Some properties of the eigenvalues of the volume Wentzell-Laplace operator

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Abstract: In this paper, we consider the volume Wentzell-Laplace eigenvalues problem. In order to illustrate the behavior of the spectrum, we consider an explicit example about the ball in $\mathbb{R}^2$, where we find explicitly the eigenvalues and their corresponding eigenfunctions. After showing the analyticity of the eigenvalues λ and their corresponding eigenfunctions u in an open neighborhood of $t = 0$, we derive, in the sense of Hadamard, the first and second-order shape derivatives of the eigenvalues at time $t = 0$. Moreover, after deriving two new Pohožaev's identities, we show that the ball is not a critical shape of this eigenvalues problem, with and without a volume constraint, in any dimension.

Keywords: Wentzell-Laplace operator, first and second-order shape derivatives, Pohožaev's identities, non-critical shapes.

Contents

1 Introduction and statement of the main results

In this paper, we consider the eigenvalues problem for the Laplace operator with Wentzell boundary conditions defined on a connected open bounded subset Ω of $\mathbb{R}^d$ with C^2 boundary,

$$\left\{ \begin{array}{rll} -\Delta u & = \lambda u & \text{in } \Omega, \\ -\Delta_\tau u + \partial_{\mathbf{n}} u & = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where λ is an eigenvalue of the Laplace operator, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$, $\Delta_\tau := div_\tau(\nabla_\tau)$ is the Laplace-Beltrami operator defined on $\partial\Omega$, and the corresponding eigenfunction u belongs to the variational space

$$\mathcal{H}(\Omega) := \{u | u \in H^1(\Omega), u_{|\partial\Omega} \in H^1(\partial\Omega)\},$$

equipped with the $\|.\|_{\mathcal{H}(\Omega)}$ norm defined by:

$$\|u\|_{\mathcal{H}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} + \|\nabla_\tau u\|_{L^2(\partial\Omega)}.$$

The spectrum of the Laplacian with volume Wentzell conditions is discrete, i.e the spectrum consists in an increasing sequence

$$\lambda_0(\Omega) = 0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) ... \to \infty \tag{2}$$

with corresponding real orthonormal eigenfunctions $u_0, u_1, u_2, ...$ in $L^2(\Omega)$. The first eigenvalue is zero with constants as corresponding eigenfunctions. As usual, we adopt the convention that each eigenvalue is repeated according to its multiplicity. Hence, the first eigenvalue of interest is λ_1. A variational characterization of the eigenvalues is available.

We define on $\mathcal{H}(\Omega)$ the two bilinear forms

$$A(u,v) = \int_\Omega \nabla u . \nabla v \; dx + \int_{\partial\Omega} \nabla_\tau u . \nabla_\tau v \; d\sigma, \qquad B(u,v) = \int_\Omega uv, \tag{3}$$

where ∇_τ is the tangential gradient. Now, the two bilinear forms are positive and the variational characterization for the k-th eigenvalue is

$$\lambda_k(\Omega) = \min \left\{ \frac{A(v,v)}{B(v,v)}, \; v \in \mathcal{H}(\Omega), \; \int_\Omega v u_i = 0, \; i = 0, ..., k-1 \right\} \tag{4}$$

In particular, when $k = 1$, the minimum is taken over the functions orthogonal to the eigenfunctions associated to $\lambda_0 = 0$, i.e constant functions.

In order to illustrate the behaviour of the spectrum, we consider an explicit example: we study the eigenpairs of Wentzell-Laplacian operator in the case of a disk B_R of radius R centered at O in dimension two. Actually, one can refer to Antoine Henrot's book to see an example of the disk for the Laplacian with Dirichlet boundary conditions in dimension two (see [9, Proposition 1.2.14, p.11]). Now, we can state:

Proposition 1 *Let $\Omega = B_R$ be a disk of radius R in dimension two; then its eigenvalues and eigenfunctions for the Wentzell-Laplacian operator are:*

$$\lambda_{0,k} = \frac{j'^2_{0,k}}{R^2}, \qquad u_{0,k}(r,\theta) = J_0\left(\frac{j'_{0,k}}{R}r\right), \qquad k \geq 1, \tag{5}$$

$$\lambda_{n,k} = \frac{j'^2_{n,k}}{R^2} \text{ double eigenvalue}, \qquad u_{n,k}(r,\theta) = \begin{cases} J_n\left(\frac{j'_{n,k}}{R}r\right)\cos n\theta, \\ J_n\left(\frac{j'_{n,k}}{R}r\right)\sin n\theta, \end{cases} \qquad n,k \geq 1, \tag{6}$$

where $j'_{n,k}$ is the k-th zero of the Bessel function J'_n (the derivative of J_n).

In a second step, we start by studying the analyticity of the eigenvalues λ and their corresponding eigenfunctions u in an open neighborhood of $t = 0$, then we derive, in the sense of Hadamard, the first and second-order shape derivatives of the eigenvalues at time $t = 0$.

First of all, In order to study the regularity of the eigenpairs of the boundary problem (1), we introduce some fundamental notations (see subsection 2.1 for more details):

We consider the map $t \mapsto \varphi_t = Id + tV$ where $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ and t is small enough. We denote by

$$\Omega_t = \varphi_t(\Omega) = \{x + tV(x), \quad x \in \Omega\}.$$

After that, we study the transport of the Laplace, Laplace-Beltrami and the Dirichlet-to-Neumann operators from the moving domains onto a fixed one (at time zero). In fact, these transports are used in proving the regularity of the eigenpairs around zero of the Wentzell-Laplace problem (1) in Theorem 7 (see subsection 2.2), that says:

Let Ω be an open smooth bounded domain of $\mathbb{R}^d$. Assume that λ is an eigenvalue of multiplicity m of the Wentzell-Laplace operator. We suppose that $\varphi_t = I_d + tV$ for some $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ and denote $\Omega_t = \varphi_t(\Omega)$. Then there exists m real-valued continuous functions $t \mapsto \lambda_i(t)$, $i = 1, 2, ..., m$ and m functions $t \mapsto u_i^t \in H^{5/2}(\Omega)$ such that the following properties hold

1. $\lambda_i(0) = \lambda, \quad i = 1, ..., m,$
2. the functions $t \mapsto \lambda_i(t)$ and $t \mapsto u_i^t$, $i = 1, 2, ..., m$ are analytic in a neighborhood of $t = 0$.
3. The functions $u_{i,t}$ defined by $u_{i,t} \circ \varphi_t = u_i^t$are normalized eigenfunctions associated to $\lambda_i(t)$ on the moving domain Ω_t. If one considers a compact subset K such that $K \subset \Omega_t$ for all t small enough, then $t \mapsto (u_{i,t})_{|K}$ is also an analytic function of t in a neighborhood of $t = 0$.
4. Let $I \subset \mathbb{R}$ be an interval such that I contains only the eigenvalue λ of the Wentzell problem of multiplicity m. Then there exists a neighborhood of $t = 0$ such that $\lambda_i(t)$ $i = 1, ..., m$ are the only eigenvalues of Ω_t which belongs to I.

Actually, to prove Theorem 7, we follow the strategy of M. Dambrine, D. Kateb, and J. Lamboley in [7, Theorem 3.4].

Now, we are aiming to study the first order shape derivative calculus for the volume Wentzell-Laplace problem. If we refer to the literature, M. Dambrine, D. Kateb, and J. Lamboley in [7] have studied the shape derivatives of simple and multiple eigenvalues for the surface Wentzell-Laplace problem, where the authors Fabien Caubet, Marc Dambrine, and Rajesh Mahadevan in [8] have computed the shape derivatives of eigenvalues problem for elliptic operators associated to various boundary conditions, that is Dirichlet, Neumann, Robin and

Wentzell boundary conditions. In this paper, we represent the first order shape derivative calculus in another way.

Let us now begin to present our main theorems in this chapter. First, we derive in Theorem 1 the local variations of the eigenvalues and the eigenfunctions of our problem:

Theorem 1 ***(Characteristic equation in*** u'***).*** *With the notations of Theorem 7, if* $t \mapsto (\lambda(t), u_t)$ *is one of the smooth eigenpair path* $(\lambda_i(t), u_{i,t})$ *of* Ω_t *for the Wentzell problem* (1)*, then the shape derivative* $u' = (\partial_t u_t)_{|t=0}$ *of the eigenfunction* u_t *satisfies*

$$\begin{aligned} -\Delta u' &= \lambda' u + \lambda u' \qquad \text{in } \Omega, & (7)\\ -\Delta_\tau u' + \partial_n u' &= div_\tau(V_n \nabla_\tau u) + \Delta_\tau(V_n \partial_n u) \\ &\quad + div_\tau(V_n(HId - 2D^2 b)\nabla_\tau u) + \lambda V_n u \qquad \text{on } \partial\Omega, & (8)\end{aligned}$$

where $D^2 b$ *is the Weingarten operator defined on* $\partial\Omega$ *and* H *is the sum of the principle curvatures of the boundary.*

Let λ be a simple eigenvalue of the volume Wentzell-Laplace equation and let u be the corresponding normalized eigenfunction. We give in this subsection the explicit formula for the shape derivative of the volume eigenvalue of the Wentzell-Laplace operator.

On $\Omega_t = (Id + tV)(\Omega)$ with t small, there is a unique eigenvalue $\lambda(t)$ near λ which is an analytic function with respect to the parameter t. The associated eigenfunction $u_t(x) = u(t, x)$ is the solution of the problem . The shape derivative denoted u' is the partial derivative $\partial_t u(t, x)$ evaluated at $t = 0$ and solves (1). Let us deduce the analytic expression of $\lambda'(0)$:

Theorem 2 ***(First-order derivative of a simple eigenvalue).*** *If* (λ, u) *is an eigenpair (with* u *normalized) for the volume Wentzell problem with the additional assumption that* λ *is simple then the application* $t \mapsto \lambda(t)$ *is analytic and its derivative at* $t = 0$*, for all deformation* $V \in W^{3,\infty}(\mathbb{R}^d)$*, is*

$$\lambda'(0) = \int_{\partial\Omega} V_n(|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + \nabla_\tau u^T(HId - 2D^2 b)\nabla_\tau u) \tag{9}$$

where $D^2 b$ *is the Weingarten operator defined on* $\partial\Omega$ *and* H *is the sum of the principle curvatures of the boundary.*

If we suppose that λ is an eigenvalue of multiplicity m. For smooth deformation $t \mapsto \Omega_t$, there will be m eigenvalues close to λ (counting their multiplicities) for small values of t. We know that such a multiple eigenvalue is no longer differentiable in the classical sense. We are then led to compute the directional derivative of $t \mapsto \lambda_i(t)$ at $t = 0$ where $\lambda_i(t)$, $j = 1, ..., m$ are given by Theorem 7 (see subsection 2.2).

Theorem 3 ***(First-order derivative of a multiple eigenvalue).*** *Let* λ *be a multiple eigenvalue of order* $m \geq 2$*. Then each* $t \mapsto \lambda_i(t)$ *for* $i = 1, 2, ..., m$ *given by Theorem 7 has a derivative near* 0*, and the values of* $\lambda_i'(0)$*,* $i = 1, 2, ..., m$ *are the eigenvalues of the matrix* $M(V_n) = (M_{j,k})_{1 \leq j,k \leq m}$ *defined by*

$$M_{j,k} = \int_{\partial\Omega} V_n(\nabla_\tau u_j . \nabla_\tau u_k - \partial_n u_j \partial_n u_k - \lambda u_j u_k + (HId - 2D^2 b)\nabla_\tau u_j . \nabla_\tau u_k) \tag{10}$$

where $D^2 b$ *is the Weingarten operator defined on* $\partial\Omega$ *and* H *is the sum of the principle curvatures of the boundary.*

Now, we study some second-order shape derivative computations. This kind of studies has been dealt with in [7] for the surface version of the Wentzell-Laplace problem.

Let us begin with deriving the characteristic equation in $u'' = \left(\dfrac{d^2 u}{dt^2}\right)_{|t=0}$ in the direction $V \in W^{3,\infty}(\mathbb{R}^d)$.

Theorem 4 ***(Characteristic equation in*** u''***).*** *With the notations of Theorem 7, if* $t \mapsto (\lambda(t), u_t)$ *is one of the smooth eigenpair path* $(\lambda_i(t), u_{i,t})$ *of* Ω_t *for the Wentzell problem* (1)*, then the second shape derivative* $u'' = (\partial_t^2 u_t)_{|t=0}$ *of the eigenfunction* u_t *satisfies*

$$-\Delta u'' = \lambda'' u + 2\lambda' u' + \lambda u'' \qquad \text{in } \Omega,$$

$$\begin{aligned}-\Delta_\tau u'' + \partial_n u'' =& div_\tau(A) + \Delta_\tau(B) + \nabla_\tau u'.\nabla_\tau V_n + V_n\lambda' u + V_n\lambda u' + V_n\Delta_\tau u' \\ &+ V_n'\lambda u + V_n\lambda' u + V_n\lambda u' + V_n\lambda\partial_n V_n u + V_n^2\lambda u\partial_n u + V_n^2 H\lambda u \qquad \text{on } \partial\Omega.\end{aligned}$$

Where

$$\begin{aligned}A :=& V_n[\nabla_\tau(\partial_n u') - 2D^2b\nabla_\tau u' + H\nabla_\tau u' + \nabla_\tau u' + \partial_n u\nabla_\tau V_n] \\ &+ V_n[\partial_n V_n\nabla_\tau u + \partial_n u\mathcal{A}\nabla_\tau V_n + \mathcal{A}\nabla_\tau u' + \partial_n u\mathcal{A}\nabla_\tau V_n + \mathcal{A}'\nabla_\tau u] \\ &+ V_n^2[\nabla_\tau(\partial_n u) - 2D^2b\nabla_\tau u + H\nabla_\tau u + 2\mathcal{A}\nabla_\tau(\partial_n u) - 2D^2b\mathcal{A}\nabla_\tau u + \tilde{\mathcal{A}}\nabla_\tau u + H\mathcal{A}\nabla_\tau u] \\ &+ \nabla_\tau V_n\partial_n u' + V_n'\nabla_\tau u + V_n'\mathcal{A}\nabla_\tau u + \partial_n V_n\mathcal{A}\nabla_\tau u,\end{aligned}$$

and

$$B := V_n'\partial_n u + V_n(\partial_n u' - \nabla_\tau V_n.\nabla_\tau u + \partial_n V_n\partial_n u) - V_n^2(\lambda u + \partial_n u + H\lambda u),$$

where D^2b *is the Weingarten operator defined on* $\partial\Omega$*,* H *is the sum of the principle curvatures of the boundary,* $\mathcal{A} := (HId - 2D^2b)$ *and we denote by* $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1\le i,j\le d}$ *the matrix whose elements are the normal derivatives of* $a_{i,j}$*, where* $a_{i,j}$*,* $i, j = 1, ..., d$*, are the components of the matrix* $\mathcal{A}$*.*

