Risk-sharing and optimal contracts with large exogenous risks

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A Class of Explicit optimal contracts in the face of shutdown

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Abstract

What type of delegation contract should be offered when facing a risk of the magnitude of the pandemic we are currently experiencing and how does the likelihood of an exogenous early termination of the relationship modify the terms of a full-commitment contract? We study these questions by considering a dynamic principal-agent model that naturally extends the classical Holmström-Milgrom setting to include a risk of shutdown before the maturity of the contract. We obtain an explicit characterization of the optimal wage along with the optimal action provided by the agent when the shutdown risk is independent of the inherent agency problem. The optimal contract is linear by offering both a fixed share of the output which is similar to the standard shutdown-free Holmström-Milgrom model and a linear prevention mechanism that is proportional to the random lifetime of the contract. We then extend the model in two directions. We first allow the agent to control the intensity of the shutdown risk. We also consider a structural agency model where the shutdown risk materializes when the state process hits zero.

***Keywords*:** Principal-Agent problems, shutdown risk, Hamilton-Jacobi Bellman equations.

1 Introduction

Without seeking to oppose public health and economic growth, there is no doubt that the management of the Covid crisis had serious consequences on entire sectors of the economy. The first few months of 2020 will go down in world history as a period of time characterized by massive layoffs, forced closures of non-essential companies, disruption of cross-border transportation whilst populations were subject to lockdown and/or social distancing measures and hospitals and the medical

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world struggled to get a grasp on the Sars-Cov-2 pandemic. Whilst the immediate priority was saving lives, decongesting hospitals and preventing the spread of the disease, many extraordinary economic support measures were taken to help businesses and individuals stay afloat during these unprecedented times and in the hope of tempering the economic crisis that would follow. Although the world has lived through many crises over the past centuries, from several Panics in the 1800s and the Great Depression of the 1930s to the more recent Financial Crisis of 2008, never before has the global economy as a whole come to such a standstill due to an external event. Such large shutdown risks do not only materialize during pandemics but throughout many major other large events. The massive bushfires that affected Australia towards the end of 2019, temporarily halting agriculture, construction activity and tourism in some areas of the country are another recent example. As we begin to see a glimpse of hope for a way out through vaccination, the focus is turning to building the world of tomorrow with the idea that we must learn to live with such risks. This paper tries to make its contribution by focusing on a simple microeconomic issue. In a world subject to moral hazard, how can we agree to an incentive contract whose obligations could be made impossible or at least very difficult because of the occurrence of a risk of the nature of the Covid19 pandemic? Including such a shutdown risk-sharing in contracts seems crucial going forward for at least two reasons. First, it is not certain that public authority will be able to continue to take significant economic support measures to insure the partners of a contract if the frequency of such global risks were to increase. On the other hand, the private insurance market does not offer protection against the risk of a pandemic which makes pooling too difficult. It therefore seems likely that we will have to turn to an organized form of risk sharing between the contractors.

Since the foundational work of Holmström and Milgrom [13], contract theory has a well-developed set of mathematical tools to analyze incentive and risk-sharing problems using expected-utility theory. The model considered by Holmström and Milgrom has subsequently been widely extended[1], and some important contributors are Schättler and Sung [24] and [25], Sannikov [23] and Cvitanic et al [6] and [7]. One can distinguish two distinct methods for dealing with the principal-agent problem. The first, developed by Sung [27], [28]; Müller [18] and Hellwig and Schmidt [12], extends the popular first-order approach to solving discrete-time problems. More recently, Williams [29] and Cvitanić, Wan and Zhang [5] used the stochastic maximum principle and a coupled system of direct and backward stochastic differential equations (SDEs). This rigorous approach is very general, but results in complicated systems from which it is difficult to extract an explicit shape for the optimal contract. The second method was introduced by Sannikov [23] who extended to continuous time the approach developed by Spear and Srivastava [26] in discrete time. This extension leads to a control problem for the principal. The main idea of this approach is to take the agent's continuation utility as a state variable for the principal problem. More precisely, by using the martingale optimality principle, it is possible to associate the agent's continuation utility to a backward stochastic differential equation (BDSE). In contrast to the first method, this one allows to focus without loss of generality on a particular form of contract, which reduces the principal problem to a stochastic control problem the solution of which can be characterized through a Hamilton-Jacobi-Bellman equation. We followed the second approach in this paper and solved the HJB equation which has, in our context, an explicit solution that is closely linked to a Bernouilli ODE yielding to an optimal linear contract.

In this paper, we extend the Holmstrom-Milgrom framework by introducing an uncontrolled shut-

[1]For a detailed review of the literature, one refers to the book by Cvitanić and Zhang [4].

down risk upon which the whole of the output process comes to a halt. Unlike previous studies that introduced jump risks in the Principal-agent problem as Biais et al. [2] and Capponi and Frei [3], the shutdown time is first assumed to not be a stopping time with respect to the output process filtration. However, because the failure time is contractible, this leads us to consider the principal-agent problem in the framework of the enlargement of filtration. A model close to ours is that of Pagès and Possamai [20] which studies the contagion in bank loan shutdowns. In [20], the principal and the agent are assumed to be risk-neutral and the uncertainty is only modeled by the failure counting process. Similarly to [3], our work considers a risk-averse principal and agent with exponential utility and combine a Brownian diffusion with a jump risk. To the best of our knowledge, our toy model is the first to study intensity-based principal-agent problems with a Brownian diffusion, in both a first-best (also called full Risk-Sharing) and second-best (also called Moral Hazard) setting. We do not claim that this model with CARA preferences is general enough to come up with robust economic facts, but it has the remarkable advantage of being explicitly creditworthy, which allows us to find an explicit optimal contract that disentangles the incentives from external risk-sharing and allows us to understand the sensitivity of the optimal contract to the different exogenous parameters of the model. A key feature of our study is that the shape of the optimal contract is linear both in the output and the random lifetime of the production process. While the linearity in the output process is in line with the existing literature on continuous-time Principal-Agent problems without shutdown under exponential utilities, the linearity in the lifetime deserves some clarification. The contract exposes both agents to a risk of exogenous interruption but it has two different regimes that are determined by an explicit relation between the risk-aversions and the agent's effort cost. Under the first regime, the agent is more sensitive to the risk of shutdown than the principal. In this case, the principal deposits on the date 0 a positive amount onto an escrow account whose balance will then decrease over time at a constant rate. It is crucial to observe that the later the shutdown arrives, the more the amount in the escrow account decreases to a point where it may even become negative. If the shutdown occurs, the principal transfers the remaining balance to the agent. Under the second regime, the principal is more sensitive to the risk of shutdown. In this case, the principal deposits a negative amount into the escrow account, which now grows at a constant rate and symmetrical reasoning applies.

Finally, this paper proposes two extensions that are very different in nature. In the first one, the agent can control the intensity of the shutdown time by exerting an additional effort in the spirit of the paper by Capponi and Frei [3]. Interestingly, the insurance part of the optimal contract has different regimes depending on the distance to contract maturity. In the second extension, we characterize explicitly the optimal contract when the shutdown risk is the first time the output process hits zero. Unlike the previous models, the shutdown risk is now a stopping time whose distribution depends on the effort that drives the output.

The rest of the document is structured as follows. In Section 2, we present the model and the Principal-Agent problems that we consider. In Section 3, we analyse the first-best case where the principal observes the agent's effort. Then in Section 4, we give our main results and analysis. In Section 5, we extend our model to include a possibility for mitigation upon a halt.

2 The Model

The model is inherited from the classical work of Holmström and Milgrom [13]. A principal contracts with an agent to manage a project she owns. The agent influences the project's profitability by exerting an unobservable effort. For a fixed effort policy, the output process is still random and the idiosyncratic uncertainty is modeled by a Brownian motion.
We assume that the contract matures at time $T > 0$ and both principal and agent are risk-averse with CARA preferences. The departure from the classical model is as follows: we assume the project is facing some external risk that could interrupt the production at some random time τ. The probability distribution of τ is first assumed to be independent of the Brownian motion that drives the uncertainty of the output process and also independent of the agent's actions. Finally, we assume that the contract offers a transfer W at time T from the principal to the agent that is a functional of the output process.

2.1 Probability setup

Let $T > 0$ be some fixed time horizon. The key to modeling our Principal-Agent problems under an agency-free external risk of shutdown is the simultaneous presence over the interval $[0, T]$ of a continuous random process and a jump process as well as the ability to extend the standard mathematical techniques used for dynamic contracting to this mixed setting. Thus, we shall deal with two kinds of information : the information from the output process, denoted as $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and the information from the shutdown time, i.e. the knowledge of the time where the shutdown occurred in the past, if the shutdown has appeared. This construction is not new and occurs frequently in mathematical finance[2].

The complete probability space that we consider will be denoted as $(\Omega, \mathcal{G}, \mathbb{P}^0)$, with two independent stochastic processes :

- B a standard one-dimensional $\mathbb{F}$-Brownian motion,
- N the right-continuous single-jump process defined as $N_t = \mathbf{1}_{\tau\leq t}$, t in $[0, T]$ where τ is some positive random variable independent of B that models the shutdown time.

N will also be referred to as the shutdown indicator process. We therefore use the standard approach of progressive enlargement of filtration by considering $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ the smallest complete right-continuous extension of $\mathbb{F}$ that makes τ a $\mathbb{G}$-stopping time. Because τ is independent of B, B is a $\mathbb{G}$-Brownian motion under $\mathbb{P}^0$ according to Proposition 1.21 p 11 in [1]. We also suppose that there exists a bounded deterministic compensator of N, $\Lambda_t = \int_0^t \lambda(s)\, ds$ for some bounded function $\lambda(.)$ called the intensity implying that:

$$M_t = N_t - \int_0^t \lambda(s)(1 - N_s)ds, \quad t \in [0, T]$$

is a $\mathbb{G}$-compensated martingale. Note that through knowledge of the function λ, the principal and agent can compute at time 0 the probability of shutdown happening over the contracting period $[0, T]$. Indeed :

$$\mathbb{P}(\tau \leq T) = 1 - \exp(-\Lambda_T).$$

[2]We refer the curious reader to the two important references [1] and [10].

We first suppose for computational ease that the intensity λ is a constant but our results may easily be generalized to general deterministic compensators.

