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Continuity of the solution of the Wentzell-Laplace boundary problem under shape deformations

Ataa Al Kheir

July 18, 2023

Abstract: We investigate the attitude of the solution of the Wentzell-Laplace boundary value problem with respect to shape deformations. On one hand, we prove the continuity of the solution under Lipschitz deformation of uniformly Lipschitz domains in the Euclidean space \mathbb{R}^d . On the other hand, we demonstrate the continuity of the solution under the $W^{2,\infty}$ convergence of $(2, \infty)$ -Sobolev domains in \mathbb{R}^d . After that, we discuss the convergence of the signed distance function in $W^{1,p}(B)$ with $1 \leq p \leq \infty$ with respect to the Hausdorff convergence of less regular domains, where B is a large ball containing the sequence of domains $(\Omega_n)_{n \in \mathbb{N}}$ and the limit set Ω , and we derive the following results: the uniformly Lipschitz case is enough for proving the convergence of the first order differential in L^p with $1 \leq p < \infty$; while in case $p = \infty$, we need to assume more regularity, the "positive reach" one. Moreover, to prove the convergence of the first order differential of the projection function on $\partial\Omega_n$ in L^p with $1 \leq p \leq \infty$, we have to inject the $(2, \infty)$ -Sobolev regularity with the necessity of $W^{2,\infty}$ convergence of these domains.

Keywords: Wentzell-Laplace operator, continuity of the solution, shape deformation, Hausdorff convergence, $W^{2,\infty}$ convergence, convergence of the signed distance function and its differential, convergence of the differential of the projection function.

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1 Introduction and Statement of the results

In this paper, we consider the Wentzell-Laplace boundary problem defined on a connected open bounded subset Ω of \mathbb{R}^d with Lipschitz boundary. Let us give the following Wentzell-Laplace boundary problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\Delta_\tau u + \partial_{\mathbf{n}} u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

with $f \in L^2(\Omega)$, and let B be a ball containing the set Ω . We denote by \mathbf{n} the outward unit normal vector on $\partial\Omega$ and $\Delta_\tau := \operatorname{div}_\tau(\nabla_\tau)$ is the Laplace-Beltrami operator defined on $\partial\Omega$, where the solution u belongs to the variational space

$$\mathcal{H}_s(\Omega) := \{u \mid u \in H^1(\Omega), u|_{\partial\Omega} \in H^1(\partial\Omega)\}$$

equipped with the graph norm $\|\cdot\|_{\mathcal{H}_s(\Omega)}$ defined by:

$$\|u\|_{\mathcal{H}_s(\Omega)} := \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} + \|\nabla_\tau u\|_{L^2(\partial\Omega)}.$$

Let's set $\mathcal{H}(\Omega)$ as

$$\mathcal{H}(\Omega) := \{u \in \mathcal{H}_s(\Omega) \mid \int_{\partial\Omega} u = 0\}$$

which is closed in $\mathcal{H}_s(\Omega)$ and inherits the norm from $\mathcal{H}_s(\Omega)$. In fact, this set is necessary in order to prove the existence and uniqueness of the solution (see section 3).

In fact, the Wentzell factor $\Delta_\tau u = \partial_n u$ is well defined in the weak sense (see Definition 2.1) everywhere on $\partial\Omega$ if Ω is uniformly Lipschitz.

This problem has been studied a lot recently, especially in the last decade, and it is important to continue discovering some new studies of this problem. So, as a kind of motivation, we are interested in studying the continuity of the solution of problem (1) with respect to shape deformations, to a variety of convergence senses, and to the regularity of the admissible domains.

First of all, we study the continuity of the solution of (1) under Lipschitz deformations of uniformly Lipschitz domains. To go deeper, we can refer to the article of M. Dabrine and D. Kateb on the "Persistence of wellposedness of Ventcel's boundary value problem under shape deformation" (see [11]), where they studied the persistence of the solvability of the boundary value problem under **smooth** domain deformation.

Moreover, we deal with the continuity of the solution of (1) with respect to the convergence of $W^{2,\infty}$ of $(2,\infty)$ -Sobolev domains (see the definition of Sobolev spaces in Definition 2.4). Recall that the convergence $W^{2,\infty}$ of the domains means that we have the convergence of the signed distance functions in $W^{2,\infty}$.

Let us give the first result of the paper:

Theorem 1 (continuity with respect to Lipschitz deformation). *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open domains and let Ω be a Lipschitz open domain that are included in \mathbb{R}^d . Consider the Wentzell-Laplace problem (1). Let $u_n \in \mathcal{H}(\Omega_n)$ and $u \in \mathcal{H}(\Omega)$ be the solutions of (1) on Ω_n and Ω , respectively. Suppose that the Micheletti distance*

$$d(\Omega_n, \Omega) = \inf_{\Phi \in W^{1,\infty}(\mathbb{R}^d), \Omega_n = \Phi(\Omega)} \|\Phi\|_{W^{1,\infty}} \xrightarrow{n} 0,$$

then $u_n \xrightarrow{n} u$ in $\mathcal{H}(\Omega)$.

First of all, we have to show the existence and the uniqueness of the solution of the problem (1). Now, to prove this theorem, we need to introduce a sequence of Lipschitz vector fields $(h_n)_{n \in \mathbb{N}}$. We define the sequence of maps $(\Phi_n)_n$, by $\Phi_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with $\Phi_n := Id + h_n$ so that we have $\Omega_n = \Phi_n(\Omega) = (Id + h_n)(\Omega)$. In fact, we divide the proof into three steps. In the first step, we formulate the variational formulation of (1) in Ω_n , then we reduce our work to the fixed domain Ω , using the inverse of the transformation Φ_n , and by defining the transport back functions $v_n := u_n \circ \Phi_n$ and $f := f_n \circ \Phi_n$. The second step consists in showing that $(v_n)_n$ is bounded in $\mathcal{H}(\Omega)$. Finally, we demonstrate the convergence of v_n to v (the solution on Ω) in $\mathcal{H}(\Omega)$ in the last step.

Now let's move on to the second main result, which is **independent** of the previous theorem:

Theorem 2 (continuity with respect to $W^{2,\infty}$ convergence). *Let $(\Omega_n)_n$ be a sequence of $(2, \infty)$ -Sobolev open domains and let Ω be a $(2, \infty)$ -Sobolev open domain, that are included in a ball B in \mathbb{R}^d . Consider the Wentzell-Laplace problem (1), and let $u_n \in H^{\frac{3}{2}}(\Omega_n)$ be the solution of (1) on Ω_n . Suppose that $\Omega_n \xrightarrow{W^{2,\infty}} \Omega$. Then there exists a limit $u \in H^{\frac{3}{2}}(\Omega)$ such that $u_n \xrightarrow{H^1(B)} u$ and u is the solution of (1) on Ω .*

Remark 1.1 *We remark that the last theorem is not valid for the Hausdorff convergence of the sets, and we give a counter example in dimension two (see Remark 6.1).*

The proof of the second theorem is much more difficult than the previous one. In fact, the Ω domain should be at least $(2, \infty)$ -Sobolev, where a $(2, p)$ -Sobolev with $1 \leq p \leq \infty$ is defined such that the second-order differential of the signed distance function at the boundary $D^2 b_\Omega$ belongs to $L^p_{loc}(U_h)$, with U_h being a tubular neighborhood around $\partial\Omega$ of radius $h > 0$. It is clear that our sets will be in particular Lipschitz domains or in the $W^{1,\infty}$ class. This remark allows to find Lipschitz constants, in some proofs of our results.

First, we need to show the existence and uniqueness of the solution to problem (1). Next, we derive uniform bound estimates for the solutions, then we show the existence of an extension operator from $H^{\frac{3}{2}}(\Omega)$ into $H^{\frac{3}{2}}(B)$. Actually, we want to extend $\mathcal{H}(\Omega)$ to $\mathcal{H}(B)$, but it's not clear at all. The difficulty with the set \mathcal{H} is that the restriction to Ω of a function in $\mathcal{H}(B)$ has no reason to belong to $\mathcal{H}(\Omega)$. So, we notice that it is interesting to study the convergence in the space $H^{\frac{3}{2}}(\Omega)$, thanks to the Necas property (see [4]) which says: *If $u \in H^1(\Omega)$ and $u \in H^1(\partial\Omega)$ with $\Delta u = -f \in L^2(\Omega) \subset (H^{\frac{1}{2}}(\Omega))'$, then $u \in H^{\frac{3}{2}}(\Omega)$ (with a uniform constant in the Lipschitz norm of Ω).* Moreover, if $(\Omega_n)_n$ is a sequence of open bounded uniformly Lipschitz domains that converges to Ω in some topology \mathcal{T} , and u_n being the solution of problem (1) on Ω_n , we prove the existence of a subsequence of $(\tilde{u}_n)_{n \in \mathbb{N}}$, with \tilde{u}_n being the extension of u_n in B , that converges weakly in $H^{\frac{3}{2}}(B)$ to a limit \tilde{u} . Therefore, we obtain the strong convergence in $H^1(B)$ thanks to the Compact injection of $H^{\frac{3}{2}}(B)$ in $H^1(B)$. Set u to be the restriction of \tilde{u} to Ω . We have $u \in H^{\frac{3}{2}}(\Omega)$. The next step shows that we have $-\Delta u = f$ in Ω . The question is: does u satisfy the boundary condition? To answer, we will give several lemmas. In Lemma 5.5, we show that for all $v \in W^{1,\infty}(B)$, we have:

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow - \int_{\partial\Omega} \partial_n u v.$$

Now, to continue the proof, we have to prove Proposition 7.1 which says: for all $v \in W^{2,\infty}(B)$, we have:

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v.$$

To overcome this obstacle, we need the assumption of the convergence of b_n to b in $W^{2,p}$, $1 \leq p \leq \infty$. In fact, a problem will appear in the differentials of the first (case $p = \infty$) and of the second order for the uniformly Lipschitz shapes. Therefore, we need more regularity to prove that Db_n goes to Db in $L^\infty(B)$, like the "positive

reach" sets (see [7, Definition 6.1, p.303]) where it suffices to have the Hausdorff convergence of these domains, and even more regular for the second-order calculus as the $(2, \infty)$ -Sobolev domains (or simply $W^{2,\infty}$) with the necessity of $W^{2,\infty}$ convergence of these domains (for more details, one can refer to [7, Definition 9.1, p. 374]). Indeed, to avoid working with Radon measures as in [7], [9], and [10], we would like to truncate around the boundaries of the domains to get rid of singular points in B ; in other words, we try to keep the skeletons far enough away from the boundaries $\partial\Omega_n$ and $\partial\Omega$, so we do the studies in a uniform tubular neighborhood U_h , for some $h > 0$ (see subsection 6.4). Moreover, if $Dp_n(x)$ and $Dp(x)$ represent the differentials of the projection functions p_n and p at the point $x \in U_h$ on $\partial\Omega_n$ and $\partial\Omega$, respectively, we will deal with the convergence of $Dp_n(x)$ to $Dp(x)$ in which we have to inject the regularity of $(2, \infty)$ -Sobolev with the need for $W^{2,\infty}$ convergence of these domains, which will be presented in Proposition 6.5. This proposition could be proved thanks to the article of Gunther Leobacher and Alexander Steinicke (see [8, Theorem C]), where they derived an explicit formula for $Dp(x)$, $x \in U_h$.

