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► To cite this version:

Tuan-anh Bui, Rachel Gelet, Didier Marot. Modelling of internal erosion based on mixture theory: General framework and a case study of soil suffusion. *International Journal for Numerical and Analytical Methods in Geomechanics*, 2019, 43 (15), pp.2407-2430. 10.1002/nag.2981 . hal-04167972

HAL Id: hal-04167972

<https://hal.science/hal-04167972>

Submitted on 21 Jul 2023

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Modelling of Internal Erosion based on Mixture Theory:

General Framework and a case study of soil suffusion

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Abstract:

A general thermo-hydro-mechanical framework for the modelling of internal erosion is proposed based on the theory of mixtures applied to two-phase porous media. The erodible soil is partitioned in two phases: one solid phase and one fluid phase. The solid phase is composed of non-erodible grains and erodible particles. The fluid phase is composed of water and fluidized particles. Within the fluid phase, species diffuse. Across phases, species transfer. The modelling of internal erosion is contributed directly by mass transfer from the solid phase toward the fluid phase. The constitutive relations governing the thermomechanical behaviour, generalized diffusion and transfer are structured by the dissipation inequality.

The particular case of soil suffusion is investigated with a focus on constitutive laws. A new constitutive law for suffusion is constructed based on thermodynamic condition and experimental investigations. This erosion law is linearly related to the power of seepage flow and to the erosion resistance index. Owing to its simplicity, this law tackles the overall trend of the suffusion process and permits the formulation of an analytical solution. This new model is then applied to simulate laboratory experiments, by both analytical and numerical methods. The comparison shows that the newly developed model, which is theoretically consistent, can reproduce correctly the overall trend of the cumulated eroded mass when the permeability evolution is small. In addition, the results are provided for four different materials, two different specimen sizes and various hydraulic loading paths to demonstrate the applicability of the new proposed law.

Keywords: internal erosion; suffusion; Multi-species mixture; erosion resistance index; poromechanics; finite element, analytical approach

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1 Introduction

Internal erosion of geomaterials (soil, rock...) is a common phenomenon related to the progressive degradation of material microstructure induced by the effect of fluid flow. This phenomenon may be involved in various engineering applications, such as stability of hydraulic earth structures (embankment dams, dikes etc.), sand production in petroleum engineering, or radionuclide migration through rock fractures due to erosion of buffer materials (Bonelli, 2012) (Veeken, et al., 1991) (Baik, et al., 2007).

Depending on the nature of different materials, the mechanism of internal erosion may be a complex coupling between mechanical, hydrodynamic, and chemical effects. For instance, erosion of bentonite as a buffer material in radioactive waste repository is highly affected by the chemical interactions between groundwater and clay particles (Baik, et al., 2007). In soil, piping erosion concerns the detachment of soil grains from an established pipe wall within a cohesive soil, whereas *suffusion* concerns the detachment of fine particles preferably in cohesionless materials (Marot, et al., 2016). The latter is of particular importance for the risk assessment of hydraulic structures since it involves diffused and complex processes such as detachment, transport and filtration of fine grains which are difficult to observe and predict.

Internal erosion, of whatever nature, may lead to serious consequences such as strength degradation and hydraulic conductivity change of geomaterials. Predicting internal erosion is thus crucial for many geotechnical engineering structures. As the process is heterogeneous and time-dependent, structure analyses require not only laboratory and field measurements, but also advanced hydromechanical modelling. However, there are still quite few numerical models for the simulation of internal erosion. Some of them employ discrete approaches and/or computation fluid dynamics (Frishfelds, et al., 2001) (Zhou, et al., 2011) (Sibille, et al., 2015). These approaches are generally very time-consuming and still limited to laboratory scale modelling. Continuum approaches, mainly based on Finite Element (FE) Method (FEM) and Mixture Theory, are more favourable for large scale applications (Rahmati, et al., 2013). For instance, Vardoulakis *et al.* (Vardoulakis, et al., 2001) (Papamichos & Vardoulakis, 2005) constructed a finite-element poromechanical model with a porosity-based erosion law. Detournay (Detournay, 2008) used a flow rate-driven erosion law to simulate sand production process. Zhang, *et al.* (Zhang, et al., 2013) also developed a coupled hydro-mechanical model for internal erosion. Yet, the erosion law used in their work, which was proposed by (Brivois, et al., 2007), is best suited for piping erosion in soils, as all soil particles, including coarse grains can be detached.

Apart from hydromechanical effects, temperature also has some influences on erodibility of geomaterials (Zreik, et al., 1998). Physical explanations may be based on the temperature dependency of fluid dynamic viscosity (thereby viscous stresses) and other physic-chemical processes especially in cohesive materials (Gularte, et al., 1980). Thermal effect may be significant depending on each considered application, for instance sanding of deep reservoir (Vaziri & Byrne, 1990) (Skjaerstein, et al., 1997) or dam monitoring by thermal optical fibre (Radzicki & Bonelli, 2012). Particular attention for these effects may lie on thermo-hydraulic couplings, since erosion usually involves heat and mass transfer due to fluid flow (Uchida, et al., 2016). Also, temperature may be used to monitor the flow in field structures by using thermal sensors (Smith and Konrad, 2011). Larger temperature gradients point towards preferential flow paths and contribute to alleviate the uncertainty on the permeability

measurement, i.e. temperature may be used to back fit in-situ permeability. In these applications, thermal effect is therefore needed in order to perform consistent theoretical and numerical simulations.

In this context, this paper presents a general Thermo-Hydro-Mechanical (THM) modelling framework for internal erosion of saturated geomaterials based on Mixture Theory. The proposed framework is inspired by previous developments (Papamichos & Vardoulakis, 2005) (Zhang, et al., 2013). Temperature effect is incorporated so as to keep the framework applicable to a larger range of internal erosion-related applications and geomaterials including both soils and rocks. Based on the proposed framework, a case study of soil suffusion modelling is developed. Thermodynamic implications for suffusion constitutive modelling are discussed. The newly developed constitutive model is then verified and tested against laboratory experiments using both analytical and finite element simulations. Throughout the paper, continuum mechanics convention is used (positive tensile stress), scalars are denoted in normal character while vectors and tensors are denoted in bold.

2 Basic assumptions and concepts

2.1 Assumptions

As aforementioned, the proposed framework is based on the concepts of Mixture Theory (Eringen & Ingram, 1965) (Biot, 1977) applied particularly to multiphase multi-species media (Loret & Simoes, 2005) (Gelet, 2011). A saturated erodible material is viewed as a two-phase multi-species mixture. Each Representative Elementary Volume (REV) is composed of a solid phase denoted “S”, which can be eroded under the flow effect of a fluid phase denoted “F”. In some situations, erosion only concerns a part of the solid phase (for instance fine particles in the case of suffusion presented later in this paper). Hence, the solid phase S is partitioned in non-erodible grains denoted “gS” and erodible/filtrated particles denoted “pS”. The fluid phase F contains pure water denoted “wF” and fluidized particles “pF”. Due to erosion, mass transfer can therefore occur between the two phases, in particular only between species “pS” and “pF”.

We define the Eulerian porosities of each phase, denoted by $n^\alpha, \alpha \in \{F, S\}$ and those of each species $n^\beta, \beta \in \{wF, pF, gS, pS\}$, which are the fraction of their volume $V_i, i \in \{\alpha, \beta\}$ with respect to the current total volume V_t of the REV. It is obvious to show that:

$$n^{wF} + n^{pF} = n^F; n^{gS} + n^{pS} = n^S; n^F + n^S = 1 \quad (1)$$

For later developments, it is also convenient to define v^F the Lagrangian fluid porosity (volumetric fraction of the fluid phase with respect to the initial total volume V_0), as well as the erosion-related particle concentration within each phase:

$$c_{pF} = \frac{n^{pF}}{n^F}; c_{pS} = \frac{n^{pS}}{n^S} \quad (2)$$

Fig. 1 illustrates the idealisation of an erodible saturated porous medium with different mechanisms of internal erosion: detachment and transport of particles and possibly filtration, i.e. when the transport is blocked somewhere in the porous medium. Mass transfer may thus occur from erodible solid particles to fluidised particles when detachment takes place, and vice versa, from fluidized particles to solid phase, when filtration process happens. These processes are somewhat similar to the notions Damage and Healing of Continuum Damage Mechanics (CDM). In practice, it is the final result of these processes, called “general erosion”, which is important for assessing internal erosion, since it is related

to the degradation of the microstructure. Moreover, general erosion can be experimentally measured in an easy way, while it is quite difficult to quantify each separate process of detachment and filtration. In the framework of this paper, we assume that detachment is predominant compared to filtration (at the scale of REV) and only “general erosion” is considered.

