $x \neq 0$, by multiplying the strict inequality (12) at left by $z^\top$ and at right by z leads to the existence of a scalar $\delta_1 > 0$ such that

$$z^\top \mathcal{M}_1 z = \dot{V}(x(t)) - \tau_{0,1}(1 - V(x)) - \tau_{1,1} z^\top M_{1,1} z - \tau_{2,1} z^\top M_{2,1} z - \tau_{3,1} z^\top M_{3,1} z - \tau_{4,1} z^\top M_{4,1} z - \tau_{5,1} z^\top M_{5,1} z < -\delta_1 \|x\|^2 < 0. \quad (17)$$

The nonlinearities Φ and Ψ satisfying the condition of zone 1 and the conditions in Lemma 1, we can conclude that on zone 1, we have $\dot{V}(x(t)) < -\delta_1 \|x\|^2$, $x \neq 0$ outside the ellipsoid $\mathcal{A}$.

We can reach the same conclusion in the zone i thanks to $z^\top \mathcal{M}_i z < -\delta_i \|x\|^2$, $x \neq 0$, $i \in \{1, 2, 3, 4\}$, leading in zone i to $\dot{V}(x(t)) \leq -\delta_i \|x\|^2$. Due to the quadratic form of the function V, we have $V(x) \leq \lambda_{\max}(P)\|x\|^2$. It can be deduced that $\dot{V}(x(t)) \leq -\frac{\delta}{\lambda_{\max}(P)} V(x(t))$, with $\delta = \min_{i \in \{1,2,3,4\}} \delta_i > 0$. The uniform ultimate boundedness time T can be taken such that it satisfies $T \geq \frac{\lambda_{\max}(P)}{\delta} \ln V(x(0))$. To make the link with Definition 1, we have $t_0 = 0$, $\|x(0)\| \leq a$ that yields to $b \leq \frac{1}{\sqrt{\lambda_{\min(P)}}}$ and $T \geq \frac{\lambda_{\max}(P)}{\delta} \ln\left\{\lambda_{\max}(P) a^2\right\}$. $\dot{V}(x(t))$ being negative definite outside the set $\mathcal{A}$, which is a level set of V, the set $\mathcal{A}$ is positively invariant by the dynamics (1). It completes the proof of UUB property for the trajectories of the nonlinear system (1). ■

C. Optimization problem

The conditions in Theorem 1 allow to prove the existence of an ellipsoid $\mathcal{A}$, defined by the unitary level set of V (see equation (16)), which is an attractor in finite-time convergence. In order to have the set $\mathcal{A}$ as small as possible, we need to introduce a size for an ellipsoid. We adopt the convention that the size of the ellipsoid $\mathcal{A}$ is given by $\text{Tr}(P^{-1})$. In order to have an expression linear in the variables, we introduce the auxiliary variable $Q \in^{n \times n}$ with $Q = Q^\top > 0_n$ and minimize $\text{Tr}(Q)$, over Q and P, under the additional condition

$$\begin{bmatrix} Q & I_n \\ I_n & P \end{bmatrix} > 0_{2n}, \quad (18)$$

which is equivalent to $Q > P^{-1}$, thanks to a standard Schur complement.

It should be noticed that when $\tau_{0,i}$, $i \in \{1, 2, 3, 4\}$ are fixed, the inequalities in Theorem 1 are LMIs with respect to the other variables (P, Q, and $\tau_{j,i}$, $j \in \{1, 2, 3, 4, 5\}$ and $i \in \{1, 2, 3, 4\}$). The LMIs are convex and can be solved thanks to numerical routines.

Remark 1: A necessary condition for the feasibility of the LMIs is that the first diagonal block is negative definite. Otherwise stated $(A - Bl_iK + \tau_{0,j}/2)^\top P + P(A - Bl_iK + \tau_{0,j}/2) \prec 0$, $i \in \{1, 2\}$, $j \in \{1, \cdots, 4\}$. It is necessary to select $0 < \tau_{0,j} < \tau_{0,\max}$, $j \in \{1, \cdots, 4\}$, where $\tau_{0,\max} = \max_{i \in \{1,2\}}(-2\text{Re}(A - Bl_iK))$.

The optimization problem solving Problem 1 is formulated as follows:

$$\text{OP1}: \quad \min_{P, Q, \{\tau_{j,i}\}_{(j,i) \in \{1,2,3,4,5\} \times \{1,2,3,4\}}} \text{Tr}(Q), \quad \text{under } (12) - (15), \ (18).$$

IV. Numerical illustration

In order to illustrate our result, we consider an example inspired from [8, Example 1], but which is modified to take into account the asymmetry of the backlash and also the presence of the dead-zone. The linear system is a double integrator and the gain K satisfies Assumption 1.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ K = \begin{pmatrix} -2 & -3 \end{pmatrix}.$$

$$l_1 = 1, \ l_2 = 1.2, \ \rho_1 = 0.1, \ \rho_2 = 0.2, \ h = 0.2.$$

Here, we compute $\tau_{0,\max} = 1.76$. When fixing $0 < \tau_{0,j} < \tau_{0,\max}$, we can solve OP1. It is clear on Figure 4 that the optimal solution of $\text{Tr}(Q)$ is reached for $\tau_{0,j} = 0.8$.

