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# A POINCARÉ-LEFSCHETZ THEOREM FOR CELLULAR COSHEAVES AND AN APPLICATION TO THE TROPICAL HOMOLOGY OF ORBIFOLD TORIC VARIETIES

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## Abstract

In a first time we present a version of the Poincaré-Lefschetz theorem for certain cellular cosheaves on a particular subdivision of a CW-complex  $K$ . To that end we construct a cellular sheaf on  $K$  whose cohomology with compact support is isomorphic to the homology of the initial cosheaf. In a second time we use the first result to generalise the tropical version of the Lefschetz hyperplane section theorem to singular tropical toric varieties and singular tropical hypersurfaces.

## Introduction

Given a regular CW-complex  $K$ , we define *dihomologic cosheaves* on  $K$  to be a mild generalisation of the concept of cellular cosheaves on a CW-complex. These objects can be seen as a system of coefficients that associate a group to every pairs of adjacent cells of the complex  $K$ . For a broad variety of examples these cosheaves correspond to classical cellular cosheaves on a suitable subdivision of  $K$ . In a first time we tackled the question: *Can we compute the homology of  $F$  from a reduced quantity of data ?* Given a set of hypotheses about the *local homology* of  $F$  we are able to construct a cellular sheaf whose cohomology with compact support is isomorphic to the homology of  $F$ . We went from data carried by adjacent pairs of cells in  $K$  to data carried by individual cells.

**Theorem 1** (Cellular Poincaré-Lefschetz Theorem). *Let  $K$  be a finite dimensional, locally finite and regular CW-complex,  $n \in \mathbb{N}$ , and  $F$  a dihomologic cosheaf whose local homologies  $H_*(K; F_e)$  vanish at all cells  $e$  in dimension different from  $n$ . Then, for  $0 \leq k \leq n$ , we have a canonical isomorphism between  $H_k(X; F)$  and  $H_c^{n-k}(X; H_n(F_*))$ . In particular  $H_k(K; F)$  vanishes for  $k > n$ . If in addition,  $K$  has dimension  $n$ , this isomorphism comes from an injective quasi-isomorphism  $C_c^{n-*}(K; H_n(F_*)) \rightarrow \Omega_*(K; F)$ .*

This statement reminded us of the Poincaré-Lefschetz duality which can be found as one of its direct corollaries.

**Corollary 1.** *If  $X$  is a homology  $n$ -manifold then  $H_k(X; \mathbb{Z}) \cong H_c^{n-k}(X; \partial X; o_{\mathbb{Z}})$  for  $o_{\mathbb{Z}}$  the system of local orientations defined on  $X \setminus \partial X$  by  $x \mapsto H_n(X; X - x; \mathbb{Z})$ .*

This is the application of Theorem 1 to the constant cosheaf  $\mathbb{Z}$  and in this special case the proof is the same as the one given by Zeeman in [Zee63]<sup>1</sup>. We want to emphasise that we chose the name *dihomologic* in reference to Zeeman's theory of dihomology [Zee62a, Zee62b, Zee63]. A statement similar to Theorem 1 could be derived the assumptions on the local homology of  $F$ . However, in this case we would associate a complex of cellular sheaves to  $F$  whose cohomology with compact support (or more precisely hypercohomology) would be isomorphic to the homology of  $F$ . If  $F$  was the subdivision of a cosheaf on  $K$  the statement would be close to Verdier duality of cellular cosheaves given by Curry in [Cur12]. Another corollary of Theorem 1 is a version of Serre duality for flat vector bundles on a homology manifolds.

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<sup>1</sup>E. C. Zeeman. *Dihomology III. A Generalization of the Poincaré Duality for Manifolds*, Theorem 1 p.159.

**Corollary 2.** *If  $X$  is a homology  $n$ -manifold and  $E$  is a flat bundle of  $\mathbb{F}$ -vector spaces of finite rank over  $X$  then:*

$$H^k(X; E) \cong (H_c^{n-k}(X; \partial X; \mathcal{O}_{\mathbb{F}} \otimes_{\mathbb{F}} E^*))^*.$$

The second theorem is a generalistaion of the tropical version of the Lefschetz hyperplane section theorem given by C. Arnal, A. Renaudineau and K. Shaw in [ARS21]<sup>2</sup> and its extension by E. Brugallé, L. Lopez de Medrano and J. Rau in [BLdMR22]<sup>3</sup> to non-convex triangulations. Tropical homology is defined as the homology of certain dihomologic cosheaves on a convex polyhedral subdivision  $K$  of a convex polytope  $P$ . If we consider an hypersurface of the toric variety  $Y$  associated with  $P$  in [IKMZ19] I. Itenberg, Katzarkov, G. Mikhalkin and I. Zharkov gave two families of cosheaves:

$$F_p^{(1)} \subset F_p^{(0)}, p \in \mathbb{N},$$

whose homologies are respectively the tropical homology groups of the hypersurface and of the toric variety  $Y$ . The tropical version of the Lefschetz hyperplane section theorem describes the nature of the morphisms induced in homology by the inclusions:

$$H_q(K; F_p^{(1)}) \rightarrow H_q(K; F_p^{(0)}).$$

C. Arnal, A. Renaudineau and K. Shaw showed in [ARS21] that when the toric variety  $Y$  associated with  $P$  is smooth and  $K$  is an unimodular triangulation these morphisms are isomorphisms when  $p + q < \dim P - 1$  and surjektivite when  $p + q = \dim P - 1$ . Considering a tropical hypersurface of  $Y$  implies the convexity of the subdivision  $K$ . However, the definition of the cosheaves  $F_p^{(1)}$  and  $F_p^{(0)}$ ,  $p \in \mathbb{N}$ , still makes sense for non-convex subdivisions and E. Brugallé, L. Lopez de Medrano and J. Rau showed that this statement remains true when the convexity hypothesis is dropped. We also state our result without assuming the subdivision to be convex. Using Theorem 1 we are able to extend the statement to orbifold toric varieties modulo a change of coefficients. The toric variety associated with  $P$  is orbifold when the polytope  $P$  is simple. When this the case we define two integers  $\delta(P), \theta(K) \geq 1$  respectively associated with the polytope and the subdivision and we show that:

**Theorem 2.** *Let  $R$  be a ring in which both  $\delta(P)$  and  $\theta(K)$  are invertible, the homological morphisms:*

$$i_{p,q}: H_q(K; F_p^{(1)} \otimes R) \rightarrow H_q(K; F_p^{(0)} \otimes R),$$

induced by the inclusions  $i_p: F_p^{(1)} \rightarrow F_p^{(0)}$  are:

- isomorphisms for all  $p + q < \dim P - 1$  ;
- surjective morphisms for all  $p + q = \dim P - 1$ .

The number  $\delta(P)$  is linked to the singularities of  $Y$ . Its value is 1 if and only if  $Y$  is smooth. On the other hand, even if the number  $\theta(K)$  is determined by the proper singularities<sup>4</sup> of the tropical hypersurface, i.e. of the subdivision  $K$ , it is less fine than  $\delta(P)$  as  $\theta(K) = 1$  on every unimodular triangulations but the converse does not even imply that  $K$  is a triangulation.

In addition of Theorem 2 we recover the formulæ giving the dimensions of the homology groups of the sheaves  $F_p^{(0)}$ , corresponding to the rational Betti numbers of the toric variety  $Y$ .

**Proposition 3.7 1.** *For every ring  $R$  in which  $\delta(P)$  is invertible, and every  $p \in \mathbb{N}$ , the only non-trivial homology group of the cosheaf  $F_p^{(0)} \otimes R$  is  $H_p(K; F_p^{(0)} \otimes R)$ . Moreover this module is free of rank  $h_p(P^\circ)$ , the  $p$ -th  $h$ -number of the polar polytope  $P^\circ$  of the simple polytope  $P$ . More precisely:*

$$\mathrm{rk}_R H_p(K; F_p^{(0)} \otimes R) = \sum_{k=0}^p (-1)^{p-k} \binom{n-k}{p-k} f_{n-k}(P),$$

where  $f_k(P)$  is the number of  $k$ -faces of  $P$ .

We divide this text into three parts. The first is devoted to the introduction of the objects of cellular homology we study here. In particular we introduce the dihomologic pseudo-subdivision of a regular CW-complex  $K$  and study its properties. In the second part we state and prove Theorem 1 and its corollaries. In the last section we apply the previous results to prove Theorem 2.

<sup>2</sup>C. Arnal, A. Renaudineau and K. Shaw. *Lefschetz Section Theorems for Tropical Hypersurfaces*, Theorem 1.2 .1349.

<sup>3</sup>E. Brugallé, L. Lopez de Medrano and J. Rau. *Combinatorial Patchworking: Back from Tropical Geometry*, Proposition 3.2 p.15.

<sup>4</sup>In the sense : non-inherited from the singularities of  $Y$ .

# 1 CW-Complexes and Cellular Homology

CW-complexes were introduced by J. H. C. Whitehead in *Combinatorial homotopy. I*, [Whi49]. Their underlying topological spaces, their supports, form a broad family of spaces usually considered well-behaved. Some of the sheaves defined on their support are particularly adapted to their structure. They can be described by a relatively small amount of data and their cohomology can be computed the techniques of cellular cohomology. In the following paragraphs we give a succinct presentation of the objects at play in this text.

## CW-Complexes

**Definition 1.1** (CW-complex). A *CW-complex*  $K$  is the data of a Hausdorff topological space  $|K|$ , called the *support* of  $K$ , filtered by closed subsets  $\emptyset = K^{(-1)} \subset K^{(0)} \subset \dots \subset K^{(k)} \subset \dots \subset |K|$  called the *skeleta* of  $K$  whose union covers  $|K|$ . Such filtration has to satisfied the additional properties:

1. For every  $k \geq 0$  and every connected component  $e^k$  of  $K^{(k)} \setminus K^{(k-1)}$ , called an *open  $k$ -cell*, there exists a surjective continuous map from the closed  $k$ -dimensional ball onto the closure  $\bar{e}^k$  carrying homeomorphically the open ball onto  $e^k$ , such a map is called a *characteristic map* of the open cell  $e^k$ ;
2.  $|K|$  has the weak topology : a subset  $A \subset |K|$  is closed if and only if its intersection  $A \cap \bar{e}^k$  with every closed cell is closed;
3. Every skeleton  $K^{(k)}$  has the weak topology in the same sense as in point 2.

We call the dimension of  $K$ ,  $\dim K$ , the smallest integer from which the filtration  $(K^{(k)})_{k \geq -1}$  is stationary. It might be  $\infty$ . A sub-complex  $L$  of  $K$  is determined by a closed subset  $|L|$  for which the induced filtration:

$$\emptyset = L^{(-1)} \subset L^{(0)} \subset \dots \subset L^{(k)} \subset \dots \subset |L| \text{ with } L^{(k)} = |L| \cap K^{(k)} \text{ for all } k \in \mathbb{N},$$

turns it into a CW-complex of its own right. The intersection of sub-complexes is again a sub-complex. For  $A$  any subset of  $|K|$  we set  $K(A)$  to be the smallest sub-complex containing  $A$  in its support i.e. the intersection of all sub-complexes containing  $A$  in their support.

**Examples.** 1. The most basic examples are given by simplices and all the geometric realisations of simplicial complexes as defined in [Whi39]. More generally, a polyhedral complex is an example of CW-complex. By a polyhedral complex we mean a collection  $K$  of polytopes<sup>5</sup> in a real vector space that contains all the faces of its polytopes and in which two distinct polytopes intersect on a common face (which might be empty). In a polyhedral complex the open cell corresponding to a polytope is its relative interior, that is to say the topological interior of the polytope in the affine space it spans.

2. An extremely classical example is given by the real projective spaces. They filter themselves  $\mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \dots \subset \mathbb{R}P^n$  by inclusion on the first coordinates and the partition of  $\mathbb{R}P^n$  into open cells corresponds to a decomposition into affine spaces, one for every  $0 \leq k \leq n$ . By extension, the inductive limit  $\mathbb{R}P^\infty$  is also a CW-complex for the induced filtration.

**Definition 1.2.** A CW-complex is called *locally finite* if all of its points has a neighbourhood that meets only finitely many open cells.

Any *finite* (with finitely many cells) CW-complex is obviously locally finite. Among our examples,  $\mathbb{R}P^\infty$  is not locally finite as the neighbourhood of a point in the open cell  $\mathbb{R}^k$  will meet all the open cells  $\mathbb{R}^n$  for  $n \geq k$ .

**Proposition 1.1** (J. H. C. Whitehead. *Combinatorial homotopy. I* [Whi49], (G), pp.225-227, (M), pp.230-231.). *A CW-complex is a normal and locally contractible topological space.*

**Definition 1.3.** A *regular* CW-complex is one that admits for each cell a characteristic map that is a homeomorphism on the entire closed ball.

It implies in particular that every closed cell of a regular CW-complex is homeomorphic to a closed ball. It excludes the CW-complex structure of the real projective spaces (apart from the trivial case  $\mathbb{R}P^0$ ) given by affine spaces as every positive dimensional closed cell is a projective space, different from a closed ball. An important

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<sup>5</sup>a convex hull of a finite number of vertices.

example of regular CW-complex is given by geometric realisations of simplicial complexes for which every closed cell is a closed simplex, hence topologically a closed ball. Likewise a polyhedral complex is necessarily a regular CW-complex.

For a general CW-complex the formula  $e_1 \leq e_2 \iff \bar{e}_1 \subset \bar{e}_2$  defines an order on the cells. When the CW-complex is regular this order shares the same properties as the inclusion of faces in a simplicial complex.

**Lemma 1.2.** *For any two open cells  $e_1, e_2$  of  $K$  a regular CW-complex,  $e_1$  meets the closure of  $e_2$  if and only if it is fully contained in it:*

$$e_1 \cap \bar{e}_2 \neq \emptyset \iff e_1 \subset \bar{e}_2.$$

One can find a proof in [CF67]<sup>6</sup>. Therefore, in a regular CW-complex whenever a cell  $e_1$  meets the closure of another one  $e_2$  we have  $e_1 \leq e_2$  and we say that  $e_1$  is a *face* of  $e_2$ . If  $e_1$  is distinct from  $e_2$  we say that  $e_1$  is a *proper face* of  $e_2$  and denote it  $e_1 < e_2$ . Furthermore, if  $e_1 \leq e_2$  or  $e_2 \leq e_1$  we say that  $e_1$  and  $e_2$  are *adjacent*.

**Lemma 1.3.** *In a regular CW-complex  $K$  the support of the sub-complex  $K(e)$  for any open cell  $e$  is its closure  $\bar{e}$ .*

**Lemma 1.4.** *Let  $k \in \mathbb{N}$  and  $e^{k+2}$  be an open cell of  $K$  a regular CW-complex. For all faces of codimension 2  $e^k$  of  $e^{k+2}$  there are exactly two cells of codimension 1 between  $e^k$  and  $e^{k+2}$ :*

$$\text{card}\{e^{k+1} \mid e^k < e^{k+1} < e^{k+2}\} = 2.$$

**Lemma 1.5 (Open Star).** *For  $e$  a cell of a regular CW-complex  $K$  the union of all the cells having  $e$  as a face, called the open star of  $e$ , is an open subset of  $|K|$ .*

Proofs of these statements are given in [CF67]<sup>7</sup>. In a locally finite CW-complex  $K$  the open star of a cell is a finite union of cells so its closure, the *closed star* of the cell, is a finite sub-complex of  $K$ . The collection  $K - e$  of all the cells whose closure avoid  $e$  is the complement of the open star of  $e$  and is a sub-complex of  $K$ . Its underlying topological space is a deformation retract of  $|K| \setminus e$  and is the largest sub-complex of  $K$  contained in the complement  $|K| \setminus e$ . Likewise we define for a subset  $A$  of  $|K|$ ,  $K - A$  to be the largest sub-complex of  $K$  contained in  $|K| \setminus A$ .

**Definition 1.4 (Subdivisions).** A *subdivision*  $K'$  of a CW-complex  $K$  is a CW-complex on the same support  $|K'| = |K|$  in which every cell  $e' \in K'$  is contained in a cell  $e \in K$ . Another way of saying it is that the partition of  $|K|$  into open cells of  $K'$  is finer than the partition given by  $K$ .

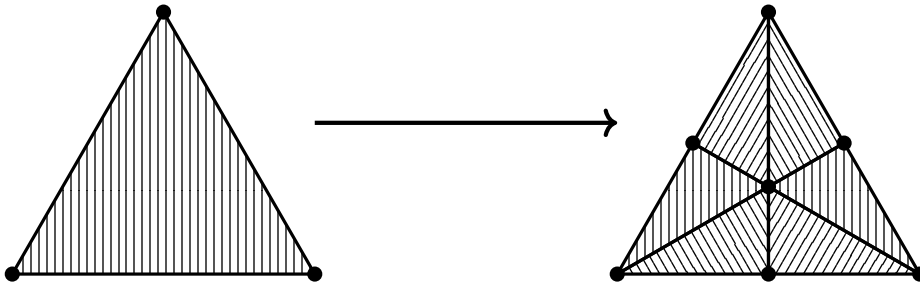


Figure 1: The barycentric subdivision of the triangle.

A common example of subdivision is given by the *barycentric subdivision*  $\text{Sd } S$  of a simplicial complex  $S$ , see for instance Figure 1. It is described abstractly as follows : the vertices of  $\text{Sd } S$  are given by the simplices of  $S$  and the simplices of  $\text{Sd } S$  by the flags of simplices of  $S$ . More concretely a collection of simplices of  $S$  say  $\{\sigma_0, \dots, \sigma_k\}$  is a simplex of  $\text{Sd } S$  if and only if they can be totally ordered by adjacency i.e. there is a permutation  $\pi$  of the indices  $\{0, \dots, k\}$  such that:

$$\sigma_{\pi(0)} < \dots < \sigma_{\pi(k)}.$$

One can define an homeomorphism between the geometric realisation of  $\text{Sd } S$  and the geometric realisation of  $S$  by sending each vertex of  $\text{Sd } S$  to the barycenter of its corresponding simplex in  $S$  and extending such map by linearity. An example of such homeomorphism is depicted in Figure 2.

<sup>6</sup>G. Cooke and R. Finney. *Homology of cell complexes*, pp.229-230, R.R.1.

<sup>7</sup>Ibid. in order, Proposition 1.6, p.30, Theorem 4.2, pp.231-232, Lemma 4.1, p.230.

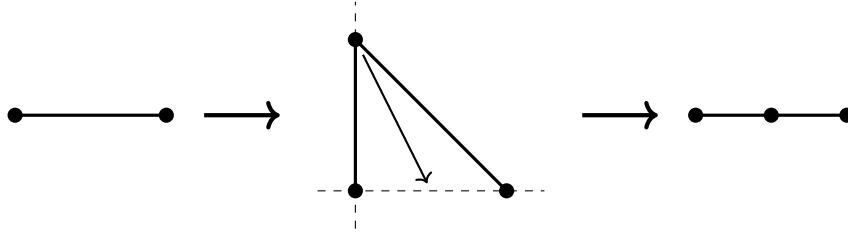


Figure 2: The homeomorphism from the barycentric subdivision of the segment to the initial segment.

The image of the skeletal filtration of  $\text{Sd } S$  under such homeomorphism defines a subdivision of the CW-complex induced by  $S$  in the sense of the previous definition. Note that if instead we chose to send each vertex of  $\text{Sd } S$  to an arbitrary point in the open cell defined by the corresponding simplex in  $S$  (and then extending the map by linearity) we would also have defined a subdivision of  $S$ , equivalent in a combinatorial way, to the previous one. We could say that the barycentric subdivision is only defined unequivocally on the abstract level. The same abstract procedure can be performed with a regular CW-complex  $K$ . Its *barycentric subdivision*  $\text{Sd } K$  is defined to be the following simplicial complex:

1. Every cell of  $K$  corresponds to a vertex of  $\text{Sd } K$ ;
2. A finite set of cells of  $K$  corresponds to a simplex of  $\text{Sd } K$  if and only if it is totally ordered by adjacency.

**Proposition 1.6.** *For every regular CW-complex  $K$ , the geometric realisation of  $\text{Sd } K$  is homeomorphic to  $|K|$  in such a way that  $\text{Sd } K$  can be seen as a subdivision of  $K$ .*

This proposition, proven in [LW69]<sup>8</sup>, allows us to see a regular CW-complex as “a simplicial complex in which the simplexes are more efficiently combined into closed cells.”<sup>9</sup> Another feature of simplicial complexes shared by regular complex is the following:

**Proposition 1.7.** *In a regular CW-complex the open stars of cells are contractible.*

*Proof.* The geometric realisation of  $\text{Sd } K$  lives in the real vector space  $V$  spanned by the cells of  $K$ . We use the same symbol to denote a cell  $e^k$  and its associated generator in  $V$ . Hence an element of  $V$  is a formal finite linear combinations of the cells of  $K$ . We endow this vector space with the norm 1:

$$\left\| \sum_{e \in K} x_e e \right\|_1 = \sum_{e \in K} |x_e|.$$

The geometric realisation  $|\text{Sd } K|$  is the union of the convex hulls of the sets of cells  $\{e^{k_0}, \dots, e^{k_n}\}$  corresponding to barycentric simplices i.e. flags of cells. It is a subset of the intersection of the unit sphere with the positive ortant  $V_+ := \{\sum_{e \in K} x_e e \in V \mid x_e \geq 0\}$ . For a flag of cells  $e^{k_0} < \dots < e^{k_n}$  let us denote here the corresponding open simplex by:

$$(e^{k_0}; \dots; e^{k_n}) := \left\{ \sum_{i=0}^n x_i e^{k_i} \mid \forall i, x_i > 0 \text{ and } \sum_{i=0}^n x_i = 1 \right\}.$$

An open cell  $e^k$  of  $K$  corresponds under the homeomorphism of Proposition 1.6 to the union of the open barycentric simplices  $(e^{k_0}; \dots; e^{k_n})$  for which  $e^{k_n} = e^k$ . Therefore the open star  $S$  of  $e^k$  is in this context:

$$S = \bigcup_{\substack{e^{k_0} < \dots < e^{k_n} \\ e^k \leq e^{k_n}}} (e^{k_0}; \dots; e^{k_n}).$$

Now define the family of bounded linear operators  $(\Phi_t : V \rightarrow V)_{0 \leq t \leq 1}$  by  $\Phi_t = (\text{id} - \pi) + t\pi$  for  $\pi$  the projection to the sub-space spanned by the  $e^p$  that does not contain  $e^k$  parallelly to the sub-space spanned by those that contain it. Let  $U$  be the open set of  $V_+$  of vectors that have at least one coordinate indexed by a cell that contains  $e^k$  strictly positive. We have  $S = |\text{Sd } K| \cap U$ . For all  $t \in [0; 1]$  the map:

<sup>8</sup>A. Lundell and S. Weingram. *The Topology of CW Complexes*, Theorem 1.7, pp.80-81.