Small deformation hypothesis, i.e. small displacements, strains and porosity variations, is adopted. Local thermal equilibrium is assumed, meaning that all the phases and species have the same local temperature denoted by the same notation T . It is also assumed that the fluidised particles are surrounded by water species and therefore possess the same pressure as the fluid phase. It is obvious that the species in the solid phase possess the same velocity $\mathbf{V}_S$. Furthermore, we assume that the fluid species diffuse with the same velocity $\mathbf{V}_F$. These assumptions have been discussed in previous works (Papamichos & Vardoulakis, 2005) (Vardoulakis, et al., 2001):

$$p_{pF} = p_{wF} = p_F; \quad \mathbf{V}_{pF} = \mathbf{V}_{wF} = \mathbf{V}_F; \quad \mathbf{V}_{pS} = \mathbf{V}_{gS} = \mathbf{V}_S \quad (3)$$

Internal erosion may lead to changes of both skeleton deformation and porous space, comprising of mechanical loading-induced and erosion-induced parts. As this proposed framework focuses on the erosion process, the mechanical loading-induced deformation is assumed to be elastic. Plastic deformation will be considered in future publications. Moreover, the skeleton deformation is assumed to be reversible, even when the material is degraded due to erosion. This hypothesis is similar to that has been sometimes adopted in CDM where damage strain is assumed to be recoverable upon unloading (Lai, et al., 2009) (Wu & Cervera, 2016) (Bui, et al., 2017). By contrast, the volume content evolution is assumed to contain a reversible part due to mechanical loading, and explicitly an irreversible part due to erosion. This concept of irreversible porosity was also proposed in poro-elasticity with dissolution by (Coussy, 2004). To sum up, the above partition assumption can be written as follows:

$$\begin{aligned} d\boldsymbol{\varepsilon} &= d\boldsymbol{\varepsilon}^e \\ dv^F &= dv^{Fe} + dv^{Fi} \end{aligned} \quad (4)$$

Where the superscripts “e” and “i” stand for incremental change of poro-elastic reversible (but degradable) parts due to mechanical loading and irreversible erosion-induced part, respectively. Equation (4) amounts to saying that in the absence of mechanical loading, the solid skeleton does not experience any deformation but the fluid content may still be modified due to erosion.

2.2 Fundamental Balance Equations

In the framework of Mixture Theory, the basic concepts of continuum mechanics (balance of momentum, mass, energy and entropy) are applied to each species and also to the whole mixture. In the following, fundamental balance equations will be presented.

The **mass balance** equations are an important element of modelling when considering mass transfer. The mass conservation of a generic species $\beta \in \{wF, pF, gS, pS\}$ writes (Loret & Simoes, 2005):

$$\frac{d(n^\beta \rho_\beta)}{dt} + n^\beta \rho_\beta \operatorname{div}(\mathbf{V}_s) + \operatorname{div} \mathbf{M}_\beta = \hat{\rho}^\beta \quad (5)$$

where ρ_β is the intrinsic density of the species β , $\hat{\rho}^\beta$ is the mass supply rate to this species per unit current volume, and $\mathbf{M}_\beta$ is its mass flux per current unit current area:

$$\mathbf{M}_\beta = \rho_\beta \mathbf{J}_\beta \quad (6)$$

in which the volumetric flux (Darcy velocity) $\mathbf{J}_\beta$ is defined by:

$$\mathbf{J}_\beta = n^\beta (\mathbf{V}_\beta - \mathbf{V}_s) \quad (7)$$

It is obvious that the flux of the species in the solid phase are null $\mathbf{M}_{ps} = \mathbf{M}_{gs} = \mathbf{0}$. Owing to the closure condition, i.e. no mass exchange with the surrounding is accounted for and mass transfer is only caused by mass exchange between the species, we get:

$$\hat{\rho}^{pF} = -\hat{\rho}^{pS} = \hat{\rho}^F; \quad \hat{\rho}^{wF} = \hat{\rho}^{gS} = 0 \quad (8)$$

The **momentum balance** can be written for each species β as follows (Loret & Simoes, 2005):

$$\operatorname{div}(n^\beta \boldsymbol{\sigma}^\beta) + \rho^\beta (\mathbf{g} - \boldsymbol{\gamma}^\beta) = \hat{\rho}^\beta (\mathbf{V}_\beta - \tilde{\mathbf{V}}_\beta) - \hat{\mathbf{p}}_\beta \quad (9)$$

where $\mathbf{g}$ is the body force which is reduced here the gravity force, $\boldsymbol{\sigma}^\beta$ is the partial stress tensor of the species, $\boldsymbol{\gamma}^\beta = d\mathbf{V}_\beta/dt$ is the acceleration, $\tilde{\mathbf{V}}_\beta$ is the velocity of the mass just before the deposit, and $\hat{\mathbf{p}}_\beta$ the momentum supply by the rest of the mixture. Similar to (8), since only internal mass transfer is considered herein (closure condition), the whole net momentum supply should vanish:

$$\sum_{\beta} (\hat{\rho}^\beta \tilde{\mathbf{V}}_\beta + \hat{\mathbf{p}}_\beta) = 0 \quad (10)$$

Although internal erosion may be related to interacting (viscous) forces between the solid and the fluid phases, we assume that the shear component of the fluid stress tensors is negligible compared to the pressure component when considering the momentum balance (Coussy, 2004). Therefore we assume:

$$\boldsymbol{\sigma}_{pF} = \boldsymbol{\sigma}_{wF} = -p_F \boldsymbol{\delta} \quad (11)$$

The total stress tensor, within the framework of infinitesimal deformations, is defined as:

$$\boldsymbol{\sigma} = \sum_{\beta} n^\beta \boldsymbol{\sigma}_\beta = \sum_{\beta} \boldsymbol{\sigma}^\beta \quad (12)$$

The **energy balance** (first law of thermodynamics), accounting for mass transfer, can be written for a generic species β as follows:

$$\begin{aligned} \frac{d(n^\beta \rho_\beta U_\beta)}{dt} + \mathbf{div}(U_\beta \mathbf{M}_\beta) + n^\beta \rho_\beta U_\beta \mathbf{div}(\mathbf{V}_s) - \boldsymbol{\sigma}^\beta : \nabla \mathbf{V}_\beta + \mathbf{div} \mathbf{q}_\beta - r_\beta \\ = \hat{\rho}^\beta \left(\tilde{U}_\beta - U_\beta + \frac{1}{2} (\mathbf{V}_\beta - \tilde{\mathbf{V}}_\beta)^2 \right) + \hat{u}_\beta \end{aligned} \quad (13)$$

where U_β is the internal energy of the species β , $\mathbf{q}_\beta$ and r_β are the heat flux and source terms, $\tilde{U}_\beta$ is the internal energy of the transferred mass just before the deposit, and $\hat{u}_\beta$ is the energy supply by the rest of the mixture. The closure condition infers that the whole net energy supply should be null:

$$\sum_\beta \left(\hat{\rho}^\beta \left(\tilde{U}_\beta + \frac{1}{2} (\tilde{\mathbf{V}}_\beta)^2 \right) + \mathbf{V}_\beta \cdot \hat{\mathbf{p}}_\beta + \hat{u}_\beta \right) = 0 \quad (14)$$

