# INVERSE STABLE RECONSTRUCTION OF 3 COEFFICIENTS FOR THE HETEROGENEOUS MAXWELL EQUATIONS BY FINITE NUMBER OF PARTIAL INTERIOR OBSERVATIONS

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# INVERSE STABLE RECONSTRUCTION OF 3 COEFFICIENTS FOR THE HETEROGENEOUS MAXWELL EQUATIONS BY FINITE NUMBER OF PARTIAL INTERIOR OBSERVATIONS

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ABSTRACT. We consider an inverse problem of determining the isotropic inhomogeneous electromagnetic coefficients of the non-stationary Maxwell's equations in a bounded domain of $\mathbb{R}^3$ by means of a finite number of interior data of as less as possible components of the solutions. Our main result is a Lipschitz stability estimate for the inverse problem and our proof relies on a Carleman estimate for the heterogeneous Maxwell's equations.
**Key words:** Inverse problems, Maxwell system, Carleman estimates.

## 1. INTRODUCTION

In this paper, we discuss the uniqueness and stability in determining the isotropic electromagnetic coefficients of the dynamical Maxwell's equations by local measurement of as less as possible components of their solutions. Let $\Omega \subset \mathbb{R}^3$ be a non-empty simply-connected bounded domain with $\mathcal{C}^2$ boundary $\Gamma = \partial\Omega$, $T > 0$, and let $\omega$ be a non-empty subdomain of $\Omega$ such that

$$\partial\omega \supset \partial\Omega, \tag{1.1}$$

that is, $\omega$ is a neighborhood of $\partial\Omega$. We introduce the following notations : $\Omega_T := \Omega \times (-T, T)$, $\Sigma := \Gamma \times (-T, T)$ and $\omega_T := \omega \times (-T, T)$.

We understand that $\Omega$ is occupied by an inhomogeneous medium with dielectric permittivity $\varepsilon$, magnetic permeability $\mu$ and conductivity $\sigma$ which depend on $x$ but are time-independent. Throughout this paper, we assume that $\rho$ is given.

We consider the problem for the linear system of Maxwell's equations

$$\begin{cases} \partial_t \mathrm{D} - \mathrm{curl}\,(\frac{1}{\mu}\mathrm{B}) + \frac{\sigma}{\varepsilon}\mathrm{D} = 0, & \text{in } \Omega_T, \\ \partial_t \mathrm{B} + \mathrm{curl}\,(\frac{1}{\varepsilon}\mathrm{D}) = 0, & \text{in } \Omega_T, \\ \mathrm{div}\,\mathrm{D} = \rho,\ \mathrm{div}\,\mathrm{B} = 0, & \text{in } \Omega_T, \\ \mathrm{D} \times \nu = q(x,t), & \text{on } \Sigma, \\ \mathrm{B}(0,x) = \mathrm{B}_0(x), \quad \mathrm{D}(0,x) = \mathrm{D}_0(x), & \text{in } \Omega. \end{cases} \tag{1.2}$$

Here the electric flux density $\mathrm{D}(t,x)$ and the magnetic flux density $\mathrm{B}(t,x)$ are three-dimensional vector-valued functions of the time $t$ and the space variable $x = (x_1, x_2, x_3)$. By $\nu = \nu(x,t)$ we denotes the unit outward normal vector to $\Gamma$. Moreover $B_0$, $D_0$ and $q$ are given vector-valued functions and $\rho$ denotes the electric charge density. We assume that $B_0$, $D_0$, $\rho$ and $q$ are sufficiently smooth and satisfy compatibility conditions so that the solution $(D, B)$ to (1.2) is in

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$((W^{2,\infty}(\Omega_T))^3)^2$ (see e.g., [5], [11]).

Henceforth the subscript $j \in \{1,2,3\}$ denotes the $j^{\text{th}}$ component of a vector under consideration.