<sup>9</sup>Ibid. p.77.

$$\begin{aligned} \Psi : [0; 1] \times U &\longrightarrow U \\ u &\longmapsto \frac{\|u\|_1}{\|\Phi_t(u)\|_1} \Phi_t(u), \end{aligned}$$

is continuous and every partial map  $\Psi(t; -)$  stabilises  $S$ . The image  $\Psi(0; S)$  is the union of the open barycentric simplices  $(e^{k_0}; \dots; e^{k_n})$  for which  $e^k \leq e^{k_0}$ . Note that the restriction of every map  $\Psi(t; -)$  is constant on this set. Therefore  $\Psi(0; S)$  is a deformation retract of  $S$ . Now this set retracts on the barycentre of  $e^k$  by simple convex interpolation  $(t; u) \in [0; 1] \times \Psi(0; S) \mapsto (1-t)u + te^k$  thus  $S$  is contractible.  $\square$

When we consider the barycentric subdivision of a finite regular CW-complex we increase considerably the number of cells. There is however a less expensive procedure that builds for any regular CW-complex  $K$  a “pseudo-subdivision” that sits between  $K$  and  $\text{Sd}K$ . By pseudo-subdivision we mean a certain recombination of the barycentric simplices that looks a lot like a regular subdivision.

**Definition 1.5** (Dihomologic Pseudo-subdivision). For  $K$  a regular CW-complex and  $e^p \leq e^q$  a pair of adjacent cells we define its associated *dihomologic pseudo-cell* to be the union of the open barycentric simplices<sup>10</sup> associated the flags  $e^{k_1} < \dots < e^{k_n}$  for which  $e^p = e^{k_1}$  and  $e^{k_n} = e^q$ . These “open” pseudo-cells partition the simplicial complex  $\text{Sd}K$ . We call such partition the *dihomologic pseudo-subdivision* of  $K$ . We say that the dihomologic pseudo-cell associated with the pair  $e^p \leq e^q$  has dimension  $q-p$ . Figure 3 illustrates this procedure on a disc.

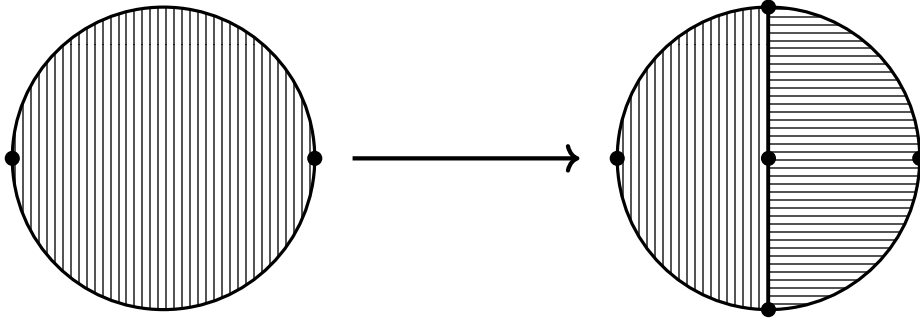


Figure 3: The dihomologic pseudo-subdivision of a regular CW-complex structure of the disc.

This partition shares many properties with a regular subdivision of  $K$  but may fail to define a CW-complex structure on  $|K|$ . In full generality the closure of dihomologic pseudo-cells might not be homeomorphic to closed balls. However, for a broad variety of examples, this is indeed a regular subdivision of  $K$ .

**Remark 1.1.** We chose to use the terminology “dihomologic” because Zeeman’s dihomology bicomplex, c.f. [Zee62a, Zee62b, Zee63], would be the cellular chain complex of this pseudo-subdivision. This bicomplex was latter (dually) rediscovered by Forman in [For02] under the name of “combinatorial differential forms”.

**Proposition 1.8.** *Let  $e^p \leq e^q$  be a pair of adjacent cells of  $K$ . The closure of the pseudo-cell associated with  $e^p \leq e^q$ , is a pure collapsible simplicial complex of dimension  $q-p$ . Moreover, any codimension 1 simplex of the open pseudo-cell is exactly contained in two of its maximal simplices.*

*Proof.* Let  $e^p \leq e^q$  be a pair of adjacent cells. If  $e^p = e^q$  the associated pseudo-cell is a single vertex and the statement of the proposition is true. Now suppose  $e^p$  is a proper face of  $e^q$ . The closure of the associated pseudo-cell is the union of the barycentric simplices whose flags  $e^{k_1} < \dots < e^{k_n}$  satisfy  $e^p \leq e^{k_1} < \dots < e^{k_n} \leq e^q$ . This is a simplicial sub-complex of  $\text{Sd}K$ . A maximal simplex of this closed pseudo-cell is given by a maximal chain of adjacent cells of  $K$  starting from  $e^p$  and ending at  $e^q$ . Since  $K$  is regular such chain necessarily has length  $q-p+1$ , so the corresponding simplex has dimension  $q-p$ . Hence, the closed pseudo-cell is a pure  $(q-p)$ -dimensional simplicial complex. If  $q-p=1$ , this closed pseudo-cell corresponds to the closed barycentric edge  $e^p < e^q$ . This is a collapsible complex. If  $q-p > 1$ , the maximal simplices of the closed pseudo-cell correspond to flags  $e^p < e^{p+1} < \dots < e^{q-1} < e^q$ . They all have a codimension 2 free face, associated with  $e^{p+1} < \dots < e^{q-1}$ . If we collapse all such simplices we obtain the union of the closed  $(q-p-1)$ -simplices associated with the flags of the form  $e^p < e^{k_1} < \dots < e^{k_{q-p-2}} < e^q$ . All such simplices are now maximal and contain each a free face of codimension 2, namely  $e^{k_1} < \dots < e^{k_{q-p-2}}$ . We can recursively perform such collapses to end up with the

<sup>10</sup>The open simplex on a vertex set  $v_0, \dots, v_n$  is the set  $\{\sum_{i=0}^n t_i v_i \mid \forall i, t_i > 0 \text{ and } \sum_{i=0}^n t_i = 1\}$ .

barycentric edge  $e^p < e^q$ . Thus the closed pseudo-cell of the pair  $e^p \leq e^q$  is collapsable. For the final part, a simplex of codimension 1 of the open pseudo-cell is given by a flag  $e^p < e^{k_1} < \dots < e^{k_{q-p-2}} < e^q$  of length  $q-p$  and has the form  $e^p < \dots < e^i < e^{i+2} < \dots < e^q$ . Since  $K$  is regular there is exactly two  $(i+1)$ -cells between  $e^i$  and  $e^{i+1}$ , hence two  $(q-p)$ -simplices.  $\square$

**Proposition 1.9.** *The open pseudo-cell associated with the pair  $e^p \leq e^q$  meets the closed pseudo-cell indexed by the pair  $e^{p'} \leq e^{q'}$  if and only if it is fully contained in it. This only happens when  $e^{p'} \leq e^p \leq e^q \leq e^{q'}$ . Moreover, if  $\epsilon_0$  is an open pseudo-cell of dimension  $k$  included in a closed pseudo-cell  $\bar{\epsilon}_2$  of dimension  $k+2$  then there is exactly two open pseudo-cells  $\epsilon_1$  of dimension  $k+1$  such that  $\epsilon_0 \subset \bar{\epsilon}_1$  and  $\epsilon_1 \subset \bar{\epsilon}_2$ .*

*Proof.* The first part is a consequence of the definition. For the second part choose  $\epsilon_0$  corresponding to a pair  $e^{p+a} \leq e^{p+a+k}$  and  $\epsilon_2$  to a pair  $e^p \leq e^{p+a+k+b}$  satisfying  $e^p \leq e^{p+a} \leq e^{p+a+k} \leq e^{p+a+k+b}$ . By assumption, we have  $a+b=2$  and three different cases can occur : one of the two numbers  $a, b$  is 2 and the other 0 or both equal 1. The two first cases are symmetric. If it's  $a$  that equals 2 we have  $e^p \leq e^{p+2} \leq e^{p+2+k} \leq e^{p+2+k}$  and the two  $(k+1)$ -pseudo-cells between  $\epsilon_0$  and  $\epsilon_1$  are those associated with the two pairs  $e^{p+1} \leq e^{p+2+k}$  with  $e^p \leq e^{p+1} \leq e^{p+2}$ . The symmetric case is similar. If both  $a$  and  $b$  equal 1 then we have  $e^p \leq e^{p+1} \leq e^{p+1+k} \leq e^{p+2+k}$  and the two  $(k+1)$ -pseudo-cells between  $\epsilon_0$  and  $\epsilon_1$  are those associated with the two pairs  $e^p \leq e^{p+1+k}$  and  $e^{p+1} \leq e^{p+2+k}$ .  $\square$

In the light of this property it makes sense to talk about *adjacent* pseudo-cells as we do for cells of regular CW-complexes. As in the regular case, if  $\epsilon$  and  $\epsilon'$  are adjacent pseudo-cells with  $\epsilon \subset \bar{\epsilon}'$  we say that  $\epsilon$  is a *face* of  $\epsilon'$ . If in addition  $\epsilon \neq \epsilon'$  we say that  $\epsilon$  is a *proper face* of  $\epsilon'$ . Also we see here that two dihomologic pseudo-cells meet on a common faces if their intersection is not empty. Note that this property might not be true in  $K$ , two closed cells meet on the union of their common faces if the intersection is non-empty. Let  $e^p < e^q$  be a proper adjacent pair of cells of  $K$  and denote  $\epsilon$  their associated pseudo-cell. The simplicial complex supported on the closure,  $(\text{Sd } K)(\epsilon)$ , is the join of the barycentric edge  $e^p < e^q$  and a sub-complex  $A(e^p; e^q) \subset \text{Sd } K$ . This sub-complex is the collection of all the barycentric simplices indexed with the flags  $e^{k_1} < \dots < e^{k_n}$  for which  $e^p < e^{k_1}$  and  $e^{k_n} < e^q$ .

**Proposition 1.10.** *The support of  $A(e^p; e^q)$  is a connected  $(q-p-2)$ -dimensional homology manifold whose  $(p+1)$ -fold suspension is homeomorphic to a  $(q-1)$ -sphere.*

Let us recall the definition:

**Definition 1.6** (Homology Manifold). A *homology manifold* of dimension  $n \in \mathbb{N}$  is the support  $X$  of a regular, finite dimensional, locally finite CW-complex for which the graded local homology group  $H_*(X; X \setminus \{x\}; \mathbb{Z})$  of every point  $x \in X$  is isomorphic to either  $H_*(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$  or 0. The *boundary* of  $X$  denoted  $\partial X$  is the set of points  $x \in X$  for which  $H_*(X; X \setminus \{x\}; \mathbb{Z}) = 0$ .

*Proof.* Let  $B$  denote the simplicial complex  $(\text{Sd } K)(\bar{e}^q \setminus e^q)$  and  $e^0 < \dots < e^p$  be a complete flag of cells of  $K$ . The simplicial complex  $A(e^p; e^q)$  is the link in  $B$  of the barycentric simplex associated with  $e^0 < \dots < e^p$ .  $B$  is a simplicially triangulated  $(q-1)$ -sphere thus in application of Proposition 1.3 from [GS80]<sup>11</sup> the  $(p+1)$ -fold suspension of  $A(e^p; e^q)$  is homeomorphic to a  $(q-1)$ -sphere. The Mayer-Vietoris long exact sequence in singular homology implies that if  $\Sigma A$  is the suspension of a topological space  $A$  we have  $H_0(\Sigma A; \mathbb{Z}) = \mathbb{Z}$ ,  $H_k(\Sigma A; \mathbb{Z}) \cong H_{k-1}(A; \mathbb{Z})$  for all  $k \geq 2$  and the exact sequence:

$$0 \rightarrow H_1(\Sigma A; \mathbb{Z}) \rightarrow H_0(A; \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0.$$

So  $A$  has the homology of a  $k$ -sphere if and only if the  $l$ -fold suspension  $\Sigma^l A$  has the homology of an  $(l+k)$ -sphere. Therefore the simplicial complex  $A(e^p; e^q)$  has the integral homology of a  $(q-p-2)$ -sphere. For the remaining part of the proposition we note that if the barycentric  $n$ -simplex with indexing flag  $e^{k_0} < \dots < e^{k_n}$  belongs to  $A(e^p; e^q)$  then its link  $L$  in this complex is the join:

$$A(e^p; e^{k_0}) * A(e^{k_0}; e^{k_1}) * \dots * A(e^{k_{n-1}}; e^{k_n}) * A(e^{k_n}; e^q).$$

Hence, its  $(p+n+2)$ -fold suspension is homeomorphic to a  $(q-1)$ -sphere and  $L$  has the integral homology of a  $(q-p-n-3)$ -sphere. Now  $A(e^p; e^q)$  is a pure simplicial complex of dimension  $(q-p-2)$  in which the link of every  $n$ -dimensional simplex has the homology of a  $(q-p-n-3)$ -sphere. This is a homology

<sup>11</sup>D. Galewski and R. Stern. *Classification of Simplicial Triangulations of Topological Manifolds*, Proposition 1.3 p.5.



manifold of dimension  $(q-p-2)$ . Indeed, if  $x$  is a point of  $|A(e^p; e^q)|$  that belongs to the relative interior of the barycentric  $n$ -simplex  $\sigma$ , then, by excision,  $H_k(|A(e^p; e^q)|; |A(e^p; e^q)| \setminus \{x\}; \mathbb{Z}) = H_k(|S|; |S| \setminus \{x\}; \mathbb{Z})$  for all  $k$  with  $S$  the closed star of  $\sigma$ . Note that  $|S|$  is contractible and  $|S| \setminus \{x\}$  is non-empty so  $H_0(|S|; |S| \setminus \{x\}; \mathbb{Z}) = 0$ ,  $H_k(|S|; |S| \setminus \{x\}; \mathbb{Z}) = H_{k-1}(|S| \setminus \{x\}; \mathbb{Z})$  for  $k \geq 2$  and :

$$0 \rightarrow H_1(|S|; |S| \setminus \{x\}; \mathbb{Z}) \rightarrow H_0(|S| \setminus \{x\}; \mathbb{Z}) \rightarrow \mathbb{Z} \rightarrow 0 .$$

Since  $|S|$  is homeomorphic to the topological join  $\sigma * |L|$  for  $L$  the link of  $\sigma$ ,  $|S| \setminus \{x\}$  is homotopic to the  $n$ -fold suspension of  $|L|$ . By assumption  $|L|$  has the homology of a  $(q-p-n-3)$ -sphere so  $|S| \setminus \{x\}$  has the homology of a  $(q-p-3)$ -sphere. Combining this with the relation observed by the relative homology of  $(|S|; |S| \setminus \{x\})$  with the homology of  $|S| \setminus \{x\}$  we find that  $H_k(|A(e^p; e^q)|; |A(e^p; e^q)| \setminus \{x\}; \mathbb{Z}) \cong H_k(\mathbb{R}^{q-p-2}; \mathbb{R}^{q-p-2} \setminus \{0\}; \mathbb{Z})$  for all  $k$  and  $A(e^p; e^q)$  is a compact  $(q-p-2)$ -homology manifold without boundary.  $\square$

As a direct consequence we get that:

**Proposition 1.11.** *The closed pseudo-cells associated with the adjacent pairs of the form  $e^0 \leq e^p$  are homeomorphic to closed balls.*

*Proof.* It follows from the previous observation that the support of this closed pseudo-cell is homeomorphic to the topological join  $[0; 1] * |A(e^0; e^p)|$  which is the cone over the suspension of  $|A(e^0; e^p)|$ . By the last proposition this suspension is a  $(p-1)$ -sphere so the closed pseudo-cell is actually a closed ball.  $\square$

For  $\epsilon$  a dihomologic pseudo-cell associated with a pair  $e^p \leq e^q$ , its “boundary”  $\bar{\epsilon} \setminus \epsilon$  is the support of the simplicial join of the union of the barycenters of  $e^p$  and  $e^q$  with  $A(e^p; e^q)$  (so the suspension of  $A(e^p; e^q)$ ). From that description we see that it is the union of dihomologic pseudo-cells that are faces of  $\epsilon$ .

We will see further that the dihomologic pseudo-subdivision also shares a lot of homological features with a regular subdivision. Now we state a condition that ensures the regularity of this pseudo-complex.

**Proposition 1.12.** *If  $K$  is not only regular but also satisfies that the induced CW-complex on every closed cell  $K(e)$  is shellable in the sense of [Bjö84] then every closed dihomologic pseudo-cell is also shellable. As a consequence, the geometric realisation of every closed pseudo-cell is actually homeomorphic to a closed ball making the dihomologic pseudo-subdivision a regular subdivision of  $K$ .*

*Proof.* Let  $e^q$  be a cell of  $K$ . From the first part of Proposition 4.4 of [Bjö84]<sup>12</sup> we know that the barycentric subdivision of  $K(e^q)$  is a shellable simplicial complex. For  $e^p \leq e^q$  a face of  $e^q$  we expressed the associated closed dihomologic pseudo-cell  $\bar{\epsilon}$  as the simplicial join of a closed interval and the simplicial complex  $A(e^p; e^q)$ . As in the proof of Proposition 1.10 we can write this complex as the link of a barycentric simplex  $\sigma$  with indexing flag  $e^0 < e^1 < \dots < e^p < e^q$  in  $\text{Sd}(K(e^q))$ . The link of  $\sigma$  is shellable by Lemma 8.7 of [Zie95]<sup>13</sup>. The complex  $A(e^p; e^q)$  is a pure  $(q-p-2)$ -dimensional shellable simplicial complex in which every codimension 1 simplex belongs to exactly two maximal simplices, hence it is homeomorphic to a sphere by Proposition 4.3 of [Bjö84]<sup>14</sup>. Finally, the closed dihomologic pseudo-cell  $\bar{\epsilon}$  is the support of a shellable simplicial complex homeomorphic to a closed ball.  $\square$

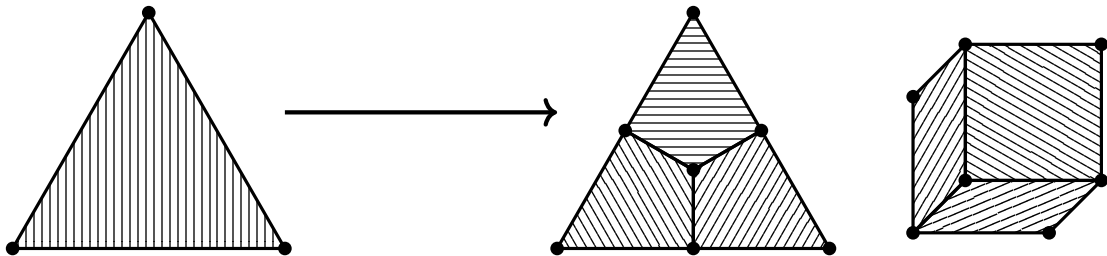


Figure 4: The dihomologic subdivision of the triangle.

When  $K$  is a polyhedral complex the theorem of Bruggesser and P. Mani [BM71]<sup>15</sup> ensures that it satisfies the hypotheses of the last proposition and the dihomologic pseudo-subdivision is an actual regular subdivision.

<sup>12</sup>A. Björner. *Posets, Regular CW Complexes and Bruhat Order*, Proposition 4.4 p.12.

<sup>13</sup>G. Ziegler, *Lectures on Polytopes*, Lemma 8.7 p.237.

<sup>14</sup>A. Björner. op. cit., Proposition 4.3 p.12.

<sup>15</sup>H. Bruggesser and P. Mani. *Shellable decompositions of cells and spheres*, Corollary p.203.

This is especially the case when  $K$  is a simplicial complex. In this particular case the associated dihomologic subdivision even has the structure of a cubical complex, c.f. Figure 4. It comes from the following triangulation of the cube  $[0; 1]^n$ : order its vertex set  $\{0; 1\}^n$  with the product order<sup>16</sup> and consider the convex hulls of the flags of such vertices as the simplices of the triangulation. The triangulation of the 3-dimensional cube is illustrated in Figure 5.

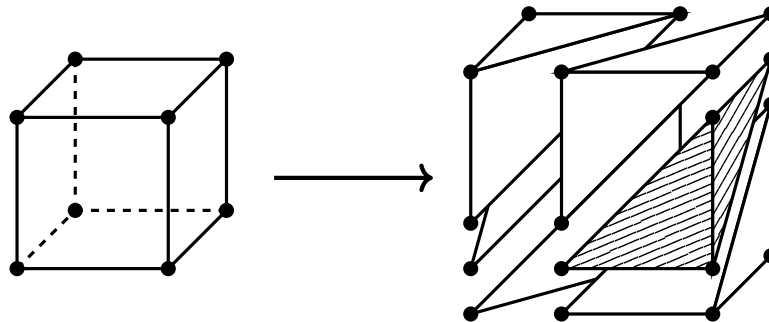


Figure 5: The subdivision of a cube into six tetrahedra, the convex hull of  $\{(0; 0; 0); (1; 0; 0); (1; 0; 1); (1; 1; 1)\}$  is marked.

It produces a triangulation of the  $n$ -cube into  $n!$  simplices. Observe now that the ordered set of vertices of  $[0; 1]^n$  is naturally isomorphic to the lattice of subsets of a set with  $n$  elements. Moreover, if  $\sigma \leq \tau$  are a pair of adjacent simplices of relative codimension  $n$ , the lattice of intermediary simplices  $\{\sigma \leq \nu \leq \tau\}$  is the same as the lattice of faces of the link of  $\sigma$  in  $\tau$  (empty face included) i.e. the lattice of subsets of a set with  $\dim(\tau) + 1 - (\dim(\sigma) + 1) = n$  elements. For more general polyhedral complexes the shapes of the pseudo-cells can be different, as shown in Figure 6.

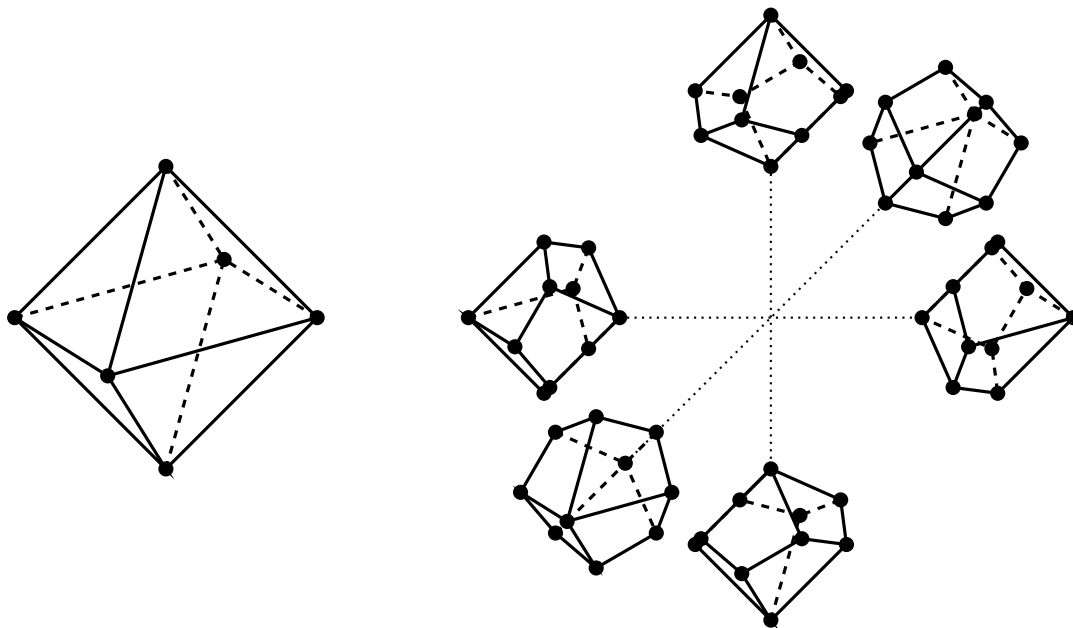


Figure 6: The dihomologic subdivision of an octahedron.