We give in Theorem 5 an explicit formula for the second-order shape derivative of a simple eigenvalue λ at $t = 0$, for all $V \in W^{3,\infty}(\mathbb{R}^d)$, that is:

Theorem 5 ***(Second-order derivative of a simple eigenvalue).*** *Let* $(u_t, \lambda(t))$ *be a smooth eigenpair path of* (1) *on* Ω_t*. If* $\lambda(t)$ *is a simple eigenvalue, then the second-order derivative of* $\lambda(t)$ *at* $t = 0$ *in direction* $V \in W^{3,\infty}(\mathbb{R}^d)$ *is:*

$$\begin{aligned}\lambda''(0) =& \int_{\partial\Omega}(V_n' + V_n\partial_n V_n + V_n^2 H)\left[|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + \mathcal{A}\nabla_\tau u.\nabla_\tau u\right] \\ &+ \int_{\partial\Omega} V_n\left[2(Id + \mathcal{A})\nabla_\tau u.\nabla_\tau u' + 2(2Id + \mathcal{A})\nabla_\tau V_n.\nabla_\tau u\partial_n u\right] \\ &+ \int_{\partial\Omega} V_n\left[-2\partial_n u\partial_n u' - \lambda'|u|^2 - 2\lambda u u' + \mathcal{A}'\nabla_\tau u.\nabla_\tau u\right] \\ &+ \int_{\partial\Omega} V_n^2\left[2(Id + \mathcal{A})\nabla_\tau(\partial_n u).\nabla_\tau u - (2D^2b + 2D^2b\mathcal{A} - \tilde{\mathcal{A}})\nabla_\tau u.\nabla_\tau u + 2(1 + H)|\partial_n u|^2\right]\end{aligned}$$

where D^2b *is the Weingarten operator defined on* $\partial\Omega$*,* H *is the sum of the principle curvatures of the boundary,* $\mathcal{A} := (HId - 2D^2b)$ *and we denote by* $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1\le i,j\le d}$ *the matrix whose elements are the normal derivatives of* $a_{i,j}$*, where* $a_{i,j}$*,* $i, j = 1, ..., d$*, are the components of the matrix* $\mathcal{A}$*.*

In the last step, we give two Pohožaev's identities, which will be used for proving that the ball is not a critical shape of this eigenvalues problem (1), with and without a volume constraint, in any dimension. We remark that in [7], the authors have shown that the balls are critical domains, in any dimension, where they have used the second-order shape derivative calculus in order to investigate that they are local maximizers in dimension 2 and 3, using an order two sensitivity analysis.

The following theorem allows us to state two new Pohožaev's identities.

Theorem 6 ***(Pohožaev's identities).*** *Let Ω be a C^2 domain, and let u be an eigenfunction of problem* (1).

1. *Let X be a translation vector field, i.e $X(x) = x + \alpha$ for all $x, \alpha \in \mathbb{R}^d$, and let D^2u be the Hessian matrix of u. We have*
$$\int_{\partial\Omega} (D^2u + \partial_n u Id)X.\nabla_\tau u + \int_{\partial\Omega} |\nabla_\tau u|^2 + \int_{\partial\Omega} |\partial_n u|^2 (X.n) = 0. \tag{11}$$
2. *If D^2b is the Hessian matrix of the signed distance function b to $\partial\Omega$, then we have*
$$\int_{\partial\Omega} D^2b \nabla_\tau u.\nabla_\tau u + \int_{\partial\Omega} D^2u\ n.\nabla_\tau u + \int_{\partial\Omega} |\partial_n u|^2 = 0. \tag{12}$$

Now, let us give the definition of a critical shape:

Definition 1 *A domain Ω is said to be critical if there exist an eigenvalue λ such that its derivative $\lambda'(\Omega, V)$, in any direction $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$, is zero.*

The following corollary shows the non-criticality of the balls in any dimension:

Corollary 1 *The balls in $\mathbb{R}^d$, with and without volume constraint, are not critical.*

To illustrate, we show the plan of this paper in Figure 1.

2 Prerequisites

2.1 Notations and preliminaries for shape deformation

We adopt the formalism of Hadamard's shape calculus and consider the map $t \mapsto \varphi_t = I_d + tV$ where $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$ and t is small enough. We denote by

$$\Omega_t = \varphi_t(\Omega) = \{x + tV(x), \quad x \in \Omega\}.$$

Remark 1 *More generally the results and computations from this section are valid if $t \mapsto \varphi_t$ satisfies:*

1. *$\varphi_0 = I_d$,*
2. *for every t near 0, φ_t is a $W^{3,\infty}$-diffeomorphism from Ω onto its image $\Omega_t = \varphi_t(\Omega)$.*
3. *The application $t \mapsto \varphi_t$ is real-analytic near $t = 0$.*

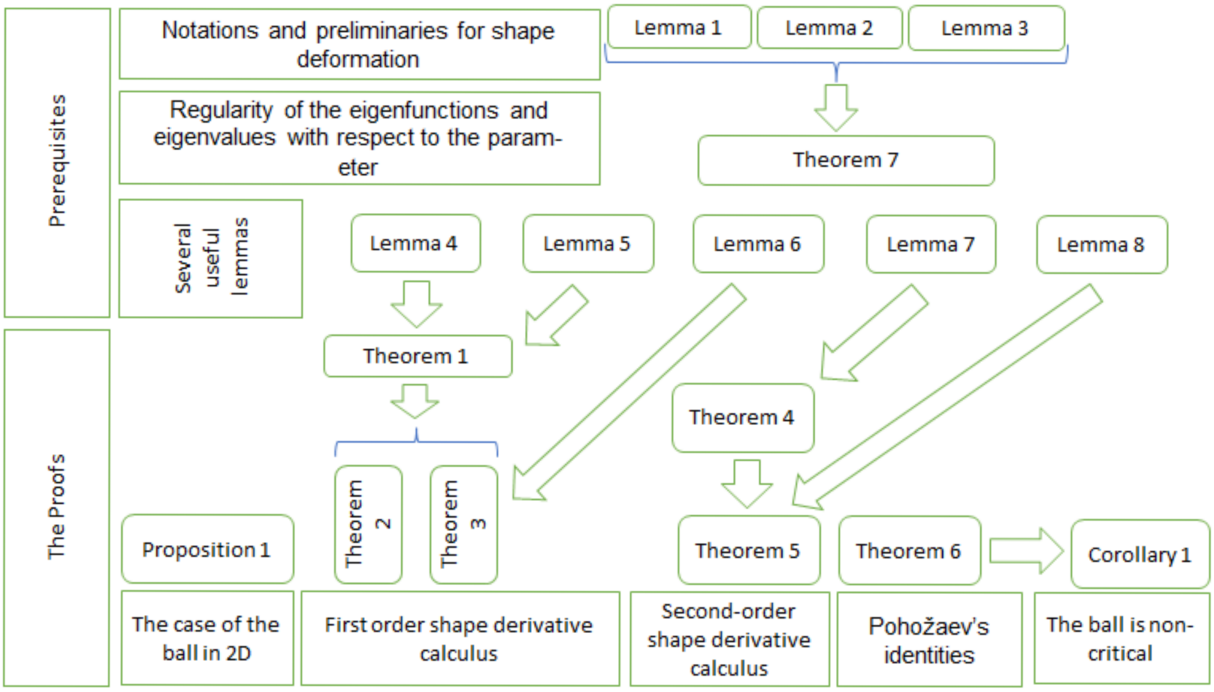

Figure 1: The map of the paper

We need to introduce the surface jacobian w_t defined as

$$w_t(x) = det(D\varphi_t(x))\|(D\varphi_t(x)^T)^{-1}\ n(x)\|,$$

and the functions

$$A_t(x) = (D\varphi_t(x))^{-1}(D\varphi_t(x))^{-1}, \quad J_t(x) = det(D\varphi_t(x))$$

$$\tilde{\mathcal{A}}_t(x) = J_t(x)A_t(x), \quad C_t(x) = w_t(x)A_t(x).$$

We have to study the transport of the considered eigenvalue problem on the deformed domain Ω_t. To that end, we first rewrite the deformed equation on the fixed domain Ω and its boundary $\partial\Omega$: we have to describe how are transported the Laplace-Beltrami and the Dirichlet-to-Neumann operators.

Transport of the Dirichlet-to-Neumann map. Let us consider the Dirichlet-to-Neumann operator defined on its natural space $D_t : H^{1/2}(\partial\Omega_t) \mapsto H^{-1/2}(\partial\Omega_t)$. It maps a function ϕ_t in $H^{1/2}(\partial\Omega_t)$ onto the normal derivative of its harmonic expansion in Ω_t, that is to say $D_t(\phi_t) = \partial_{n_t} u_t$, where u_t solves the boundary values problem:

$$\left\{ \begin{array}{rcl} -\Delta u_t & = & 0 \text{ in } \Omega_t, \\ u_t & = & \phi_t \text{ on } \partial\Omega_t. \end{array} \right. \tag{13}$$

To compute the quantity $\mathcal{D}_t$ such that $\mathcal{D}_t(\phi_t \circ \varphi_t) = [D_t(\phi_t)] \circ \varphi_t$, we transport the boundary value problem (13) back on the domain Ω.

To be more precise, we have the following result proved in [4].

Lemma 1 *Given $\psi \in H^{1/2}(\partial\Omega)$, we denote v^t the solution of the boundary value problem*

$$\left\{ \begin{array}{rcl} -div(\tilde{\mathcal{A}}_t \nabla v^t) & = & 0 \text{ in } \Omega, \\ v^t & = & \psi \text{ on } \partial\Omega. \end{array} \right. \tag{14}$$

and then define $\mathcal{D}_t\psi \in H^{-1/2}(\partial\Omega)$ as

$$\mathcal{D}_t\psi : f \in H^{1/2}(\partial\Omega) \mapsto \int_\Omega \tilde{\mathcal{A}}_t(x)\nabla v^t(x).\nabla E(f)(x)dx,$$

where E is a continuous extension operator from $H^{1/2}(\partial\Omega)$ to $H^1(\Omega)$. Then the relation

$$(D_t\psi) \circ \varphi_t = \mathcal{D}_t[\psi \circ \varphi_t]$$

holds for all functions $\psi \in H^{1/2}(\Omega_t)$.

Setting $u^t = u_t \circ \varphi_t$, we check from the variational formulation, that the function u^t is the unique solution of the transported boundary value problem:

$$\left\{ \begin{array}{rcl} -div(\tilde{\mathcal{A}}_t \nabla u^t) & = & 0 \text{ in } \Omega, \\ u^t & = & \phi_t \circ \varphi_t \text{ on } \partial\Omega. \end{array} \right. \tag{15}$$

Hence, setting $y = \varphi_t(x)$, $x \in \Omega$ we get formally

$$D_t(\phi_t)(y) = \nabla u_t(y).n_t(y) = (D\varphi_t(x))^{-1}\nabla v_t(x).\frac{(D\varphi_t(x))^{-1}n(x)}{\|(D\varphi_t(x))^{-1}n(x)\|} = \frac{A_t(x)n(x).\nabla u^t(x)}{\|(D\varphi_t(x))^{-1}n(x)\|}.$$

Here again, we can give a sense to the co-normal derivative $A_t n.\nabla u^t$ thanks to the boundary value problem (15): this quantity is defined in a weak sense as the previous Dirichlet-to-Neumann operator D_t.

Transport of the Laplace-Beltrami operator. We recall now the expression of the transported Laplace-Beltrami operator, relying on the relation

$$\forall \phi \in H^2(\partial\Omega_t), \quad (\Delta_\tau \phi) \circ \varphi_t = \frac{1}{w_t(x)} div_\tau [C_t(x) \nabla_\tau (\phi \circ \varphi_t)(x)] \quad on\ \partial\Omega.$$

Let us denote by $\mathcal{L}_t$ the operator defined as

$$\mathcal{L}_t[\phi \circ \varphi_t](x) = \frac{1}{w_t(x)} div_\tau \left\{ C_t(x) \nabla (\phi \circ \varphi_t)(x) - \frac{C_t(x) \nabla (\phi \circ \varphi_t)(x).n(x)}{A_t(x) n(x).n(x)} A_t(x) n(x) \right\}$$

for $\phi \in H^{5/2}(\Omega_t)$. In [4], they have showed the following lemma:

Lemma 2 *The identity*

$$[\Delta_\tau \phi] \circ \varphi_t = \mathcal{L}_t[\phi \circ \varphi_t]$$

holds for all functions ϕ belonging to $H^{5/2}(\Omega_t)$.

Transport of the Laplace operator. In the following, we transport back the Laplace operator Δ defined on $H^{5/2}(\Omega_t)$ into a stable one L_t that takes variables in $H^{5/2}(\Omega)$, where

$$\forall x \in \Omega, \quad L_t(x) := \frac{1}{J_t(x)} div[\tilde{A}_t(x) \nabla \phi^t(x)]$$

and for $\phi_t \in H^{5/2}(\Omega_t)$, $\phi^t = \phi_t \circ \varphi_t$ is the back transport of ϕ_t onto $H^{5/2}(\Omega)$.

Actually, we can prove the following lemma:

Lemma 3 *The identity*

$$[\Delta \phi_t] \circ \varphi_t = L_t[\phi_t \circ \varphi_t]$$

holds for all functions ϕ_t belonging to $H^{5/2}(\Omega_t)$.

Proof: Let $\phi_t \in H^{5/2}(\Omega_t)$. We fix a test function $\psi_t \in H^{5/2}(\Omega_t)$.

