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HARNACK PARTS FOR 4-BY-4 TRUNCATED SHIFT WITH NUMERICAL RADIUS ONE

GILLES CASSIER^{1*} MEHDI NAIMI², MOHAMMED BENHARRAT³

ABSTRACT. Let S be a n -by- n truncated shift whose numerical radius equal one. First, Cassier, Benharrat and Belmouhoub in [12] proved that the Harnack part of S is trivial if $n = 2$, while, if $n = 3$, is an orbit associated with the action of a group of unitary diagonal matrices, see [12, Theorem 3.1 and Theorem 3.3]. Second, Cassier and Benharrat in [7] described elements of the Harnack part of the truncated n -by- n shift S under an extra assumption. In Section 2, we present useful results in the general finite dimensional situation. In Section 3, we give a complete description of the Harnack part of S for $n = 4$, the answer is surprising and instructive. It shows that, even when the dimension is an odd number, the Harnack part is bigger than conjectured in [7, Question 2.]. We also give a negative answer to [7, Question 1.] when $\rho = 2$.

1. INTRODUCTION

Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded operators acting on H . If $T \in B(H)$, as usually we denote by $\sigma(T)$ the spectrum of T and by $r(T)$ its spectral radius and we write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null space and the range of T , respectively. Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H . When H is of finite dimension n , $B(H)$ can be identified with the algebra of $n \times n$ complex matrices. For $\rho > 0$, an operator $T \in B(H)$ is called a ρ -contraction if T admits a *unitary ρ dilation* (see for instance [18] and [19]). It means that there is a Hilbert space \mathcal{H} containing H as a closed subspace and a unitary operator $U \in B(\mathcal{H})$ such that

$$T^n = \rho P_H U^n|_H, \quad n \in \mathbb{N}^*, \quad (1.1)$$

where P_H is the orthogonal projection onto the subspace H of \mathcal{H} . Denote by $C_\rho(H)$ the set of all ρ -contractions. The classes C_ρ were introduced by B. Sz.-Nagy and C. Foiaş in [18] (see also [19]). The class $C_1(H)$ is exactly the class of all contractions, i.e., operators T such that $\|T\| \leq 1$, [17]. The class $C_2(H)$ is precisely the class of all operators $T \in B(H)$ whose the *numerical radius*

$$w(T) = \sup\{|\langle Tx, x \rangle| : x \in H, \|x\| = 1\}$$

is less or equal to one, [4]. More generally, $T \in C_\rho(H)$ if and only if $w_\rho(T) \leq 1$, where $w_\rho(T)$ is the ρ -numerical radius (or the *operator radii*) of an operator $T \in B(H)$ defined

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by

$$w_\rho(T) := \inf\{\gamma > 0 : \frac{1}{\gamma}T \in C_\rho(H)\}, \quad (1.2)$$

see [15] and [20]. Hence operators in $C_\rho(H)$ are contractions with respect to the ρ -numerical radius. Note that $w_1(T) = \|T\|$, $w_2(T) = w(T)$ and $\lim_{\rho \rightarrow \infty} w_\rho(T) = r(T)$, where $r(T)$ is the spectral radius of T .

Some properties of the classes $C_\rho(H)$ become more clear (see for instance, [5], [8],[9], [10] and [6]) due to the use of the following operatorial ρ -kernel for a bounded operator T having its spectrum in the closed unit disc $\overline{\mathbb{D}}$:

$$K_z^\rho(T) = (I - \bar{z}T)^{-1} + (I - zT^*)^{-1} + (\rho - 2)I, \quad (z \in \mathbb{D}). \quad (1.3)$$

The ρ -kernels are related to ρ -contraction by the next result. An operator T is in the class $C_\rho(H)$ if and only if, $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $K_z^\rho(T) \geq 0$ for any $z \in \mathbb{D}$ (see [9]).

Let T_0, T_1 in $C_\rho(H)$, recall that T_1 is Harnack dominated by T_0 , denoted by $T_1 \stackrel{H}{\prec} T_0$, (see [10]) if there exists a constant $c \geq 1$ such that

$$\Re p(T_1) \leq c^2 \Re p(T_0) + (c^2 - 1)(\rho - 1) \Re p(O_H)$$

for any polynomial p with $\Re p \geq 0$ on $\overline{\mathbb{D}}$, where $\Re z$ is the real part of a complex number z . A detailed description of these Harnack domination and other equivalent definitions are given in [10, Theorem 3.1]. In particular, it is proved in [10] that the Harnack domination is equivalent to

$$K_z^\rho(T_1) \leq c^2 K_z^\rho(T_0) \quad \text{for all } z \in \mathbb{D} \quad (1.4)$$

for some constant $c \geq 1$. Thus the operatorial ρ -kernel plays a central role in Harnack analysis of operators (see for instance [6], [10], [12] and [7]). It is essentially due to the fact that it allows us to use directly harmonic analysis methods in the operator theory setting.

The relation $\stackrel{H}{\prec}$ is a preorder relation (reflexive and transitive) in $C_\rho(H)$ and induces an equivalent relation, called Harnack equivalence. The associated equivalence classes are called the Harnack parts of $C_\rho(H)$. So, we say that T_1 and T_0 are Harnack equivalent and we write $T_1 \stackrel{H}{\sim} T_0$, if they belong to the same Harnack parts. Classifying the equivalence classes induced by this preorder relation is a complicated question and is an important topic discussed by many authors. Foiaş proved in [13] that, in the case of $\rho = 1$, the Harnack parts of $C_1(H)$ containing the null operators O_H is exactly the class of all strict contractions. This work was extended to $C_\rho(H)$ by Cassier and Suciu; [10, Theorem 4.4]. It is proved that the equivalence class of the null operators O_H is exactly the class of all strict ρ -contractions (that is $T \in C_\rho(H)$ such that $w_\rho(T) < 1$). An interesting question is now to describe the Harnack parts of ρ -contractions T with ρ -numerical radius one. A few answers in the literature of the previous question are given, essentially in $C_1(H)$ with norm one, see [1] [16] and [3]. The question is treated for $\rho \neq 1$ in [12] and [7]. A nice result of [12] is the description of the Harnack part of some nilpotent matrices with numerical radius one, in three cases: a nilpotent matrix of order two in the two dimensional case, a nilpotent matrix of order two in \mathbb{C}^n and a nilpotent matrix of order three in the three dimensional case. In particular, it was showed that in the first case the Harnack part is trivial, while in the third case the Harnack part is an orbit associated with the action of a group of unitary diagonal matrices, see [12, Theorem 3.1] and [12,