Finally, the **entropy balance** is written for each species:

$$n^\beta \rho_\beta \frac{d(S_\beta)}{dt} + \nabla S_\beta \cdot \mathbf{M}_\beta + \mathbf{div} \frac{\mathbf{q}_\beta}{T} - \frac{r_\beta}{T} - \hat{\rho}^\beta (\tilde{S}_\beta - S_\beta) - \hat{s}_\beta = n^\beta \rho_\beta \tilde{S}_\beta \quad (15)$$

where S_β is the entropy of the species β , $\tilde{S}_\beta$ the entropy of the transferred mass just before the deposit, $\hat{s}_\beta$ the entropy supply by the rest of the mixture, while $\tilde{S}_\beta$ is the specific entropy production rate. The closure condition also applies so that the whole net entropy supply of the system should be zero:

$$\sum_\beta (\hat{\rho}^\beta \tilde{S}_\beta + \hat{s}_\beta) = 0 \quad (16)$$

The second law of thermodynamics implies that the entropy production rate is always non-negative. However, only a less restrictive condition on the negativity of the whole mixture is considered in the proposed framework:

$$\sum_\beta n^\beta \rho_\beta \tilde{S}_\beta \geq 0 \quad (17)$$

2.3 Clausius-Duhem inequality

The classic combination of the first law (13) with the second law (15) and (17) for the whole mixture, using the closure conditions (14) and (16), leads to the fundamental Clausius-Duhem inequality:

$$\Phi_{total} = \Phi_{sk} + \Phi_F + \Phi_T \geq 0 \quad (18)$$

in which Φ_{sk} is the dissipation related the solid skeleton, Φ_F is the dissipation due to fluid flow and Φ_T is that related to heat flow, defined as:

$$\begin{aligned} \Phi_{sk} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - v^{Fe} \dot{p}_F + p_F \dot{v}^{Fi} - m^S S_S \dot{T} - \hat{\rho}^F G_{pF} - \dot{\psi}_s \\ \Phi_F = -J_F [\nabla p_F - \rho_F \mathbf{g}]; \quad \Phi_T = -\mathbf{q} \frac{\nabla T}{T} \end{aligned} \quad (19)$$

where $\mathbf{q} = \sum_{\beta} \mathbf{q}_{\beta}$ is the total heat flux supplied to the mixture, m^S is the density of the solid phase, and G_{pF} the specific Gibbs free energy of the fluidized particles. Detail of the derivation leading to (18) can be found in Appendix A. The thermodynamic potential is postulated as a function of the state variables $\psi_s = \psi_s(\boldsymbol{\varepsilon}, p_F, v^{Fi}, T)$ and defined as:

$$\psi_s = \psi_s(\boldsymbol{\varepsilon}, p_F, v^{Fi}, T) = m^S E_S - v^{Fe} p_F \quad (20)$$

Note that the above derivation is quite different from that proposed by (Bowen & Garcia, 1970) who developed a formulation for a mixture where each constituent has its own temperature field. In the work of (Bowen & Garcia, 1970), a Massieu function for each species was introduced, whereas (Truesdell, 1984) used the Helmotz free energy as potential. Later (Bowen, 1976) (Bowen, 1982) used the free energy per unit current volume as a thermodynamic potential which simplifies the expressions. Detailed derivations and comments may be found in (Loret & Simões, 2017). Besides the multi-temperature approach, Bowen's theory also contains the effects of non-linear elasticity, non-linear heat conduction, non-linear viscosity and coupled generalized diffusion. In contrast, the present derivation aims at highlighting some specific couplings between the linear hydraulic diffusion and erosion, see later Eq. (34), and assumes a common temperature for all species. In this sense, the thermodynamic analysis is an adaptation of the classical works on mixture theory by (Truesdell & Toupin, 1960), (Eringen & Ingram, 1965), (Haase, 1969), (Atkin & Craine, 1976), (Bowen, 1976), (Truesdell, 1984) to consider erosion within permeable erodible porous media (intrinsic permeability lower than $10^{-18} m^2$).

The dissipation Φ_{sk} has been discussed in the context of phase change in porous media (Coussy, 2004). In this internal erosion problem, it represents the irrecoverable energy related to the skeleton during both mechanical loading (mechanical dissipation) and erosion induced degradation (erosion dissipation). Further manipulations help to distinguish these two processes as follows:

$$\Phi_{sk} = \Phi_m + \Phi_{er} \quad (21)$$

$$\Phi_m = \left(\boldsymbol{\sigma} - \frac{\partial \psi_s}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \left(v^{Fe} + \frac{\partial \psi_s}{\partial p_F} \right) \dot{p}_F - \left(m^S S_S + \frac{\partial \psi_s}{\partial T} \right) \dot{T} \quad (22)$$

$$\Phi_{er} = p_F \dot{v}^{Fi} - \hat{p}^F G_{pF} - \frac{\partial \psi_s}{\partial v^{Fi}} \dot{v}^{Fi} \quad (23)$$

The dissipations $\Phi_m, \Phi_{er}, \Phi_F, \Phi_T$ have different physical natures. However, although the dissipative physical processes are different, they are generally coupled. Those couplings may be of high or small importance depending on each application. For example, in geothermal energy, fluid flow and heat transfer may be strongly coupled, leading to non-isothermal heat flow induced by fluid pressure gradient and thermo-osmosis fluid flow induced by temperature gradient (Gelet, 2011). Therefore, the general condition (18) is always valid while imposing separate non-negativity condition on each dissipation is sometimes too restrictive.

In our general framework, apart from (18), two other assumptions are adopted:

$$\Phi_m = 0 \quad (24)$$

$$\Phi_T \geq 0 \quad (25)$$

That is to say no energy is transferred into heat during pure mechanical loading (elasticity framework) and heat flow always induces dissipated energy independently of other processes. Further specific assumptions will be made for each specific case in order to construct appropriate constitutive laws.

The proposed decoupling between the thermal and hydraulic dissipation does not permit coupled flow phenomena such as thermos-osmosis and isothermal heat flow. In the literature, there is a consensus that those coupled phenomena should not be neglected for very low permeability porous media (intrinsic permeability lower than 10^{-18}m^2) (Ghassemi & Diek, 2003) (Chen, et al., 2013) (Chen, et al., 2018). Hence, the present approach may not be applicable for those particular materials.

3 Constitutive and field equations

3.1 Thermo-Poro-elastic and thermal flow constitutive equations

The assumptions (24) and (25) made on the mechanical and heat flow dissipations help to postulate the mechanical and thermal flow constitutive equations.

The mechanical constitutive equations can be derived directly from (24) and

(22). The classic poro-elastic equations (Coussy, 2004) (Gelet, 2011) may be generalised as follows:

$$\begin{aligned} d\boldsymbol{\sigma} &= \mathbb{C}:d\boldsymbol{\epsilon}^e - \xi \delta dp_F - \gamma_T dT \\ dv^{Fe} &= \xi d\epsilon^e + (\xi - n^F)c_S dp_F - (\xi - n^F)c_T dT \\ d(\rho^S S_S) &= \gamma_T d\epsilon^e - (\xi - n^F)c_T dp_F + C_p^S \frac{dT}{T} \end{aligned} \quad (26)$$

where $\mathbb{C}$ denotes the fourth-order drained elastic stiffness tensor; $\xi = 1 - \frac{c_S}{c}$ is the Biot's effective stress parameter, in which c is the drained compressibility of the porous medium and c_S is that of the solid matrix; $\gamma_T = \frac{c_T}{c}$ is a thermal coefficient with c_T the thermal dilation of the porous medium; ϵ^e is the volumetric strain; C_p^S is the apparent heat capacity at constant strain and fluid pressure, per unit initial volume of the mixture. Note that as the thermodynamic potential ψ_s is also dependent on v^{Fi} as a separate internal variable, the thermo-poro-elastic parameters $\mathbb{C}, c, \xi, c_T$ could be functions of v^{Fi} . With this regard, Equations (26) are similar to that often postulated in CDM, where v^{Fi} plays an identical role as a damage variable (Bui, et al., 2016). This is also why the incremental form should be used, since nonlinearity may be involved.

It is also assumed from (25) that thermal flow is independent of other processes and governed by the classic Fourier's law:

$$\mathbf{q} = -\lambda \frac{\nabla T}{T} \quad (27)$$

where λ is the thermal conductivity coefficient.

3.2 Erosion and Hydraulic flow constitutive equations

The erosion process is described by v^{Fi} and $\hat{\rho}^F$. Firstly we derive a relationship between these variables. Summing up the mass balance equations (5) for the species in the solid phase (i.e. $\beta \in \{gS, pS\}$) and using (8), we obtain the mass balance equation for the whole solid phase:

$$\frac{d(n^S \rho_S)}{dt} + n^S \rho_S \text{div}(\mathbf{V}_S) = -\hat{\rho}^F \quad (28)$$

Assuming a thermo-elastic behavior of the solid grains, we obtain:

$$\frac{1}{\rho_S} d(\rho_S) = -\frac{c_S}{3n^S} d\sigma_m^S - c_T dT \quad (29)$$

where σ_m^S is the mean stress of the partial stress tensor of the solid phase. Noting that $n^S = 1 - n^F$ and infinitesimal deformation is assumed, and using (12) and (26)a, we get:

$$\frac{dv^F}{dt} = \xi d\epsilon^e + c_S(\xi - n^F) dp_F - (\xi - n^F) c_T dT + \frac{\hat{\rho}^F}{\rho_S} \quad (30)$$

By means of (4)b, (26)b and (30), the following simple relation is obtained:

$$v^{Fi} = \frac{\hat{\rho}^F}{\rho_S} \quad (31)$$

This allows to rewrite the dissipation due to erosion (23) as follows:

$$\Phi_{er} = \hat{\rho}^F \left(-E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}} \right) \quad (32)$$

where E_{pF} is the free energy of the fluidized particles, defined by:

$$E_{pF} = G_{pF} - \frac{p_F}{\rho_S} \quad (33)$$

Indeed, internal erosion is always strongly coupled with hydraulic flow (Chang & Zhang, 2013) (Li & Fannin, 2008). Therefore, it is assumed here that the dissipated energy originating from fluid flow and erosion processes are coupled into a general term Φ_H called the total dissipated energy due to hydrodynamic effects, using (19) and (32):

$$\Phi_H = \Phi_F + \Phi_{er} = -J_F[\nabla p_F - \rho_F \mathbf{g}] + \hat{\rho}^F \left(-E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}} \right) \geq 0 \quad (34)$$

The non-negativity of Φ_H results from the fact that the purely mechanical dissipation is assumed to be null, whereas the thermal dissipation is decoupled from other processes, as indicated in (24) and (25).