<sup>16</sup> $(x_i)_{1 \leq i \leq n} \leq (y_i)_{1 \leq i \leq n}$  if and only if  $x_i \leq y_i$ , for all  $1 \leq i \leq n$ .

However, even when  $K$  doesn't satisfy the hypotheses of the Proposition 1.12 all the 2-dimensional dihomologic pseudo-cell are squares because of Lemma 1.4, c.f. Figure 7. Finally in low dimension the pseudo subdivision is always regular:

**Proposition 1.13.** *Let  $e^p \leq e^q$  be an adjacent pair of cells of  $K$ . If,  $2 \leq q - p \leq 4$ , then  $A(e^p; e^q)$  is homeomorphic to a sphere. If  $q = p + 5$ ,  $A(e^p; e^q)$  is a 3-dimensional integral homology sphere.*

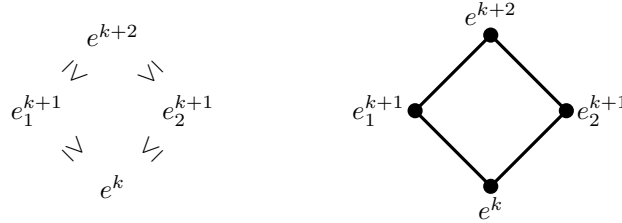


Figure 7: The lattice of faces and the dihomologic square associated with a pair of relative codimension 2.

*Proof.* As shown in Figure 7 the simplicial complex  $A(e^p; e^{p+2})$  consists of two vertices and therefore is a 0-sphere. If we look at  $A(e^p; e^{p+3})$  the link of every simplex is either empty or a  $A(e^k; e^{k+2})$  so  $A(e^p; e^{p+3})$  is actually a manifold by Proposition 1.3 of [GS80]<sup>17</sup>. Therefore by Proposition 1.10, it is a 1-dimensional integral homology sphere, so a circle. Now for  $A(e^k; e^{k+4})$  we have from the proof of Proposition 1.10 that the link of every simplex is either empty or a join of a  $A(e^k; e^{k+2})$  with a  $A(e^k; e^{k+3})$  which we have just shown to be spheres. Therefore,  $A(e^k; e^{k+4})$  is a 2-dimensional integral homology sphere. By classification of compact orientable 2-dimensional manifolds it is homeomorphic to a 2-sphere. For the last part our previous arguments show that the  $A(e^p; e^{p+5})$  are 3-dimensional integral homology spheres.  $\square$

**Remark 1.2.** The 3-dimensional closed pseudo-cells are not only closed balls but even *trapezohedra* i.e. similar to Figure 8. The family of such polyhedra is indexed by an integer  $n$  at least equal to 3 for which we find “the cube”.

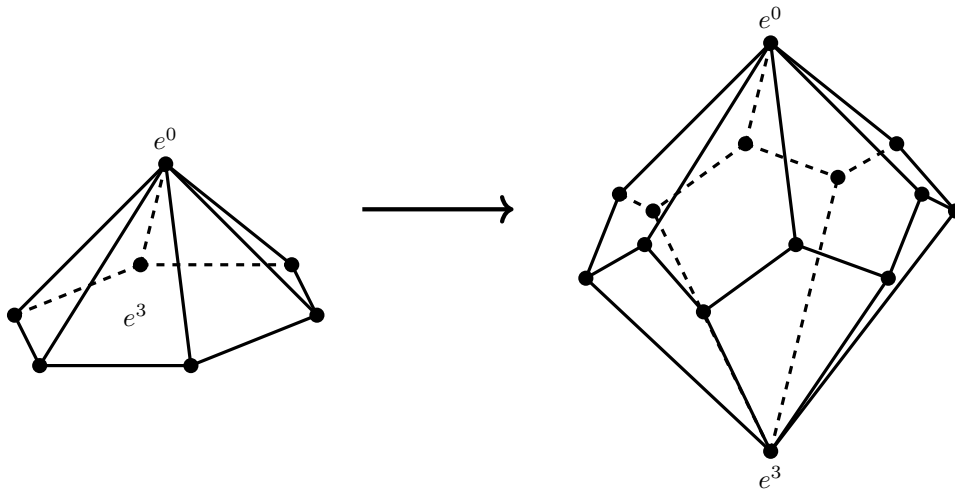


Figure 8: The shape of the dihomologic cell associated with the adjacent pair  $e^0 \leq e^3$ .

A direct consequence of Proposition 1.13 and Proposition 1.11 is:

**Proposition 1.14.** *If  $\dim K$  is at most 5 its dihomologic pseudo-subdivision is a regular subdivision.*

<sup>17</sup>D. Galewski and R. Stern, *Classification of Simplicial Triangulations of Topological Manifolds*, Proposition 1.3 p.5.

## Cellular Sheaves and Cosheaves

In this paragraph  $K$  denotes a regular CW-complex.

**Definition 1.7** (Constructible Sheaves). A sheaf  $F$  on  $|K|$  is called *constructible* with respect to the CW-complex structure if its restriction to every open cell is constant.

Such sheaves are among the “simplest ones” on  $|K|$  as they are reducible to combinatorial data. The knowledge of their section groups above every open star as well as the restrictions morphisms between these stars is enough to characterise the sheaf completely up to isomorphism. The key facts about these sheaves is the following : for every cell  $e$  of  $K$ , if  $S$  denotes its open star and  $x \in e$  then the two following morphisms are isomorphisms:

$$F(S) \xrightarrow{\text{rest.}} F|_e(e) \xrightarrow{\text{stalk}} F_x.$$

See for instance [Kas84]<sup>18</sup>. Since  $e$  is connected and locally connected the fact that  $F|_e(e) \rightarrow F_x$  is an isomorphism follows from the constance of  $F|_e$ . Let  $F(e)$  denote the group  $F|_e(e)$  of values of  $F|_e$ . From the previous observation we gain a “restriction map” between  $F(e^q)$  and  $F(e^p)$  for all pairs of adjacent cells  $e^q \leq e^p$ , coming from the commutative diagram:

$$\begin{array}{ccc} F(e^q) & \longrightarrow & F(e^p) \\ \text{rest. to } F|_{e^q}(e^q) \uparrow \cong & & \cong \uparrow \text{rest. to } F|_{e^p}(e^p) \\ F(S_{e^q}) & \xrightarrow[\text{rest. } S_{e^p} \subset S_{e^q}]{} & F(S_{e^p}) \end{array}$$

with  $S_{e^q}$  and  $S_{e^p}$  the respective open stars of  $e^q$  and  $e^p$ . The data of the groups  $F(e^p)$  together with the morphisms connecting them, defined from the topological sheaf  $F$ , is called a cellular sheaf:

**Definition 1.8** (Cellular Sheaf). A *cellular sheaf* on  $K$  is the data of a covariant functor:

$$F : \mathbf{Cell} K \rightarrow \mathbf{Mod}_R,$$

from the category of cells of  $K$  with arrows given by adjacency to the category of  $R$ -modules (for some commutative ring  $R$ ). We call the images of the arrows by such functor its *restriction morphisms*. For two adjacent cells  $e^p \leq e^q$  and  $f \in F(e^p)$  we will denote by  $f|_{e^p}^{e^q}$  the image of  $f$  in  $F(e^q)$  by the restriction morphism.

A typical example of cosheaf on a topological space is the assignment to every open set its ring of continuous functions with compact support. The definition of cellular cosheaf is dual to the definition of cellular sheaves.

**Definition 1.9** (Cellular Cosheaf). A *cellular cosheaf* on  $K$  is the data of a contravariant functor:

$$F : (\mathbf{Cell} K)^{\text{op}} \rightarrow \mathbf{Mod}_R,$$

from the category of cells of  $K$  with arrows given by adjacency to the category of  $R$ -modules (for some commutative ring  $R$ ). We call the images of the arrows by such functor its *extension morphisms*. For two adjacent cells  $e^p \leq e^q$  and  $f \in F(e^q)$  we will denote by  $f|_{e^p}^{e^q}$  the image of  $f$  in  $F(e^p)$  by the extension morphism.

Every functorial operation performed on Abelian groups, or more generally on modules over a given commutative ring, such as direct sums, products, tensor products, etc. can be performed as well on cellular sheaves and cosheaves by performing it group by group over every cell. Also we can construct a cosheaf from a sheaf  $F$  by considering for  $G$  a fixed group the contravariant functor  $e \mapsto \text{Hom}(F(e); G)$  with adjoint arrows. This construction also goes the other way around when one start with a cosheaf.

**Definition 1.10** (Morphisms of Sheaves and Cosheaves). A *morphism of cellular sheaves* (or *cosheaves*)  $f : F \rightarrow F'$  is a natural transformation. Such morphism is said to be *injective* (resp. *surjective*, resp. *invertible*) if the associated morphisms  $f_e : F(e) \rightarrow F'(e)$  are injective (resp. surjective, resp. invertible) for all cells  $e$ . The *kernel*, *image*, and *cokernel* of such morphism  $f$  are the “cell-wise” kernel, image, and cokernel. They are (co)sheaves themselves with the induced restriction/extension morphisms because  $f$  is a natural transformation.

<sup>18</sup>M. Kashiwara, *The Riemann-Hilbert Problem for Holonomic Systems*, Proposition 1.3 p.323.

The most basic example of such objects are given by *local system of coefficients*. We see such local systems as fibre bundles of discrete groups above  $|K|$ . Since every cell is connected and contractible the restriction of its sheaf of continuous sections to any cell is constant. Therefore, it satisfies the hypothesis of the definition and induces a cellular sheaf. It has the property that all its restriction morphisms are invertible. This is even a way to characterise such local systems. This special property also allows us to see it as a cellular cosheaf by inverting every arrows. Indeed, the commutativity conditions on the composition of such morphisms are automatically satisfied from the ones given by the cellular sheaf structure. Another family of examples is given by the *characteristic cosheaves* associated with sub-complexes:

**Definition 1.11.** Let  $K'$  be a sub-complex of  $K$  and  $G$  be an Abelian group (or a module over a commutative ring) we denote by  $[K'; G]$  the cellular cosheaf defined by:

$$e \in K \mapsto \begin{cases} G & \text{if } e \in K' \\ 0 & \text{otherwise} \end{cases},$$

with extension morphisms given either by the identity of  $G$  or the zero morphism whenever one of the two groups involved is trivial. If a cell belongs to  $K'$  then all of its faces belong to it too. As a consequence the commutativity conditions are satisfied for every triplet of adjacent cells gives rise to one of the following commutative diagrams:

$$\begin{array}{ccc} \begin{array}{ccc} G & & \\ \downarrow \text{id} & \searrow \text{id} & \\ & & G \\ \downarrow \text{id} & \swarrow \text{id} & \\ G & & \end{array} & \begin{array}{ccc} 0 & & \\ \downarrow 0 & \searrow 0 & \\ & & G \\ \downarrow 0 & \swarrow \text{id} & \\ G & & \end{array} & \begin{array}{ccc} 0 & & \\ \downarrow 0 & \searrow 0 & \\ & & 0 \\ \downarrow 0 & \swarrow 0 & \\ G & & \end{array} & \begin{array}{ccc} 0 & & \\ \downarrow 0 & \searrow 0 & \\ & & 0 \\ \downarrow 0 & \swarrow 0 & \\ G & & \end{array} \end{array}$$

Whenever  $K''$  is a sub-complex of  $K'$  we have a natural injective morphism of cosheaves  $[K''; G] \rightarrow [K'; G]$  either given by the 0 morphism or the identity. We will denote the resulting quotient by  $[K'; K''; G]$ . It is  $G$  on the cells of  $K'$  not contained in  $K''$  and 0 elsewhere. A sub-family of these examples will be of particular interest. They are the “local” cosheaves  $[K; K - e; G]$ , for  $e$  a cell of  $K$ , whose value is  $G$  only on the cells containing  $e$ . For a cell  $e$ , the cosheaves  $[K(e); G]$  and  $[K; K - e; G]$  are the dual constructions of the elementary cellular sheaves considered by A. Shepard in his thesis [She85]. They were also considered later by J. Curry in [Cur12] for instance.

**Definition 1.12** (Localisation of a cellular cosheaf). Let  $F$  be a cellular cosheaf on  $K$  and  $e$  a cell. We denote by  $F_e$  the tensor product  $F \otimes_{\mathbb{Z}} [K; K - e; \mathbb{Z}]$  and call it the *localisation* of  $F$  at  $e$ . For  $e'$  another cell  $F_e(e')$  is  $F(e')$  if  $e' \geq e$  and 0 otherwise, its extension morphisms are then appropriately given by the extension morphisms of  $F$  or 0. Moreover, the natural projection  $[K; K - e; \mathbb{Z}] \rightarrow [K; K - e'; \mathbb{Z}]$  for adjacent cells  $e \leq e'$  induces a surjective localisation morphism  $F_e \rightarrow F_{e'}$ .

**Definition 1.13** (Subdivision). If  $K'$  is a subdivision of  $K$  there is a subdivision functor from the category of cellular cosheaves of  $K$  (resp. cellular sheaves). If  $F$  is a cosheaf (resp. sheaf) on  $K$  its *subdivision*  $F'$  is given for all cell  $e' \in K'$  by:

$$F'(e') = F(e),$$

for  $e$  the only cell of  $K$  containing  $e'$ . The extension (resp. restriction) morphisms are the adequately derived from those of  $F$ . If  $e'_0 \leq e'_1$  are contained in the same cell the morphism is the identity. If they are not it is given by the morphism associated with the only pair of cells  $e_0 \leq e_1$  of  $K$  satisfying  $e'_0 \subset e_0$  and  $e'_1 \subset e_1$ .

**Definition 1.14** (Dihomologic Cellular Sheaves and Cosheaves). A *dihomologic cellular cosheaf* (resp. *sheaf*) on  $K$  is the data of a contravariant (resp. covariant) functor  $F$  from the category associated with the set of dihomologic pseudo-cells of  $K$  ordered by adjacency to the category of  $R$ -modules. As in the case of cellular sheaves and cosheaves a morphism of such objects is defined to be a natural transformation of functors. The notions of injectivity, surjectivity and invertibility are also defined “cell-wise” and so are the kernels, images and cokernels.

Formally, a dihomologic cellular cosheaf consists of an assignment of a module  $F(e^p; e^q)$  to every pair of adjacent cells  $e^p \leq e^q$  and morphisms connecting them. Because of the commutativity conditions on the

compositions of such morphisms it is only necessary to define them on elementary adjacency relations. By that we mean that if the dihomologic pseudo-cell of the pair  $e^p \leq e^q$  is a face of the pseudo-cell  $e^{p'} \leq e^{q'}$  then we have  $e^{p'} \leq e^p \leq e^q \leq e^{q'}$  and the commutative diagram of extension morphisms:

$$\begin{array}{ccccc}
 & & F(e^{p'}; e^{q'}) & & \\
 & \swarrow (1) & \downarrow (5) & \searrow (3) & \\
 F(e^p; e^{q'}) & & & & F(e^{p'}; e^q) \\
 & \searrow (2) & \downarrow & \swarrow (4) & \\
 & & F(e^p; e^q) & & 
 \end{array}$$

Knowing the morphism (5) only amounts to knowing the composition of (1) and (2) or (3) and (4). So to describe such  $F$  completely we can only provide the groups and the extension morphisms when we “increase the first coordinate” and “decrease the second one” and verify that these satisfy the commutative diagram:

$$\begin{array}{ccc}
 & F(e^{p'}; e^{q'}) & \\
 \swarrow & & \searrow \\
 F(e^p; e^{q'}) & & F(e^{p'}; e^q) \\
 \searrow & & \swarrow \\
 & F(e^p; e^q) & 
 \end{array}$$

**Definition 1.15** (Dihomologic Subdivision of Cellular Sheaves and Cosheaves). Let  $F$  be a cellular cosheaf (resp. sheaf) on  $K$  its *dihomologic subdivision*  $F'$  is the dihomologic cellular cosheaf (resp. sheaf) that associates to every pair of adjacent cells  $e^p \leq e^q$  the module:

$$F'(e^p; e^q) := F(e^q),$$

with extension (resp. restriction) morphisms coming from those of  $F$  and illustrated in the following commutative diagram (resp. with opposite arrows) for elementary adjacency relations  $e^{p'} \leq e^p \leq e^q \leq e^{q'}$ :

$$\begin{array}{ccccc}
 & & F(e^{q'}) & & \\
 & \swarrow \text{id} & \parallel & \searrow \begin{array}{c} | \\ e^q \\ | \\ e^{q'} \end{array} & \\
 & & F(e^{p'}; e^{q'}) & & \\
 \swarrow & & \swarrow & \searrow & \\
 F(e^{q'}) & \xlongequal{\quad} & F(e^p; e^{q'}) & & F(e^{p'}; e^q) \xlongequal{\quad} F(e^q) \\
 & \searrow & \swarrow & \searrow & \swarrow \\
 & & F(e^p; e^q) & & \\
 \swarrow & & \parallel & \searrow \text{id} & \\
 \begin{array}{c} | \\ e^q \\ | \\ e^{q'} \end{array} & & F(e^q) & & 
 \end{array}$$

Whenever the dihomologic pseudo-subdivision of  $K$  is a regular subdivision this construction corresponds to the usual subdivision of cosheaves (resp. sheaves) of Definition 1.13. The open cell  $e^q$  is covered by the open dihomologic (pseudo)-cells associated with the adjacent pairs of the form  $e^p \leq e^q$ .

**Definition 1.16** (Localisation by fixing the first coordinate). Let  $F$  be a dihomologic cosheaf on  $K$ . For  $e$  a cell of  $K$  we define the local cellular cosheaf  $F_e$  on  $K$  by the formula:

$$e' \in K \mapsto \begin{cases} F(e; e') & \text{if } e' \geq e \\ 0 & \text{otherwise} \end{cases},$$

with extension morphisms either 0 or given by  $F$ . We call it local as it is invariant by the operation of localisation at  $e$  :  $(F_e)_e = F_e$ . Moreover, if we apply this process to a dihomologic cosheaf  $F'$  obtained by subdividing a cellular cosheaf  $F$ , we recover the localisation operation previously defined. The situation is illustrated in the following commutative diagram:

$$\begin{array}{ccc}
\{\text{Cosheaves of } K\} & \xrightarrow{\text{subd.}} & \{\text{Dihomologic cosheaves of } K\} \\
\text{loc. at } e \downarrow & \swarrow \text{fix. loc. at } e & \downarrow \text{loc. at } (e \leq e) \\
\{\text{Cosheaves of } K\} & \xrightarrow{\text{subd.}} & \{\text{Dihomologic cosheaves of } K\}
\end{array} \tag{D1}$$

## Cellular Homology and Cohomology

In this paragraph,  $K$  denotes a locally finite regular CW-complex. When one computes the homology of the CW-complex  $K$  cellularily by filtering, the singular chain complex for instance, by its skeleta, one ends up on the  $E^1$ -page with the cellular chain complex of  $K$  : the  $k$ -th group in this complex is given by the direct sum of free Abelian groups of rank 1, one for each  $k$ -cell. These groups, that we redefine below, are a key ingredient in cellular homology. Its two generators correspond to the two orientations of the cell.

**Definition 1.17** (Oriented Cells, J. Munkers, *Elements of Algebraic Topology* [Mun84], §39. pp.222-231.). Let  $e$  be a  $k$ -cell of  $K$ . We call an *orientation* of  $e$  a generator of the group  $\mathbb{Z}(e) := H_k(|K|; |K| \setminus e; \mathbb{Z}) = H_k(\bar{e}; \bar{e} \setminus e; \mathbb{Z})$  computed with the singular homology. We will call the latter group the *group of oriented coefficients* of  $e$  and say that  $[e]$  is an *oriented  $k$ -cell* when  $[e]$  is an orientation of the  $k$ -cell  $e$ . Whenever  $e^{p-1}$  is a codimension 1 face of  $e^p$  we have a *boundary morphism*  $\mathbb{Z}(e^p) \rightarrow \mathbb{Z}(e^{p-1})$  defined by the composition :

$$H_k(\bar{e}^p; \bar{e}^p \setminus e^p) \xrightarrow{(1)} H_{k-1}(\bar{e}^p \setminus e^p) \xrightarrow{(2)} H_{k-1}(\bar{e}^p \setminus e^p; \bar{e}^p \setminus (e^p \cup e^{p-1})) \xrightarrow{(3)} H_{k-1}(\bar{e}^{p-1}; \bar{e}^{p-1} \setminus e^{p-1}),$$

with all four homology groups computed with integer coefficients. The morphism (1) is the connection morphism of the homological long exact sequence associated with the pair  $(\bar{e}^p \setminus e^p) \subset \bar{e}^p$ , (2) is the reduction modulo  $\bar{e}^p \setminus (e^p \cup e^{p-1})$ , and (3) is the inverse of the excision isomorphism. The image of an orientation  $[e^p]$  under this morphism is nothing but the  $\mathbb{Z}(e^{p-1})$ -component of its boundary when seen as a relative cellular chain. It is a generator of  $\mathbb{Z}(e^{p-1})$ . The first map, the connection morphism, comes from the boundary operator of the singular homology chain complex and therefore relies on the canonical orientation of  $\mathbb{R}^n$ . For a singular simplex  $\sigma$  on  $\text{Conv}(\{0, \dots, n\})$  we have the formula:

$$\partial\sigma = \sum_{i=0}^n (-1)^i \sigma_i,$$

with  $\sigma_i$  the restriction of  $\sigma$  to  $\text{Conv}(\{0, \dots, n\} \setminus \{i\})$ . The convention on the orientation of such restriction  $\sigma_i$  is then given by “outward pointing normal vector” as illustrated in Figure 9.

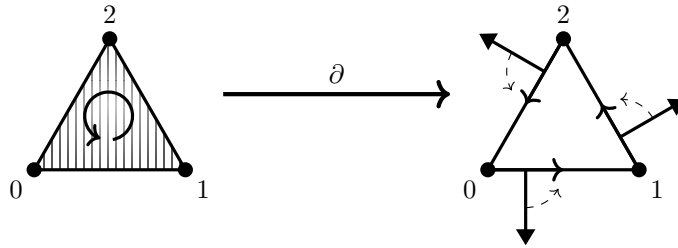


Figure 9: The orientation of the boundary.

These boundary morphisms satisfy the well known property that if  $e^p$  is a codimension 2 face of some cell  $e^{p+2}$  and  $e_1^{p+1}, e_2^{p+1}$  denote the two codimension 1 faces of  $e^{p+2}$  adjacent to  $e^p$  then the two compositions of boundary morphisms  $\mathbb{Z}(e^{p+2}) \rightarrow \mathbb{Z}(e_1^{p+1}) \rightarrow \mathbb{Z}(e^p)$  and  $\mathbb{Z}(e^{p+2}) \rightarrow \mathbb{Z}(e_2^{p+1}) \rightarrow \mathbb{Z}(e^p)$  are opposite of each other. In particular, composing boundary morphisms between oriented coefficients does not define a cellular cosheaf.