Theorem 3.3], respectively. Remarkably, we notice that the structure of the Harnack part of the truncated shift are of different nature when $\rho = 1$ and $\rho = 2$. For the first case the Harnack part of the Jordan block of size $n + 1$ as a contraction is completely determined in [12, Collorary 2.27] independently of the value of n while, in the later case, depend on the parity of dimension. Thus, the problem of determining the Harnack part of the truncated shift viewed as a ρ -contraction with ρ -numerical radius one seems to be hard. In fact, the question is still unsettled for $\rho = 2$ and the dimension more than three. Recently, Cassier and Benharrat, [7], study the structure of the Harnack part w_ρ -normalized truncated shift S of size $n + 1$ defined in the canonical basis by

$$S = S_{n+1}(a) \tag{1.5}$$

where

$$S_{n+1}(b) = \begin{bmatrix} 0 & b & & 0 \\ & 0 & \ddots & \\ & & \ddots & b \\ 0 & & & 0 \end{bmatrix}.$$

and $a = w_\rho(S_{n+1})^{-1} = w_\rho(S_{n+1}(1))^{-1}$. It should be noted that the sequence $a_k(\rho)$ when $\rho = 2$ is completely described by U. Haagerup and P. De la Harpe in [14] and given by

$$\left(\left(\cos \frac{\pi}{n+1} \right)^{-1} \right)_{n \geq 2}.$$

Unfortunately, we don't have an explicit formula for $a_k(\rho)$ when $\rho \neq 2$, but as pointed out in [7] this ubiquitous sequence is fundamental in order to analysis the w_ρ -normalized truncated shift S . A fundamental result of [7] is the description the generators vector of the null space of the ρ -operatorial kernel of any elements in the Harnack part of S . This result play an important role to determine the Harnack parts of the w_ρ -normalized truncated shift. When $n + 1$ is an even number, one interesting consequence of this result, is that the unitary orbit of S in the Harnack part of S is trivial, and that any elements of this Harnack part is irreducible. This used to give a complete description of an element of $C_2(\mathbb{C}^{n+1})$ Harnack equivalent to the w_2 -normalized truncated shift S with norm equal those of S . More precisely, it proved that if $T \stackrel{H}{\sim} S$ in $C_2(\mathbb{C}^{n+1})$ such that $\|T\| = \|S\| = \left(\cos \left(\frac{\pi}{n+2} \right) \right)^{-1}$ then $T = S$ if $n + 1$ is an odd natural number, and is a unitary orbit of S if $n + 1$ is an even natural number, see Theorem 3.7 of [7]. According to this fact, it is naturally to ask about the structure of the Harnack part of the truncated shift S in the case of the dimension is more than three and without the the norm equal condition. At the moment, it is probably a too general question because we don't have an explicit formula for $a_k(\rho)$ when $\rho \neq 2$. We thus, restrict here to more particular case of the w_2 -normalized truncated shift S of size 4.

The purpose of this paper is to analysis the Harnack part of truncated shifts of size 4 whose numerical radius equal one (Section 3). Before this done, we give some interesting results which play a fundamental role to determine the Harnack parts of an operator with numerical radius equal one (Section 2).

2. AUXILIARY RESULTS AND HARNACK PART OF S IN $C_2(\mathbb{C}^{n+1})$

In this section, we give results having their own interest and leading to a first characterization of the operators belonging to the Harnack part of the w_2 -normalized truncated shift S of size $n + 1$ (see Theorem 2.4 bellow). Some of them can be used to make more clear how to proceed in the determination of the Harnack part of some operator in $C_2(H)$. We begin by the following useful theorem of general interest.

Theorem 2.1. *Let $H = E_1 \oplus E_2 \oplus E_3$ be a orthogonal decomposition of a separable Hilbert space H , and let L be a self-adjoint operator in H which admits the following block representation according to this decomposition :*

$$L = \begin{bmatrix} I_1 & X & 0 \\ X^* & N & Y \\ 0 & Y^* & I_3 \end{bmatrix}.$$

Then we have,

- (i) L is a positive operator if and only if $N - X^*X - YY^*$ is a positive operator.
- (ii) The null space of L is given by

$$\mathcal{N}(L) = \{(-Xu, u, -Y^*u) : u \in \mathcal{N}(N - X^*X - YY^*)\},$$

in particular we have $\dim \mathcal{N}(L) = \dim \mathcal{N}(N - X^*X - YY^*)$. Moreover, when H is of finite dimension then $\det(L) = \det(N - X^*X - YY^*)$.

Proof. (i) We can see that

$$L = \begin{bmatrix} I_1 & 0 & 0 \\ X^* & I_2 & Y \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & N - X^*X - YY^* & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} I_1 & X & 0 \\ 0 & I_2 & 0 \\ 0 & Y^* & I_3 \end{bmatrix} := A^* \tilde{L} A.$$

We notice that if $\lambda \neq 1$, then $A - \lambda I$ is invertible and

$$(A - \lambda I)^{-1} = \frac{1}{1 - \lambda} \begin{bmatrix} I_1 & -\frac{1}{1-\lambda}X & 0 \\ 0 & I_2 & 0 \\ 0 & -\frac{1}{1-\lambda}Y^* & I_3 \end{bmatrix},$$

and consequently $\sigma(A) = \{1\}$. In particular, A is invertible and it follows that $L \geq 0$ if and only if $N - X^*X - YY^* \geq 0$.

- (ii) Using the previous notations, we see that $\mathcal{N}(L) = A^{-1}(\mathcal{N}(\tilde{L}))$, which means that for every $x \in \mathcal{N}(L)$, we have

$$x = \begin{bmatrix} I_1 & -X & 0 \\ 0 & I_2 & 0 \\ 0 & -Y^* & I_3 \end{bmatrix} \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix} = \begin{bmatrix} -Xu \\ u \\ -Y^*u \end{bmatrix} \text{ with } u \in \mathcal{N}(N - X^*X - YY^*).$$

From this, we derive that

$$\dim(\mathcal{N}(L)) = \dim(\mathcal{N}(\tilde{L})) = \dim(\mathcal{N}(N - X^*X - YY^*)).$$

In the finite dimensional case ($\dim(H) < \infty$), as we have $\sigma(A) = \{1\}$ it follows that $\det(A) = 1$, and consequently

$$\det(L) = \overline{\det(A)} \det(\tilde{L}) \det(A) = \det(N - X^*X - YY^*).$$

□

In the following we introduce the class \mathcal{M} of operators T which admit a block representation of the form

$$T = \begin{bmatrix} 0 & A & 0 \\ 0 & R & B \\ 0 & 0 & 0 \end{bmatrix}$$

with respect to an orthogonal decomposition of the type $H = E_1 \oplus E_2 \oplus E_3$. Notice that elements contained in the Harnack part of the w_2 -normalized truncated shift belong to \mathcal{M} .

The following corollary has its own interest, it characterizes operators $T \in \mathcal{M}$ with numerical radius less or equal to 1 in terms of their entries A , B and R , and gives additional useful properties.