In the proof of Theorem 2, we use the uniqueness of the limit to prove that u is the unique solution of (1). Moreover, since u is the only accumulation point of the sequence $(u_n)_{n \in \mathbb{N}}$, the whole sequence converges to u in $H^1(B)$, which completes the proof.

The plan of this paper is as follows (see Figure 1): in order to study the existence and uniqueness of the solution of problem (1) in section 3, we define the set of ε -cone in Definition 2.3, then we conclude that a Lipschitz domain has this property thanks to Theorem 3. Furthermore, we can deduce in Corollary 2.1 that if Ω is $(2, \infty)$ -Sobolev domain then it also has the ε -cone property. Actually, we use the Lax-Milgram Theorem (see Proposition 3.1).

Concerning Theorem 1, the proof is in subsection 8.1.

For a proof of Theorem 2, we follow the following plan: After using Necas property stated above, we desire to derive several uniform estimates, with respect to domains in the set \mathcal{O} which is introduced in Definition 4.1, on the function $u \in H^{\frac{3}{2}}(\Omega)$ (see Propositions 3.2 and 4.1). Then in Proposition 4.2, we show the existence of an extension operator from $H^{\frac{3}{2}}(\Omega)$ to $H^{\frac{3}{2}}(B)$. After that, we give some uniform estimations (with respect to ε) on the extended function \tilde{u} in $H^{\frac{3}{2}}(B)$ (Corollary 4.1), and if we consider a sequence of uniformly Lipschitz domains $(\Omega_n)_n$ that converges to Ω under certain topology \mathcal{T} , we show the strong convergence of the sequence of solutions $(\tilde{u}_n)_n$ to a limit \tilde{u} in $H^1(B)$ (Corollary 4.2). In Lemma 5.1, we show that the restriction $u \in H^{\frac{3}{2}}(\Omega)$ of the limit \tilde{u} to Ω satisfies $-\Delta u = f$. But we still need to show that u solves the problem (1), i.e. does u satisfies $-\Delta_{\tau} u = \partial_{\mathbf{n}} u$ on $\partial\Omega$? To answer this question, we give several lemmas (from Lemma 5.2 to 5.5), in addition to Proposition 7.1 (see section 7). In fact, in this proposition, we need to deal with the convergence of the signed distance functions in $W^{2,p}$, $p \geq 1$, in a uniform tubular neighborhood U_h for some $h > 0$, which always exists due to Proposition 6.4 (see subsection 6.4), and then Theorem 2 is proved.

In section 6, we are going to study the behavior of the convergence in $W^{1,p}(B)$, with $1 \leq p \leq \infty$, of the sequence of the oriented distance functions $(b_n)_{n \in \mathbb{N}}$ or $(b_{\Omega_n})_{n \in \mathbb{N}}$ with respect to the Hausdorff convergence of less regular domains (see Propositions 6.1, 6.2 and 6.3). Also, we study the convergence of the differential of the projection functions p_n on $\partial\Omega_n$ in L^p for $1 \leq p \leq \infty$ with respect to the $W^{2,\infty}$ convergence of $(2, \infty)$ -Sobolev domains (see Proposition 6.5).

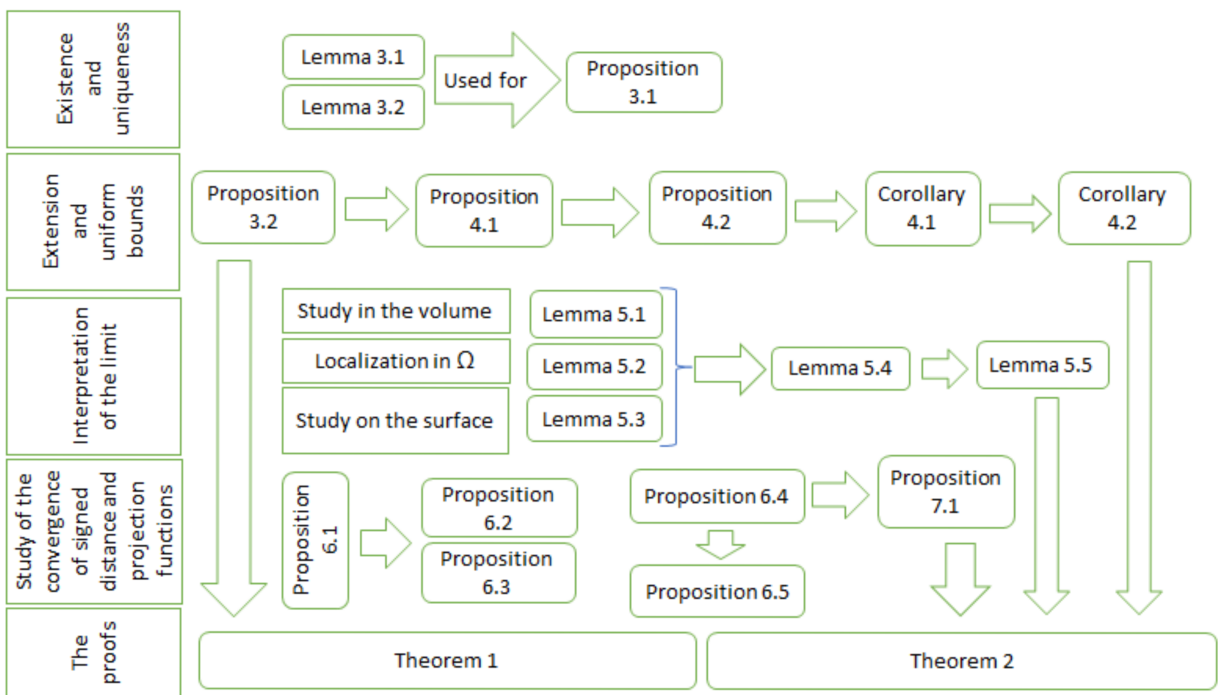


Figure 1: The map of the paper

2 Definitions and preliminaries

The first definition is for claiming that the normal derivative $\partial_{\mathbf{n}}u$ is well defined in $H^{-\frac{1}{2}}(\partial\Omega)$:

Definition 2.1 *If $\Delta u \in L^2(\Omega)$, then the normal derivative $\partial_{\mathbf{n}}u$ is well defined in $H^{-\frac{1}{2}}(\partial\Omega)$, where \mathbf{n} denotes the outward unit normal vector field on the boundary, in the following weak sense:*

Let us define the linear continuous form \mathbf{L} of $H^{\frac{1}{2}}(\partial\Omega)$:

$$\forall v \in H^{\frac{1}{2}}(\partial\Omega), \quad \mathbf{L}(v) := \int_{\Omega} \Delta u \tilde{v} + \nabla u \cdot \nabla \tilde{v} = \int_{\partial\Omega} \partial_{\mathbf{n}}u v \quad (2)$$

where $\tilde{v} := \gamma_0^{-1}(v) \in H^1(\Omega)$ is the inverse image of $v \in H^{\frac{1}{2}}(\partial\Omega)$ by the linear bounded trace mapping γ_0 from $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\partial\Omega)$. By Riesz representation theorem (see [5, Theorem 4.12, p.40]), there exists a unique y in $(H^{\frac{1}{2}}(\partial\Omega))' = H^{-\frac{1}{2}}(\partial\Omega)$, such that $\mathbf{L}(v) = \langle y, v \rangle_{H^{-\frac{1}{2}} \times H^{\frac{1}{2}}}$ for all $v \in H^{\frac{1}{2}}(\partial\Omega)$, then we can conclude from (2) that $\partial_{\mathbf{n}}u = y$ in a weak sense on the boundary $\partial\Omega$. ◇

From the definition of $\mathcal{H}(\Omega)$, $\nabla_{\tau}u$ is defined on $\partial\Omega$ in a weak sense. For the study of this chapter, we need to define an extension of $\nabla_{\tau}u$ in B :

Let b_{Ω} (or simply b if no confusion) be the signed distance function to the set Ω , that is defined by

$$b_{\Omega}(x) = b(x, \partial\Omega) := \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega, \\ d(x, \partial\Omega) & \text{if } x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

where d is the usual metric in \mathbb{R}^d , and for any $x \in \mathbb{R}^d$,

$$d(x, \partial\Omega) = \inf_{y \in \partial\Omega} d(x, y) = \inf_{y \in \partial\Omega} |x - y|$$

Now, as the skeletons of Ω are of zero d -dimension, we deduce that b is differentiable almost everywhere in B , thus $\nabla b \in L^2(B)$. In addition, we have outside the skeletons: its gradient satisfies $|\nabla b| = 1$ and $\nabla b = \mathbf{n}$ on $\partial\Omega$, where \mathbf{n} is the outward normal vector field.

Definition 2.2 *We define the extension vector $\nabla_{\tau}u \in L^2(B)$ outside $\partial\Omega$ as*

$$\nabla_{\tau}u := \nabla u - (\nabla u \cdot \nabla b) \nabla b.$$
◇

Definition 2.3 *For a fixed $\varepsilon > 0$, let us introduce the class of open sets:*

$$O_{\varepsilon} = \{\Omega \text{ open set, } \Omega \text{ has the } \varepsilon\text{-cone property}\}.$$
◇

One can refer to [2, p.54] to see more details about this set. Moreover, we have the following theorem proved in [2, p.56].

Theorem 3 *An open set Ω with a bounded boundary has the ε -cone property if and only if it has a Lipschitz boundary.*

Let us now state a result in A. Henrot and M. Pierre book (see [2, Theorem 2.4.10, p.59]).

Theorem 4 Let Ω_n be a sequence of open sets in the class O_ε . Then there exists an open set $\Omega \in O_\varepsilon$ and a subsequence Ω_{n_k} that converges to Ω in the sense of Hausdorff, in the sense of characteristic functions, and in the sense of compacts. Moreover, Ω_{n_k} and $\partial\Omega_{n_k}$ converge in the sense of Hausdorff respectively to Ω and $\partial\Omega$.

Let us introduce the following domains:

Definition 2.4 (Sobolev domains).

Given $m > 1$ and $p \geq 1$, a subset Ω of \mathbb{R}^d is said to be an (m, p) -Sobolev domain or simply a $W^{m,p}$ -domain if $\partial\Omega \neq \emptyset$ and there exists $h > 0$ such that

$$b_\Omega \in W_{loc}^{m,p}(U_h(\partial\Omega)),$$

where U_h is a tubular neighborhood of $\partial\Omega$ of radius h .

◇

We can deduce the following corollary:

Corollary 2.1 If Ω is a $(2, \infty)$ -Sobolev open bounded domain, then it has the ε -cone property, with $\varepsilon > 0$ is uniform on $\partial\Omega$.

3 Existence and uniqueness of the solution of (1)

In order to prove the existence and uniqueness of the solution of problem (1), we give the variational formulation of this problem:

For all $v \in \mathcal{H}(\Omega)$, we have

$$\begin{cases} \int_\Omega -\Delta u v = \int_\Omega f v, \\ \int_{\partial\Omega} -\Delta_\tau u v + \int_{\partial\Omega} \partial_{\mathbf{n}} u v = 0. \end{cases} \quad (3)$$

Hence, we get

$$\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v = \int_\Omega f v. \quad (4)$$

Actually, we aim to use the Lax-Milgram Theorem, by showing the continuity of A and B on $\mathcal{H}(\Omega)$ (even $\mathcal{H}_s(\Omega)$ is true), where

$$A(u, v) := \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v, \quad B(v) := \int_\Omega f v, \quad (5)$$

and that A is coercive on $\mathcal{H}(\Omega)$. Let us remark that the coercivity cannot be valid on $\mathcal{H}_s(\Omega)$, since $A(1, 1) = 0$ and $\|1\|_{\mathcal{H}_s(\Omega)} = \text{Volume}(\Omega) + \text{Perimeter}(\Omega)$.