**Definition 1.18** (Dihomologic Orientations). Let  $\epsilon$  be the dihomologic pseudo-cell associated with an adjacent pair  $e^p \leq e^q$ , we define its *group of oriented coefficients* to be:

$$\mathbb{Z}(\epsilon) = \mathbb{Z}(e^p; e^q) := \text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}(e^q)).$$

We call a generator of such group an *orientation* of  $\epsilon$  or a *relative orientation* of the pair  $e^p \leq e^q$  and denote such element by the symbol  $[e^p; e^q]$ . If  $\epsilon'$  is a codimension 1 face of the pseudo-cell  $\epsilon$  we also have a *boundary morphism*  $\mathbb{Z}(\epsilon) \rightarrow \mathbb{Z}(\epsilon')$  defined by the boundary morphisms between the groups of oriented cells of  $K$  and, up to sign, by the functorial properties of the bifunctor  $\text{Hom}$ , as shown in the diagram below.

$$\begin{array}{ccccc} & \mathbb{Z}(e^p; e^q) & & \mathbb{Z}(e^{p+1}) & \mathbb{Z}(e^q) \\ & \swarrow^{(-1)^{p+1}\text{Hom}(\partial; \text{id})} & \searrow^{(-1)^p\text{Hom}(\text{id}; \partial')} & \downarrow \partial & \downarrow \partial' \\ \mathbb{Z}(e^{p+1}; e^q) & & \mathbb{Z}(e^p; e^{q-1}) & \mathbb{Z}(e^p) & \mathbb{Z}(e^{q-1}) \end{array} \quad (\text{D2})$$

Note that if  $\epsilon$  is a dihomologic pseudo-cell of dimension  $k$  its closure can be expressed as the cone over a space that has the homology of a  $(k-1)$ -sphere. In that description  $\epsilon$  would correspond to the open cone. Therefore, the singular homology of  $\bar{\epsilon}$  relatively to  $\bar{\epsilon} \setminus \epsilon$  is isomorphic to singular homology of a  $k$ -ball relatively to its boundary. Thus, we could equivalently define  $\mathbb{Z}(\epsilon)$  to be  $H_k(\bar{\epsilon}; \bar{\epsilon} \setminus \epsilon; \mathbb{Z})$  as in the case of a regular CW-complex. Indeed, for an adjacent pair  $e^p \leq e^q$  of  $K$  indexing the pseudo-cell  $\epsilon$ , there is a canonical isomorphism between the group  $H_{q-p}(\bar{\epsilon}; \bar{\epsilon} \setminus \epsilon; \mathbb{Z})$  and  $\text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}(e^q))$ . If we represent a cell  $e^p$  by a  $p$ -dimensional real vector space inside a  $q$ -dimensional vector space (representing  $e^q$ ) then the dihomologic pseudo-cell  $\epsilon$  indexed by  $e^p \leq e^q$  represents a supplementary sub-space of the former in the latter. An orientation of such supplementary space allows by wedge product to orient the  $q$ -dimensional vector space from an orientation of the  $p$ -dimensional one. The situation is explained by the cohomological bilinear cup product:

$$\cup : H^p(\bar{e}^p; \bar{e}^p \setminus e^p) \otimes H^{q-p}(\bar{e}; \bar{e} \setminus \epsilon) \rightarrow H^q(\bar{e}^q; \bar{e}^q \setminus e^q).$$

It is non-degenerate. To see it one can notice that all the spaces considered are supports of simplicial complexes so each of these groups can be computed simplicially. A generator of  $H^p(\bar{e}^p; \bar{e}^p \setminus e^p)$  is represented by the simplicial cocycle whose value is 1 on the barycentric simplex indexed by a complete flag  $e^0 < \dots < e^p$  and 0 elsewhere. Likewise, a generator of  $H^{q-p}(\bar{e}; \bar{e} \setminus \epsilon)$  is represented by the simplicial cocycle whose value is 1 on the barycentric simplex indexed by a complete flag  $e^p < \dots < e^q$  and 0 elsewhere. Their cup product is the simplicial cocycle whose value is 1 on the barycentric simplex indexed by the complete flag  $e^0 < \dots < e^p < \dots < e^q$  and 0 elsewhere. It represents a generator of  $H^q(\bar{e}^q; \bar{e}^q \setminus e^q)$ . It gives rise to an isomorphism:

$$H^{q-p}(\bar{e}; \bar{e} \setminus \epsilon) \cong \text{Hom}\left(H^p(\bar{e}^p; \bar{e}^p \setminus e^p); H^q(\bar{e}^q; \bar{e}^q \setminus e^q)\right).$$

Using the universal coefficients theorem [CE56]<sup>19</sup> three times this isomorphism becomes:

$$\mathbb{Z}(\epsilon) \cong \text{Hom}\left(\text{Hom}\left(\text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}); \text{Hom}(\mathbb{Z}(e^q); \mathbb{Z})\right); \mathbb{Z}\right).$$

Since all the groups involved are free we can compose it with the musical isomorphism of the trace scalar product:

$$\text{Hom}\left(\text{Hom}\left(\text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}); \text{Hom}(\mathbb{Z}(e^q); \mathbb{Z})\right); \mathbb{Z}\right) \cong \text{Hom}\left(\text{Hom}(\mathbb{Z}(e^q); \mathbb{Z}); \text{Hom}(\mathbb{Z}(e^p); \mathbb{Z})\right),$$

and then with the transposition:

$$\text{Hom}\left(\text{Hom}(\mathbb{Z}(e^q); \mathbb{Z}); \text{Hom}(\mathbb{Z}(e^p); \mathbb{Z})\right) \cong \text{Hom}\left(\mathbb{Z}(e^p); \mathbb{Z}(e^q)\right),$$

to finally find our desired isomorphism  $\mathbb{Z}(\epsilon) \cong \text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}(e^q))$ . Note that the signs in the Definition 1.18, diagram D2, of the boundary morphisms between the groups of oriented coefficients of dihomologic pseudo-cells come from this description and the relations:

$$d_1\alpha \cup \beta + (-1)^p\alpha \cup d_2\beta = 0 \text{ and } d_3(\alpha \cup \gamma) = 0 + (-1)^p\alpha \cup d_4\gamma,$$

emanating from the graded Leibniz rule and satisfied by all  $\alpha \in H^p(\bar{e}^p; \bar{e}^p \setminus e^p)$ ,  $\beta \in H^{q-p-1}(\bar{e}_1; \bar{e}_1 \setminus \epsilon_1)$  and  $\gamma \in H^{q-p-1}(\bar{e}_2; \bar{e}_2 \setminus \epsilon_2)$ . Where:

<sup>19</sup>H. Cartan and S. Eilenberg. *Homological Algebra*, Theorem 3.3 p.113.



1.  $\epsilon_1$  and  $\epsilon_2$  are adjacent dihomologic pseudo-cells respectively indexed by the pairs  $e^{p+1} \leq e^q$  and  $e^p \leq e^{q-1}$ ;
2.  $d_1$  and  $d_3$  are respectively transpose of the orientation boundary morphisms  $\partial_1 : \mathbb{Z}(e^{p+1}) \rightarrow \mathbb{Z}(e^p)$  and  $\partial_3 : \mathbb{Z}(e^q) \rightarrow \mathbb{Z}(e^{q-1})$ ;
3.  $d_2 : H^{q-p-1}(\bar{\epsilon}_1; \bar{\epsilon}_1 \setminus \epsilon_1) \rightarrow H^{q-p}(\bar{\epsilon}_3; \bar{\epsilon}_3 \setminus \epsilon_3)$  and  $d_4 : H^{q-p-1}(\bar{\epsilon}_2; \bar{\epsilon}_2 \setminus \epsilon_2) \rightarrow H^{q-p}(\bar{\epsilon}_3; \bar{\epsilon}_3 \setminus \epsilon_3)$  with  $\epsilon_3$  indexed by the pair  $e^p \leq e^q$ .

If  $[e^p] \in \mathbb{Z}(e^p)$  is an orientation of  $e^p$  and  $[e^p; e^q] \in \mathbb{Z}(e^p; e^q)$  is a relative orientation, we denote  $[e^p][e^p; e^q]$  the associated orientation of  $e^q$ . For any pair  $e^p \leq e^{p+1}$  of relative codimension 1 the inverse of the boundary morphism  $\partial : \mathbb{Z}(e^{p+1}) \rightarrow \mathbb{Z}(e^p)$  defines a canonical relative orientation. We will always denote it by the symbol  $[e^p; e^{p+1}]$  but we should emphasise that for any other positive dimensional dihomologic pseudo-cell the similar notation denotes an arbitrary orientation, possibly subject to conditions, as there are no canonical orientation for them. We can compose relative orientations in their morphism representations, we adopt the convention  $[e^p; e^q][e^q; e^r]$  to denote  $[e^q; e^r] \circ [e^p; e^q]$ . In these notations, one can rewrite the anti-commutativity of the boundary morphisms as follows: for all adjacent pair  $e^p \leq e^{p+2}$  of relative codimension 2 we have:

$$\sum_{e^p \leq e^{p+1} \leq e^{p+2}} [e^p; e^{p+1}][e^{p+1}; e^{p+2}] = 0.$$

**Definition 1.19** (Cellular Chain and Cochain Complexes). Let  $F$  be a cellular cosheaf on  $K$  and  $k \in \mathbb{N}$ . We define the group of *cellular  $k$ -chains* of  $K$  with coefficients in  $F$  to be :

$$C_k(K; F) := \bigoplus_{\dim e=k} F(e) \otimes_{\mathbb{Z}} \mathbb{Z}(e).$$

Moreover, if  $[e]$  is an oriented  $k$ -cell of  $K$  and  $c$  a  $k$ -chain with coefficients in  $F$  we denote  $\langle c, [e] \rangle$  the unique element of  $F(e)$  satisfying  $c_e = \langle c, [e] \rangle \otimes [e]$ . We also define the *boundary operator*  $\partial : C_k(K; F) \rightarrow C_{k-1}(K; F)$  by the formula :

$$\partial f := \sum_{e^{k-1} < e^k} \langle f, [e^k] \rangle_{e^{k-1}}^{e^k} \otimes [e^{k-1}],$$

for all  $f \in F(e^k) \otimes_{\mathbb{Z}} \mathbb{Z}(e^k)$  with  $[e^{k-1}]$  the image of  $[e^k]$  by the boundary morphism, which can be written  $[e^{k-1}][e^{k-1}; e^k] = [e^k]$  with  $[e^{k-1}; e^k]$  the canonical relative orientation. The property of the compositions of the boundary morphisms between oriented coefficients implies directly that  $\partial^2 = 0$  and that  $(C_k(K; F); \partial)$  is a chain complex.

Dually, when  $F$  is a cellular sheaf, the *cochain complex* with coefficients in  $F$ , is defined, for  $k \in \mathbb{N}$ , by the groups:

$$C^k(K; F) := \prod_{\dim e=k} \text{Hom}(\mathbb{Z}(e); F(e)).$$

We write for  $\alpha$  a  $k$ -cochain with coefficients in  $F$  and  $[e]$  an oriented  $k$ -cell,  $\alpha([e]) \in F(e)$  for the value of the map  $\alpha(e)$  at  $[e]$ . The *coboundary operator*  $d : C^k(K; F) \rightarrow C^{k+1}(K; F)$  is given for all  $k$ -cochains  $\alpha$  and all oriented  $(k+1)$ -cells  $[e^{k+1}]$  by:

$$d\alpha([e^{k+1}]) = \sum_{e^k < e^{k+1}} \alpha([e^k])_{e^k}^{e^{k+1}},$$

for  $[e^k]$  the unique orientation of  $e^k$  whose image by the boundary morphism is  $[e^{k+1}]$ , i.e.  $[e^k][e^k; e^{k+1}] = [e^{k+1}]$ .

The last cochain complex we define here is the complex of *cellular cochains with compact support*. It is a sub-complex of  $(C^k(K; F))_{k \geq 0}$  whose groups are given for all  $k \in \mathbb{N}$ , by:

$$C_c^k(K; F) := \bigoplus_{\dim e=k} \text{Hom}(\mathbb{Z}(e); F(e)).$$

For  $k \in \mathbb{N}$ , the image of  $C_c^k(K; F)$  under  $d$  is contained in  $C_c^{k+1}(K; F)$  only because we assumed  $K$  to be locally finite. A morphism of cosheaves or sheaves gives rise to a morphism of chain or cochain complexes, respectively. In a more categorical language, the association  $F \mapsto (C_k(K; F))$  for  $F$  a cosheaf and  $G \mapsto (C_c^k(K; G))$  for  $G$  a sheaf are all covariant functors.

**Definition 1.20** (Dihomologic chain complex). Let  $F$  be a dihomologic cellular cosheaf on  $K$ . We define the group of *dihomologic cellular  $k$ -chains* of  $K$  with coefficients in  $F$  to be :

$$\Omega_k(K; F) := \bigoplus_{\dim \epsilon = k} F(\epsilon) \otimes_{\mathbb{Z}} \mathbb{Z}(\epsilon),$$

where  $\epsilon$  runs through the set of  $k$ -dimensional dihomologic pseudo-cells of  $K$ . If  $[\epsilon]$  is an oriented  $k$ -pseudo-cell of  $K$  and  $c$  a  $k$ -chain with coefficients in  $F$  we denote  $\langle c, [\epsilon] \rangle$  the unique element of  $F(\epsilon)$  satisfying  $c_\epsilon = \langle c, [\epsilon] \rangle \otimes [\epsilon]$ . We also define the boundary operator  $\partial : \Omega_k(K; F) \rightarrow \Omega_{k-1}(K; F)$  by the formula :

$$\partial f := \sum_{\epsilon^{k-1} < \epsilon^k} \langle f, [\epsilon^k] \rangle_{\epsilon^{k-1}}^{\epsilon^k} \otimes [\epsilon^{k-1}],$$

for all  $f \in F(\epsilon^k) \otimes_{\mathbb{Z}} \mathbb{Z}(\epsilon^k)$  with  $[\epsilon^{k-1}]$  the image of  $[\epsilon^k]$  by the boundary morphism.

**Remark 1.3.** If the dihomologic pseudo-subdivision of  $K$  is a regular subdivision of  $K$  we find that the last definition is identical to the definition of the cellular chain complex associated with a cellular cosheaf on a CW-complex.

**Definition 1.21.** For  $F$  a cellular cosheaf (resp. a cellular sheaf) we define its *homology* (resp. *cohomology*, resp. *cohomology with compact support*) to be the homology of its cellular chain complex (resp. cohomology of its cellular cochain complex, resp. cohomology of cellular its cochain complex with compact support). We also define the homology of a dihomologic cosheaf to be the homology of the associated chain complex.

**Example.** Given two sub-complexes  $K_1 \leq K_2$  of  $K$  and an Abelian group  $G$  the homology of  $[K_2; K_1; G]$  is exactly the same as the cellular homology of  $K_2$  relatively to  $K_1$ .

The following propositions illustrate the usefulness of these cellular constructions:

**Proposition 1.15.** *If  $F$  is a sheaf of Abelian groups on  $|K|$ , constructible with respect to the skeletal filtration, then the sheaf cohomology of  $F$  is isomorphic to the cellular cohomology of the cellular sheaf associated with  $F$ .*

The proof of this fact is easily derived from the fact that the open cover of  $|K|$  by the open stars of its vertices is a Leray cover for  $F$  and that the cellular cochain complex of its associated cellular sheaf is the complex of its Čech cochains, see [God58]<sup>20</sup> for instance.

**Proposition 1.16.** *Whenever  $F$  is the cellular cosheaf arising from a local system of coefficients  $L$  on  $|K|$  there is a canonical isomorphism from the cellular homology of  $F$  to the singular homology of  $|K|$  with coefficients in  $L$ .*

The result follows directly from the description in [Hat00]<sup>21</sup> of the complex of singular chains with coefficients in  $L$  and an adaptation of the classical proof relating singular homology and cellular homology.

**Proposition 1.17.** *Let  $K'$  be a subdivision of  $K$  and  $F$  a cellular cosheaf on  $K$ . If  $F'$  denotes the subdivision of  $F$  we have the following injective morphism of chain complexes:*

$$\begin{aligned} C_k(K; F) &\longrightarrow C_k(K'; F') \\ c &\longmapsto \sum_{e^k \in K} \left( \sum_{\substack{e'^k \subset e^k \\ e'^k \in K'}} \langle c, [e^k] \rangle \otimes [e'^k] \right) \end{aligned}$$

with the orientations defined as follows : if  $[e^k]$  is an orientation of  $e^k$  then  $[e'^k]$  is the image of such orientation under the isomorphism  $H_k(\bar{e}^k; \bar{e}^k \setminus e^k) \rightarrow H_k(\bar{e}'^k; \bar{e}'^k \setminus e'^k)$  inverse of the map induced by the inclusion  $(\bar{e}'^k \setminus e^k; \bar{e}'^k) \subset (\bar{e}^k \setminus e'^k; \bar{e}^k)$ . This is a quasi-isomorphism of chain complexes.

A proof of this can be found in [She85]<sup>22</sup> in the dual context of cellular sheaves. The same proposition holds for dihomologic subdivisions of cellular cosheaves on  $K$ . A proof of the specific case of the constant sheaf on the dihomologic subdivision is given in [For02]<sup>23</sup>.

<sup>20</sup>R. Godement. *Topologie Algébrique et Théorie des Faisceaux*, Théorème 5.10.1 p.228.

<sup>21</sup>A. Hatcher. *Algebraic Topology*, Section 3.H. Local Coefficients pp.327-337.

<sup>22</sup>A. Shepard. *A Cellular Description of the Derived Category of a Stratified Space*, Theorem 1.5.2 p.31.

<sup>23</sup>R. Forman, *Combinatorial Novikov–Morse Theory*, Theorem 1.2 p.12.

**Proposition 1.18.** *Let  $F$  be a cellular cosheaf on  $K$ . If  $F'$  denotes the dihomologic subdivision of  $F$  we have the following injective morphism of chain complexes:*

$$\begin{aligned} C_k(K; F) &\longrightarrow \Omega_k(K; F') \\ c &\longmapsto \sum_{e^0 \leq e^k} \langle c; [e^k] \rangle \otimes [e^0; e^k] \end{aligned}$$

with the orientations defined as follows : if  $[e^k]$  is an orientation of  $e^k$  then  $[e^0; e^k]$  is the relative orientation defined by the relation  $[e^0][e^0; e^k] = [e^k]$  with  $[e^0]$  the canonical orientation of the vertex  $e^0$  of  $K$ . This is a quasi-isomorphism of chain complexes.

This is an adaptation of the proof given by R. Forman.

*Proof.* Denote by  $f$  the chain complexes morphism given in the statement of the proposition. We have natural inclusions of the dihomologic complexes associated with the restrictions of  $F'$  to the skeleta of  $K$ :

$$\Omega_*(K^{(q)}; F') \subset \Omega_*(K^{(q+1)}; F'),$$

for all  $q \in \mathbb{N}$ . Note that for every  $k \in \mathbb{N}$ ,  $f$  maps  $C_k(K; F)$  to  $\Omega_k(K^{(k)}; F')$ . Now consider the spectral sequence associated with this filtration:

$$E_{p,q}^r := \frac{Z_{p,q}^r + \Omega_{q-p}(K^{(q-1)}; F')}{\partial Z_{p+r-2, q+r-1}^{r-1} + \Omega_{q-p}(K^{(q-1)}; F')},$$

with  $Z_{p,q}^r = \{c \in \Omega_{q-p}(K^{(q)}; F') \mid \partial c \in \Omega_{q-p-1}(K^{(q-r)}; F')\}$ . The boundary operator  $\partial_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r+1, q-r}^r$  applied to an element  $c \in E_{p,q}^r$  is given by computing  $\partial c' \in \Omega_{q-p-1}(K; F')$  for any lift  $c' \in Z_{p,q}^r$  and then projecting to  $E_{p-r+1, q-r}^r$ . The term  $E_{p,q}^0$  is written:

$$E_{p,q}^0 = \bigoplus_{e^p \leq e^q} F'(e^p; e^q) \otimes \mathbb{Z}(e^p; e^q),$$

with  $F'(e^p; e^q) = F(e^q)$  by assumption. The boundary  $\partial_{p,q}^0$  acts on the element  $f \otimes [e^p; e^q] \in F'(e^p; e^q) \otimes \mathbb{Z}(e^p; e^q) = F(e^q) \otimes \mathbb{Z}(e^p; e^q)$  as follows:

$$\partial_{p,q}^0 f \otimes [e^p; e^q] = \sum_{e^p \leq e^{p+1} \leq e^q} f \otimes [e^{p+1}; e^q] \in \bigoplus_{e^p \leq e^{p+1} \leq e^q} F'(e^{p+1}; e^q) \otimes \mathbb{Z}(e^{p+1}; e^q),$$

for  $[e^{p+1}; e^q]$  the image of  $[e^p; e^q]$  by the boundary morphism  $\partial : \mathbb{Z}(e^p; e^q) \rightarrow \mathbb{Z}(e^{p+1}; e^q)$ . Remember that this morphism is given by  $(-1)^{p+1} \text{Hom}(\partial'; \text{id})$  with  $\partial' : \mathbb{Z}(e^{p+1}) \rightarrow \mathbb{Z}(e^q)$ . Therefore, if we decide to write  $[e^p; e^q]$  in the form  $\alpha \otimes [e^q] \in \text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}) \otimes \mathbb{Z}(e^q)$  we have  $[e^{p+1}; e^q] = (-1)^{p+1}(\alpha \circ \partial') \otimes [e^q]$ . As a consequence we see that the line of index  $q$  of the  $E^0$ -page of the spectral sequence splits into the direct sum of the cellular cochain complexes of the  $K(e^q)$  with coefficients in the constant cellular sheaf  $F(e^q) \otimes \mathbb{Z}(e^q)$  with coboundary operator  $d$  twisted by  $(-1)^{p+1}$ :

$$\cdots \longrightarrow E_{p,q}^0 \xrightarrow{\partial^0} E_{p+1,q}^0 \longrightarrow \cdots$$

$$\bigoplus_{e^q \in K} C^p(K(e^q); F(e^q) \otimes \mathbb{Z}(e^q)) \xrightarrow{\oplus (-1)^{p+1} d} \bigoplus_{e^q \in K} C^{p+1}(K(e^q); F(e^q) \otimes \mathbb{Z}(e^q))$$

Since the support of  $K(e^q)$  is the closure of  $e^q$ , the  $E^1$ -page of the spectral sequence is concentrated on the 0-th column and satisfies:

$$E_{0,q}^1 = \bigoplus_{e^q \in K} H^0(K(e^q); F(e^q) \otimes \mathbb{Z}(e^q)).$$

Note that  $E_{0,q}^1 \subset E_{0,q}^0$  and is precisely the image of  $f$ . It defines an isomorphism between this 0-th column and  $C_*(K; F)$  and therefore between the homology of  $F$  and the homology of its subdivision  $F'$ . Moreover, we deduce that our filtration is adapted to the cokernel of  $f$  and that computing the first page of the induced spectral sequence amounts to replace in our computations the cohomology groups of the  $K(e^q)$ 's with coefficients in the group  $F(e^q) \otimes \mathbb{Z}(e^q)$  with the reduced cohomology. Since  $|K(e^q)|$  is contractible they all vanish. Thus, the cokernel of  $f$  has trivial homology and  $f$  is a quasi-isomorphism.  $\square$

## 2 A Poincaré-Lefschetz Theorem for Dihomologic Cellular Cosheaves

Let  $K$  be a locally finite regular CW-complex and  $F$  a dihomologic cosheaf on  $K$ . The chain complex  $(\Omega_k(K; F))_{k \geq 0}$  is actually the total complex of a bicomplex. The  $k$ -dimensional dihomologic pseudo-cells being represented by adjacent pairs of cells  $e^p \leq e^q$  with  $q - p = k$  their set is partitioned into sets of cells of different types. If we say that a dihomologic pseudo-cell indexed by  $e^p \leq e^q$  has type  $(p, q)$ , the set of  $k$ -dimensional dihomologic pseudo-cells is the disjoint union of the set of pseudo-cells of type  $(p, q)$  for  $q - p = k$ . Similarly the group  $\Omega_k(K; F)$  splits into the direct sum:

$$\Omega_k(K; F) = \bigoplus_{q-p=k} \Omega_{p,q}(K; F),$$

with:

$$\Omega_{p,q}(K; F) = \bigoplus_{e^p \leq e^q} F(e^p; e^q) \otimes \mathbb{Z}(e^p; e^q),$$

for all  $p, q \in \mathbb{N}$ . A codimension 1 face of  $e^p \leq e^q$  either starts with  $e^p$  or ends with  $e^q$  and therefore the restriction of the boundary operator to  $\Omega_{p,q}(K; F)$  has values in the sum  $\Omega_{p+1,q}(K; F) \oplus \Omega_{p,q-1}(K; F)$ . It splits into the sum of an operator  $\partial_1$  of bidegree  $(+1; 0)$  and an operator  $\partial_2$  of bidegree  $(0; -1)$ . With the additional structure we can consider the two canonical filtrations and associated spectral sequences. We will only look at the (decreasing) horizontal filtration, that is to say the one filtered by the index  $p$ . The filtering pieces are, for  $l, k \in \mathbb{N}$ :

$$\Omega_k^{(l)}(K; F) := \bigoplus_{\substack{q-p=k \\ p \geq l}} \Omega_{p,q}(K; F).$$

The associated spectral sequence is given for all  $p, q \in \mathbb{Z}$  by :

$$E_{p,q}^r := \frac{Z_{p,q}^r + \Omega_{q-p}^{(p+1)}(K; F)}{\partial Z_{p-r+1, q-r+2}^{r-1} + \Omega_{q-p}^{(p+1)}(K; F)},$$

with  $Z_{p,q}^r = \{c \in \Omega_{q-p}^{(p)}(K; F) \mid \partial c \in \Omega_{q-p-1}^{(p+r)}(K; F)\}$ . The boundary operators  $\partial_{p,q}^r : E_{p,q}^r \rightarrow E_{p+r, q+r-1}^r$  applied to an element  $c \in E_{l,k}^r$  is given by computing  $\partial c' \in F^{p+r} C_{q-p-1}(K; F)$  for any lift  $c' \in Z_{p,q}^r$  and then projecting to  $E_{p+r, q+r-1}^r$ .