Those assumptions do not mean that these processes are independent of erosion, but that they constitute additional physically-based constraints for constructing constitutive equations.

It is observed that the dissipated energy caused by erosion Φ_{er} is related to the product between a “flux” term $\hat{\rho}^F$ and a “force” term $\left(-E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}}\right)$. The latter can be defined as the difference between the “erosion energy release rate” owing to the loss of particles during an internal erosion process $\frac{\partial \psi_S}{\partial v^{Fi}}$, and the free energy of the fluidized particles per unit volume E_{pF} .

The dissipation form (34) suggests that one may construct at the same time constitutive laws for both fluid flow and erosion as two coupled processes, by postulating appropriate laws relating the flux and force vectors:

$$\begin{pmatrix} \mathbf{J}_F \\ \hat{\rho}^F \end{pmatrix} = \mathcal{F} \begin{pmatrix} -[\nabla p_F - \rho_F \mathbf{g}] \\ -E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}} \end{pmatrix} \quad (35)$$

For instance, (35) suggests that an additional term may be added into the classical Darcy law to describe the influence of erosion which is expected to increase the fluid flow rate. In particular, mass transfer rate $\hat{\rho}^F$ should probably be a function of both hydraulic gradient $-\nabla p_F - \rho_F \mathbf{g}$ and the degradation energy $\left(-E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}}\right)$. Note that (35) is postulated in an effort to identify the relevant forces rather than to enforce a new constraint.

3.3 Principal Field Equations

The main field equations are formulated on the basis of the balance and constitutive equations for a closed system. Details of the derivation of these equations can be found in Appendix B.

First, applying (5) for both solid species, one of which is directly related to internal erosion, the first field equation (**fluid porosity**) is obtained:

$$\frac{dn^F}{dt} + A_n = 0 \quad (36)$$

$$A_n = a_{nu} \frac{d\epsilon^e}{dt} + a_{np} \frac{dp_F}{dt} + a_{nT} \frac{dT}{dt} - \frac{\hat{\rho}^F}{\rho_S} \quad (37)$$

in which the coupling coefficients writes:

$$a_{nu} = -(\xi - n^F); \quad a_{np} = -c_S(\xi - n^F); \quad a_{nT} = c_T(\xi - n^F) \quad (38)$$

Second, applying (5) to the fluidized particles, which are directly related to internal erosion, the second field equation (**concentration field equation**) is obtained:

$$\text{div}(c_{pF} \mathbf{J}_F) + A_c = 0 \quad (39)$$

where:

$$A_c = n^F \frac{dc_{pF}}{dt} + a_{cu} \frac{d\epsilon^e}{dt} + a_{cp} \frac{dp_F}{dt} + a_{cT} \frac{dT}{dt} + c_{pF} J_F (c_{SH} \nabla p_F - c_{ST} \nabla T) - (1 - c_{pF}) \frac{\hat{\rho}^F}{\rho_S} \quad (40)$$

in which the coupling coefficients writes:

$$a_{cu} = \xi c_{pF}; \quad a_{cp} = \xi c_{pF} c_S; \quad a_{cT} = -\xi c_{pF} c_T \quad (41)$$

Next, applying (5) for both fluid species, noting that those species are governed by the same pressure field, the **pressure field equation** is obtained:

$$\text{div} J_F + A_F = 0 \quad (42)$$

$$A_F = a_{pu} \frac{d\epsilon^e}{dt} + a_{pp} \frac{dp_F}{dt} + a_{pT} \frac{dT}{dt} + J_F (c_{FH} \nabla p_F - c_{FT} \nabla T) \quad (43)$$

where the coupling coefficients read:

$$a_{pu} = \xi; \quad a_{pp} = c_S (\xi - n^F) + c_{FH} n^F; \quad a_{pT} = -c_T (\xi - n^F) - c_{FT} n^F \quad (44)$$

and the averaged compressibility and expansion coefficient of the whole fluid mixture are defined:

$$c_{FH} = c_{pF} c_{SH} + (1 - c_{pF}) c_{WH}; \quad c_{FT} = c_{pF} c_{ST} + (1 - c_{pF}) c_{WT} \quad (45)$$

In addition, applying (9) to the whole mixture, using the relations (8), (10) and (12), and limiting to quasi-static analysis, one may easily get the **mechanical field equation**:

$$\text{div}(\sigma) + \rho g = 0 \quad (46)$$

in which the momentum supply due to mass transfer vanishes for the case of a closed system (Loret & Simoes, 2005) (Gelet, 2011).

Finally, the **temperature field equation** for the whole mixture (governed by a unique temperature field) is derived using the energy balance equation (13) and the closure condition (14):

$$\text{div} q + A_T = 0 \quad (47)$$

$$A_T = b_{Tu} \frac{d\epsilon^e}{dt} + b_{Tp} \frac{dp_F}{dt} + b_{TT} \frac{dT}{dt} + J_F \cdot [c_{pF} \rho_S \nabla H_{pF} + (1 - c_{pF}) \rho_w \nabla H_{wF} + c_{FT} \nabla p_F - C_{p,F} \nabla T] + \hat{\rho}^F \left[U_{pF} + \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^F i} \right] \quad (48)$$

where $H_{pF} = U_{pF} + \frac{p_F}{\rho_S}$ is the free enthalpy of the species “pF” and the coupling coefficients are defined:

$$b_{Tu} = T \gamma_T; \quad b_{Tp} = -T c_T (\xi - n^F) - T c_{FT} n^F; \quad b_{TT} = n^F C_{p,F} + n^S C_{p,S} \quad (49)$$

with $C_{p,F}$ and $C_{p,S}$ are the specific volumetric heat capacities, at constant pressure, of the fluid and the solid phase, respectively [J/m³/K].

It is observed that three field equations contain a source term representing the mass transfer effect on the fluid porosity, the particle concentration and the temperature field.

4 A case study for soil suffusion

4.1 A new constitutive law for soil suffusion

To illustrate the applicability of the proposed framework, a specific case of soil suffusion will be studied hereafter. As aforementioned in Section 1, soil suffusion is an internal erosion process concerning particularly the detachment and the transport of fine particles. Its evolution depends on various factors, including grain size distribution, grain angularity, chemical activity of soil (especially for clay), effective stress state of the soil, and certainly, hydraulic loading (Marot & Benamar, 2012) (Le, et al., 2017).

So far, very few attempts on modelling of soil suffusion have been achieved. Most of existing research aims at evaluating the suffusion sensitivity with some criteria based on either the critical hydraulic gradient (Skempton & Brogan, 1994), or the particle size distribution (Kenney & Lau, 1985) (Li & Fannin, 2008). Recently, Marot, et al (Marot, et al., 2016) have proposed a new criterion based on the flow energy and derived an intrinsic material suffusion resistance index I_α .

All the above models are only confined to suffusion strength but not the kinetic of the process. To better characterize suffusion evolution, we propose hereafter a new simple suffusion constitutive law. This is based on the flow energy approach proposed by Marot and co-workers presented above. They observed that at a complete suffusion state, the total eroded mass is always linearly proportional to the total energy dissipated by seepage flow. The complete suffusion state is defined towards the end of an experimental test by a simultaneous decrease of the erosion rate and by a stable hydraulic conductivity. This state is interpreted as being the end of the suffusion process. The ratio between the total eroded dry mass and the total energy is related to an intrinsic material property I_α that is named the erosion resistance index (Marot, et al., 2016).