Lemma 3.1 A and B are continuous on $\mathcal{H}(\Omega)$.

Proof: We have:

$$A(u, u) = \|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_\tau u\|_{L^2(\Omega)}^2 \leq \|u\|_{\mathcal{H}(\Omega)}^2$$

and

$$|B(u)| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{\mathcal{H}(\Omega)}$$

thanks to the Cauchy-Schwartz inequality. Therefore, A and B are continuous on $\mathcal{H}(\Omega)$.

□

Lemma 3.2 *A is coercive on $\mathcal{H}(\Omega)$.*

Proof: Let us divide our proof into five parts:

1. Set λ_{LB} to be the lowest non-zero eigenvalue of Laplacian-Beltrami operator, then $\forall u \in H^1(\partial\Omega)$ with $\int_{\partial\Omega} u = 0$, we have

$$\|u\|_{L^2(\partial\Omega)} \leq \lambda_{LB} \|\nabla_{\tau} u\|_{L^2(\partial\Omega)} \quad (6)$$

since we have by Courant-Hilbert lemma

$$\lambda_{LB} = \inf_{u \in H^1(\partial\Omega), u \perp 1} \frac{\|\nabla_{\tau} u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\partial\Omega)}}.$$

2. Since we have

$$\|u\|_{L^2(\partial\Omega)} \leq \lambda_{LB} \|\nabla_{\tau} u\|_{L^2(\partial\Omega)},$$

we get by interpolation between $L^2(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ that

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\partial\Omega) \|\nabla_{\tau} u\|_{L^2(\partial\Omega)} \leq C(\partial\Omega) \|u\|_{H^1(\partial\Omega)}. \quad (7)$$

3. Let us split $u = v + w$ where v solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = u & \text{on } \partial\Omega \end{cases} \quad (8)$$

and w solves

$$\begin{cases} \Delta w = \Delta u & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (9)$$

Now, for $v \in H^{\frac{3}{2}}(\Omega)$ we have

$$\|v\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)} = \text{constant} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\varepsilon) \|v\|_{H^{\frac{1}{2}}(\partial\Omega)} = C(\varepsilon) \|u\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad (10)$$

where $C(\varepsilon)$ is a constant that depends on the equivalence of norms between $H^{\frac{1}{2}}(\partial\Omega)$ (the trace of $H^1(\Omega)$) and the intrinsic Sobolev norm $H^{\frac{1}{2}}$ defined as a double integral. In fact it depends also on the uniform Lipschitz constant or ε -cone property (for more details, see Maz'ya's book in Sobolev spaces [12]).

Thus, by (7) we get

$$\|v\|_{L^2(\Omega)} \leq C(\varepsilon) \|u\|_{H^1(\partial\Omega)}. \quad (11)$$

Concerning w , we have using integration by parts in Ω :

$$-\int_{\Omega} w \Delta w = \int_{\Omega} |\nabla w|^2. \quad (12)$$

On the other hand, we have:

$$-\int_{\Omega} w \Delta w = -\int_{\Omega} w \Delta u = \int_{\Omega} \nabla u \cdot \nabla w \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}. \quad (13)$$

Hence, combining between (12) and (13) we obtain

$$\|\nabla w\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}. \quad (14)$$

But by Poincaré inequality we have:

$$\exists C_p = C_{\text{Poincaré}}, \text{ such that } \|w\|_{L^2(\Omega)} \leq C_p \|\nabla w\|_{L^2(\Omega)}. \quad (15)$$

Therefore, we get

$$\|w\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}. \quad (16)$$

4. As $u = v + w$, we deduce that

$$\|u\|_{L^2(\Omega)}^2 \leq 2(\|v\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2) \leq 2(\|\nabla u\|_{L^2(\Omega)}^2 + C(\varepsilon)\|u\|_{H^1(\partial\Omega)}^2). \quad (17)$$

5. In this step, we will gather all the information above. So we have

$$\begin{aligned} \|u\|_{\mathcal{H}(\Omega)}^2 &= \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2 + \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \\ &\leq 2 \left[\|\nabla u\|_{L^2(\Omega)}^2 + C(\varepsilon) \left(\|u\|_{L^2(\partial\Omega)}^2 + \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \right) \right] + \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\partial\Omega)}^2 + \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \\ &\leq 3\|\nabla u\|_{L^2(\Omega)}^2 + 2C(\varepsilon) (\lambda_{LB}^2(\partial\Omega) + 1) \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 + (\lambda_{LB}^2(\partial\Omega) + 1) \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \\ &= 3\|\nabla u\|_{L^2(\Omega)}^2 + (2C(\varepsilon) + 1) (\lambda_{LB}^2(\partial\Omega) + 1) \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \\ &\leq \max [3, (2C(\varepsilon) + 1) (\lambda_{LB}^2(\partial\Omega) + 1)] \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \right) \end{aligned}$$

Therefore, there exists a constant $\alpha := \max [3, (2C(\varepsilon) + 1) (\lambda_{LB}^2(\partial\Omega) + 1)]$ such that

$$\|u\|_{\mathcal{H}(\Omega)}^2 \leq \alpha \left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_{\tau} u\|_{L^2(\partial\Omega)}^2 \right). \quad (18)$$

If we set $\beta := \frac{1}{\alpha}$

$$A(u, u) \geq \beta \|u\|_{\mathcal{H}(\Omega)}^2. \quad (19)$$

Hence A is coercive on $\mathcal{H}(\Omega)$.

□

Remark 3.1 *The constant α of coercivity of A depends only on:*

1. $C(\varepsilon)$ the constant in the equivalence of norms between $H^{\frac{1}{2}}(\partial\Omega)$ (seen as the trace of $H^1(\Omega)$) and the intrinsic Sobolev norm $H^{\frac{1}{2}}$ defined as a double integral. It is uniform on the class of ε -cone domains (see Maz'ya's book in Sobolev spaces [12]).
2. The first non-trivial eigenvalue of the Laplace-Beltrami operator on $\partial\Omega$.
*In dimension two, it depends only on the perimeter and decreases with respect to it; if there exists $\delta > 0$ such that $\text{Perimeter}(\partial\Omega) \geq \delta > 0$, then $\lambda_{LB}(\partial\Omega)$ is bounded by a constant depending only on δ and not Ω .
 In dimension three, bounds on curvature of $\partial\Omega$ could be needed (to investigate).*

Now, in order to end the proof of the existence and uniqueness of the solution, let us give this crucial proposition:

Proposition 3.1 (*Existence and uniqueness of the solution of (1)*). *Let Ω be a Lipschitz domain. Then for any $f \in \mathcal{H}'(\Omega)$ (the dual of $\mathcal{H}(\Omega)$), problem (1) has a unique solution u in $\mathcal{H}(\Omega)$.*

Proof: In order to prove this proposition, we just have to apply Lax-Milgram Theorem. □

From the work above, we can show this proposition that will be useful in the next subsection:

Proposition 3.2 (*A priori bounds for u_Ω*). *For all $f \in \mathcal{H}'(\Omega)$, the solution u_Ω of (1) on Ω satisfies*

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega) \|f\|_{\mathcal{H}'(\Omega)}. \quad (20)$$

Moreover, if $f \in L^2(\Omega)$, then we have

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega) \|f\|_{L^2(\Omega)}. \quad (21)$$

Proof: Let $f \in \mathcal{H}'(\Omega)$. We use $u_\Omega \in \mathcal{H}(\Omega)$ as a test function so that we get from (18)

$$\|u_\Omega\|_{\mathcal{H}(\Omega)}^2 \leq \alpha(\Omega) A(u_\Omega, u_\Omega) = \alpha(\Omega) b(u_\Omega) \leq \alpha(\Omega) \|f\|_{\mathcal{H}'(\Omega)} \|u_\Omega\|_{\mathcal{H}(\Omega)},$$

hence we get

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega) \|f\|_{\mathcal{H}'(\Omega)}.$$

If $f \in L^2(\Omega)$, then we have

$$\|u_\Omega\|_{\mathcal{H}(\Omega)} \leq \alpha(\Omega) \|f\|_{L^2(\Omega)}.$$

□

4 Extension operator and uniform bound estimations

The difficulty with the set \mathcal{H} is that the restriction to Ω of a function in $\mathcal{H}(B)$ has no reason to belong to $\mathcal{H}(\Omega)$. Therefore, we consider the space $H^{\frac{3}{2}}$ to overcome this difficulty. That's the reason we deal with the right hand side f to be in $L^2(\Omega)$ and not just in $\mathcal{H}'(\Omega)$.

In this section, our goal is to show the existence of an extension operator from $H^{\frac{3}{2}}(\Omega)$ into $H^{\frac{3}{2}}(B)$. Furthermore, we would like to study some uniform bound estimation in $H^{\frac{3}{2}}(\Omega)$ and $H^{\frac{3}{2}}(B)$ in order to show the existence of a weakly convergent sequence of solutions $(u_n)_n$, with respect to the convergence of a sequence of domains $(\Omega_n)_n$ to the limit set Ω in some sense, to a limit u in the extended set $H^{\frac{3}{2}}(B)$, and in particular we get the strong convergence of u_n to u in $H^1(B)$, thanks to the compact injection of $H^{\frac{3}{2}}(B)$ into $H^1(B)$.

In order to set ourselves in the uniform case with respect to Ω , we give this definition:

Definition 4.1 We set \mathcal{O} to be a class of domains $\Omega \in \mathcal{O}_\varepsilon$ such that there exists $M > 0$ and $\alpha(\Omega) \leq M$ (the constant M is uniform with respect to $\Omega \in \mathcal{O}_\varepsilon$). In other words, the domains in \mathcal{O} are considered to be uniformly Lipschitz with a uniform constant M . ◇

In order to have an extension operator, we pass to $H^{\frac{3}{2}}(\Omega)$ thanks to Necas property (see [4]):

If $u \in H^1(\Omega)$ and $u \in H^1(\partial\Omega)$ with $\Delta u = -f \in L^2(\Omega) \subset (H^{\frac{1}{2}}(\Omega))'$, then $u \in H^{\frac{3}{2}}(\Omega)$ (with a uniform constant in the Lipschitz norm of Ω).

Proposition 4.1 On the set \mathcal{O} , there exist $M > 0$ such that $\|u\|_{H^{\frac{3}{2}}(\Omega)} \leq M$.

Proof: We use the uniform bound in $\mathcal{H}(\Omega)$ and Necas property. □

Proposition 4.2 Let $\Omega \in \mathcal{O}$. Then there exists an extension operator from $H^{\frac{3}{2}}(\Omega)$ into $H^{\frac{3}{2}}(B)$.