We assume that $\mu$, $\varepsilon$ and $\sigma$ are scalar functions in $\mathcal{C}^2(\overline{\Omega})$ verifying

$$\mu(x) \geq \alpha_0, \quad \varepsilon(x) \geq \alpha_0, \quad \sigma(x) \geq \sigma_0, \qquad x \in \overline{\Omega} \tag{1.3}$$

for some fixed constant $\alpha_0 > 0$.

In the following, we consider the inverse problem of determining simultaneously the dielectric permittivity $\varepsilon = \varepsilon(x)$, the magnetic permeability $\mu = \mu(x)$ and the conductivity $\sigma = \sigma(x)$ from a finite number of observations in $\omega_T$ of the solution $(\mathrm{B}, \mathrm{D})$ to (1.2). Furthermore, our goal is to reduce the number of components to be observed and we prove a reconstruction result of three coefficients $(\mu, \lambda, \sigma)$ by the knowledge of only selected components.

1.1. **Notations and main result.** Let us choose two suitable sets $\mathrm{B}_0^k$, $\mathrm{D}_0^k$, $k = 1, 2$ of initial data. By $\mathrm{B}^k = (B_1^k, B_2^k, B_3^k)$ and $\mathrm{D} = (D_1^k, D_2^k, D_3^k)$ we denote the magnetic flux density and the electric flux density respectively with initial values $B_0^k, D_0^k$. In this paper, we consider the following inverse problem:*Determine $\varepsilon(x)$, $\mu(x)$ and $\sigma(x)$ for $x \in \Omega$ from the observations for two sets of initial conditions of the 2 components:* $\mathrm{D}_j^k(t,x)$, $(t,x) \in \omega \times (-T,T)$, $k = 1,2$, *and* $j = 2,3$.

We now introduce some notations used throughout this paper and state our results.
We consider $x_0 \in \mathbb{R}^3 \backslash \overline{\Omega}$ and a non-empty subdomain $\widehat{\omega}$ of $\Omega$ such that $\widehat{\omega} = \Omega \cap \mathcal{O}$ for some neighbourhood $\mathcal{O}$ of $\Gamma$ in $\mathbb{R}^3$ and $\omega \subset \widehat{\omega}$. Then, for a fixed function $\widehat{\varepsilon} \in \mathcal{C}^2(\overline{\widehat{\omega}})$ and given constants $M_0, M_1 > 0$, $0 < C_0 < 1$, we define an admissible set of unknown coefficients $\mu$, $\varepsilon$ and $\sigma$:

$$\begin{aligned} \mathcal{A} = \big\{ (\mu,\varepsilon,\sigma) \ \in (\mathcal{C}^2(\overline{\Omega}))^3 ; \ &\text{(1.3) holds,} \\ \|\mu\|_{\mathcal{C}^2(\overline{\Omega})}, \ \|\varepsilon\|_{\mathcal{C}^2(\overline{\Omega})}, \ \|\sigma\|_{\mathcal{C}^2(\overline{\Omega})} \leq M_0, \ \ &\varepsilon = \widehat{\varepsilon} \text{ in } \widehat{\omega} \\ |\nabla(\varepsilon(x)\mu(x))| < M_1 \text{ and } \frac{(x-x_0)\cdot\nabla(\varepsilon(x)\mu(x))}{2\varepsilon(x)\mu(x)} &> -C_0, \ \forall x \in \overline{\Omega}, \ x_0 \in \mathbb{R}^3 \setminus \overline{\Omega} \Big\}. \end{aligned} \tag{1.4}$$

Condition (1.4) is one sufficient condition for a key Carleman estimate (Lemma 2.2), and does not necessarily exclude other choices of conditions (e.g., Amirov and Yamamoto [1]).

Our main result, Theorem 1.1, establishes Lipschitz stability for the three coefficients $\mu$, $\varepsilon$ and $\sigma$ in suitable norms by partial measurement data of two components of $\mathrm{D}(t,x)$.
Henceforth $\cdot^\top$ denotes the transpose of a matrix under consideration and we set $e_1 = (1,0,0)^\top$, $e_2 = (0,1,0)^\top$ and $e_3 = (0,0,1)^\top$. From now, $C > 0$ denotes generic constants which may change from line to line.