Remark 2.1. *Here we will suppose that the compensator of N is common knowledge to both the Principal and the Agent. We could imagine settings where the Principal and Agent's beliefs regarding the risk of shutdown may differ : this natural extension of our work would call for analysis of the dynamic contracting problem under hidden information which is left for future research.*

2.2 Principal-Agent Problem

We suppose that the agent agrees to work for the principal over a time period $[0, T]$ and provide up to the shutdown time a costly action $(a_t)_{t\in[0,T]}$ belonging to $\mathcal{B}$, where $\mathcal{B}$ denotes the set of admissible $\mathbb{F}$-predictable strategies that will be specified later on. The Principal-Agent problem models the realistic setting where the principal cannot observe the agent's effort. As such the agent chooses his action in order to maximize his own utility. The principal must offer a wage based on the information driven by the output process up to the shutdown time that incentivizes the agent to work efficiently and contribute positively to the output process. Mathematically, the unobservability of the agent's behaviour is modeled through a change of measure. Under $\mathbb{P}^0$, we assume that the project's profitability evolves as

$$X_t := x_0 + \int_0^t (1 - N_s)dB_s.$$

Thus, $\mathbb{P}^0$ corresponds to the probability distribution of the profitability when the agent makes no effort over $[0, T]$. When the agent makes an effort $a = (a_t)_t$, we shall assume that the project's profitability evolves as

$$X_t := x_0 + \int_0^t a_s(1 - N_s)ds + \int_0^t (1 - N_s)dB^a_s,$$

where B^a is a $\mathbb{F}$-Brownian motion under some measure $\mathbb{P}^a$ that will be specified later. The agent fully observes the decomposition of the production process under a measure $\mathbb{P}^a$ whilst the principal only observes the realization of X_t. In order for the model to be consistent, the probabilities $\mathbb{P}^0$ and $\mathbb{P}^a$ must be equivalent for all $(a_t)_{t\in[0,T\wedge\tau]}$ belonging to $\mathcal{B}$. Therefore, for a fixed $A > 0$ that can be as large as we need, we introduce the following set of actions.

$$\mathcal{B} = \Big\{a = (a_t)_t : \quad \mathbb{F}\text{-predictable and taking values in } [-A, A]\Big\}.$$

For $a \in \mathcal{B}$, we define $\mathbb{P}^a$ as

$$\frac{d\mathbb{P}^a}{d\mathbb{P}^0}|\mathcal{G}_T = \exp\left(\int_0^T a_s(1 - N_s)dB_s - \frac{1}{2}\int_0^T |a_s|^2(1 - N_s)ds\right) := L_T.$$

Because $\mathbb{E}^0(L_T) = 1$, $(B^a_t)_{t\in[0,T]}$ with $B^a_t = B_t - \int_0^t a_s(1-N_s)ds, t \in [0, T]$ is a $\mathbb{G}$-Brownian motion under $\mathbb{P}^a$ according to Proposition 3.6 c) p 55 in [1]. It is key to note that if halt occurs, i.e. if $\tau \leq T$, then the production process is halted[3] before T meaning that : $X^a_{t\wedge\tau} = X^a_t, \quad t \in [0, T]$. Let

[3]Our study falls within the framework of random horizon agency problems. However, contrary to the paper [16], the horizon here is not a control variable that the principal can choose.

us then observe that an action $a = (a_t)_t$ of $\mathcal{B}$ can be extended to a $\mathbb{G}$-predictable process $(\tilde{a}_t)_{t\in[0,T]}$ by setting $\tilde{a}_t = a_t \mathbb{1}_{t\leq\tau}$.
The cost of effort for the agent is modeled through a quadratic cost function : $\kappa(a) := \kappa\frac{a^2}{2}$, for $\kappa > 0$ some fixed parameter. As a reward for the agent's effort, the principal pays him a wage W at time T. W is assumed to be a $\mathcal{G}_{T\wedge\tau}$ random variable which means that the payment at time T in case of an early shutdown is known at time $T \wedge \tau$. The principal and the agent are considered to be risk averse and risk aversion is modeled through two CARA utility functions :

$$U_P(x) := -\exp(-\gamma_P x) \text{ and } U_A(x) := -\exp(-\gamma_A x),$$

where $\gamma_P > 0$ and $\gamma_A > 0$ are two fixed constants modeling the principal's and the agent's risk aversion.

In this setting and for any given wage W, the agent maximizes his own utility and solves :

$$V_0^A(W) = \sup_{a\in\mathcal{B}} \mathbb{E}^a \left[U_A \left(W - \int_0^T \kappa(a_s(1 - N_s))ds \right) \right]. \tag{2.1}$$

A wage W is said to be incentive compatible if there exists an action policy $a^*(W) \in \mathcal{B}$ that maximises (2.1) and thus satisfies

$$V_0^A(W) = \mathbb{E}^{a^*(W)} \left[U_A \left(W - \int_0^T \kappa(a_s^*(W)(1 - N_s))ds \right) \right].$$

When the principal is able to offer an incentive compatible wage W, she knows what the agent's best reply will be. As such the principal establishes a set $\mathcal{A}^*(W) \subset \mathcal{B}$ of best replies for the agent for any incentive compatible W. Therefore, the first task is to characterize the set of incentive-compatible wages $\mathcal{W}_{IC}$. Only then may the principal consider maximizing his own utility by solving :

$$\sup_{W\in\mathcal{W}_{IC}} \sup_{a^*\in\mathcal{A}^*(W)} \mathbb{E}^{a^*(W)} \left[U_P \left(X_T^{a^*(W)} - W \right) \right] \tag{2.2}$$

under the participation constraint

$$\mathbb{E}^{a^*(W)} \left[U_A \left(W - \int_0^T \kappa(a_s^*(W)(1 - N_s))ds \right) \right] \geq U_A(y_0), \tag{2.3}$$

where y_0 is a monetary reservation utility for the agent.

Remark 2.2. *Problem (2.2) has been thoroughly analyzed in a setting where the output process may not shutdown (see the pioneer papers [13], [24]). Setting $\kappa = 1$ for simplicity, the optimal action is constant and given by :*

$$a^* = \frac{\gamma_P + 1}{\gamma_P + \gamma_A + 1},$$

and the optimal wage is linear in the output:

$$W = y_0 + a^* X_T + \left(\frac{\gamma_A - 1}{2} (a^*)^2 \right) T.$$

We may naturally expect to encounter an extension of these results in our setting.

3 Optimal First-best contracting

We begin with analysis of the first-best benchmark (the full Risk-Sharing problem) which leads to a simple optimal sharing rule. Of course this problem is not the most realistic when it comes to modeling dynamic contracting situations. However it provides a benchmark to which we can compare the more realistic Moral Hazard situation. Indeed, the principal's utility in the full Risk-Sharing problem is the best that the principal will ever be able to obtain in a contracting situation as he may observe (and it is thus assumed that he may dictate) the agent's action.

To write the first-best problem, we assume that both the principal and the agent observe the variations of the same production process $(X_t^a)_{t\in[0,T]}$ under $\mathbb{P}^0$:

$$X_t^a := x_0 + \int_0^t a_s(1-N_s)ds + \int_0^t (1-N_s)dB_s. \quad t\in[0,T] \tag{3.1}$$

The agent is guaranteed a minimum value of expected utility through the participation constraint :

$$\mathbb{E}\left[U_A\left(W - \int_0^T \kappa(a_s(1-N_s))ds\right)\right] \geq U_A(y_0), \tag{3.2}$$

but has no further say on the wage or action. Consider the admissible set :

$\mathcal{A}_{PC} := \{(W,a)$ such that W is $\mathcal{G}_{T\wedge\tau}$ measurable with $\mathbb{E}\left[\exp(-2\gamma_A W)\right] < +\infty, (a_t)_t \in \mathcal{B},$ and (3.2) is satisfied$\}$.

The full Risk-Sharing problem involves maximizing the principal's utility across $\mathcal{A}_{PC}$:

$$\sup_{(W,a)\in\mathcal{A}_{PC}} \mathbb{E}\left[U_P\left(X_T^a - W\right)\right]. \tag{3.3}$$

3.1 Tackling the Participation Constraint

A first step to optimal contracting in this first-best setting involves answering the following question: can we characterize the set $\mathcal{A}_{PC}$? Following the standard route, we will first establish a necessary condition. For a given pair $(W,a)\in\mathcal{A}_{PC}$, let us introduce the agent's continuation utility $(U_t^{(W,a)})_t$ as follows:

$$U_t^{(W,a)} := \mathbb{E}_t\left[U_A\left(W - \int_t^T \kappa(a_s(1-N_s))ds\right)\right],$$

where we use the shorthand notation : $\mathbb{E}_t[.] := \mathbb{E}[.|\mathcal{G}_t]$. We may write the Agent's continuation value process as the product :

$$U_t^{(W,a)} = \mathcal{M}_t^{(W,a)}\mathcal{D}_t^{(W,a)},$$

where :

$$\mathcal{M}_t^{(W,a)} := \mathbb{E}_t\left[U_A\left(W - \int_0^T \kappa(a_s(1-N_s))ds\right)\right] \quad \text{and} \quad \mathcal{D}_t^{(W,a)} := \exp\left(-\gamma_A\int_0^t \kappa(a_s(1-N_s))ds\right).$$

Observe that for any admissible pair $(W,a)\in\mathcal{A}_{PC}$, the process $\mathcal{M} = (\mathcal{M}_t^{(W,a)})_t$ is a $\mathbb{G}$-square integrable martingale. According to the Martingale Representation Theorem for $\mathbb{G}$-martingales

(see [1], Theorem 3.12 p. 60), there exists some predictable pair (z_s, l_s) in $\mathbb{H}^2 \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the set of $\mathbb{F}$-predictable processes Z with $\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < +\infty$, such that :

$$\mathcal{M}_t^{(W,a)} := \mathcal{M}_0^{(W,a)} + \int_0^t z_s(1-N_s)dB_s + \int_0^t l_s(1-N_s)dM_s.$$

Integration by parts yields the dynamic of U, noting that $\mathcal{D}$ has finite variation :

$$dU_t^{(W,a)} = -\gamma_A \kappa(a_t(1-N_s))U_t^{(W,a)}dt + \mathcal{D}_t^{(W,a)} z_t(1-N_s)dB_t + \mathcal{D}_t^{(W,a)} l_t(1-N_s)dM_t.$$

Setting $Z_t^{(W,a)} := \mathcal{D}_t^{(W,a)} z_t \in \mathbb{H}^2$ and $K_t^{(W,a)} := \mathcal{D}_t^{(W,a)} l_t \in \mathbb{H}^2$, we obtain:

$$dU_t^{(W,a)} = -\gamma_A \kappa(a_t(1-N_s))U_t^{(W,a)}dt + Z_t^{(W,a)}(1-N_s)dB_t + K_t^{(W,a)}(1-N_s)dM_t.$$

By construction, we have that $U_T^{(W,a)} = U_A(W)$. It follows that $\left(U_t^{(W,a)}, Z_t^{(W,a)}, K_t^{(W,a)}\right)$ is a solution to the BSDE:

$$-dU_t^{(W,a)} = -Z_t^{(W,a)}(1-N_s)dB_t - K_t^{(W,a)}(1-N_s)dM_t + \gamma_A \kappa(a_t(1-N_s))U_t^{(W,a)}dt, \tag{3.4}$$

with $U_T^{(W,a)} = U_A(W)$. Therefore, (3.2) is satisfied if and only if $U_0^{(W,a)} \geq U_A(y_0)$.