In the following, a new constitutive law for soil suffusion based on both experimental evidences and the above thermodynamic framework (Section 3) is proposed. The new model focuses on the erosion process of soil in an isothermal state. Thus, *thermal effects are neglected, and the incompressibility of both solid grains and fluid is assumed*. Based on experimental data, small concentration is also assumed, namely $c_{pF} \ll 1$. Inspired by the energy approach of Marot and co-workers and also by the thermodynamic implication (35), we suggest the following local suffusion law:

$$\hat{\rho}^F = k_\alpha \frac{k_F}{\rho_F g} [\nabla p_F - \rho_F \mathbf{g}]^2 = k_\alpha P_{seepage} \quad (50)$$

where $k_\alpha = 10^{-I_\alpha}$ is the coefficient of erodibility of the material, which is an intrinsic property, k_F is the hydraulic conductivity and $\mathbf{g}$ is the gravity acceleration vector. The right-hand-side term of (50) is in fact the product between k_α and the volumetric power dissipated by the flow $P_{seepage}$ (Marot, et al., 2011).

This law (50) assumes that the local eroded mass is a function of the hydraulic gradient $[\nabla p_F - \rho_F \mathbf{g}]$, while the effect of the second term in (35) $\left(-E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}}\right)$ is disregarded. A reasoning for this simplification is that the suffusion process is assumed to be quite slow compared to fluid flow. The energy release of the solid skeleton due to suffusion-induced degradation may therefore have enough time to transform into the free energy of the fluidized particles. This means that local equilibrium may

be reached and this potential energy difference contribution is assumed small compared to the hydraulic gradient contribution. Note that this adiabatic assumption should be validated when more advanced research on the suffusion-induced degradation rate are carried out.

Furthermore, the above law also infers that the suffusion law is governed by an indirect force (the volumetric power dissipated by the flow) rather than by its direct thermodynamic force (energy difference, as described in (34)). This is partly in accordance with laboratory tests, and also similar to other well-known practical problems (Mitchell & Soga, 2005). For instance, the chloride ion flux initiated by the electric gradient (electrophoresis) is probably larger than that induced by the chemical gradient in migration testing of chloride ions in concrete (Djerbi, et al., 2008).

The above assumption also allows to neglect the effect of $\left(-E_{pF} - \frac{1}{\rho_S} \frac{\partial \psi_S}{\partial v^{Fi}}\right)$ in (35), in the fluid flow equation. The fluid flux may be dependent solely on the hydraulic gradient. Also assuming laminar flow, the classic Darcy law is assumed:

$$\mathbf{J}_F = -\frac{k_F}{\rho_F g} [\nabla p_F - \rho_F \mathbf{g}] \quad (51)$$

Note that the combination of (50) and (51) lead to the fact that $\hat{\rho}^F \propto (-\mathbf{J}_F [\nabla p_F - \rho_F \mathbf{g}])$, namely we assume that the local mass transfer rate is proportional to the volumetric seepage power, which is inspired by the approach of Marot and co-workers (Marot, et al., 2011) (Marot, et al., 2012). Let's now introduce the volumetric seepage energy as the integration of the volumetric seepage power over time:

$$E_{seepage} = \int_{t_0}^t -\mathbf{J}_F [\nabla p_F - \rho_F \mathbf{g}] dt \quad (52)$$

Based on experimental data, the suffusion process initiates and develops until a final stable state which represents the end of the suffusion process. Beyond this point in time, the rate of eroded mass decreases significantly and the hydraulic conductivity remains constant. This final stable state is repeatable and is characterised by a threshold volumetric seepage energy E_{max} . Beyond this value, we assume that $\hat{\rho}^F = 0$, so we write:

$$\begin{cases} k_\alpha = 10^{-I_\alpha} & \text{if } E_{seepage} < E_{max} \\ k_\alpha = 0 & \text{otherwise} \end{cases} \quad (53)$$

At variance with the work of (Sibille et al., 2015), the proposed approach uses only one material parameter, namely I_α . In consequence, the proposed law targets the smoothed trend of the suffusion process. This simplistic approach permits the development of an analytical solution that will improve our understanding of the suffusion process. Moreover, the degradation of mechanical properties caused by suffusion is generally not negligible. Nonetheless, lack of experimental data and also for simplicity, it is assumed that this mechanical degradation is neglected and the material stiffness parameters are constant in (26).

To sum up, the constitutive equations (26), (31), (50) and (51) complement the general field equations (36), (39), (42), (46) to constitute a general system of equations for numerical modelling. Re-call that all temperature-related terms are disregarded owing to the isothermal assumption, and due to the small concentration assumptions ($c_{pF} \ll 1$) and incompressibility, the system is simplified as follows:

$$\begin{aligned} d\boldsymbol{\sigma} &= \mathbb{C}:d\boldsymbol{\varepsilon}^e - dp_F \boldsymbol{\delta} \\ dv^{Fe} &= d\epsilon^e \end{aligned}$$

$$\begin{aligned}
 \hat{\rho}^F &= \frac{k_\alpha k_F}{\rho_F g} [\nabla p_F - \rho_F \mathbf{g}]^2 \\
 \frac{dn^F}{dt} - (1 - n^F) \frac{d\epsilon^e}{dt} - \frac{\hat{\rho}^F}{\rho_S} &= 0 \\
 c_{pF} \operatorname{div} \left(-\frac{k_F}{\rho_F g} [\nabla p_F - \rho_F \mathbf{g}] \right) - \frac{k_F}{\rho_F g} [\nabla p_F - \rho_F \mathbf{g}] \nabla c_{pF} + n^F \frac{dc_{pF}}{dt} + c_{pF} \frac{d\epsilon^e}{dt} - \frac{\hat{\rho}^F}{\rho_S} \\
 &= 0 \\
 \operatorname{div} \left(-\frac{k_F}{\rho_F g} [\nabla p_F - \rho_F \mathbf{g}] \right) + \frac{d\epsilon^e}{dt} &= 0 \\
 \mathbf{div}(\boldsymbol{\sigma}) + \rho \mathbf{g} &= 0
 \end{aligned} \tag{54}$$

Note that in this case, the Biot's coefficient $\xi = 1$. The suffusion law (50) states that the erosion process depends solely on the pressure field variable. In the case of soil suffusion, no evident boundary condition can be imposed on the porosity field nor on the concentration field. Hence, both fields (Eq.

(54)d and

(54)e) will be partly decoupled from the other field equations Eq.

(54)f and

(54)g. Therefore the porosity and the concentration fields can be solved separately from the main system of equations, which contains only the pressure and the displacement fields. The porosity and the concentration can now be seen as state variables. The size of the system of equations is thus equal to a classic hydro-mechanical coupling.

4.2 Analytical study

To illustrate the above newly developed formulation, some laboratory suffusion tests, carried out in the team, are simulated. The tests were performed under oedometeric conditions while macroscopic hydraulic gradients were applied. Different water pressures were imposed at the inlet and outlet sections of the samples, and cumulated (eroded masses) were measured during the tests (Rochim et al., 2017) (Zhong et al., 2018). Under these conditions, the tests can be considered one-dimensional and all the field variables (pressure, displacement, concentration) are only dependent on axial coordinate z and time t . In this 1D case, the problem can be solved analytically. The displacement, strain and stress fields write:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & & \\ & \sigma_x & \\ & & \sigma_z \end{bmatrix}; \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}; \quad \boldsymbol{\epsilon} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & \frac{\partial u}{\partial z} \end{bmatrix}; \quad \epsilon = \operatorname{tr}(\boldsymbol{\epsilon}) = \epsilon_z = \frac{\partial u}{\partial z} \tag{55}$$

where $u = u(z, t)$ is the axial displacement.

The initial stress-strain-pressure write:

$$\begin{aligned}
 \boldsymbol{\sigma}(z, 0) &= \boldsymbol{\sigma}_0(z); \quad \sigma_z(z, 0) = -\rho g z; \quad u(z, 0) = 0; \quad \epsilon(z, 0) = 0; \\
 p_F(z, 0) &= p_{F0}(z) = \rho_F g z; \quad c_{pF}(z, 0) = 0.
 \end{aligned} \tag{56}$$

The boundary conditions are illustrated in Fig. 2. At the inlet section A, an imposed pressure p_{FA} is applied, corresponding to an equal compressive total axial stress $\sigma_z(0, t)$. In this case the Biot's

coefficient $\xi = 1$, this boundary condition infers that a zero effective stress is imposed. Since pure water (zero concentration) is supplied to the sample at this section, the mass conservation implies that the fluidized particle concentration should also be zero at this section. At the outlet section B, a zero displacement and an imposed pressure p_{FB} are applied. The concentration at this section, due to mass conservation, should be equal to the cumulated mass rate during the tests, but this can be computed based on the solving of the system of equations, rather than a boundary condition to be satisfied. Note that only one boundary condition is needed for the first-order partial derivative concentration equation

(54)e.