The following corollary investigates a uniform bound on the extended function \tilde{u} in $H^{\frac{3}{2}}(B)$:

Corollary 4.1 Let $\Omega \in \mathcal{O}$. Then there exists $M > 0$ and \tilde{u} an extension in $H^{\frac{3}{2}}(B)$ of u the solution of (1) in Ω , such that

$$\|\tilde{u}\|_{H^{\frac{3}{2}}(B)} \leq M. \quad (22)$$

As a consequence of Corollary 4.1, we have

Corollary 4.2 Assume, for a given topology \mathcal{T} on \mathcal{O} , that $\Omega_n \xrightarrow{\mathcal{T}} \Omega$, where Ω_n and Ω are included in B and belong to the set \mathcal{O} . Let u_n be the solution of (1) on Ω_n and \tilde{u}_n its extension to B . Then $\|\tilde{u}_n\|_{H^{\frac{3}{2}}(B)}$ is uniformly bounded. By consequence, there exists $\tilde{u} \in H^{\frac{3}{2}}(B)$ such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } H^{\frac{3}{2}}(B). \quad (23)$$

In particular, we have the strong convergence in $H^1(B)$, in other words, we have

$$\tilde{u}_n \longrightarrow \tilde{u} \quad \text{in } L^2(B), \quad (24)$$

$$\nabla \tilde{u}_n \longrightarrow \nabla \tilde{u} \quad \text{in } L^2(B). \quad (25)$$

Set u to be the restriction of \tilde{u} to Ω . One has $u \in H^{\frac{3}{2}}(\Omega)$. The question now is: does u solves (1) on Ω ? To answer, we have to pass to the limit in the variational formulations.

Remark 4.1 In the following, we will denote by u_n (respectively u) the restriction to Ω_n (respectively Ω) of $\tilde{u}_n \in H^{\frac{3}{2}}(B)$ (respectively $\tilde{u} \in H^{\frac{3}{2}}(B)$), where Ω_n and Ω are in \mathcal{O} .

5 Interpretation of the limit

In this section, we deal firstly with the volume case, so that we show that the limit u is a solution of the volume equation $-\Delta u = f$. But in order to prove that u satisfies the boundary conditions of (1), we give several lemmas, in addition to using Proposition 7.1. We will divide this section into many subsections.

5.1 The differential equation in the volume

Lemma 5.1 *Given a sequence of uniformly Lipschitz domains $(\Omega_n)_n$ that converges to a Lipschitz domain Ω , let $(u_n)_n$ be a sequence of solutions of (1) defined in $H^{\frac{3}{2}}(\Omega_n)$, and let u be its limit in $H^{\frac{3}{2}}(\Omega)$. Then, for $f \in L^2(\Omega)$, we have on Ω :*

$$-\Delta u = f.$$

Proof: Let $\varphi \in \mathcal{D}(\Omega)$, we have:

$$\begin{aligned} \langle -\Delta u_n, \varphi \rangle_{\mathcal{D}'(\Omega_n) \times \mathcal{D}(\Omega)} &= \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} \Rightarrow \langle \nabla u_n, \nabla \varphi \rangle_{\mathcal{D}'(\Omega_n) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} \\ &\Rightarrow \langle \nabla u_n, \nabla \varphi \rangle_{L^2(B) \times L^2(\Omega)} = \langle f, \varphi \rangle_{L^2(B) \times L^2(\Omega)} \\ &\Rightarrow \langle \nabla u, \nabla \varphi \rangle_{L^2(B) \times L^2(\Omega)} = \langle f, \varphi \rangle_{L^2(B) \times L^2(\Omega)} \quad \text{by (25)} \\ &\Rightarrow \langle \nabla u, \nabla \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} \Rightarrow \langle -\Delta u, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}. \end{aligned}$$

□

5.2 Localization in Ω (the limit set)

As a consequence of Theorem 4, we can derive this lemma:

Lemma 5.2 *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded domain Ω . Let $f \in L^2(B)$ and $v \in L^2(B)$. Then we have*

$$\int_B \mathcal{X}_{\Omega_n} f v \longrightarrow \int_B \mathcal{X}_{\Omega} f v$$

where \mathcal{X}_{Ω_n} and \mathcal{X}_{Ω} are the classical characteristic functions defined in $L^1(B)$.

Proof: From Theorem 3, we deduce that $\Omega_n \in \mathcal{O}_\varepsilon$. Thus, we can apply Theorem 4 that provides the existence of a subsequence that converges in three different senses, in particular we have the characteristic convergence of Ω_n to Ω (i.e. $|\Omega_n| \rightarrow |\Omega|$ or $\Omega_n \rightarrow \Omega$ in $L^1(B)$). Let $f \in L^2(B)$ and $v \in L^2(B)$. Then we have:

$$\mathcal{X}_{\Omega_n} \xrightarrow{L^1(B)} \mathcal{X}_{\Omega} \tag{26}$$

Since \mathcal{X}_{Ω_n} and the limit \mathcal{X}_{Ω} take values in $\{0, 1\}$, we get

$$\mathcal{X}_{\Omega_n} \xrightarrow{L^\infty(B)} \mathcal{X}_{\Omega}. \tag{27}$$

Hence we have

$$\mathcal{X}_{\Omega_n} f v \xrightarrow{L^1(B)} \mathcal{X}_{\Omega} f v,$$

that is to say

$$\int_B \mathcal{X}_{\Omega_n} f v \longrightarrow \int_B \mathcal{X}_{\Omega} f v.$$

□

Lemma 5.3 *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded set Ω . Let $(u_n)_{n \geq 1}$ be a sequence of solutions of (1) defined in $H^{\frac{3}{2}}(\Omega_n)$, and let u be its limit in $H^{\frac{3}{2}}(\Omega)$. Then, for all v in $W^{1,\infty}(B)$, we have:*

$$\int_B \mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \longrightarrow \int_B \mathcal{X}_{\Omega} \nabla u \cdot \nabla v$$

where \mathcal{X}_{Ω_n} and \mathcal{X}_{Ω} are the classical characteristic functions defined in $L^1(B)$.

Proof: As $\nabla u_n \longrightarrow \nabla u$ and $\mathcal{X}_{\Omega_n} \longrightarrow \mathcal{X}_{\Omega}$ in $L^2(B)$, we get $\nabla u_n \mathcal{X}_{\Omega_n} \longrightarrow \nabla u \mathcal{X}_{\Omega}$ in $L^1(B)$. Since $\nabla v \in L^\infty(B)$, we deduce that $\mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \longrightarrow \mathcal{X}_{\Omega} \nabla u \cdot \nabla v$ in $L^1(B)$. Hence we obtain

$$\int_B \mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \longrightarrow \int_B \mathcal{X}_{\Omega} \nabla u \cdot \nabla v.$$

□

5.3 Study of the surface term

Lemma 5.4 *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded domain Ω . Let $(u_n)_{n \geq 1}$ be a sequence of solutions of (1) defined in $H^{\frac{3}{2}}(\Omega_n)$, and let u be its limit in $H^{\frac{3}{2}}(\Omega)$. Then, for all v in $W^{1,\infty}(B)$, we have:*

$$\int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v \longrightarrow \int_{\partial\Omega} \partial_{\mathbf{n}} u v.$$

Proof: We have:

$$\begin{aligned} \int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v &= \int_{\Omega_n} \Delta u_n v + \int_{\Omega_n} \nabla u_n \cdot \nabla v = - \int_{\Omega_n} f v + \int_{\Omega_n} \nabla u_n \cdot \nabla v \\ &= - \int_B \mathcal{X}_{\Omega_n} f v + \int_B \mathcal{X}_{\Omega_n} \nabla u_n \cdot \nabla v \\ &\longrightarrow - \int_B \mathcal{X}_{\Omega} f v + \int_B \mathcal{X}_{\Omega} \nabla u \cdot \nabla v \quad \text{by Lemmas 5.2 and 5.3} \\ &= - \int_{\Omega} f v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} \Delta u v + \int_{\Omega} \nabla u \cdot \nabla v \quad \text{by Lemma 5.1} \\ &= \int_{\partial\Omega} \partial_{\mathbf{n}} u v. \end{aligned}$$

□

To end this subsection, we prove the following lemma which will be crucial in the proof of Theorem 2:

Lemma 5.5 *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open bounded sets that converges in the Hausdorff sense to a Lipschitz open bounded set Ω . Let $(u_n)_{n \geq 1}$ be a sequence of solutions of (1) defined in $H^{\frac{3}{2}}(\Omega_n)$, and let u be its limit in $H^{\frac{3}{2}}(\Omega)$. Then, for any $v \in W^{1,\infty}(B)$, we have:*

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow - \int_{\partial\Omega} \partial_{\mathbf{n}} u v.$$

Proof: The proof can be easily done. From the boundary conditions in (3), we have:

$$\int_{\partial\Omega_n} -\Delta_\tau u_n v + \int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v = 0.$$

Using integration by parts, we obtain

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v = - \int_{\partial\Omega_n} \partial_{\mathbf{n}} u_n v.$$

Applying the limit to the right-hand side as n tends to infinity, we get using Lemma 5.4

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow - \int_{\partial\Omega} \partial_{\mathbf{n}} u v.$$

□

In order to complete the path concerning the surface terms, we still have to show that for all v in $W^{2,\infty}(B)$, we have

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v$$

which will be proved in Proposition 7.1 (see section 7). In fact, to prove this proposition, we need to study locally the attitude of the convergence of the signed distance function $b_n := b_{\Omega_n}$ to $b := b_\Omega$ in $W^{2,p}$, with $p \geq 1$, in a uniform tubular neighborhood U_h of the boundaries for some $h > 0$, and actually we need the $W^{2,\infty}$ convergence of $(2, \infty)$ -Sobolev domains (here the $(2, \infty)$ -Sobolev regularity has to be injected into the domains Ω_n and Ω , which is much smoother than the Lipschitz one).

6 Study of the convergence of the signed distance and projection functions in $W^{1,p}$

In this section, we are going to study the behavior of the convergence in $W^{1,p}(B)$, with $1 \leq p \leq \infty$, of the sequence of the oriented distance functions $(b_n)_{n \in \mathbf{N}}$ or $(b_{\Omega_n})_{n \in \mathbf{N}}$ with respect to the Hausdorff convergence of less regular domains (uniformly Lipschitz and positive reach regularities). After that, assuming that b_n converges toward b in $W^{2,\infty}$, we show the existence of a uniform tubular neighborhood U_h of the boundaries for some $h > 0$, for $(2, \infty)$ -Sobolev domains. Also, we study the convergence of the differential of the projection functions p_n on $\partial\Omega_n$ in $L^p(U_h)$, $1 \leq p \leq \infty$, with the assumption of $W^{2,\infty}$ convergence of $(2, \infty)$ -Sobolev domains.

6.1 L^p convergence of the signed distance functions for uniformly Lipschitz domains, for $1 \leq p \leq \infty$

Proposition 6.1 *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open bounded sets, that converges to a uniformly Lipschitz open bounded set Ω , in the Hausdorff distance sense, and let B be a large ball containing Ω_n and Ω . Then we have:*

$$b_{\Omega_n} \xrightarrow{L^\infty(B)} b_\Omega.$$

In particular, for $p \geq 1$ we have

$$b_{\Omega_n} \xrightarrow{L^p(B)} b_\Omega.$$

Proof: If $p = \infty$, the the proof is done for the complementary of Ω_n and Ω in M. Dambrine and B. Puig article (see [6], p.5, Proposition 2.7, Step 2.)