**Proposition 2.1.** *If there is an  $n \in \mathbb{N}$  such that for every cell  $e$  of  $K$  the local cosheaf  $F_e$  has its homology concentrated in dimension  $n$  then the horizontal spectral sequence of  $\Omega(K; F)$  degenerates at second page.*

*Proof.* We have :

$$E_{p,q}^0 \cong \Omega_{p,q}(K; F),$$

with  $\partial_{p,q}^0$  corresponding to the vertical component,  $\partial_2$ , of the total boundary operator,  $\partial$ , of  $(\Omega_k(K; F))_{k \in \mathbb{N}}$ . The following page  $(E_{p,q}^1)_{p,q \in \mathbb{N}}$  is given by the homology groups of the column complexes of the bicomplex  $(\Omega_{p,q}(K; F))_{p,q \in \mathbb{N}}$ . For  $p, q \in \mathbb{N}$ , let's consider the morphism:

$$\Phi_{p,q} : \Omega_{p,q}(K; F) \rightarrow \bigoplus_{e^p \in K} \text{Hom}(\mathbb{Z}(e^p); C_q(K; F_{e^p})),$$

for which the  $(e^p)$ -component of the image of an element  $c$  is given by the linear map:

$$[e^p] \mapsto (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^q > e^p} \langle c, [e^p; e^q] \rangle \otimes [e^q],$$

with  $[e^p; e^q], [e^q]$  some choices of orientations satisfying  $[e^p][e^p; e^q] = [e^q]$  (the map does not depend on such choices). For a fixed  $p \in \mathbb{N}$ , the collection  $(\Phi_{p,q})_{q \in \mathbb{N}}$  is almost a chain complex isomorphism between the  $p$ -th column of the bicomplex of  $F$ -valued dihomologic chains  $\Omega_{p,*}(K; F)$  and the direct sum over the  $p$ -cells  $e^p$  of the complexes  $\text{Hom}(\mathbb{Z}(e^p); C_*(K; F_{e^p}))$ . We have  $\partial \Phi_{p,q} = (-1)^{q-p} \Phi_{p,q-1} \partial_2$  so  $(\Phi_{p,q})_{q \in \mathbb{N}}$  still defines an isomorphism in homology. Each of the  $\Phi_{p,q}$  being individually an isomorphism is a matter of bookkeeping. On the left hand side we sum the groups  $F(e^p, e^q) \otimes \mathbb{Z}(e^p, e^q)$  over all ordered pairs of cells  $e^p < e^q$  and on the right hand side we

sum the groups  $\text{Hom}(\mathbb{Z}(e^p); F(e^p, e^q) \otimes \mathbb{Z}(e^q))$  over the same index set but in a different order,  $\Phi_{p,q}$  then sends bijectively each one of the former summands to one of the latter by means of the composition:

$$F(e^p, e^q) \otimes \mathbb{Z}(e^p, e^q) \longleftarrow F(e^p, e^q) \otimes \text{Hom}(\mathbb{Z}(e^p); \mathbb{Z}(e^q)) \xrightarrow{\pm 1} \text{Hom}(\mathbb{Z}(e^p); F(e^p, e^q) \otimes \mathbb{Z}(e^q)).$$

For the ‘‘commutativity’’ relation with the boundary operators, let  $c$  be a  $(p, q)$ -chain. On the one hand, for  $e^p$  a  $p$ -cell of  $K$ , the value of the  $(e^p)$ -component of  $\Phi_{p,q-1}(\partial_2 c)$  on an oriented cell  $[e^p]$  is given by the formula:

$$\Phi_{p,q-1}(\partial_2 c)[e^p] = (-1)^{\frac{p(p+1)}{2} + \frac{q(q-1)}{2}} \sum_{e^{q-1} > e^p} \left( \sum_{e^q > e^{q-1}} \langle c, [e^p; e^q] \rangle \Big|_{e^p, e^{q-1}}^{e^p, e^q} \right) \otimes [e^{q-1}],$$

for  $[e^p][e^p, e^{q-1}] = [e^{q-1}]$  and  $[e^p, e^q] = (-1)^p [e^p, e^{q-1}][e^{q-1}, e^q]$ . That is to say:

$$\Phi_{p,q-1}(\partial_2 c)[e^p] = (-1)^{\frac{p(p-1)}{2} + \frac{q(q-1)}{2}} \sum_{e^{q-1} > e^p} \left( \sum_{e^q > e^{q-1}} \langle c, [e^p; e^q] \rangle \Big|_{e^p, e^{q-1}}^{e^p, e^q} \right) \otimes [e^{q-1}],$$

for  $[e^p][e^p, e^q] = [e^{q-1}][e^{q-1}, e^q]$ . Note that we changed the equation defining the orientation which resulted in the multiplication by a factor  $(-1)^p$  in the computation of  $\Phi_{p,q-1}(\partial_2 c)[e^p]$  and hence turned  $(-1)^{\frac{p(p+1)}{2}}$  into  $(-1)^{\frac{p(p-1)}{2}}$ . On the other hand, the value of the  $(e^p)$ -component of  $\partial \Phi_{p,q}(c)$  on  $[e^p]$  is the same as the boundary of  $\Phi_{p,q}(c)[e^p]$ :

$$\partial \Phi_{p,q}(c)[e^p] = (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^q > e^p} \left( \sum_{e^{q-1} < e^q} \langle c, [e^p, e^q] \rangle \Big|_{e^{q-1}}^{e^q} \otimes [e^{q-1}] \right),$$

for  $[e^p][e^p, e^q] = [e^q]$  and  $[e^{q-1}][e^{q-1}, e^q] = [e^q]$ , i.e.  $[e^p][e^p, e^q] = [e^{q-1}][e^{q-1}, e^q]$ . Note that the element  $\langle c, [e^p, e^q] \rangle$  of  $F(e^p, e^q)$  in the last formula is understood as an element of  $F_{e^p}(e^q)$  hence the extension morphisms of  $F_{e^p}$  apply to it and this is why we wrote  $\langle c, [e^p, e^q] \rangle \Big|_{e^{q-1}}^{e^q}$ . However, these extension morphisms are zero whenever  $e^{q-1}$  doesn't contain  $e^p$  and identical to those of  $F$  in the opposite case. Therefore we can write :

$$\begin{aligned} \partial \Phi_{p,q}(c)[e^p] &= (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^q > e^p} \left( \sum_{e^p < e^{q-1} < e^q} \langle c, [e^p, e^q] \rangle \Big|_{e^p, e^{q-1}}^{e^p, e^q} \otimes [e^{q-1}] \right) \\ &= (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^{q-1} > e^p} \left( \sum_{e^q > e^{q-1}} \langle c, [e^p; e^q] \rangle \Big|_{e^p, e^{q-1}}^{e^p, e^q} \right) \otimes [e^{q-1}] \\ &= (-1)^{q-p} \Phi_{p,q-1}(\partial_2 c)[e^p]. \end{aligned}$$

For each  $p$ -cell, the group  $\mathbb{Z}(e^p)$  is free so the universal coefficient theorem ensures that  $E_{p,q}^1$  is isomorphic to the direct sum of the  $\text{Hom}(\mathbb{Z}(e^p); H_q(K; F_{e^p}))$ 's. By assumptions  $H_q(K; F_{e^p})$  is trivial as soon as  $q$  is not  $n$ . As a consequence all pages following  $E^1$  are concentrated on the horizontal line  $\{q = n\}$  and since  $\partial_{p,q}^r$  has bidegree  $(r, r-1)$  the spectral sequence degenerates at second page.  $\square$

**Theorem 1** (Cellular Poincaré-Lefschetz Theorem). *Let  $K$  be a finite dimensional, locally finite and regular CW-complex and  $F$  satisfy the hypotheses of Proposition 2.1. Then for every  $0 \leq k \leq n$ ,  $H_k(K; F)$  and  $H_c^{n-k}(K; H_n(F_*))$  are canonically isomorphic. In particular  $H_k(K; F)$  vanishes for  $k > n$ . If in addition  $K$  has dimension  $n$ , then this isomorphism comes from an injective quasi-isomorphism:*

$$C_c^{n-*}(K; H_n(F_*)) \rightarrow \Omega_*(K; F).$$

*Proof.* Let  $d = \dim K$  and consider the horizontal filtration of the homology of  $F$  :

$$0 \subset H_k(K; F)^{(d)} \subset \cdots \subset H_k(K; F)^{(n-k)} \subset \cdots \subset H_k(K; F)^{(0)},$$

whose graded pieces:

$$E_{d,k+d}^\infty \quad \cdots \quad E_{n-k,n}^\infty \quad \cdots \quad E_{0,k}^\infty,$$

satisfy, in light of the last proposition,  $E_{p,q}^\infty = E_{p,q}^2 = 0$  as soon as  $q$  is not  $n$ . Therefore,  $H_k(K; F) = H_k(K; F)^{(n-k)} = E_{n-k,n}^\infty = E_{n-k,n}^2$ . Now because of the isomorphisms  $(\Phi_{p,q})_{p,q \in \mathbb{N}}$  given in the last proof, we

recognise that  $E_{p,q}^1 \cong C_c^p(K; H_q(F_*))$ . Then it remains only to show that the boundary operator  $\partial_{p,q}^1$  of the spectral sequence is mapped to the coboundary operator  $d$ . If  $c' \in \Omega_{p,q}(K; F)$  is a  $\partial_2$ -cycle representing an element  $c \in E_{p,q}^1$  then  $\partial_1 c'$  is a  $\partial_2$ -cycle representing  $\partial_{p,q}^1 c$ . We have:

$$\langle \partial_1 c', [e^{p+1}; e^q] \rangle = \sum_{e^p < e^{p+1}} \langle c', [e^p; e^q] \rangle \Big|_{e^{p+1}; e^q}^{e^p, e^q},$$

for  $[e^p; e^q] = (-1)^{p+1} [e^p; e^{p+1}] [e^{p+1}; e^q]$ . So the image of  $\partial_1 c'$  under  $\Phi$  satisfies:

$$\Phi_{p+1,q}(\partial_1 c') [e^{p+1}] = (-1)^{\frac{(p+2)(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^q > e^{p+1}} \left( \sum_{e^p < e^{p+1}} \langle c', [e^p; e^q] \rangle \Big|_{e^{p+1}; e^q}^{e^p, e^q} \right) \otimes [e^q],$$

for  $[e^{p+1}] [e^{p+1}; e^q] = [e^q]$  and  $[e^p; e^q] = (-1)^{p+1} [e^p; e^{p+1}] [e^{p+1}; e^q]$ . Which can be written as:

$$\begin{aligned} \Phi_{p+1,q}(\partial_1 c') [e^{p+1}] &= (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^p < e^{p+1}} \left( \sum_{e^q > e^{p+1}} \langle c', [e^p; e^q] \rangle \Big|_{e^{p+1}; e^q}^{e^p, e^q} \otimes [e^q] \right) \\ &= (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^p < e^{p+1}} \Psi_{e^p}^{e^{p+1}} \left( \sum_{e^q > e^p} \langle c', [e^p; e^q] \rangle \otimes [e^q] \right), \end{aligned}$$

for  $[e^{p+1}] [e^{p+1}; e^q] = [e^q]$ ,  $[e^p; e^q] = [e^p; e^{p+1}] [e^{p+1}; e^q]$  and  $\Psi_{e^p}^{e^{p+1}} : C_q(K; F_{e^p}) \rightarrow C_q(K; F_{e^{p+1}})$  associated with the cosheaf morphism  $F_{e^p} \rightarrow F_{e^{p+1}}$ . Therefore,  $\Phi_{p+1,q}(\partial_{p,q}^1 c)$  is the  $(p+1)$ -cochain with compact support that associates to an oriented cell  $[e^{p+1}]$  the sum over its codimension 1 faces of the images in  $H_q(K; F_{e^{p+1}})$  of the homology classes  $\left[ (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^q > e^p} \langle c', [e^p; e^q] \rangle \otimes [e^q] \right] \in H_q(K; F_{e^p})$  for  $[e^p; e^q] = [e^p; e^{p+1}] [e^{p+1}; e^q]$  and  $[e^p; e^q] = [e^p; e^{p+1}] [e^{p+1}; e^q]$ . Finally we check that this is precisely  $d\Phi_{p,q}(c)$  for:

$$d\Phi_{p,q}(c) [e^{p+1}] = \sum_{e^p < e^{p+1}} \Phi_{p,q}(c) [e^p],$$

with  $[e^p] [e^p; e^{p+1}] = [e^{p+1}]$  and  $\Phi_{p,q}(c) [e^p] = \left[ (-1)^{\frac{p(p+1)}{2} + \frac{q(q+1)}{2}} \sum_{e^q > e^p} \langle c', [e^p; e^q] \rangle \otimes [e^q] \right]$ .

For the second part of the statement, when  $\dim K = n$ , we need to remember that  $\Phi$  provided us with an isomorphism between the chain complexes  $(E_{n-*}^1; \partial^1)$  and  $(C_c^{n-*}(K; H_n(F_*)); d)$ . Also, in the special context of spectral sequences of bicomplex we know that  $E_{n-p,n}^1$  is the  $n$ -th homology group of the  $(n-p)$ -th column of  $\Omega_{*,*}(K; F)$ . With the dimensional assumption, this is the homology group of highest dimension and therefore the same as the group of cycles. We have an inclusion of  $E_{n-p,n}^1$  in  $\Omega_{n-p,n}(K; F)$  as the kernel of the vertical part of the boundary operator, namely  $\partial_2$ . With this description it is clearly an injective chain complexes morphism  $(E_{n-*}^1; \partial^1) \rightarrow (\Omega_*(K; F); \partial)$  whose cokernel inherits a bicomplex structure from  $\Omega_{*,*}(K; F)$ . By construction all the columns of this bicomplex are exact and so is its total complex. Our injective quasi-isomorphism is then the composition of  $\Phi$  with the inclusion of  $(E_{n-*}^1; \partial^1)$  in the total complex  $(\Omega_*(K; F); \partial)$ .  $\square$

**Remark 2.1.** In the proof of Theorem 1 we did not actually used that  $c'$  was a  $\partial_2$ -cycle and have actually proved that  $\Phi$  is a bicomplex isomorphism. Indeed we have  $(\sum_{q-p=k} \Phi_{p,q}) \circ (\partial_1 + \partial_2) = (d + (-1)^{q-p} \partial) \circ (\sum_{q-p=k+1} \Phi_{p,q})$ . So if we no longer assume the vanishing hypotheses on the local homology of  $F$  what we get instead is a complex of cellular sheaves  $e \mapsto C^*(K; F_e)$  whose cohomology (or hypercohomology) with compact support corresponds to the homology of  $F$ .

A direct consequence of the last corollary is the already known Poincaré-Lefschetz theorem:

**Corollary 1.** *If  $X$  is a homology  $n$ -manifold in the sense of Definition 1.6 then  $H_k(X; \mathbb{Z}) \cong H_c^{n-k}(X; \partial X; o_{\mathbb{Z}})$  for  $o_{\mathbb{Z}}$  the system of local orientations defined on  $X \setminus \partial X$  by  $x \mapsto H_n(X; X - x; \mathbb{Z})$ .*

*Proof.* Let's consider the local system  $o_{\mathbb{Z}}$  on  $X \setminus \partial X$  given by  $x \mapsto H_n(X; X \setminus \{x\}; \mathbb{Z})$ . Denote  $K$  a locally finite, regular, finite dimensional CW-complex whose support is  $X$ . We have by hypotheses, for every cell  $e$  of  $K$ , that  $H_k(K; K - e; \mathbb{Z})$  is zero as soon as  $k$  does not equal  $n$ . Then by Theorem 1 we have:

$$H_k(K; \mathbb{Z}) \cong H_c^{n-k}(X; H_n(\mathbb{Z}_*)).$$

The cellular sheaf  $H_n(\mathbb{Z}_*)$  vanishes on the boundary and corresponds to the local system  $o_{\mathbb{Z}}$  elsewhere hence the corollary follows.  $\square$

An interesting corollary is a version of Serre duality for flat vector bundles over a field  $\mathbb{F}$  with  $o_{\mathbb{F}} = o_{\mathbb{Z}} \otimes \mathbb{F}$  playing the role of the canonical line bundle:

**Corollary 2.** *If  $X$  is a homology  $n$ -manifold in the sense of Definition 1.6 and  $E$  is a flat bundle of  $\mathbb{F}$ -vector spaces of finite rank over  $X$  then:*

$$H^k(X; E) \cong \left( H_c^{n-k}(X; \partial X; o_{\mathbb{F}} \otimes_{\mathbb{F}} E^*) \right)^*.$$

*Proof.* Let  $K$  denotes a locally finite, regular, finite dimensional CW-complex whose support is  $X$ . By universal coefficients theorem (c.f. [CE56]<sup>24</sup>) we have, after noticing that  $(C^k(K; E))_{k \geq 0}$  is dual to the complex  $(C_k(K; E^*))_{k \geq 0}$ , that  $H^k(X; E) \cong (H_k(X; E^*))^*$ . For  $e$  a cell of  $K$  we define an isomorphism of cosheaves  $\phi_e : [K; K - e; E^*(e)] \rightarrow E_e^*$  given by the inverses of the extension morphisms, i.e. its restriction morphisms.  $E^*(e') \rightarrow E^*(e)$  for all cells  $e' \geq e$ . This being done, we have the isomorphism:

$$H_k(K; K - e; E^*) \cong H_k(K; K - e; E^*(e)) \cong H_k(K; K - e; \mathbb{F}) \otimes_{\mathbb{F}} E^*(e).$$

By hypotheses these groups are all 0 as soon as  $k$  does not equal  $n$  hence the dihomologic pseudo-subdivision of the cosheaf  $E^*$  satisfies the conditions of Proposition 2.1. Therefore, Theorem 1 applies. In addition, for  $e^p \leq e^q$  we have the commutative square:

$$\begin{array}{ccc} [K; K - e^p; E^*(e^p)] & \xrightarrow{\phi_{e^p}} & E_{e^p}^* \\ \downarrow & & \downarrow \text{loc. at } e^q \\ [K; K - e^q; E^*(e^q)] & \xrightarrow{\phi_{e^q}} & E_{e^q}^* \end{array}$$

with the unlabelled morphism on the left given by the tensor product of the localisation morphism  $[K; K - e^p; \mathbb{F}] \rightarrow [K; K - e^q; \mathbb{F}]$  with the extension  $E^*(e^p) \rightarrow E^*(e^q)$ . It appears that the cellular sheaf  $e \mapsto H_n(K; K - e; E^*)$  is isomorphic to the tensor product  $e \mapsto H_n(K; K - e; \mathbb{F}) \otimes_{\mathbb{F}} E^*$ . By Theorem 1, we get:

$$H^k(X; E) \cong \left( H_c^{n-k}(X; H_n(E^*)) \right)^* \cong \left( H_c^{n-k}(X; H_n(\mathbb{F}_*) \otimes_{\mathbb{F}} E^*) \right)^*.$$

The sheaf  $H_n(\mathbb{F}_*)$  vanishes on  $\partial X$  and its restriction to  $X \setminus \partial X$  is given by the local system  $o_{\mathbb{F}}$  so finally:

$$H^k(X; E) \cong \left( H_c^{n-k}(X; \partial X; o_{\mathbb{F}} \otimes_{\mathbb{F}} E^*) \right)^*.$$

□

### 3 Application to Tropical Homology : Lefschetz Hyperplane Section Theorem

In this section we apply Theorem 1 to the cosheaves arising from tropical geometry. In a first paragraph we will state and prove four preliminary lemmata. Then we will define the objects we consider. Finally, we will state and prove Theorem 2.

#### Four Lemmata

**Definition 3.1.** Let  $V$  be an  $R$ -module,  $G \subset V$  a finite subset and  $p \in \mathbb{N}$ . We define the complex of  $R$ -modules  $C(V; G; p) := (C_*, \partial) :$

$$C_k := \bigoplus_{\substack{F \subset G \\ |F|=k}}^{p-k} \bigwedge V_F,$$

with  $V_F$  the quotient of  $V$  by the sub-module spanned by  $F$ . We set the map  $\partial_k : C_k \rightarrow C_{k-1}$  to be the sum of the maps :

<sup>24</sup>H. Cartan and S. Eilenberg, *Homological Algebra*, Theorem 3.3 p.113.

$$\begin{aligned} \bigwedge^{p-k} V_F &\longrightarrow \bigoplus_{f \in F} \bigwedge^{p-(k-1)} V_{F \setminus f} \\ v &\longmapsto \sum_{f \in F} f \wedge v. \end{aligned}$$

Because of the antisymmetry of the wedge product,  $\partial^2$  vanishes. Moreover, it is worth noticing that when  $G'$  is a subset of  $G$  the complex  $C(V; G'; p)$  is naturally a sub-complex of  $C(V; G; p)$ .