Remark 3.1. *Let $\mathbb{S}^2$ be the set of $\mathbb{G}-$adapted RCLL processes U such that*

$$\mathbb{E}[\sup_{0\leq t\leq T} |U_t|^2] < +\infty.$$

Through Proposition 2.6 of [8], the solution to (3.4) *is unique in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$. Indeed, the driver $g(\omega, U) = \gamma_A \kappa(a_t(1-N_t))U$ is uniformly Lipschitz in U because $(a_t)_t$ is bounded and the terminal condition is in L^2.*

To sum up, we have the following necessary condition for admissibility.

Lemma 3.1. *If $(W,a) \in \mathcal{A}_{PC}$ then there exists a unique solution $\left(U_t^{(W,a)}, Z_t^{(W,a)}, K_t^{(W,a)}\right)$ in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$ to the BSDE* (3.4) *such that $U_0^{(W,a)} \geq U_A(y_0)$.*

To obtain a sufficient condition, we introduce, for $\pi = (y, a, \beta, H) \in \mathbb{R} \times \mathcal{B} \times \mathbb{H}^2 \times \mathbb{H}^2$, the wage process $(W_t^\pi)_t$ defined as

$$W_t^\pi := y + \int_0^t \beta_s(1-N_s)dB_s + \int_0^t H_s(1-N_s)dM_s + \int_0^t \Big\{ \frac{\gamma_A}{2}\beta_s^2(1-N_s) + \kappa(a_s(1-N_s)) \\ \frac{\lambda}{\gamma_A}[\exp(-\gamma_A H_s) - 1 + \gamma_A H_s](1-N_s) \Big\} ds, \tag{3.5}$$

and consider the set

$$\Gamma := \left\{ (y,a,\beta,H) \in \mathbb{R} \times \mathcal{B} \times \mathbb{H}^2 \times \mathbb{H}^2 \text{ such that } y \geq y_0 \text{ and } \mathbb{E}\left[\exp(-2\gamma_A W_T^\pi)\right] < +\infty. \right\}.$$

We have the following result.

Lemma 3.2. *For any $\pi \in \Gamma$, the pair (W_T^π, a) belongs to $\mathcal{A}_{PC}$.*

Proof. We apply Itô's formula to the process $Y_t^\pi = U_A(W_t^\pi)$ to obtain

$$dY_t^\pi = -\gamma_A Y_t^\pi \beta_t (1-N_t)\, dB_t + Y_t^\pi \left(e^{-\gamma_A H_t} - 1\right)(1-N_t)\, dM_t - \gamma_A \kappa(a_t(1-N_t)) Y_t^\pi\, dt.$$

Moreover, because $\pi \in \Gamma$, $Y_T^\pi = U_A(W_T^\pi)$ is square-integrable. Remark 3.1 yields the triplet $\left(Y_t^\pi, -\gamma_A Y_t^\pi \beta_t, Y_t^\pi(e^{-\gamma_A H_t} - 1)\right)$ is the unique solution in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$ to BSDE (3.4) with terminal condition $U_A(W_T^\pi)$ when $\pi \in \Gamma$. Therefore,

$$Y_0^\pi = U_A(y) = \mathbb{E}\left[U_A\left(W_T^\pi - \int_0^T \kappa(a_s(1-N_s))\, ds\right)\right] \geq U_A(y_0),$$

and thus (3.2) is satisfied. □

Remark 3.2. *The admissible contracts are essentially the terminal values of the controlled processes* (3.5) *for $\pi \in \Gamma$. The difficulty is that we do not know how to characterize the β and H processes that guarantee that π belongs to Γ. Nevertheless, it is easy to check by a standard application of the Gronwall lemma that if β and H are bounded then $\pi \in \Gamma$. This last observation will prove to be crucial in the explicit resolution of our problem.*

3.2 First-best Dynamic contracting

Using Lemma 3.2, the full Risk-Sharing problem under shutdown writes as the Markovian control problem :

$$V_P^{FB} := \sup_{\pi=(y,a,Z,K)\in\Gamma} \mathbb{E}\left[U_P\left(X_T^{(x_0,a)} - W_T^\pi\right)\right], \tag{3.6}$$

where $X_t^{(x_0,a)}$ is given by :

$$dX_s^{(x_0,a)} = a_s(1-N_s)ds + (1-N_s)dB_s,$$

with $X_0^{(x_0,a)} = x_0$ and the wage process is given by :

$$\begin{aligned} dW_s^\pi = {} & Z_s(1-N_s)dB_s + K_s(1-N_s)dM_s \\ & + \left\{\frac{\gamma_A}{2} Z_s^2(1-N_s) + \kappa(a_s(1-N_s)) + \frac{\lambda}{\gamma_A}[\exp(-\gamma_A K_s) - 1 + \gamma_A K_s](1-N_s)\right\} ds, \end{aligned}$$

with $W_0^\pi = y$. We have the following key theorem for the first-best problem.

Theorem 3.1. *Let $a_t^* = \dfrac{1}{\kappa}, Z_t^* = \dfrac{\gamma_P}{\gamma_P + \gamma_A}$, and let :*

$$K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

where :

$$\Phi_0(t) := \left(\frac{c_1 + c_2}{c_1} \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A}(T-t)\right) - \frac{c_2}{c_1}\right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}}, \tag{3.7}$$

with :

$$c_1 := \frac{\gamma_P^2 \gamma_A}{2(\gamma_P+\gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda \frac{\gamma_P+\gamma_A}{\gamma_A} \quad \textit{and} \quad c_2 := \lambda \frac{\gamma_P+\gamma_A}{\gamma_A}.$$

Then $\pi^* = (y_0, a^*, Z^*, K^*) \in \Gamma$ *parameterizes the optimal contract* $(W_T^{\pi^*}, a^*)$ *for the first-best problem.*

The rest of this subsection is dedicated to the proof of this Theorem. We first make the following observation. As X remains constant after τ, the principal has no further decision to make after the shutdown time. Thus, its value function is constant and equal to $U_P(X_\tau - W_\tau)$ on the interval $[\tau, T]$.
We now focus on the control part of the problem (i.e. computation of the optimal control triplet $\tilde{\pi} = (a, Z, K)$ for a given pair (x_0, y)). To do so, we follow the dynamic programming approach developed in [21], Section 4 to define the value function

$$V(0, x_0, y_0) = \sup_{\tilde{\pi} \in \tilde{\Gamma}} \mathbb{E}\left[U_P(X_T^a - W_T^{\tilde{\pi}})(1 - N_T) + \int_0^T U_P(X_t^a - W_t^{\tilde{\pi}}) \lambda e^{-\lambda t}\, dt \right], \tag{3.8}$$

where

$$\tilde{\Gamma} = \left\{ \tilde{\pi} \in \mathcal{B} \times \mathbb{H}^2 \times \mathbb{H}^2 \right\},$$

Because $\Gamma \subset \mathbb{R} \times \tilde{\Gamma}$, we have

$$V_P^{FB} \leq \sup_{y \geq y_0} V(0, x_0, y).$$

According to stochastic control theory, the Hamilton-Jacobi-Bellman equation associated to the stochastic control problem (3.8) is the following (see [19]):

$$\partial_t v(t,x,y) + \sup_{a,Z,K} \left\{ \partial_x v(t,x,y) a + \partial_y v(t,x,y) \left[\frac{\gamma_A}{2} Z^2 + \kappa(a) + \frac{\lambda}{\gamma_A} [\exp(-\gamma_A K) - 1] \right] \right.$$
$$\left. + \lambda \left[U_P(x - y - K) - v_0(t,x,y) \right] + \partial_{yy} v(t,x,y) \frac{Z^2}{2} + \frac{1}{2} \partial_{xx} v(t,x,y) + \partial_{xy} v(t,x,y) Z \right\} = 0, \tag{3.9}$$

with the boundary condition :

$$v(T, x, y) = U_P(x - y).$$

It happens that the HJB equation (3.9) is explicitly solvable by exploiting the separability property of the exponential utility function.

Lemma 3.3. *The function* $v(t,x,y) = U_P(x-y)\Phi_0(t)$ *with :*

$$\Phi_0(t) = \left(\frac{c_1 + c_2}{c_1} \exp\left(c_1 \frac{\gamma_A}{\gamma_P + \gamma_A} (T - t) \right) - \frac{c_2}{c_1} \right)^{\frac{\gamma_P + \gamma_A}{\gamma_A}},$$

where :

$$c_1 = \frac{\gamma_P^2 \gamma_A}{2(\gamma_P+\gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda \frac{\gamma_P+\gamma_A}{\gamma_A} \quad \textit{and} \quad c_2 = \lambda \frac{\gamma_P+\gamma_A}{\gamma_A},$$

solves (in the classical sense) the HJB partial differential equation (3.9).
Furthermore $a_t^* = \frac{1}{\kappa}, Z_t^* = \frac{\gamma_P}{\gamma_P + \gamma_A}$ *and* $K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t))$ *are the optimal controls.*

Proof. We search for a solution to Equation (3.9) for a v of the form :

$$v(t,x,y) = U_P(x-y)\Phi_0(t),$$

with Φ_0 a positive mapping. Such a v satisfies (3.9) if and only if $\Phi_0(t)$ solves the PDE :

$$\Phi_0'(t) + \inf_{a,Z,K}\left\{-\gamma_P\Phi_0(t)a + \gamma_P\Phi_0(t)\left(\frac{\gamma_A}{2}Z^2 + \kappa(a) + \frac{\lambda}{\gamma_A}\left\{\exp(-\gamma_A K) - 1\right\}\right)\right.$$
$$\left. +\gamma_P^2\Phi_0(t)\frac{Z^2}{2} + \frac{\gamma_P^2}{2}\Phi_0(t) - \gamma_P^2\Phi_0(t)Z + \lambda\left(\exp(\gamma_P K) - \Phi_0(t)\right)\right\} = 0,$$

with the boundary condition $\Phi_0(T) = 1$. As Φ_0 is a positive mapping, the infimum is well defined. We derive the following first-order conditions that must be satisfied by the optimal controls :

$$\begin{cases} \gamma_P\Phi_0(t) = \gamma_P\kappa a\Phi_0(t) \\ \gamma_P\Phi_0(t)Z(\gamma_A+\gamma_P) = \gamma_P^2\Phi_0(t) \\ \gamma_P\Phi_0(t)\lambda\exp(-\gamma_A K) = \gamma_P\lambda\exp(\gamma_P K), \end{cases}$$

equating to :

$$a^* = \frac{1}{\kappa}, \quad Z^* = \frac{\gamma_P}{\gamma_P+\gamma_A}, \quad K^* = \frac{\log(\Phi_0(t))}{\gamma_P+\gamma_A}.$$