Corollary 2.2. *Let T be an operator in \mathcal{M} such that $\sigma(T) \subset \mathbb{D}$, then we have*

(i) $w(T) \leq 1$ if and only if

$$I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}R) \geq 0 \quad \text{for all } z \in \mathbb{T}, \quad (2.1)$$

where I_2 denotes the identity operator of E_2 .

(ii) $w(T) = 1$ if and only if property (2.1) holds and there exists $z_0 \in \mathbb{T}$ such that

$$I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}_0R) \text{ is not invertible.}$$

(iii) $w(T) = 1$ and $\dim(\mathcal{N}(K_z^2(T))) = d$ for all $z \in \mathbb{T}$ if and only if property (2.1) is satisfied and $\dim(\mathcal{N}(I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}R))) = d$ for all $z \in \mathbb{T}$.

Proof. (i) Recall that a classical result due to C. A. Berger asserts that $T \in C_2(H)$ if and only if $w(T) \leq 1$, see [4]. We know by [9] that $T \in C_\rho(H)$ if and only if, $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $K_z^\rho(T) \geq 0$ for any $z \in \mathbb{D}$. For $\rho = 2$, we can write

$$K_z^2(T) = (I - zT^*)^{-1}(2I - \bar{z}T - zT^*)(I - \bar{z}T)^{-1}.$$

Consequently $w_2(T) \leq 1$ if and only if $2I - \bar{z}T - zT^* \geq 0$ for any $z \in \mathbb{T}$. Since

$$I - \Re(\bar{z}T) = \begin{bmatrix} I_1 & -\frac{\bar{z}}{2}A & 0 \\ -\frac{z}{2}A^* & I_2 - \Re(\bar{z}R) & -\frac{\bar{z}}{2}B \\ 0 & -\frac{z}{2}B^* & I_3 \end{bmatrix},$$

then by Theorem 2.1, we see that for all $z \in \mathbb{T}$ the positivity of the operator $I - \Re(\bar{z}T)$ is equivalent to the positivity of the operator $I_2 - \frac{1}{4}|z|^2A^*A - \frac{1}{4}|z|^2BB^* - \Re(\bar{z}R)$.

(ii) By [11, Lemma 3.], we know that for every z in the open disk $D(0, w(T)^{-1})$ the operator $K_z^2(T)$ is invertible. From the assumption $I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}_0R)$ is not invertible for some $z_0 \in \mathbb{T}$ we derive that $K_{z_0}^2(T)$ is not invertible, and hence $w(T) \geq 1$. But then, by (2.1) we get $w(T) = 1$.

Conversely, let us remark that (2.1) holds by the first item. Assume that $K_z^2(T)$ invertible in $\overline{\mathbb{D}}$ and $\sigma(T) \subset \mathbb{D}$. Then $I - \Re(\bar{z}T) > 0$ for all $z \in \overline{\mathbb{D}}$, then by using continuity and compactness arguments, we see that there exists $r > 1$ such that

$I_2 - \Re(\bar{z}rT) \geq 0$ on $\overline{\mathbb{D}}$. Thus, $w(rT) = rw(T) \leq 1$ and so $w(T) < 1$, which is a contradiction. Consequently, there exists $z_0 \in \mathbb{T}$ such that $I - \Re(\bar{z}_0T)$ is not invertible, and again by Theorem 2.1, we get $I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}_0R)$ is not invertible.

- (iii) This assertion follows directly from (ii) and the fact that $d = \dim(\mathcal{N}(K_z^2(T))) = \dim(\mathcal{N}(I - \Re(\bar{z}T))) = \dim(\mathcal{N}(I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}R)))$. □

The next result gives relationships between the spectrum of $T \in \mathcal{M}$ and the spectrum of the entry R .

Proposition 2.3. *If $T \in \mathcal{M}$, then*

$$\sigma(T) = \sigma(R) \cup \{0\}.$$

Proof. Let $T \in \mathcal{M}$, then

$$T = \begin{bmatrix} 0 & A & 0 \\ 0 & R & B \\ 0 & 0 & 0 \end{bmatrix}$$

according to the orthogonal decomposition $H = E_1 \oplus E_2 \oplus E_3$. We see that $(T - \lambda I)x = y$ is equivalent to the following system

$$\begin{cases} y_1 &= -\lambda x_1 + Ax_2 \\ y_2 &= (R - \lambda I_2)x_2 + Bx_3 \\ y_3 &= -\lambda x_3. \end{cases}$$

Let $\lambda \in \rho(T) \setminus \{0\}$. Set $y_3 = 0$, take an arbitrary y_2 and use the two last equations, since $(T - \lambda I)$ is invertible, the existence of $x_2 \in E_2$ such that $y_2 = (R - \lambda I_2)x_2$ is guaranteed, showing that $(R - \lambda I_2)$ is surjective. Now, let x_2 be in $\mathcal{N}(R - \lambda I_2)$, then

$$(T - \lambda I) \begin{bmatrix} A \begin{pmatrix} x_2 \\ \lambda \end{pmatrix} \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives $x_2 = 0$ since $T - \lambda I$ is one-to-one, and hence $(R - \lambda I_2)$ is injective. We have already prove that $\rho(T) \setminus \{0\} \subseteq \rho(R) \setminus \{0\}$.

Conversely, assume that $\lambda \in \rho(R) \setminus \{0\}$, then we consider the operator C defined by

$$C = \begin{bmatrix} -\frac{I_1}{\lambda} & \frac{1}{\lambda}A(R - \lambda I_2)^{-1} & \frac{1}{\lambda^2}A(R - \lambda I_2)^{-1}B \\ 0 & (R - \lambda I_2)^{-1} & \frac{1}{\lambda}(R - \lambda I_2)^{-1}B \\ 0 & 0 & -\frac{I_3}{\lambda} \end{bmatrix}.$$

It is well defined and verifies $C(T - \lambda I) = (T - \lambda I)C = I$, so $\rho(T) \setminus \{0\} = \rho(R) \setminus \{0\}$, and hence $\rho(R) \setminus \{0\} \subseteq \rho(T) \setminus \{0\}$. Finally, we have shown that $\sigma(T) = \sigma(R) \cup \{0\}$. □

As consequence, the following theorem play an important role in the description of the elements of the Harnack part of w_2 -normalized truncated shift S for any size. Recall that for the w_2 -normalized truncated shift S of size $n + 1$, we have for all $z \in \mathbb{T}$

$$\mathcal{N}(K_z^2(S)) = \mathbb{C}v(z) \text{ where } v(z) = (v_0, zv_1, \dots, z^n v_n) = (v_0, zw(z), z^n v_n),$$

with $v_0 \neq 0$ and $v_k = -v_{n-k}$.