**Lemma 3.1.** *If  $V$  is free of finite rank,  $G$  is linearly independent and spans a free summand of  $V$  then  $(C; \partial)$  can only have non-trivial homology in dimension 0 and this  $H_0$  is equal to the  $p$ -th exterior power of  $V_G$ . In other words:*

$$0 \leftarrow \bigwedge^p V_G \leftarrow C_0 \leftarrow \dots \leftarrow C_{|G|} \leftarrow 0,$$

is a free resolution of  $\bigwedge^p V_G$ . (The augmenting morphism is the reduction modulo the module spanned by  $G$ .)

*Proof.* By hypotheses one can find  $G' \subset V$  disjoint from  $G$  such that  $G \cup G'$  is a basis of  $V$ . Let us write  $G = \{g_1, \dots, g_n\}$  and  $G' = \{g'_1, \dots, g'_{n'}\}$  then for  $F = \{g_i : i \in I\} \subset G$  with  $|I| = k$  a basis of the  $(p-k)$ -th exterior power of  $V_F$  is given by  $g_{P \setminus I} \wedge g'_Q$  for  $I \subset P \subset \{1, \dots, n\}$ ,  $Q \subset \{1, \dots, n'\}$  such that  $|P| + |Q| = p$ . Moreover the image of the generator  $g_{P \setminus I} \wedge g'_Q$  under the boundary map is:

$$\partial g_{P \setminus I} \wedge g'_Q = \sum_{i \in I} g_i \wedge g_{P \setminus I} \wedge g'_Q \in \bigoplus_{i \in I} \bigwedge^{p-(k-1)} V_{F \setminus g_i}.$$

Therefore, if we fix  $P$  and  $Q$  with  $|P| + |Q| = p$  and see  $P$  as an abstract simplex we have an injective morphism of chain complexes from the reduced simplicial chain complex of  $P$  to  $C$  given by:

$$\begin{aligned} \tilde{C}_k(P; R) &\longrightarrow C_{k+1} \\ I &\longmapsto g_{P \setminus I} \wedge g'_Q, \end{aligned}$$

By construction,  $C$  is the direct sum of the images of these complexes, hence it can only have homology in dimension 0 and the only summands that contribute are the ones for which  $P = \emptyset$ : there is exactly one free summand of rank 1 for every basis element of  $\bigwedge^p V_G$  although a very quick computation show that  $B_0 = \langle G \rangle \wedge \bigwedge^{p-1} V$ .  $\square$

**Lemma 3.2** (Homology Shift). *Let  $n \in \mathbb{N}$  be a natural number and:*

$$0 \rightarrow C(n) \xrightarrow{a_n} C(n-1) \xrightarrow{a_{n-1}} \dots \xrightarrow{a_2} C(1) \xrightarrow{a_1} C(0) \rightarrow 0, \quad (\text{ES})$$

be an exact sequence of chain complexes. If for all  $1 \leq i \leq n-1$ ,  $C(i)$  is exact then the homology of  $C(0)$  is the one of  $C(n)$  shifted by  $1-n$ :

$$H(0) = H(n)[1-n].$$

*Proof.* For all  $1 \leq i \leq n$ , let  $A(i)$  denote the kernel complex of  $a_i$ . Since (ES) is an exact sequence the following diagram has all of its columns exact:

$$\begin{array}{ccccccc} & & & 0 & & 0 & & 0 & & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & & A(n-1) & \dots & A(i) & \dots & A(1) & & A(0) \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C(n) & \xrightarrow{a_n} & C(n-1) & \xrightarrow{a_{n-1}} & \dots & \xrightarrow{a_{i+1}} & C(i) & \xrightarrow{a_i} & \dots & \xrightarrow{a_2} & C(1) & \xrightarrow{a_1} & C(0) & \longrightarrow & 0 \\ & & \downarrow a_n & & \downarrow a_{n-1} & & & & \downarrow a_i & & & & \downarrow a_1 & & \downarrow & & \\ & & A(n-1) & & A(n-2) & & \dots & & A(i-1) & & \dots & & A(0) & & 0 & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & & & 0 & & & & 0 & & 0 & & \end{array}$$



By assumption, for all  $1 \leq i \leq n-1$ ,  $H(i) = 0$  so  $H(A(i-1)) = H(A(i))[-1]$  by use of the long exact sequence associated with  $0 \rightarrow A(i) \rightarrow C(i) \rightarrow A(i-1) \rightarrow 0$ . Hence by a finite recursion we get that:

$$H(0) = H(A(0)) = H(A(n-1))[1-n] = H(n)[1-n].$$

□

**Lemma 3.3** (Double Localisation). *Assume  $F$  is a cellular cosheaf on  $K$  and  $K'$  is a subdivision of  $K$ . If  $e$  is a cell of  $K$  and  $e'$  is a cell of  $K'$  contained in  $e$  then the local homology of  $F$  at  $e$  is identical to the local homology of the subdivision of  $F$  at  $e'$ . More precisely, if  $F'$  is the subdivision of  $F$  there is a canonical quasi-isomorphism between  $C_*(K; F_e)$  and  $C_*(K'; F'_{e'})$ .*

*Proof.* If we denote by  $(F_e)'$  the subdivision of  $F_e$  then we have a commutative square of cellular cosheaves on  $K'$ :

$$\begin{array}{ccc} F' & \longrightarrow & (F')_{e'} \\ \downarrow & & \downarrow \\ (F_e)' & \longrightarrow & (F_e)'_{e'} \end{array}$$

where all four morphisms come from localisation and subdivision. The two cosheaves on the right hand column are actually equal and the arrow is rigorously the identity between them. To see it we only need to check it for all cells  $\tilde{e}'$  that contains  $e'$  since both these cosheaves vanish elsewhere. If  $\tilde{e}$  is the unique cell of  $K$  containing  $\tilde{e}'$  then they satisfy:

$$\begin{array}{ccc} \tilde{e} & \geq & e \\ \cup & & \cup \\ \tilde{e}' & \geq & e' \end{array}$$

therefore  $(F')_{e'}(\tilde{e}') = F(\tilde{e}) = F_e(\tilde{e}) = (F_e)'_{e'}(\tilde{e}')$  with our morphism given by the central equality. On the level of cellular chain complexes we have the following diagram:

$$\begin{array}{ccc} & & C_*(K'; (F')_{e'}) \\ & \nearrow h & \parallel \\ C_*(K; F_e) & \xrightarrow[\text{subd.}]{f} C_*(K'; (F_e)') & \xrightarrow[\text{loc.}]{g} C_*(K'; (F_e)'_{e'}) \end{array}$$

with  $h$  the canonical quasi-isomorphism given in the statement of the proposition. Since  $f$  is a subdivision morphism it is automatically a quasi-isomorphism. We only need to show that the localisation morphism  $g$  is a quasi-isomorphism. This is a surjective morphism since localisations are quotients. We have, by definition, the short exact sequence of cosheaves:

$$0 \rightarrow (F_e)' \otimes [K' - e'; \mathbb{Z}] \rightarrow (F_e)' \rightarrow (F_e)'_{e'} \rightarrow 0,$$

so showing that  $g$  is a quasi-isomorphism amounts to showing that the kernel cosheaf has trivial homology.  $(F_e)' = F' \otimes [K'; K' - e; \mathbb{Z}]$  for  $K' - e$  is the subdivision of the sub-complex  $K - e$ . Consequently:

$$(F_e)' \otimes [K' - e'; \mathbb{Z}] = F' \otimes [K'; K' - e; \mathbb{Z}] \otimes [K' - e'; \mathbb{Z}] = F' \otimes [K' - e'; K' - e; \mathbb{Z}].$$

By a process similar to excision we see that:

$$[K' - e'; K' - e; \mathbb{Z}] = [K'(e) - e'; K'(e) - e; \mathbb{Z}].$$

Indeed,  $K'(e)$  is the smallest sub-complex of  $K'$  containing  $e$  in its support and any cell not in it falls into  $K' - e$ . Note that  $[K'(e) - e'; K'(e) - e; \mathbb{Z}]$  can be non-zero only on cells of  $K'$  contained in  $e$  so:

$$(F_e)' \otimes [K' - e'; \mathbb{Z}] = F' \otimes [K' - e'; K' - e; \mathbb{Z}] = [K'(e) - e'; K'(e) - e; F(e)].$$

Now because  $K$  is regular,  $|K'(e)|$  is a closed ball  $B$ ,  $|K'(e) - e|$  is its boundary  $\partial B$  and  $|K'(e) - e'|$  sits somewhere between  $\partial B$  and  $B$  punctured on one point. Since the latter retracts by deformation on the former the homology of  $[K'(e) - e'; K'(e) - e; F(e)]$  is trivial. □

**Lemma 3.4.** *Let  $M$  be a lattice of finite rank with some linear forms  $\alpha_1, \dots, \alpha_k \in \text{Hom}(M; \mathbb{Z}) \setminus \{0\}$ . If  $\omega$  is a generator of the last exterior power of the lattice spanned by the  $\alpha_i$ 's then for all  $p \in \mathbb{N}$ , the quotient:*

$$G := \left\{ v \in \bigwedge^p M \mid \omega \cdot v = 0 \right\} \Big/ \sum_{i=1}^k \bigwedge^p \ker(\alpha_i),$$

with  $\omega \cdot v$  the contraction of  $v$  by  $\omega$ , is a finite group.

*Proof.* Let  $p \in \mathbb{N}$ . First we remind that  $\omega \cdot v = 0$  if  $p < \deg \omega$  and is, otherwise, the unique  $(p - \deg \omega)$ -element satisfying  $\alpha(\omega \cdot v) = (\alpha \wedge \omega)(v)$  for all  $(p - \deg \omega)$ -form  $\alpha$ . Notice that for each  $1 \leq i \leq k$  the form  $\alpha_i$  divides a non-zero multiple of  $\omega$  in the exterior algebra of  $\text{Hom}(M; \mathbb{Z})$ . It follows that all  $p$ -elements of  $\ker(\alpha_i)$  contract to 0 against  $\omega$ . Assume now that  $r \geq 1$  is the rank of the sub-lattice of  $\text{Hom}(M; \mathbb{Z})$  spanned by the  $\alpha_i$ 's and that  $\alpha_1 \wedge \dots \wedge \alpha_r = m\omega$  with  $m \neq 0$ . Then we have:

$$\left\{ v \in \bigwedge^p M \mid \omega \cdot v = 0 \right\} = \left\{ v \in \bigwedge^p M \mid (\alpha_1 \wedge \dots \wedge \alpha_r) \cdot v = 0 \right\}.$$

We can complete the set  $\{\alpha_1, \dots, \alpha_r\}$  with a set  $\{\beta_1, \dots, \beta_s\} \subset \text{Hom}(M; \mathbb{Z})$  into a basis of the rational vector space  $\text{Hom}_{\mathbb{Q}}(M; \mathbb{Q})$ . For a  $p$ -element  $v \in \bigwedge^p M$ , seen as a  $p$ -vector in  $\bigwedge_{\mathbb{Q}}^p(M \otimes \mathbb{Q})$ , with  $\{e_1, \dots, e_r, f_1, \dots, f_s\}$  the dual basis of  $\{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$ , we have:

$$(\alpha_1 \wedge \dots \wedge \alpha_r) \cdot v = \sum_{|J|=p-r} (\alpha_1 \wedge \dots \wedge \alpha_r \wedge \beta_J)(v) f_J,$$

and thus:

$$\left\{ v \in \bigwedge^p M \mid \omega \cdot v = 0 \right\} \otimes \mathbb{Q} = \left\langle e_I \wedge f_J : |I| < r \text{ and } |I| + |J| = p \right\rangle_{\mathbb{Q}}.$$

Now,  $\ker(\alpha_i) \otimes \mathbb{Q} = \ker(\alpha_i \otimes 1) = \langle e_j, f_k : j \neq i \text{ and } 1 \leq k \leq s \rangle_{\mathbb{Q}}$ , so:

$$\left\{ v \in \bigwedge^p M \mid \omega \cdot v = 0 \right\} \otimes \mathbb{Q} = \left( \sum_{i=1}^k \bigwedge^p \ker(\alpha_i) \right) \otimes \mathbb{Q},$$

then  $G \otimes \mathbb{Q} = 0$ . Since  $G$  is finitely generated, it is finite.  $\square$

## Hodge Theory in Tropical Toric Geometry

Let  $P$  be a full dimensional integer polytope<sup>25</sup> in a finite dimensional real vector space  $\mathfrak{t}^*(\mathbb{R})$  endowed with a lattice  $\mathfrak{t}^*(\mathbb{Z})$ . Its corresponding toric variety is a projective algebraic variety defined over the integers. The tropical locus  $Y$  of such a toric variety we can be seen as a compactification of the tropical torus  $\mathfrak{t}(\mathbb{R}) = \text{Hom}_{\mathbb{R}}(\mathfrak{t}^*(\mathbb{R}); \mathbb{R})$ . Moreover, the moment map provides an isomorphism between  $Y$  and the polytope  $P$  itself. The real or complex toric variety defined by an integer polytope comes equipped with an ample line bundle. The space of global sections of this line bundle is naturally isomorphic to the vector space of Laurent polynomials whose exponents are integer points of  $P$ . The ‘‘tropical sections’’ of this line bundle are likewise defined as tropical Laurent polynomials whose exponents are integer points of  $P$  i.e. the convex piecewise affine functions of the form:

$$\begin{aligned} f : \mathfrak{t}(\mathbb{R}) &\longrightarrow \mathbb{T} = \mathbb{R} \cup \{-\infty\} \\ v &\longmapsto \max_{\alpha \in P \cap \mathfrak{t}^*(\mathbb{Z})} (a_{\alpha} + \alpha(v)), \end{aligned}$$

where the  $a_{\alpha}$ 's are tropical numbers. The tropical hypersurface  $X$  of  $Y$  defined by this equation is the topological closure of the non-differentiability locus of  $f$  in  $Y$ . As usual the Newton polytope of  $f$  is the convex hull of the  $\alpha \in P \cap \mathfrak{t}^*(\mathbb{Z})$  for which the associated coefficient  $a_{\alpha} \neq -\infty$ . By Theorem 2.3.7 of G. Mikhalkin and J. Rau [MR18]<sup>26</sup>, any tropical hypersurface  $X$  of  $Y$  defined by an equation  $f$  whose Newton polytope is  $P$ , is dual to an integer convex polyhedral subdivision  $K$  of  $P$ . This means that  $X$  is homeomorphic to the sub-complex of the dihomologic subdivision<sup>27</sup> of  $K$  consisting of the union of the closed dihomologic cells indexed the adjacent

<sup>25</sup>the convex hull of a finite number of vertices.

<sup>26</sup>G. Mikhalkin and J. Rau, *Tropical Geometry*, Theorem 2.3.7 p.44.

<sup>27</sup>Here  $K$  is polyhedral so its dihomologic pseudo-subdivision is an actual regular subdivision of  $K$ .

pairs  $e^1 \leq e^p$ . The situation is illustrated in Figure 10. The theory of tropical homology defined by I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov for both  $X$  and  $Y$  can be expressed as the homology of some dihomologic cosheaves on  $K$ . Moreover, the data of the polyhedral subdivision  $K$  alone is enough to define these cosheaves. We will adopt this point of view and state our result in terms of cosheaves associated to an integer polyhedral subdivision  $K$  of an integer polytope  $P$ . As noted by E. Brugallé, L. López de Medrano and J. Rau in [BLdMR22], the tropical cosheaves can be associated to any integer polyhedral subdivision regardless of its convexity and most of the results about the tropical homology of tropical hypersurfaces apply to them. Theorem 2 is no exception to that observation. We will not assume the subdivision to be convex and therefore the theorem will not be stated in the framework of tropical hypersurfaces.

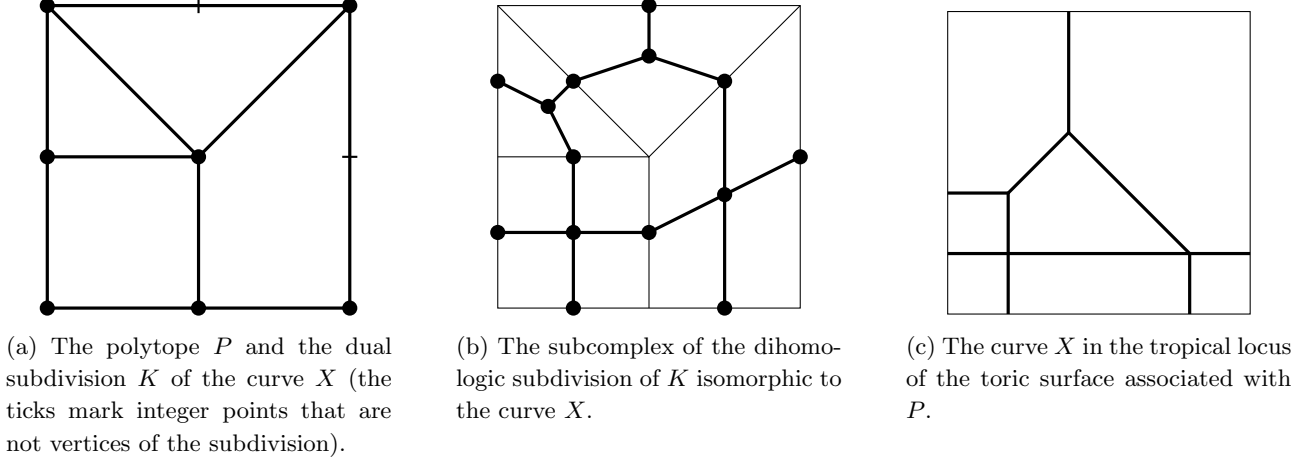


Figure 10: A singular curve  $X$  of bidegree  $(2;2)$  in the tropical locus of  $\mathbb{P}^1 \times \mathbb{P}^1$ .

- Notations.**
1.  $\mathfrak{t}^*(\mathbb{Z})$  is a lattice of finite rank  $n \in \mathbb{N}$  with dual lattice  $\mathfrak{t}(\mathbb{Z})$  ;
  2. For  $R$  a commutative ring with unit,  $\mathfrak{t}^*(R)$  (resp.  $\mathfrak{t}(R)$ ) is the associated free  $R$ -module  $\mathfrak{t}^*(\mathbb{Z}) \otimes R$  (resp.  $\mathfrak{t}(\mathbb{Z}) \otimes R$ ) ;
  3.  $P$  is a full dimensional polytope of  $\mathfrak{t}^*(\mathbb{R})$  whose vertices lie in the lattice  $\mathfrak{t}^*(\mathbb{Z})$  ;
  4.  $K$  is a polyhedral subdivision of  $P$ , with  $K^{(0)} \subset \mathfrak{t}^*(\mathbb{Z})$ . (Note that every cell  $e^q$  of  $K$  is the relative interior of a  $q$ -dimensional polytope whose tangent space  $Te^q$  is rational relatively to  $\mathfrak{t}^*(\mathbb{Z})$ , i.e.  $Te^q \cap \mathfrak{t}^*(\mathbb{Z})$  is free of rank  $q$  and in particular, the quotient of  $\mathfrak{t}^*(\mathbb{Z})$  by this sub-lattice is free of rank  $n - q$ .)

**Definition 3.2** (Tropical Cosheaves). The first cosheaf we define is called the *sedentarity*. It represents the stabilisers of the action of the tropical torus. We denote it by  $\text{Sed}$ . It is defined on the CW-complex associated with the polytope  $P$ . If  $Q$  is a face of  $P$  we set:

$$\text{Sed}(Q) := \{v \in \mathfrak{t}(\mathbb{Z}) \mid \alpha(v) = 0, \forall \alpha \in TQ\} \subset \mathfrak{t}(\mathbb{Z}),$$

with  $TQ$  denoting the tangent space of  $Q$  i.e. the vectorial direction of the affine space spanned by the polytope  $Q$ . The extension morphisms are simply given by inclusions.  $\text{Sed}(Q)$  consists of the integral vectors orthogonal to  $TQ$ . So, whenever  $Q'$  is a face of  $Q$ ,  $\text{Sed}(Q)$  is a sub-module of  $\text{Sed}(Q')$ .

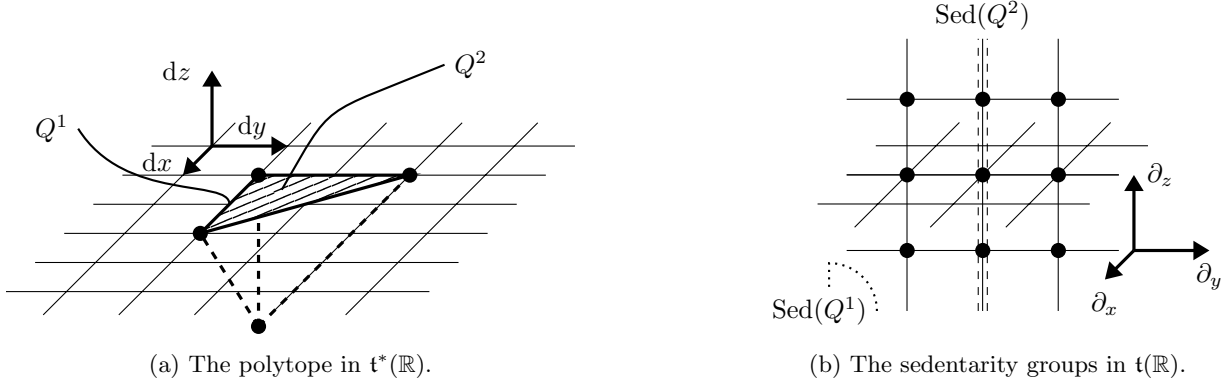
The tropical cosheaves that would be associated with  $Y$  are also defined on  $P$ , by the formula,  $p \in \mathbb{N}$ :

$$F_p^{(0)} := \bigwedge^p \mathfrak{t}(\mathbb{Z}) / \text{Sed}.$$

We will abuse the notations and denote by the same symbols  $F_p^{(0)}$  all the subdivisions of these cosheaves, in particular its subdivisions on  $K$  and the dihomologic subdivision of  $K$ .

We denote the tropical cosheaves that would be associated with a tropical hypersurface  $X$  by  $F_p^{(1)}$ ,  $p \in \mathbb{N}$ . They consist of dihomologic cosheaves on  $K$ . For  $e^q < e^{q'}$  two cells of  $K$  and  $p \in \mathbb{N}$  the group  $F_p^{(1)}(e^q; e^{q'})$  is:

$$F_p^{(1)}(e^q; e^{q'}) = \sum_{e^1 \leq e^q} \bigwedge^p (Te^1)^\perp \cap \mathfrak{t}(\mathbb{Z}) / \text{Sed}(e^{q'}). \quad (\text{A})$$



(a) The polytope in  $t^*(\mathbb{R})$ .

(b) The sedentarity groups in  $t(\mathbb{R})$ .

Figure 11: An edge  $Q^1$  of a 2-dimensional face  $Q^2$  and their respective sedentarity.