It follows that :

$$\inf_{a,Z,K}\left\{-\gamma_P\Phi_0(t)a + \gamma_P\Phi_0(t)\left(\frac{\gamma_A}{2}Z^2 + \kappa(a) + \frac{\lambda}{\gamma_A}\left\{\exp(-\gamma_A K) - 1\right\}\right)\right.$$
$$\left. +\gamma_P^2\Phi_0(t)\frac{Z^2}{2} + \frac{\gamma_P^2}{2}\Phi_0(t) - \gamma_P^2\Phi_0(t)Z + \lambda\left(\exp(\gamma_P K) - \Phi_0(t)\right)\right\}$$
$$= -\gamma_P\Phi_0(t)a^* + \gamma_P\Phi_0(t)\left(\frac{\gamma_A}{2}Z^{*2} + \kappa(a^*) + \frac{\lambda}{\gamma_A}\left\{\exp(-\gamma_A K^*) - 1\right\}\right)$$
$$+\gamma_P^2\Phi_0(t)\frac{Z^{*2}}{2} + \frac{\gamma_P^2}{2}\Phi_0(t) - \gamma_P^2\Phi_0(t)Z^* + \lambda\left(\exp(\gamma_P K^*) - \Phi_0(t)\right)$$
$$= \underbrace{\Phi_0(t)\frac{\gamma_P^2\gamma_A}{2(\gamma_P+\gamma_A)}}_{\text{terms with } Z^*} \underbrace{-\Phi_0(t)\frac{\gamma_P}{2\kappa}}_{\text{terms with } a^*} \underbrace{-\lambda\frac{\gamma_P+\gamma_A}{\gamma_A}\Phi_0(t) + \lambda\frac{\gamma_P+\gamma_A}{\gamma_A}\Phi_0(t)^{\frac{\gamma_P}{\gamma_P+\gamma_A}}}_{\text{terms with } K^*}.$$

We may inject this expression back into the PDE on Φ_0. Doing so yields the following Bernoulli equation :

$$\Phi_0'(t) + c_1\Phi_0(t) + c_2\Phi_0(t)^{\frac{\gamma_P}{\gamma_P+\gamma_A}} = 0, \quad \Phi_0(T) = 1,$$

where

$$c_1 = \frac{\gamma_P^2\gamma_A}{2(\gamma_P+\gamma_A)} - \frac{\gamma_P}{2\kappa} - \lambda\frac{\gamma_P+\gamma_A}{\gamma_A} \quad \text{and} \quad c_2 = \lambda\frac{\gamma_P+\gamma_A}{\gamma_A}.$$

The unique solution to this equation is (see for instance [30]) :

$$\Phi_0(t) = \left(\frac{c_1+c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right) - \frac{c_2}{c_1}\right)^{\frac{\gamma_P+\gamma_A}{\gamma_A}},$$

and the result follows. □

Proof of Theorem 3.1. The value function $v(t,x,y) = U_P(x-y)\Phi_0(t)$ is a classical solution to the HJB equation (3.9). A standard verification theorem yields that $v = V$. Through Lemma 3.3, the optimal controls for the full Risk-Sharing problem are :

$$a_t^* = \frac{1}{\kappa}, Z_t^* = \frac{\gamma_P}{\gamma_P + \gamma_A} \text{ and } K_t^* = \frac{1}{\gamma_P + \gamma_A} \log(\Phi_0(t)),$$

with Φ_0 as defined in Lemma 3.3. These controls are free of y and it follows that :

$$V(0,x_0,y_0) = E\left[U_P\left(X_T^{(x_0,a^*)} - W_T^{(y_0,a^*,Z^*,K^*)}\right)\right],$$

is a decreasing function of y_0. Thus we obtain

$$\sup_{y_0 \geq y_{PC}} V_0(0,x_0,y_0) = E\left[U_P\left(X_T^{(x_0,a^*)} - W_T^{(y_{PC},a^*,Z^*,K^*)}\right)\right].$$

Finally, we observe that the optimal controls are bounded and thus Remark (3.2) yields $\pi^* = (y_{PC}, a^*, Z^*, K^*) \in \Gamma$. As a consequence,

$$\sup_{y \geq y_0} V(0,x_0,y) = \mathbb{E}\left[U_P\left(X_T^{(x_0,a)} - W_T^{(y),a^*,Z^*,K^*)}\right)\right] \leq V_P^{FB}.$$

Because the reverse inequality holds, the final result follows. □

4 Optimal contracting under shutdown risk

4.1 Main results

The following is dedicated to our main result for the Moral Hazard problem. We shall state our main theorem with the explicit optimal contract before turning to some analysis of the effect of the shutdown on dynamic contracting. In the case of moral hazard, one is forced to make a stronger assumption about the nature of a contract. This stronger hypothesis will naturally appear to justify the martingale optimality principle. In our setting, a contract is a $\mathbb{G}_{T\wedge\tau}$ measurable random variable W such that for every $\beta \in \mathbb{R}$, we have

$$\mathbb{E}\left[\exp(\beta W)\right] < +\infty.$$

A first step to optimal contracting involves answering the preliminary question: can we characterize incentive compatible wages and if so what is the related optimal action for the agent? The characterization of incentive compatible contracts relies on the martingale optimality principle (see [14] and [22]) that we recall below.

Lemma 4.1 (Martingale Optimality Principle)**.** *Given a contract W, consider a family of stochastic processes $R^a(W) := (R_t^a)_{t\in[0,T]}$ indexed by a in $\mathcal{B}$ that satisfies :*

1. *$R_T^a = U_A(W - \int_0^T \kappa(a_s(1-N_s))ds)$ for any a in $\mathcal{B}$*
2. *$R_\cdot^a$ is a $\mathbb{P}^a$-supermartingale for any a in $\mathcal{B}$*

3. R_0^a is independent of a.

4. There exists a^* in $\mathcal{B}$ such that R^{a^*} is a $\mathbb{P}^{a^*}$-martingale.

Then,

$$R_0^{a^*} = \mathbb{E}^{a^*}\left[U_A(W - \int_0^T \kappa(a_s^*)ds)\right] \geq \mathbb{E}^{a}\left[U_A(W - \int_0^T \kappa(a_s(1-N_s))ds)\right],$$

meaning that a^* is the optimal agent's action in response to the contract W.

We will construct such a family following the standard route. Consider a given contract W, we define the family $R^a(W) := (R_t^a)_{t\in[0,T]}$ by

$$R_t^a := -\exp\left(-\gamma_A\left(Y_t(W) - \int_0^t \kappa(a_s(1-N_s))ds\right)\right),$$

where $(Y(W), Z(W), K(W))$ in $(\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2)$ is the unique solution of the following BSDE under $\mathbb{P}^0$

$$Y_t(W) = W - \int_t^T f(Z_s(W), K_s(W))(1-N_s)\,ds - \int_t^T Z_s(W)(1-N_s)\,dB_s - \int_t^T K_s(W)(1-N_s)\,dM_s, \tag{4.1}$$

with

$$f(z,k) := \frac{1}{2}\gamma_A z^2 + \lambda k + \frac{\lambda}{\gamma_A}(e^{-\gamma_A k} - 1) + \inf_{a\in\mathcal{B}}\left\{\kappa(a) - az\right\}.$$

Remark 4.1. *The theoretical justification of the well-posedness of the BSDE* (4.1) *deserves some comments. The first results were obtained in [15] and [9] when the contract W is assumed to be bounded. The necessary extension in our model when W admits an exponential moment has been treated recently in the paper [17].*

By construction, $R_T^a = U_A(W - \int_0^T \kappa(a_s(1-N_s))ds)$ for any a in $\mathcal{B}$. Moreover, $R_0^a = Y_0(W)$ is independent of the agent's action a. We have

$$\begin{aligned}
dR_s^a &= -\gamma_A R_s^a Z_s(1-N_s)dB_s + R_s^a(e^{-\gamma_A K_s} - 1)(1-N_s)dM_s \\
&\quad + R_s^a\gamma_A\left\{\frac{1}{2}\gamma_A Z_s^2 - f(Z_s, K_s) + \kappa(a_s(1-N_s)) + \lambda K_s + \frac{\lambda}{\gamma_A}(e^{-\gamma_A K_s} - 1)\right\}(1-N_s)ds. \\
&= -\gamma_A R_s^a Z_s(1-N_s)dB_s^a + R_s^a(e^{-\gamma_A K_s} - 1)(1-N_s)dM_s \\
&\quad + R_s^a\gamma_A\left\{\frac{1}{2}\gamma_A Z_s^2 - f(Z_s, K_s) + \kappa(a_s(1-N_s)) + \lambda K_s + \frac{\lambda}{\gamma_A}(e^{-\gamma_A K_s} - 1) - a_s Z_s\right\}(1-N_s)ds.
\end{aligned}$$

Thus R^a is a $\mathbb{P}^a$-super-martingale for every a in $\mathcal{B}$, the function

$$a^*(z) = -A\mathbb{1}_{z\leq -\kappa A} + \frac{z}{\kappa}\mathbb{1}_{-\kappa A\leq z\leq \kappa A} + A\mathbb{1}_{z\geq \kappa A}$$

is a unique minimizer for f and R^{a^*} is a $\mathbb{P}^{a^*}$-martingale. As a consequence, every contract W is incentive compatible which a unique best reply $a^*(Z(W))$. Finally, a contract W satisfies the participation constraint if and only if $Y_0(W) \geq y_0$.