For the identification of $(\varepsilon, \mu, \sigma)$, we need to take observations twice, that is, we consider two sets of initial data $(D_0^k, B_0^k)$, $k = 1, 2$:

$$\mathrm{D}_0^k(x) := \big(D_1^k(x), D_2^k(x), D_3^k(x)\big)^\top, \quad \mathrm{B}_0^k(x) := \big(B_1^k(x), B_2^k(x), B_3^k(x)\big)^\top. \tag{1.5}$$

For the statement of the main results, we define the $12 \times 7$ matrix

$$\mathcal{G}(x) = \begin{pmatrix} e_1 \times \mathrm{B}_0^1 & e_2 \times \mathrm{B}_0^1 & e_3 \times \mathrm{B}_0^1 & 0 & 0 & 0 & \mathrm{D}_0^1 \\ 0 & 0 & 0 & e_1 \times \mathrm{D}_0^1 & e_2 \times \mathrm{D}_0^1 & e_3 \times \mathrm{D}_0^1 & 0 \\ e_1 \times \mathrm{B}_0^2 & e_2 \times \mathrm{B}_0^2 & e_3 \times \mathrm{B}_0^2 & 0 & 0 & 0 & \mathrm{D}_0^2 \\ 0 & 0 & 0 & e_1 \times \mathrm{D}_0^2 & e_2 \times \mathrm{D}_0^2 & e_3 \times \mathrm{D}_0^2 & 0 \end{pmatrix} \tag{1.6}$$

for $x \in \Omega$, where $a \times b$ denotes the exterior product: $a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)^\top$ for $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$.

Moreover by $(\mathrm{B}^k(t,x), \mathrm{D}^k(t,x))$ and $(\widetilde{\mathrm{B}}^k(t,x), \widetilde{\mathrm{D}}^k(t,x))$ we denote the solutions to (1.2) with initial data $(\mathrm{B}_0^k, \mathrm{D}_0^k)$, $k = 1, 2$, associated to $(\mu, \varepsilon, \sigma)$ and $(\widetilde{\mu}, \widetilde{\varepsilon}, \widetilde{\sigma})$, respectively.

**Theorem 1.1.** *Let $(\mathrm{B}^k(t,x), \mathrm{D}^k(t,x))$, $k = 1, 2$, be the solutions to (1.2) with initial data $(\mathrm{B}_0^k, \mathrm{D}_0^k) \in (H^2(\Omega)^3)^2$, $k = 1, 2$ such that there exists a $7 \times 7$ minor $m(x)$ of the matrix $\mathcal{G}(x)$ defined by* (1.6) *such that*

$$m(x) \neq 0 \quad \text{for } x \in \overline{\Omega \backslash \widehat{\omega}}. \tag{1.7}$$

*We choose $((\mu, \varepsilon, \sigma), (\widetilde{\mu}, \widetilde{\varepsilon}, \widetilde{\sigma})) \in \mathcal{A}^2$ such that*

$$\mu = \widetilde{\mu},\ \varepsilon = \widetilde{\varepsilon} \text{ on } \partial\Omega \tag{1.8}$$

*and*

$$\left\| \left(\mathrm{B}^k, \mathrm{D}^k\right) \right\|_{\mathcal{C}^3([-T,T];W^{2,\infty}(\Omega))}, \left\| \left(\widetilde{\mathrm{B}}^k, \widetilde{\mathrm{D}}^k\right) \right\|_{\mathcal{C}^3([-T,T];W^{2,\infty}(\Omega))} \leq M_1,\ k = 1, 2, \tag{1.9}$$