To sum up, the boundary conditions write:

$$\begin{aligned}\sigma_z(0, t) &= -p_{FA}; \quad u(L, 0) = 0; \\ p_F(0, t) &= p_{FA}; \quad p_F(L, t) = p_{FB}; \\ c_{pF}(0, t) &= 0.\end{aligned}\tag{57}$$

Also noteworthy is the fact that the inlet boundary conditions are applied in two successive phases: the first one applies (very quickly) the stress and water pressure boundary conditions, and the second maintains those quantities constant over time. From the analytical point of view, the first phase is considered instantaneous and only affects initial stress and strain states (by static equilibrium and constitutive relations) but not water pressure and concentration fields (which need time to propagate). Only the second phase will be considered hereafter to study the erosion rate so that the initial conditions correspond to an initial equilibrium without any erosion. The system solving will therefore only consider the out-of-balance variations, i.e. $\bar{\sigma} = \sigma - \sigma_0$ and $\bar{p}_F = p_F - p_{F0}$. In terms of these variables and using (58), the z-component of the constitutive equations

(54)a write:

$$\bar{\sigma}_z = (\lambda + 2\mu)\epsilon - \bar{p}_F\tag{58}$$

where λ and μ are Lamé's elastic coefficients. Taking the z-derivative of both sides of (58), assuming that $|\rho - \rho_0| \ll 1$ and using the only non-null component (in z coordinate) of the mechanical equation

(54)g, we get:

$$(\lambda + 2\mu)\frac{\partial \epsilon}{\partial z} - \frac{\partial \bar{p}_F}{\partial z} = 0\tag{59}$$

This equation (59) is a well-known equation of poroelasticity, which can be combined with

(54)f (water mass balance) to give the classic equation:

$$\frac{\partial \bar{p}_F}{\partial t} - \omega \frac{\partial^2 \bar{p}_F}{\partial z^2} = 0 \quad (60)$$

where the diffusivity coefficient is defined as:

$$\omega = (\lambda + 2\mu) \frac{k_F}{\rho_F g} \quad (61)$$

Different from the consolidation equation, Equation (60) for our case of suffusion test has different boundary conditions as described in (56). Its solution writes (see (Polyanin, 2001)):

$$\bar{p}_F(z, t) = \bar{p}_{FA} + (\bar{p}_{FB} - \bar{p}_{FA}) \frac{z}{L} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \bar{p}_{FB} - \bar{p}_{FA}}{n} \sin\left(\frac{n\pi z}{L}\right) e^{\left(\frac{-\omega n^2 \pi^2}{L^2} t\right)} \quad (62)$$

Equation (62) describes the transient evolution of the pressure. It is also straightforward to deduce the steady-state solution:

$$\bar{p}_F(z) = \bar{p}_{FA} + (\bar{p}_{FB} - \bar{p}_{FA}) \frac{z}{L} \quad (63)$$

Substituting (62) into the concentration equation

(54)e provides us a very complex partial differential equation. To simplify, we will only consider the steady-state solution (63), i.e. the concentration evolution when water flow already reaches a permanent state, to get the following equation:

$$n_F \frac{\partial c_{pF}}{\partial t} + \frac{k_F}{\rho_F g} \left(\frac{\bar{p}_{FA} - \bar{p}_{FB}}{L} \right) \frac{\partial c_{pF}}{\partial z} = \frac{k_\alpha k_F}{\rho_F g \rho_S} \left(\frac{\bar{p}_{FA} - \bar{p}_{FB}}{L} \right)^2 \quad (64)$$

Using the classic method of characteristics for linear partial derivative equations, Equation (63), on account of the boundary conditions (56), has the following solution:

$$c_{pF} = \frac{k_\alpha}{\rho_S} \left(\frac{\bar{p}_{FA} - \bar{p}_{FB}}{L} \right) z \quad (65)$$

The rate of eroded mass (mass obtained at the outlet section per unit of time) can be calculated based on (63) and (69):

$$\dot{m}_{er} = \hat{p}^F(z) V_{sample} = \rho_S S_{sample} (c_{pF} \mathbf{J}_F) \big|_{z=L} \quad (66)$$

where V_{sample} and S_{sample} are the specimen's volume and cross section area, respectively. Finally, the total cumulated eroded mass is integrated over time until $E_{seepage} \leq E_{max}$:

$$m_{er} = \int_t \dot{m}_{er} dt \quad (67)$$

Similarly, the porosity evolution (for the steady state solution) is obtained by substituting the solution (63) into the porosity equation

(54)d:

$$\frac{\partial n^F(z)}{\partial t} - (1 - n^F) \frac{\partial \epsilon(z)}{\partial t} = \frac{\hat{\rho}^F(z)}{\rho_S} = k_\alpha \frac{k_F}{\rho_F g \rho_S} \left(\frac{\bar{p}_{FA} - \bar{p}_{FB}}{L} \right)^2 \quad (68)$$

The equation (68) integrated over the time step Δt , on account of the steady state strain-pressure relation (57) $\epsilon(z) = \bar{p}_F(z)/(\lambda + 2\mu)$, gives the following simple solution:

$$\Delta n_F(z) = \frac{k_\alpha}{\rho_S} \frac{k_F}{\rho_F g} \left(\frac{\bar{p}_{FA} - \bar{p}_{FB}}{L} \right)^2 \Delta t \quad (69)$$

Recall that the porosity increment and the cumulated eroded mass increase until $E_{seepage} \leq E_{max}$. Beyond this threshold value, we assume that the rate of eroded mass becomes instantaneously zero which is a crude but realistic description of experimental observations (Marot, et al., 2012) (Rochim, et al., 2017) (Zhong, et al., 2018).

In this work, the rate of mass transfer is smoothed out, owing to the assumed constant coefficient of erodibility k_α in (50) (before reaching the threshold E_{max}), which explains that the porosity increase (68) depends on the time step until $E_{seepage} = E_{max}$. Although crude, this approach permitted the development of the above analytical solution that will improve our understanding of the suffusion process.

It is worth stressing that k_α , i.e. I_α , and E_{max} are the only material parameters which are used in the suffusion law. No other fitting parameters are introduced here. In addition, grain size distribution parameters are not included either. Yet, a link has been found between grain size based parameters and the resistance suffusion index I_α (Le, et al., 2017).

5 Suffusion tests

Suffusion has been investigated in the past (Zhong, et al., 2018) with tests on two different apparatuses under a flow in downward direction. It is worth stressing that for these tests, different histories of hydraulic loading were applied and two sizes of specimens were used.

5.1 Experimental devices and procedures

The smaller device consists essentially of a triaxial cell which is modified in order to inject a downward flow through the specimen. Specimen sizes are 50 mm in diameter and up to a 100 mm in height. With the objective to test specimens in oedometric conditions, the membrane of the triaxial erodimeter is surrounded by a steel mold. The larger device, named the oedopermeameter is composed of a 285 mm inner diameter rigid wall cylinder cell, and the specimen height can reach 600 mm. For both devices, the fluid circulates into the top cap, which contains a layer of gravel or glass beads to diffuse the fluid flow uniformly on the specimen top surface. Both cell bases have a vertical funnel shaped draining system specially designed to avoid clogging. Each draining system is connected to an effluent tank containing a rotating support with eight beakers to catch the eroded particles during testing. All specimens are placed on a sieve with 1.2 mm pore opening size that is fixed on a 10 mm mesh screen. According to the apparatus used, the range of flow rate varies; thus two configurations are used: at the overflow outlet of the triaxial erodimeter, water falls in a beaker that is continuously weighed, whereas a flowmeter is used in the case of the oedopermeameter. The hydraulic-controlled system is composed of a pressure controller connected to one upstream water tank for the triaxial erodimeter and two 200 L tanks, alternatively used in the oedopermeameter apparatus. The differential pore water pressure across the specimen is measured using a differential pressure transducer connected to the top cap and

the pedestal base. For each device, a dedicated computer operates the data acquisition thanks to a LabVIEW program developed by the authors.

For triaxial erodimeter tests, the specimens were prepared using a single-layer semistatic compaction technique to reach a fixed initial dry density. Within the oedopermeameter cell, the specimens were placed in three layers, and each layer was compacted, again targeting a prescribed initial dry density. The upward saturation of all specimens started by injection of carbon dioxide to improve dissolution of gases into water, and finally, it was completed by adding water. This step was performed with a low hydraulic gradient by increasing the level of the dedicated water tank until the water reached the air-release valve.