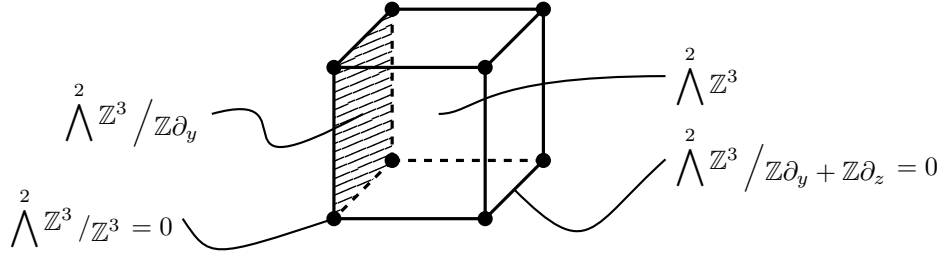


Figure 12: A cube and the groups associated by  $F_2^{(0)}$  to some of its faces.

As pointed out in Definition 1.14, the extension morphisms need only be defined on elementary adjacencies. For  $e^{q_1} \leq e^{q_2} \leq e^{q_3} \leq e^{q_4}$  four cells of  $K$  (maybe with repetitions) the elementary extension morphisms are depicted in the following diagram:

$$\begin{array}{ccc}
 & F_p^{(1)}(e^{q_1}; e^{q_4}) & \\
 f \swarrow & & \searrow g \\
 F_p^{(1)}(e^{q_2}; e^{q_4}) & & F_p^{(1)}(e^{q_1}; e^{q_3}) \\
 h \searrow & & \swarrow t \\
 & F_p^{(1)}(e^{q_2}; e^{q_3}) &
 \end{array}$$

where:

1. The morphisms  $f$  and  $t$  are basic inclusions coming from the definition of the groups, c.f. (A);
2. The morphisms  $g$  and  $h$  are reductions modulo  $\text{Sed}(e^{q_3})$ . More precisely they correspond to the canonical projection  $\bigwedge^p (Te^1)^\perp \cap t(\mathbb{Z}) / \text{Sed}(e^{q_4}) \rightarrow \bigwedge^p (Te^1)^\perp \cap t(\mathbb{Z}) / \text{Sed}(e^{q_3})$  on every summand.

By the nature of the morphisms involved the diagram is commutative. The Figure 13 illustrates the values taken by the cosheaf  $F_1^{(1)}$  on a triangle with trivial subdivision  $K$ .

**Definition 3.3** (Tropical Homology Groups). Let  $R$  be a commutative ring. Let  $X$  denote a tropical hypersurface of  $Y$  dual to an integer convex polyhedral subdivision  $K$  of  $P$ . The tropical homology groups of  $Y$  are defined, for  $p, q \in \mathbb{N}$ , by:

$$H_{p,q}(Y; R) := H_q(K; F_p^{(0)} \otimes R).$$

Likewise, the tropical homology groups of  $X$  are given for  $p, q \in \mathbb{N}$ , by:

$$H_{p,q}(X; R) := H_q(K; F_p^{(1)} \otimes R).$$

Moreover, the inclusions  $F_p^{(1)} \subset F_p^{(0)}$ , for  $p \in \mathbb{N}$ , induce morphisms in homology:

$$i_{p,q} : H_{p,q}(X; R) \longrightarrow H_{p,q}(Y; R).$$

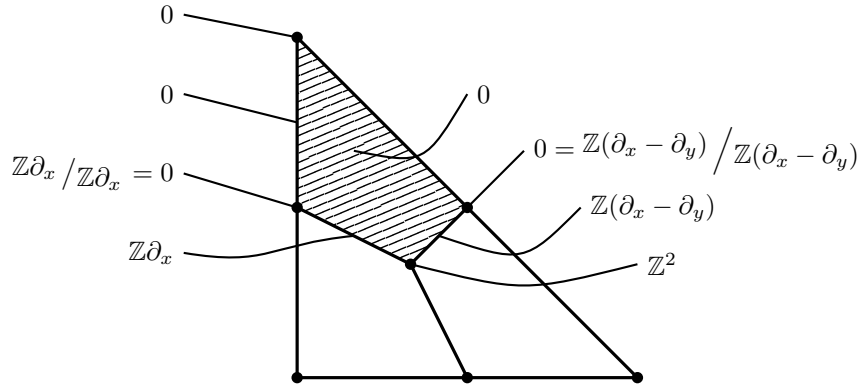


Figure 13: A triangle and the groups associated by  $F_1^{(1)}$  to some of its dihomologic cells.

## Lefschetz Hyperplane Section Theorem in Tropical Orbifold Toric Varieties

**Definition 3.4.** Let  $P$  be a simple polytope, we denote by  $\text{Sed}_{(1)}$  the cosheaf:

$$\text{Sed}_{(1)} := \bigoplus_{\substack{Q < P \\ \text{codim} Q = 1}} [Q; \text{Sed}(Q)].$$

If  $Q$  is a codimension 1 face of  $P$  there is a natural injective cosheaf morphism  $[Q; \text{Sed}(Q)] \rightarrow \text{Sed}$ . If  $Q'$  is a face of  $P$  the group  $[Q; \text{Sed}(Q)](Q')$  is either  $\text{Sed}(Q)$  or 0. The former only happens when  $Q'$  is a face of  $Q$  and in this case the morphism is given by the inclusion. Summing all these yields a morphism:

$$\text{Sed}_{(1)} \rightarrow \text{Sed}.$$

It is injective because the polytope  $P$  is simple : if  $Q' < P$  has codimension  $q$  it is the intersection of exactly  $q$  faces of codimension 1 and:

$$(TQ')^\perp = \bigoplus_{\substack{Q < P \\ \text{codim} Q = 1}} TQ^\perp,$$

so the  $\text{Sed}(Q)$ 's are in direct sum inside  $\text{Sed}(Q')$ . For the same reason the quotient  $\Delta$  of  $\text{Sed}$  by  $\text{Sed}_{(1)}$  is a cosheaf of finite groups. We denote by  $\delta(P)$  the least common multiple of the exponents<sup>28</sup> of the groups  $\Delta(Q)$  for all  $Q \leq P$ .

**Remarks.** For  $P$  a simple polytope:

1. The cosheaf of finite groups  $\Delta$  encodes the singularities of the toric variety associated with  $P$ . The set of complex points of the affine open set associated with the face  $Q \leq P$  is quotient of  $(\mathbb{C}^\times)^k \times \mathbb{C}^{n-k}$  by an algebraic action of the group  $\Delta(Q)$  (c.f. [Ful93]<sup>29</sup>).
2. For two adjacent faces  $Q_1 < Q_2$  of  $P$ , the snake lemma implies that the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Sed}_{(1)}(Q_2) & \longrightarrow & \text{Sed}(Q_2) & \longrightarrow & \Delta(Q_2) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & \text{Sed}_{(1)}(Q_1) & \longrightarrow & \text{Sed}(Q_1) & \longrightarrow & \Delta(Q_1) & \longrightarrow & 0 \end{array}$$

gives rise to an exact sequence:

<sup>28</sup>The exponent of a group  $G$ , with group law denoted multiplicatively, is the smallest, if any, positive integer  $e$  for which  $g^e = 1$  for all  $g \in G$ .

<sup>29</sup>W. Fulton, *Introduction to Toric Varieties*, Section 2.2 p.34.

$$0 \rightarrow \underbrace{0}_{\ker(f)} \rightarrow \underbrace{0}_{\ker(g)} \rightarrow \ker(h) \rightarrow \underbrace{\bigoplus_{\substack{Q_1 < Q < P \\ Q \succ Q_2 \\ \text{codim } Q=1}} \text{Sed}(Q)}_{\text{coker}(f)} \rightarrow \underbrace{\text{Sed}(Q_1)/\text{Sed}(Q_2)}_{\text{coker}(g)} \rightarrow \text{coker}(h) \rightarrow 0.$$

However,  $\ker(h)$  is a finite group and  $\text{coker}(f)$  is free, so  $\ker(h) = 0$  and  $h$  is injective. As a consequence,  $\delta(P)$  is the least common multiple of the exponents of the groups  $\Delta(V)$ ,  $V$  running amongst all vertices of  $P$ .

- For  $R$  a commutative ring, the classification of finite Abelian groups implies that the vanishing of both  $\Delta(V) \otimes R$  and  $\text{Tor}(\Delta(V); R)$  is equivalent to the invertibility of the exponent of the group  $\Delta(V)$  in  $R$ . When  $R$  is a field this is equivalent to the coprimality of this exponent with the characteristic of  $R$ .

**Definition 3.5.** Let  $Q$  be an integer polytope in  $\mathfrak{t}^*(\mathbb{R})$ . Its tangent space is rational, thus contains a lattice  $TQ \cap \mathfrak{t}^*(\mathbb{Z})$  and we denote  $\omega(Q)$  one of the two generators of its last exterior power. The contraction against this element defines an endomorphism of degree  $(-\dim Q)$  of the exterior algebra  $\bigwedge^* \mathfrak{t}(\mathbb{Z})$  and we denote its kernel by  $\ker(\omega(Q) \cdot -)$ . Because of Lemma 3.4 the sub-module:

$$\sum_{\substack{E \leq Q \\ \dim E=1}} \ker(\omega(E) \cdot -) \subset \ker(\omega(Q) \cdot -),$$

has the same rank. We denote by  $\theta(Q)$  the exponent of the finite quotient:

$$\ker(\omega(Q) \cdot -) \Big/ \sum_{\substack{E \leq Q \\ \dim E=1}} \ker(\omega(E) \cdot -).$$

By definition this number is 1 for all polytopes of dimension at most 1. For  $K$  a compact integer polyhedral complex of  $\mathfrak{t}^*(\mathbb{R})$  we define  $\theta(K)$  to be the least common multiple of the  $\theta(e)$ , for  $e$  a cell of  $K$ .

**Proposition 3.5.** *Let  $P$  be a simple integer polytope endowed with an integer polyhedral subdivision  $K$ . For all ring  $R$  in which  $\delta(P)$  is invertible, the images of the cosheaves  $(F_p^{(0)} \otimes R)_{p \geq 0}$  under the dihomologic subdivision of  $K$  satisfy the hypotheses of Proposition 2.1 i.e. for all polyhedral cells  $e \in K$  we have:*

$$H_k \left( K; (F_p^{(0)} \otimes R)_e \right) \neq 0 \Rightarrow k = n.$$

*Proof.* Let  $F$  denote one of the cosheaves  $(F_p^{(0)})_{0 \leq p \leq n}$  on  $P$ . The polyhedral subdivision  $K$  of  $P$  first and then its dihomologic subdivision give rise to two consecutive subdivisions of cosheaves  $F \mapsto F'$  then  $F' \mapsto F''$ . For  $e$  a cell of  $K$ , the localisation process for dihomologic cosheaves consisting in “fixing the first coordinate”  $G \mapsto G_e$  applied to a subdivided cosheaf is equivalent to localising directly the cosheaf that was subdivided, c.f. diagram D1. Consequently, the cosheaf  $F''_e$  is the same as  $F'_e$ . In the light of Lemma 3.3 on double localisation, showing that  $F'_e$  can only have non-trivial homology in degree  $n$  for all cell  $e \in K$  is strictly equivalent to showing that  $F_Q$  can only have non-trivial homology in degree  $n$  for all face  $Q < P$ . Because  $P$  is simple, we have the exact sequence of cosheaves of abelian groups:

$$0 \rightarrow \text{Sed}_{(1)} \rightarrow \text{Sed} \rightarrow \Delta \rightarrow 0,$$

which is tensorised to the exact sequence of cosheaves of  $R$ -modules:

$$0 \rightarrow \text{Tor}(\Delta; R) \rightarrow \text{Sed}_{(1)} \otimes R \rightarrow \text{Sed} \otimes R \rightarrow \Delta \otimes R \rightarrow 0.$$

By hypotheses,  $\text{Sed} \otimes R$  is then isomorphic to  $\text{Sed}_{(1)} \otimes R$  and  $F_p^{(0)} \otimes R$  is isomorphic to:

$$\bigwedge^p \mathfrak{t}(R) \Big/ (\text{Sed}_{(1)} \otimes R).$$

The cosheaf  $\text{Sed}_{(1)}$  can be described as follows : choose, for every codimension 1 face  $Q_{(1)}$  of  $P$ , a generator  $g_{Q_{(1)}}$  of  $\text{Sed}(Q_{(1)}) \cong \mathbb{Z}$ , and associate to all faces  $Q$  of  $P$  the set  $G(Q)$  of the  $g_{Q_{(1)}}$ 's for which  $Q < Q_{(1)}$ . The association  $G : Q \mapsto G(Q)$  is a cosheaf of sets with inclusions as extension maps. In these notations,  $\text{Sed}_{(1)}$  is

the cosheaf that associate to  $Q$  the sub-module of  $\mathfrak{t}(\mathbb{Z})$  spanned by  $G(Q)$  and with extension morphisms given by inclusions. We can consider the cosheaf of chain complexes  $Q \mapsto C(\mathfrak{t}(\mathbb{Z}); G(Q); p)$  of Definition 3.1. It has an augmentation morphism:

$$0 \leftarrow \bigwedge^p \mathfrak{t}(\mathbb{Z}) / \text{Sed}_{(1)} \leftarrow C(\mathfrak{t}(\mathbb{Z}); G; p),$$

and becomes a resolution once we tensorise every group by  $R$ . Indeed, for all  $Q$  the set  $G(Q) \otimes R$  is linearly independent in  $\mathfrak{t}(R)$  and  $\text{Sed}_{(1)}(Q) \otimes R = \text{Sed}(Q) \otimes R$  is a free summand of  $\mathfrak{t}(R)$  since  $\mathfrak{t}(R)/(\text{Sed}(Q) \otimes R) \cong \text{Hom}(TQ_{\mathbb{Z}}; R)$  is free, so Lemma 3.1 applies. For that reason, we set  $\text{Res}_p := C(\mathfrak{t}(\mathbb{Z}); G; p) \otimes R$  and we have a resolution of cosheaves:

$$0 \leftarrow F_p^Y \otimes R \leftarrow \text{Res}_p.$$

Localising at a face  $Q$  amounts to tensorisation by a cosheaf of free modules, therefore it is an exact endofunctor of the category of cellular cosheaves. To avoid confusion we will denote, from now on, the CW-complex defined by the polytope  $P$  by  $\Pi$  and every face  $Q \leq P$  is meant to be open. In particular,  $\Pi(Q)$  is the smallest sub-complex of  $\Pi$  containing the open face  $Q$ , that is to say the collection of its faces. For a face  $Q$ , we have the local resolution:

$$0 \leftarrow (F_p^{(0)} \otimes R)_Q \leftarrow (\text{Res}_p)_Q,$$

and for  $0 \leq q$ :

$$\begin{aligned} (\text{Res}_{p,q})_Q &= \bigoplus_{\substack{Q_{(q)} < P \\ \text{codim } Q_{(q)} = q}} \left[ \Pi(Q_{(q)}); \left( \bigwedge^{p-q} \mathfrak{t}(\mathbb{Z}) / \text{Sed}_{(1)}(Q_{(q)}) \right) \otimes R \right] \otimes_R [\Pi; \Pi - Q; R] \\ &= \bigoplus_{\substack{Q < Q_{(q)} < P \\ \text{codim } Q_{(q)} = q}} \left[ \Pi(Q_{(q)}); \Pi(Q_{(q)}) - Q; \left( \bigwedge^{p-q} \mathfrak{t}(\mathbb{Z}) / \text{Sed}_{(1)}(Q_{(q)}) \right) \otimes R \right]. \end{aligned}$$

If  $Q$  is a proper face of  $Q_{(q)}$  then  $|\Pi(Q_{(q)})|$ , the closure of  $Q_{(q)}$ , retracts to  $|\Pi(Q_{(q)}) - Q|$  and the cosheaf:

$$\left[ \Pi(Q_{(q)}); \Pi(Q_{(q)}) - Q; \left( \bigwedge^{p-q} \mathfrak{t}(\mathbb{Z}) / \text{Sed}_{(1)}(Q_{(q)}) \right) \otimes R \right],$$

has trivial homology. When  $Q_{(q)}$  equals  $Q$ , we are computing the homology of a closed  $(n - q)$ -ball relatively to its boundary. The homology is concentrated in top dimension  $n - q$ . In application of Lemma 3.2, the homology of  $(F_p^{(0)} \otimes R)_Q$  is the shift by  $q$  of  $(\text{Res}_{p,q})_Q$  and therefore has homology concentrated in dimension  $(n - q) + q = n$ .  $\square$

**Remark 3.1.** In the previous proof we have defined a resolution of the cosheaf  $F_p^Y \otimes R$ , namely:

$$0 \leftarrow F_p^Y \otimes R \leftarrow \text{Res}_p.$$

**Proposition 3.6.** *Under the same hypotheses as Proposition 3.5 and if  $\theta(K)$  is invertible in  $R$ , the dihomologic cosheaves  $(F_p^{(1)} \otimes R)_{0 \leq p \leq n-1}$  of  $K$  satisfy the hypotheses of Proposition 2.1 i.e. for all polyhedral cells  $e \in K$ :*

$$H_k \left( K; (F_p^{(1)} \otimes R)_e \right) \neq 0 \Rightarrow k = n.$$

*Proof.* Let  $p$  be a non-negative integer. Proposition 3.6 follows from Proposition 3.5 and the fact that  $F_p^{(1)} \otimes R$  is locally a direct summand of  $F_p^{(0)} \otimes R$ . More precisely, we will show that for all cells  $e^q$  of  $K$ , the cosheaf  $(F_p^{(1)} \otimes R)_{e^q}$  is a direct summand of the cosheaf  $(F_p^{(0)} \otimes R)_{e^q}$ . This statement implies that the homology of the former is a direct summand of the latter. By Proposition 3.5, the latter can only have non-vanishing homology in dimension  $n$  and so can the former.

Let  $e^q$  be a cell of  $K$  and set  $A := (Te^q)^\perp \cap \mathfrak{t}(\mathbb{Z})$ . Now choose  $B$  a supplementary sub-module<sup>30</sup> of  $A$  in  $\mathfrak{t}(\mathbb{Z})$  and consider  $A/\text{Sed}$  and  $B/\text{Sed}$  the respective images of the constant cosheaves  $[K; A]$  and  $[K; B]$

<sup>30</sup>It can be done for  $(Te^q)^\perp$  is a rational sub-space of  $\mathfrak{t}(\mathbb{R})$ .

in the quotient  $\mathfrak{t}(\mathbb{Z})/\text{Sed}$ . Notice that if  $e^r$  is a cell containing  $e^q$  then  $\text{Sed}(e^r) \subset A$  and the projection  $B \rightarrow B/\text{Sed}(e^r)$  is an isomorphism. Since the commutativity conditions of these projections are satisfied they induce an isomorphism of cosheaves:

$$[K; K - e^q; B] \xrightarrow{\cong} (B/\text{Sed})_{e^q}.$$

Therefore, the cosheaf  $\mathfrak{t}(\mathbb{Z})/\text{Sed}$  splits around  $e^q$  into the direct sum:

$$\left(\mathfrak{t}(\mathbb{Z})/\text{Sed}\right)_{e^q} = \left(A/\text{Sed}\right)_{e^q} \oplus \left(B/\text{Sed}\right)_{e^q},$$

and for all  $p \geq 0$  we have the decomposition:

$$\left(\bigwedge^p \mathfrak{t}(\mathbb{Z})/\text{Sed}\right)_{e^q} = \bigoplus_{p_A+p_B=p} \left(\bigwedge^{p_A} A/\text{Sed}\right)_{e^q} \otimes \left(\bigwedge^{p_B} B/\text{Sed}\right)_{e^q}.$$

All the cosheaves involved being made of free groups this decomposition remains valid after tensorisation by  $R$ :

$$\left(\bigwedge^p \mathfrak{t}(\mathbb{Z})/\text{Sed}\right)_{e^q} \otimes R = \bigoplus_{p_A+p_B=p} \left(\bigwedge^{p_A} A/\text{Sed}\right)_{e^q} \otimes \left(\bigwedge^{p_B} B/\text{Sed}\right)_{e^q} \otimes R.$$

By assumption  $\theta(K)$  is invertible in  $R$ . As  $\theta(e^q)$  divides  $\theta(K)$ , this number is invertible in  $R$  as well. It implies that if  $\omega$  is a generator of  $\bigwedge^q(Te^q \cap \mathfrak{t}^*(\mathbb{Z}))$ , the group  $F_p^{(1)}(e^q; e^r) \otimes R$ , for  $e^q < e^r$ , consists of the  $p$ -elements of  $\bigwedge^p \mathfrak{t}(\mathbb{Z})/\text{Sed}(e^r) \otimes R$  whose contraction against  $\omega \otimes 1$  vanishes. Indeed, we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \sum_{e^1 \leq e^q} \bigwedge^p ((Te^1)^\perp \cap \mathfrak{t}(\mathbb{Z})) & \longrightarrow & \{v \in \bigwedge^p \mathfrak{t}(\mathbb{Z}) \mid \omega \cdot v = 0\} & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_p^{(1)}(e^q; e^r) & \longrightarrow & \{v \in \bigwedge^p \mathfrak{t}(\mathbb{Z})/\text{Sed}(e^r) \mid \omega \cdot v = 0\} & \longrightarrow & G' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

By definition the exponent of  $G$  divides  $\theta(e^q)$  and therefore the exponent of  $G'$  divides it as well. Then it follows that both  $G' \otimes R$  and  $\text{Tor}(G'; R)$  vanishes and  $F_p^{(1)}(e^q; e^r) \otimes R = \{v \in \bigwedge^p \mathfrak{t}(\mathbb{Z})/\text{Sed}(e^r) \mid \omega \cdot v = 0\} \otimes R$ . Thus the localisation  $(F_p^{(1)})_{e^q} \otimes R$  is expressed as:

$$\left(F_p^{(1)}\right)_{e^q} \otimes R = \bigoplus_{\substack{p_A+p_B=p \\ p_B < q}} \left(\bigwedge^{p_A} A/\text{Sed}\right)_{e^q} \otimes \left(\bigwedge^{p_B} B/\text{Sed}\right)_{e^q} \otimes R,$$

and is a direct summand of  $(F_p^{(0)})_{e^q} \otimes R$ . □

Before moving on to express the relations between the homologies of the cosheaves  $(F_p^{(1)} \otimes R)_{p \in \mathbb{N}}$  and  $(F_p^{(0)} \otimes R)_{p \in \mathbb{N}}$ , we note that the proof of Proposition 3.5 contains all the necessary ingredients to completely compute the homology of the cosheaves  $(F_p^{(0)} \otimes R)_{p \in \mathbb{N}}$ .

**Proposition 3.7.** *For all ring  $R$  in which  $\delta(P)$  is invertible, and  $p \in \mathbb{N}$ , the only non-trivial homology group of the cosheaf  $F_p^{(0)} \otimes R$  is  $H_p(K; F_p^{(0)} \otimes R)$ . Moreover this module is free of rank  $h_p(P^\circ)$ , the  $p$ -th  $h$ -number of the polar polytope  $P^\circ$  of the simple polytope  $P$ . More precisely:*

$$\text{rk}_R H_p(K; F_p^{(0)} \otimes R) = \sum_{k=0}^p (-1)^{p-k} \binom{n-k}{p-k} f_{n-k}(P),$$

where  $f_k(P)$  denotes the number of  $k$ -faces of  $P$ .