Theorem 2.4. *Let $T \in C_2(\mathbb{C}^{n+1})$, then the next two assertions are equivalent:*

- (i) *The operator T is in the Harnack parts of w_2 -normalized truncated shift S given by (1.5).*
- (ii) *According to the decomposition $\mathbb{C}^{n+1} = \mathbb{C}e_0 \oplus (\mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_{n-1}) \oplus \mathbb{C}e_n$, the operator $T \in \mathcal{M}$, with $A \in M_{1,n-1}(\mathbb{C})$, $B \in M_{n-1,1}(\mathbb{C})$, $R \in M_{n-1}(\mathbb{C})$ and $\sigma(R) \subset \mathbb{D}$. Moreover, $M(z) := I_{n-1} - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}R)$ is positive, $\dim(\mathcal{N}(M(z))) = 1$ and for all $z \in \mathbb{T}$ we can write*

$$\begin{bmatrix} 2I_1 & \bar{z}A(I_2 - \bar{z}R)^{-1} & \bar{z}^2A(I_2 - \bar{z}R)^{-1}B \\ z(I_2 - zR^*)^{-1}A^* & K_z^2(R) & \bar{z}(I_2 - \bar{z}R)^{-1}B \\ z^2B^*(I_2 - zR^*)^{-1}A^* & zB^*(I_2 - zR^*)^{-1} & 2I_3 \end{bmatrix} \begin{bmatrix} v_0 \\ zw(z) \\ -v_0z^n \end{bmatrix} = 0.$$

Proof. We assume first that $T \in C_2(\mathbb{C}^{n+1})$ is such that $T \stackrel{H}{\sim} S$. By [7, Proposition 3.4] we know that

$$T = \begin{bmatrix} 0 & * & \dots & * & 0 \\ 0 & \vdots & & \vdots & * \\ 0 & \vdots & & \vdots & \vdots \\ 0 & * & \dots & * & * \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A & 0 \\ 0 & R & B \\ 0 & 0 & 0 \end{bmatrix} \text{ in } \mathbb{C}^{n+1} = E_1 \oplus E_2 \oplus E_3, \quad (2.2)$$

where $E_1 = \mathbb{C}e_0$, $E_2 = (\mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_{n-1})$ and $E_3 = \mathbb{C}e_n$. Further, from Corollary 2.2 in [12] it follows that $\sigma(T) \cap \mathbb{T} = \sigma(S) \cap \mathbb{T} = \emptyset$, and hence $\sigma(T) \subseteq \mathbb{D}$. Therefore, Proposition 2.3 implies that $\sigma(R) \subset \mathbb{D}$. Since the Harnack part of the null operator consists of all operators with numerical radius strictly less than 1, see [10, Theorem 4.4], we have necessarily $w_2(T) = 1$. Then, by [7, Theorem 2.1], we have

$$\mathcal{N}(K_z^2(T)) = \mathbb{C}v(z), \quad \text{for all } z \in \mathbb{T}. \quad (2.3)$$

By Corollary 2.2 (iii), we deduce that $M(z)$ is a positive operator and $\dim(\mathcal{N}(M(z))) = 1$ for all $z \in \mathbb{T}$. Further, since $\sigma(R) \subset \mathbb{D}$, we can represent $K_z^2(T)$ as follows

$$K_z^2(T) = \begin{bmatrix} 2I_1 & \bar{z}A(I_2 - \bar{z}R)^{-1} & \bar{z}^2A(I_2 - \bar{z}R)^{-1}B \\ z(I_2 - zR^*)^{-1}A^* & K_z^2(R) & \bar{z}(I_2 - \bar{z}R)^{-1}B \\ z^2B^*(I_2 - zR^*)^{-1}A^* & zB^*(I_2 - zR^*)^{-1} & 2I_3 \end{bmatrix} \quad (2.4)$$

and using Theorem 2.1 in [7] we get

$$v(z) = \begin{bmatrix} v_0 \\ zw(z) \\ -v_0z^n \end{bmatrix} \in \mathcal{N}(K_z^2(T)).$$

This complete the prove of the implication (i) \implies (ii).

Conversely, assume all conditions of (ii) are satisfied, then $T \in \mathcal{M}$ and $\sigma(T) = \sigma(R) \cup \{0\} \subset \mathbb{D}$ (Proposition 2.3). Also, the operatorial 2-kernel $K_z^2(T)$ associated with T takes

the form given by (2.4). The last condition in (ii) implies that $v(z)$ is in $\mathcal{N}(K_z^2(T))$ for all $z \in \mathbb{T}$. Then, as $M(z) \geq 0$ and $\dim(\mathcal{N}(K_z^2(T))) = 1$ for all $z \in \mathbb{T}$, using Corollary 2.2 we get that $w_2(T) = 1$ (so $T \in C_2(\mathbb{C}^{n+1})$) and $\mathcal{N}(K_z^2(T)) = \mathbb{C}v(z)$ for all $z \in \mathbb{T}$. Thus, by [12, Corollary 2.23] we obtain that $T \stackrel{H}{\sim} S$. It ends the proof. \square

We end this section by giving a proposition that has an interesting connection with the norm equality condition in Theorem 3.7 of [7].

Proposition 2.5. *Let T be a nilpotent operator in the Harnack parts of w_ρ -normalized truncated shift S given by (1.5). Then we have*

$$\|T\| \geq \|S\|.$$

Proof. So, we know that $T \stackrel{H}{\sim} S$ in $C_\rho(H)$ and $T^{n+1} = 0$. On the one hand, since $w_\rho(S) = 1$ using Theorem 4.4 of [10] we see that we necessarily have $w_\rho(T) = 1$. On the other hand, set $R = \frac{T}{\|T\|}$. Then R is a nilpotent contraction such that $R^{n+1} = 0$, so using Theorem 3.1 or Corollary 4.1 of [2] we get

$$\frac{1}{\|T\|} = w_\rho(R) \leq w_\rho(S_{n+1}) = \frac{1}{\|S\|}$$

which leads to the desired inequality. \square

3. THE HARNACK PARTS OF THE TRUNCATED SHIFT OF SIZE 4

Let S be the w_2 -normalized truncated shift in $C_2(\mathbb{C}^4)$ given in the canonical basis of \mathbb{C}^4 by

$$S = a \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.1)$$

where $a = \frac{1}{\cos(\frac{\pi}{5})}$ ($w(S) = 1$).

In the following we give a complete description the Harnack parts of S . We will see that this Harnack part is surprising compared to what was known.

Let T in the Harnack part of S_4 . By Proposition 3.4 of [7], according to the orthogonal decomposition $\mathbb{C}^4 = \mathbb{C}e_0 \oplus (\mathbb{C}e_1 \oplus \mathbb{C}e_2) \oplus \mathbb{C}e_3$, T is necessarily of the form

$$T = \begin{bmatrix} 0 & a_1 & a_2 & 0 \\ 0 & r_{11} & r_{12} & b_1 \\ 0 & r_{21} & r_{22} & b_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A & 0 \\ 0 & R & B \\ 0 & 0 & 0 \end{bmatrix} \quad (3.2)$$

where

$$A = [a_1 \ a_2], \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}.$$

The following result is a consequence of Theorem 2.4, it will be useful in the proof of next theorem (Theorem 3.3).