Relying on the idea of Sannikov [23] and its recent theoretical justification by Cvitanic, Possamai and Touzi [7], we will consider the agent promised wage $Y(W)$ as a state variable to embed the principal's problem into the class of Markovian problems, by considering the sensitivities of the agent's promised wage $Z(W)$ and $K(W)$ as control variables. For $\pi = (y, Z, K) \in [y_0; +\infty) \times \mathbb{H}^2 \times \mathbb{H}^2$, we define under $\mathbb{P}^0$, the control process called the agent continuation value

$$W_t^{(y_0,Z,K)} = y + \int_0^t Z_s(1 - N_s)dB_s + \int_0^t K_s(1 - N_s)dM_s + \int_0^t f(Z_s, K_s)(1 - N_s)\, ds. \tag{4.2}$$

Under $\mathbb{P}^* := \mathbb{P}^{(a^*(Z))}$, we thus have

$$\begin{aligned} W_t^{(y_0,Z,K)} &= y + \int_0^t Z_s(1 - N_s)dB_s^* + \int_0^t K_s(1 - N_s)dM_s \\ &+ \int_0^t \left\{ \frac{\gamma_A}{2} Z_s^2 + \kappa(a^*(Z_s)) + \frac{\lambda}{\gamma_A}[\exp(-\gamma_A K_s) - 1 + \gamma_A K_s] \right\} (1 - N_s)ds \\ &= y + \int_0^t Z_s(1 - N_s)dB_s^* + \int_0^t K_s(1 - N_s)dN_s \\ &+ \int_0^t \left\{ \frac{\gamma_A}{2} Z_s^2 + \kappa(a^*(Z_s)) + \frac{\lambda}{\gamma_A}[\exp(-\gamma_A K_s) - 1] \right\} (1 - N_s)ds \end{aligned} \tag{4.3}$$

Now, we consider the set

$$\zeta = \left\{ \pi = (Z, K) \in \mathbb{H}^2 \times \mathbb{H}^2 \text{ such that } \forall \beta \in \mathbb{R},\, \mathbb{E}\left[\exp(\beta W_T^{(y,Z,K)}) \right] < +\infty \text{ for } y \in \mathbb{R} \right\}.$$

By construction, $W_T^{(y,\pi)}$ is a contract that satisfies the participation constraint for every $\pi \in \zeta$ and $y \geq y_0$. Moreover, by the well-posedness of the BSDE (4.1) , every contract W that satisfies the participation constraint can be written $W_T^{(Y_0(W),Z(W),K(W))}$ with $\pi(W) = (Z(W), K(W)) \in \zeta$. Therefore, the problem of the principal can now be rewritten as the following optimisation problem

$$V_P := \sup_{y \geq y_0} v(0, x, y),$$

where

$$v(0, x, y) = \sup_{\pi \in \zeta} \mathbb{E}^* \left[U_P(X_{T \wedge \tau} - W_{T \wedge \tau}^{\pi}) \right] \tag{4.4}$$

To characterize the optimal contract, we will proceed analogously as in the full risk sharing case by constructing a smooth solution to the HJB equation associated to the Markov control problem (4.4) given by

$$\begin{aligned} 0 = \partial_t v(t, x, y) + \sup_{Z,K} \Bigg\{ & \partial_x v(t, x, y) \frac{Z}{\kappa} + \partial_y v(t, x, y) \left[\frac{\gamma_A}{2} Z^2 + \kappa(a^*(Z)) + \frac{\lambda}{\gamma_A}[\exp(-\gamma_A K) - 1] \right] \\ & + \lambda \left[U_P(x - y - K) - v(t, x, y) \right] + \partial_{yy} v_0(t, x, y) \frac{Z^2}{2} + \frac{1}{2} \partial_{xx} v(t, x, y) + \partial_{xy} v(t, x, y) Z \Bigg\}, \end{aligned} \tag{4.5}$$

Lemma 4.2. *Assume the constant A in the definition of the set of admissible efforts $\mathcal{B}$ satisfies*

$$A > \frac{\gamma_P + \kappa^{-1}}{\kappa(\gamma_P + \gamma_A) + 1}\,.$$

Then, the function $U_P(x-y)\Phi_0(t)$, with

$$\Phi_0(t) = \left(\frac{c_1 + c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right) - \frac{c_2}{c_1}\right)^{\frac{\gamma_P+\gamma_A}{\gamma_A}},$$

where

$$c_1 = \frac{\gamma_P^2\gamma_A}{2(\gamma_P+\gamma_A+\kappa^{-1})} - \frac{\gamma_P\kappa^{-1}(\gamma_P+\kappa^{-1})}{2(\gamma_P+\gamma_A+\kappa^{-1})} - \lambda\frac{\gamma_P+\gamma_A}{\gamma_A} \quad \textit{and} \quad c_2 = \lambda\frac{\gamma_P+\gamma_A}{\gamma_A}.$$

solves in the classical sense the HJB equation (4.5). In particular $Z_t^* = \dfrac{\gamma_P+\kappa^{-1}}{\gamma_P+\gamma_A+\kappa^{-1}}$ and $K_t^* = \dfrac{1}{\gamma_P+\gamma_A}\log(\Phi_0(t))$,

Proof. Because the assumption on A implies $a^*(z) = z/\kappa$, the proof of this lemma is a direct adaptation of the proof of Lemma 3.3 to which we refer the reader. □

We are in a position to prove the main result of this section

Theorem 4.1. *We have the following explicit characterizations of the optimal contracts. Let A as in the Lemma 4.2 and let $Z_t^* = \dfrac{\gamma_P+\kappa^{-1}}{\gamma_P+\gamma_A+\kappa^{-1}}$ and $K_t^* = \frac{1}{\gamma_P+\gamma_A}\log(\Phi_0(t))$, where Φ_0 is defined as in (3.7) with the constants :*

$$c_1 := \frac{\gamma_P^2\gamma_A}{2(\gamma_P+\gamma_A+\kappa^{-1})} - \frac{\gamma_P\kappa^{-1}(\gamma_P+\kappa^{-1})}{2(\gamma_P+\gamma_A+\kappa^{-1})} - \lambda\frac{\gamma_P+\gamma_A}{\gamma_A} \quad \textit{and} \quad c_2 := \lambda\frac{\gamma_P+\gamma_A}{\gamma_A}.$$

Then (y_0, Z^*, K^*) parametrizes the optimal wage for the Moral Hazard problem. The Agent performs the optimal action $\frac{Z^*}{\kappa}$.

Proof. Because the function $U_P(x-y)\Phi_0(t)$ is a classical solution to the HJB equation (4.5) and the optimal controls are bounded and free of y, we proceed analogously as in the proof of Theorem 3.1. Finally, we have to prove that the optimal wage $W^* = Y_T^{(y_0,Z^*,K^*)}$ admits exponential moments to close the loop. According to (4.3), we have

$$W^* = y_0 + Z^*B^*_{T\wedge\tau} + \frac{1}{2}\left(\gamma_A + \frac{1}{\kappa}\right)(Z^*)^2(T\wedge\tau) + K^*_{\tau_-}\mathbf{1}_{\tau\le T} + \int_0^T \frac{\lambda}{\gamma_A}[\exp(-\gamma_A K_s^*) - 1](1-N_s)ds.$$

Because $(B_t^*)_t$ is a Brownian motion and K_t^* is deterministic, it is straightforward to check that W^* admits exponential moments. □

4.2 Model analysis

The optimal contract includes two components. One is linear in the output with an incentivizing slope that is similar to the classical optimal contract found in [13]. This is necessary to implement a desirable level of effort. The other is unrelated to the incentives but linked to the shutdown risk sharing. It is key to observe that this second term is nonzero even if the shutdown risk does not materialize before the termination of the contract.
The characterization of the optimal contracts in Theorem 4.1 sparks an immediate observation: the two parties only need to be committed to the contracting agreement up until $T \wedge \tau$. Therefore in this simple model, using an expected-utility related reasoning and without considering mechanisms such as employment law, the occurence of the agency-free external risk, halting production, leads to early contract terminations. This is in line with what actually happened during the Covid pandemic. Indeed in the USA and in eight weeks of the pandemic, 36.5 million people applied for unemployment insurance. In more protective economies, mass redundancies were only prevented through the instauration of furlough type schemes allowing private employees' wages to temporarily be paid by gouvernements. This phenomena makes fundamental sense : a principal whose output process is completely halted cannot enforce the agent to work hard because she has no revenue to provide the incentives. Let's focus on the second term:

$$K^*_{\tau_-}\mathbf{1}_{\tau\leq T} + \int_0^T \frac{\lambda}{\gamma_A}[\exp(-\gamma_A K^*_s) - 1](1 - N_s)ds, \tag{4.6}$$

Understanding the effect of these extra terms is crucial to fully understand the sharing of the agency-free shutdown risk. First, we show that the sign of the control K^* is constant.

Lemma 4.3. *Let c_1 and c_2 be the relevant constants given in Theorem 4.1 then the optimal control $(K^*_t)_{t\in[0,T]}$ can be expressed as :*

$$K^*_t = \frac{1}{\gamma_A}\log\left(\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)((T-t)\wedge\tau)\right)\right]\right) \quad t\in[0,T]. \tag{4.7}$$

Proof. We have that :

$$K^*_t = \frac{1}{\gamma_P+\gamma_A}\log(\Phi_0(t)),$$

with

$$\Phi_0(t) = \left(\frac{c_1+c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right) - \frac{c_2}{c_1}\right)^{\frac{\gamma_P+\gamma_A}{\gamma_A}}.$$

The aim here is to link this expression for Φ_0 to that of an expected value. As such, we consider the following expected value that decomposes as shown :

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)((T-t)\wedge\tau)\right)\right]\\ &= \mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)(T-t)\right)\mathbf{1}_{\tau>T-t}\right] + \mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)\tau\right)\mathbf{1}_{\tau\leq T-t}\right].\end{aligned}$$

Using $c_2 = \dfrac{\gamma_P + \gamma_A}{\gamma_A}\lambda$, the first term of the expected value rewrites as follows :

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)(T-t)\right)\mathbf{1}_{\tau>T-t}\right] &= \exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)(T-t)\right)\exp\left(-\lambda(T-t)\right)\\ &= \exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right).\end{aligned}$$

It remains to compute the second term. We obtain :

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)\tau\right)\mathbf{1}_{\tau\leq T-t}\right] &= \int_0^{T-t}\lambda\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)s\right)\exp\left(-\lambda s\right)ds\\ &= \int_0^{T-t}\lambda\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)s-\lambda s\right)ds\\ &= \int_0^{T-t}\lambda\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}c_1 s\right)ds\\ &= \left[\frac{\lambda}{c_1}\frac{\gamma_P+\gamma_A}{\gamma_A}\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}c_1 s\right)\right]_0^{T-t}\\ &= \left[\frac{c_2}{c_1}\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}c_1 s\right)\right]_0^{T-t}\\ &= \frac{c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-\frac{c_2}{c_1}.\end{aligned}$$

Combining both terms we reach the final expression :

$$\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)((T-t)\wedge\tau)\right)\right] = \frac{c_1+c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-\frac{c_2}{c_1}.$$

Therefore we identify that :

$$\Phi_0(t) = \left(\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)((T-t)\wedge\tau)\right)\right]\right)^{\frac{\gamma_P+\gamma_A}{\gamma_A}}.$$

As a consequence, we may also rewrite K_t^*. Indeed :

$$K_t^* = \frac{1}{\gamma_P+\gamma_A}\log(\Phi_0(t)),$$

and with the new expression for Φ_0 we obtain the result :

$$K_t^* = \frac{1}{\gamma_A}\log\left(\mathbb{E}\left[\exp\left(\frac{\gamma_A}{\gamma_P+\gamma_A}(c_1+c_2)((T-t)\wedge\tau)\right)\right]\right).$$

□

Remark 4.2. *We have the same expression for the optimal control K^* in the first-best case, using for c_1 and c_2 the relevant constants given in Theorem 3.1.*

As a consequence, this alternative form for K^* leads to easy analysis of the sign of the control, given in the following lemma.

