Some Systemic Risk Indicators

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Abstract. This paper aims to introduce novel systemic risk indicators based on risk allocation methods employed in actuarial science. We present diverse general approaches for constructing these indicators and utilize them to derive indicators based on commonly used risk measures such as Value at Risk, Tail Value at Risk, and Expectiles. Furthermore, we analyze the influence of the dependence structure on the behavior of these indicators using a range of copula models. To support our findings, we provide numerical illustrations.

Introduction

Since the last global financial crisis, the concept of systemic risk has become extremely important in finance. The subprime financial crisis highlighted the need to improve understanding and modeling of systemic risk. Financial institutions are considered systemically dangerous if their bankruptcy situations have the potential to damage the entire financial system. The bankruptcy of Lehman Brothers on September 15, 2018, and its impact on the world financial system serve as a perfect illustration of the danger of systemic risk, emphasizing the importance of its modeling and management.

Several systemic risk measures have been proposed in the literature since the last financial crisis. The marginal expected shortfall, denoted as MES, is a systemic risk measure introduced by Acharya et al. (2010) [3], and another approach to MES was presented by Acharya et al. (2012) [2]. It is defined for two random variables X and Y as follows:

$$MES_\alpha(Y \mid X) = \mathbb{E}[Y \mid X \geq VaR_\alpha(X)].$$

We recall the definition of the *Value at Risk* (VaR) as a risk measure of a random variable X at the confidence level α:

$$VaR_\alpha(X) = F_X^{-1}(\alpha) := \inf\{x \in \mathbb{R} | F_X(x) \geq \alpha\},$$

where F_X is the distribution function of X.

Another measure of systemic risk, $\Delta CoVaR$, was introduced by Adrian and Brunnermeier (2011) [4]. Mainik and Schaanning (2014) [24] provided explicit formulas for $\Delta CoVaR$ in particular cases and presented a new approach to this risk measure that takes into account the dependence between risks. Its definition is given by:

$$\Delta CoVaR_{\alpha,\beta}(Y \mid X) = CoVaR^{=}_{\alpha,\beta}(Y \mid X) - VaR_\beta(Y),$$

where

$$CoVaR^{=}_{\alpha,\beta}(Y \mid X) = VaR_\beta(Y \mid X = VaR_\alpha(X)).$$

This measure was also studied in Löffler and Raupach (2013) [23] and Castro and Ferrari (2014) [12].

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Similarly, Mainik and Schaanning (2014) [24] defined the *Conditional Expected Shortfall* (CoES), based on the definition of the Expected Shortfall risk measure:

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du.$$

The CoES is defined as:

$$CoES^{=}_{\alpha,\beta}(Y \mid X) = \frac{1}{1-\beta}\int_\beta^1 CoVaR^{=}_{\alpha,u}(X)du.$$

A second version is given by:

$$CoES_{\alpha,\beta}(Y \mid X) = \frac{1}{1-\beta}\int_\beta^1 CoVaR_{\alpha,u}(X)du,$$

where

$$CoVaR_{\alpha,\beta}(Y \mid X) = VaR_\beta(Y \mid X \geq VaR_\alpha(X)).$$

Other measures based on financial constructions have been introduced in the literature. For example, the tail risk gamma, presented by Knaup and Wagner (2012) [22], utilizes the sensitivity of an institution's equity return to changes in put options on system prices.

In actuarial science, the issue of capital allocation is well known and studied. Once the solvency capital requirement is calculated using risk aggregation methodologies, it must be distributed among different branches or business lines. Capital allocation is then an exercise of redistributing capital based on marginal risk levels measured using a top-down approach.

The starting point of this paper is the remarkable similarity between measuring systemic risk in a financial context and allocating economic capital in insurance. The goal of this work is to introduce general forms of systemic risk indicators. Several indicators can then be derived from these forms using different risk measures.

This paper is organized as follows: in the first section, we explain the methodology used to construct our indicators, utilizing various capital allocation methods. In section 2, we derive systemic risk indicators using the Euler method and common risk measures such as VaR, TVaR, and Expectiles. Section 3 presents the expressions for the obtained systemic risk indicators in dependence models based on copulas. Finally, section 4 is dedicated to numerical illustrations.

1. Modeling systemic risk using capital allocation methods

In the insurance industry, capital allocation plays a crucial role for multi-branch companies. Once the solvency capital requirement is calculated using risk aggregation methodologies, the task at hand is to determine the actual contribution of each branch to the overall risk level. This essentially involves distributing the benefits of diversification among different business lines, taking into account the marginal distribution and dependence structure. Similarly, in the context of systemic risk, the ultimate goal remains the same: quantifying the impact of individual components on the overall risk level of the system. This is why we believe that capital allocation methods can be utilized as effective tools for modeling systemic risk.

1.1. **Methodology.**

We assume that our financial system consists of d institutions. Let $X = (X_1, \ldots, X_d)$ be a random vector, where X_i represents the risk associated with the i^{th} financial institution, such as the P&L (Profit and Loss). From a risk perspective, we consider the random variables X_i to be positive in the case of losses and negative in the case of profits. The aggregate risk, denoted by S, is defined as the sum of all individual risks: $\sum_{k=1}^d X_k$, representing the overall

risk of the system.
In general, we can express the aggregate risk as $\sum_{k=1}^{d} \alpha_k X_k$, where α_i represents the marginal weight assigned to each individual risk in the aggregate risk calculation.

From a local perspective, it is possible to select a univariate risk measure to quantify each risk independently of the rest of the financial system. In this case, the contribution of a risk X_i can be assessed using the following quantity:

$$\omega_i^{loc} = \frac{\rho(X_i)}{\sum_{k=1}^{d} \rho(X_k)},$$

where ρ represents the chosen univariate risk measure.
This approach overlooks the interdependence among the system's components and assumes, for the purpose of risk aggregation, that they are comonotonic. Comonotonic risks refer to cases of perfect dependence and have been studied by Hoeffding (1940) [19] and Fréchet (1951) [17]. The definition of comonotonic risks as first introduced in actuarial literature by Borch (1962) [11] is adopted here.
A vector of random variables $(X_1, X_2, \ldots, X_n)$ is considered comonotonic if and only if there exists a random variable Y and non-decreasing functions $\varphi_1, \ldots, \varphi_n$ such that:

$$(X_1, \ldots, X_n) \overset{d}{=} (\varphi_1(Y), \ldots, \varphi_n(Y)).$$

When the risks $X_1, \ldots, X_d$ are comonotonic, there exists a uniform random variable U such that $X_i = F_{X_i}^{-1}(U)$ for all $i \in 1, \ldots, d$, and $S = \sum_{i=1}^{d} F_{X_i}^{-1}(U) = \varphi(U)$, where $\varphi(t) = \sum_{i=1}^{d} F_{X_i}^{-1}(t)$ and φ is a non-decreasing function.
When the chosen risk measure ρ satisfies the property of additive-comonotonicity, we have:

$$\rho(S) = \sum_{i=1}^{d} \rho(X_i).$$

In this case, ω_i^{loc} represents the marginal contribution of X_i to the aggregated risk.

Now, by adopting a multivariate approach and utilizing capital allocation methods, we can obtain a more comprehensive understanding of the risk associated with each component of the financial system, considering their dependence interactions. Let $\rho(X_i | \sum_{k=1}^{d} X_k)$ represent the allocation of risk X_i within a capital allocation framework based on the amount $\rho(\sum_{k=1}^{d} X_k)$. In this context, the actual contribution of X_i to the system's overall risk can be expressed as:

$$\omega_i^{sys} = \frac{\rho(X_i | \sum_{k=1}^{d} X_k)}{\rho(\sum_{k=1}^{d} X_k)} = \frac{\rho(X_i|S)}{\rho(S)}.$$

This ratio, ω_i^{sys}, quantifies the relative significance of X_i with respect to the aggregated risk of the system, taking into account the dependence structure.

In this paper, our objective is to quantitatively assess the systemic risk associated with each component of a financial system using the following quantity:

$$SRI_{\rho}(X_i|S) = \omega_i^{loc} - \omega_i^{sys} = \frac{\rho(X_i)}{\sum_{k=1}^{d} \rho(X_k)} - \frac{\rho(X_i|S)}{\rho(S)}.$$

It is important to note that the measures chosen for ω_i^{loc} and ω_i^{sys} can be different. In such cases, the systemic risk indicator is defined as follows:

$$SRI_{\rho,\Pi}(X_i|S) = \frac{\rho(X_i)}{\sum_{k=1}^{d} \rho(X_k)} - \frac{\Pi(X_i|S)}{\Pi(S)}. \tag{1.1}$$

The systemic risk is measured by comparing the weight of the individual risk component as evaluated independently within the financial system (locally) to its risk weight considering the dependence structure within the system (systemically).

When the SRI value is negative, it indicates that the branch is systemic, and its systemic risk level is underestimated. Conversely, when the SRI value is positive, it indicates an overestimation of systemic risk. The systemic risk level is inversely related to the SRI indicator, meaning that as the SRI value increases, the systemic risk decreases.

In the case where we choose to use the same risk measure ($\rho = \Pi$), the resulting risk indicator may inherit certain properties of the selected risk measure. It is worth noting that for full allocation principles (allocations that satisfy $\rho(S) = \sum_{i=1}^{d} \rho(X_i|S)$), the sum of marginal systemic risk indicators is indeed zero. This means that the system as a whole is not inherently risky. Instead, it is the interactions and interdependencies among its components that give rise to systemic risk.

1.2. **Incremental indicator.** The incremental allocation method is grounded in measuring the marginal impact of each individual risk on the overall risk. Utilizing this fundamental actuarial technique, we define the following risk indicator:

Definition 1.1 (The incremental systemic risk indicator)**.** Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a non-negative random vector in $\mathbb{R}^d$, and ρ be a risk measure. The incremental systemic risk indicator of X_i with respect to ρ is given by:

$$SRI_{\rho}^{Inc}(X_i|S) = \frac{\rho(X_i)}{\sum_{\ell=1}^{d} \rho\left(X_\ell\right)} - \frac{\rho(S) - \rho(S - X_i)}{\sum_{\ell=1}^{d} \rho(S) - \rho(S - X_\ell)},$$

where $S = \sum_{i=1}^{d} X_i$.

The main drawback of the incremental method is that it does not account for the dependence structure among the risks $X_{i,i=1,\ldots,d}$ or between each risk and a subset of the others. It solely relies on the marginal contribution of each risk to the overall sum of risks.

1.3. **Shapley's indicator.** In this method, the Systemic Risk (SR) index is derived using the Shapley capital allocation approach. We begin by revisiting the principle of the Shapley method for capital allocation and subsequently introduce a Shapley indicator for measuring systemic risk.