*Proof.* Since  $F_p^{(0)} \otimes R$  is originally defined on the CW-structure induced by the faces of  $P$  we can compute its homology on  $\Pi$  the CW-complex induced on  $P$  by its faces. Note, on the one hand, that from the Definition 3.2 the group  $F_p^{(0)}(Q) = 0$  for all faces  $Q$  of dimension  $q < p$ . Therefore  $H_q(\Pi; F_p^{(0)} \otimes R) = 0$  for all  $q < p$ . On the other hand we have the resolution of Remark 3.1:

$$0 \leftarrow F_p^{(0)} \otimes R \leftarrow \text{Res}_p .$$

This is an acyclic<sup>31</sup> resolution<sup>32</sup>. The  $\text{Res}_{p,q}$ 's are sums of elementary cosheaves  $[\Pi(Q); M]$  for  $Q$  a face of  $P$  and  $M$  a free  $R$ -module. All these sub-complexes  $\Pi(Q)$  are contractible so these cosheaves only have homology in dimension 0. The cosheaf resolution becomes a resolution of chain complexes:

$$0 \leftarrow C_*(\Pi; F_p^{(0)} \otimes R) \leftarrow C_*(\Pi; \text{Res}_{p,0}) \leftarrow C_*(\Pi; \text{Res}_{p,1}) \leftarrow \cdots .$$

Since the complexes  $C_*(\Pi; \text{Res}_{p,q})_{q \leq 0}$  are acyclic it is well known that the homology of the complex  $C_*(\Pi; F_p^{(0)} \otimes R)$  is isomorphic to the homology of the complex:

$$0 \leftarrow H_0(\Pi; \text{Res}_{p,0}) \leftarrow H_0(\Pi; \text{Res}_{p,1}) \leftarrow \cdots .$$

The cosheaves  $\text{Res}_{p,q}$  vanish for all  $q > p$ . Indeed, we have:

$$\text{Res}_{p,q} = \bigoplus_{\substack{Q_{(q)} < P \\ \text{codim } Q_{(q)} = q}} \left[ \Pi(Q_{(q)}); \underbrace{\left( \bigwedge^{p-q} \mathfrak{t}(\mathbb{Z}) / \text{Sed}_{(1)}(Q_{(q)}) \right)}_{=0 \text{ if } q > p} \otimes R \right].$$

Therefore, the only possibly non-trivial homology group of  $F_p^{(0)} \otimes R$  is the  $p$ -th. Using again the resolution to compute this group we see that it coincides with a group of cycles:

$$H_p(\Pi; F_p^{(0)} \otimes R) \cong \ker(\partial: H_0(\Pi; \text{Res}_{p,p}) \rightarrow H_0(\Pi; \text{Res}_{p,p-1})).$$

This boundary  $\partial$  is defined as the tensor product with  $R$  of a map  $\partial_{\mathbb{Z}}: M \rightarrow N$  between two free Abelian groups. Since the image of such  $\partial_{\mathbb{Z}}$  is a free Abelian group,  $H_p(\Pi; F_p^{(0)} \otimes R) \cong \ker(\partial_{\mathbb{Z}}) \otimes R$  is free of rank  $r = \text{rk}_{\mathbb{Z}} \ker(\partial_{\mathbb{Z}})$ . In particular this rank does not depend on the chosen ring  $R$ . Since  $\delta(P)$  is always invertible in  $\mathbb{Q}$  we can compute this rank using the rational coefficients. Because  $F_p^{(0)} \otimes \mathbb{Q}$  can only have non-trivial homology in dimension  $p$  we have:

$$\dim_{\mathbb{Q}} H_p(\Pi; F_p^{(0)} \otimes \mathbb{Q}) = (-1)^p \sum_{k=0}^n (-1)^k \dim_{\mathbb{Q}} H_0(\Pi; \text{Res}_{p,k}^{\mathbb{Q}}) = \sum_{k=0}^p (-1)^{p-k} \binom{n-k}{p-k} f_{n-k}(P) = h_p(P^\circ),$$

denoting by  $f_k(P)$  the number of  $k$ -faces of  $P$  and by  $\text{Res}_{p,k}^{\mathbb{Q}}$  the cosheaves  $\text{Res}_{p,k}$  defined for the ring  $\mathbb{Q}$ . Note that  $f_{k-1}(P^\circ) = f_{n-k}(P)$ .  $\square$

**Theorem 2.** *Let  $R$  be a ring in which both  $\delta(P)$  and  $\theta(K)$  are invertible, the homological morphisms:*

$$i_p: H_q(K; F_p^{(1)} \otimes R) \rightarrow H_q(K; F_p^{(0)} \otimes R),$$

*induced by the inclusions  $i_p: F_p^{(1)} \rightarrow F_p^{(0)}$  are:*

- *isomorphisms for all  $p + q < n - 1$  ;*
- *surjective morphisms for all  $p + q = n - 1$ .*

*Proof.* In the light of Theorem 1, Proposition 3.5, and Proposition 3.6 we can write:

$$H_q(K; F_p^{(k)} \otimes R) \cong H_c^{n-q}(K; H_n((F_p^{(k)} \otimes R)_*)) \text{ for } k \in \{0, 1\}.$$

Note that since  $K$  is finite ( $P$  being compact), cohomology with compact support is the same as cohomology. Let us denote by  $G_p^{(k)}$ ,  $k \in \{0, 1\}$ , the sheaf  $H_n((F_p^{(k)} \otimes R)_*)$ . The cosheaf inclusion  $i_p: F_p^{(1)} \rightarrow F_p^{(0)}$  induces a

<sup>31</sup>A cosheaf is acyclic if it has trivial homology in dimension at least 1.

<sup>32</sup>This is even a projective resolution, c.f. [She85] in the dual setting of cellular sheaves.

morphism of sheaves  $j_p : G_p^{(1)} \rightarrow G_p^{(0)}$ . Since  $K$  has dimension  $n$ , the long exact sequence in homology associated with:

$$0 \rightarrow F_p^{(1)} \otimes R \xrightarrow{i_p} F_p^{(0)} \otimes R \rightarrow \text{coker}(i_p) \rightarrow 0,$$

implies that  $j_p$  is also injective. We have the following commutative square relating the homological and cohomological counterparts of  $i_p$  and  $j_p$ :

$$\begin{array}{ccc} H_q(K; F_p^{(1)} \otimes R) & \xrightarrow{i_p} & H_q(K; F_p^{(0)} \otimes R) \\ \cong \uparrow & & \uparrow \cong \\ H^{n-q}(K; G_p^{(1)}) & \xrightarrow{j_p} & H^{n-q}(K; G_p^{(0)}) \end{array}$$

The vertical isomorphisms are induced by the quasi-isomorphisms given by the second part of Theorem 1. The commutativity is already satisfied on the level of chain and cochain complexes:

$$\begin{array}{ccc} \Omega_q(K; F_p^{(1)} \otimes R) & \xrightarrow{i_p} & \Omega_q(K; F_p^{(0)} \otimes R) \\ \text{q.i.} \uparrow & & \uparrow \text{q.i.} \\ \Omega_{n-q,n}(K; F_p^{(1)} \otimes R) \cap \ker(\partial_2) & \xrightarrow[\text{q.i.}]{\text{rest.}} & \Omega_{n-q,n}(K; F_p^{(0)} \otimes R) \cap \ker(\partial_2) \\ \parallel & & \parallel \\ E_{n-q,n}^1 & & E_{n-q,n}^1 \\ \Phi \uparrow & & \uparrow \Phi \\ C^{n-q}(K; G_p^{(1)}) & \xrightarrow{j_p} & C^{n-q}(K; G_p^{(0)}) \end{array}$$

The two vertical isomorphisms in the previous diagram are the homological and cohomological counterparts of compositions of the isomorphism  $\Phi$  and the quasi-isomorphic inclusions of Proposition 2.1. For  $F$  either  $F_p^{(1)} \otimes R$  or  $F_p^{(0)} \otimes R$  and  $G$  respectively designating  $G_p^{(0)}$  or  $G_p^{(1)}$ , we have:

$$\begin{aligned} \Omega_{n-q,n}(K; F) &= \bigoplus_{e^{n-q} \leq e^n} F(e^{n-q}; e^n) \otimes \mathbb{Z}(e^{n-q}; e^n), \\ \text{and } \Omega_{n-q,n-1}(K; F) &= \bigoplus_{e^{n-q} \leq e^{n-1}} F(e^{n-q}; e^n) \otimes \mathbb{Z}(e^{n-q}; e^{n-1}). \end{aligned}$$

In this description, both these groups have a splitting indexed by the  $(n-q)$ -cells of  $K$ . Both the morphisms  $i_p$  and  $\partial_2$  respect these splittings which explains the commutativity of the upper square of the last diagram. Also in that setting, the value  $\text{Hom}(\mathbb{Z}(e^{n-q}; e^n), G(e^{n-q}))$ , on some cell  $e^{n-q}$ , is, modulo the action of the isomorphism  $\Phi$ , the kernel of  $\partial_2$  restricted to the  $(e^{n-q})$ -component of  $\Omega_{n-q,n}(K; F)$ . By construction,  $j_p$  is the restriction of  $i_p$  to the  $(e^{n-q})$ -component of the kernel of  $\partial_2$ , so the bottom square also commutes.

Let us now prove that whenever  $r > p$  the map  $j_p(e^r) : G_p^{(1)}(e^r) \rightarrow G_p^{(0)}(e^r)$  is an isomorphism for all  $r$ -cells  $e^r$ . This implies that the cokernel of  $j_p$  is trivial in dimension greater than  $p$ . By means of the long exact sequence induced in cohomology by the injective morphism of sheaves  $j_p$ , we find that  $j_p : H^r(K; G_p^{(1)}) \rightarrow H^r(K; G_p^{(0)})$  is surjective for  $r = p + 1$  and invertible for  $r > p + 1$ . Theorem 2 follows after performing the change of variables  $r = n - q$ .

Let  $e^r$  be a cell of  $K$ . We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_p^{(1)}(e^r) & \longrightarrow & \bigoplus_{e^n \geq e^r} F_p^{(1)}(e^r; e^n) \otimes R \otimes \mathbb{Z}(e^n) & \xrightarrow{\partial} & \bigoplus_{e^{n-1} \geq e^r} F_p^{(1)}(e^r; e^{n-1}) \otimes R \otimes \mathbb{Z}(e^{n-1}) \\ & & \downarrow j_p(e^r) & & \downarrow \bigoplus i_p(e^r; e^n) & & \downarrow \bigoplus i_p(e^r; e^{n-1}) \\ 0 & \longrightarrow & G_p^{(0)}(e^r) & \longrightarrow & \bigoplus_{e^n \geq e^r} F_p^{(0)}(e^r; e^n) \otimes R \otimes \mathbb{Z}(e^n) & \xrightarrow{\partial} & \bigoplus_{e^{n-1} \geq e^r} F_p^{(0)}(e^r; e^{n-1}) \otimes R \otimes \mathbb{Z}(e^{n-1}) \end{array}$$

Furthermore, for all cells  $e^q \geq e^r$  we have the exact sequence:

$$0 \longrightarrow F_p^{(1)}(e^r; e^q) \otimes R \xrightarrow{i_p(e^r; e^q)} \underbrace{\bigwedge^p \mathfrak{t}(\mathbb{Z}) / \text{Sed}(e^q)}_{F_p^{(0)}(e^r; e^q) \otimes R} \otimes R \xrightarrow{\omega} \bigwedge^{p-r} \mathfrak{t}(\mathbb{Z}) / \text{Sed}(e^q) \otimes R,$$

with the last morphism given by the contraction against  $\omega$  a generator of  $\bigwedge^r (Te^r \cap \mathfrak{t}^*(\mathbb{Z}))$ . By definition the contraction of the  $r$ -form  $\omega$  against a  $p$ -vector is 0 whenever  $r > p$ . Therefore, in this case  $i_p(e^r; e^q)$  is the identity for all pairs  $e^r < e^q$ . Consequently,  $\bigoplus i_p(e^r; e^{n-1})$  and  $\bigoplus i_p(e^r; e^n)$  are also the identity and so is  $j_p(e^r)$ .  $\square$

From the last theorem and Definition 3.3 we deduce the following corollary:

**Corollary 3** (Lefschetz Hyperplane Section Theorem). *Let  $Y$  be the orbifold tropical variety associated with the simple polytope  $P$  and  $X$  be a tropical hypersurface of  $Y$  dual to a convex integer polyhedral subdivision  $K$  of  $P$ . For every ring  $R$  in which both  $\delta(P)$  and  $\theta(K)$  are invertible, the homological morphisms:*

$$i_{p,q}: H_{p,q}(X; R) \rightarrow H_{p,q}(Y; R),$$

induced by the inclusions  $i_p: F_p^{(1)} \rightarrow F_p^{(0)}$ ,  $p \in \mathbb{N}$ , are:

- isomorphisms for all  $p + q < n - 1$  ;
- surjective morphisms for all  $p + q = n - 1$ .

The hypothesis on the coefficients cannot be dropped in Theorem 2 or Corollary 3 as the following two examples show where Theorem 2 fails for  $R = \mathbb{Z}$  because either  $\delta(P) \neq 1$  or  $\theta(K) \neq 1$ .

**Examples.** 1. Let the triangle  $T$  be the convex hull of 0,  $dx$ , and  $2dy$  in  $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2; \mathbb{R})$  endowed with the canonical lattice  $\mathbb{Z}dx + \mathbb{Z}dy$ , c.f. Figure 14a. On this polytope we consider its unique unimodular triangulation  $K$  depicted in Figure 14a. We have  $\delta(P) = 2$  and  $\theta(K) = 1$  and the groups  $H_q(K; F_p^{(1)})$  and  $H_q(K; F_p^{(0)})$  are given, along with some of the homological morphisms induced by inclusion, in the Figure 14c. In this example, the morphism  $i_{1,0}: H_1(K; F_0^{(1)}) \rightarrow H_1(K; F_0^{(0)})$  is not surjective.

2. Consider the cube with edges of length 2 in  $\text{Hom}_{\mathbb{R}}(\mathbb{R}^3; \mathbb{R})$ , endowed with its canonical lattice  $\mathbb{Z}dx + \mathbb{Z}dy + \mathbb{Z}dz$ , and subdivided by  $K$  as in Figure 15a. Its associated tropical surfaces are singular and we have  $\delta(P) = 1$  and  $\theta(K) = 2$ . Here the map  $i_{1,1}$  is not surjective. Its image is  $\mathbb{Z}(e_1 + e_2) + \mathbb{Z}(e_1 - e_2) + \mathbb{Z}e_3$  for  $e_1, e_2, e_3$  the standard basis of  $H_1(K; F_1^{(0)})$  given by the three independent copies of  $\mathbb{P}^1$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , the toric variety associated with the cube.

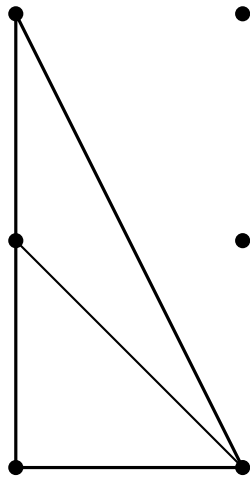
We would like to conclude few remarks on the numbers  $\delta(P)$  and  $\theta(K)$  and the definition of the cosheaves  $(F_p^{(1)})_{p \in \mathbb{N}}$ . From its definition the number  $\delta(P)$  equals 1 if and only if the toric variety associated with  $P$  is smooth. Therefore, assuming  $\delta(P) = 1$  puts us closer to the tropical Lefschetz hyperplane section theorem of C. Arnal, A. Renaudineau and K. Shaw [ARS21]<sup>33</sup>. However, assuming  $\delta(P) = \theta(K) = 1$  does not implies that  $X$  is smooth. The hypersurface  $X$  is said to be smooth when  $K$  is an unimodular triangulation and in this case  $\theta(K) = 1$ . Therefore, assuming both  $\delta(P)$  and  $\theta(K)$  to be 1 already includes in the statement some singular hypersurfaces. For a general polytope  $Q$ , the number  $\theta(Q)$  seems difficult to compute. However, it seems computable for simplices. For segments it is 1. For a triangle  $T$ , a direct computation yields:

$$\theta(T) = \frac{2 \cdot \text{vol}_{\mathbb{Z}}(T) \cdot \text{GCD}\{\text{vol}_{\mathbb{Z}}(E) : E \text{ edge of } T\}}{\prod_{E \leq T} \text{vol}_{\mathbb{Z}}(E)},$$

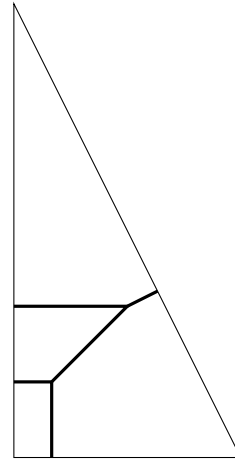
where  $\text{vol}_{\mathbb{Z}}(Q)$  for an integer polytope  $Q$  is its integer volume, i.e. its Lebesgue measure in the affine sub-space it spans normalised so that a parallelogram on a basis of the induced lattice has measure 1.

When  $\theta(K)$  equals 1, Lemma 3.4 and Definition 3.5 describe the cosheaf  $\bigoplus_{p \in \mathbb{N}} F_p^{(1)}$  as the kernel of a contraction. When  $\theta(K)$  is greater than 1 the latter is the saturation of the former. Theorem 2 suggests that if we alternatively defined the the cosheaf  $\bigoplus_{p \in \mathbb{N}} F_p^{(1)}$  as the kernel of a contraction then every tropical hypersurface

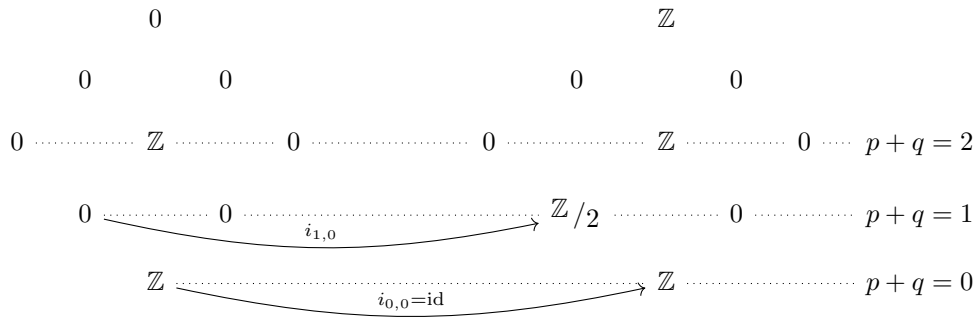
<sup>33</sup>C. Arnal, A. Renaudineau and K. Shaw. *Lefschetz Section Theorems for Tropical Hypersurfaces*, Theorem 1.2 p.1349.



(a) The triangle  $T$  associated with the weighted projective space  $\mathbb{P}(1, 1, 2)$  and its subdivision  $K$  into two unimodular triangles.



(b) A tropical curve whose dual subdivision is  $K$ .

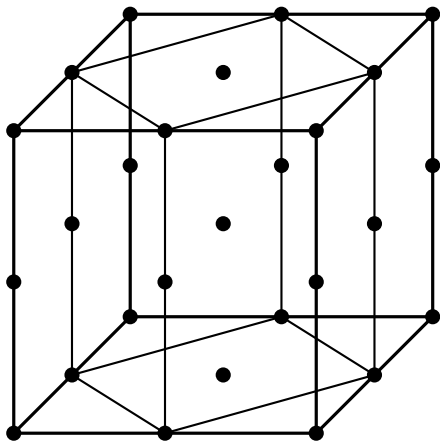


(c) The Hodge diamond of the curve on the left and the Hodge diamond of  $\mathbb{P}(1, 1, 2)$  on the right. The parameter  $p$  increases in the north-east direction ( $\nearrow$ ) and  $q$  in the north-west direction ( $\nwarrow$ ).

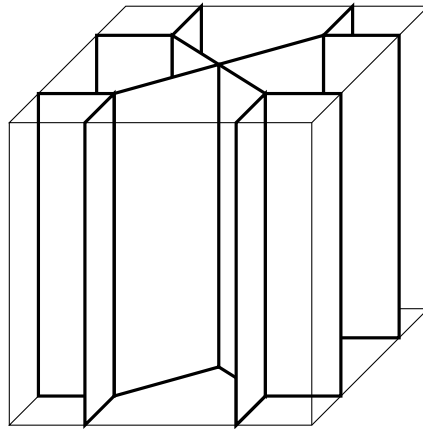
Figure 14: A curve in the weighted projective plane  $\mathbb{P}(1, 1, 2)$  whose integral tropical homology don't satisfies the tropical version of the Lefschetz hyperplane section theorem.

dual to a polyhedral subdivision (combinatorially ample in the terminology of [ARS21]) in a projective non-singular tropical toric variety would satisfy the tropical Lefschetz hyperplane section theorem with integral coefficients.

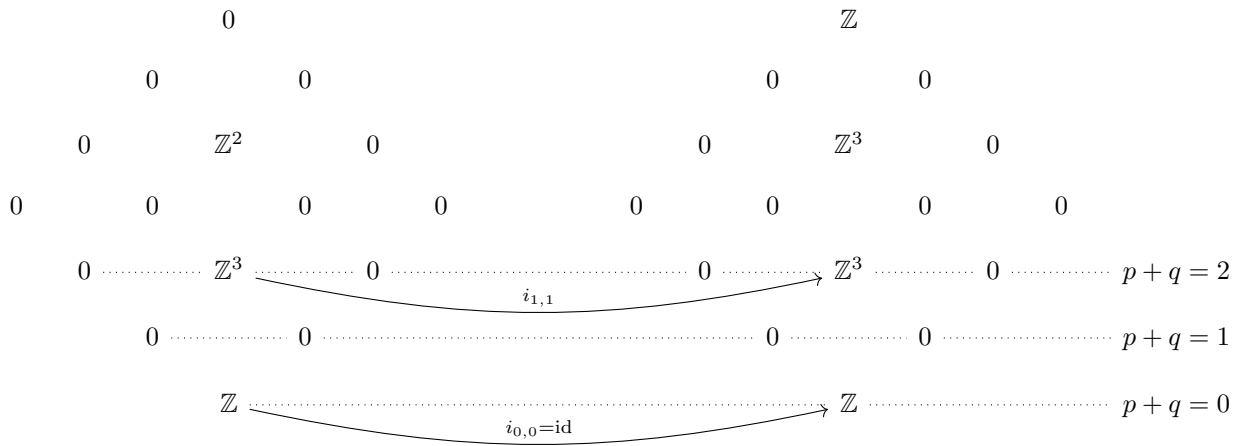
Finally, we want to acknowledge that even if the Lefschetz hyperplane section theorem with rational coefficients is a consequence of the Hard Lefschetz theorem with rational coefficients is it usually not the case when one considers coefficient rings of positive characteristic. For instance a product of at least three copies of  $\mathbb{P}^1$  does not have any cohomology class over  $\mathbb{F}_2$ , the field with two elements, satisfying the Hard Lefschetz property. However the Lefschetz hyperplane theorem with  $\mathbb{F}_2$  coefficient remains valid for any non-singular hypersurface of this product.



(a) A convex subdivision  $K$  of the cube.



(b) A singular tropical surface of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  dual to the subdivision  $K$ .



(c) The Hodge diamond of the surface on the left and the Hodge diamond of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  on the right. The parameter  $p$  increases in the north-east direction ( $\nearrow$ ) and  $q$  in the north-west direction ( $\nwarrow$ ).

Figure 15: A singular surface in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  whose integral tropical homology does not satisfy the tropical version of the Lefschetz hyperplane section theorem.

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