Moreover, if $p < \infty$, we have:

$$\|b_n - b\|_{L^p(B)}^p = \int_B |b_n(x) - b(x)|^p \leq \int_B (\sup_{x \in B} |b_n(x) - b(x)|)^p = \|b_n - b\|_{L^\infty(B)}^p \text{vol}(B) \xrightarrow{n} 0.$$

□

6.2 $W^{1,\infty}$ convergence of the signed distance functions for domains of positive reach

Proposition 6.2 *Let $(\Omega_n)_n$ be a sequence of open bounded sets of uniform positive reach (i.e. each skeleton is distant from its corresponding boundary) that converges to an open bounded domain Ω of positive reach, in the Hausdorff distance sense. Let h_n be a positive real number and let U_h be a uniform tubular neighborhood, so that $0 < h < h_n, \forall n \in \mathbb{N}$, such that $U_h \subset \cap_n U_{h_n}$, where*

$$U_{h_n} := \{x \in B; |b_{\partial\Omega_n}(x)| \leq h_n\}.$$

Then we have

$$b_{\Omega_n} \xrightarrow{W^{1,\infty}(U_h)} b_\Omega.$$

Proof: First of all, by [7, Theorem 3.3, (vii), p.347], we compute the gradient of b_Ω :

$$\nabla b_\Omega(x) = \left\{ \begin{array}{ll} \nabla d_\Omega(x) = \frac{x - p(x)}{|x - p(x)|}, & x \in \mathbb{R}^d \setminus \bar{\Omega}, \\ \nabla d_{\mathbb{C}\Omega}(x) = -\frac{x - p(x)}{|x - p(x)|}, & x \in \Omega \end{array} \right\}, \quad |\nabla b_\Omega(x)| = 1, \quad (28)$$

where d_Ω is the distance function to Ω , knowing that $b_\Omega(x) = d_\Omega(x) - d_{\mathbb{C}\Omega}(x)$, and p is the orthogonal projection of $x \in U_h$ on $\partial\Omega$. Note that $p(x)$ is unique in U_h . We will denote by p_n the projection on $\partial\Omega_n$ instead of p_{Ω_n} .

Then we have for all $x \in U_h \setminus \cup_{n \in \mathbb{N}} (\partial\Omega_n \cup \partial\Omega)$:

$$|\nabla b_{\Omega_n}(x) - \nabla b_{\Omega}(x)| = \left| \frac{x - p_n(x)}{|x - p_n(x)|} - \frac{x - p(x)}{|x - p(x)|} \right|.$$

Since the function $\varphi(y) = \frac{y}{|y|}$ is Lipschitz continuous on $\mathbb{R}^n \setminus \{0\}$, we obtain

$$|\nabla b_{\Omega_n}(x) - \nabla b_{\Omega}(x)| \leq |x - p_n(x) - x + p(x)| = |p_n(x) - p(x)|. \quad (29)$$

Also, from Proposition 6.1, we get for all $x \in U_h$:

$$d(x, p_n(x)) = b_{\Omega_n}(x) \longrightarrow b_{\Omega}(x) = d(x, p(x)). \quad (30)$$

Now, since U_h is compact, $\exists y \in U_h$, and there exists a subsequence $(p_k(x))_{k \in \mathbb{N}}$ of the sequence $(p_n(x))_{n \in \mathbb{N}}$, still denoted by $(p_n(x))_{n \in \mathbb{N}}$, such that $p_n(x) \longrightarrow y$ in U_h . Then it remains to show that $y \in \bar{\Omega}$ and $y = p(x)$.

Step 1. Setting $y_n := p_n(x)$, we have

$$d(y_n, \bar{\Omega}) \leq d(y_n, \bar{\Omega}_n) + d_H(\bar{\Omega}_n, \bar{\Omega}) = 0 + d_H(\bar{\Omega}_n, \bar{\Omega}) = d(\bar{\Omega}_n, \bar{\Omega}).$$

Therefore, passing to the limit, taking into consideration the Hausdorff convergence and from the continuity of the distance function $d(\cdot, \bar{\Omega})$, we get:

$$d(y, \bar{\Omega}) = \lim_n d(y_n, \bar{\Omega}) \leq \lim_n d(\bar{\Omega}_n, \bar{\Omega}) = 0.$$

So $y \in \bar{\Omega}$.

Step 2. If $p_n(x) \longrightarrow y$, then $d(x, p_n(x)) \longrightarrow d(x, y)$ by continuity of the distance function. Thus by (30), we get $d(x, y) = d(x, p(x))$ by the uniqueness of the limit. Since $x \in U_h$, $y \mapsto d(x, y)$ has the unique minimizer $p(x)$ on $\bar{\Omega}$. Hence $y = p(x)$.

Therefore $|p_n(x) - p(x)| \longrightarrow 0$ for all $x \in U_h$, this means that

$$\|p_n - p\|_{L^\infty(U_h)} \longrightarrow 0, \quad (31)$$

and by (29), we get

$$\|\nabla b_{\Omega_n} - \nabla b_{\Omega}\|_{L^\infty(U_h)} \longrightarrow 0. \quad (32)$$

Finally, by Proposition 6.1, we obtain the result. □

6.3 $W^{1,p}$ convergence of the signed distance functions for uniformly Lipschitz domains, $1 \leq p < \infty$

Proposition 6.3 *Let $(\Omega_n)_n$ be a sequence of uniformly Lipschitz open bounded sets in \mathbb{R}^d , that converges to a Lipschitz open bounded set Ω , in the Hausdorff distance sense. Let B be a large ball containing all these domains. Then, we have for $1 \leq p < \infty$*

$$b_{\Omega_n} \xrightarrow{W^{1,p}(B)} b_{\Omega}.$$

Proof: The proof here is much sharper than the precedent one. First of all, let us remark that since we don't have the positive reach condition, the points on the Skeletons (here the skeletons exist at least locally in a tabular neighborhood of $\partial\Omega$) do not have a unique projection on the boundary $p(x)$. In other word, the function $|p_n(x) - p(x)|$ is not defined on all B ; hence to overcome this problem, we look for a new procedure in the following:

Let $x \in B \setminus \cup_{n \in \mathbb{N}}(\partial\Omega_n \cup \partial\Omega)$. Set

$$\Theta_n := \arg \min_{y \in \partial\Omega_n} d(x, y) = \{y \in \partial\Omega_n \mid d(x, y) = d(x, \partial\Omega_n)\},$$

and

$$\Theta := \arg \min_{y \in \partial\Omega} d(x, y) = \{y \in \partial\Omega \mid d(x, y) = d(x, \partial\Omega)\},$$

and define the functions g_n , $n \in \mathbb{N}$, on B , such that:

$$\forall x \in B, \quad g_n(x) := \inf_{y_n \in \Theta_n, y \in \Theta} d(y_n, y).$$

In fact, we have for $x \in B \setminus \cup_{n \in \mathbb{N}}\partial\Omega_n \cup \partial\Omega$,

$$|\nabla b_n(x) - \nabla b(x)| \leq |p_n(x) - p(x)| = g_n(x).$$

Therefore, if we prove that g_n goes to zero pointwisely, and since $|\nabla b_n(x) - \nabla b(x)| \leq |\nabla b_n(x)| + |\nabla b(x)| = 1 + 1 = 2$ which is integrable on B , then $\nabla b_{\Omega_n} \xrightarrow{L^1(B)} \nabla b_{\Omega}$, thanks to the dominated convergence theorem, and hence $b_{\Omega_n} \xrightarrow{W^{1,1}(B)} b_{\Omega}$. Now, by M. C. Delfour and J.-P. Zolésio book (see [7 p.350, Theorem 4.1 (i)]), we have the convergence in $W^{1,p}(B)$, for all integers $1 \leq p < \infty$.

Pointwise convergence: From the Hausdorff converges of Ω_n to Ω , we have:

$$\forall \varepsilon > 0, \exists N_{\varepsilon} > 0, \text{ such that } \forall n \geq N_{\varepsilon}, d(\Omega_n, \Omega) \leq \varepsilon. \quad (33)$$

Then we get (see Figure 2.2):

1. $B(p_n(x), \varepsilon) \cap \Omega_n \neq \emptyset$, i.e.

$$\exists y_n \in \partial\Omega_n, \text{ such that } d(p_n(x), x) \leq d(y_n, x) \leq d(p(x), x) + \varepsilon \leq d(y, x) + \varepsilon.$$

2. $B(p(x), \varepsilon) \cap \Omega_n \neq \emptyset$, i.e.

$$\exists y \in \partial\Omega_n, \text{ such that } d(p(x), x) \leq d(y, x) \leq d(p_n(x), x) + \varepsilon \leq d(y_n, x) + \varepsilon.$$

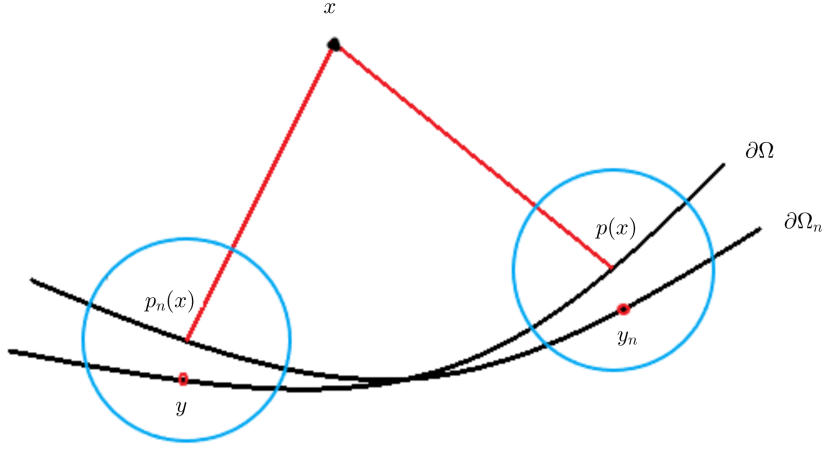


Figure 2: The projection of point x on $\partial\Omega_n$ and $\partial\Omega$.

Therefore, we have on one side:

$$|d(y_n, x) - d(y, x)| \leq \varepsilon. \quad (34)$$

On the other side, we get

$$|d(p_n(x), x) - d(p(x), x)| \leq \varepsilon,$$

that is equivalent to say

$$|d(\partial\Omega_n, x) - d(\partial\Omega, x)| \leq \varepsilon. \quad (35)$$

so, in order to finish the proof of the pointwise convergence, we have to prove that

$$\exists y_n \in \Theta_n, \exists y \in \Theta, \text{ such that } d(y_n, y) \longrightarrow 0. \quad (36)$$

To prove that, let $(y_n)_{n \in \mathbb{N}}$, with $y_n \in \Theta_n$, be a sequence in B . Since B is compact, there exists a subsequence $(y_k)_k$ (still denoted by $(y_n)_n$ since no confusion) of $(y_n)_n$ in B , and there exist an element y in B , such that $y_n \longrightarrow y$ in B . Now, it is sufficient to prove that $y \in \partial\Omega$.