*for some $M_1 > 0$. Then there exists an open set $\omega$ and a constant $C > 0$ depending on $\Omega, \omega, T, C_0, \alpha, M_0$ and $M_1$, such that we have*

$$\begin{aligned} &\|\mu - \widetilde{\mu}\|_{H^1(\Omega)} + \|\varepsilon - \widetilde{\varepsilon}\|_{H^1(\Omega)} + \|\sigma - \widetilde{\sigma}\|_{H^1(\Omega)} \\ &\leq C \sum_{k=1}^{2} \sum_{j=2}^{3} \left\| \mathrm{D}_j^k - \widetilde{\mathrm{D}}_j^k \right\|_{H^1(-T,T;L^2(\omega))}. \end{aligned}$$

Theorems 1.1 asserts Lipschitz stability in determining coefficients of Maxwell's equations within the class defined by (1.4). The proof is based on a Carleman estimate stated in Lemma 2.2.

Concerning inverse problems for Maxwell's equations, we can refer to Romanov [15], [16], Romanov and Kabainikhin [17], Sun and Uhlmann [19], Yamamoto [20], [21]. However, there are few results on inverse problems of determining the coefficients of Maxwell's equations with a reduced number of measurements. Li and Yamamoto [12], [13] establish Lipschitz stability for the determination of the dielectric permittivity $\varepsilon$ and the magnetic permeability $\mu$ with a finite number of measurements provided that unknown coefficients satisfy some a priori conditions. They extend their results in [14] to non-stationary media. In [3], Bellassoued, Cristofol and Soccorsi give a Hölder stability estimate for the determination of the dielectric permittivity and the magnetic permeability in heterogeneous Maxwell's equations by some partial boundary measurements, and in [2], the authors established a similar stability inequality using less observations and provide numerical simulations. Recently in [18] the case of uniaxially anisotropic medium was addressed.

The remainder of the paper is organized as follows: a Carleman estimate for Maxwell's equations (1.2) is given in Section 2. In Section 3, we prove our main Theorem 1.1.

## 2. Carleman estimate for Maxwel's equations

We now consider the solutions $(\mathrm{B}^k, \mathrm{D}^k, \mu, \varepsilon, \sigma)$ and $(\widetilde{\mathrm{B}}^k, \widetilde{\mathrm{D}}^k, \widetilde{\mu}, \widetilde{\varepsilon}, \widetilde{\sigma})$, $k = 1, 2$ to (1.2) associated with initial data $(\mathrm{B}_0^k, \mathrm{D}_0^k) \in (H^2(\Omega)^3)^2$, and we set

$$\mathbb{D}^k(t,x) = \mathrm{D}^k - \widetilde{\mathrm{D}}^k,\ \mathbb{B}^k(t,x) = \mathrm{B}^k - \widetilde{\mathrm{B}}^k,\ d_{mp} = \frac{1}{\mu} - \frac{1}{\widetilde{\mu}},\ d_{dp} = \frac{1}{\varepsilon} - \frac{1}{\widetilde{\varepsilon}},\ d_c = \frac{\sigma}{\varepsilon} - \frac{\widetilde{\sigma}}{\widetilde{\varepsilon}}.$$

In the following, for simplicity, we will omit the superscript $k$ and we deal with the system :

$$\begin{cases} \partial_t \mathbb{D} - \mathrm{curl}\,(\frac{1}{\mu}\mathbb{B}) + \frac{\sigma}{\varepsilon}\mathbb{D} = R_1, & \text{in } \Omega_T, \\ \partial_t \mathbb{B} + \mathrm{curl}\,(\frac{1}{\varepsilon}\mathbb{D}) = R_2, & \text{in } \Omega_T, \\ \mathrm{div}\,\mathbb{D} = 0, \quad \mathrm{div}\,\mathbb{B} = 0, & \text{in } \Omega_T, \\ \mathbb{D} \times \nu = 0, & \text{on } \Sigma, \\ \mathbb{B}(x,0) = 0, \quad \mathbb{D}(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $R_1 = \mathrm{curl}\,(d_{mp}\widetilde{\mathrm{B}}) - d_c\widetilde{\mathrm{D}}$ and $R_2 = -\mathrm{curl}\,(d_{dp}\widetilde{\mathrm{D}})$.
To this form of Maxwel's equations, we apply a Carleman estimate in the $H^{-1}$ norm for a second-order hyperbolic equation. Indeed, we can reduce Maxwell's equations to a weakly coupled system of hyperbolic equations. More precisely, taking the time derivative in (2.1) this leads to the following system:

$$\begin{cases} \varepsilon\mu\partial_t^2\mathbb{D} - \Delta\mathbb{D} = f(\mathbb{B}, \mathbb{D}, R_1, R_2, \varepsilon, \mu, \sigma), \\ \varepsilon\mu\partial_t^2\mathbb{B} - \Delta\mathbb{B} = g(\mathbb{B}, \mathbb{D}, R_1, R_2, \varepsilon, \mu, \sigma), \end{cases} \tag{2.2}$$

where

$$f(\mathbb{B}, \mathbb{D}, R_1, R_2, \varepsilon, \mu, \sigma) = \varepsilon\mu\partial_t R_1 + \varepsilon\mu\mathrm{curl}\,(\frac{1}{\mu}R_2)$$
$$+\sigma\mu\left(-\mathrm{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{\sigma}{\varepsilon}\mathbb{D} - R_1\right) - \varepsilon\mu\left(\nabla(\frac{1}{\mu}) \times (\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) + \nabla(\frac{1}{\mu}) \times (\frac{1}{\varepsilon}\mathrm{curl}\,\mathbb{D})\right)$$
$$-\varepsilon\left(\mathrm{curl}\,(\nabla(\frac{1}{\varepsilon}) \times \mathbb{D} + \nabla(\frac{1}{\varepsilon}) \times \mathrm{curl}\,\mathbb{D}\right)$$

and

$$g(\mathbb{B}, \mathbb{D}, R_1, R_2, \varepsilon, \mu, \sigma) = \varepsilon\mu\partial_t R_2 + \varepsilon\mu\left(\mathrm{curl}\,(\frac{\sigma}{\epsilon^2}\mathbb{D}) - \mathrm{curl}\,(\frac{R_1}{\epsilon})\right)$$
$$-\varepsilon\mu\left(\nabla(\frac{1}{\epsilon}) \times \mathrm{curl}\,(\frac{1}{\mu}\mathbb{B}) + \frac{1}{\epsilon}\mathrm{curl}\,(\nabla(\frac{1}{\mu}) \times \mathbb{B}) + \frac{1}{\epsilon}\nabla(\frac{1}{\mu}) \times \mathrm{curl}\,\mathbb{B}\right).$$

We detail the calculation for $f(\mathbb{B}, \mathbb{D}, R_1, R_2, \varepsilon, \mu, \sigma)$ as follows. In $\partial_t^2\mathbb{D} - \mathrm{curl}\,(\frac{1}{\mu}\partial_t\mathbb{B}) + \frac{\sigma}{\varepsilon}\partial_t\mathbb{D} = \partial_t R_1$, we substitute

$\partial_t\mathbb{D} = \mathrm{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{\sigma}{\varepsilon}\mathbb{D} + R_1$ and $\partial_t\mathbb{B} = -\mathrm{curl}\,(\frac{1}{\varepsilon}\mathbb{D}) + R_2$,

to obtain

$\partial_t^2\mathbb{D} - \text{curl}\,(-\frac{1}{\mu}\text{curl}\,(\frac{1}{\varepsilon}\mathbb{D}) + \frac{1}{\mu}R_2) + \frac{\sigma}{\varepsilon}(\text{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{\sigma}{\varepsilon}\mathbb{D} + R_1) = \partial_t R.$

Thus

$\partial_t^2\mathbb{D} + \text{curl}\,(\frac{1}{\mu}\text{curl}\,(\frac{1}{\varepsilon}\mathbb{D})) - \text{curl}\,(\frac{1}{\mu}R_2)\frac{\sigma}{\varepsilon}\text{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{\sigma^2}{\varepsilon^2}\mathbb{D} + \frac{\sigma}{\varepsilon}R_1 = \partial_t R.$