5.2 Testing materials and testing program

Soils that are likely to suffer from suffusion have a grain-size distribution curve either discontinuous or upwardly concave (Fell & Fry, 2007). Thus, both types of gradations were selected: upwardly concave (soil number 3) and gap graded (soils number 4, 5 and 6). With the objective to improve the readability, the first number of each test name is related to a soil gradation (Fig. 3). The letter indicates the used apparatus: O for oedopermeameter tests and T for triaxial erodimeter tests, and the last number details the specimen number. Table 1 indicates for the four tested specimens (named: 3T1, 4O, 5O and 6O2), the length of the specimen, the initial values of dry density and hydraulic conductivity. This table also specifies the values of the applied hydraulic gradient and the test duration.

5.3 Test results

The hydraulic conductivity of tested specimens and the erosion rate per unit cross section are shown on Figures 4 and 5, respectively.

The behavior of specimens 4O, 5O and 6O2 is comparable with erosion rates that vary between $5 \cdot 10^{-5} \text{ kg.s}^{-1}.\text{m}^{-2}$ and $2 \cdot 10^{-3} \text{ kg.s}^{-1}.\text{m}^{-2}$ and hydraulic conductivities that are all above 10^{-3} m/s . Specimens 4O and 6O2 display a slightly decreasing hydraulic conductivity which suggests a filtration, concomitant to the detachment process. The higher fine percentage of specimen 6O2, compared with 4O, seems to amplify this filtration process, as the hydraulic conductivity decreases more.

It can be observed that specimen 3T1 is a less permeable specimen (i.e. initial hydraulic conductivity is equal to $6 \cdot 10^{-5} \text{ m.s}^{-1}$) which can be explained by its gradation that reveals 8% of silt. At the beginning of the test, the rate of erosion mass is fairly low (smaller than $2 \cdot 10^{-5} \text{ kg.s}^{-1}.\text{m}^{-2}$) and little eroded mass is collected. Most of the eroded mass is detached towards the end of the test which is sometimes called a burst of particles. This is in agreement with the rate of eroded mass which becomes larger towards the end of the test (greater than $1 \cdot 10^{-4} \text{ kg.s}^{-1}.\text{m}^{-2}$). This result may be attributed to localized filtration within the specimen and also to a delayed measurement. Since the silty fines are often in suspension within the interstitial fluid, a certain time is required for those fine particles to exit and to settle in the outlet beakers so that the measured eroded mass is often delayed compared with the instantaneous eroded mass. This shortcoming mainly arise for silty and clay fines, and hence for specimen 3T1.

Finally these results highlight the complexity of suffusion, which appears as the combination of detachment, filtration and transport processes. Due to this coupling between erosion and filtration, the time evolutions of the hydraulic conductivity and of the erosion rate can be complex.

6 Numerical simulations and validation

To validate our rather simple linear suffusion law (50), our analytical solution is compared with the four laboratory suffusion tests. This linear law targets the smoothed evolution of the cumulated eroded mass. The specimen parameters used for the finite element and analytical solutions are reported in Tables 2 and 3. The finite element results are also presented as a preliminary validation of our numerical code. This code is been developed to investigate real dams or dikes-on-foundation problems that are often characterized by several soil layers with complex spatial distributions of the susceptibility and the hydraulic loading (Zhang, et al., 2018) for which no analytical solution can be found. Furthermore, finite element models provide a suitable environment to account for permeability reduction of soil due to the partial or definite blockage of some particles and for plastic deformations.

For each tests and for all plots, the finite element results match with the analytical solutions (Fig. 6 to 9). This is due to the fact that the hydraulic steady state is reached very quickly owing to the relatively large permeabilities and small seepage lengths. Regarding the hydraulic loading, the imposed inlet pressure is computed based on the measured hydraulic gradient, averaged over each loading stage. The ramp between two stages is imposed to two minutes which corresponds to the experimental observations. This loading procedure mimics quite well the experimentally imposed pressures to within the experimental scatter.

For each specimen, the numerical permeability is assumed to be constant and its value is detailed in Table 2. Based on these loadings and on *constant* numerical permeabilities, the Darcy velocities are now compared for each specimens. The differences observed between the analytical solutions and the experimental results are attributed to the evolving experimental permeabilities (Fig. 4), a phenomenon that is not accounted for in the analytical solutions. In fact, the Darcy velocities are best reproduced when experimental permeabilities are constant or slightly increasing (specimens 3T1 and 50). When filtration occurs, i.e. the permeability decreases with time in spite of an increase of the cumulated eroded mass, the Darcy velocities are not precisely reproduced (specimen 40) or even poorly reproduced (specimen 602). These conclusions raise the need of developing new permeability evolution laws that would improve our predictions. So far, the filtration susceptibility seems to be influenced by not only porosity (as represented via Kozeny-Carman type of relation) but also the initial grain size distribution curve (Figure 3). In fact, the larger the fine content and the smaller the gap ratio, the larger the filtration susceptibility. This will be step by step tackled in future work.

Since each power dissipated by the flow is computed from the pressure gradient and the Darcy velocity, the power history inherits the Darcy velocity's imperfection. Now, when the power dissipated by the flow is well reproduced (specimens 3T1 and 50), the total cumulated eroded mass is quite well predicted by the analytical solution. Conversely, if the power increase is over estimated, the total cumulated eroded mass is also over estimated. This tendency is particularly obvious when filtration is significant (specimen 602).

The porosity kinetic is the same as that of the cumulated eroded mass, to within a scaling factor, so that these results are presented in a compact way in Figures 6d, 7d, 8d and 9d. The experimental porosity evolution has been inferred from the initial material parameters, the measured eroded mass and the measured specimen's axial deformation. The finite element and analytical results of the porosity history account for both the rate of mass transfer term and for the volumetric deformation term. For both types of result, numerical or experimental, the deformation contribution is found negligible in front of the eroded mass contribution. Alike the cumulated eroded mass, the porosity history depends on the

correct estimation of the power. In addition, the magnitude of the porosity change due to suffusion, ranging from 1% (3T1) to 5,4% (6O2), is nice to have since it can be incorporated into a constitutive mechanical model (Scholtès, et al., 2010) (Rousseau, et al., 2018).

Overall, the new suffusion law allows a relatively good prediction of the cumulated eroded mass, i.e. all predictions lie within the experimental range of magnitude. In addition, the results are provided for four different materials, two different specimen sizes and various hydraulic loading paths to demonstrate the applicability of the new proposed law. This positive achievement is mainly due to the intrinsic character of the material parameter I_α that is independent of the specimen size (Zhong et al., 2018) and of the hydraulic loading path (Rochim et al., 2017).

7 Conclusions

In the present paper, a new poromechanical framework is proposed for the modelling of internal erosion in geomaterials. The framework is built on the basis of the mixture theory where internal erosion is modelled by mass transfer process. It takes into account different phenomena, such as the influence of mass transfer in constitutive and balance equations (including not only mass but also energy and momentum balance), thermal effect and grain compressibility. In particular, constitutive erosion law may be constructed based on thermodynamic condition and damage mechanics. The modelling framework is thus expected to be applicable to different internal erosion-related problems (sand production, soil suffusion...) in a wide range of geomaterials (soils, rocks...).

To illustrate the capability of this framework, a new model is developed particularly for soil suffusion phenomena. A new suffusion constitutive model is suggested based on both thermodynamic implication and experimental observation. The erosion evolution law is a function of the power of seepage flow. This new model is then applied to simulate laboratory experiments, by both analytical and finite element method. The comparison shows that the newly developed model can reproduce correctly the main effects of soil suffusion while theoretical consistency and simplicity (for analytical study) are still satisfied.

Here, the rate of eroded mass is linearly related to the power dissipated by the flow (per unit volume) to tackle a smoothed evolution. A more general model would account for a realistic rate of eroded mass kinetic, i.e. a sharp increase at the beginning of each loading step followed by a strong decrease (Sibille et al., 2015). A subsequent non-linear suffusion law will be addressed in a future work.

8 Acknowledgments

The authors thank the company EDF, France, for providing financial support for this work.