By the variational definition of the Laplace operator, if v_t is the trace of ϕ_t on $\partial\Omega_t$, we get

$$\begin{aligned} \int_{\Omega_t} \Delta \phi_t(y) \psi_t(y) &= -\int_{\Omega_t} \nabla \phi(y).\nabla \psi_t(y) + \int_{\partial\Omega_t} \partial_n v_t(y) \psi_t(y) \\ &= -\int_{\Omega_t} \nabla \phi_t(y).\nabla \psi_t(y) + \int_{\partial\Omega_t} D_t \phi_t(y) \psi_t(y). \end{aligned}$$

By the change of variables $y = \varphi_t(x)$, we get integrals defined in the fixed domain Ω and its boundary $\partial\Omega$. Therefore, thanks to the back transport of the Dirichlet-to-Neumann operator $\mathcal{D}_t$, we get

$$\begin{aligned} \int_\Omega ((\Delta \phi_t) \circ \varphi_t)(x) \psi^t(x) J_t(x) &= -\int_\Omega \nabla \phi^t(x).\nabla \psi^t(x) \tilde{A}_t(x) + \int_{\partial\Omega} \mathcal{D}_t(\phi_t \circ \varphi_t)(x) \psi^t(x) w_t(x) \\ &= -\int_\Omega \nabla \phi^t(x).\nabla \psi^t(x) \tilde{A}_t(x) + \int_{\partial\Omega} \frac{A_t(x)\ \mathbf{n}(x).\nabla \phi^t(x)}{\|(D\varphi_t(x))^{-1}\ \mathbf{n}(x)\|} \psi^t(x) w_t(x) \\ &= -\int_\Omega \nabla \phi^t(x).\nabla \psi^t(x) \tilde{A}_t(x) + \int_{\partial\Omega} \tilde{A}_t(x) \nabla \phi^t(x).\mathbf{n}(x) \psi^t(x). \end{aligned}$$

Thus, by the divergence theorem, we obtain

$$\begin{aligned}\int_\Omega ((\Delta\phi_t)\circ\varphi_t)(x)\psi^t(x)J_t(x) &= -\int_\Omega \nabla\phi^t(x).\nabla\psi^t(x)\tilde{A}_t(x) + \int_\Omega div\left[\tilde{A}_t(x)\nabla\phi^t(x)\psi^t(x)\right]\\ &= -\int_\Omega \nabla\phi^t(x).\nabla\psi^t(x)\tilde{A}_t(x) + \int_\Omega div\left(\tilde{A}_t(x)\nabla\phi^t(x)\right)\psi^t(x)\\ &+ \int_\Omega \tilde{A}_t(x)\nabla\phi^t(x).\nabla\psi^t(x)\\ &= \int_\Omega div\left(\tilde{A}_t(x)\nabla\phi^t(x)\right)\psi^t(x) = \int_\Omega L_t[\phi_t\circ\varphi_t](x)J_t(x)\psi^t(x).\end{aligned}$$

Hence, we get the result:

$$\forall\psi^t \in H^{5/2}(\Omega), \quad [\Delta\phi_t]\circ\varphi_t = L_t[\phi_t\circ\varphi_t].$$

□

2.2 Regularity of the eigenfunctions and eigenvalues with respect to the parameter

We present now a standard theorem of the regularity of eigenvalues and associated eigenfunctions with respect to the parameter t. The section is a slight variation of a theorem due to Ortega and Zuazua on the existence and regularity of eigenvalues and associated eigenfunctions in the case of the Stokes system [6]. The difficulty comes from the possible multiple eigenvalues. The main result is, for a fixed deformation field $V \in W^{3,\infty}(\Omega,\mathbb{R}^d)$, the existence of smooth branches of eigenvalue. In other words, the eigenvalues are not regular when sorted in increasing order, but can be locally relabeled around the multiple point in order to remain smooth. The restriction is that this labeling depends on the deformation field V hence one cannot hope to prove Frechet-differentiability:

Theorem 7 *Let Ω be an open smooth bounded domain of $\mathbb{R}^d$. Assume that λ is an eigenvalue of multiplicity m of the Wentzell-Laplace operator. We suppose that $\varphi_t = I_d + tV$ for some $V \in W^{3,\infty}(\Omega,\mathbb{R}^d)$ and denote $\Omega_t = \varphi_t(\Omega)$. Then there exists m real-valued continuous functions $t \mapsto \lambda_i(t)$, $i = 1,2,...,m$ and m functions $t \mapsto u_i^t \in H^{5/2}(\Omega)$ such that the following properties hold*

1. *$\lambda_i(0) = \lambda, \quad i = 1,...,m,$*
2. *the functions $t \mapsto \lambda_i(t)$ and $t \mapsto u_i^t$, $i = 1,2,...,m$ are analytic in a neighborhood of $t = 0$.*
3. *The functions $u_{i,t}$ defined by $u_{i,t}\circ\varphi_t = u_i^t$ are normalized eigenfunctions associated to $\lambda_i(t)$ on the moving domain Ω_t. If one considers a compact subset K such that $K \subset \Omega_t$ for all t small enough, then $t \mapsto (u_{i,t})_{|K}$ is also an analytic function of t in a neighborhood of $t = 0$.*
4. *Let $I \subset \mathbb{R}$ be an interval such that $\bar{I}$ contains only the eigenvalue λ of the Wentzell problem of multiplicity m. Then there exists a neighborhood of $t = 0$ such that $\lambda_i(t)$ $i = 1,...,m$ are the only eigenvalues of Ω_t which belongs to I.*

Proof: We follow the method used by M. Dambrine, D. Kateb and J. Lamboley (see [7, Theorem 3.4]), adapting ourselves to our volume Wentzell-Laplace problem, where we focus on the Laplace operator studied in the volume of Ω_t. To start with, Let λ be an eigenvalue of multiplicity m and let $u_1,...,u_m$ be the orthonormal eigenfunctions associated to λ. Let $(\lambda(t),u_t)$ be an eigenpair satisfying

$$\begin{aligned} -\Delta u_t &= \lambda(t) u_t \quad \text{in } \Omega_t, \\ -\Delta_\tau u_t + \partial_n u_t &= 0 \quad \text{on } \partial\Omega_t. \end{aligned} \tag{16}$$

Setting $u^t = u_t \circ \varphi_t$, Lemma 3 (transport of the Laplace operator) shows that the system (16) above is equivalent to the following equation set in the volume

$$-L_t u^t = \lambda(t) J_t(x) u^t \quad \text{in } \Omega, \tag{17}$$

Consider the operator $S(t)$ defined on $H^{5/2}(\Omega)$ by

$$v \mapsto S(t)v = -L_t v \tag{18}$$

From the regularity assumption on φ_t, the operator A_t is analytic in a neighborhood of $t = 0$. Since $J_t > 0$ for t small enough, we deduce that all the expressions involved in L_t are analytic in a neighborhood of $t = 0$. This enables us to conclude that $S(t)$ is also analytic in a neighborhood of zero.

To show that the eigenvalues and the corresponding eigenfunctions are analytic in a neighborhood of zero, we apply the Lyapunov-Schmidt reduction in order to treat a problem on a finite dimensional space, namely the kernel of $S(0) - \lambda I$. To that end, we rewrite the problem (16) in the fixed domain Ω as

$$S(t)(u^t) - \lambda(t) J_t(x) u^t = 0.$$

From the decomposition

$$(S(0) - \lambda)(u^t) = [(S(0) - S(t)) + [(\lambda(t) - \lambda)J_t + \lambda(J_t - 1)]u^t,$$

u^t is solution of the equation

$$(S(0) - \lambda)(u^t) = W(t, \lambda(t) - \lambda)u^t, \tag{19}$$

where we have set $R(t) = S(0) - S(t) + \lambda(J_t - 1)$ and $W(t, \alpha) = R(t) + \alpha J_t I$. From the Lyapunov-Schmidt Theorem (see [6, Lemma 3-2, p. 999]), we obtain that $S(0) - \lambda$ has a right inverse operator denoted by K. Hence the equation above implies that

$$u^t = KW(t, \lambda(t) - \lambda)u^t + \psi_t$$

where $\psi_t \in \ker(S(0) - \lambda)$, i.e $\psi_t = \sum_{k=1}^m c_k(t)\phi_k$ where (ϕ_k) is a basis of $\ker(S(0) - \lambda)$. Notice that $I - KW(t, \lambda(t) - \lambda)$ is invertible on $\ker(S(0) - \lambda I)$, the inverse of his operator restricted to this kernel will be denoted by $(I - KW(t, \lambda(t) - \lambda))^{-1}$ so that

$$u^t = (I - KW(t, \lambda(t) - \lambda))^{-1}\psi_t.$$

From (19), $W(t, \lambda(t) - \lambda)u^t$ belongs to $\mathrm{Im}(S(0) - \lambda) = \ker^\perp(S(0) - \lambda)$ since $S(0)$ is a Fredholm self-adjoint operator, and then

$$\sum_{k=1}^m c_k(t)\langle W(t, \lambda(t) - \lambda)(I - KW(t, \lambda(t) - \lambda))^{-1}\phi_k, \phi_i\rangle = 0, \quad i = 1, 2, ..., m, \tag{20}$$

where $\langle ., . \rangle$ denote the scalar product of $L^2(\Omega)$. This shows that a vector of coefficients $C = (c_j)_{j=1,...,m} \neq 0$ is a solution if and only if the determinant of the $m \times m$ matrix $M(t, \lambda(t) - \lambda)$ with entries

$$M(t, \alpha)_{i,j} = \langle W(t, \alpha)(I - KW(t, \alpha))^{-1}\phi_j, \phi_i\rangle$$

satisfies

$$\det(M(t,\lambda(t)-\lambda))=0.$$

Hence $\lambda(t)$ is an eigenvalue of our problem if and only if $\det(M(t,\lambda(t)-\lambda))=0$. Note that $t\mapsto M(t,\lambda(t))$ is analytic around $t=0$.

For small values of t the operator $(I-KW(t,\alpha))^{-1}$is well defined since $I-KW(0,0)=I$ and $t\mapsto (I-KW(t,\alpha))^{-1}$ is analytic around $t=0$. On the other hand, if $\det M(t,\alpha)=0$ then (20) has a nontrivial solution $c_1(t),...,c_m(t)$ and this means that $\lambda(t)=\lambda+\alpha$ is an eigenvalue of (16).

We focus now on $\det(M(t,\alpha))$ for $\alpha\in\mathbb{R}$. From the fact that $W(0,\alpha)=\alpha I$, it comes that for sufficiently small values of α, the operator $I-KW(0,\alpha)$ is invertible on $\ker(S(0)-\lambda I)$ and from the Von Neumann expansion we write

$$\langle W(0,\alpha)(I-KW(0,\alpha))^{-1}\phi_i,\phi_j\rangle=\alpha[\delta_{ij}+\sum_{k=1}^{\infty}\alpha^k\langle K^k\phi_i,\phi_j\rangle];$$

hence

$$\det(M(0,\alpha))=\alpha^m+\sum_{i=1}^{\infty}\beta_i\alpha^{m+1}=\alpha^m(1+\sum_{i=1}^{\infty}\beta_i\alpha^m).$$

Since $\det(M(0,\alpha))\neq 0$ is the restriction on $t=0$ of $\det(M(t,\alpha))$, we deduce from the Weierstrass preparation theorem that there is neighborhood of $(0,0)$ such that $\det(M(t,\alpha))$ is uniquely representable as

$$\det(M(t,\alpha))=P_m(t,\alpha)h(t,\alpha)$$

where

$$P_m(t,\alpha)=\alpha^m+\sum_{i=1}^{\infty}\beta_i\alpha^{m+1}=\alpha^m+\sum_{k=1}^{m}a_k(t)\alpha^{m-k}$$

and where

$$h(t,\alpha)\neq 0.$$

Furthermore, the coefficients $a_k(t)$, $k=1,...,m$ are real and analytic in a neighborhood of $t=0$. Then $\det(M(t,\alpha))=0$ if and only if $P_m(t,\alpha)=0$. If $a_k(t),k=1,...,m$ are the real roots of the polynomial, we take $\lambda_1(t)=\lambda+\alpha_1(t)$ if $\alpha_1(t)$ is not identically equal to zero.

We now have to find the $(m-1)$ other branches $\lambda_i(t)$ and the corresponding eigenfunction $u_{i,t}$ for $i=2,...,m$. We use the idea of the deflation method by considering the operator

$$S_2(t)=S(t)-\lambda_1 P_1(t)$$

where P_1 is the orthogonal projection on the subspace spanned by $u_{1,t}$. At $t=0$, we obtain

$$S_2(0)u_j=S(0)u_j-\lambda\delta_{1j}u_j$$

in other terms $S_2(0)u_j=\lambda u_j, j=2,...,m$ while $S_2(0)u_1=0$. This shows that λ is an eigenvalue of multiplicity $m-1$ of $S_2(0)$ with eigenvalues $u_2,...,u_m$. One can show that these functions are the only linearly independent eigenfunctions associated to λ. Now we can apply the same recipe used before to the operator S_2 instead of S. We then get a branch $\lambda_2(t)$ such that $t\mapsto\lambda_2(t)$ is analytic in a neighborhood of $t=0$.

Iterating the process, we get at the end the m-branches $\lambda_i(t)$, $i=1,...,m$ such that each branch is analytic in a neighborhood of $t=0$ and m corresponding eigenfunctions forming an orthonormal set of functions in $H^{5/2}(\Omega_t)$.

The proof of the last item follows the same lines as the proof of Ortega and Zuazua for the Stokes system, see [6].

□

2.3 Several useful lemmas

In this subsection, we will give several lemmas that are needed in our work. Beginning from the first order shape derivative calculations for a simple eigenvalues, passing through the multiple eigenvalue case with its corresponding orthonormal eigenfunctions in $L^2(\Omega)$, and reaching the second-order shape derivative computations for simple eigenvalues, where some additional derivation equations are derived.

Lemma 4 *Let $t \mapsto (\lambda(t), u_t)$ is a smooth eigenpair path on Ω_t for the volume Wentzell problem (1), and $u' = (\partial_t u_t)_{|t=0}$ is the derivative of the eigenfunction u, then for all $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$, we have at $t = 0$:*

$$2\int_\Omega u\ u' + \int_{\partial\Omega} |u|^2 V_n = 0, \tag{21}$$

$$\int_{\partial\Omega} \partial_n u\ u\ V_n = -\int_{\partial\Omega} \nabla_\tau V_n\ \nabla_\tau u\ u - \int_{\partial\Omega} |\nabla_\tau u|^2 V_n, \tag{22}$$

and

$$\int_{\partial\Omega} \nabla_\tau(V_n \partial_n u).\nabla_\tau u = \int_{\partial\Omega} \partial_n u \nabla_\tau V_n.\nabla_\tau u + \int_{\partial\Omega} V_n \nabla_\tau(\partial_n u).\nabla_\tau u, \tag{23}$$

where V_n denotes the usual scalar product $\langle V, n\rangle$.

Proof: The first line comes from the fact that u_t is normalize in $L^2(\Omega_t)$, for all $t > 0$, i.e. we have

$$\int_{\Omega_t} |u_t|^2 = 1.$$

Making the shape derivative we obtain

$$\frac{d}{dt}\left(\int_{\Omega_t} |u_t|^2\right)_{|t=0} = 0,$$

that is

$$2\int_\Omega u\ u' + \int_{\partial\Omega} |u|^2 V_n = 0.$$

For the second line, it is sufficient to use the boundary condition $\Delta_\tau u = \partial_n u$ in (1) and make integration by parts:

$$\int_{\partial\Omega} \partial_n u\ u\ V_n = \int_{\partial\Omega} \Delta_\tau u\ u\ V_n = -\int_{\partial\Omega} \nabla_\tau u\ \nabla_\tau(uV_n) = -\int_{\partial\Omega} \nabla_\tau V_n\ \nabla_\tau u\ u - \int_{\partial\Omega} |\nabla_\tau u|^2 V_n.$$

The last line is trivial.