Therefore,

$$\begin{aligned}
& \text{curl}\,(\frac{1}{\mu}(\text{curl}\,(\frac{1}{\varepsilon}\mathbb{D}))) \\
= \quad & \text{curl}\,(\frac{1}{\mu}(\nabla(\frac{1}{\varepsilon}) \times \mathbb{D} + \frac{1}{\varepsilon}\text{curl}\,\mathbb{D})) \\
= \quad & \nabla(\frac{1}{\mu}) \times (\nabla(\frac{1}{\varepsilon}) \times \mathbb{D} + \frac{1}{\varepsilon}\text{curl}\,\mathbb{D}) + \frac{1}{\mu}\text{curl}\,(\nabla(\frac{1}{\varepsilon}) \times \mathbb{D} + \frac{1}{\varepsilon}\text{curl}\,\mathbb{D}) \\
= \quad & \nabla(\frac{1}{\mu}) \times (\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) + \nabla(\frac{1}{\mu}) \times (\frac{1}{\varepsilon}\text{curl}\,\mathbb{D}) + \frac{1}{\mu}\text{curl}\,(\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) + \frac{1}{\mu}\text{curl}\,(\frac{1}{\varepsilon}\text{curl}\,\mathbb{D}),
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{\mu}\text{curl}\,(\frac{1}{\varepsilon}\text{curl}\,\mathbb{D}) = \quad & \frac{1}{\mu}(\nabla(\frac{1}{\varepsilon}) \times \text{curl}\,\mathbb{D} + \frac{1}{\varepsilon}\text{curl}\,\text{curl}\,\mathbb{D}) \\
= \quad & \frac{1}{\mu}\nabla(\frac{1}{\varepsilon}) \times \text{curl}\,\mathbb{D} + \frac{1}{\varepsilon\mu}(-\Delta\mathbb{D} + \nabla\text{div}\,\mathbb{D}).
\end{aligned}$$

As $\text{div}\,\mathbb{D} = 0$, we reach

$$\begin{aligned}
\text{curl}\,(\frac{1}{\mu}(\text{curl}\,(\frac{1}{\varepsilon}\mathbb{D}))) = \quad & \nabla(\frac{1}{\mu}) \times (\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) + \nabla(\frac{1}{\mu}) \times (\frac{1}{\varepsilon}\text{curl}\,\mathbb{D}) + \frac{1}{\mu}\text{curl}\,(\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) \\
+ \quad & \frac{1}{\mu}\nabla(\frac{1}{\varepsilon}) \times \text{curl}\,\mathbb{D} - \frac{1}{\varepsilon\mu}\Delta\mathbb{D}.
\end{aligned}$$

Finally,

$$\begin{aligned}
& \partial_t^2\mathbb{D} - \frac{1}{\varepsilon\mu}\Delta\mathbb{D} \\
+ \quad & \nabla(\frac{1}{\mu}) \times (\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) + \nabla(\frac{1}{\mu}) \times (\frac{1}{\varepsilon}\text{curl}\,\mathbb{D}) + \frac{1}{\mu}\text{curl}\,(\nabla(\frac{1}{\varepsilon}) \times \mathbb{D}) \\
+ \quad & \frac{1}{\mu}\nabla(\frac{1}{\varepsilon}) \times \text{curl}\,\mathbb{D} + \frac{\sigma}{\varepsilon}\text{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{\sigma^2}{\varepsilon^2}\mathbb{D} \\
= \quad & \partial_t R_1 - \frac{\sigma}{\varepsilon}R_1 + \text{curl}\,(\frac{1}{\mu}R_2).
\end{aligned}$$

We obtain $g(\mathbb{B}, \mathbb{D}, R_1, R_2, \varepsilon, \mu, \sigma)$ similarly, after taking the time derivative of

