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Valentin Ritzenthaler, Xavier Ferrières, Sébastien Pernet, Guillaume Puigt. Compatible Discrete Operators scheme applied to Maxwell's Equation in and its link with FDTD. ACES 2023, Mar 2023, MONTEREY, United States. 10.23919/ACES57841.2023.10114701 . hal-04165249

**HAL Id: hal-04165249**

**<https://hal.science/hal-04165249>**

Submitted on 18 Jul 2023

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# Compatible Discrete Operators scheme applied to Maxwell's Equation in and its link with FDTD

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**Abstract**—We present an approach to discretize Maxwell's equations called Compatible Discrete Operators (CDO). It is a low-order discretization that belongs to the large family of mimetic schemes. This method is a generalization of the FDTD method. It discretizes the closure relations in a geometrically robust way that allows to deal with distorted and non-conformal polyhedral meshes. We briefly introduce the scheme, its link with the FDTD method and we give a first numerical result.

**Index Terms**—Maxwell's equations, time domain, low-order method, hybrid mesh

## I. INTRODUCTION

Geometrical accuracy, computational time and the size of the discrete problem are variables to optimize when dealing with simulations. The use of Cartesian meshes is a good way to increase the performances in terms of computational time and memory storage, but they are not well-adapted to capture geometrical details. The latter can be caught by using an unstructured mesh, leading to more complex numerical scheme. In this perspective, using polyhedral meshes seems to be a good trade-off between the accuracy and the computational cost. In this paradigm there are two solutions. The first one, is to hybridize meshes, with unstructured and structured zones, and schemes [1], by assigning a scheme to each zone of the mesh and ensure the energy conservation at zones interfaces through time. The second one, which will be considered in this paper, is to use schemes that are able to deal with hybrid mesh. The Compatible Discrete Operators (CDO) spatial discretization is one of these methods. It is a robust generalization of the finite difference method [2] that allows to deal with highly distorted and non-conformal polyhedral meshes. The presentation is done with the formalism proposed in [3], and lies on the work of [4], [2], [5].

## II. CDO DISCRETIZATION

We recall Maxwell's equations in time-domain

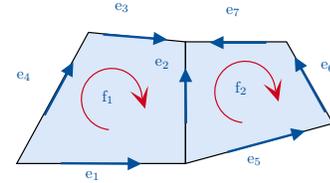
$$\partial_t \mathbf{B} - \nabla \times \mathbf{E} = 0, \quad (1)$$

$$\partial_t \mathbf{D} + \nabla \times \mathbf{H} = -\mathbf{J}, \quad (2)$$

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad (3)$$

$$\mathbf{B} = \mu \mathbf{H}. \quad (4)$$

In the CDO discretization, each geometrical entity has a fixed orientation (Fig. 1) and the degrees of freedom are chosen according to the physical representation of the fields. Flux variables  $\mathbf{D}$ ,  $\mathbf{B}$  and  $\mathbf{J}$  are attached to faces, and circulation variables  $\mathbf{E}$  and  $\mathbf{H}$  are attached to edges. The discretization lies on the decomposition of the equations into two parts: the topological law (1), (2) (independent of the material) and the constitutive equations (3), (4) (dependent on the material, possibly non-isotropic) [4].



$$\text{CURL} = \begin{pmatrix} -1 & -1 & +1 & +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

Fig. 1. Example of a 2D mesh composed of 2 quadrangular cells and its incidence matrix  $\text{CURL} \in \mathbb{R}^{2 \times 7}$  defined by the orientation of the faces and the edges.

### A. Topological laws

The topological relations are discretized exactly using the Stokes theorem applied to circulations and fluxes (in three dimensions). It gives rise to discrete differential operators represented by incidence matrices [4], which verify the same physical properties as the continuous operators (for instance Poincaré's relations and adjunction properties). An example of  $\text{CURL}$  matrix, the discrete counter-part of the continuous  $\nabla \times$  operator, for a 2D mesh is given in Fig. 1. Discretized

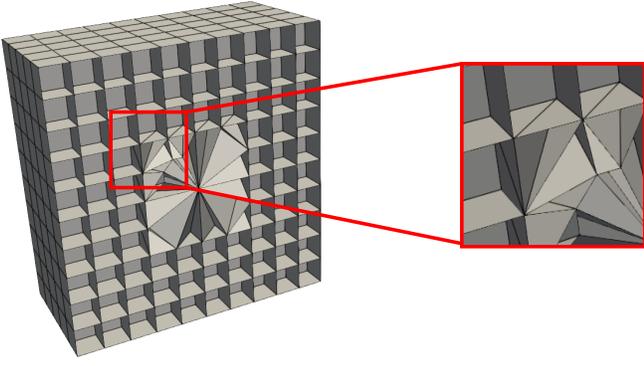


Fig. 2. Unstructured non-conformal mesh of 1178 volumes. It is composed of 840 cubes, 242 tetrahedra and 96 polyhedra.

topological laws take the form:

$$\partial_t \underline{\mathbf{B}} - \text{CURL } \underline{\mathbf{E}} = 0, \quad (5)$$

$$\partial_t \underline{\mathbf{D}} + \text{CURL}^T \underline{\mathbf{H}} = -\underline{\mathbf{J}}, \quad (6)$$

with  $\underline{\mathbf{B}}, \underline{\mathbf{D}}, \underline{\mathbf{J}} \in \mathbb{R}^{N_f}$ ,  $\underline{\mathbf{E}}, \underline{\mathbf{H}} \in \mathbb{R}^{N_e}$  and  $\text{CURL} \in \mathbb{R}^{N_f \times N_e}$ .  $N_f$  and  $N_e$  are respectively the number of faces and the number of edges that compose the mesh.