□

Lemma 5 *Let $t \mapsto (\lambda(t), u_t)$ is a smooth eigenpair path of Ω_t for the volume Wentzell problem (1), and $u' = (\partial_t u_t)_{|t=0}$ is the derivative of the eigenfunction u, then we have at $t = 0$:*

$$\lambda'(0) = \int_{\partial\Omega} \partial_n u\ u' - \partial_n u'\ u. \tag{24}$$

Proof: We have in Ω:

$$-\Delta u' = \lambda u' + \lambda'(0)u.$$

Then, multiplying by u, and integrating over Ω we get:

$$-\int_\Omega \Delta u' u = \lambda \int_\Omega u'\ u + \lambda' \int_\Omega |u|^2.$$

Using integration by parts twice and the normalization of u, we obtain

$$-\int_\Omega u' \Delta u + \int_{\partial\Omega} \partial_n u\ u' - \int_{\partial\Omega} \partial_n u'\ u = \lambda \int_\Omega u' u + \lambda'(0).$$

Since $-\Delta u = \lambda u$ in Ω, we obtain $\lambda \int_\Omega u'u$ on both sides, which allow us to cancel this term. Therefore, we get the result

$$\lambda'(0) = \int_{\partial\Omega} \partial_n u\ u' - \partial_n u'\ u.$$

□

Lemma 6 *Let $(u_i)_i$ and $(u_j)_j$, $i, j = 1, ..., m$, be two families of smooth eigenfunctions of (1) such that u_i and u_j are orthogonal in $L^2(\Omega_t)$ for all $i \neq j$ and $\forall t$, and denote by u_i' and u_j' the derivative of the eigenfunction u_i and u_j at $t = 0$, respectively. Then for all $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$, we have at $t = 0$:*

$$\int_\Omega u_i'\ u_j + \int_\Omega u_i\ u_j' + \int_{\partial\Omega} u_i u_j V_n = 0.$$

Proof: We use the $L^2(\Omega_t)$ orthogonality of u_i and u_j with each other:

$$\forall t > 0, \quad \int_{\Omega_t} u_i u_j = 0.$$

Applying the shape derivative, we get at $t = 0$

$$\int_\Omega (u_i u_j)' + \int_{\partial\Omega} u_i u_j V_n = 0$$

that is

$$\int_\Omega u_i'\ u_j + \int_\Omega u_i\ u_j' + \int_{\partial\Omega} u_i u_j V_n = 0$$

which gives the result.

□

In the following lemma, we give some derivation formulas:

Lemma 7 *Let v be a C^2 function defined on a C^2 domain Ω. Then we have the following derivation formulas on its boundary $\partial\Omega$ for some deformation vector fields $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$:*

$$(\partial_n v)' = \partial_n v' - \nabla_\tau v \nabla_\tau V_n, \tag{25}$$

and

$$(\nabla_\tau v)' = \nabla_\tau v' + (\nabla_\tau v.\nabla_\tau V_n)n + \nabla_\tau V_n \partial_n v. \tag{26}$$

Proof: To start with the first equation, we write $\partial_n v$ as $\nabla v.n$, then we have

$$(\partial_n v)' = (\nabla v.n)' = \nabla v'.n + \nabla v.n',$$

and using (81) we get

$$(\partial_n v)' = \nabla v'.n - \nabla_\tau V_n.\nabla v.$$

Again, by replacing $\nabla v'$ by the tangential and normal components, we obtain

$$(\partial_n v)' = \nabla_\tau v.'n + \partial_n v' n.n - \nabla_\tau V_n.\nabla_\tau v - (\partial_n v)\ n.\nabla_\tau V_n = \partial_n v' - \nabla_\tau v.\nabla_\tau V_n.$$

Moving on to the second equation, we remark that $\nabla_\tau v = \nabla v - \partial_n v n$ on $\partial\Omega$. Therefore, we have

$$(\nabla_\tau v)' = (\nabla v - \partial_n v n)' = \nabla v' - (\partial_n v)' n - \partial_n v n',$$

and by (25), we get

$$(\nabla_\tau v)' = \nabla v' - (\partial_n v')n + (\nabla_\tau v.\nabla_\tau V_n)n - \partial_n v n'.$$

Now, using (81), we obtain

$$(\nabla_\tau v)' = \nabla v' - (\partial_n v')n + (\nabla_\tau v.\nabla_\tau V_n)n + \partial_n v \nabla_\tau V_n.$$

Finally, writing $\nabla v'$ as $(\partial_n v')n + \nabla_\tau v'$, we get

$$(\nabla_\tau v)' = (\partial_n v')n + \nabla_\tau v' - (\partial_n v')n + (\nabla_\tau v.\nabla_\tau V_n)n + \partial_n v \nabla_\tau V_n = \nabla_\tau v' + (\nabla_\tau v.\nabla_\tau V_n)n + \partial_n v \nabla_\tau V_n.$$

□

The next Lemma is used to derive the second order shape derivative of the eigenvalues of (1):

Lemma 8 *Let $t \mapsto (\lambda(t), u_t)$ is a smooth eigenpair path on Ω_t for the Wentzell-Laplace problem (1). Let u' and u'' be the first and second-order derivatives of u at $t = 0$, respectively. Then we have at $t = 0$, for all deformation vector fields $V \in W^{3,\infty}(\Omega, \mathbb{R}^d)$:*

$$\lambda''(0) = \int_{\partial\Omega} \partial_n u u'' - \int_{\partial\Omega} \partial_n u'' u + \int_{\partial\Omega} V_n \lambda' |u|^2. \tag{27}$$

Proof: We follow the same way as in lemma 5. In fact, we have in Ω:

$$-\Delta u'' = \lambda'' u + 2\lambda' u' + \lambda u''$$

then, multiplying by u, and integrating over Ω we get:

$$-\int_\Omega \Delta u''\ u = \lambda \int_\Omega u''\ u + \lambda'' \int_\Omega |u|^2 + 2\lambda' \int_\Omega u'\ u$$

using integration by parts twice and the normalization of the solution u,

$$\lambda'' = -\int_\Omega u''\Delta u + \int_{\partial\Omega} \partial_n u\ u'' - \int_{\partial\Omega} \partial_n u''\ u - 2\lambda' \int_\Omega u'\ u - \int_\Omega \lambda u''\ u$$

by (21) and since $-\Delta u = \lambda u$ in Ω, we get

$$\lambda''(0) = \int_{\partial\Omega} \partial_n u u'' - \int_{\partial\Omega} \partial_n u'' u + \int_{\partial\Omega} V_n \lambda' |u|^2.$$

□

3 The proofs

In this section, we prove all the main theorems, propositions and corollaries of this chapter.

3.1 Proof of Proposition 1

Let us consider the disk B_R of radius R centered at O. In polar coordinates (r, θ), looking for an eigenfunction u of D of the kind $u(r,\theta) = v(r)w(\theta)$ leads us to solve the ordinary differential equations:

$$w''(\theta) + kw(\theta) = 0,\ w\ 2\pi\text{-periodic}, \qquad v''(r) + \frac{1}{r}v'(r) + (\lambda - \frac{k}{r^2})v(r) = 0,$$

$$v'(0) = 0, \qquad -w''(\theta)v(R) + v'(R)w(\theta) = 0.$$

The periodicity condition for the first one implies that $k = n^2$ where n is an integer. Thus, the general solution of the first differential equation is

$$w(\theta) = a_n \cos n\theta + b_n \sin n\theta. \tag{28}$$

Then, replacing k by n^2 in the second equation allows us to recognize the classical Bessel differential equation, for which its general solutions (bounded at 0 as our domain is a disk $D = \{r < R\}$ and $v'(0) = 0$) are the modified Bessel functions

$$v(r) = J_n(\sqrt{\lambda}r). \tag{29}$$

By the boundary conditions, we have $v'(R) = -n^2 v(R)$. Therefore, we have

$$\sqrt{\lambda}J_n'(\sqrt{\lambda}R) + n^2 J_n(\sqrt{\lambda}R) = 0. \tag{30}$$

Now, we use the following two relations about the Bessel functions:

$$J_n(\sqrt{\lambda}R) = \frac{\sqrt{\lambda}R}{2n}\left[J_{n-1}(\sqrt{\lambda}R) + J_{n+1}(\sqrt{\lambda}R)\right], \qquad \text{for } n \geq 1, \tag{31}$$

$$J_n'(\sqrt{\lambda}R) = \frac{1}{2}\left[J_{n-1}(\sqrt{\lambda}R) - J_{n+1}(\sqrt{\lambda}R)\right], \tag{32}$$

We propagate by induction. For $n = 0$, By (30) we get $J_0'(\sqrt{\lambda}R) = 0$. Therefore, the eigenvalues $\lambda_{0,k}$ belongs to the set $\{\lambda \mid J_0'(\sqrt{\lambda}R) = 0\}$, that is $\lambda_{0,k} = \frac{j_{0,k}'^2}{R^2}$, $k = 1, 2, ...$, where $j_{0,k}'$ is the k-th zero of the Bessel function J_0'.

Let us assume that $\lambda_{n,k} = \frac{j'^2_{n,k}}{R^2}$ is true for n, and we have to show that we have $\lambda_{n+1,k} = \frac{j'^2_{n+1,k}}{R^2}$. To do so, it is sufficient to prove that $J'_{n+1}(\sqrt{\lambda}R) = 0$ for $n+1$.

Now, since $J'_n(\sqrt{\lambda}R) = 0$ for n, we obtain from (30) that $J_n(\sqrt{\lambda}R) = 0$. Hence, using the two equations (31) and (32), we have

$$\begin{pmatrix} \frac{\sqrt{\lambda}R}{2n} & \frac{\sqrt{\lambda}R}{2n} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} J_{n-1}(\sqrt{\lambda}R) \\ J_{n+1}(\sqrt{\lambda}R) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{33}$$

that is $J_{n-1}(\sqrt{\lambda}R) = 0 = J_{n+1}(\sqrt{\lambda}R)$ for n. But we know that (30) is valid for $n+1$, thus we have

$$\sqrt{\lambda}J'_{n+1}(\sqrt{\lambda}R) + (n+1)^2 J_{n+1}(\sqrt{\lambda}R) = 0, \tag{34}$$

and then we get $J'_{n+1}(\sqrt{\lambda}R) = 0$ for $n+1$.

Finally, we can deduce that the eigenvalues are $\lambda_{n,k} = \frac{j'^2_{n,k}}{R^2}$, $k = 1, 2, ...$, where $j'_{n,k}$ is the k-th zero of the Bessel function J'_n.

3.2 Proof of Theorem 1

The expression of u' in the volume domain in the first line is trivial. To derive the boundary condition satisfied by u', we use a test function φ_t defined in Ω_t with $\partial_n \varphi_t = 0$ and $\varphi'_t = 0$ on $\partial\Omega_t$, then we get the following weak formulation valid for all t small enough:

$$\int_{\Omega_t} \nabla u(t,x).\nabla\varphi_t + \int_{\partial\Omega_t} \nabla_\tau u(t,x).\nabla_\tau\varphi_t = \lambda(t)\int_{\Omega_t} u(t,x)\varphi_t.$$

We take the derivative with respect to t and get at $t = 0$:

$$\frac{d}{dt}\left(\int_{\Omega_t} \nabla u(t,x).\nabla\varphi_t\right)_{|t=0} + \frac{d}{dt}\left(\int_{\partial\Omega_t} \nabla_\tau u(t,x).\nabla_\tau\varphi_t\right)_{|t=0} = \frac{d}{dt}\left(\lambda(t)\int_{\Omega_t} u(t,x)\varphi_t\right)_{|t=0}.$$

Now, using Appendix A and making some integrations by parts, the first term of the left hand side can be computed:

$$\frac{d}{dt}\left(\int_{\Omega_t} \nabla u(t,x).\nabla\varphi_t\right)_{|t=0} = \int_\Omega (\nabla u.\nabla\varphi)' + \int_{\partial\Omega} V_n\ \nabla u.\nabla\varphi$$

$$= \int_\Omega \nabla u'.\nabla\varphi + \int_{\partial\Omega} V_n\ \nabla u.\nabla\varphi = -\int_\Omega \Delta u'\varphi + \int_{\partial\Omega} \partial_n u'\ \varphi + \int_{\partial\Omega} V_n\ \nabla u.\nabla\varphi$$

but $\nabla u = \nabla_\tau u + \partial_n u\ n$ on $\partial\Omega$, and $\nabla\varphi = \nabla_\tau\varphi$ as $\partial_n\varphi = 0$ on $\partial\Omega$. Then

$$\frac{d}{dt}\left(\int_{\Omega_t} \nabla u(t,x).\nabla\varphi_t\right)_{|t=0} = -\int_\Omega \Delta u'\varphi + \int_{\partial\Omega} \partial_n u'\ \varphi + \int_{\partial\Omega} V_n \nabla_\tau u.\nabla_\tau\varphi$$

$$= -\int_\Omega \Delta u'\varphi + \int_{\partial\Omega} \partial_n u'\ \varphi - \int_{\partial\Omega} div_\tau(V_n\nabla_\tau u)\varphi.$$

Where the second term is

$$\frac{d}{dt}\left(\int_{\partial\Omega_t}\nabla_\tau u(t,x).\nabla_\tau\varphi_t\right)_{|t=0}=\int_{\partial\Omega}(\nabla_\tau u.\nabla_\tau\varphi)'+\int_{\partial\Omega}V_n[\partial_n(\nabla_\tau u.\nabla_\tau\varphi)+H\nabla_\tau u.\nabla_\tau\varphi]$$

$$=\int_{\partial\Omega}(\nabla_\tau u)'.\nabla_\tau\varphi+\int_{\partial\Omega}V_n[\partial_n(\nabla_\tau u.\nabla_\tau\varphi)+H\nabla_\tau u.\nabla_\tau\varphi].$$

By Equation (26) (see Lemma 7), that says

$$(\nabla_\tau u)'=\nabla_\tau u'+(\nabla_\tau u.\nabla_\tau V_n)n+\nabla_\tau V_n\partial_n u,$$

we get

$$\frac{d}{dt}\left(\int_{\partial\Omega_t}\nabla_\tau u(t,x).\nabla_\tau\varphi_t\right)_{|t=0}=\int_{\partial\Omega}\nabla_\tau u'.\nabla_\tau\varphi+\int_{\partial\Omega}(\nabla_\tau V_n.\nabla_\tau u)\ n.\nabla_\tau\varphi$$

$$+\int_{\partial\Omega}\partial_n u\nabla_\tau V_n.\nabla_\tau\varphi+\int_{\partial\Omega}V_n[\partial_n(\nabla_\tau u.\nabla_\tau\varphi)+H\nabla_\tau u.\nabla_\tau\varphi]$$

Notice that we have

$$\int_{\partial\Omega}(\nabla_\tau V_n.\nabla_\tau u)\ n.\nabla_\tau\varphi=0 \tag{35}$$

since n is orthogonal to $\nabla_\tau\varphi$ in $\mathbb{R}^d$.
Moreover, using Proposition 6 in Appendix D, that says

$$\nabla_\tau\varphi.\nabla_\tau(\partial_n u)+\nabla_\tau u.\nabla_\tau(\partial_n\varphi)=2\nabla_\tau u^T D^2 b\nabla_\tau\varphi+\partial_n(\nabla_\tau u.\nabla_\tau\varphi), \tag{36}$$

we get

$$\frac{d}{dt}\left(\int_{\partial\Omega_t}\nabla_\tau u(t,x).\nabla_\tau\varphi_t\right)_{|t=0}=\int_{\partial\Omega}\nabla_\tau u'.\nabla_\tau\varphi+\int_{\partial\Omega}\partial_n u\nabla_\tau V_n.\nabla_\tau\varphi$$

$$+\int_{\partial\Omega}V_n H\nabla_\tau u.\nabla_\tau\varphi-2\int_{\partial\Omega}V_n\nabla_\tau u D^2 b\nabla_\tau\varphi+\int_{\partial\Omega}V_n\nabla_\tau(\partial_n u).\nabla_\tau\varphi+\int_{\partial\Omega}V_n\nabla_\tau(\partial_n\varphi).\nabla_\tau u$$

by (23) and since $\partial_n\varphi=0$ on $\partial\Omega$, we get

$$\frac{d}{dt}\left(\int_{\partial\Omega_t}\nabla_\tau u(t,x).\nabla_\tau\varphi_t\right)_{|t=0}=\int_{\partial\Omega}\nabla_\tau u'.\nabla_\tau\varphi+\int_{\partial\Omega}\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi+\int_{\partial\Omega}V_n\nabla_\tau u(HI_d-D^2b)\nabla_\tau\varphi$$

$$=\int_{\partial\Omega}(-\Delta_\tau u'-\Delta_\tau(V_n\partial_n u)-div_\tau(V_n\nabla_\tau u(HI_d-2D^2b)))\varphi.$$

On the other hand, the right hand side can be easily calculated by

$$\frac{d}{dt}\left(\lambda(t)\int_{\Omega_t}u(t,x)\varphi_t\right)_{|t=0}=\int_{\Omega_t}\frac{d}{dt}\left(\lambda(t)\ u(t,x)\ \varphi_t\right)_{|t=0}+\int_{\partial\Omega}V_n\ \lambda(0)\ u(0,x)\ \varphi$$

$$=\lambda'(0)\int_\Omega u\ \varphi+\lambda\int_\Omega u'\ \varphi+\lambda\int_{\partial\Omega}V_n\ u\ \varphi.$$

To end the proof, it suffices to gather the relations.