The Shapley allocation method finds its roots in cooperative game theory and a detailed description can be found in the paper by Micheal Denault [13]. The Shapley value was originally introduced by Shapley (1953) [30] and has been studied by various researchers, such as Aumann and Myerson (1988) [5] and Winter (2002) [36]. While initially used to allocate total costs in cooperative games, this method can be effectively adapted to address capital allocation problems within a multivariate context. In this context, we consider risks as players within the framework of a cooperative game. The chosen univariate risk measure, denoted as ρ, serves as its characteristic function, which is defined for any coalition $Z \subset \{1, \ldots, d\}$ as $\rho\left(\sum_{i \in Z} X_i\right)$.

The Shapley method provides us with the contribution of each risk X_i to the overall risk, given by the Shapley value:

$$\omega_i(\rho) = \sum_{\substack{Z \subset D \\ i \in Z}} \frac{(d-z)!\,(z-1)!}{d!} \left[\rho\left(\sum_{j \in Z} X_j\right) - \rho\left(\sum_{j \in Z \setminus \{i\}} X_j\right)\right],$$

where $D = \{1, \ldots, d\}$ and $z = \text{card}(Z)$. It is important to note that, for any choice of risk measure ρ, the Shapley allocation constitutes a full allocation, satisfying $\sum_{i=1}^{n} \omega_i = \Pi\left(\sum_{i=1}^{n} X_i\right)$. This property arises from the Pareto optimality axiom fulfilled by the Shapley value.

Definition 1.2 presents the systemic risk indicator derived from the Shapley risk allocation method.

Definition 1.2 (Shapley's systemic risk indicator)**.** Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a non-negative random vector in $\mathbb{R}^d$, and ρ be a risk measure. The Shapley's systemic risk indicator of X_i according to ρ is given by:

$$SRI_{\rho}^{Shapley}(X_i|S) = \frac{\rho(X_i)}{\sum_{\ell=1}^{d} \rho\left(X_\ell\right)} - \sum_{\substack{Z \subset D \\ i \in Z}} \frac{(d-z)!\,(z-1)!}{d!} \frac{\rho\left(\sum_{\ell \in Z} X_\ell\right) - \rho\left(\sum_{\ell \in Z \setminus \{i\}} X_\ell\right)}{\rho\left(\sum_{\ell=1}^{d} X_\ell\right)},$$

where $D = \{1, \ldots, d\}$ and $z = \text{card}(Z)$.

The Shapley indicator can be considered an improvement over the incremental method as it not only considers the marginal impact of each risk X_i on the overall risk but also incorporates its marginal impact on the risk of all subsets containing it. However, the primary drawback of this indicator is its computational cost, which grows exponentially with the number of components, approximately on the order of 2^d.

1.4. **Euler's indicator.** The Euler's capital allocation method has been extensively studied in the works of Tasche (2007) [33] and Tasche (2008) [34]. This technique is based on the concept of allocating capital according to the infinitesimal marginal impact of each risk, which represents the decrease in overall risk achieved by an infinitely small decrement in risk X_i. It assumes that risks can be infinitely divided.

Given an initial vector $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we consider a portfolio with a value of $X(\mathbf{x}) = \sum_{i=1}^{d} x_i X_i$. For a univariate risk measure ρ, we define the function $f_\rho(\mathbf{x}) = \rho(X(\mathbf{x}))$, assuming that f_ρ is continuously differentiable. We denote by $\rho(X_i|S)$ the contribution of risk X_i to the overall risk. This contribution can be obtained using Euler's principle:

$$\rho(X_i|S) = \lim_{h \to 0} \frac{\rho(S) - \rho(S - hX_i)}{h} = \frac{\partial f_\rho}{\partial x_i}(1, \ldots, 1).$$

If the risk measure ρ is positively homogeneous, Euler's theorem states:

$$f_\rho(\mathbf{x}) = \sum_{i=1}^{d} x_i \frac{\partial f_\rho}{\partial x_i}(\mathbf{x}),$$

which ensures that capital allocation based on Euler's principle becomes a full allocation by construction:

$$\rho(S) = \rho\left(\sum_{i=1}^{d} X_i\right) = f_\rho(1, ..., 1) = \sum_{i=1}^{d} \frac{\partial f_\rho}{\partial x_i}(1, ..., 1) = \sum_{i=1}^{d} \rho(X_i|S).$$

Euler's method has been extensively studied in the allocation literature, examining its properties such as coherence and compatibility with Return on Risk-Adjusted Capital (RORAC), under different assumptions. Notable works include Balog (2011) [6], Tasche (2000) [31], and Tasche (2004) [32]. The economic interpretation of Euler's method makes it a relevant approach to capital allocation and explains its popularity as an actuarial practice.

Utilizing Euler's principle, we define a systemic risk indicator as follows:

Definition 1.3 (Euler's systemic risk indicator)**.** Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a non-negative random vector in $\mathbb{R}^d$, and ρ be a positively homogeneous risk measure. The Euler's systemic risk indicator of X_i according to ρ is given by:

$$SRI_{\rho}^{Euler}(X_i|S) = \frac{\rho(X_i)}{\sum_{\ell=1}^{d} \rho\left(X_\ell\right)} - \lim_{h\to 0} \frac{\rho\left(\sum_{\ell=1}^{d} X_\ell\right) - \rho\left(\sum_{\ell=1}^{d} X_\ell - hX_i\right)}{\rho\left(\sum_{\ell=1}^{d} X_\ell\right) h}. \tag{1.2}$$

Starting from the upcoming section, our focus will be solely on examining indicators constructed using the Euler method.

1.5. **Other constructions.** We can define additional measures of systemic risk based on various capital allocation methods found in the actuarial literature. For example, we mention the optimal allocation method presented by Dhaene et al. (2012) [14] and studied by Maume-Deschamps et al. (2016) [26]. In this case, the systemic risk (SR) index is obtained as follows:

$$SRI_{opt} = \frac{\rho(X_i)}{\sum_{\ell=1}^{d} \rho\left(X_\ell\right)} - \frac{< \mathbf{x}, \mathbf{e_i} >}{\rho\left(\sum_{\ell=1}^{d} X_\ell\right)},$$

where $< \mathbf{x}, \mathbf{e_i} >$ denotes the i^{th} coordinate of $\mathbf{x}$, and

$$\mathbf{x} \in \underset{\mathbf{v}\in\mathcal{U}_u^d}{\arg\inf}\ \mathbb{E}\left[S\left(\mathbf{X}, v\right)\right],$$

with $S : \mathbb{R}^{+d} \times \mathbb{R}^{+d} \to \mathbb{R}^+$ being a scoring function, and $\mathcal{U}_u^d = \{\mathbf{v} \in [0,u]^d, \sum_{i=1}^d v_i = u\}$ where $u = \rho\left(\sum_{\ell=1}^d X_\ell\right)$.

Other systemic risk measures can be constructed using the concept of cointegration, which was introduced by Engel and Granger (1983) as a measure of dependence between time series (see Engle and Granger (1987) [16]). The adjustment coefficient defined in ruin probabilities can also be utilized to quantify systemic risk.

The indicators constructed using these methods can be utilized in either static or dynamic form, depending on specific requirements. They can serve as valuable decision support tools for risk control authorities, enabling them to establish tolerance thresholds for the systemic level of a component within the financial system and take appropriate actions when these thresholds are exceeded. For instance, these authorities may request capital increases in the form of safety margins. When the objective is to minimize marginal systemic risk, the precise systemic safety margin can be determined by solving the following equation:

$$SRI_{\rho}(X_i + a|S + a) = 0.$$

The choice of risk measures utilized for constructing these indicators is of utmost importance. It allows for the modeling of systemic risk aversion and provides an economic interpretation of the resulting indicators. Standard risk measures are typically the preferred choices for these purposes. In the following section, we will focus on indicators derived from such measures.

2. Systemic risk indicators using usual risk measures

In this section, our focus is on the Euler's systemic risk indicator defined in 1.3. For the incremental and Shapley's indicators, we only need the expression of the chosen risk measure since these indicators are composed of aggregated risk measures. However, in the case of Euler's version, the expression is not straightforward. The objective of this section is to provide the expressions of the Euler's systemic risk indicator for common risk measures such as Value-at-Risk, Tail Value-at-Risk, and Expectiles.

2.1. **Covariance.**

A straightforward choice for the allocation risk measure is variance. Lemma 2.1 provides the expression of the covariance-based systemic risk indicator.

Lemma 2.1 (Covariance systemic risk indicator)**.** *Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a non-negative random vector in $\mathbb{R}^d$, and ρ be a risk measure. The covariance systemic risk indicator of X_i is given by:*

$$SRI_{Cov}(X_i|S) = \frac{Var(X_i)}{\sum_{k=1}^{d} Var(X_k)} - \frac{Cov(X_i, S)}{Var(S)}.$$

Proof. In this case,

$$f_{Var}(u) = Var(X(u)) = Var\left(\sum_{i=1}^{d} u_i X_i\right) = \sum_{i=1}^{d} u_i^2 Var(X_i) + 2 \sum_{1 \leq i < j \leq d} u_i u_j Cov(X_i, X_j).$$

Using Euler's method, the risk contribution of X_i is given by:

$$Var(X_i|S) = \frac{\partial f_{Var}}{\partial u_i}(1, ..., 1).$$

Differentiating f_{Var} with respect to u_i, we obtain:

$$\frac{\partial f_{Var}}{\partial u_i}(u) = 2u_i Var(X_i) + 2 \sum_{j=1, j \neq i}^{d} u_j Cov(X_i, X_j),$$

thus,

$$Var(X_i|S) = 2Var(X_i) + 2 \sum_{j=1, j \neq i}^{d} Cov(X_i, X_j) = 2Cov(X_i, S),$$

and in order to achieve a full allocation ($\sum_{i=1}^{d} Var(X_i|S) = Var(S)$), we normalize this expression to obtain the risk contribution of each X_i as:

$$Var(X_i|S) = Cov(X_i, S).$$

Finally, from Definition 1.3, we deduce the construction of the given covariance systemic risk indicator:

$$SRI_{Cov}(X_i|S) = \frac{Var(X_i)}{\sum_{k=1}^{d} Var(X_k)} - \frac{Cov(X_i, S)}{Var(S)}.$$

□

However, it is important to note that variance is not a tail risk measure and not a coherent risk measure. The use of covariance limits the ability to incorporate dependence beyond linear forms.

2.2. **Wang's risk measures.** Recall the definitions of two commonly used risk measures: Value at Risk (VaR) and Tail Value at Risk (TVaR). The VaR risk measure at level α is defined for a random variable X as:

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq \alpha\} = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\} = F_X^{-1}(\alpha),$$

which represents the quantile of the same level. The TVaR at level α is defined as the mean of the VaRs exceeding $VaR_\alpha(X)$:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\mu(X)\mathrm{d}\mu.$$

Well-known VaR and TVaR-based allocation rules are examples of methods obtained from this approach. They are specific cases of capital allocation using Wang's risk measures introduced in [35], and can be defined as:

$$\rho(X) = \int_0^1 VaR_\alpha(X)\ \mathrm{dg}(\alpha),$$

where g is an increasing distortion function satisfying $g(0) = 0$ and $g(1) = 1$. These measures are homogeneous, translation-invariant, and monotone.