Corollary 3.1. *Let $T \in C_2(\mathbb{C}^4)$, then the next two assertions are equivalent:*

- (i) *T is in the Harnack parts of the w_2 -normalized truncated shift S in $C_2(\mathbb{C}^4)$.*
- (ii) (a) *The matrix T has a block representation as given in (3.2), with $\sigma(R) \subset \mathbb{D}$;*
 (b) *For all $z \in \mathbb{T}$*

$$K_z^2(T)v(z) = K_z^2(T) \begin{bmatrix} v_0 \\ v_1 z \begin{pmatrix} 1 \\ -z \end{pmatrix} \\ -v_0 z^3 \end{bmatrix} = 0;$$

(c) *We have*

$$\det(R) = 0, \tag{3.3}$$

$$\det \begin{pmatrix} R\vec{i} & M\vec{j} \end{pmatrix} + \det \begin{pmatrix} M\vec{i} & R\vec{j} \end{pmatrix} = 0, \tag{3.4}$$

$$\det(M) + \frac{1}{4} \det \begin{pmatrix} R^*\vec{i} & R\vec{j} \end{pmatrix} + \frac{1}{4} \det \begin{pmatrix} R\vec{i} & R^*\vec{j} \end{pmatrix} = 0, \tag{3.5}$$

$$2 - \frac{1}{4}(\|A\|^2 + \|B\|^2) > |\text{Tr}(R)|, \tag{3.6}$$

$$\text{where } \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } M := I_2 - \frac{1}{4}(A^*A + BB^*).$$

Proof. Set $M(z) = I_2 - \frac{1}{4}(A^*A + BB^*) - \Re(\bar{z}R)$ for every $z \in \mathbb{T}$. Since $M(z)$ is a self-adjoint 2 by 2 complex matrix, then $M(z) \geq 0$ and $\dim(\mathcal{N}(M(z))) = 1$ if and only if $\det(M(z)) = 0$ and $\text{Tr}(M(z)) > 0$. It follows that for every $z \in \mathbb{T}$ we have

$$\begin{aligned} 0 &= \det(M(z))\vec{i} \wedge \vec{j} = (M - \Re(\bar{z}R))\vec{i} \wedge (M - \Re(\bar{z}R))\vec{j} \\ &= M\vec{i} \wedge M\vec{j} - \Re(\bar{z}R)\vec{i} \wedge M\vec{j} - M\vec{i} \wedge \Re(\bar{z}R)\vec{j} + \Re(\bar{z}R)\vec{i} \wedge \Re(\bar{z}R)\vec{j} \\ &= \left[\det(M) + \frac{1}{4} \det \begin{pmatrix} R^*\vec{i} & R\vec{j} \end{pmatrix} + \frac{1}{4} \det \begin{pmatrix} R\vec{i} & R^*\vec{j} \end{pmatrix} + \frac{\bar{z}^2}{4} \det(R) + \frac{z^2}{4} \overline{\det(R)} \right. \\ &\quad \left. - \frac{\bar{z}}{2} \left(\det \begin{pmatrix} R\vec{i} & M\vec{j} \end{pmatrix} + \det \begin{pmatrix} M\vec{i} & R\vec{j} \end{pmatrix} \right) - \frac{z}{2} \left(\det \begin{pmatrix} R^*\vec{i} & M\vec{j} \end{pmatrix} + \det \begin{pmatrix} M\vec{i} & R^*\vec{j} \end{pmatrix} \right) \right] \vec{i} \wedge \vec{j}. \end{aligned}$$

As M is self-adjoint, notice that

$$\det \begin{pmatrix} R^*\vec{i} & M\vec{j} \end{pmatrix} + \det \begin{pmatrix} M\vec{i} & R^*\vec{j} \end{pmatrix} = \overline{\det \begin{pmatrix} R\vec{i} & M\vec{j} \end{pmatrix} + \det \begin{pmatrix} M\vec{i} & R\vec{j} \end{pmatrix}}.$$

Therefore, by identification we see that $\det(M(z)) = 0$ is equivalent to (3.3), (3.4) and (3.5).

Moreover, since we have $\text{Tr}(M(z)) = \text{Tr}(M) - \Re(\bar{z}\text{Tr}(R)) \geq \text{Tr}(M) - |\text{Tr}(R)|$ and $\Re(\bar{z}\text{Tr}(R)) = |\text{Tr}(R)|$ for some $z \in \mathbb{T}$, we see that (3.6) is equivalent to the fact that $\text{Tr}(M(z)) > 0$ for all $z \in \mathbb{T}$. Then, we can conclude by applying Theorem 2.4. \square

Remark 3.2. In any dimension, we could expand $\det(M(z))$ in powers of z and \bar{z} , and then obtain some necessary conditions. In particular, under the assumption that $T \stackrel{H}{\sim} S$ (see Theorem 2.4), by examining the coefficients of higher degree we already find that R is not invertible.

The following result gives a complete description of the Harnack part of S of size $n = 4$ with numerical radius equal one.

Theorem 3.3. *The Harnack part of S is the set of all matrices T of $C_2(\mathbb{C}^4)$ of the form*

$$T = S \quad \text{or} \quad T = 2 \begin{bmatrix} 0 & -\frac{1}{a} & 0 & 0 \\ 0 & 0 & (1 - \frac{1}{a^2}) & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $a = \frac{1}{\cos(\pi/5)}$.

Proof. Let $T \in C_2(\mathbb{C}^4)$, if T in the Harnack parts of S then, by Corollary 3.1, T takes the form (3.2), with $\sigma(R) \subset \mathbb{D}$ and $\det(R) = 0$. By Proposition 2.3, we have $\sigma(T) = \sigma(R) \cup \{0\}$, therefore $\sigma(R) = \{0, \lambda\}$, with $\lambda \in \mathbb{D}$ (by [12, Corollary 2.2]). It follows that $R^2 = \lambda R$, and consequently

$$(I_2 - \bar{z}R)^{-1} = I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R \quad \text{for all } z \in \bar{\mathbb{D}}.$$