3.3 Proof of Theorem 2

We start with the result of Theorem 1. Let us multiply the two sides of the boundary condition satisfied by u' (8) by the eigenfunction u and integrate over the boundary $\partial\Omega$, so that we get

$$\int_{\partial\Omega} \partial_n u' \; u - \int_{\partial\Omega} \Delta_\tau u' \; u' = \int_{\partial\Omega} V_n \Delta_\tau u \; u + \int_{\partial\Omega} \Delta_\tau (V_n \partial_n u) \; u \\ + \int_{\partial\Omega} div_\tau (V_n (HI_d - 2D^2 b) \nabla_\tau u) \; u + \int_{\partial\Omega} \lambda V_n u \; u. \tag{37}$$

Integrating by parts the second term on the left hand side twice, and knowing that $\Delta_\tau u = \partial_n u$ on $\partial\Omega$, we obtain:

$$\int_{\partial\Omega} \partial_n u' \; u - \int_{\partial\Omega} \partial_n u \; u' = \int_{\partial\Omega} V_n \Delta_\tau u \; u + \int_{\partial\Omega} \Delta_\tau (V_n \partial_n u) \; u \\ + \int_{\partial\Omega} div_\tau (V_n (HI_d - 2D^2 b) \nabla_\tau u) \; u + \int_{\partial\Omega} \lambda V_n u \; u. \tag{38}$$

Using Lemma 5 and making some integration by parts on the right hand side, we obtain the result

$$\lambda'(0) = \int_{\partial\Omega} V_n (|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda |u|^2 + (HI_d - 2D^2 b) \nabla_\tau u . \nabla_\tau u).$$

3.4 Proof of Theorem 3

Let $t \mapsto (u(t,x), \lambda(t) = \lambda(\Omega_t))$ be a smooth path of eigenpair of the Laplace-Wentzell problem, so that it satisfies

$$-\Delta u(t,x) = \lambda(t) u(t,x) \qquad \text{in } \Omega_t, \tag{39}$$

$$-\Delta_\tau u(t,x) + \partial_n u(t,x) = 0 \qquad \text{on } \partial\Omega_t. \tag{40}$$

We have proved that $u' = \partial_t u(0,x)$ satisfies the volume equation (7) in Ω and the boundary condition (8) on $\partial\Omega$. We use the decomposition of $u = u(0,x)$ in the eigenspace as

$$u = \sum_{j=1}^{m} c_j u_j \tag{41}$$

for some $c = (c_1, c_2, ..., c_m)^T \neq 0$. Multiplying the two sides equation of (41) by u_k, we get after some integration by parts the eigenvalue equation

$$\lambda'(0) c = M c$$

where $M = (M_{jk})_{1 \leq i,j \leq m}$ is defined by (10). From this, we deduce that the set of derivatives $(\lambda_i'(0))_{i=1,...,m}$ is exactly the set of eigenvalues of the matrix M, which achieves the proof of the theorem.

3.5 Proof of Theorem 4

The expression of u'' in the volume domain in the first line is trivial. To derive the boundary condition satisfied by u'', we multiply (8) by a test function φ_t defined on $\partial\Omega_t$ with $\partial_n \varphi_t = 0$ and $\varphi_t' = 0$. Let us set $\mathcal{A} := (HId - 2D^2 b)$. Then we get for all t small enough:

$$\int_{\partial\Omega_t}(-\Delta_\tau u'\varphi+\partial_n u')\varphi=\int_{\partial\Omega_t} div_\tau(V_n\nabla_\tau u)\varphi+\int_{\partial\Omega_t}\Delta_\tau(V_n\partial_n u)\varphi \tag{42}$$
$$+\int_{\partial\Omega_t} V_n\lambda u\varphi+\int_{\partial\Omega_t} div_\tau(V_n\mathcal{A}\nabla_\tau u)\varphi.$$

Let us do the shape derivative at time zero of each term of this equation. To simplify the notations, we set:

$$E(u,u',u'',\varphi):=\frac{d}{dt}\left(\int_{\partial\Omega_t}(-\Delta_\tau u'\varphi+\partial_n u')\varphi\right)_{|t=0}$$
$$A(u,u',\varphi):=\frac{d}{dt}\left(\int_{\partial\Omega_t} div_\tau(V_n\nabla_\tau u)\varphi\right)_{|t=0}$$
$$B(u,u',\varphi):=\frac{d}{dt}\left(\int_{\partial\Omega_t}\Delta_\tau(V_n\partial_n u)\varphi\right)_{|t=0}$$
$$C(u,u',\varphi):=\frac{d}{dt}\left(\int_{\partial\Omega_t} V_n\lambda u\varphi\right)_{|t=0}$$
$$D(u,u',\varphi):=\frac{d}{dt}\left(\int_{\partial\Omega_t} div_\tau(V_n\mathcal{A}\nabla_\tau u)\varphi\right)_{|t=0}.$$

Thus we have:

$$E(u,u',u'',\varphi)=A(u,u',\varphi)+B(u,u',\varphi)+C(u,u',\varphi)+D(u,u',\varphi). \tag{43}$$

We are going to study each of the five terms of this equation:

Study of $E(u,u',u'',\varphi)$:

$$E(u,u',u'',\varphi):=\frac{d}{dt}\left(\int_{\partial\Omega_t}(-\Delta_\tau u'\varphi+\partial_n u')\varphi\right)_{|t=0}=\frac{d}{dt}\left(\int_{\partial\Omega_t}-\Delta_\tau u'\varphi\right)_{|t=0}+\frac{d}{dt}\left(\int_{\partial\Omega_t}\partial_n u'\varphi\right)_{|t=0}$$
$$=\frac{d}{dt}\left(\int_{\partial\Omega_t}\nabla_\tau u'.\nabla_\tau\varphi\right)_{|t=0}+\frac{d}{dt}\left(\int_{\partial\Omega_t}\partial_n u'\varphi\right)_{|t=0}.$$

Applying Proposition 3 (see Appendix B), we get

$$E(u,u',u'',\varphi)=\int_{\partial\Omega}\frac{d}{dt}\left(\nabla_\tau u'.\nabla_\tau\varphi\right)+\int_{\partial\Omega}V_n\left[\partial_n(\nabla_\tau u'.\nabla_\tau\varphi)+H\nabla_\tau u'.\nabla_\tau\varphi\right]$$
$$+\int_{\partial\Omega}\frac{d}{dt}\left(\partial_n u'\varphi\right)+\int_{\partial\Omega}V_n\left[\partial_n(\partial_n u'\varphi)+H\partial_n u'\varphi\right].$$

Since $\partial_n\varphi=0$ and $\varphi'=0$ on $\partial\Omega$, in addition to applying Proposition 6 (see Appendix C) and equation (25), we get

$$E(u,u',u'',\varphi)=\int_{\partial\Omega}\left(\nabla_\tau u'\right)'.\nabla_\tau\varphi+\int_{\partial\Omega}V_n\left[\left(\nabla_\tau(\partial_n u').\nabla_\tau\varphi-2D^2b\nabla_\tau u'.\nabla_\tau\varphi\right)+H\nabla_\tau u'.\nabla_\tau\varphi\right]$$

$$+\int_{\partial\Omega}\left(\partial_n u'' - \nabla_\tau u'\nabla_\tau V_n\right)\varphi + \int_{\partial\Omega} V_n\left[\frac{\partial^2 u'}{\partial n^2}\varphi + H\partial_n u'\varphi\right].$$

Now, using equation (26) and Proposition 7 (see Appendix C), we obtain

$$\begin{aligned}
E(u,u',u'',\varphi) &= \int_{\partial\Omega}\left[\nabla_\tau u'' + \nabla_\tau V_n\partial_n u'\right].\nabla_\tau\varphi + \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau\varphi \\
&\quad -2\int_{\partial\Omega} V_n^2 D^2 b\nabla_\tau u'.\nabla_\tau\varphi + \int_{\partial\Omega} V_n H\nabla_\tau u'.\nabla_\tau\varphi + \int_{\partial\Omega}\partial_n u''\varphi \\
&\quad -\int_{\partial\Omega}\nabla_\tau u'.\nabla_\tau V_n\varphi + \int_{\partial\Omega} V_n\left[\Delta u' - \Delta_\tau u' - H\partial_n u'\right]\varphi + \int_{\partial\Omega} V_n H\partial_n u'\varphi.
\end{aligned}$$

Using the volume expression of u', i.e $-\Delta u' = \lambda' u + \lambda u'$, we get

$$\begin{aligned}
E(u,u',u'',\varphi) &= \int_{\partial\Omega}\nabla_\tau u''.\nabla_\tau\varphi + \int_{\partial\Omega}\partial_n u'\nabla_\tau V_n.\nabla_\tau\varphi + \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau\varphi \\
&\quad -2\int_{\partial\Omega} V_n^2 D^2 b\nabla_\tau u'.\nabla_\tau\varphi + \int_{\partial\Omega} V_n H\nabla_\tau u'.\nabla_\tau\varphi + \int_{\partial\Omega}\partial_n u''\varphi \\
&\quad -\int_{\partial\Omega}\nabla_\tau u'.\nabla_\tau V_n\varphi - \int_{\partial\Omega} V_n\lambda' u\varphi - \int_{\partial\Omega} V_n\lambda u'\varphi \\
&\quad -\int_{\partial\Omega} V_n\Delta_\tau u'\varphi - \int_{\partial\Omega} V_n H\partial_n u'\varphi + \int_{\partial\Omega} V_n H\partial_n u'\varphi.
\end{aligned}$$

Finally, in order to isolate the function φ, we make integration by parts to obtain

$$\begin{aligned}
E(u,u',u'',\varphi) &= -\int_{\partial\Omega} div_\tau(\nabla_\tau u'')\varphi - \int_{\partial\Omega} div_\tau(\nabla_\tau V_n\partial_n u')\varphi - \int_{\partial\Omega} div_\tau(V_n\nabla_\tau(\partial_n u'))\varphi \\
&\quad +2\int_{\partial\Omega} div_\tau(V_n D^2 b\nabla_\tau u')\varphi - \int_{\partial\Omega} div_\tau(V_n H\nabla_\tau u')\varphi + \int_{\partial\Omega}\partial_n u''\varphi \\
&\quad -\int_{\partial\Omega}\nabla_\tau u'.\nabla_\tau V_n\varphi - \int_{\partial\Omega} V_n\lambda' u\varphi - \int_{\partial\Omega} V_n\lambda u'\varphi - \int_{\partial\Omega} V_n\Delta_\tau u'\varphi.
\end{aligned} \tag{44}$$

Now, let us move to the right hand side of equation (43):

Study of $A(u,u',\varphi)$:

$$A(u,u',\varphi) := \frac{d}{dt}\left(\int_{\partial\Omega_t} div_\tau(V_n\nabla_\tau u)\varphi\right)_{|t=0} = -\frac{d}{dt}\left(\int_{\partial\Omega_t} V_n\nabla_\tau u.\nabla_\tau\varphi\right)_{|t=0}.$$

Applying Proposition 3 (see Appendix B), we get

$$A(u,u',\varphi) = -\int_{\partial\Omega}\frac{d}{dt}\left(V_n\nabla_\tau u.\nabla_\tau\varphi\right) - \int_{\partial\Omega} V_n\left[\partial_n(V_n\nabla_\tau u.\nabla_\tau\varphi) - V_n\nabla_\tau u.\nabla_\tau\varphi\right].$$

By equation (26) and using Proposition 6, and as $\partial_n\varphi = 0$ and $\varphi' = 0$, we obtain

$$A(u,u',\varphi) = -\int_{\partial\Omega} V_n'\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega} V_n\nabla_\tau u'.\nabla_\tau\varphi - \int_{\partial\Omega} V_n\partial_n u\nabla_\tau V_n.\nabla_\tau\varphi - \int_{\partial\Omega} V_n\partial_n V_n\nabla_\tau u.\nabla_\tau\varphi$$

$$-\int_{\partial\Omega} V_n^2 \nabla_\tau(\partial_n u).\nabla_\tau\varphi + 2\int_{\partial\Omega} V_n^2 D^2 b\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega} V_n^2 H\nabla_\tau u.\nabla_\tau\varphi.$$

Making integration by parts we obtain

$$\begin{aligned} A(u,u',\varphi) &= \int_{\partial\Omega} div_\tau(V_n'\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n\nabla_\tau u')\varphi + \int_{\partial\Omega} div_\tau(V_n\partial_n u\nabla_\tau V_n)\varphi \\ &+ \int_{\partial\Omega} div_\tau(V_n\partial_n V_n\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n^2\nabla_\tau(\partial_n u))\varphi - 2\int_{\partial\Omega} div_\tau(V_n^2 D^2 b\nabla_\tau u)\varphi \\ &+ \int_{\partial\Omega} div_\tau(V_n^2 H\nabla_\tau u)\varphi. \end{aligned} \tag{45}$$