Indeed, we have:

$$\begin{aligned} d(x, y) &= \lim_n d(x, y_n) && \text{by estimation (34)} \\ &= \lim_n d(x, \partial\Omega_n) && \text{since } y_n \in \Theta_n \\ &= d(x, \partial\Omega) && \text{by estimation (35)} \end{aligned}$$

hence, $y \in \partial\Omega$, and the proof of (36) is finished. □

6.4 L^p convergence of the differential of the projection function for $(2, \infty)$ -Sobolev domains, for $1 \leq p \leq \infty$

In the following, as mentioned in the introduction, we are going to deal with the second-order derivatives of the signed distance function. In fact, we need such an extra regularity on Ω_n and Ω , which is the (m, p) -Sobolev assumption, with $m > 1$ and $p \geq 1$, in which via these regularity, the m -th order derivatives can be defined now locally. In particular, it is sufficient to choose $m = 2$ to deal with the second-order derivatives. Let us remark that in order to avoid working with Radon measures as in [7], [9], and [10], we would like to make a truncation around the boundaries of the domains to get rid of the singular points of the signed distance functions in B . In other words, we separate the skeletons from the boundaries of $\partial\Omega_n$ and $\partial\Omega$. Actually, we have to deal with the Hessian matrices D^2b_n and D^2b in a uniform tubular neighborhood U_h , which always exists (see Proposition 6.4). Moreover, If we set p_n and p to be the projection on $\partial\Omega_n$ and $\partial\Omega$, respectively, we will study the convergence of $Dp_n(x)$ to $Dp(x)$ for all $x \in U_h$, which will be presented in Proposition 6.5. This proposition could be proved thanks to Gunther Leobacher and Alexander Steinicke paper (see [8, Theorem C]), where they investigated an explicit formula for $Dp(x)$, $x \in U_h$.

To do so, let us introduce the notion of a tubular neighborhoods of $\partial\Omega_n$ for all $n \in \mathbb{N}$ (see Figure 3), then we show the existence of a uniform one, in the following proposition:

Proposition 6.4 *Let $(\Omega_n)_n$ be a sequence of $(2, \infty)$ -Sobolev domains that converges toward a $(2, \infty)$ -Sobolev domain Ω in the $W^{2, \infty}$ sense. For small $h_n > 0$ and $\forall n \in \mathbb{N}$, we set*

$$U_{h_n} := \{x \in B; |b_{\partial\Omega_n}(x)| \leq h_n\}.$$

Then there exists $h > 0$, such that $h_n > h$, $\forall n \in \mathbb{N}$, and a uniform tubular neighborhood U_h , such that $U_h \subset \cap_n U_{h_n}$, so that we guarantee that the chosen domain U_h will not include any of the skeletons of Ω_n and Ω , for all $n \in \mathbb{N}$.

Proof: Suppose by contrary that there exists $n_0 \in \mathbb{N}$ such that $h_{n_0} < h$, for all h small enough, then the skeleton of this set Ω_{n_0} will approach from the boundary $\partial\Omega_{n_0}$, this means that the radius of curvature of this boundary at a point x_{n_0} is going to zero. This is equivalent to saying that: at least one of the principle curvature of this boundary at x_{n_0} is going to infinity. But we know that Ω_{n_0} is converging to Ω in the $W^{2, \infty}$ sense, i.e. the eigenvalues of the Hessian matrix $D^2b_{n_0}$ are converging toward those of D^2b . Hence, we get a contradiction. (see Figure 2.4). □

Remark 6.1 *We remark that Proposition 6.4 is not true for the Hausdorff convergence of the domains (see a counter example in dimension two below).*

Counter example: Let us give a counter example in dimension two for Hausdorff convergence of sets. Let $\gamma : s \in \mathbb{R} \longrightarrow \mathbb{R}^2$ be a parametrization of the boundary $\partial\Omega$. We make a small perturbation γ_ε , for all $\varepsilon > 0$ small enough, around a point $x_0 \in \partial\Omega$ in the following way:

$$\gamma_\varepsilon(s) := \gamma(s) + \varepsilon \varphi\left(\frac{s - s_0}{\varepsilon}\right)$$

where φ is a mollifier and $\gamma(s_0) = x_0$. One can see that as ε goes to zero, the perturbed domain Ω_ε converges to Ω in the Hausdorff sense. But the distance h_ε between the skeleton $Sk(\Gamma_\varepsilon)$ of the perturbed section Γ_ε of $\partial\Omega_\varepsilon$ and the curve Γ_ε itself will tend to zero as $\varepsilon \longrightarrow 0$ (see Figure 5). This contradict the fact that there exist $h > 0$ such that $h_\varepsilon > h$ for all $\varepsilon > 0$.

Now, let us make a uniform cut-off of size h .

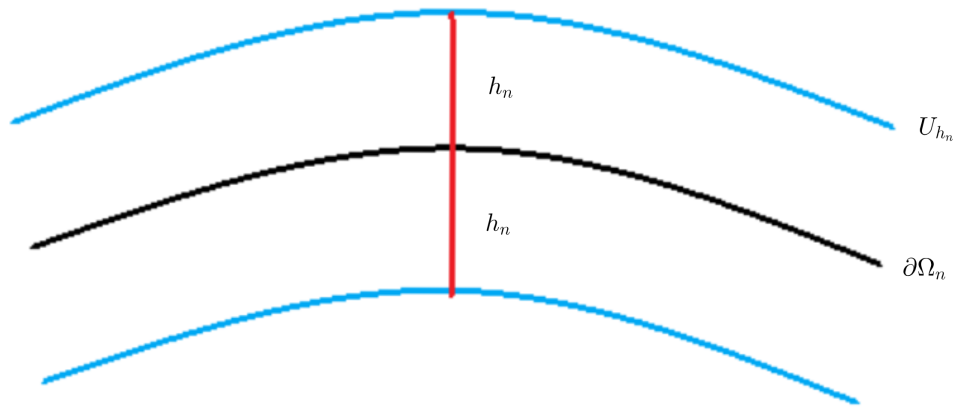


Figure 3: Tabular neighborhood of the boundary $\partial\Omega_n$ locally.

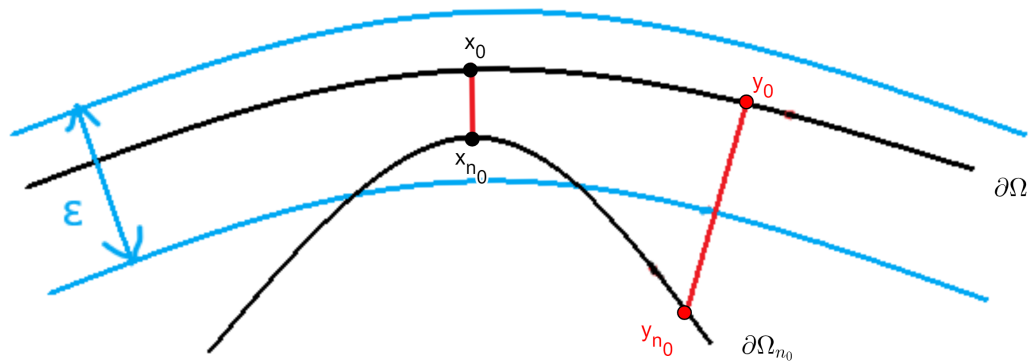


Figure 4: Curvatures of $\partial\Omega_{n_0}$ and $\partial\Omega$ at points x_{n_0} and x_0 respectively.

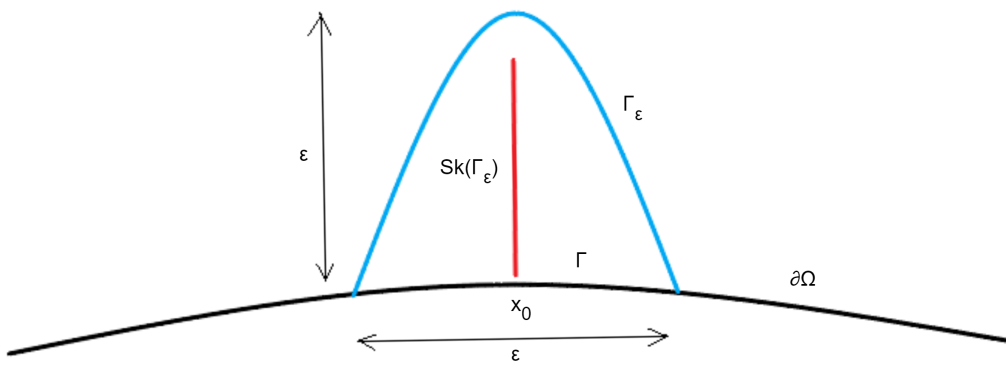


Figure 5: The perturbed domain

Definition 6.1 We define a C^∞ uniform cut-off function ψ as:

$$\psi := \ell \left(\frac{d}{h} \right)$$

where d is the distance function to the limit set Ω , and ℓ is a C^∞ function defined on \mathbb{R}^+ as the following:

$$\ell(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1. \end{cases}$$

◇

Now, we prove the convergence of the sequence $(Dp_n)_n$ to Dp in $L^p(U_h)$ for $p \geq 1$, where Dp_n and Dp represent the differential of the projection functions p_n and p , respectively.

Proposition 6.5 Let $(\Omega_n)_n$ be a sequence of $(2, \infty)$ -Sobolev open bounded domains in \mathbb{R}^d , that converges to a $(2, \infty)$ -Sobolev open bounded domain Ω , in the $W^{2, \infty}$ sense. Then, for $1 \leq p \leq \infty$, we have

$$Dp_n \xrightarrow{L^p(U_h)} Dp,$$

where U_h is a uniform tubular neighborhood defined in subsection 6.4 and $Dp_n \xrightarrow{L^p(U_h)} Dp$, $1 \leq p \leq \infty$, means that the ij -component $D_n^{ij} \xrightarrow{L^p(U_h)} D^{ij}$ for all $i, j = 1, \dots, d$.

Proof: First of all, let us compute explicitly the matrix $Dp(x)$ at the point $x \in U_h$. To do so, we refer to [8, Theorem C], where the differential of p at $x \in U_h$ is computed as

$$Dp(x) = \left(Id_{T_{p(x)}(\partial\Omega)} - |x - p(x)|L_{p(x),v} \right)^{-1} P_{T_{p(x)}(\partial\Omega)}, \quad (37)$$

where $v = |x - p(x)|^{-1}(x - p(x))$ and $L_{p(x),v}$ is the shape operator in the direction v at $p(x)$, and $P_{T_{p(x)}(\partial\Omega)}$ is the projection operator on $T_{p(x)}(\partial\Omega)$.

Let us explicit the shape and projection operator in local coordinates. We define the basis \mathcal{B} of \mathbb{R}^d around $p(x)$ like this: the first $d - 1$ vectors are the principle vectors of the tangent space $T_{p(x)}(\partial\Omega)$ at $p(x)$ and the last vector is the normal unit vector v . Then we have:

$$P_{T_{p(x)}(\partial\Omega)} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

and the shape operator in the same basis can be written as

$$L_{p(x),v} = \begin{pmatrix} k_1(x) & 0 & \dots & \dots & 0 \\ 0 & k_2(x) & \dots & \dots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & k_{d-1}(x) & 0 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

where $k_i(x)$, $i = 1, \dots, d-1$, are the principle curvatures of $\partial\Omega$ at $p(x)$.