Thus,

$$K_z^2(T) = \begin{bmatrix} 2I_1 & \bar{z}A(I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R) & \bar{z}^2A(I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R)B \\ z(I_2 + \frac{z}{1 - \lambda z}R^*)A^* & 2I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R + \frac{z}{1 - \lambda z}R^* & \bar{z}(I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R)B \\ z^2B^*(I_2 + \frac{z}{1 - \lambda z}R^*)A^* & zB^*(I_2 + \frac{z}{1 - \lambda z}R^*) & 2I_3 \end{bmatrix},$$

for all $z \in \mathbb{T}$. Now, by (2.3), we have for all $z \in \mathbb{T}$,

$$K_z^2(T)(v_0, w(z), -v_0z^3)^T = 0 \quad \text{where } w(z) = v_1z(1, -z)^T, \quad \text{and } v_k \neq 0, \text{ for } k = 0, 1,$$

which is equivalent to

$$\left\{ \begin{array}{l} 2v_0 + v_1A(I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R) \begin{bmatrix} 1 \\ -z \end{bmatrix} - zv_0A(I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R)B = 0 \\ v_0z(I_2 + \frac{z}{1 - \lambda z}R^*)A^* + v_1z(2I_2 + \frac{z}{1 - \lambda z}R^* + \frac{\bar{z}}{1 - \lambda\bar{z}}R) \begin{bmatrix} 1 \\ -z \end{bmatrix} \\ -v_0z^2(I_2 + \frac{\bar{z}}{1 - \lambda\bar{z}}R)B = 0 \\ v_0B^*(I_2 + \frac{z}{1 - \lambda z}R^*)A^* + v_1B^*(I_2 + \frac{z}{1 - \lambda z}R^*) \begin{bmatrix} 1 \\ -z \end{bmatrix} - 2v_0z = 0, \end{array} \right.$$

which in turn is equivalent to

$$\left\{ \begin{array}{l} 2v_0(z - \lambda) + v_1 A [(z - \lambda)I_2 + R] \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0 z A [(z - \lambda)I_2 + R] B = 0 \\ v_0(z - \lambda) [(1 - \bar{\lambda}z)I_2 + zR^*] A^* + v_1 [2(1 - \bar{\lambda}z)(z - \lambda)I_2 + z(z - \lambda)R^* \\ + (1 - z\bar{\lambda})R] \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0 z(1 - z\bar{\lambda}) [(z - \lambda)I_2 + R] B = 0 \\ v_0 B^* [(1 - \bar{\lambda}z)I_2 + zR^*] A^* + v_1 B^* [(1 - \bar{\lambda}z)I_2 + zR^*] \begin{bmatrix} 1 \\ -z \end{bmatrix} - 2v_0 z(1 - \bar{\lambda}z) = 0. \end{array} \right. \quad (3.7)$$

Further, by analytic extension, we can see that the system (3.7) is true for all $z \in \mathbb{C}$. The third equation of (3.7) is equivalent to

$$v_0 A((1 - \bar{z}\lambda)I_2 + \bar{z}R)B + v_1 [1 \quad -\bar{z}] ((1 - \bar{z}\lambda)I_2 + \bar{z}R)B - 2v_0 \bar{z}(1 - \lambda\bar{z}) = 0,$$

which means that

$$v_0 z A((z - \lambda)I_2 + R)B + v_1 [z \quad -1] ((z - \lambda)I_2 + R)B - 2v_0(z - \lambda) = 0.$$

Summing this to the first equation of (3.7), we get

$$A((z - \lambda)I_2 + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} + [z \quad -1] ((z - \lambda)I_2 + R)B = 0, \text{ for all } z \in \mathbb{C}.$$

It implies that the condition

$$K_z^2(T)(v_0, \tilde{v}(z), -v_0 z^3)^T = 0$$

is equivalent to the following system denoted by \mathcal{S}

$$\left\{ \begin{array}{l} 2v_0(z - \lambda) + v_1 A [(z - \lambda)I_2 + R] \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0 z A [(z - \lambda)I_2 + R] B = 0 \quad (3.8a) \\ v_0(z - \lambda) [(1 - \bar{\lambda}z)I_2 + zR^*] A^* + v_1 [2(1 - \bar{\lambda}z)(z - \lambda)I_2 + z(z - \lambda)R^* \\ + (1 - z\bar{\lambda})R] \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0 z(1 - z\bar{\lambda}) [(z - \lambda)I_2 + R] B = 0 \quad (3.8b) \\ \left\{ \begin{array}{l} v_0 B^* [(1 - \bar{\lambda}z)I_2 + zR^*] A^* + v_1 B^* [(1 - \bar{\lambda}z)I_2 + zR^*] \begin{bmatrix} 1 \\ -z \end{bmatrix} \\ -2v_0 z(1 - \bar{\lambda}z) = 0 \\ \text{or} \\ A [(z - \lambda)I_2 + R] \begin{bmatrix} 1 \\ -z \end{bmatrix} + [z \quad -1] [(z - \lambda)I_2 + R] B = 0. \end{array} \right. \quad (3.8c) \end{array} \right.$$

The coefficient of z^2 in (3.8c) gives $A \begin{bmatrix} 0 \\ -1 \end{bmatrix} + [1 \quad 0] B = 0$, so

$$b_1 = a_2, \quad (3.9)$$

and looking at the coefficient of z^2 in (3.8a) we get

$$\begin{aligned} 0 &= v_1 A \begin{bmatrix} 0 \\ -1 \end{bmatrix} - v_0 AB = -v_1 a_2 - v_0 [a_1 \ a_2] \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \\ &= -v_1 a_2 - v_0(a_1 a_2 + a_2 b_2) \end{aligned}$$

then

$$a_2(v_1 + v_0(a_1 + b_2)) = 0. \quad (3.10)$$

Moreover, we have

$$v_1 \lambda = v_1 \text{Tr}(R) = \langle v_1 R \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle + \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_1 R^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle. \quad (3.11)$$

On the one hand, letting $z = 0$ in (3.8b), we find

$$-\lambda v_0 A^* + v_1(-2\lambda I_2 + R) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0,$$

and so

$$v_1 R \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda v_0 A^* + 2\lambda v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (3.12)$$

On the other hand, the coefficient of z^3 in (3.8b) leads to the equation

$$v_1(-2\bar{\lambda} I_2 + R^*) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + v_0 \bar{\lambda} B = 0,$$

and therefore

$$v_1 R^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1 2\bar{\lambda} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \bar{\lambda} B. \quad (3.13)$$

Then, the equations (3.11), (3.12) and (3.13) can be combined to yield the following calculation

$$\begin{aligned} v_1 \lambda &= v_1 \text{Tr}(R) = \left\langle \lambda v_0 A^* + 2\lambda v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, 2\bar{\lambda} v_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \bar{\lambda} v_0 B \right\rangle \\ &= \left\langle 2\lambda v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_0 \lambda A^*, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, 2v_1 \bar{\lambda} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \bar{\lambda} B \right\rangle \\ &= 2\lambda v_1 + v_0 \lambda \bar{a}_1 + 2\lambda v_1 + v_0 \lambda \bar{b}_2. \end{aligned}$$

We deduce that

$$0 = \bar{\lambda}(3v_1 + v_0(a_1 + b_2)) \quad (3.14)$$

$$0 = \bar{\lambda}(2a_2 v_1 + a_2 v_1 + v_0 a_2(a_1 + b_2))$$

$$0 = \bar{\lambda} a_2 \quad (\text{by (3.10)}). \quad (3.15)$$

Let assume now that $\lambda \neq 0$ so by (3.15) $b_1 = a_2 = 0$, and since the coefficient of z^2 in (3.8c) has to be zero, we find

$$v_1 B^*(R^* - \bar{\lambda} I_2) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + 2\bar{\lambda} v_0 = 0.$$