Study of $B(u,u',\varphi)$:

$$B(u,u',\varphi) := \frac{d}{dt}\left(\int_{\partial\Omega_t} \Delta_\tau(V_n\partial_n u)\varphi\right)_{|t=0} = -\frac{d}{dt}\left(\int_{\partial\Omega_t} \nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi\right)_{|t=0}.$$

Applying Proposition 3 (see Appendix B), we get

$$B(u,u',\varphi) = -\int_{\partial\Omega}\frac{d}{dt}\left(\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi\right) - \int_{\partial\Omega} V_n[\partial_n(\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi) + H\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi].$$

By equation (26) and Proposition 6, as $\partial_n\varphi = 0$ and $\varphi' = 0$, we obtain

$$B(u,u',\varphi) = -\int_{\partial\Omega}\nabla_\tau((V_n\partial_n u)').\nabla_\tau\varphi - \int_{\partial\Omega}\partial_n(V_n\partial_n u)\nabla_\tau V_n.\nabla_\tau\varphi - \int_{\partial\Omega} V_n\nabla_\tau(\partial_n(V_n\partial_n u)).\nabla_\tau\varphi$$

$$+2\int_{\partial\Omega} V_n D^2 b\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi - \int_{\partial\Omega} V_n H\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi.$$

By equation (25) and integrating by parts the term $\int_{\partial\Omega} V_n\nabla_\tau(\partial_n(V_n\partial_n u)).\nabla_\tau\varphi$, we obtain

$$B(u,u',\varphi) = -\int_{\partial\Omega}\nabla_\tau(V_n'\partial_n u).\nabla_\tau\varphi - \int_{\partial\Omega}\nabla_\tau(V_n\partial_n u').\nabla_\tau\varphi + \int_{\partial\Omega}\nabla_\tau(V_n\nabla_\tau V_n\nabla_\tau u).\nabla_\tau\varphi$$

$$-\int_{\partial\Omega}\partial_n(V_n\partial_n u)\nabla_\tau V_n.\nabla_\tau\varphi + \int_{\partial\Omega}\partial_n(V_n\partial_n u)\nabla_\tau V_n.\nabla_\tau\varphi + \int_{\partial\Omega} V_n(\partial_n(V_n\partial_n u))\Delta_\tau\varphi$$

$$-\int_{\partial\Omega} V_n\mathcal{A}\nabla_\tau(V_n\partial_n u).\nabla_\tau\varphi.$$

By expanding the terms and some cancellations, in addition to using Proposition 6, we obtain

$$B(u,u',\varphi) = -\int_{\partial\Omega}\nabla_\tau(V_n'\partial_n u).\nabla_\tau\varphi - \int_{\partial\Omega}\nabla_\tau(V_n\partial_n u').\nabla_\tau\varphi + \int_{\partial\Omega}\nabla_\tau(V_n\nabla_\tau V_n\nabla_\tau u).\nabla_\tau\varphi$$

$$+\int_{\partial\Omega} V_n\partial_n V_n\partial_n u\Delta_\tau\varphi + \int_{\partial\Omega} V_n^2[\Delta u - \Delta_\tau u - H\partial_n u]\Delta_\tau\varphi - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau\varphi$$

$$-\int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau\varphi.$$

Now, by splitting the fifth term and since $-\Delta u = \lambda u$, we get

$$\begin{aligned} B(u,u',\varphi) &= -\int_{\partial\Omega} \nabla_\tau(V_n'\partial_n u).\nabla_\tau\varphi - \int_{\partial\Omega} \nabla_\tau(V_n\partial_n u').\nabla_\tau\varphi + \int_{\partial\Omega} \nabla_\tau(V_n\nabla_\tau V_n\nabla_\tau u).\nabla_\tau\varphi \\ &+ \int_{\partial\Omega} V_n\partial_n V_n\partial_n u\Delta_\tau\varphi - \int_{\partial\Omega} V_n^2\lambda u\Delta_\tau\varphi - \int_{\partial\Omega} V_n^2\Delta_\tau u\Delta_\tau\varphi - \int_{\partial\Omega} V_n^2 H\partial_n u\Delta_\tau\varphi \\ &- \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau\varphi - \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau\varphi. \end{aligned}$$

Making integration by parts, we get

$$\begin{aligned} B(u,u',\varphi) &= \int_{\partial\Omega} \Delta_\tau(V_n'\partial_n u)\varphi + \int_{\partial\Omega} \Delta_\tau(V_n\partial_n u')\varphi - \int_{\partial\Omega} \Delta_\tau(V_n\nabla_\tau V_n\nabla_\tau u)\varphi \\ &+ \int_{\partial\Omega} \Delta_\tau(V_n\partial_n V_n\partial_n u)\varphi - \int_{\partial\Omega} \Delta_\tau(V_n^2\lambda u)\varphi - \int_{\partial\Omega} \Delta_\tau(V_n^2\Delta_\tau u)\varphi \\ &- \int_{\partial\Omega} \Delta_\tau(V_n^2 H\partial_n u)\varphi + \int_{\partial\Omega} div_\tau(V_n\partial_n u\mathcal{A}\nabla_\tau V_n)\varphi + \int_{\partial\Omega} div_\tau(V_n^2\mathcal{A}\nabla_\tau(\partial_n u))\varphi. \end{aligned} \tag{46}$$

Study of $C(u,u',\varphi)$:

$$C(u,u',\varphi) := \frac{d}{dt}\left(\int_{\partial\Omega_t} V_n\lambda u\varphi\right)_{|t=0}$$

Applying Proposition 3 (see Appendix B), we get

$$C(u,u',\varphi) = \int_{\partial\Omega} \frac{d}{dt}\left(V_n\lambda u\varphi\right) + \int_{\partial\Omega} V_n[\partial_n(V_n\lambda u\varphi) + HV_n\lambda u\varphi].$$

As $\partial_n\varphi = 0$ and $\varphi' = 0$, we get

$$\begin{aligned} C(u,u',\varphi) &= \int_{\partial\Omega} V_n'\lambda u\varphi + \int_{\partial\Omega} V_n\lambda' u\varphi + \int_{\partial\Omega} V_n\lambda u'\varphi \\ &+ \int_{\partial\Omega} V_n\partial_n V_n\lambda u\varphi + \int_{\partial\Omega} V_n^2\lambda\partial_n u\varphi + \int_{\partial\Omega} V_n^2 H\lambda u\varphi. \end{aligned} \tag{47}$$

Study of $D(u,u',\varphi)$: Denote by $\tilde{\mathcal{A}} := (\tilde{a}_{i,j})_{1\le i,j\le m}$ the matrix whose elements are the normal derivatives of $a_{i,j}$, where $a_{i,j}$, $i,j = 1,....,m$, are the components of the matrix $\mathcal{A}$.

Thus, we have:

$$D(u,u',\varphi) := \frac{d}{dt}\left(\int_{\partial\Omega_t} div_\tau(V_n\mathcal{A}\nabla_\tau u)\varphi\right)_{|t=0} = -\frac{d}{dt}\left(\int_{\partial\Omega_t} V_n\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi\right)_{|t=0}.$$

Applying Proposition 3 (see Appendix B), we get

$$D(u,u',\varphi) = -\int_{\partial\Omega} \frac{d}{dt}\left(V_n\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi\right) - \int_{\partial\Omega} V_n[\partial_n(V_n\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi) + HV_n\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi].$$

By equation (26) and Proposition 6, as $\partial_n\varphi = 0$ and $\varphi' = 0$, we obtain

$$D(u,u',\varphi) = -\int_{\partial\Omega} V_n'\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega} V_n\mathcal{A}\nabla_\tau u'.\nabla_\tau\varphi - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau\varphi$$
$$-\int_{\partial\Omega} V_n\mathcal{A}'\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega_t} V_n\partial_n V_n\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau\varphi$$
$$+2\int_{\partial\Omega} V_n^2 D^2 b\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega} V_n^2\tilde{\mathcal{A}}\nabla_\tau u.\nabla_\tau\varphi - \int_{\partial\Omega} V_n^2 H\mathcal{A}\nabla_\tau u.\nabla_\tau\varphi.$$

Making integrations by parts, we get

$$\begin{aligned} D(u,u',\varphi) &= \int_{\partial\Omega} div_\tau(V_n'\mathcal{A}\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n\mathcal{A}\nabla_\tau u')\varphi + \int_{\partial\Omega} div_\tau(V_n\partial_n u\mathcal{A}\nabla_\tau V_n)\varphi \\ &+ \int_{\partial\Omega} div_\tau(V_n\mathcal{A}'\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n\partial_n V_n\mathcal{A}\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n^2\mathcal{A}\nabla_\tau(\partial_n u))\varphi \\ &-2\int_{\partial\Omega} div_\tau(V_n^2 D^2 b\mathcal{A}\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n^2\tilde{\mathcal{A}}\nabla_\tau u)\varphi + \int_{\partial\Omega} div_\tau(V_n^2 H\mathcal{A}\nabla_\tau u)\varphi. \end{aligned} \tag{48}$$

3.6 Proof of Theorem 5

First of all, we have seen in Lemma 8 that the expression of λ'' contains the second order derivative u''. So in order to get rid of this term, we try to find an explicit formula that depends only on u and u'. To do so, we multiply the boundary condition equation obtained in Theorem 4 by a special test function which is chosen to be an eigenfunction u corresponding to eigenvalue λ, and then we make integration by parts to obtain the following equation:

$$\mathcal{E}(u,u',u'') = \mathcal{F}(u,u') - \mathcal{B}(u,u') - \mathcal{C}(u,u') + \mathcal{D}(u,u'), \tag{49}$$

where we pose:

$$\begin{aligned} \mathcal{E}(u,u',u'') &:= E(u,u',u'',u), \\ \mathcal{A}(u,u') &:= A(u,u',u), \\ \mathcal{B}(u,u') &:= B(u,u',u), \\ \mathcal{C}(u,u') &:= C(u,u',u), \end{aligned}$$

and

$$\mathcal{D}(u,u') := D(u,u',u).$$

We start with the left hand side term of equation (49):

Calculation of $\mathcal{E}(u,u',u'')$:

We replace the function φ by an eigenfunction u in equation (44), so that we get:

$$\begin{aligned}\mathcal{E}(u,u',u'') &= -\int_{\partial\Omega} div_\tau(\nabla_\tau u'')u - \int_{\partial\Omega} div_\tau(\nabla_\tau V_n\partial_n u')u - \int_{\partial\Omega} div_\tau(V_n\nabla_\tau(\partial_n u'))u\\ &\quad +2\int_{\partial\Omega} div_\tau(V_n D^2 b\nabla_\tau u')u - \int_{\partial\Omega} div_\tau(V_n H\nabla_\tau u')u + \int_{\partial\Omega}\partial_n u''u\\ &\quad -\int_{\partial\Omega}(\nabla_\tau u'.\nabla_\tau V_n)u - \int_{\partial\Omega} V_n\lambda' uu - \int_{\partial\Omega} V_n\lambda u'u - \int_{\partial\Omega} V_n\Delta_\tau u'u.\end{aligned}$$

Making integration by parts, we get

$$\begin{aligned}\mathcal{E}(u,u',u'') &= \int_{\partial\Omega}\nabla_\tau u''.\nabla_\tau u + \int_{\partial\Omega}\partial_n u'\nabla_\tau V_n.\nabla_\tau u + \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau u\\ &\quad -2\int_{\partial\Omega} V_n D^2 b\nabla_\tau u'.\nabla_\tau u + \int_{\partial\Omega} V_n H\nabla_\tau u'.\nabla_\tau u + \int_{\partial\Omega}\partial_n u''u\\ &\quad -\int_{\partial\Omega}(\nabla_\tau u'.\nabla_\tau V_n)u - \int_{\partial\Omega} V_n\lambda' uu - \int_{\partial\Omega} V_n\lambda u'u - \int_{\partial\Omega} V_n\Delta_\tau u'u.\end{aligned}$$

Let us remark that, using integration by parts, we have:

$$\int_{\partial\Omega}\nabla_\tau u''.\nabla_\tau u = -\int_{\partial\Omega} div_\tau(\nabla_\tau u)u'' = -\int_{\partial\Omega}\Delta_\tau u\ u'' = -\int_{\partial\Omega}\partial_n u\ u'', \tag{50}$$

$$\int_{\partial\Omega}\partial_n u'\nabla_\tau V_n.\nabla_\tau u = -\int_{\partial\Omega} V_n div_\tau(\partial_n u'\nabla_\tau u) = -\int_{\partial\Omega} V_n\partial_n u'\Delta_\tau u - \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau u, \tag{51}$$

and

$$-\int_{\partial\Omega} V_n\Delta_\tau u'u = \int_{\partial\Omega}\nabla_\tau(V_n u).\nabla_\tau u' = \int_{\partial\Omega}(\nabla_\tau V_n.\nabla_\tau u')u + \int_{\partial\Omega} V_n(\nabla_\tau u.\nabla_\tau u'). \tag{52}$$

Therefore, by equations (50), (51) and (52), we get

$$\begin{aligned}\mathcal{E}(u,u',u'') &= -\int_{\partial\Omega}\partial_n u\ u'' - \int_{\partial\Omega} V_n\partial_n u'\Delta_\tau u - \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau u + \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau u\\ &\quad -2\int_{\partial\Omega} V_n D^2 b\nabla_\tau u'.\nabla_\tau u + \int_{\partial\Omega}\partial_n u''u + \int_{\partial\Omega} V_n H\nabla_\tau u'.\nabla_\tau u - \int_{\partial\Omega}(\nabla_\tau u'.\nabla_\tau V_n)u\\ &\quad -\int_{\partial\Omega} V_n\lambda'|u|^2 - \int_{\partial\Omega} V_n\lambda u'u + \int_{\partial\Omega}(\nabla_\tau V_n.\nabla_\tau u')u + \int_{\partial\Omega} V_n(\nabla_\tau u.\nabla_\tau u').\end{aligned}$$

Using Lemma 8, we obtain

$$\begin{aligned}\mathcal{E}(u,u',u'') &= -\lambda''(0) + \int_{\partial\Omega} V_n\lambda'|u|^2 - \int_{\partial\Omega} V_n\partial_n u'\Delta_\tau u - \int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau u\\ &\quad +\int_{\partial\Omega} V_n\nabla_\tau(\partial_n u').\nabla_\tau u - 2\int_{\partial\Omega} V_n D^2 b\nabla_\tau u'.\nabla_\tau u + \int_{\partial\Omega} V_n H\nabla_\tau u'.\nabla_\tau u\\ &\quad -\int_{\partial\Omega}(\nabla_\tau u'.\nabla_\tau V_n)u - \int_{\partial\Omega} V_n\lambda'|u|^2 - \int_{\partial\Omega} V_n\lambda u'u\\ &\quad +\int_{\partial\Omega}(\nabla_\tau V_n.\nabla_\tau u')u + \int_{\partial\Omega} V_n(\nabla_\tau u.\nabla_\tau u').\end{aligned}$$