Lemma 2.2 provides a general expression for the VaR-based systemic risk indicator.

Lemma 2.2 (VaR systemic risk indicator)**.** *Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a non-negative random vector in $\mathbb{R}^d$. The VaR systemic risk indicator of X_i is given by:*

$$SRI_{VaR_\alpha}(X_i|S) = \frac{VaR_\alpha(X_i)}{\sum_{\ell=1}^{d} VaR_\alpha(X_\ell)} - \frac{\mathbb{E}[X_i|S = VaR_\alpha(S)]}{VaR_\alpha(S)}. \tag{2.1}$$

Proof. Tasche (2000) [31] provides an expression for the derivative of the quantile function, given by:

$$\frac{\partial VaR_\alpha}{\partial x_i}(X(\mathbf{x})) = \mathbb{E}[X_i|X(\mathbf{x}) = VaR_\alpha(X(\mathbf{x}))],\ \forall \mathbf{x} \in \mathbb{R}^d.$$

From this, we directly obtain the expression of $\frac{\partial f_{VaR_\alpha}}{\partial x_i}(\mathbf{x})$:

$$\frac{\partial f_{VaR_\alpha}}{\partial x_i}(\mathbf{x}) = \mathbb{E}\left[X_i|X(\mathbf{x}) = VaR_\alpha\left(X(\mathbf{x})\right)\right], \forall \mathbf{x} \in \mathbb{R}^d.$$

The risk contribution of each risk in the overall one, according to the VaR-based allocation rule, is given by:

$$VaR_\alpha(X_i|S) = \frac{\partial f_{VaR_\alpha}}{\partial x_i}(1, ..., 1) = \mathbb{E}[X_i|S = VaR_\alpha(S)].$$

Hence, the corresponding systemic risk indicator is:

$$SRI_{VaR_\alpha}(X_i|S) = \frac{VaR_\alpha(X_i)}{\sum_{\ell=1}^{d} VaR_\alpha(X_\ell)} - \frac{\mathbb{E}[X_i|S = VaR_\alpha(S)]}{VaR_\alpha(S)}.$$

□

Since Wang's risk measures are mixtures of VaRs, Lemma 2.2 can be used to find a general expression for systemic risk indicators derived from Wang's risk measures.

Proposition 2.3 (The Wang's systemic risk indicator)**.** *Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a non-negative random vector in* $\mathbb{R}^d$*, and* ρ *be a Wang's risk measure. The systemic risk indicator corresponding to a Wang's risk measure is defined as follows:*

$$SRI_g(X_i|S) = \frac{\int_0^1 VaR_\alpha(X_i)\ \mathrm{dg}(\alpha)}{\int_0^1 \sum_{k=1}^d VaR_\alpha(X_k)\ \mathrm{dg}(\alpha)} - \frac{\int_0^1 \mathbb{E}[X_i|S = VaR_\alpha(S)]\mathrm{dg}(\alpha)}{\int_0^1 VaR_\alpha(S)\ \mathrm{dg}(\alpha)}, \tag{2.2}$$

where g *is the distortion function associated with* ρ*.*

Proof. For Wang's measures, the function f_ρ takes the following form:

$$f_\rho(\mathbf{x}) = \int_0^1 VaR_\alpha\left(X(\mathbf{x})\right)\ \mathrm{dg}(\alpha),\ \forall \mathbf{x} \in \mathbb{R}^d.$$

Its derivative is then given by

$$\frac{\partial f_\rho}{\partial x_i}(\mathbf{x}) = \frac{\partial}{\partial x_i}\int_0^1 VaR_\alpha(X(\mathbf{x}))\ \mathrm{dg}(\alpha) = \int_0^1 \frac{\partial}{\partial x_i} VaR_\alpha(X(\mathbf{x}))\ \mathrm{dg}(\alpha). \tag{2.3}$$

Combining (2.3) and (2.1), we deduce that

$$\frac{\partial f_\rho}{\partial x_i}(\mathbf{x}) = \int_0^1 \mathbb{E}[X_i|X(\mathbf{x}) = VaR_\alpha(X(\mathbf{x}))]\mathrm{dg}(\alpha), \quad \forall \mathbf{x} \in \mathbb{R}^d,$$

from which we obtain the expression of risk contributions in an Euler capital allocation with any Wang's risk measure:

$$\rho(X_i|S) = \int_0^1 \mathbb{E}[X_i|S = VaR_\alpha(S)]\mathrm{dg}(\alpha), \tag{2.4}$$

where g is the distortion function associated with ρ. The systemic risk indicator corresponding to a Wang's risk measure is then defined as follows:

$$SRI_g(X_i|S) = \frac{\int_0^1 VaR_\alpha(X_i)\ \mathrm{dg}(\alpha)}{\int_0^1 \sum_{k=1}^d VaR_\alpha(X_k)\ \mathrm{dg}(\alpha)} - \frac{\int_0^1 \mathbb{E}[X_i|S = VaR_\alpha(S)]\mathrm{dg}(\alpha)}{\int_0^1 VaR_\alpha(S)\ \mathrm{dg}(\alpha)},$$

where g is the distortion function associated with ρ. □

Since TVaR is also a Wang's risk measure, we can use Proposition 2.3 to obtain the expression of the TVaR-based systemic risk indicator.

Lemma 2.4 (The TVaR systemic risk indicator)**.** *Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a non-negative random vector in* $\mathbb{R}^d$*. The TVaR systemic risk indicator for* X_i *is given by:*

$$SRI_{TVaR_\alpha}(X_i|S) = \frac{\mathbb{E}[X_i|X_i \geq VaR_\alpha(X_i)]}{\sum_{k=1}^d \mathbb{E}[X_k|X_k \geq VaR_\alpha(X_k)]} - \frac{\mathbb{E}[X_i|S \geq VaR_\alpha(S)]}{\mathbb{E}[S|S \geq VaR_\alpha(S)]}. \tag{2.5}$$

Proof. Since TVaR is a Wang's risk measure, we can directly deduce the allocation rule associated with this measure from (2.4). In fact, $TVaR_\alpha$ is a Wang's risk measure associated with the following distortion function:

$$g_\alpha(x) = \frac{x - \alpha}{1 - \alpha}\mathbb{1}_{\{x \in [\alpha, 1]\}}.$$

Note that for a fixed $\alpha \in [0, 1]$, g_α is the uniform distribution function with support $[\alpha, 1]$. By using the expression of the derivative $\frac{\partial f_\rho}{\partial x_i}(\mathbf{x})$ for Wang's risk measures, we have:

$$\frac{\partial f_{TVaR_\alpha}}{\partial x_i}(\mathbf{x}) = \int_0^1 \mathbb{E}\left[X_i|X(\mathbf{x}) = VaR_u\left(X(\mathbf{x})\right)\right]\mathrm{dg}_\alpha(\mathrm{u}).$$

By making a simple substitution, $t = VaR_u(X(\mathbf{x})) = F^{-1}_{X(\mathbf{x})}(u)$, and using the analytical expression for the expectation, we obtain:

$$\frac{\partial f_{TVaR_\alpha}}{\partial x_i}(\mathbf{x}) = \mathbb{E}\left[X_i \frac{1}{1-\alpha} \mathbb{1}_{\{X(\mathbf{x}) \geq VaR_{X(\mathbf{x})}(\alpha)\}}\right], \ \forall \mathbf{x} \in \mathbb{R}^d.$$

Thus, the risk contribution expression for the case of continuous distributions, using TVaR and Euler's method, is given by:

$$TVaR_\alpha\left(X_i|S\right) = \mathbb{E}[X_i|S \geq VaR_\alpha(S)],$$

which corresponds exactly to the systemic risk measure $MES_\alpha(X_i \mid S)$ defined by Acharya et al. (2012) [2]. The expression of the systemic risk indicator derived from TVaR is given by:

$$SRI_{TVaR_\alpha}(X_i|S) = \frac{\mathbb{E}[X_i|X_i \geq VaR_\alpha(X_i)]}{\sum_{k=1}^d \mathbb{E}[X_k|X_k \geq VaR_\alpha(X_k)]} - \frac{\mathbb{E}[X_i|S \geq VaR_\alpha(S)]}{\mathbb{E}[S|S \geq VaR_\alpha(S)]}.$$

□

The VaR is not a coherent risk measure since it is not sub-additive. On the other hand, TVaR is coherent but not an elicitable risk measure. Expectiles are the only risk measures that are both coherent and elicitable. Therefore, we are also interested in the expectile-based systemic risk indicator.

2.3. **Expectiles.** Elicitability is a desirable statistical property for risk measures, and its importance has been highlighted in recent works on risk theory since Gneiting's paper [18]. According to Bellini and Bignozzi (2015) [8], a risk measure ρ is considered elicitable with respect to the class $\mathcal{P}$ if there exists a scoring function $S : \mathbb{R}^2 \to \mathbb{R}^+$ such that

$$\rho(\mathbb{P}) = \arg\min_{x \in \mathbb{R}} \int S(x, y) d\mathbb{P}(y), \quad \forall \mathbb{P} \in \mathcal{P}.$$

They demonstrate in the same paper that expectiles are the only risk measures that are both coherent and elicitable.
Expectiles were initially introduced in the context of statistical regression models by Newey and Powell (1987) [21]. For a random variable X with a finite second-order moment, the expectile of level α is defined as

$$e_\alpha(X) = \arg\min_{x \in \mathbb{R}} \mathbb{E}[\alpha(X - x)_+^2 + (1 - \alpha)(x - X)_+^2], \tag{2.6}$$

where $(x)_+ = \max(x, 0)$.
Expectile risk measures are elicitable by construction. They are coherent for all $\alpha \geqslant 1/2$. However, for $\alpha < 1/2$, expectiles are super-additive and therefore not coherent. When $\alpha = 1/2$, the expectile coincides with the mean. In the rest of this paper, we only consider the case where $\alpha > 1/2$.
Alternatively, the expectile can be defined for any random variable with a finite first-order moment as the unique solution of the following equation:

$$\alpha\mathbb{E}[(X - x)_+] = (1 - \alpha)\mathbb{E}[(x - X)_+]. \tag{2.7}$$

This equation is obtained as an optimality condition using the strict convexity of the scoring function. It can also be written as

$$\frac{1-\alpha}{\alpha} = \frac{\mathbb{E}[(X - x)_+]}{\mathbb{E}[(x - X)_+]}.$$

From this definition, one can give an economic interpretation to the expectile risk measure as a threshold that provides a profits/losses ratio of value $\frac{1-\alpha}{\alpha}$, and construct its acceptance set directly

as

$$\mathcal{A}_{e_\alpha} := \left\{ X \;\middle|\; \frac{\mathbb{E}[(X)_+]}{\mathbb{E}[(X)_+ - X]} \leq \frac{1-\alpha}{\alpha} \right\},$$

which characterizes the expectile risk measure as

$$e_\alpha(X) = \inf \left\{ m | X - m \in \mathcal{A}_{e_\alpha} \right\}.$$

The properties of expectile risk measures have been studied in several papers. Interested readers can refer to [15] and [9] for more information. A multivariate extension of expectiles is proposed in [27], and their asymptotic behavior is studied in [28].