### B. Closure relations

The constitutive laws, represented by the Hodge operator  $\star$  in differential geometry, are approximated by a linear operator whose algebraic representation is a symmetric positive definite matrix [2], [6]. Discretized closure relations take the form:

$$\underline{\mathbf{D}} = \mathbf{H}_\varepsilon \underline{\mathbf{E}}, \quad (7)$$

$$\underline{\mathbf{H}} = \mathbf{H}_{\mu^{-1}} \underline{\mathbf{B}}. \quad (8)$$

Different strategies can be applied to build these Hodge matrices, giving for each of them, a different scheme. For instance, if  $\mathbf{I}_n$  is the identity matrix and if we consider a 3D orthogonal grid with uniform step  $\Delta x$  in each direction, the choice of Hodge matrices as:

$$\mathbf{H}_\varepsilon := \varepsilon \Delta x \mathbf{I}_n \quad \text{and} \quad \mathbf{H}_{\mu^{-1}} := \frac{1}{\mu \Delta x} \mathbf{I}_n,$$

leads exactly to the FDTD scheme. On polyhedral meshes they are defined [2] as:

$$(\mathbf{H}_\varepsilon)_{i,j} := \int_{\Omega} \varepsilon \ell_i(\mathbf{x}) \ell_j(\mathbf{x}) d\mathbf{x},$$

$$(\mathbf{H}_{\mu^{-1}})_{i,j} := \int_{\Omega} \frac{1}{\mu} h_i(\mathbf{x}) h_j(\mathbf{x}) d\mathbf{x},$$

where the basis functions  $\ell_i$  and  $h_i$  are defined in order to take into account the geometrical properties of the polyhedral cells [7].

As a result, on a mesh composed of a Cartesian part and a polyhedral one, it is possible to combine the FDTD method with a more general CDO method.

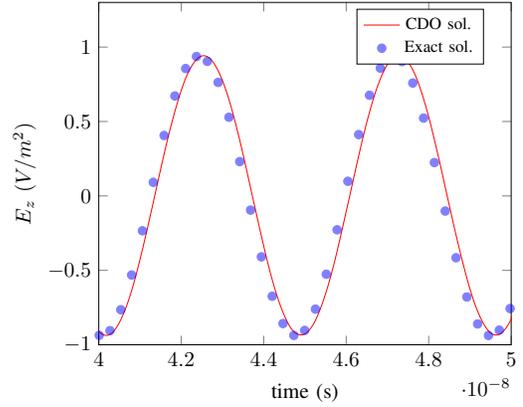


Fig. 3. Comparison of the numerical solution given by the CDO method (red curve), and the analytical solution (blue dots) for the mode (1, 1, 0).

### III. PRIMARY RESULTS

When using a leap-frog time integration strategy, this scheme at the discrete level conserves the electro-magnetic energy under the CFL condition:

$$\Delta t \leq \frac{2}{\|\mathbf{H}_\varepsilon^{-\frac{1}{2}} \text{CURL}^T \mathbf{H}_{\mu^{-1}}^{\frac{1}{2}}\|}. \quad (9)$$

Figure 3 shows the comparison between the numerical solution resulting from the CDO scheme and the analytical solution. The source is the mode (1, 1, 0) inside a  $1m \times 1m \times 1m$  cavity. The mesh used for the simulation is shown in Fig. 2. The time-step given by (9) is  $\Delta t = 5.24 \cdot 10^{-11} s$  while the FDTD time-step would have been  $1.75 \cdot 10^{-10} s$  for a mesh with similar number of volumes. This is the cost of the hybridization for a case that fits perfectly the FDTD framework. But, in the tetrahedral part, the reference mesh length is  $h = 3.33 \cdot 10^{-2} m$  (the minimal height of a tetrahedron). To obtain the same degree of accuracy with a Cartesian grid, the number of volumes should have been 7 times higher than the number of volumes in the actual mesh.

### REFERENCES

- [1] P. Bonnet, X. Ferrières, B. L. Michielsen, P. Klotz, and J. L. Roumiguères, "Finite-volume time domain method," in *Time Domain Electromagnetics*, 1999.
- [2] A. Bossavit, "Computational electromagnetism and geometry," *Journal of Japanese Society of Applied Electromagnetics and Mechanics*, 2000.
- [3] J. Bonelle, "Compatible discrete operator schemes on polyhedral meshes for elliptic and stokes equations," PhD Thesis, Université Paris-Est, 2014.
- [4] P. Bochev and J. Hyman, *Principles of Mimetic Discretizations of Differential Operators*, 2007.
- [5] L. Codecasa and F. Trevisan, "Convergence of electromagnetic problems modelled by discrete geometric approach," *Computer Modeling in Engineering & Sciences*, 2010.
- [6] L. Codecasa, R. Specogna, and F. Trevisan, "Symmetric positive-definite constitutive matrices for discrete eddy-current problems," *IEEE Transactions on Magnetics*, 2007.
- [7] —, "A new set of basis functions for the discrete geometric approach," *J. Comput. Phys.*, 2010.