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Appendix A. Derivation of C-D inequality

From (13) we get:

$$\begin{aligned} T \left(\operatorname{div} \frac{\mathbf{q}_\beta}{T} - \frac{r_\beta}{T} \right) + \nabla T \cdot \operatorname{div} \mathbf{q}_\beta \\ = \hat{\rho}^\beta \left(\tilde{U}_\beta - U_\beta + \frac{1}{2} (\mathbf{V}_\beta - \tilde{\mathbf{V}}_\beta)^2 \right) + \hat{u}_\beta + \boldsymbol{\sigma}^\beta : \nabla \mathbf{V}_\beta - \frac{d(n^\beta \rho_\beta U_\beta)}{dt} \\ - \operatorname{div} (U_\beta \mathbf{M}_\beta) - n^\beta \rho_\beta U_\beta \operatorname{div} (\mathbf{V}_s) \end{aligned} \quad (70)$$

Substituting the quantity $\left(\operatorname{div} \frac{\mathbf{q}_\beta}{T} - \frac{r_\beta}{T} \right)$ from (70) into (15) then taking the summation over all the phases, we obtain after some classic derivations under the small deformation assumption (see for instance (Gelet, 2011)):

$$\begin{aligned} \Phi_{total} = - \frac{d(m^S E_S)}{dt} + p_F \frac{dv^F}{dt} + \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - m^S S_S \frac{dT}{dt} - \hat{\rho}^F \left(G_{pF} + \frac{1}{2} (\mathbf{V}_F - \mathbf{V}_S)^2 \right) - \mathbf{q} \frac{\nabla T}{T} \\ - J_F [\nabla p_F - \rho_F \mathbf{g}] \geq 0 \end{aligned} \quad (71)$$

Using assumption (4)b and definition (20) of the thermodynamic potential, we may rewrite easily:

$$\begin{aligned} \Phi_{total} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - v^{Fe} \dot{p}_F + p_F \dot{v}^{Fi} - m^S S_S \dot{T} - \hat{\rho}^F G_{pF} - \dot{\psi}_s - \frac{1}{2} (\mathbf{V}_F - \mathbf{V}_S)^2 \hat{\rho}^F - \mathbf{q} \frac{\nabla T}{T} \\ - J_F [\nabla p_F - \rho_F \mathbf{g}] \geq 0 \end{aligned} \quad (72)$$

Neglecting the dynamic term, we get the classic Clausius-Duhem (18).

Appendix B. Derivation of the field equations

Applying (5) to the solid phase, and accounting for (30), we may easily simplify (28) to obtain:

$$\frac{dn^F}{dt} - (\xi - n^F) \frac{d\epsilon^e}{dt} - c_s (\xi - n^F) \frac{dp_F}{dt} + c_T (\xi - n^F) \frac{dT}{dt} - \frac{\hat{\rho}^F}{\rho_s} = 0 \quad (73)$$

This is the porosity field equation which may be written in the concise form (36).

Similarly, applying (5) to the fluidized particle species, we get:

$$\frac{d(n^F c_{pF} \rho_s)}{dt} + n^{pF} \rho_s \operatorname{div} (\mathbf{V}_s) + \operatorname{div} (\rho_s c_{pF} \mathbf{J}_F) = \hat{\rho}^F \quad (74)$$

Which may be simplified after some manipulations:

$$\frac{d(n^F c_{pF})}{dt} + n^F c_{pF} \frac{d\rho_S}{\rho_S dt} + n^{pF} \text{div}(\mathbf{V}_s) + \text{div}(c_{pF} \mathbf{J}_F) + c_{pF} \mathbf{J}_F \frac{\nabla \rho_S}{\rho_S} = \frac{\hat{\rho}^F}{\rho_S} \quad (75)$$

Now recalling the classic *fluid state equations* (see (Gelet, 2011) for more detail):

$$\frac{\nabla \rho_S}{\rho_S} = c_{SH} \nabla p_F - c_{ST} \nabla T; \quad \frac{d\rho_S}{\rho_S} = c_{SH} dp_F - c_{ST} dT \quad (76)$$

Substituting (76) and (73) into (75) we obtain the concentration field equation (39).

Now using the same procedure for the pure water species:

$$\begin{aligned} \frac{d(n^F(1 - c_{pF}))}{dt} + n^F(1 - c_{pF}) \frac{d\rho_S}{\rho_S dt} + n^{wF} \text{div}(\mathbf{V}_s) + \text{div}((1 - c_{pF}) \mathbf{J}_F) + (1 - c_{pF}) \mathbf{J}_F \frac{\nabla \rho_S}{\rho_S} \\ = 0 \end{aligned} \quad (77)$$

Summing up the balance equations (75) and (77) for the fluid species, we derive the following equation:

$$\frac{d(n^F)}{dt} + n^F \frac{d\rho_S}{\rho_S dt} + n^F \text{div}(\mathbf{V}_s) + \text{div}(\mathbf{J}_F) + \mathbf{J}_F \frac{\nabla \rho_S}{\rho_S} = 0 \quad (78)$$

By means of (76), (73) and (26), the last equation may be rewritten to get the concise form of the pressure field equation (42).

The balance of energy (13), without neither any heat source supply r_β nor energy supply $\hat{u}_\beta$, may be written firstly for the solid phase as a whole:

$$\frac{d(m^S E_S)}{dt} + T \frac{d(m^S S_S)}{dt} + m_S S_S \frac{dT}{dt} - \boldsymbol{\sigma}^S : \frac{d\boldsymbol{\varepsilon}}{dt} + \text{div} \mathbf{q}_S = \hat{\rho}^\beta \frac{1}{2} (\mathbf{V}_S)^2 \quad (79)$$

By means of the definition of the potential (20), equations (22), (24), the volume content decomposition (4)b and the small deformation assumption, one may further expand the above equation:

$$T \frac{d(\rho^S S_S)}{dt} + T \rho^S S_S \text{div}(\mathbf{V}_S) + p_F \frac{dn^F}{dt} - \left(p_F - \frac{\partial \psi_s}{\partial v^{Fi}} \right) \frac{dv^{Fi}}{dt} + \text{div} \mathbf{q}_S = \hat{\rho}^\beta \frac{1}{2} (\mathbf{V}_S)^2 \quad (80)$$

Similar procedure applied to each fluid species “k” leads to:

$$\begin{aligned} T \frac{d(\rho^k S_k)}{dt} + T \rho^k S_k \text{div}(\mathbf{V}_S) + T \text{div}(\mathbf{M}_k S_k) - p_k \frac{dn^k}{dt} + \mathbf{J}_F (\nabla p_k - \rho_k \mathbf{g}) + \text{div} \mathbf{q}_S \\ = \hat{\rho}^\beta \left(\frac{1}{2} (\mathbf{V}_S)^2 - \frac{1}{2} (\mathbf{V}_k - \mathbf{V}_S)^2 - G_k \right) \end{aligned} \quad (81)$$

Note that the inertial terms may be neglected due to quasi-static condition. This writing allows to make the identifiable terms (via constitutive equations for solid and fluid phases) appear. Recall the fluid constitutive equation writes (see (Gelet, 2011)):

$$T \frac{d(S_k)}{dt} = -c_{kT} \frac{1}{\rho_k} T \frac{dp_k}{dt} + c_{p,k} \frac{1}{\rho_k} \frac{dT}{dt} \quad (82)$$

Summing up the above energy equations for solid and fluid phases, we obtain the final form of the energy balance for the whole mixture (47).

Specimen reference in paper	Specimen length (mm)	*Initial dry density γ_d (kN/m ³)	Applied hydraulic gradient i (-)	Initial hydraulic conductivity 10 ⁻³ (m/s)	Test duration (min)
3T1	100	17.00	0.11 to 4.65	0.06	215
40	437	15.87	0.04 to 0.16	37.83	167
50	440	16.65	0.04 to 0.26	12.70	147
602	435	16.73	0.04 to 0.42	5.64	310

Table 1. Properties of tested specimens and summary of testing program. *The initial dry density accounts for the mass loss and the volume change occurring during the saturation phase.

Specimen reference in paper	Initial porosity n_F	Averaged permeability k_F	Suffusion resistance index I_α	Maximum volumetric energy $E_{\max}$
	(-)	(m ²)	(s ² /m ²)	(J/m ³)
3T1	0.3462	$7.45 \cdot 10^{-12}$	3.60	36 853.02
40	0.3964	$3.6 \cdot 10^{-9}$	3.06	36 087.56
50	0.3598	$1.94 \cdot 10^{-9}$	3.36	42 886.16
602	0.3678	$3.75 \cdot 10^{-10}$	3.08	75 435.14

Table 2. Specimen parameters used for the finite element and the analytical solutions.

Material parameter	Value	Unit	Reference
Young modulus E	20	MPa	(Prat, et al., 1995)
Poisson ratio ν	0.3	-	(Prat, et al., 1995)
Water density ρ_w	999.7	Kg/m ³	*
Fluid dynamic viscosity μ_F	$1.0 \cdot 10^{-3}$	Pa.s	*
Solid grains density ρ_s	2600	Kg/m ³	*

Table 3. Material parameters used for the finite element and the analytical solutions. *Estimated parameters for water at 20°C and for cohesionless soil grains at medium density.