Lemma 2.5 (The expectile systemic risk indicator)**.** *Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a non-negative random vector in* $\mathbb{R}^d$*. The expectile systemic risk indicator of* X_i *is defined as follows:*

$$SRI_{e_\alpha}(X_i|S) = \frac{e_\alpha(X_i)}{\sum_{k=1}^d e_\alpha(X_k)} - \frac{\alpha\mathbb{E}\left[X_i \mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[X_i \mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{E}\left[S\mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[S\mathbb{1}_{\{S<e_\alpha(S)\}}\right]}. \tag{2.8}$$

Proof. Emmer et al. (2015) [15] showed that the contribution of risk X_i in the sum $S = \sum_{\ell=1}^d X_\ell$ is given by

$$e_\alpha(X_i|S) = \frac{\alpha\mathbb{E}\left[X_i \mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[X_i \mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{P}\left(S > e_\alpha(S)\right) + (1-\alpha)\mathbb{P}\left(S < e_\alpha(S)\right)}, \tag{2.9}$$

for $\alpha \in [1/2, 1[$.
We can also express the contribution $e_\alpha(X_i|S)$ as

$$e_\alpha(X_i|S) = \frac{\alpha\mathbb{E}\left[X_i \mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[X_i \mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{E}\left[S\mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[S\mathbb{1}_{\{S<e_\alpha(S)\}}\right]} e_\alpha(S), \tag{2.10}$$

since the expectile $e_\alpha(S)$ satisfies

$$e_\alpha(S) = \frac{\alpha\mathbb{E}\left[S \mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[S \mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{P}\left(S > e_\alpha(S)\right) + (1-\alpha)\mathbb{P}\left(S < e_\alpha(S)\right)}.$$

The allocation percentage $e_\alpha(X_i|S)/e_\alpha(S)$ can be directly obtained from (2.10). Therefore, the expectile systemic risk indicator is defined as

$$SRI_{e_\alpha}(X_i|S) = \frac{e_\alpha(X_i)}{\sum_{k=1}^d e_\alpha(X_k)} - \frac{\alpha\mathbb{E}\left[X_i \mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[X_i \mathbb{1}_{\{S<e_\alpha(S)\}}\right]}{\alpha\mathbb{E}\left[S\mathbb{1}_{\{S>e_\alpha(S)\}}\right] + (1-\alpha)\mathbb{E}\left[S\mathbb{1}_{\{S<e_\alpha(S)\}}\right]}.$$

□

The main difference between the indicator based on TVaR and the one derived from expectiles is that expectiles take into account not only the participation of a marginal risk in the system risk, but also its participation in positive scenarios. This means that the expectile systemic risk indicator does not neglect the positive side of dependence when it contributes to financial performance.

Other expressions of systemic risk indicators can be derived in a similar manner for spectral risk measures, as developed by Acerbi (2002) [1], or for generalized quantiles defined by Bellini et al. (2014) [10].

3. The systemic risk indicators in some dependence models

The indicators derived in the previous section can be estimated using statistical methods available in the literature on risk measures. In the case of a Gaussian distribution, their expressions become simpler due to the probabilistic stability of a Gaussian vector when conditioning on its components.
The aim of this section is to investigate the influence of dependence on the behavior of indicators based on common risk measures such as VaR, TVaR, and Expectiles. To achieve this, we introduce three bivariate risk models. The findings obtained from these models can be extended to the multivariate case. The selection of these models is motivated by the variation in the nature of dependence. The first model assumes simple independence, where the diversification gain is non-zero, leading to non-trivial expressions for the indicators. The second model incorporates a structure of weak dependence, modeled using an FGM copula. The third model exhibits stronger dependence, particularly at the asymptotic level.

3.1. **Bivariate independent exponential model.**

We consider a bivariate independent exponential random vector (X_1, X_2) with $X_i \in \mathcal{E}(\beta_i)$, where $i \in \{1, 2\}$. Let S denote the aggregated sum of risks $X_1 + X_2$. In the case where $\beta_1 = \beta_2$, the systemic risk indicator is trivial and null for both risks due to symmetry. Without loss of generality, we assume $\beta_1 < \beta_2$.
Lemmas 3.1, 3.2, and 3.3 provide the expressions for the conventional systemic risk indicators.

Lemma 3.1 (VaR-SRI, EI Model)**.** *The VaR systemic risk indicator of X_i is given by:*

$$SRI_{VaR_\alpha}(X_i|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2} \frac{1}{\beta_\ell}} - \frac{\gamma\left(s^*, \beta_i, \beta_{3-i}\right)}{s^* h(s^*, \beta_i, \beta_{3-i})}, \tag{3.1}$$

where s^* is the unique solution to the equation:

$$\bar{H}(s, \beta_1, \beta_2) = 1 - \alpha,$$

and $\bar{H}$, h, and γ are defined as follows:

$$\bar{H}(s, \beta_1, \beta_2) = \sum_{k=1}^{2} \left(\prod_{\ell=1, \ell\neq k}^{2} \frac{\beta_\ell}{\beta_\ell - \beta_k} \right) \mathrm{e}^{-\beta_k x},$$

$$h(s, \beta_1, \beta_2) = \sum_{k=1}^{2} \left(\prod_{\ell=1, \ell\neq k}^{2} \frac{\beta_\ell}{\beta_\ell - \beta_k} \right) \beta_k \mathrm{e}^{-\beta_k x},$$

and

$$\gamma\left(s, \beta_1, \beta_2\right) = \frac{\beta_1 \beta_2}{\beta_2 - \beta_1} \left[s e^{-\beta_1 s} + \frac{1}{\beta_2 - \beta_1} \left(e^{-\beta_2 s} - e^{-\beta_1 s} \right) \right].$$

Proof. Since $X_i \sim \mathcal{E}(\beta_i)$ for all $i \in \{1, 2\}$, we have:

$$VaR_\alpha(X_i) = -\frac{1}{\beta_i} \ln\left(1 - \alpha\right).$$

The conditional expectation $\mathbb{E}[X_i | S = VaR_\alpha(S)]$ can be expressed as:

$$\mathbb{E}[X_i | S = VaR_\alpha(S)] = \frac{\mathbb{E}[X_i \mathbb{1}_{\{S = VaR_\alpha(S)\}}]}{f_S\left(VaR_\alpha(S)\right)}.$$

The unique solution s^* to the equation $\bar{H}(s,\beta_1,\beta_2) = 1-\alpha$ is determined, where:

$$\bar{H}(s,\beta_1,\beta_2) = \bar{F}_S(s) = \frac{\beta_1\beta_2}{\beta_2-\beta_1}\left(\frac{e^{-\beta_1 s}}{\beta_1} - \frac{e^{-\beta_2 s}}{\beta_2}\right) = \sum_{k=1}^{2}\left(\prod_{\ell=1,\ell\neq k}^{2}\frac{\beta_\ell}{\beta_\ell-\beta_k}\right)\mathrm{e}^{-\beta_k x}.$$

Let

$$h(s,\beta_1,\beta_2) = f_S(s) = \frac{\beta_1\beta_2}{\beta_2-\beta_1}\left(e^{-\beta_1 s} - e^{-\beta_2 s}\right) = \sum_{k=1}^{2}\left(\prod_{\ell=1,\ell\neq k}^{2}\frac{\beta_\ell}{\beta_\ell-\beta_k}\right)\beta_k\mathrm{e}^{-\beta_k x}.$$

We can then calculate $\mathbb{E}[X_1\mathbb{1}_{\{S=s\}}]$ as follows:

$$\begin{aligned}\mathbb{E}\left[X_1\mathbb{1}_{\{S=s\}}\right] &= \gamma\left(s,\beta_1,\beta_2\right)\\ &= \int_0^s x f_{X_1,S}\left(x,s\right)dx = \int_0^s x f_{X_1,X_2}\left(x,s-x\right)dx\\ &= \frac{\beta_1\beta_2}{\beta_2-\beta_1}\left[se^{-\beta_1 s} + \frac{1}{\beta_2-\beta_1}\left(e^{-\beta_2 s} - e^{-\beta_1 s}\right)\right].\end{aligned}$$

Thus, the systemic risk indicator can be expressed as:

$$SRI_{VaR_\alpha}(X_i|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2}\frac{1}{\beta_\ell}} - \frac{\gamma\left(s^*,\beta_i,\beta_{3-i}\right)}{s^* h(s^*,\beta_i,\beta_{3-i})}.$$

□

Lemma 3.2 (TVaR-SRI, EI Model)**.** *The TVaR systemic risk indicator of X_i is given by*

$$SRI_{TVaR}(X_i|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2}\frac{1}{\beta_\ell}} - \frac{\xi\left(s^*;\beta_i,\beta_{3-i}\right)}{\zeta\left(s^*;\beta_1,\beta_2\right)}, \tag{3.2}$$

where $s^* = VaR_\alpha(S)$, and ζ and ξ are the functions defined as follows:

$$\zeta\left(x;\beta_1,\beta_2\right) = \sum_{k=1}^{2}\left(\prod_{\ell=1,\ell\neq k}^{2}\frac{\beta_\ell}{\beta_\ell-\beta_k}\right)\left(x\mathrm{e}^{-\beta_k x} + \frac{\mathrm{e}^{-\beta_k x}}{\beta_k}\right),$$

and

$$\xi\left(x;\beta_i,\beta_j\right) = \frac{\beta_j e^{-\beta_i x}\left(x+\frac{1}{\beta_i}\right)}{\left(\beta_j-\beta_i\right)} - \left(\frac{\beta_j e^{-\beta_i x}}{\left(\beta_i-\beta_j\right)^2} - \frac{\beta_i e^{-\beta_j x}}{\left(\beta_i-\beta_j\right)^2}\right).$$

Proof. Since $X_i \sim \mathcal{E}(\beta_i)$ for all $i \in \{1,\ldots,d\}$, we have

$$TVaR_\alpha(X_i) = \frac{1}{\beta_i}\left(1 - \ln\left(1-\alpha\right)\right).$$