Hence, replacing $|x - p(x)|$ by $b(x)$, we get

$$\begin{aligned} Dp(x) &= (Id_{T_{p(x)}(\partial\Omega)} - b(x)L_{p(x),v})^{-1}P_{T_{p(x)}(\partial\Omega)} \\ &= \begin{pmatrix} \frac{1}{1-bk_1(x)} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{1-bk_2(x)} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{1}{1-bk_{d-1}(x)} & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}. \end{aligned} \quad (38)$$

Notice that the denominators $1 - bk_i(x)$, $i = 1, \dots, d$, cannot be zero since we are working in a bounded volume U_h . Clearly, the matrix $Dp(x)$ is diagonal with respect to the basis \mathcal{B} , so it is sufficient to study the diagonal terms. Therefore, if we denote by $Dp^{ij}(x)$ the ij -th component of the matrix $Dp(x)$, we have for all $i = j$:

1) **Convergence in $L^\infty(U_h)$:**

$$\begin{aligned} \|Dp_n^{ii} - Dp^{ii}\|_{L^\infty(U_h)} &= \sup_{x \in U_h} |Dp_n^{ii}(x) - Dp^{ii}(x)| = \sup_{x \in U_h} \left| \frac{1}{1-b_n k_{n,i}(x)} - \frac{1}{1-bk_i(x)} \right| \\ &= \sup_{x \in U_h} \left| \frac{b_n k_{n,i}(x) - bk_i(x)}{(1-b_n k_{n,i}(x))(1-bk_i(x))} \right| = \sup_{x \in U_h} \left| \frac{b_n k_{n,i}(x) - bk_i(x) + b_n k_i(x) - b_n k_i(x)}{(1-b_n k_{n,i}(x))(1-bk_i(x))} \right| \\ &\leq \sup_{x \in U_h} \left| \frac{(b_n - b)k_i(x)}{(1-b_n k_{n,i}(x))(1-bk_i(x))} \right| + \sup_{x \in U_h} \left| \frac{b_n(k_{n,i} - k_i)(x)}{(1-b_n k_{n,i}(x))(1-bk_i(x))} \right| \end{aligned}$$

In fact, since D^2b_n tends to D^2b in $L^\infty(U_h)$, we have the convergence of the eigenvalues of the Hessian matrix D^2b_n toward those of D^2b in $L^\infty(U_h)$, in other words, the principle curvatures $k_{n,i}(x)$ for all $i = 1, \dots, d$ converge uniformly toward $k_i(x)$ (see Figure 2.4). Therefore, we have $\|k_{n,i} - k_i\|_{L^\infty(U_h)} = \sup_{x \in U_h} |k_{n,i}(x) - k_i(x)| \xrightarrow{n} 0$. In particular, the principle curvatures $k_{n,i}$, $i = 1, \dots, d$, are uniformly bounded with respect to $n \in \mathbb{N}$. In addition, the oriented distance functions b_n go to b in $L^\infty(U_h)$, hence they are also uniformly bounded with respect to $n \in \mathbb{N}$. Therefore, we get:

$$\|Dp_n^{ii} - Dp^{ii}\|_{L^\infty(U_h)} \leq C_1 \|b_n - b\|_{L^\infty(U_h)} + C_2 \|k_{n,i} - k_i\|_{L^\infty(U_h)} \xrightarrow{n} 0$$

where C_1 and C_2 are two positive constants independent of $n \in \mathbb{N}$. Finally, we get the desired result:

$$Dp_n \xrightarrow{L^\infty(U_h)} Dp. \quad (39)$$

2) **Convergence in $L^p(U_h)$, $p < \infty$:**

For all $i = j$, we have:

$$\begin{aligned} \|Dp_n^{ii} - Dp^{ii}\|_{L^p(U_h)}^p &= \int_{U_h} |Dp_n^{ii}(x) - Dp^{ii}(x)|^p \\ &\leq \int_{U_h} (\sup_{x \in U_h} |Dp_n^{ii}(x) - Dp^{ii}(x)|)^p \\ &= \|Dp_n^{ii} - Dp^{ii}\|_{L^\infty(U_h)}^p \text{vol}(U_h) \longrightarrow 0. \end{aligned}$$

Hence, we have it for all $i, j = 1, \dots, d$. Therefore, we get the result:

$$Dp_n \xrightarrow{L^p(U_h)} Dp. \quad (40)$$

□

7 The remaining convergence on the surface

In order to prove Theorem 2, we have to give the following crucial proposition:

Proposition 7.1 *If $u_n \in H^{\frac{3}{2}}(\Omega_n)$ and $u \in H^{\frac{3}{2}}(\Omega)$ are defined as in Remark 4.1, then for all $v \in W^{2,\infty}(B)$, we have:*

$$\int_{\partial\Omega_n} \nabla_\tau u_n \cdot \nabla_\tau v \longrightarrow \int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v.$$

Proof: We begin by transferring our domains of the integrals to the large one B which is containing all of Ω_n and Ω , then we proceed by applying the truncation on the uniform domain U_h .

First of all, let us recall that the tangential vector $\nabla_\tau u$ can be extended to the vector $\nabla u - (\nabla u \cdot \nabla b) \nabla b$ in the uniform tubular neighborhood U_h . Now, without loss of generality, we choose the test function $v \in W^{2,\infty}(B)$ so that the normal derivative vanishes on $\partial\Omega$ (i.e. we get $\nabla v = \nabla_\tau v$ in B). Then, we have:

$$\int_{\partial\Omega} \nabla_\tau u \cdot \nabla_\tau v = \int_{\partial\Omega} [\nabla u - (\nabla u \cdot \nabla b) \nabla b] \cdot \nabla v = \int_{\partial\Omega} \nabla u \cdot \nabla v - \int_{\partial\Omega} (\nabla u \cdot \nabla b) \nabla b \cdot \nabla v$$

Now, as the normal vector \mathbf{n} is unitary and using the fact that ∇b is the extension of \mathbf{n} in U_h , we obtain by applying the divergence theorem: for all $v \in W^{2,\infty}(B)$,

$$\begin{aligned} \int_{\partial\Omega} \nabla u \cdot \nabla v &= \int_{\partial\Omega} \nabla u \cdot \nabla v (\mathbf{n} \cdot \mathbf{n}) = \int_{\partial\Omega} \psi (\nabla u \cdot \nabla v) (\mathbf{n} \cdot \mathbf{n}) \\ &= \int_{\partial\Omega} [\psi (\nabla u \cdot \nabla v) \mathbf{n}] \cdot \mathbf{n} = \int_{\Omega} \text{div} [\psi (\nabla u \cdot \nabla v) \mathbf{n}] = \int_{U_h} \text{div} [\psi (\nabla u \cdot \nabla v) \nabla b] \end{aligned}$$

where the cut-off function ψ is introduced in Definition 6.1. Hence we get

$$\int_{\partial\Omega} \nabla u \cdot \nabla v = \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi \nabla (\nabla u \cdot \nabla v) \cdot \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b$$

$$\begin{aligned}
&= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi (\nabla (\nabla u^T) \nabla v^T) \cdot \nabla b + \int_{U_h} \psi \nabla u^T \nabla (\nabla v^T) \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\
&= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi \nabla v^T \nabla (\nabla u^T) \nabla b + \int_{U_h} \psi \nabla u^T \nabla (\nabla v^T) \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\
&= \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b + \int_{U_h} \psi \nabla v^T D^2 u \nabla b + \int_{U_h} \psi \nabla u^T D^2 v \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b.
\end{aligned}$$

Also, we have:

$$\begin{aligned}
\int_{\partial\Omega} (\nabla u \cdot \nabla b) \nabla b \cdot \nabla v &= \int_{\partial\Omega} \psi (\nabla u \cdot \nabla b) \nabla v \cdot \mathbf{n} = \int_{\Omega} \operatorname{div} [\psi (\nabla u \cdot \nabla b) \nabla v] = \int_{U_h} \operatorname{div} [\psi (\nabla u \cdot \nabla b) \nabla v] \\
&= \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v + \int_{U_h} \psi \nabla (\nabla u \cdot \nabla b) \cdot \nabla v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v \\
&= \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v + \int_{U_h} \psi \nabla b^T \nabla (\nabla u^T) \nabla v + \int_{U_h} \psi \nabla u^T \nabla (\nabla b^T) \nabla v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v \\
&= \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v + \int_{U_h} \psi \nabla b^T D^2 u \nabla v + \int_{U_h} \psi \nabla u^T D^2 b \nabla v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v.
\end{aligned}$$

Therefore, due to cancellation of some terms, we get:

$$\begin{aligned}
&\int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v = \\
&\int_{U_h} \psi \nabla u^T D^2 v \nabla b + \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b + \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b \\
&- \int_{U_h} \psi \nabla u^T D^2 b \nabla v - \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v - \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v.
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
&\int_{\partial\Omega_n} \nabla_{\tau} u_n \nabla_{\tau} v - \int_{\partial\Omega} \nabla_{\tau} u \nabla_{\tau} v = \\
&\int_{U_h} \psi \nabla u_n^T D^2 v \nabla b_n - \int_{U_h} \psi \nabla u^T D^2 v \nabla b + \int_{U_h} \psi (\nabla u_n \cdot \nabla v) \Delta b_n - \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b \\
&- \int_{U_h} \psi \nabla u_n^T D^2 b_n \nabla v + \int_{U_h} \psi \nabla u^T D^2 b \nabla v - \int_{U_h} \psi (\nabla u_n \cdot \nabla b_n) \Delta v + \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v \\
&+ \int_{U_h} (\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b - \int_{U_h} (\nabla u_n \cdot \nabla b_n) \nabla \psi \cdot \nabla v + \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v \\
&= (A_n - A) + (B_n - B) - (C_n - C) - (D_n - D) + (E_n - E) - (F_n - F).
\end{aligned}$$

Let us study the convergence of each of these six terms:

1) **Convergence of $A_n - A$ to 0:**

We have:

$$\begin{aligned} & \int_{U_h} \psi \nabla u_n^T D^2 v \nabla b_n - \int_{U_h} \psi \nabla u^T D^2 v \nabla b \\ &= \int_{U_h} \psi \left[\sum_{i=1}^d \sum_{j=1}^d \partial_i u_n \partial_{ij} v \partial_j b_n - \sum_{i=1}^d \sum_{j=1}^d \partial_i u \partial_{ij} v \partial_j b \right] \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} v \partial_j b_n - \partial_i u \partial_{ij} v \partial_j b]. \end{aligned}$$

As $\partial_i u_n \xrightarrow{n} \partial_i u$ and $\partial_j b_n \xrightarrow{n} \partial_j b$ in $L^2(U_h)$, we get $\partial_i u_n \partial_j b_n \xrightarrow{n} \partial_i u \partial_j b$ in $L^1(U_h)$. Since $\psi \partial_{ij} v \in L^\infty(U_h)$, we obtain $\psi \partial_i u_n \partial_{ij} v \partial_j b_n \xrightarrow{n} \psi \partial_i u \partial_{ij} v \partial_j b$ in $L^1(U_h)$, therefore we get

$$\int_{U_h} \psi \nabla u_n^T D^2 v \nabla b_n - \int_{U_h} \psi \nabla u^T D^2 v \nabla b = \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} v \partial_j b_n - \partial_i u \partial_{ij} v \partial_j b] \xrightarrow{n} 0.$$

Hence, we get the aimed convergence:

$$A_n - A \xrightarrow{n} 0. \quad (41)$$

2) **Convergence of $B_n - B$ to 0:**

In this point, we have:

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla v) \Delta b_n - \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b = \int_{U_h} \psi [(\nabla u_n \cdot \nabla v) \Delta b_n - (\nabla u \cdot \nabla v) \Delta b].$$