Lemma 4.4. *The sign of K^* over the contracting period $[0,T]$ is constant and entirely determined by the model's risk aversions γ_P and γ_A, and the Agent's effort cost κ. Indeed, the sign of K^* is equal to the sign of $\gamma_P\gamma_A - \gamma_P\kappa^{-1} - (\kappa^{-1})^2$. Moreover, K_t^* varies monotonously in time, with $K_T^* = 0$.*

Proof. From the expression (4.7), we easily deduce that :

- If $c_1 + c_2 = 0$ then $K_t^* = 0$ for every $t \in [0,T]$,
- if $c_1 + c_2 > 0$ then $K_t^* > 0$ for $t \leq \tau$ and the function $t \to K_t^*$ decreases,
- if $c_1 + c_2 < 0$ then $K_t^* < 0$ for $t \leq \tau$ and the function $t \to K_t^*$ increases.

Replacing c_1 and c_2 by their relevant expressions in each case leads to the result.

□

Finally, we will show that the risk-sharing component of the contract is in fact linear with respect to the shutdown time. This is a strong result of our study for which we had no ex-ante intuition.

Theorem 4.2. *The shutdown risk-sharing component of the optimal wage is linear in the shutdown time. More precisely, the optimal wage is*

$$W^* = y_{PC} + Z^* B^*_{T\wedge\tau} + \frac{1}{2}\left(\gamma_A + \frac{1}{\kappa}\right)(Z^*)^2(T\wedge\tau) + K_0^* - \left(\frac{c_1}{\gamma_P+\gamma_A} + \frac{\lambda}{\gamma_A}\right)(T\wedge\tau).$$

Proof. Because $K_T^* = 0$, the optimal wage can be written

$$W^* = y_{PC} + Z^* B^*_{T\wedge\tau} + \frac{1}{2}\left(\gamma_A + \frac{1}{\kappa}\right)(Z^*)^2(T\wedge\tau) + f(T\wedge\tau),$$

with

$$f(t) = K_t^* + \int_0^t \frac{\lambda}{\gamma_A}\left(\exp(-\gamma_A K_s^*) - 1\right) ds, t \in [0,T].$$

Let us define

$$g(t) = \frac{c_1+c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right) - \frac{c_2}{c_1}.$$

We have $g'(t) = -(c_1+c_2)\frac{\gamma_A}{\gamma_P+\gamma_A}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)$.
Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t}K_t^* &= \frac{1}{\gamma_A}\frac{g'(t)}{g(t)} \\
&= \frac{1}{\gamma_P+\gamma_A}\left\{\frac{-(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)}{\frac{c_1+c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right) - \frac{c_2}{c_1}}\right\} \\
&= \frac{c_1}{\gamma_P+\gamma_A}\left\{\frac{-(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)}{(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right) - c_2}\right\}.
\end{aligned}$$

Also

$$\frac{\partial}{\partial t}\int_0^t \frac{\lambda}{\gamma_A}\left(\exp(-\gamma_A K_s^*)-1\right)ds = \frac{\lambda}{\gamma_A}\left(\exp(-\gamma_A K_t^*)-1\right)$$

and so :

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^t \frac{\lambda}{\gamma_A}\left(\exp(-\gamma_A K_s^*)-1\right)ds &= \frac{\lambda}{\gamma_A}\left(\exp(-\gamma_A K_t^*)-1\right)\\
&= \frac{\lambda}{\gamma_A}\left(\frac{1}{g(t)}-1\right) \quad \text{well-defined as } g(t)>0 \text{ on } [0,T]\\
&= \frac{\lambda}{\gamma_A}\left\{\frac{1}{\frac{c_1+c_2}{c_1}\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-\frac{c_2}{c_1}}-1\right\}\\
&= \frac{c_1\lambda}{\gamma_A}\left\{\frac{1}{(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-c_2}\right\}-\frac{\lambda}{\gamma_A}
\end{aligned}$$

Finally, we have :

$$\begin{aligned}
f'(t) &= \frac{\partial}{\partial t}K_t^* + \frac{\partial}{\partial t}\int_0^t \frac{\lambda}{\gamma_A}\left(\exp(-\gamma_A K_s^*)-1\right)ds\\
&= \frac{c_1}{\gamma_P+\gamma_A}\left\{\frac{-(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)}{(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-c_2}\right\}\\
&+ \frac{c_1\lambda}{\gamma_A}\left\{\frac{1}{(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-c_2}\right\}-\frac{\lambda}{\gamma_A}\\
&= \frac{-c_1}{\gamma_P+\gamma_A}\frac{1}{(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-c_2}\left\{(c_1+c_2)\exp\left(c_1\frac{\gamma_A}{\gamma_P+\gamma_A}(T-t)\right)-\lambda\frac{\gamma_A}{\gamma_P+\gamma_A}\right\}\\
&- \frac{\lambda}{\gamma_A}\\
&= -\frac{c_1}{\gamma_P+\gamma_A}-\frac{\lambda}{\gamma_A} \quad \text{as } c_2=\frac{\lambda\gamma_A}{\gamma_P+\gamma_A}\\
&= -\left(\frac{c_1}{\gamma_P+\gamma_A}+\frac{\lambda}{\gamma_A}\right).
\end{aligned}$$

□

5 Mitigating the shutdown risk

Even if this moves us away from our pandemic motivation, we will assume, similarly to Capponi and Frei's paper [3], that the agent can execute two different tasks: as before, he can exert some costly

effort a to improve the productivity of the project and he can also exert another preventive action to delay the arrival of the shutdown risk. The prevention effort is modeled in a manner similar to that proposed by Pagès and Possamai [20]. More precisely, the prevention action at time t is modeled by a measurable process $b_t \in \{0,1\}$, if agent exerts effort at time t, $b_t = 1$. The intensity of the shutdown time τ is assumed to be

$$\lambda_t^{(b)} = \frac{\lambda}{2}(2 - b_t).$$

The agent incurs disutility for exerting effort and prevention. We assume to fix ideas that it takes the form of $c(a,b) = \kappa a^2/2 + \epsilon b$, where $\epsilon > 0$.

Remark 5.1. *Unlike the paper by Capponi and Frei [3], our cost function does not capture interactions between effort provision and risk prevention. In this paper, we choose this simplifying assumption because we want to explore the impact of prevention on the linear contract in the simplest possible extension of the main model.*

We denote by $\mathcal{P}$ the set of $\mathbb{F}$-predictable processs with values in $\{0,1\}$. For $a \in \mathcal{B}$ and $b \in \mathcal{P}$, we define by Girsanov theorem (see [1], Th. 1.30 p.24) the probability measure

$$\frac{d\mathbb{P}^{a,b}}{d\mathbb{P}^0} = \frac{d\mathbb{P}^a}{d\mathbb{P}^0} \exp\left(\int_0^T \log(\lambda_t^{(b)}/\lambda_t^{(0)})\, dN_t - \int_0^T (\lambda_t^{(b)} - \lambda_t^{(0)})\, dt \right)$$

under which $N_t - \int_0^t \lambda_s^{(b)}\, ds$ is a martingale.

5.1 The agent

Given a contract W, the agent solves

$$\sup_{a \in \mathcal{B}} \sup_{b \in \mathcal{P}} \mathbb{E}^{a,b}\left[U_A(W - \int_0^T c(a_s(1-N_s), b_s(1-N_s))\, ds) \right].$$

In order to characterize the incentive-compatible contracts, we proceed analogously as in Section 4 by relying on the martingale optimality principle. Then, we introduce the process

$$R_t^{a,b} = U_A\left(Y_t - \int_0^t c(a_s(1-N_s), b_s(1-N_s))\, ds \right),$$

where, under $\mathbb{P}^0$,

$$Y_t = W - \int_t^T f(Z_s, K_s)(1-N_s)\, ds - \int_t^T Z_s(1-N_s)\, dB_s - \int_t^T K_s(1-N_{s-})\, dN_s.$$

Proceeding again as in Section 4, we have that $(R_t^{a,b})_t$ is a $\mathbb{P}^{a,b}$ super-martingale for every pair $(a,b) \in \mathcal{B} \times \mathcal{P}$ if and only if

$$f(z,k) = \frac{\gamma_A}{2} z^2 + \inf_a (\kappa \frac{a^2}{2} - az) + \inf_b \left(\frac{\lambda}{2\gamma_A}(2-b)(e^{-\gamma_A k} - 1) + \epsilon b \right).$$

The first order condition for determining the optimal effort a^* is unchanged. On the other hand, the optimal prevention is bang-bang and we have $b_t^* = \mathbb{1}_{K_t \leq \hat{K}}$ where

$$\hat{K} = -\frac{1}{\gamma_A} \log\left(1 + \frac{2\gamma_A}{\lambda}\epsilon\right) < 0.$$

Remark 5.2. *To encourage the agent to make the preventive effort, it is necessary to punish the agent financially if the risk materializes.*

Hence,

$$f(z,k) = \frac{\gamma_A - \kappa^{-1}}{2} z^2 + \left(\frac{\lambda}{2}\left(\frac{e^{-\gamma_A k} - 1}{\gamma_A}\right) + \epsilon\right) \mathbb{1}_{\{K \leq \hat{K}\}} + \lambda \left(\frac{e^{-\gamma_A k} - 1}{\gamma_A}\right) \mathbb{1}_{\{K > \hat{K}\}}.$$

Proceeding analogously as in Section 4, we introduce the agent promised wage as the forward SDE

$$\begin{aligned} W_t = y_0 &+ \int_0^t Z_s(1 - N_s) dB_s^* + \int_0^t K_s(1 - N_{s-}) dN_s \\ &+ \int_0^t \frac{\gamma_A + \kappa^{-1}}{2} Z_s^2 + \left(\frac{\lambda}{2}\left(\frac{e^{-\gamma_A K_s} - 1}{\gamma_A}\right) + \epsilon\right) \mathbb{1}_{\{K_s \leq \hat{K}\}} + \lambda \left(\frac{e^{-\gamma_A K_s} - 1}{\gamma_A}\right) \mathbb{1}_{\{K_s > \hat{K}\}} (1 - N_s) ds, \end{aligned}$$

where from now on the processes Z and K are seen as controls for the principal.