Furthermore, we can express

$$\mathbb{E}[X_i|S \geq VaR_\alpha(S)] = \frac{1}{1-\alpha}\int_{VaR_\alpha(S)}^{+\infty}\mathbb{E}[X_i\mathbb{1}_{\{S=s\}}]ds,$$

and

$$\mathbb{E}\left[X_i\mathbb{1}_{\{S=s\}}\right] = \gamma\left(s,\beta_i,\beta_{3-i}\right),$$

where γ is the function defined in Lemma 3.1.
The remaining expressions can be obtained straightforwardly from their definitions using

$$\mathbb{E}\left[S \times \mathbb{1}_{\{S \geq x\}}\right] = \zeta\left(x; \beta_1, \beta_2\right) = \sum_{i=1}^{2} \left(\prod_{j=1, j \neq i}^{2} \frac{\beta_j}{\beta_j - \beta_i} \right) \left(x \mathrm{e}^{-\beta_i x} + \frac{\mathrm{e}^{-\beta_i x}}{\beta_i} \right),$$

and

$$\mathbb{E}\left[X_i \times \mathbb{1}_{\{S \geq x\}}\right] = \xi\left(x; \beta_i, \beta_{3-i}\right),$$

where

$$\xi\left(x; \beta_1, \beta_2\right) = \frac{\beta_2 e^{-\beta_1 x}\left(x + \frac{1}{\beta_1}\right)}{\left(\beta_2 - \beta_1\right)} - \left(\frac{\beta_2 e^{-\beta_1 x}}{\left(\beta_1 - \beta_2\right)^2} - \frac{\beta_1 e^{-\beta_2 x}}{\left(\beta_1 - \beta_2\right)^2} \right).$$

□

Lemma 3.3 (Expectile-SR, EI Model)**.** *For the risk X_i, the Expectile systemic risk indicator is given by:*

$$SRI_{e_\alpha}\left(X_i | S\right) = \frac{x_i^*}{x_i^* + x_{3-i}^*} - \frac{(2\alpha - 1)\beta_i \xi\left(s^*; \beta_i, \beta_{3-i}\right) + (1 - \alpha)}{(2\alpha - 1)\bar{H}\left(s^*; \beta_1, \beta_2\right) + (1 - \alpha)} \frac{1}{s^* \beta_i},$$

where x_i^* is the unique solution to the following equation:

$$\beta_i x - \frac{2\alpha - 1}{1 - \alpha} e^{-\beta_i x} - 1 = 0,$$

and s^* is the unique solution to the following equation:

$$(2\alpha - 1)\left[\zeta\left(s; \beta_1, \beta_2\right) - s\bar{H}\left(s; \beta_1, \beta_2\right)\right] = (1 - \alpha)\left[s - \frac{1}{\beta_1} - \frac{1}{\beta_2}\right],$$

where ξ, ζ, and $\bar{H}$ are the same functions defined in Lemma 3.2.

Proof. From the expectile definition (2.7), $e_\alpha(S)$ is the unique solution to the equation:

$$\alpha \mathbb{E}[(S - s)_+] = (1 - \alpha)\mathbb{E}[(s - S)_+],$$

which can be written as:

$$(2\alpha - 1)\mathbb{E}[(S - s)_+] = (1 - \alpha)\left(s - \mathbb{E}[S]\right).$$

Furthermore, from Equation 2.9, the contribution $e_\alpha\left(X_i | S\right)$ can be written as:

$$e_\alpha(X_i | S) = \frac{(2\alpha - 1)\mathbb{E}\left[X_i \mathbb{1}_{\{S > e_\alpha(S)\}}\right] + (1 - \alpha)\mathbb{E}\left[X_i\right]}{(2\alpha - 1)\mathbb{P}\left(S > e_\alpha(S)\right) + (1 - \alpha)}.$$

The expressions are obtained straightforwardly from their definitions using the functions $\bar{H}$ and ξ defined in Lemma 3.2. □

The case of independence serves as a reference scenario, enabling us to assess the influence of marginal distributions on risk indicators and facilitating comparisons between different indicators. This basic model also helps us understand how these indicators behave based on the threshold parameter α.

3.2. **Bivariate FGM model.** Let the joint distribution of (X_1, X_2) be defined by a Farlie-Gumbel-Morgenstern (FGM) copula, given as

$$C_\theta(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1-u_1)(1-u_2), \quad -1 \leq \theta \leq 1,$$

(see e.g., Nelsen (2007) [29], Example 3.12, Section 3.2.5). The marginal distributions are exponential with parameters β_1 and β_2, respectively. This leads to the joint cumulative distribution function:

$$F_{X_1,X_2}(x_1, x_2) = \left(1 - \mathrm{e}^{-\beta_1 x_1}\right)\left(1 - \mathrm{e}^{-\beta_2 x_2}\right) + \theta\left(1 - \mathrm{e}^{-\beta_1 x_1}\right)\left(1 - \mathrm{e}^{-\beta_2 x_2}\right)\mathrm{e}^{-\beta_1 x_1}\mathrm{e}^{-\beta_2 x_2}.$$

It is important to note that the FGM construction represents a weak dependence model. The Pearson correlation coefficient is given by $\rho_P(X_1, X_2) = \frac{\theta}{4}$, which implies $\rho_P(X_1, X_2) \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. The Spearman's correlation coefficient, denoted as ρ_S, is given by $\rho_S = \frac{\theta}{3} \in \left[-\frac{1}{3}, \frac{1}{3}\right]$. We recall that Spearman's rho is a concordance measure defined for continuous bivariate distributions with copula C as the dependence structure. It can be calculated as:

$$\begin{aligned}\rho_S &= 12 \int\int_{[0,1]^2} uv dC(u,v) - 3 \\ &= 12 \int\int_{[0,1]^2} C(u,v) du dv - 3.\end{aligned}$$

The FGM construction is also considered an asymptotically independent model since its upper tail dependence coefficient is $\lambda_U = 0$. We recall the definition of the upper tail dependence coefficient as presented in Joe (1997) [20] for bivariate random variables (X, Y) with continuous marginal distributions:

$$\begin{aligned}\lambda_U &= \lim_{u \to 1^-} \mathbb{P}(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)) \\ &= \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u},\end{aligned}$$

when the limit exists. The joint density is given by:

$$f_{X_1,X_2}(x_1, x_2) = \beta_1 \mathrm{e}^{-\beta_1 x_1} \beta_2 \mathrm{e}^{-\beta_2 x_2} + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} \times i\beta_1 \mathrm{e}^{-i\beta_1 x_1} \times j\beta_2 \mathrm{e}^{-j\beta_2 x_2}. \tag{3.3}$$

Lemmas 3.4, 3.5, and 3.6 present the expressions of the marginal systemic risk indicators obtained using VaR, TVaR, and the expectile as risk measures, respectively.

Lemma 3.4 (VaR-SRI, FGM Model)**.** *The VaR systemic risk indicator of X_i is given by*

$$SRI_{VaR_\alpha}(X_i|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2} \frac{1}{\beta_\ell}} - \frac{\gamma(s^*; \beta_k, \beta_\ell) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} \gamma(s^*; i\beta_k, j\beta_\ell)}{s^* h(s^*; \beta_k, \beta_\ell) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} s^* h(s^*; i\beta_k, j\beta_\ell)}, \tag{3.4}$$

where $s^* = VaR_\alpha(S)$ is the unique solution to the following equation:

$$\bar{H}(s^*; \beta_k, \beta_\ell) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} \bar{H}(s^*; i\beta_k, j\beta_\ell) = 1 - \alpha,$$

and $\bar{H}$, h, and γ are the same functions defined in Lemma 3.1.

Proof. Using Equation 3.3 and the functions h and γ defined in Lemma 3.1, we can easily calculate the following expressions:

$$f_S(s) = h(s;\beta_1,\beta_2) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} h(s;i\beta_1,j\beta_2),$$

and

$$\mathbb{E}\left[X_k \times \mathbb{1}_{\{S=s\}}\right] = \gamma(s;\beta_k,\beta_{3-k}) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} \gamma(s;i\beta_k,j\beta_{3-k}), \quad k=1,2.$$

These expressions are sufficient to derive the expression of the VaR-based systemic risk indicator as announced in Lemma 2.2. □

Lemma 3.5 (TVaR-SRI, FGM Model)**.** *The TVaR systemic risk indicator of X_k is given by*

$$SRI_{TVaR}(X_k|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2}\frac{1}{\beta_\ell}} - \frac{\xi(s^*;\beta_k,\beta_{3-k}) + \theta\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\xi(s^*;i\beta_k,j\beta_{3-k})}{\zeta(s;\beta_1,\beta_2) + \theta\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\zeta(s;i\beta_1,j\beta_2)}, \tag{3.5}$$

where $s^* = VaR_\alpha(S)$, and ζ, ξ are the functions defined in Lemma 3.2.

Proof. For the FGM model, the expressions for the contributions in the TVaR allocation are given in Bargès et al. (2009) [7]. From those expressions, we can derive the following expressions:

$$\mathbb{E}\left[S \times \mathbb{1}_{\{S\geq s\}}\right] = \zeta(s;\beta_1,\beta_2) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} \zeta(s;i\beta_1,j\beta_2),$$

and

$$\mathbb{E}\left[X_k \times \mathbb{1}_{\{S\geq s\}}\right] = \xi(s;\beta_k,\beta_{3-k}) + \theta \sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{i+j} \xi(s;i\beta_k,j\beta_{3-k}),$$

where ζ and ξ are the functions defined in Lemma 3.2. By combining these expressions with Lemma 2.4, we obtain the expression for the TVaR-based systemic risk indicator. □

Lemma 3.6 (Expectile-SRI, FGM Model)**.** *For the risk X_i, the Expectile systemic risk indicator is given by:*

$$SRI_{e_\alpha}(X_i|S) = \frac{x_i^*}{x_i^* + x_{3-i}^*} - \frac{(2\alpha-1)\beta_i\left[\xi(s^*;\beta_i,\beta_{3-i}) + \theta\sum_{k=1}^{2}\sum_{\ell=1}^{2}(-1)^{k+\ell}\xi(s^*;k\beta_i,\ell\beta_{3-i})\right] + 1 - \alpha}{(2\alpha-1)\left[\bar{H}(s^*;\beta_1,\beta_2) + \theta\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}\bar{H}(s^*;i\beta_1,j\beta_2)\right] + 1 - \alpha}\frac{1}{s^*\beta_k},$$

where x_i^* is the unique solution to the equation:

$$\beta_i x - \frac{2\alpha-1}{1-\alpha}e^{-\beta_i x} - 1 = 0,$$

and s^* is the unique solution to the equation:

$$(2\alpha-1)\left[T(s;\beta_1,\beta_2) + \theta\sum_{i=1}^{2}\sum_{j=1}^{2}(-1)^{i+j}T(s;i\beta_1,j\beta_2)\right] = (1-\alpha)\left[s - \left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)\right],$$

and ξ, ζ, and $\bar{H}$ are the functions defined in Lemma 3.2, and T is the function defined by:

$$T(s;i\beta_1,j\beta_2) = \zeta(s;i\beta_1,j\beta_2) - s\bar{H}(s;i\beta_1,j\beta_2), \ \forall s\in\mathbb{R}^+, \ \forall (i,j)\in\{1,2\}^2.$$

Proof. The expectile-based risk indicators are obtained directly using Equation 2.9 and the functions defined in Lemma 3.2. □

We observe that the FGM model includes the case of independence ($\theta = 0$). However, this dependency structure does not capture tail dependencies ($\lambda_U = 0$). In the next subsection, we study the systemic risk indicators in a model that specifically incorporates tail dependence, addressing this limitation.