By this last equation and (3.13), we get

$$\begin{aligned}
0 &= v_1 B^*(R^* - 2\bar{\lambda}I_2) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + v_0 \bar{\lambda} B^* B \\
&= v_1 B^*(R^* - \bar{\lambda}I_2) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + v_1 B^* \bar{\lambda} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_0 \bar{\lambda} B^* B \\
&= -2\bar{\lambda}v_0 + v_1 \bar{\lambda} \overline{b_2} + v_0 \bar{\lambda} |b_2|^2.
\end{aligned}$$

Consequently $0 = v_0 |b_2|^2 + v_1 \bar{b_2} - 2v_0$ so $b_2 \in \mathbb{R}$ and is a solution of

$$v_0 x^2 + v_1 x - 2v_0 = 0. \quad (3.16)$$

By taking $z = 0$ in (3.8a), we find that

$$-2\lambda v_0 + v_1 A(R - \lambda I_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

From this equation and (3.12), we derive that

$$\begin{aligned}
0 &= -\lambda v_0 A A^* + v_1 A(R - \lambda I_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \lambda v_1 A \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= -\lambda v_0 A A^* + 2\lambda v_0 - \lambda v_1 a_1.
\end{aligned}$$

Then, we have $v_0 |a_1|^2 + v_1 a_1 - 2v_0 = 0$ which implies that a_1 is necessarily a real solution of (3.16).

If we assume that $a_1 \neq b_2$, then we must have $a_1 + b_2 = -\frac{v_1}{v_0}$ (a_1 and b_2 are solution (3.16)). On the other hand, by (3.14) we can deduce that $a_1 + b_2 = -3\frac{v_1}{v_0}$ which leads to a contradiction. This forces $a_1 = b_2$

From (3.12), we get

$$0 = -\lambda v_0 A^* + v_1 (R - 2\lambda I_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda v_0 a_1 + v_1 (r_{1,1} - 2\lambda) \\ v_1 r_{21} \end{bmatrix},$$

which means that

$$\begin{cases} v_1 (r_{11} - 2\lambda) = \lambda v_0 a_1 \\ r_{21} = 0. \end{cases} \quad (3.17)$$

Suppose that $r_{11} = 0$, then $a_1 = -2\frac{v_1}{v_0}$ by (3.17), but we know that $2a_1 = a_1 + b_2 = -3\frac{v_1}{v_0}$, which gives a contradiction. Therefore $r_{11} \neq 0$ and $0 = \det(R) = r_{11} r_{22}$, so $r_{22} = 0$ and $\lambda = \text{Tr}(R) = r_{11}$, which also leads to a contradiction again. By (3.17), we get that $-\lambda v_1 = \lambda v_0 a_1$, thus $a_1 = \frac{v_1}{v_0}$, so we conclude that $\lambda = 0$.

Consequently, the system \mathcal{S} becomes,

$$\left\{ \begin{array}{l} 2v_0z + v_1A(zI_2 + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0zA(zI_2 + R)B = 0 \\ v_0z(I_2 + zR^*)A^* + v_1(2zI_2 + z^2R^* + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0z(zI_2 + R)B = 0 \\ \left\{ \begin{array}{l} v_0B^*(I_2 + zR^*)A^* + v_1B^*(I_2 + zR^*) \begin{bmatrix} 1 \\ -z \end{bmatrix} - 2v_0z = 0 \\ \text{or} \\ A(zI_2 + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} + \begin{bmatrix} z & -1 \end{bmatrix} (zI_2 + R)B = 0. \end{array} \right. \end{array} \right. \quad \begin{array}{l} (3.18a) \\ (3.18b) \\ (3.18c) \end{array}$$

Now by taking $z = 0$ in (3.18b), it comes that $v_1R \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$ so $R = \begin{bmatrix} 0 & r_{12} \\ 0 & r_{22} \end{bmatrix}$, but $\sigma(R) = \{0\}$, therefore $r_{22} = 0$ and $R = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$. Notice that (3.9) and (3.10) are independent of the choice of λ .

The coefficient of z in (3.18b) provides

$$0 = v_0A^* + 2v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_1R \begin{bmatrix} 0 \\ -1 \end{bmatrix} - v_0RB = \begin{bmatrix} v_0\bar{a}_1 + 2v_1 - v_1r - v_0rb_2 \\ v_0\bar{a}_2 \end{bmatrix}.$$

This clearly forces $b_1 = a_2 = 0$ and

$$v_0\bar{a}_1 + 2v_1 - v_1r - v_0rb_2 = 0. \quad (3.19)$$

In the same manner, the coefficient of z in (3.18c) should be 0, thus

$$0 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + AR \begin{bmatrix} 0 \\ -1 \end{bmatrix} + [1 \ 0] RB - [0 \ 1] B = a_1 - a_1r + rb_2 - b_2.$$

Therefore, we have

$$a_1 - b_2 = r(a_1 - b_2).$$

Let assume that $a_1 \neq b_2$, then $r = 1$ and from (3.19) we get $v_1 = v_0(b_2 - \bar{a}_1)$. Since the coefficient of z^2 in (3.18b) has to be 0, we obtain

$$\begin{aligned} 0 &= v_0R^*A^* + 2v_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + v_1R^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} - v_0B \\ &= v_0 \begin{bmatrix} 0 & 0 \\ \bar{r} & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ 0 \end{bmatrix} + 2v_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + v_1 \begin{bmatrix} 0 & 0 \\ \bar{r} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - v_0 \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ v_0\bar{r}\bar{a}_1 - 2v_1 + v_1\bar{r} - v_0b_2 \end{bmatrix}, \end{aligned}$$

and hence

$$v_0ra_1 - 2v_1 + v_1r - v_0\bar{b}_2 = 0.$$

As $r = 1$, it follows that $v_0(a - \bar{b}_2) = v_1$, and therefore $v_0(a_1 - \bar{b}_2) = v_1 = v_0(b_2 - \bar{a}_1)$, which gives

$$\Re(a_1) = \Re(b_2).$$

By considering the coefficient of z in (3.18a), we find

$$\begin{aligned} 0 &= 2v_0 + v_1 A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_1 AR \begin{bmatrix} 0 \\ -1 \end{bmatrix} - v_0 ARB = 0 \\ &= 2v_0 + v_1 a_1 - v_1 a_1 r - v_0 a_1 b_2 r. \end{aligned} \quad (3.20)$$

Here we have $r = 1$, so we get $a_1 b_2 = 2$, thus $v_1 = v_0(a_1 - \overline{b_2}) = v_0(a_1 - \frac{2}{a_1})$, which implies that $0 = v_0|a_1|^2 - 2v_0 - v_1 \overline{a_1}$. Consequently, a_1 and b_2 must both be real. It implies that $a_1 = \Re(a_1) = \Re(b_2) = b_2$, a contradiction with the assumption $a_1 \neq b_2$. Hence, from now on we can assume that $a_1 = b_2$.