5.2 The Principal

Still following Sannikov's methodology, the principal problem can be written as follows

$$V_P := \sup_{y \geq y_0} v(0, x, y),$$

where

$$v(0,x,y) = \sup_{Z,K} \mathbb{E}^{(a^*, b^*)} \left[U_P(X_{T \wedge \tau} - W_{T \wedge \tau})\right] \tag{5.1}$$

To characterize the optimal contract, we will proceed analogously as in Section 4 by constructing a smooth solution to the HJB equation associated to the Markov control problem (5.1) given by

$$\begin{aligned} 0 = v_t(t,x,y) & \qquad (5.2) \\ + \sup_{Z,K} \Bigg\{ & v_x(t,x,y)\frac{Z}{\kappa} + v_y(t,x,y)\left[\frac{\gamma_A + \kappa^{-1}}{2} Z^2 + \left(\frac{\lambda}{2}\left(\frac{e^{-\gamma_A K} - 1}{\gamma_A}\right) + \epsilon\right) \mathbb{1}_{\{K \leq \hat{K}\}} + \lambda \left(\frac{e^{-\gamma_A K} - 1}{\gamma_A}\right) \mathbb{1}_{\{K > \hat{K}\}}\right] \\ & + \left(\frac{\lambda}{2}\mathbb{1}_{\{K \leq \hat{K}\}} + \lambda \mathbb{1}_{\{K > \hat{K}\}}\right)\left[U_P(x - y - K) - v(t,x,y)\right] + v_{yy}(t,x,y)\frac{Z^2}{2} + \frac{1}{2} v_{xx}(t,x,y) + v_{xy}(t,x,y) Z \Bigg\}, \end{aligned}$$

Taking advantage of the separability of the problem from the control variables, we once again make the guess that $v(t,x,y) = \phi(t) U(x - y)$, where ϕ is a positive function, to get

$$\begin{aligned} 0 = \phi' &+ \inf_Z \left\{-\gamma_P \frac{Z}{\kappa} + \gamma_P \frac{\gamma_A + \kappa^{-1}}{2} Z^2\} + \gamma_P \frac{Z^2}{2} + \gamma_P^2 - \gamma_P^2 Z\right\} \phi \\ &+ \inf_K \left\{f_1(K, \phi) \mathbb{1}_{\{K \leq \hat{K}\}} + f_0(K, \phi) \mathbb{1}_{\{K > \hat{K}\}}\right\}, \end{aligned}$$

where

$$f_1(K,\phi) = \left(\frac{\lambda}{2}\left(\frac{e^{-\gamma_A K}-1}{\gamma_A}\right)+\epsilon\right)\gamma_P\phi + \frac{\lambda}{2}(e^{\gamma_P K}-\phi),$$

and

$$f_0(K,\phi) = \lambda\left(\frac{e^{-\gamma_A K}-1}{\gamma_A}\right)\gamma_P\phi + \lambda(e^{\gamma_P K}-\phi),$$

The first observation is that the minimization problem with respect to Z is similar to that of Section 4. Keeping the same notations as in the theorem 4.1, we obtain for $t \in [0,T)$

$$0 = \phi'(t) + (c_1+c_2)\phi + \inf_K \left\{ f_1(K,\phi(t))\mathbb{1}_{\{K\leq\hat{K}\}} + f_0(K,\phi(t))\mathbb{1}_{\{K>\hat{K}\}} \right\}.$$

Hereafter, denote $f(\phi) = \inf_K \left\{ f_1(K,\phi)\mathbb{1}_{\{K\leq\hat{K}\}} + f_0(K,\phi)\mathbb{1}_{\{K>\hat{K}\}} \right\}$.
The second observation is that, for a given ϕ, the functions $f_0(.,\phi)$ and $f_1(.,\phi)$ are convex and attain their minimum on $\mathbb{R}$ at the point

$$\bar{K}(\phi) = \frac{1}{\gamma_P+\gamma_A}\log(\phi).$$

Therefore, two cases have to be considered.

- $\underline{\bar{K}(\phi) > \hat{K}}$: for which, we have

$$f(\phi) = \min(f_1(\hat{K},\phi), f_0(\bar{K}(\phi),\phi)) = \min\left((2\gamma_P\epsilon - \frac{\lambda}{2})\phi + \frac{\lambda}{2}e^{\gamma_P\hat{K}}, \frac{\lambda(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi)\right). \tag{5.3}$$

- $\underline{\bar{K}(\phi) \leq \hat{K}}$: for which, we have

$$f(\phi) = \min(f_0(\hat{K},\phi), f_1(\bar{K}(\phi),\phi)) = \min\left((2\gamma_P\epsilon - \lambda)\phi + \lambda e^{\gamma_P\hat{K}}, \frac{\lambda}{2}\frac{(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi) + \gamma_P\epsilon\phi\right). \tag{5.4}$$

Our third observation is that in the neighborhood of ϕ =1, it is always optimal not to induce the agent to take preventive actions. Indeed, because $\bar{K}(1) > \hat{K}$, we have from Equation (5.3) that

$$f(1) = 0 = f_0(\bar{K}(1),1).$$

Remembering that $\frac{\lambda}{2}\left(\frac{e^{-\gamma_A\hat{K}}-1}{\gamma_A}\right) = \epsilon$, we observe that f is continuous at $\hat{\phi} = e^{(\gamma_A+\gamma_P)\hat{K}}$ and we have

$$\begin{aligned} f(e^{(\gamma_A+\gamma_P)\hat{K}}) &= \min(f_1(\hat{K}, e^{(\gamma_A+\gamma_P)\hat{K}}), f_0((\hat{K}, e^{(\gamma_A+\gamma_P)\hat{K}})) \\ &= \min((2\gamma_P+\gamma_A)\epsilon e^{(\gamma_A+\gamma_P)\hat{K}}, (2\gamma_P+2\gamma_A)\epsilon e^{(\gamma_A+\gamma_P)\hat{K}}) \\ &= (2\gamma_P+\gamma_A)\epsilon e^{(\gamma_A+\gamma_P)\hat{K}}. \end{aligned}$$

This leads to our fourth observation. In the neighborhood of $\hat{\phi}$, it is optimal for the principal to encourage the agent to make prevention actions.
To sum up, we are looking for a smooth solution to the non-linear o.d.e.

$$\begin{cases} 1 &= \phi(T) \\ 0 &= \phi' + (c_1 + c_2)\phi + f(\phi), \end{cases}$$

where

$$f(\phi) = \begin{cases} \min\left((2\gamma_P\epsilon - \frac{\lambda}{2})\phi + \frac{\lambda}{2}e^{\gamma_P\hat{K}}, \frac{\lambda(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi)\right) \text{ if } \phi > \hat{\phi} \\ \min\left((2\gamma_P\epsilon - \lambda)\phi + \lambda e^{\gamma_P\hat{K}}, \frac{\lambda}{2}\frac{(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi) + \gamma_P\epsilon\phi\right)\text{if } \phi \leq \hat{\phi} \end{cases}$$

The idea is to build the solution using our previous observations starting from the solution ϕ_0 of Section 4 since it is known that it is not optimal to enforce prevention actions in the neighborhood of 1. Unfortunately, due to the large number of parameters, we will not give a complete description of all the possible cases but concentrate on some particular cases to give economic intuitions. The first result provides sufficient conditions that it is never optimal for the principal to give the agent incentives to take preventive actions.

Proposition 5.1. *Let c_1, c_2 and ϕ_0 given in Lemma 4.2. Assume $c_1 + c_2 \geq 0$ then $\phi = \phi_0$. The optimal contract is identical to the one signed when the shock is exogenous.*

Proof. Let's assume for a moment that we have proven that $f(\phi) = \frac{\lambda(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi)$ for all $\phi \geq 1$. Under the assumptions $c_1 + c_2 \geq 0$, the solution ϕ_0 remains above 1 for every t and consequently $K_t^* = \frac{1}{\gamma_P+\gamma_A}\log(\phi_0(t)) \geq \hat{K}$. Therefore, ϕ_0 solves the desired o.d.e..
To prove $f(\phi) = \frac{\lambda(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi)$, we study the function

$$g(\phi) = \frac{\lambda(\gamma_P + \gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi) - (2\gamma_P\epsilon - \frac{\lambda}{2})\phi + \frac{\lambda}{2}e^{\gamma_P\hat{K}}.$$

Clearly, $f(\phi) = \frac{\lambda(\gamma_P+\gamma_A)}{\gamma_A}(\phi^{\frac{\gamma_P}{\gamma_P+\gamma_A}} - \phi)$ if $g(\phi) < 0$. A straightforward computation shows that g is concave, diverges to $-\infty$ and cancels out only once at a point $\phi_\vee \in [\hat{\phi}, 1)$ which ends the proof. □

When the agent is more sensitive to shutdown risk than the principal, the Section 4 contract is already protective enough for the agent that it is not optimal for the principal to force preventive actions even if the cost of prevention is zero.

The end of the paper is devoted to the case where $c_1 + c_2 < 0$. We know that in this case, the function ϕ_0 is increasing. Suppose that the contract is long enough (T large), so that there exists $t_\vee \in [0, T]$ such that $\phi_0(t_\vee) = \phi_\vee$. Let ϕ_1 be the solution on $[0, t_\vee]$ of the linear equation

$$\begin{cases} \phi_\vee &= \phi_1(t_\vee) \\ 0 &= \phi_1' + (c_1 + c_2)\phi_1 + (2\gamma_P\epsilon - \frac{\lambda}{2})\phi_1 + \frac{\lambda}{2}e^{\gamma_P\hat{K}}. \end{cases}$$

Let $\alpha = c_1 + c_2 + 2\gamma_P\epsilon - \frac{\lambda}{2}$ and $\beta = \frac{\lambda}{2}e^{\gamma_P\hat{K}}$. Moreover, assume $\alpha < 0$. Then,

$$\phi_1(t) = (\phi_\vee + \frac{\beta}{\alpha})e^{\alpha(t_\vee - t)} - \frac{\beta}{\alpha}.$$

6 A STRUCTURAL AGENCY MODEL

In this section, we will assume that the shutdown risk is a $\mathbb{F}$-stopping time. More precisely, interpreting now the process X as the equity value of the firm, we assume that τ is the first time X hits zero. Although the process $N_t = \mathbb{1}_{\{\tau \leq t\}}$ is now $\mathbb{F}$-adapted, we can follow Section 2 to the line to introduce the agency problem. In particular, the principal commits to deliver a payment W at time $T \wedge \tau$ and the agent's problem's is

$$\sup_{a \in \mathcal{B}} \mathbb{E}^a \left[U_A \left(W - \int_0^{T \wedge \tau} \kappa(a_s) ds \right) \right].$$

To ease the exposition, we will assume a quadratic cost $k(a) = \frac{a^2}{2}$. Our first lemma gives a family of incentive-compatible contracts. For $\beta = (\beta_t)_t \in \mathcal{B}$, we define the process

$$W_t^{y,\beta} = y + \int_0^t f(\beta_s)\, ds + \int_0^t \beta_s\, dB_s, \tag{6.1}$$

where $f(\beta) = \frac{\gamma_A}{2}\beta^2 + \inf_a \left(\frac{a^2}{2} - a\beta \right)$.