3.3. **Bivariate Common Mixture Model.** This approach to constructing multivariate models is extensively described by Joe (1997) [20]. It involves selecting a random variable Θ with support S_Θ and independent random variables Y_i to construct the random variables X_i, which are conditionally independent given Θ. This construction ensures that the conditional distribution function of X_i given $\Theta = \theta$ is given by

$$\bar{F}_{X_i|\Theta=\theta}(x_i) = (\bar{F}_{Y_i}(x_i))^\theta.$$

By integrating with respect to the distribution of Θ, this construction provides both the marginal distributions and the joint distribution, as explained in Marceau (2013) [25].
In our study, we focus specifically on a bivariate exponential mixture model. We assume that the moment-generating function of the random variable Θ, denoted by M_Θ, exists. The joint density function of X_1 and X_2 is given by the integral:

$$f_{X_1,X_2}(x_1,x_2) = \int_{\theta\in S_\Theta} \beta_1\theta e^{-\beta_1\theta x_1}\beta_2\theta e^{-\beta_2\theta x_2} dF_\Theta(\theta) = \beta_1\beta_2 \frac{d^2 M_\Theta(t)}{dt^2}|_{t=-(\beta_1 x_1+\beta_2 x_2)}.$$

We consider a pair of continuous random variables (X_1, X_2) that follow a mixture of exponential distributions. Each X_i is exponentially distributed with parameter $\beta_i\theta$, where $0 < \beta_1 < \beta_2$. The random variable θ follows a Gamma distribution with shape parameter γ and scale parameter b, denoted as $\theta \sim Ga(\gamma, b)$. As a result, the survival function of X_i is given by:

$$\bar{F}_{X_i}(x) = \int_0^\infty \bar{F}_{X_i|\Theta=\theta} f_\Theta(\theta)d\theta = \int_0^\infty e^{-\beta_i\theta x} f_\Theta(\theta)d\theta = \left(1+\frac{\beta_i x}{b}\right)^{-\gamma}.$$

Consequently, X_i follows a Pareto distribution with parameters $\left(\gamma, \frac{b}{\beta_i}\right)$. Importantly, X_1 and X_2 are conditionally independent. The survival bivariate distribution, denoted as $\bar{F}_{X_1,X_2}(x_1,x_2)$, is given by the expression:

$$\bar{F}_{X_1,X_2}(x_1,x_2) = \left(\frac{1}{1+\frac{\beta_1}{b}x_1+\frac{\beta_2}{b}x_2}\right)^\gamma = \left(\bar{F}_{X_1}(x_1)^{-1/\gamma} + \bar{F}_{X_1}(x_1)^{-1/\gamma} - 1\right)^{-\gamma}.$$

This represents the survival Clayton copula with a dependence parameter $\theta = 1/\gamma$. Consequently, the upper tail dependence coefficient is given by:

$$\lambda_U = \lambda_L^{Clayton} = 2^{-\gamma},$$

where $\lambda_L^{Clayton}$ represents the lower tail dependence coefficient of the Clayton copula. This dependence model exhibits upper tail dependence.
The density function of S, denoted as $f_S(s)$, is given by:

$$f_S(s) = h_{CM}(s,\beta_1,\beta_2,b,\gamma) = \frac{\beta_1\beta_2\gamma}{(\beta_1-\beta_2)b}\left[\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^{\gamma+1} - \left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma+1}\right],$$

and its survival function, denoted as $\bar{F}_S(s)$, is given by:

$$\bar{F}_S(s) = \bar{H}_{CM}(s,\beta_1,\beta_2,b,\gamma) = \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^\gamma + \frac{\beta_2}{\beta_2-\beta_1}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^\gamma.$$

Lemma 3.7 (VaR-SRI, CM Model)**.** *The VaR systemic risk indicator of X_i is given by:*

$$SRI_{VaR_\alpha}(X_i|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2}\frac{1}{\beta_\ell}} - \frac{\gamma_{CM}\left(s^*, \beta_i, \beta_{3-i}; b, \gamma\right)}{s^* h_{CM}\left(s^*, \beta_1, \beta_2, b, \gamma\right)}, \tag{3.6}$$

where $s^* = VaR_\alpha(S)$ is the unique solution to the following equation:

$$\frac{\beta_1}{\beta_1 - \beta_2}\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^\gamma + \frac{\beta_2}{\beta_2 - \beta_1}\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^\gamma = 1-\alpha,$$

and γ_{CM} is the function defined as:

$$\gamma_{CM}\left(s, \beta_1, \beta_2; b, \gamma\right) = \frac{\beta_1\beta_2\gamma}{(\beta_2-\beta_1)b}s\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma+1} + \frac{\beta_1\beta_2}{(\beta_1-\beta_2)^2}\left[\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^\gamma - \left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^\gamma\right].$$

Proof. Since X_i follows a Pareto distribution with parameters $\left(\gamma, \frac{b}{\beta_i}\right)$, the VaR of X_i is calculated as:

$$VaR_\alpha(X_i) = \frac{b}{\beta_i}\left((1-\alpha)^{-\frac{1}{\gamma}} - 1\right).$$

To compute the VaR-based allocation contribution, we use its definition:

$$\mathbb{E}\left[X_k \times \mathbb{1}_{\{S=s\}}\right] = \gamma_{CM}\left(s, \beta_k, \beta_{3-k}; b, \gamma\right),$$

where

$$\gamma_{CM}\left(s, \beta_1, \beta_2; b, \gamma\right) = \frac{\beta_1\beta_2\gamma}{(\beta_2-\beta_1)b}s\left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^{\gamma+1} + \frac{\beta_1\beta_2}{(\beta_1-\beta_2)^2}\left[\left(\frac{1}{1+\frac{\beta_2}{b}s}\right)^\gamma - \left(\frac{1}{1+\frac{\beta_1}{b}s}\right)^\gamma\right].$$

These calculations provide us with the expression of the marginal systemic risk indicator. □

Lemma 3.8 (TVaR-SRI, CM Model)**.** *The TVaR systemic risk indicator of X_i is given by:*

$$SRI_{TVaR}(X_i|S) = \frac{\frac{1}{\beta_i}}{\sum_{\ell=1}^{2}\frac{1}{\beta_\ell}} - \frac{\xi_{CM}\left(s^*, \beta_i, \beta_{3-i}, \gamma, b\right)}{\zeta_{CM}\left(s, \beta_1, \beta_2; b, \gamma\right)}, \tag{3.7}$$

where $s^* = VaR_\alpha(S)$ and ζ_{CM}, ξ_{CM} are defined as:

$$\zeta_{CM}\left(s, \beta_1, \beta_2; b, \gamma\right) = \sum_{i=1}^{2}\frac{\beta_{3-i}}{\beta_{3-i}-\beta_i}\left[s\left(\frac{1}{1+\frac{\beta_i}{b}s}\right)^\gamma + \frac{b}{(\gamma-1)\beta_i}\left(\frac{1}{1+\frac{\beta_i}{b}s}\right)^{\gamma-1}\right],$$

$$\begin{aligned}\xi_{CM}\left(s^*, \beta_i, \beta_{3-i}, \gamma, b\right) &= \frac{\beta_{3-i}b}{(\beta_{3-i}-\beta_i)\beta_i(\gamma-1)}\left(\frac{1}{1+\frac{\beta_i}{b}s}\right)^\gamma\left(1+\gamma\frac{\beta_i}{b}s\right)\\ &+ \frac{1}{(\beta_i-\beta_{3-i})^2(\gamma-1)}\left[\beta_i b\left(\frac{1}{1+\frac{\beta_{3-i}}{b}s}\right)^{\gamma-1} - \beta_{3-i}b\left(\frac{1}{1+\frac{\beta_i}{b}s}\right)^{\gamma-1}\right].\end{aligned}$$

Proof. Since X_i follows a Pareto distribution with parameters $\left(\gamma, \frac{b}{\beta_i}\right)$ and $\gamma > 1$, the TVaR of X_i is calculated as:

$$TVaR_\alpha(X_i) = \frac{b}{\beta_i}\left(\frac{\gamma}{\gamma-1}(1-\alpha)^{-\frac{1}{\gamma}} - 1\right).$$

To compute the sum TVaR, we use its definition:

$$\mathbb{E}\left[S \times \mathbb{1}_{\{S \geq s\}}\right] = \sum_{i=1}^{2} \frac{\beta_{3-i}}{\beta_{3-i} - \beta_i} \left[s \left(\frac{1}{1 + \frac{\beta_i}{b} s} \right)^{\gamma} + \frac{b}{(\gamma - 1)\beta_i} \left(\frac{1}{1 + \frac{\beta_i}{b} s} \right)^{\gamma - 1} \right].$$

By using the expression for the truncated expectation:

$$\begin{aligned} \mathbb{E}\left[X_1 \times \mathbb{1}_{\{S \geq s\}}\right] &= \frac{\beta_2 b}{(\beta_2 - \beta_1)\beta_1(\gamma - 1)} \left(\frac{1}{1 + \frac{\beta_1}{b} s} \right)^{\gamma} \left(1 + \gamma \frac{\beta_1}{b} s \right) \\ &\quad + \frac{1}{(\beta_1 - \beta_2)^2(\gamma - 1)} \left[\beta_1 b \left(\frac{1}{1 + \frac{\beta_2}{b} s} \right)^{\gamma - 1} - \beta_2 b \left(\frac{1}{1 + \frac{\beta_1}{b} s} \right)^{\gamma - 1} \right]. \end{aligned}$$

By combining all the obtained expressions, we obtain the systemic risk indicator as stated. □

Lemma 3.9 (Expectile-SRI, CM Model)**.** *Let (X_1, X_2) follow a bivariate common Gamma mixture model. The Expectile systemic risk indicator of $X_i, i = 1, 2$ is given by:*

$$SRI_{e_\alpha}(X_i|S) = \frac{x_i^*}{x_i^* + x_{3-i}^*} - \frac{(2\alpha - 1)(\gamma - 1)\beta_i \xi_{CM}(s^*, \beta_i, \beta_{3-i}, \gamma, b) + (1 - \alpha)b}{(2\alpha - 1)\bar{H}_{CM}(s^*, \beta_1, \beta_2, \gamma, b) + (1 - \alpha)} \frac{1}{(\gamma - 1)\beta_i s^*},$$

where ξ_{CM} is defined in Lemma 3.8, x_i^* is the unique solution to the following equation:

$$\frac{\beta_i}{b}(\gamma - 1)x - \frac{2\alpha - 1}{1 - \alpha} \left(\frac{1}{1 + \frac{\beta_i}{b} x} \right)^{\gamma - 1} - 1 = 0,$$

and s^* is the unique solution to the following equation:

$$(2\alpha - 1) \left[\frac{\beta_1/\beta_2}{(\beta_1 - \beta_2)} \left(\frac{1}{1 + \frac{\beta_2}{b} s} \right)^{\gamma - 1} + \frac{\beta_2/\beta_1}{(\beta_2 - \beta_1)} \left(\frac{1}{1 + \frac{\beta_1}{b} s} \right)^{\gamma - 1} \right] = (1 - \alpha) \left[\frac{s}{b}(\gamma - 1) - \frac{1}{\beta_1} - \frac{1}{\beta_1} \right].$$