Finally, we can deduce that

$$\mathcal{E}(u,u',u'') = -\lambda''(0) + \int_{\partial\Omega} V_n(\nabla_\tau u'.\nabla_\tau u) - \int_{\partial\Omega} V_n\partial_n u'\Delta_\tau u - \int_{\partial\Omega} V_n\lambda u'u + \int_{\partial\Omega} V_n\mathcal{A}\nabla_\tau u'.\nabla_\tau u. \tag{53}$$

Let us now move to the other side of equation (49):

Calculation of $\mathcal{A}(u,u')$:

We replace φ by u in equation (45), so that we get

$$\begin{aligned}\mathcal{A}(u,u') &= \int_{\partial\Omega} div_\tau(V_n'\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n\nabla_\tau u')u + \int_{\partial\Omega} div_\tau(V_n\partial_n u\nabla_\tau V_n)u \\ &+ \int_{\partial\Omega} div_\tau(V_n\partial_n V_n\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n^2\nabla_\tau(\partial_n u))u - 2\int_{\partial\Omega} div_\tau(V_n^2 D^2 b\nabla_\tau u)u \\ &+ \int_{\partial\Omega} div_\tau(V_n^2 H\nabla_\tau u)u.\end{aligned}$$

Making integration by parts, we get

$$\begin{aligned}\mathcal{A}(u,u') &= -\int_{\partial\Omega} V_n'\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n\nabla_\tau u'.\nabla_\tau u - \int_{\partial\Omega} V_n\partial_n u\nabla_\tau V_n.\nabla_\tau u \\ &- \int_{\partial\Omega} V_n\partial_n V_n\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n^2\nabla_\tau(\partial_n u).\nabla_\tau u + 2\int_{\partial\Omega} V_n^2 D^2 b\nabla_\tau u.\nabla_\tau u \\ &- \int_{\partial\Omega} V_n^2 H\nabla_\tau u.\nabla_\tau u.\end{aligned} \tag{54}$$

Calculation of $\mathcal{B}(u,u')$:

We replace φ by u in equation (46), so that we get

$$\begin{aligned}\mathcal{B}(u,u') &= \int_{\partial\Omega} \Delta_\tau(V_n'\partial_n u)u + \int_{\partial\Omega} \Delta_\tau(V_n\partial_n u')u - \int_{\partial\Omega} \Delta_\tau(V_n\nabla_\tau V_n.\nabla_\tau u)u \\ &+ \int_{\partial\Omega} \Delta_\tau(V_n\partial_n V_n\partial_n u)u - \int_{\partial\Omega} \Delta_\tau(V_n^2\lambda u)u - \int_{\partial\Omega} \Delta_\tau(V_n^2\Delta_\tau u)u \\ &- \int_{\partial\Omega} \Delta_\tau(V_n^2 H\partial_n u)u + \int_{\partial\Omega} div_\tau(V_n\partial_n u\mathcal{A}\nabla_\tau V_n)u + \int_{\partial\Omega} div_\tau(V_n^2\mathcal{A}\nabla_\tau(\partial_n u))u.\end{aligned}$$

Making integration by parts twice, we get

$$\begin{aligned}\mathcal{B}(u,u') &= \int_{\partial\Omega} V_n'\partial_n u\Delta_\tau u + \int_{\partial\Omega} V_n\partial_n u'\Delta_\tau u - \int_{\partial\Omega} V_n\nabla_\tau V_n.\nabla_\tau u\Delta_\tau u + \int_{\partial\Omega} V_n\partial_n V_n\partial_n u\Delta_\tau u \\ &- \int_{\partial\Omega} V_n^2\lambda u\Delta_\tau u - \int_{\partial\Omega} V_n^2\Delta_\tau u\Delta_\tau u - \int_{\partial\Omega} V_n^2 H\partial_n u\Delta_\tau u - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau u \\ &- \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau u.\end{aligned}$$

Remark that we have the following identity:

$$\begin{aligned} &-\int_{\partial\Omega} V_n\nabla_\tau V_n.\nabla_\tau u\Delta_\tau u = \int_{\partial\Omega} V_n div_\tau(V_n\Delta_\tau u\nabla_\tau u)\\ &= \int_{\partial\Omega} V_n\nabla_\tau V_n.\nabla_\tau u\Delta_\tau u + \int_{\partial\Omega} V_n^2\partial_n u\Delta_\tau u + \int_{\partial\Omega} V_n^2\nabla_\tau u\nabla_\tau(\Delta_\tau u). \end{aligned} \tag{55}$$

Now, by (55) and using the boundary condition $\Delta_\tau u = \partial_n u$, we obtain

$$\begin{aligned} \mathcal{B}(u,u') &= \int_{\partial\Omega} V_n'|\partial_n u|^2 + \int_{\partial\Omega} V_n\partial_n u'\partial_n u + \int_{\partial\Omega} V_n\nabla_\tau V_n.\nabla_\tau u\partial_n u + \int_{\partial\Omega} V_n^2\nabla_\tau u.\nabla_\tau(\partial_n u)\\ &+ \int_{\partial\Omega} V_n\partial_n V_n|\partial_n u|^2 - \int_{\partial\Omega} V_n^2\lambda\partial_n uu - \int_{\partial\Omega} V_n^2 H|\partial_n u|^2 - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau u\\ &- \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau u. \end{aligned}$$

By adding and subtracting the terms $\int_{\partial\Omega} V_n^2\nabla_\tau u.\nabla_\tau(\partial_n u)$ and $\int_{\partial\Omega} V_n^2 H|\partial_n u|^2$, we get

$$\begin{aligned} \mathcal{B}(u,u') &= \int_{\partial\Omega} V_n'|\partial_n u|^2 + \int_{\partial\Omega} V_n\partial_n u'\partial_n u + \int_{\partial\Omega} V_n\nabla_\tau V_n.\nabla_\tau u\partial_n u + 2\int_{\partial\Omega} V_n^2\nabla_\tau u.\nabla_\tau(\partial_n u)\\ &- \int_{\partial\Omega} V_n^2\nabla_\tau u.\nabla_\tau(\partial_n u) + \int_{\partial\Omega} V_n\partial_n V_n|\partial_n u|^2 - \int_{\partial\Omega} V_n^2\lambda\partial_n uu - 2\int_{\partial\Omega} V_n^2 H|\partial_n u|^2\\ &+ \int_{\partial\Omega} V_n^2 H|\partial_n u|^2 - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau u - \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau u. \end{aligned}$$

Finally, using (55), we replace the term $2\int_{\partial\Omega} V_n^2\nabla_\tau u.\nabla_\tau(\partial_n u)$, so that we get

$$\begin{aligned} \mathcal{B}(u,u') &= \int_{\partial\Omega} V_n'|\partial_n u|^2 + \int_{\partial\Omega} V_n\partial_n u'\partial_n u - 3\int_{\partial\Omega} V_n\nabla_\tau V_n.\nabla_\tau u\partial_n u - 2\int_{\partial\Omega} V_n^2|\partial_n u|^2\\ &- \int_{\partial\Omega} V_n^2\nabla_\tau u.\nabla_\tau(\partial_n u) + \int_{\partial\Omega} V_n\partial_n V_n|\partial_n u|^2 - \int_{\partial\Omega} V_n^2\lambda\partial_n uu - 2\int_{\partial\Omega} V_n^2 H|\partial_n u|^2\\ &+ \int_{\partial\Omega} V_n^2 H|\partial_n u|^2 - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau u - \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau u. \end{aligned} \tag{56}$$

Calculation of $\mathcal{C}(u,u')$:

We replace φ by u in equation (47), so that we get

$$\begin{aligned} \mathcal{C}(u,u') &= \int_{\partial\Omega} V_n'\lambda|u|^2 + \int_{\partial\Omega} V_n\lambda'|u|^2 + \int_{\partial\Omega} V_n\lambda u'u + \int_{\partial\Omega} V_n\partial_n V_n\lambda|u|^2\\ &+ \int_{\partial\Omega} V_n^2\lambda\partial_n uu + \int_{\partial\Omega} V_n^2 H\lambda|u|^2. \end{aligned} \tag{57}$$

Calculation of $\mathcal{D}(u,u')$:

We replace φ by u in equation (48), so that we get

$$\begin{aligned}\mathcal{D}(u,u') &= \int_{\partial\Omega} div_\tau(V_n'\mathcal{A}\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n\mathcal{A}\nabla_\tau u')u + \int_{\partial\Omega} div_\tau(V_n\partial_n u\mathcal{A}\nabla_\tau V_n)u \\ &+ \int_{\partial\Omega} div_\tau(V_n\mathcal{A}'\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n\partial_n V_n\mathcal{A}\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n^2\mathcal{A}\nabla_\tau(\partial_n u))u \\ &-2\int_{\partial\Omega} div_\tau(V_n^2 D^2 b\mathcal{A}\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n^2\tilde{\mathcal{A}}\nabla_\tau u)u + \int_{\partial\Omega} div_\tau(V_n^2 H\mathcal{A}\nabla_\tau u)u.\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}\mathcal{D}(u,u') &= -\int_{\partial\Omega} V_n'\mathcal{A}\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n\mathcal{A}\nabla_\tau u'.\nabla_\tau u - \int_{\partial\Omega} V_n\partial_n u\mathcal{A}\nabla_\tau V_n.\nabla_\tau u \\ &- \int_{\partial\Omega} V_n\mathcal{A}'\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n\partial_n V_n\mathcal{A}\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n^2\mathcal{A}\nabla_\tau(\partial_n u).\nabla_\tau u \\ &+2\int_{\partial\Omega} V_n^2 D^2 b\mathcal{A}\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n^2\tilde{\mathcal{A}}\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega} V_n^2 H\mathcal{A}\nabla_\tau u.\nabla_\tau u.\end{aligned} \tag{58}$$

In order to finish the proof, it is sufficient to replace the five terms in equation (49).

3.7 Proof of Theorem 6

To write the first Pohožaev's identity, we multiply the boundary eigenvalue problem (1) in Ω by $X.\nabla u$, where u is an eigenfunction and X is defined on $\mathbb{R}^d$ by $X(x) := x + \alpha$ for all $x, \alpha \in \mathbb{R}^d$, so that we have

$$\int_\Omega \nabla u.\nabla(X.\nabla u) - \int_{\partial\Omega} \partial_n u(X.\nabla u) = \lambda\int_\Omega u(X.\nabla u). \tag{59}$$

Let us remark that $\nabla(X.\nabla u) = D(X)\nabla u + D^2 uX$, where $D(X)$ is the differential matrix of the vector X and D^2u is the Hessian matrix of u. Since $D(X) = Id$, we get

$$\nabla(X.\nabla u) = \nabla u + D^2 uX. \tag{60}$$

Now, using integration by parts on the boundary $\partial\Omega$, as $\Delta_\tau u = \partial_n u$, we get

$$\int_\Omega |\nabla u|^2 + \int_\Omega D^2 uX.\nabla u + \int_{\partial\Omega} \nabla_\tau u.\nabla_\tau(X.\nabla u) = \lambda\int_\Omega u(X.\nabla u). \tag{61}$$

Let us compute $\nabla_\tau(X.\nabla u)$: We write $\nabla_\tau(X.\nabla u)$ as $\nabla(X.\nabla u) - \partial_n(X.\nabla u)n$. where n is the outward unit normal field on $\partial\Omega$. So we get

$$\begin{aligned}\nabla_\tau(X.\nabla u) =& \nabla(X.\nabla u) - [\nabla(X.\nabla u).n]n \\ &= \nabla u + D^2 uX - [\partial_n u + (D^2 uX).n]n \\ &= \nabla_\tau u + D^2 uX - (D^2 uX.n)n\end{aligned} \tag{62}$$

Now, let us remark that $(D^2 uX.n)n.\nabla_\tau u = 0$ since $\nabla_\tau u$ is orthogonal to n. Therefore, replacing (62) in (61), we obtain

$$\int_\Omega |\nabla u|^2 + \int_\Omega D^2 uX.\nabla u - \lambda \int_\Omega u(X.\nabla u) = -\int_{\partial\Omega} \nabla_\tau u.\nabla_\tau (X.\nabla u) \tag{63}$$
$$= -\int_{\partial\Omega} |\nabla_\tau u|^2 - \int_{\partial\Omega} D^2 uX.\nabla_\tau u.$$

On the other hand, referring to (59), we have

$$\int_\Omega \nabla u.\nabla(X.\nabla u) - \lambda \int_\Omega u(X.\nabla u) = \int_{\partial\Omega} \partial_n u(X.\nabla u). \tag{64}$$

Hence, we get by expressing ∇u as $\nabla_\tau u + \partial_n u n$:

$$\int_\Omega |\nabla u|^2 + \int_\Omega D^2 uX.\nabla u - \lambda \int_\Omega u(X.\nabla u) = \int_{\partial\Omega} \partial_n u(X.\nabla u) \tag{65}$$
$$= \int_{\partial\Omega} \partial_n u(X.[\nabla_\tau u + \partial_n u n])$$
$$= \int_{\partial\Omega} \partial_n u(X.\nabla_\tau u) + \int_{\partial\Omega} |\partial_n u|^2 (X.n)$$

Combining between (63) and (65), we obtain

$$\int_{\partial\Omega} (D^2 u + \partial_n u Id)X.\nabla_\tau u + \int_{\partial\Omega} |\nabla_\tau u|^2 + \int_{\partial\Omega} |\partial_n u|^2 (X.n) = 0. \tag{66}$$

Let us move to the second Pohožaev's identity. Since Ω is of class C^2, there exists $h > 0$, such that the signed distance function $b := b_\Omega$ belongs to $C^2(U_h)$, where $U_h := \{x \in \Omega \mid dist(x, \partial\Omega) < h\}$ is a tubular neighborhood. Let us make a truncation on U_h so that we set $\tilde{b} = b\psi$ in U_h, where ψ is a $C^\infty(\Omega)$ cut-off function defined by

$$\psi := \ell\left(\frac{d}{h}\right)$$

where d is the distance function to the boundary, and ℓ is a C^∞ function defined on $\mathbb{R}^+$ as the following:

$$\ell(t) := \begin{cases} 1 & if\ 0 \le t \le \frac{1}{2}, \\ \\ 0 & if\ t \ge 1. \end{cases}$$

Now, we have that $\tilde{b} \in C^2(\mathbb{R}^d)$. Furthermore, we have $\tilde{b} = b$ in a neighborhood of $\partial\Omega$, in particular we get $\tilde{b} = b$ on $\partial\Omega$.

To continue the proof, we multiply the boundary eigenvalue problem (1) in Ω by $\nabla\tilde{b}.\nabla u$, where u is an eigenfunction. Then, we integrate by parts, so that we get

$$\int_\Omega \nabla u.\nabla(\nabla\tilde{b}.\nabla u) - \int_{\partial\Omega} \partial_n u(\nabla\tilde{b}.\nabla u) = \lambda \int_\Omega u(\nabla\tilde{b}.\nabla u). \tag{67}$$

We remark that $\nabla(\nabla\tilde{b}.\nabla u) = D(\nabla\tilde{b})\nabla u + D^2 u\nabla\tilde{b} = D^2\tilde{b}\nabla u + D^2 u\nabla\tilde{b}$, where $D^2\tilde{b}$ and $D^2 u$ are the Hessian matrices of $\tilde{b}$ and u respectively.