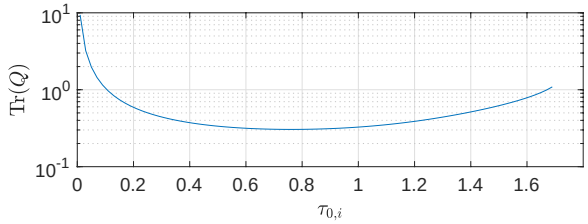

Fig. 4. Solution $\text{Tr}(Q)$ in function of $\tau_{0,i}$.

By setting $\tau_{0,i} = 0.8$, $i \in \{1, 2, 3, 4\}$, OP1 has a solution and delivers

$$P = \begin{pmatrix} 7.9111 & 3.1759 \\ 3.1759 & 7.7574 \end{pmatrix}, \ \text{Tr}(P^{-1}) = 0.3055.$$

For the trajectory, we impose $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$, which leads to the initial input $u(0) = Kx_0 = -2.8$. An admissible value of the nonlinearity is $\Phi_0 = -2.52$, such that $(Kx_0, \Phi_0) \in \Sigma$. To know precisely the behavior of the nonlinear operator, we need the time-derivative of the state-feedback: $\dot{f} = K\dot{x} = KAx_0 + KB\Phi_0 = 6.76$, that is $\dot{\Phi}(0) = 0$. On Figure 5, the trajectory on the nonlinear characteristic of the nonlinear operator is drawn in blue, while the characteristic of Figure 1 is recalled in back dashes. It can be observed that the set Σ contains all the trajectory $(Kx(t), \Phi(t))$ as expected.

On Figure 6, the phase portrait of the system is depicted. The matrix A is not Hurwitz-stable and we are faced to an

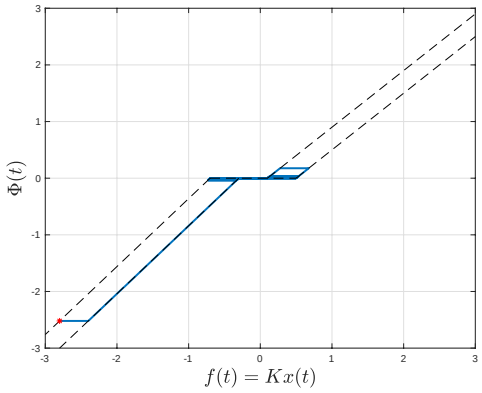

Fig. 5. Characteristic of the nonlinear operator. The starting point corresponds to $(Kx0, \Phi_0) = (-2.8, -2.52)$, depicted with a red cross. The trajectory $(Kx(t), \Phi(t))$ is in blue. The characteristic of Figure 1 is recalled in back dashes.

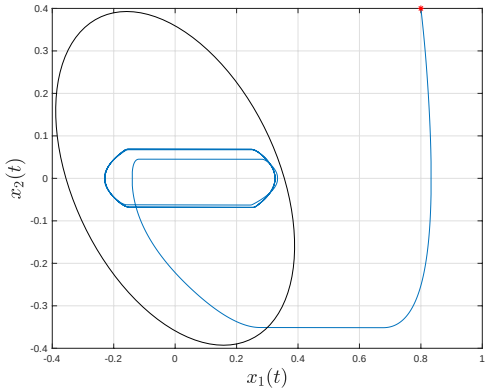

Fig. 6. Phase portrait of the system for a trajectory starting from $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$, depicted with a red cross. The ellipsoid is the attractor $\mathcal{A}$, depicted in black.

attractive limit cycle for the trajectory. The trajectory enters into the set $\mathcal{A}$ and the limit cycle belongs also to this set.

The time trajectory of the state of the system is plotted on Figure 7. We can notice that the trajectory tends to a periodic behavior associated to the limit cycle. In addition we can see on Figure 7 that the Lyapunov function $V(t)$ decreases below 1 and after that, $V(t)$ remains below 1.

V. Conclusion and perspectives

The problem of the Uniform Ultimately Boundedness of the interconnection between a linear system (possible unstable) and a nonlinear operator, which is characterized by asymmetric backlash and asymmetric dead-zone has been investigated. After recalling preliminary results and definitions, an adequate partition of the state space has been proposed leading to sufficient conditions in terms of Linear Matrix Inequalities. A numerical example has been presented to emphasize the relevance of our contribution. An interesting perspective to extend our contribution is to consider the design of a gain K to improve the set related to the UUB property.

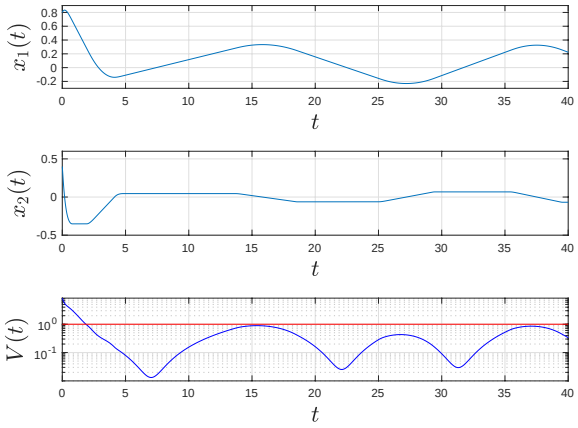

Fig. 7. Time-trajectory of the state for the system starting from $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$ and Lyapunov function $V(t)$ in function of time.

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