Proof. Since X_i follows a Pareto distribution with parameters $\left(\gamma, \frac{b}{\beta_i}\right)$ and $\gamma > 1$, the univariate expectile $e_\alpha(X_i)$ is obtained as the unique solution to the following equation:

$$\frac{\beta_i}{b}(\gamma - 1)x - \frac{2\alpha - 1}{1 - \alpha} \left(\frac{1}{1 + \frac{\beta_i}{b} x} \right)^{\gamma - 1} - 1 = 0.$$

Using the expression $\bar{F}_{CM}$, we calculate the stop-loss function:

$$\mathbb{E}[(S - s)_+] = \frac{\frac{\beta_1}{\beta_2} b}{(\beta_1 - \beta_2)(\gamma - 1)} \left(\frac{1}{1 + \frac{\beta_2}{b} s} \right)^{\gamma - 1} + \frac{\frac{\beta_2}{\beta_1} b}{(\beta_2 - \beta_1)(\gamma - 1)} \left(\frac{1}{1 + \frac{\beta_1}{b} s} \right)^{\gamma - 1}.$$

Thus, the expectile $e_\alpha(S)$ is the unique solution to the following equation:

$$(2\alpha - 1) \left[\frac{\beta_1/\beta_2}{(\beta_1 - \beta_2)} \left(\frac{1}{1 + \frac{\beta_2}{b} s} \right)^{\gamma - 1} + \frac{\beta_2/\beta_1}{(\beta_2 - \beta_1)} \left(\frac{1}{1 + \frac{\beta_1}{b} s} \right)^{\gamma - 1} \right] = (1 - \alpha) \left[\frac{s}{b}(\gamma - 1) - \frac{1}{\beta_1} - \frac{1}{\beta_1} \right].$$

By using the expressions of the Expectile-based allocation contributions given in Equation 3.3, and the function ξ_{CM} defined in Lemma 3.8, we obtain the systemic risk indicators based on the expectile as a risk measure. □

The common mixture model is particularly useful when dealing with risks that exhibit significant asymptotic behavior. The Pareto marginal distribution, known for its heavy-tailed nature, provides an effective representation of extreme events. The models presented in this section will serve as the basis for numerical illustrations in the subsequent section.

4. Numerical illustrations

In this section, we present some numerical illustrations based on the bivariate models studied in Section 3. Our aim is to analyze the behavior of the TVaR and Expectile derived systemic risk indicators. We also want to highlight the impact of dependence on these indicators. Recall that in the bivariate case, the systemic risk indicators are symmetric between the two risks composing the system.

4.1. **Independence case.** For this first example, we are considering two independent exponential random variables, namely $X_1 \sim \mathcal{E}(\beta_1)$ and $X_2 \sim \mathcal{E}(\beta_2)$, where $\beta_1 < \beta_2$. This implies, in terms of stochastic order, that X_1 is riskier than X_2. The theoretical expressions of the indicators in this case are presented in Sub-section 3.1. Figure 1 illustrates the variation of the indicators with respect to the threshold α. As α increases, representing higher risk aversion, the more risky component is considered more systemic.

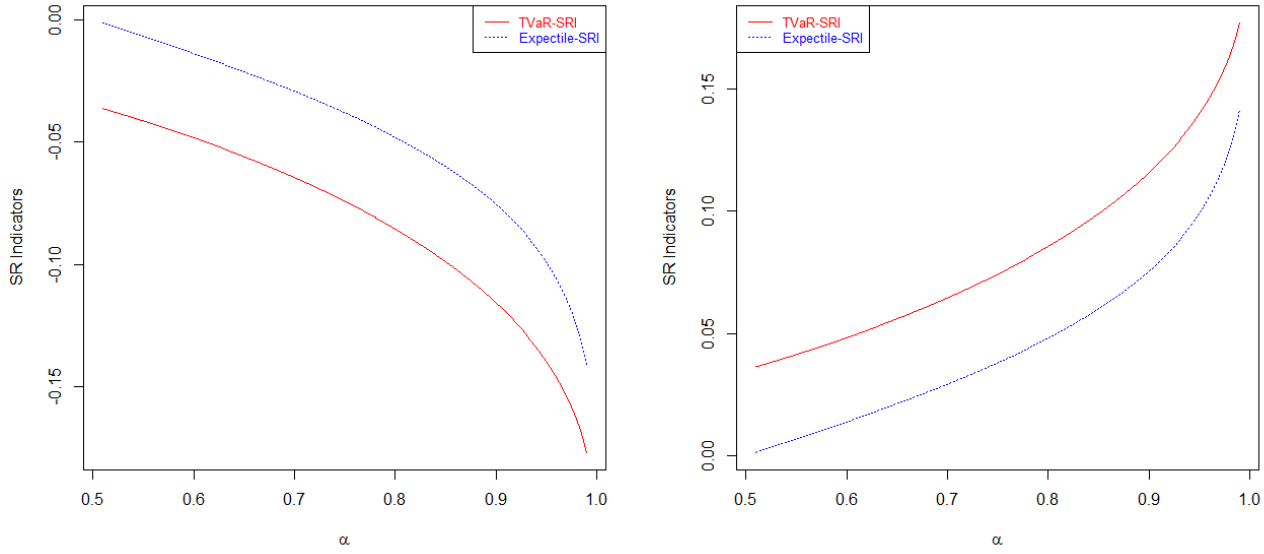

Figure 1. $SRI_{TVaR_\alpha}(X_i|S)$ and $SRI_{e_\alpha}(X_i|S)$ - Independence case
(Left: $X_1 \sim \mathcal{E}(\beta_1 = 0.10)$, Right: $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$)

Both indicators show levels above 10% for the asymptotic levels of α. The values obtained for SRI_{e_α} are consistently lower than those obtained by SRI_{TVaR_α}, which can be attributed to the construction of the expectile measure that considers both tails of the distribution.

4.2. **FGM copula.** Now, let's examine the impact of dependence on the behavior of the two indicators SRI_{TVaR_α} and SRI_{e_α}. To do this, we consider the FGM model, and the theoretical results for this model are presented in Sub-section 3.2. By setting the parameter of the FGM copula to $\theta = 1$, representing the maximum level of positive dependence captured by this copula, we observe a decrease in the values of both indicators compared to the case of independence ($\theta = 0$), as shown in Figure 1. Positive dependence increases the responsibility of the less risky component (in this case, X_2) in overall risk, thus increasing its systemic level and decreasing that of X_1.

To examine the impact of dependence on the behavior of the indicators according to its nature, we present in Figure 3 the evolution of the two indicators: SRI_{TVaR_α} (on the left) and SRI_{e_α} (on

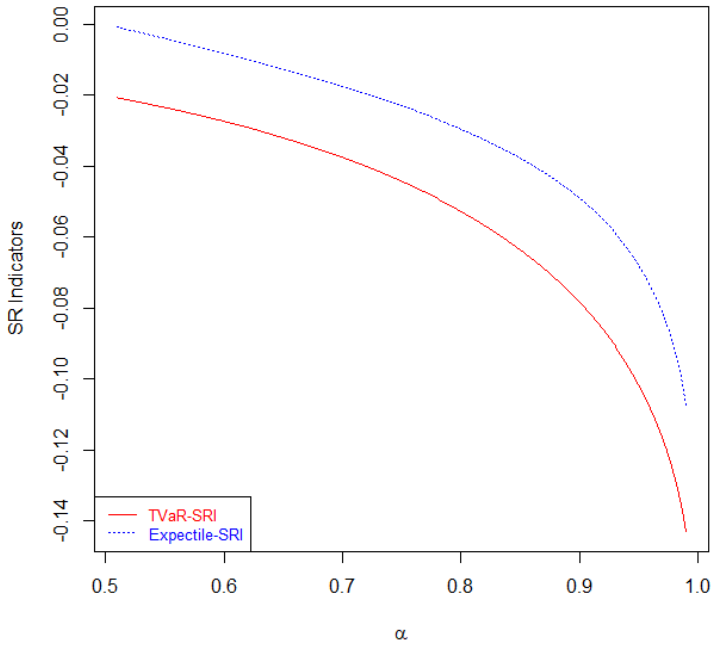

FIGURE 2. $SRI_{TVaR_\alpha}(X_1|S)$ and $SRI_{e_\alpha}(X_1|S)$ - FGM Model $\theta = 1$ ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$ and $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$)

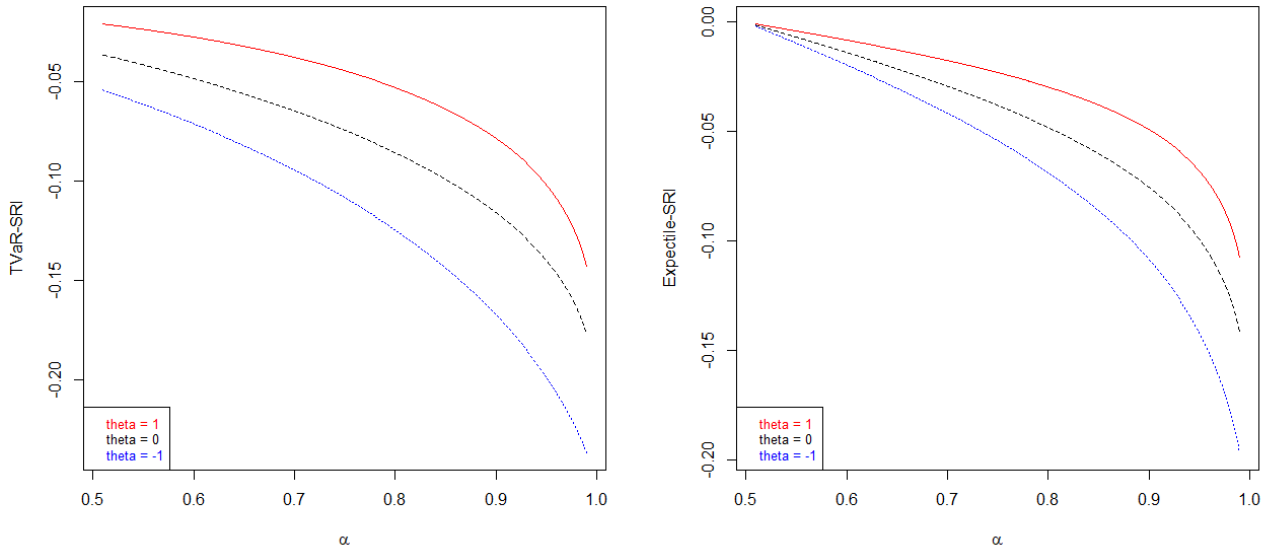

FIGURE 3. FGM Model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$ and $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$), $\theta = -1, 0, 1$
Left: $SRI_{TVaR_\alpha}(X_1|S)$ - Right: $SRI_{e_\alpha}(X_1|S)$

the right), for three values of the parameter $\theta = -1, 0, +1$. Figure 3 shows that the systemic level of the riskier component increases in the presence of negative dependence. A significant portion of the overall risk is attributed to this component, resulting in a higher absolute value of the systemic