By (3.19), we then have

$$v_0|a_1|^2 + 2v_1 a_1 - v_1 r a_1 - v_0 r a_1^2 = 0.$$

Using (3.20), we obtain

$$2v_0 + v_1 a_1 - v_0|a_1|^2 - 2v_1 a_1 = 0,$$

so

$$2v_0 - v_0|a_1|^2 - v_1 a_1 = 0,$$

which shows that a_1 is real and a solution of (3.16).

Since $T = S$ is the trivial solution of our problem, then $a = a_1$ is a root of (3.16). The second root must be $-2/a$ and we derive that $\frac{v_1}{v_0} = \frac{2}{a} - a$.

Thereby, we deduce from (3.20) that

$$0 = 2 + \frac{v_1}{v_0} a - \frac{v_1}{v_0} a^2 - a^3 = 2 + (\frac{2}{a} - a)a - (\frac{2}{a} - a)a^2 - a^3 = 4 - a^2 - 2a.$$

Since $a > 0$, we have $a = \sqrt{5} - 1$. We distinguish two cases:

Case 1. If $a_1 = a$, using (3.19) we see that $v_0 a + 2v_1 = r(v_1 + v_0 a)$, which leads to $\frac{4 - a^2}{a} = \frac{2r}{a}$, and hence $a = r$. Thus, in this case we should have $T = S$.

Case 2. If $a_1 = -\frac{2}{a}$, again using (3.19) we have $-v_0 \frac{2}{a} + 2v_1 = r(v_1 - v_0 \frac{2}{a})$. Since $\frac{v_1}{v_0} = \frac{2}{a} - a$, as we have seen before, we get finally $r = 2(1 - \frac{1}{a^2})$.

So, in this case we find that

$$T = 2 \begin{bmatrix} 0 & -\frac{1}{a} & 0 & 0 \\ 0 & 0 & (1 - \frac{1}{a^2}) & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, it remains to verify that this matrix is Harnack equivalent to S in $C_2(\mathbb{C}^4)$. This true if T satisfies all conditions given in Corollary 3.1 (ii). The condition (a) is clearly satisfied.

For (b), we have seen that the system given by $K_z^2(T)v(z) = 0$ is equivalent to \mathcal{S} . The following calculation shows that the first equation (3.18a) of \mathcal{S} is satisfied. Indeed, we have

$$\begin{aligned}
& 2v_0z + v_1A(zI_2 + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0zA(zI_2 + R)B \\
&= 2v_0 + v_1A \begin{bmatrix} z & r \\ 0 & z \end{bmatrix} \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0zA \begin{bmatrix} z & r \\ 0 & z \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \end{bmatrix} \\
&= 2v_0z + v_1 \begin{bmatrix} a_1 & 0 \end{bmatrix} \begin{bmatrix} (1-r)z \\ -z^2 \end{bmatrix} - v_0z \begin{bmatrix} a_1 & 0 \end{bmatrix} \begin{bmatrix} ra_1 \\ za_1 \end{bmatrix} \\
&= 2v_0z + v_1a_1(1-r)z - v_0zra_1^2 = z(2v_0 + v_1a_1 - v_1a_1r - v_0ra_1^2) = 0
\end{aligned}$$

which is satisfied with the entries of T thanks to (3.20). Concerning the second equation (3.18b) of the system \mathcal{S} , we see that

$$\begin{aligned}
& v_0z(I_2 + zR^*)A^* + v_1(2zI_2 + z^2R^* + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0z(zI_2 + R)B \\
&= v_0z \begin{bmatrix} 1 & 0 \\ zr & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ 0 \end{bmatrix} + v_1 \begin{bmatrix} 2z & r \\ z^2r & 2z \end{bmatrix} \begin{bmatrix} 1 \\ -z \end{bmatrix} - v_0z \begin{bmatrix} z & r \\ 0 & z \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \end{bmatrix} \\
&= \begin{bmatrix} z(v_0a_1 + (2-r)v_1 - v_0ra_1) \\ z^2(v_0ra_1 + v_1(r-2) - v_0a_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

by (3.19). The third equality (3.18c) is checked directly :

$$\begin{aligned}
A(zI_2 + R) \begin{bmatrix} 1 \\ -z \end{bmatrix} + [z \quad -1] (zI_2 + R)B &= \begin{bmatrix} a_1 & 0 \end{bmatrix} \begin{bmatrix} (1-r)z \\ -z^2 \end{bmatrix} + [z \quad -1] \begin{bmatrix} ra_1 \\ za_1 \end{bmatrix} \\
&= a_1(1-r)z + zra_1 - za_1 = 0.
\end{aligned}$$

Summarizing what has just be done in this paragraph, we see that condition (b) of Corollary 3.1 is satisfied.

Here $A = \begin{bmatrix} a_1 & 0 \end{bmatrix}$,

$$B = \begin{bmatrix} 0 \\ a_1 \end{bmatrix}, R = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \text{ et } M = I_2 - \frac{1}{4}(A^*A + BB^*) = \begin{bmatrix} 1 - \frac{|a_1|^2}{4} & 0 \\ 0 & 1 - \frac{|a_1|^2}{4} \end{bmatrix},$$

where $a_1 = -\frac{2}{a}$ and $r = 2\left(1 - \frac{1}{a^2}\right)$. It is clear that (3.3) and (3.4) are true. On the other hand, we have $\det(M) + \frac{1}{4}\det(R^*\vec{i}, R\vec{j}) + \frac{1}{4}\det(R\vec{i}, R^*\vec{j}) = \left(1 - \frac{|a_1|^2}{4}\right) - \frac{r^2}{4} = 0$ and $2 - \frac{1}{4}(\|A\|^2 + \|B\|^2) = 2\left(1 - \frac{1}{a^2}\right) = 2\sin^2\left(\frac{\pi}{5}\right) > |Tr(R)| = 0$, so condition (c) is also verified.

Finally, all the conditions of Corollary 3.1 are satisfied, which implies that $T \stackrel{H}{\sim} S$ in $C_2(\mathbb{C}^4)$. The proof of Theorem 3.3 is now complete. \square

Remark 3.4. We see that this result asserts that, even the dimension is an odd number, the Harnack part of a truncated shift can be bigger than the set given in Theorem 3.7 in [7] under the extra assumption that $\|T\| = \|S\|$ for $T \stackrel{H}{\sim} S$, so it answers negatively to [7, Question 2.]. Moreover, this result and Proposition 2.5 show that the norm equality condition in Theorem 3.7 of [7] corresponds to the minimum possible norm in the case of nilpotent operators belonging to the Harnack part of S . Also it gives a negative answer to [7, Question 1.] when $\rho = 2$.

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