Lemma 6.1. *The contract $W^{\beta}_{T\wedge\tau}$ is incentive-compatible for every $\beta = (\beta_t)_t \in \mathcal{B}$. Moreover, the agent's best reply is $a^*(\beta) = \beta$.*

Proof. Assume the principal offers $W^{y,\beta}_{T\wedge\tau}$. Then, the agent has to solve

$$\inf_{a \in \mathcal{B}} \left(\mathbb{E}^a \left[e^{-\gamma_A \left(y + \int_0^{T\wedge\tau} f(\beta_s) + a_s\beta_s - \frac{a_s^2}{2}\, ds + \int_0^{T\wedge\tau} \beta_s\, dB_s^a \right)} \right] := J(a) \right).$$

Because the process $\beta = (\beta_t)_t$ is uniformly bounded, the process $M = (M_t)_t$ with

$$M_t = \exp\left(-\gamma_A \int_0^t \beta_s\, dB_s^a - \frac{\gamma_A^2}{2} \int_0^t \beta_s^2\, ds \right)$$

is a $\mathbb{F}$-martingale. We define an equivalent probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}^a} | \mathcal{F}_T = M_T,$$

to obtain

$$J(a) = \mathbb{E}^{\mathbb{Q}^a} \left[\exp\left(-\gamma_A \left(y + \int_0^{T\wedge\tau} (f(\beta_s) + a_s\beta_s - \frac{a_s^2}{2} - \frac{\gamma_A}{2}\beta_s^2)\, ds \right) \right) \right].$$

By definition of f, we get for every $a \in \mathcal{B}$,

$$J(a) \geq e^{-\gamma_A y} = J(\beta),$$

which ends the proof. □

Then, the principal problem is $\sup_{y \geq y_0} v(0,x,y)$ with

$$v(0,x,y) = \sup_{\beta \in \mathcal{B}} \mathbb{E}^{\beta} \left[U_P(X^{\beta}_{T\wedge\tau} - W^{\beta}_{T\wedge\tau}) \right]$$

where

$$\begin{aligned} dX^{\beta}_t &= \beta_t \, dt + dB^{\beta}_t \\ dW^{\beta}_t &= \left(\frac{\gamma_A + 1}{2} \right) \beta_t^2 \, dt + \beta_t dB^{\beta}_t. \end{aligned}$$

The associated HJB equation is

$$0 = v_t + \sup_{\beta} \left(\beta v_x + \left(\frac{\gamma_A + 1}{2} \right) \beta^2 v_y + \frac{1}{2} v_{xx} + \frac{\beta^2}{2} v_{yy} + \beta v_{xy} \right),$$

with boundary conditions $v(T,x,y) = U_P(x-y)$ and $v(t,0,y) = U_P(-y)$ for $t \leq T$.
We will build a classical solution to the above HJB equation by looking for a solution in the form $v(t,x,y) = U_P(w(t,x) - y)$. We obtain

$$w_t = \inf_{\beta} \left(-(1+\gamma_P) w_x \beta + \frac{\gamma_A + \gamma_P + 1}{2} \beta^2 \right) + \frac{\gamma_P}{2} w_x^2 - \frac{1}{2} w_{xx}, \tag{6.2}$$

with boundary conditions $w(T,x) = x$ and $w(t,0) = 0$ for $t \leq T$.
The first-order condition gives

$$\beta^* = \left(\frac{1+\gamma_P}{1+\gamma_P+\gamma_A} \right) w_x. \tag{6.3}$$

Plugging (6.3) into HJB (6.2), we obtain

$$w_t + \frac{1}{2} w_{xx} = G w_x^2,$$

where

$$G = \frac{\gamma_P}{2} - \frac{(1+\gamma_P)^2}{2(1+\gamma_A+\gamma_P)}.$$

Remark 6.1. *When the CARA coefficients γ_A and γ_P satisfies $\gamma_A \gamma_P = 1 + \gamma_P$, we have $G = 0$. Therefore, the optimal contract is linear with $\beta^* = \frac{1}{\gamma_A}$ and the principal value is stationary given by $U_P(x-y)$. Observe that γ_A must be larger or equal to 1 (and thus $\beta^* \leq 1$) to have $G = 0$.*

When $G \neq 0$, we define $g(t,x) = e^{-Gw(t,x)}$ to obtain the linear PDE

$$g_t + \frac{1}{2} g_{xx} = 0$$

with boundary conditions $g(T,x) = e^{-Gx}$ and $g(t,0) = 1$ for $t \leq T$, for which we have a classical solution given by Feynman-Kac formulae. More precisely,

$$g(0,x) = \mathbb{E} \left[e^{-G(x + B_{T \wedge T_{-x}})} \right],$$

where $T_{-x} = \inf\{t \geq 0,\, B_t = -x\}$.

We are in a position to state the main result.

Proposition 6.1. *The optimal contract is given by the terminal value of Equation* (6.1)*,* $W_{T\wedge\tau}^{\beta^*}$ *where* β^* *is given by* (6.3) *and where the shutdown time is*

$$\tau = \inf\{t \geq 0,\, x + \int_0^t \beta_s^*\, ds + B_t^* = 0\}.$$

Proof. In order to apply a standard verification result, it suffices to check that β^* is bounded or equivalently that the ratio $\frac{g_x}{g}$ is bounded. Without loss of generality, we make the proof for $t = 0$. We first assume that $G < 0$ for which g is increasing. We have for $\varepsilon > 0$,

$$0 \leq g(0, x+\varepsilon) - g(0,x) \leq \mathbb{E}\left[e^{-G(x+\varepsilon+B_{T\wedge T_{-(x+\varepsilon)}})}\right] - \mathbb{E}\left[e^{-G(x+B_{T\wedge T_{-x}})}\right]$$

The strong Markov property implies that $\hat{B}_t = B_{T\wedge T_{-x}+t} - B_{T\wedge T_{-x}}$ is a Brownian motion. Therefore,

$$\mathbb{E}\left[e^{-G(x+\varepsilon+B_{T\wedge T_{-(x+\varepsilon)}})}\right] = e^{-G(x+\varepsilon)}\mathbb{E}\left[e^{-GB_{T\wedge T_{-x}}}\phi_\varepsilon(T\wedge T_{-x})\right],$$

where

$$\phi_\varepsilon(t) = \mathbb{E}\left[e^{-G\hat{B}_{(T-t)_+\wedge\hat{T}_{-\varepsilon}}}\right],$$

with $\hat{T}_{-\varepsilon} = \inf\{t \geq 0,\, \hat{B}_t = -\varepsilon\}$. Therefore,

$$0 \leq g(0,x+\varepsilon) - g(0,x) \leq e^{-Gx}\mathbb{E}\left[e^{-GB_{T\wedge T_{-x}}}\right](e^{-G\varepsilon}\phi_\varepsilon(0) - 1)$$

We now focus on the term $e^{-G\varepsilon}\phi_\varepsilon(0) - 1$. We first observe

$$\begin{aligned}
\phi_\varepsilon(0) &= \mathbb{E}\left[e^{-G\hat{B}_{T\wedge\hat{T}_{-\varepsilon}}}\right] \\
&= e^{G\varepsilon}\mathbb{P}(T_{-\varepsilon} \leq T) + \mathbb{E}\left[e^{-G\hat{B}_T \mathbb{1}_{\{T<T_{-\varepsilon}\}}}\right] \\
&\leq e^{G\varepsilon} + e^{\frac{G^2T}{2}}\mathbb{P}^{(G)}(T < T_{-\varepsilon}),
\end{aligned}$$

where $\mathbb{P}^{(G)}$ is defined on $\mathcal{F}_T$ by $\frac{d\mathbb{P}^{(G)}}{d\hat{\mathbb{P}}}|_{\mathcal{F}_T} = \exp(-G\hat{B}_T - \frac{G^2}{2}T)$. Under $\mathbb{P}^{(G)}$, the process $B_t^{(G)} = \hat{B}_t + Gt$ is a Brownian motion and

$$T_{-\varepsilon} = \inf\{t \geq 0,\, B_t^{(G)} - Gt = -\varepsilon\}.$$

Using the density function of the hitting time of a drifted Brownian motion, we obtain

$$\begin{aligned}
\phi_\varepsilon(0) &\leq e^{G\varepsilon} + e^{\frac{G^2T}{2}}\int_T^{+\infty} \frac{\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{1}{2t}(\varepsilon+Gt)^2}\, dt \\
&= e^{G\varepsilon} + e^{-G\varepsilon}\int_T^{+\infty} \frac{\varepsilon}{\sqrt{2\pi t^3}} e^{-\frac{\varepsilon^2}{2t}} e^{-\frac{G^2}{2}(t-T)}\, dt \\
&\leq e^{G\varepsilon} + e^{-G\varepsilon}\sqrt{\frac{2}{\pi}}\int_0^{\frac{\varepsilon}{\sqrt{T}}} e^{-\frac{u^2}{2}}\, du \quad \text{change of variables } u = \frac{\varepsilon}{\sqrt{t}} \\
&= 1 + \left(\sqrt{\frac{2}{\pi T}} + G\right)\varepsilon + o(\varepsilon).
\end{aligned}$$

We then deduce

$$0 \leq g(0, x+\varepsilon) - g(0,x) \leq g(x)\sqrt{\frac{2}{\pi T}}\varepsilon + o(\varepsilon),$$

which yields the result by letting ε tend to zero.
We now assume $G > 0$ for which g is a decreasing function. In the case, the proof follows from Jensen's inequality applied to the convex function e^{-Gx} which implies that the process e^{-GB_t} is a sub-martingale. Applying the optional sampling theorem for e^{-GB_t} for the bounded stopping times $T \wedge T_{-x} \leq T \wedge T_{-x-\varepsilon}$, we get

$$0 \geq g(0, x+\varepsilon) - g(0,x) \geq g(x)(e^{-G\varepsilon} - 1),$$

which implies the result by letting ε tend to zero. □

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