Now, using integration by parts on the boundary $\partial\Omega$, as $\Delta_\tau u = \partial_n u$, we get

$$\int_\Omega D^2\tilde{b}\nabla u.\nabla u + \int_\Omega D^2 u\nabla\tilde{b}.\nabla u + \int_{\partial\Omega} \nabla_\tau u.\nabla_\tau(\nabla\tilde{b}.\nabla u) = \lambda\int_\Omega u(\nabla\tilde{b}.\nabla u). \tag{68}$$

Let us compute $\nabla_\tau(\nabla\tilde{b}.\nabla u)$: We write $\nabla_\tau(\nabla\tilde{b}.\nabla u)$ as $\nabla(\nabla\tilde{b}.\nabla u) - \partial_n(\nabla\tilde{b}.\nabla u)n$. where n is the outward unit normal field on $\partial\Omega$. So we get

$$\begin{aligned}\nabla_\tau(\nabla\tilde{b}.\nabla u) =& \nabla(\nabla\tilde{b}.\nabla u) - [\nabla(\nabla\tilde{b}.\nabla u).n]n \\ =& D^2\tilde{b}\nabla u + D^2 u\nabla\tilde{b} - [\nabla(\nabla\tilde{b}.\nabla u).n]n\end{aligned} \tag{69}$$

Now, let us remark that $[\nabla(\nabla\tilde{b}.\nabla u).n]n.\nabla_\tau u = 0$ since $\nabla_\tau u$ is orthogonal to n. Therefore, replacing (69) in (68), we obtain

$$\begin{aligned}\int_\Omega D^2\tilde{b}\nabla u.\nabla u + \int_\Omega D^2 u\nabla\tilde{b}.\nabla u - \lambda\int_\Omega u(\nabla\tilde{b}.\nabla u) =& -\int_{\partial\Omega}\nabla_\tau u.\nabla_\tau(\nabla\tilde{b}.\nabla u) \\ =& -\int_{\partial\Omega} D^2\tilde{b}\nabla u.\nabla_\tau u + D^2 u\nabla\tilde{b}.\nabla_\tau u \\ =& -\int_{\partial\Omega} D^2\tilde{b}\nabla_\tau u.\nabla_\tau u - \int_{\partial\Omega}\partial_n u D^2\tilde{b}\ n.\nabla_\tau u \\ & -\int_{\partial\Omega} D^2 u\nabla\tilde{b}.\nabla_\tau u.\end{aligned} \tag{70}$$

On the other hand, referring to (67), we get by expressing ∇u as $\nabla_\tau u + \partial_n u n$:

$$\begin{aligned}\int_\Omega D^2\tilde{b}\nabla u.\nabla u + \int_\Omega D^2 u\nabla\tilde{b}.\nabla u - \lambda\int_\Omega u(\nabla\tilde{b}.\nabla u) =& \int_{\partial\Omega}\partial_n u(\nabla\tilde{b}.\nabla u) \\ =& \int_{\partial\Omega}\partial_n u(\nabla\tilde{b}.[\nabla_\tau u + \partial_n u n]) \\ =& \int_{\partial\Omega}\partial_n u(\nabla\tilde{b}.\nabla_\tau u) + \int_{\partial\Omega}|\partial_n u|^2(\nabla\tilde{b}.n).\end{aligned} \tag{71}$$

Combining between (70) and (71), we obtain

$$\begin{aligned}\int_{\partial\Omega} D^2\tilde{b}\nabla_\tau u.\nabla_\tau u + \int_{\partial\Omega}\partial_n u D^2\tilde{b}\ n.\nabla_\tau u + \int_{\partial\Omega} D^2 u\nabla\tilde{b}.\nabla_\tau u + \int_{\partial\Omega}\partial_n u(\nabla\tilde{b}.\nabla_\tau u) \\ + \int_{\partial\Omega}|\partial_n u|^2(\nabla\tilde{b}.n) = 0.\end{aligned} \tag{72}$$

As $\tilde{b} = b$ on $\partial\Omega$, we deduce the following equation

$$\begin{aligned}\int_{\partial\Omega} D^2 b\nabla_\tau u.\nabla_\tau u + \int_{\partial\Omega}\partial_n u D^2 b\ n.\nabla_\tau u + \int_{\partial\Omega} D^2 u\nabla b.\nabla_\tau u + \int_{\partial\Omega}\partial_n u(\nabla b.\nabla_\tau u) \\ + \int_{\partial\Omega}|\partial_n u|^2(\nabla b.n) = 0.\end{aligned} \tag{73}$$

Let us remark that the vector $D^2b\ n = Dn.n \equiv 0$ on $\partial\Omega$ since n is unitary and Dn is symmetric (see [1, p.223]). Also since $\nabla b = n$ on $\partial\Omega$, and as $\nabla_\tau u$ is orthogonal to n, we obtain

$$\int_{\partial\Omega} D^2b\nabla_\tau u.\nabla_\tau u + \int_{\partial\Omega} D^2u\ n.\nabla_\tau u + \int_{\partial\Omega} |\partial_n u|^2 = 0. \tag{74}$$

3.8 Proof of Corollary 1

To start the proof, Let $X \in W^{3,\infty}(\Omega, \mathbb{R}^2)$ be deformation field defined in Theorem 6. We are going to study both cases (simple and multiple cases).

1. **Simple case.** If λ is simple, then we will use the equation (9) in Theorem 2:

$$\lambda'(0) = \int_{\partial\Omega} V_n[|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + (HId - 2D^2b)\nabla_\tau u.\nabla_\tau u].$$

2. **Multiple case.** If λ is of multiplicity h, then let us define the real numbers φ_{ij}, $i, j = 1, ..., h$, by

$$\varphi_{ij} := \int_{\partial\Omega} V_n\nabla_\tau u_i\nabla_\tau u_j - V_n\lambda u_i u_j - V_n\partial_n u_i\partial_n u_j + V_n(HId - 2D^2b)\nabla_\tau u_i.\nabla_\tau u_i.$$

Now, as the $h \times h$ matrix M defined by $(\varphi_{ij})_{i,j}$ is symmetric, then M is diagonalizable with a single eigenvalue $\lambda'(0)$. Therefore, we have

$$M = \lambda'(0)Id.$$

So we get:

$$\varphi_{ii} = \lambda'(0) \quad \text{for } 1 \le i \le h. \tag{75}$$

Hence, we obtain

$$\lambda'(0) = \int_{\partial\Omega} V_n[|\nabla_\tau u_i|^2 - |\partial_n u_i|^2 - \lambda|u_i|^2 + (HId - 2D^2b)\nabla_\tau u_i.\nabla_\tau u_i], \qquad i = 1, ..., h. \tag{76}$$

In our case, the domain Ω is a ball B, hence we have $X.n = 1$. Since $X = (X.n)n$ on ∂B, then $(D^2u + \partial_n uId)X.\nabla_\tau u = [(D^2u + \partial_n uId)X.n]n.\nabla_\tau u$ is zero due to orthogonality of the vector n to $\nabla_\tau u$. Thus, we get by (12)

$$\nabla_\tau u = 0 \text{ and } \partial_n u = 0 \ \text{ on } \partial B. \tag{77}$$

First case. If we suppose that B is a critical shape, without volume constraint, then we have:

$$\lambda'(0) = 0,$$

that is: for all V we have

$$\lambda|u|^2 = 0, \ \text{ on } \partial B.$$

Since $\lambda > 0$, we have for all V:

$$u = 0, \quad \text{on } \partial B.$$

By Holmgren's Theorem, we deduce that u is null in a neighborhood of ∂B. Thus, by analyticity, we get $u = 0$ in the whole domain B, which is a contradiction since u is an eigenfunction of the problem (1).

Second case. On the other hand, if there is a volume constraint $Vol(B) =$ constant, then there exists a Lagrangian constant μ such that

$$\lambda'(0) = \mu \; Vol'(B). \tag{78}$$

So we have

$$\int_{\partial B} V_n[|\nabla_\tau u|^2 - |\partial_n u|^2 - \lambda|u|^2 + (HId - 2D^2 b)\nabla_\tau u.\nabla_\tau u] = \mu \int_{\partial B} V_n. \tag{79}$$

Therefore, by (77), we get for all V

$$\lambda|u|^2 = \mu, \quad \text{on } \partial B.$$

Therefore, we deduce that u is constant on ∂B, which means that u is so in a neighborhood of ∂B, thanks to the Holmgren's Theorem. Thus, by analyticity, the eigenfunction u is constant in the whole domain B, which is impossible.

4 Appendices

4.1 Appendix A. Differentiating in moving domains.

Let us now state one of the main tools for differentiating on intervals of $\mathbb{R}$ (see [1, Corollary 5.2.8, p.199]).

Proposition 2 *Let us assume*

$$\varphi \in C^1([0,T); W^{1,\infty}(\mathbb{R}^d)), \qquad f \in C^1([0,T); L^1(\mathbb{R}^d)) \cap C([0,T); W^{1,1}(\mathbb{R}^d))$$

We set $V(t,x) = \dfrac{\partial \varphi}{\partial t}(t, \varphi(t)^{-1}(x))$ *for all* $x \in \Omega_t$, *where* Ω_t *is a* C^2 *domain. Then* $t \in [0,T) \to I(t)$ *defined by*

$$I(t) := \int_{\Omega_t} f(t)$$

is continuously differentiable on $[0,T)$ *and we have*

$$I'(t) = \int_{\Omega_t} \frac{\partial f}{dt}(t) + div[fV](t) = \int_{\Omega_t} \frac{\partial f}{dt}(t) + \int_{\partial \Omega_t} (V.n) f(t).$$

4.2 Appendix B. Differentiating on moving boundaries

We now state a classical proposition in shape optimization, which is proved in [1] (see [1, Proposition 5.4.4, p.215]).

Proposition 3 *Assume Ω is of class C^3, that $t \in [0,T) \to \varphi(t) \in C^2$ is differentiable at 0 with $\varphi(0) = I$, $\varphi'(0) = V$, and that $t \to g(t) \in W^{1,1}(\mathbb{R}^d)$ is differentiable at 0 with $g(0) \in W^{2,1}(\mathbb{R}^d)$. Then, $t \to G(t)$ defined by*

$$G(t) := \int_{\partial\Omega_t} g(t)$$

is differentiable at 0 and we have the formula

$$G'(t) = \int_{\partial\Omega} g'(0) + (V.n)[\partial_n g(0) + Hg(0)].$$

4.3 Appendix C. Some classical results on tangential differential calculus

4.3.1 C.1. Notations and definitions

We recall some essential notations and definitions that are needed for the computations of shape derivatives. Given a smooth function $f : \partial\Omega \mapsto \mathbb{R}$, we define its tangential gradient ∇_τ as

$$\nabla_\tau f = \nabla \tilde{f} - (\nabla \tilde{f}.\mathbf{n})\mathbf{n}$$

where $\tilde{f}$ is any extension of f in a tubular neighborhood of $\partial\Omega$. An extension is easily obtained when $\partial\Omega$ is smooth. The tangential gradient does not depends on the extension. It is also useful to define the tangential gradient as the normal projection of $\nabla\tilde{f}$ to the tangent hyperplane of $\partial\Omega$; in other words

$$\nabla_\tau f = \nabla \tilde{f} - n \otimes n \nabla \tilde{f}, \text{ on } \partial\Omega.$$

We also need the definition of the tangential divergence: For regular functions we define the surface Laplacian or Laplace-Beltrami operator as

$$\Delta_\tau f := div_\tau(\nabla_\tau f).$$

We recall the definition of the oriented distance $b_{\partial\Omega}$:

$$b_{\partial\Omega}(x) := \left\{ \begin{array}{ll} d_\Omega(x) & \text{for } x \in \mathbb{R}^d \backslash \bar{\Omega} \\ \\ -d_\Omega(x) & \text{for } x \in \Omega \end{array} \right\},$$

where the notation d_Ω stands for the distance function for a subset $\Omega \subset \mathbb{R}^d$:

$$d_\Omega(x) = \inf_{y \in \Omega} |x - y|$$

We shall sometimes write b instead of $b_{\partial\Omega}$; its gradient is an extension of the normal vector field $\mathbf{n}$ in a neighborhood of $\partial\Omega$. Let $D^2 b$ be the Weingarten operator with entries $(\nabla_\tau)_i n_j$ where n_j is the j-th component of $\mathbf{n}$. The normal vector is known to be in the kernel of $D^2 b$, while the other eigenfunctions are tangential with

the corresponding eigenvalues given by the principal curvatures of $\partial\Omega$. Let $k_i, i = 1, ..., d-1$ be the non zero eigenvalues of $D^2 b$. We define the mean curvature H as

$$H = \sum_{i=1}^{d-1} k_i = Tr(D^2 b) = \Delta b, \text{ on } \partial\Omega.$$

4.3.2 C.2. Preliminary results

We restate some propositions, in which one can refer to [7]. Let us start with the following proposition:

Proposition 4 *Suppose that the boundary $\partial\Omega$ is of class C^3. Then the normal derivative of the mean curvature H is*

$$\partial_n H = -\sum_{i=1}^{d-1} k_i^2. \tag{80}$$

Shape derivative of the main curvature H and of the normal $\mathbf{n}$ in the direction of a velocity V:

Proposition 5 *Let a surface $\partial\Omega$ be of class C^2. The shape derivatives of the normal $\boldsymbol{n}$ and of the mean curvature H in the direction of the velocity vector V are*

$$\boldsymbol{n}' = -\nabla_\tau V_n, \tag{81}$$

$$H' = -\Delta_\tau V_n, \tag{82}$$

where $V_n = \langle V, n\rangle$ denotes the normal component of the vector deformation V.

4.3.3 C.3. Some commutation equations

Let us state a useful proposition that is proved in [7]. Let f and g be two smooth functions defined on U a neighborhood of $\partial\Omega$; the notation b stands for the oriented distance. Recall that its gradient is an extension of the normal field $\mathbf{n}$ on $\partial\Omega$.

Proposition 6 *Let f and g be two smooth functions defined on U a neighborhood of $\partial\Omega$. Then we have:*

$$\partial_n(\nabla_\tau f.\nabla_\tau g) + 2\nabla_\tau f^T D^2 b \nabla_\tau g = \nabla_\tau(\partial_n f).\nabla_\tau g + \nabla_\tau(\partial_n g).\nabla_\tau f.$$

In the last proposition, we give an essential equation that is proved in [1, Proposition 5.4.12].

Proposition 7 *Let Ω be an open set of class C^2 and $u : \bar{\Omega} \to \mathbb{R}$ of class C^2. Then*

$$\Delta u = \Delta_\tau u + H\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \quad on \quad \partial\Omega. \tag{83}$$

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