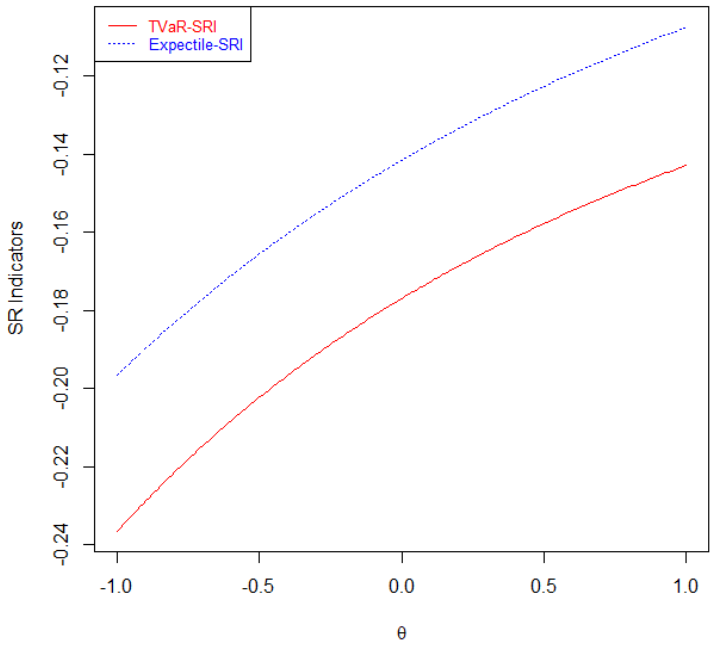

FIGURE 4. $SRI_{TVaR_{99\%}}(X_1|S)$ and $SRI_{99\%}(X_1|S)$ - FGM Model $\theta \in [-1, 1]$ ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$ and $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$)

risk indicator.

To closely examine this impact of dependence, we present in Figure 4 the values obtained for the two indicators for all possible values of θ, with the threshold α fixed at 99%. We observe a considerable difference between the values obtained depending on θ. This confirms the idea that the nature of dependence can have a significant impact on the systemic level of a component in the system.

4.3. **Survival Clayton copula.** The FGM copula represents a weak dependence structure that cannot capture all levels of dependence. Therefore, we consider the case of a Clayton survival copula constructed by common mixing, as presented in Sub-section 3.3. This structure models positive dependence and has the ability to capture asymptotic dependence. We set the parameter b to 1 and vary the parameter γ, which represents a copula parameter, to observe the impact of dependence.

Figure 5 illustrates the variation of the indicators $SRI_{TVaR_\alpha}(X_1|S)$ and $SRI_{e_\alpha}(X_1|S)$ as a function of the level α in the case of $\gamma = 1$. The impact of the dependence structure is evident in the asymptotic behavior of these indicators, which differs from that observed in Figure 2. This difference arises because the considered copula has $\lambda_U > 0$, while this measure of extreme dependence is zero for the FGM copula. In Figure 6, we explore the cases of $\gamma = 1.01, \gamma = 1.5$, and $\gamma = 2$. It is important to note that in this model, we assume $\gamma > 1$. As the dependence increases by decreasing the parameter γ, the values of the indicators decrease due to positive dependence. By decreasing the parameter γ, we approach the case of perfect dependence, which nullifies diversification and, consequently, the systemic risk indicators. This effect is particularly noticeable for $\gamma = 1.01$.

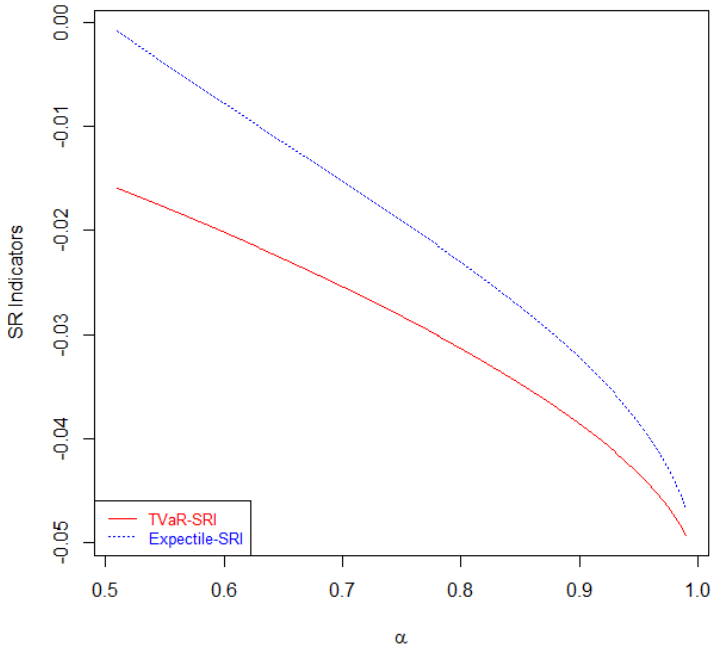

FIGURE 5. $SRI_{TVaR_\alpha}(X_1|S)$ and $SRI_{e_\alpha}(X_1|S)$ - Survival Clayton Model ($b = 1$,$\gamma = 2$, $X_1 \sim \mathcal{E}(\beta_1 = 0.10)$ and $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$)

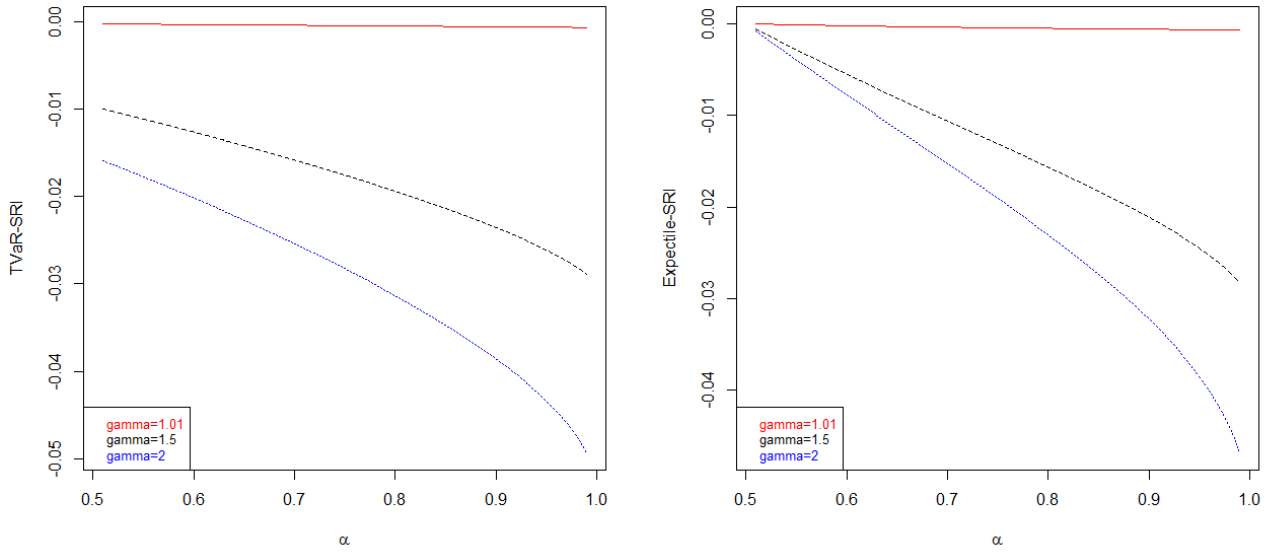

FIGURE 6. Survival Clayton Model: ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$ and $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$)
Left: $SRI_{TVaR_\alpha}(X_1|S)$ - Right: $SRI_{e_\alpha}(X_1|S)$

To examine the asymptotic behavior of the indicators, we present in Figure 7 their variation as a function of the parameter γ for $\alpha = 99\%$.

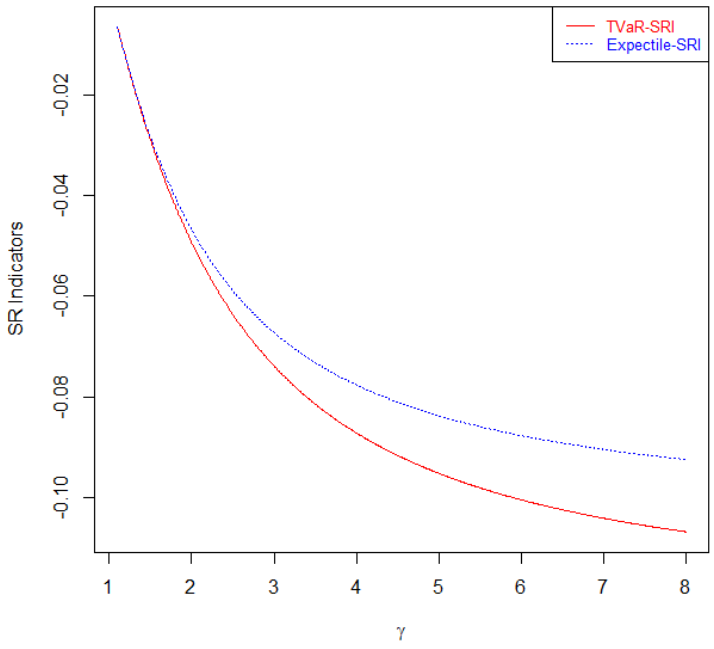

FIGURE 7. $SRI_{TVaR_{99\%}}(X_1|S)$ and $SRI_{e_{99\%}}(X_1|S)$ - Survival Clayton Model ($X_1 \sim \mathcal{E}(\beta_1 = 0.10)$ and $X_2 \sim \mathcal{E}(\beta_2 = 0.25)$)

As extreme dependence increases with decreasing γ, the systemic risk level of X_1 decreases more rapidly compared to FGM case presented in Figure 4. Hence, the nature of the dependence structure significantly affects the asymptotic behavior of these indicators.

CONCLUSION

In this article, our aim was to propose a construction of systemic risk indicators based on risk allocation methods commonly used in actuarial science. These indicators can be derived from various risk measures depending on the desired level of risk aversion. Their economic interpretation is straightforward, and their estimation follows the chosen risk measures.

Furthermore, we demonstrated the impact of dependence on the behavior of these indicators, which were derived from well-known measures such as TVaR (Value at Risk) and expectile, using a few examples of bivariate copulas. As a future perspective for this research, it would be interesting to apply these indicators to a financial system in order to identify components that represent significant systemic risk. Subsequently, it would be possible to develop a decision support tool aimed at mitigating this risk, such as by recommending capital increases.

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