As $\nabla u_n \xrightarrow{n} \nabla u$ in $L^2(U_h)$ (hence in $L^1(U_h)$) and $\psi \nabla v \in L^\infty(U_h)$, we get $\psi \nabla u_n \cdot \nabla v \xrightarrow{n} \psi \nabla u \cdot \nabla v$ in $L^1(U_h)$. Since $\Delta b_n \xrightarrow{n} \Delta b$ in $L^\infty(U_h)$, we obtain $\psi [(\Delta b_n \nabla u_n - \Delta b \nabla u) \cdot \nabla v] \xrightarrow{n} 0$ in $L^1(U_h)$, therefore we have

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla v) \Delta b_n - \int_{U_h} \psi (\nabla u \cdot \nabla v) \Delta b = \int_{U_h} \psi [(\Delta b_n \nabla u_n - \Delta b \nabla u) \cdot \nabla v] \xrightarrow{n} 0.$$

Thus, we get the desired convergence:

$$B_n - B \xrightarrow{n} 0. \quad (42)$$

3) **Convergence of $C_n - C$ to 0:**

Applying the same procedure as before, we have:

$$\int_{U_h} \psi \partial_i u_n \partial_{ij} b_n \partial_j v - \int_{U_h} \psi \partial_i u \partial_{ij} b \partial_j v$$

$$\begin{aligned}
&= \int_{U_h} \psi \left[\sum_{i=1}^d \sum_{j=1}^d \partial_i u_n \partial_{ij} b_n \partial_j v - \sum_{i=1}^d \sum_{j=1}^d \partial_i u \partial_{ij} b \partial_j v \right] \\
&= \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} b_n \partial_j v - \partial_i u \partial_{ij} b \partial_j v].
\end{aligned}$$

As $\partial_i u_n \xrightarrow{n} \partial_i u$ and $\partial_{ij} b_n \xrightarrow{n} \partial_{ij} b$ in $L^2(U_h)$, we get $\partial_i u_n \partial_{ij} b_n \xrightarrow{n} \partial_i u \partial_{ij} b$ in $L^1(U_h)$. Since $\psi \partial_j v \in L^\infty(U_h)$, we obtain $\psi \partial_i u_n \partial_{ij} b_n \partial_j v \xrightarrow{n} \psi \partial_i u \partial_{ij} b \partial_j v$ in $L^1(U_h)$, thus we get

$$\int_{U_h} \psi \partial_i u_n \partial_{ij} b_n \partial_j v - \int_{U_h} \psi \partial_i u \partial_{ij} b \partial_j v = \sum_{i=1}^d \sum_{j=1}^d \int_{U_h} \psi [\partial_i u_n \partial_{ij} b_n - \partial_i u \partial_{ij} b] \partial_j v \xrightarrow{n} 0.$$

Therefore, we reach our target:

$$C_n - C \xrightarrow{n} 0. \quad (43)$$

4) **Convergence of $D_n - D$ to 0:**

We have:

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla b_n) \Delta v - \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v = \int_{U_h} \psi [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \Delta v.$$

As $\nabla u_n \xrightarrow{n} \nabla u$ and $\nabla b_n \xrightarrow{n} \nabla b$ in $L^2(U_h)$, we get $\nabla u_n \cdot \nabla b_n \xrightarrow{n} \nabla u \cdot \nabla b$ in $L^1(U_h)$. Since $\psi \Delta v$ in $L^\infty(U_h)$, we obtain $\psi [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \Delta v \xrightarrow{n} 0$ in $L^1(U_h)$, therefore we have

$$\int_{U_h} \psi (\nabla u_n \cdot \nabla b_n) \Delta v - \int_{U_h} \psi (\nabla u \cdot \nabla b) \Delta v = \int_{U_h} \psi [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \Delta v \xrightarrow{n} 0.$$

Thus, we get:

$$D_n - D \xrightarrow{n} 0. \quad (44)$$

5) **Convergence of $E_n - E$ to 0:**

In this point, we have:

$$\int_{U_h} (\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b = \int_{U_h} \nabla \psi \cdot [(\nabla u_n \cdot \nabla v) \nabla b_n - (\nabla u \cdot \nabla v) \nabla b].$$

As $\nabla u_n \xrightarrow{n} \nabla u$ in $L^2(U_h)$ (hence in $L^1(U_h)$) and $\nabla v \in L^\infty(U_h)$, we get $\nabla u_n \cdot \nabla v \xrightarrow{n} \nabla u \cdot \nabla v$ in $L^1(U_h)$. Since $\nabla b_n \xrightarrow{n} \nabla b$ in $L^\infty(U_h)$, we get $\nabla \psi \cdot \nabla b_n \xrightarrow{n} \nabla \psi \cdot \nabla b$ in $L^\infty(U_h)$. Hence, we obtain $(\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b \xrightarrow{n} 0$ in $L^1(U_h)$. Therefore we get

$$\int_{U_h} (\nabla u_n \cdot \nabla v) \nabla \psi \cdot \nabla b_n - \int_{U_h} (\nabla u \cdot \nabla v) \nabla \psi \cdot \nabla b \xrightarrow{n} 0.$$

Hence, we get the aimed convergence:

$$E_n - E \xrightarrow{n} 0. \quad (45)$$

6) **Convergence of $F_n - F$ to 0:**

We have:

$$\int_{U_h} (\nabla u_n \cdot \nabla b_n) \nabla \psi \cdot \nabla v - \int_{U_h} (\nabla u \cdot \nabla b) \nabla \psi \cdot \nabla v = \int_{U_h} \nabla \psi \cdot \nabla v [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b].$$

As $\nabla u_n \xrightarrow{n} \nabla u$ in $L^2(U_h)$ (hence in $L^1(U_h)$) and $\nabla b_n \xrightarrow{n} \nabla b$ in $L^\infty(U_h)$, we get $\nabla u_n \cdot \nabla b_n \xrightarrow{n} \nabla u \cdot \nabla b$ in $L^1(U_h)$. Since $\nabla \psi \cdot \nabla v \in L^\infty(U_h)$, we obtain

$$\nabla \psi \cdot \nabla v [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \xrightarrow{n} 0 \text{ in } L^1(U_h).$$

Therefore, we deduce that

$$\int_{U_h} \nabla \psi \cdot \nabla v [\nabla u_n \cdot \nabla b_n - \nabla u \cdot \nabla b] \xrightarrow{n} 0.$$

Hence, we get:

$$F_n - F \xrightarrow{n} 0. \quad (46)$$

□

8 The proofs

8.1 Proof of Theorem 1

Consider the sequence of vector fields $(h_n)_{n \in \mathbb{N}}$, with $h_n \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\forall n \in \mathbb{N}$. We define the sequence of diffeomorphisms $(\Phi_n)_n$, by $\Phi_n : \mathbb{R}^d \mapsto \mathbb{R}^d$ with $\Phi_n := Id + h_n$, so that we have $\Omega_n = \Phi_n(\Omega) = (Id + h_n)(\Omega)$. The proof is divided into 3 steps:

Step 1. Transforming the solutions onto the fixed domain Ω :

First of all, let us give the variational formula of (1) on the moving domains Ω_n :

$$\forall \varphi \in \mathcal{H}(\Omega_n), \quad \int_{\Omega_n} \nabla u_n \cdot \nabla \varphi + \int_{\partial \Omega_n} \nabla_\tau u_n \cdot \nabla_\tau \varphi = \int_{\Omega_n} f \varphi. \quad (47)$$

Then, we transform the solutions of (1) into a fixed domain Ω , using the inverse of the transformation function Φ_n . Let us set the following notations:

$$v_n := u_n \circ \Phi_n, \quad A_n := (D\Phi_n^{-1})^T D\Phi_n^{-1} J_n, \quad B_n := (D\Phi_n^{-1})^T D\Phi_n^{-1} J_n^\sigma,$$

where J_n and J_n^σ are the volume and the surface Jacobian, respectively. Then, transforming the integrals in equation (47) from Ω_n into the fixed domain Ω , we get:

$$\forall \varphi \in \mathcal{H}(\Omega), \quad \int_{\Omega} A_n \nabla v_n \cdot \nabla \varphi + \int_{\partial \Omega} B_n \nabla_\tau v_n \cdot \nabla_\tau \varphi = \int_{\Omega} J_n f \varphi. \quad (48)$$

Step 2. Uniform bound of v_n in $\mathcal{H}(\Omega)$:

By the uniform bound C shown in Proposition 3.2, we can deduce a uniform bound with respect to the parameter $n \in \mathbb{N}$, with the necessity of Ω to be in \mathcal{O} defined in section 4, so that we have

$$\|v_n\|_{\mathcal{H}(\Omega)} \leq C.$$

Step 3. Showing the convergence of v_n to v (the solution on Ω) in $\mathcal{H}(\Omega)$:

Assume that it is not the case. Then there exists a subsequence $(v_k)_{k \in \mathbb{N}}$ of the sequence $(v_n)_{n \in \mathbb{N}}$, and there exists $\beta > 0$, such that $\|v_k - v\|_{\mathcal{H}(\Omega)} \geq \beta > 0$. But we know that v_k is bounded in $\mathcal{H}(\Omega)$, hence there exists $v_\infty \in \mathcal{H}(\Omega)$ and a subsequence $(v_p)_{p \in \mathbb{N}}$ of the sequence $(v_k)_{k \in \mathbb{N}}$, such that $v_p \rightharpoonup v_\infty$ weakly in $\mathcal{H}(\Omega)$ as $p \rightarrow \infty$.

Now, writing (48) as

$$\forall \varphi \in \mathcal{H}(\Omega), \quad \int_{\Omega} A_p \nabla v_p \cdot \nabla \varphi + \int_{\partial\Omega} B_p \nabla_{\tau} v_p \cdot \nabla_{\tau} \varphi = \int_{\Omega} J_p f \varphi,$$

and passing to the limit when $p \rightarrow \infty$, we get:

$$\forall \varphi \in \mathcal{H}(\Omega), \quad \int_{\Omega} \nabla v_\infty \cdot \nabla \varphi + \int_{\partial\Omega} \nabla_{\tau} v_\infty \cdot \nabla_{\tau} \varphi = \int_{\Omega} f \varphi,$$

since A_p and B_p converge to Id and J_p goes to 1 when $p \rightarrow \infty$. Therefore, v_∞ is a solution of (1) in $\mathcal{H}(\Omega)$. By uniqueness of the solution and since v is the only accumulation point, it says that $v_\infty = v$. Thus $\forall \beta > 0$, we have $\|v_p - v\|_{\mathcal{H}(\Omega)} \leq \beta$, which leads to a contradiction.

8.2 Proof of Theorem 2

To start the proof, we have to gather Lemma 5.5 with Proposition 7.1, so that we get

$$- \int_{\partial\Omega} \partial_{\mathbf{n}} u v = \int_{\partial\Omega} \nabla_{\tau} u \cdot \nabla_{\tau} v,$$

by the uniqueness of the limit. Moreover, making integration by parts on the right-hand side, we obtain:

$$\forall v \in W^{2,\infty}(B), \quad \partial_{\mathbf{n}} u = \Delta_{\tau} u \quad \text{on } \partial\Omega.$$

Hence, we deduce that u is the unique solution of problem (1) in Ω . Furthermore, as u is the only accumulation point of the sequence $(u_n)_n$, we deduce from Corollary 4.2 that the whole sequence converges strongly to u in $H^1(B)$.

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