$\partial_t\mathbb{B} + \text{curl}\,(\frac{1}{\varepsilon}\mathbb{D}) = R_2.$

Therefore

$\partial_t^2\mathbb{B} = -\text{curl}\,\left(\frac{1}{\varepsilon}[\text{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{\sigma}{\varepsilon}\mathbb{D} + R_1]\right) + \partial_t R_2.$

We obtain

$\partial_t^2\mathbb{B} = -\text{curl}\,(\frac{1}{\varepsilon}\text{curl}\,(\frac{1}{\mu}\mathbb{B})) + \text{curl}\,(\frac{\sigma}{\varepsilon^2}\mathbb{D}) - \text{curl}\,(\frac{1}{\varepsilon}R_1) + \partial_t R_2$

and

$\partial_t^2\mathbb{B} = -\nabla(\frac{1}{\varepsilon}) \times \text{curl}\,(\frac{1}{\mu}\mathbb{B}) - \frac{1}{\varepsilon}\text{curl}\,[\text{curl}\,(\frac{1}{\mu}\mathbb{B})] + \text{curl}\,(\frac{\sigma}{\varepsilon^2}\mathbb{D}) - \text{curl}\,(\frac{1}{\varepsilon}R_1) + \partial_t R_2.$

Multiplying by $\varepsilon\mu$, we obtain

$$\varepsilon\mu\partial_t^2\mathbb{B} = -\varepsilon\mu[\nabla(\frac{1}{\varepsilon}) \times \text{curl}\,(\frac{1}{\mu}\mathbb{B})] - \mu\text{curl}\,\left[\nabla(\frac{1}{\mu}) \times \mathbb{B} + \frac{1}{\mu}\text{curl}\,\mathbb{B}\right]$$
$$+\varepsilon\mu\left[\text{curl}\,(\frac{\sigma}{\varepsilon^2}\mathbb{D}) - \text{curl}\,(\frac{1}{\varepsilon}R_1) + \partial_t R_2\right].$$

Then, using $\text{curl}\,(\text{curl}\,\mathbb{B}) = \nabla(\text{div}\,\mathbb{B}) - \Delta\mathbb{B} = -\Delta\mathbb{B}$, we reach the desired result.

For $x_0 \in \mathbb{R}^3\backslash\overline{\Omega}$ and a constant $\beta_0 > 0$, we define the weight function $\psi(t,x)$ by

$$\psi(x,t) = |x - x_0|^2 - \beta t^2 + \beta_0,\ x \in \overline{\Omega},\ t \in [-T,T]. \tag{2.3}$$

Then $\min_{x\in\overline{\Omega}} \psi(x,0) \geq \beta_0$.

Furthermore we set $\varphi : \Omega \times [-T,T] \longrightarrow \mathbb{R}$ defined as

$$\varphi(x,t) = e^{\lambda\psi(x,t)},\ t \in [-T,T],\ x \in \overline{\Omega} \tag{2.4}$$

with some fixed $\lambda > 0$.

Recall the Carleman estimate in the $H^{-1}$ norm for a second-order hyperbolic equation:

**Lemma 2.1.** *Let $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $\varphi$ be given by (2.4). We assume that $a \in \mathcal{C}^2(\overline{\Omega_T})$ satisfies the following conditions*

- $0 < \beta a(t,x) \leq 1$,
- $\beta^2\left(a^2(t,x) + a(t,x)t\partial_t a(t,x) + 2\,|t|\,|\nabla a(t,x)|\right) < 1 + \dfrac{(x-x_0)\cdot\nabla a(t,x)}{a(t,x)}$

*Let $v \in H_0^2(\Omega_T)$ such that*

$$Pv(t,x) = a^2(t,x)\partial_t^2 v(t,x) - \Delta v(t,x) = g(t,x) + \sum_k \partial_k g_k, \quad x \in \Omega, \quad t \in \mathbb{R}. \tag{2.5}$$

*Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, there exist two constants $C_0 > 0$ and $s_0 > 0$ such that the following Carleman estimate holds*

$$s\int_{\Omega_T} e^{2s\varphi}\,|v|^2\,dxdt \leq C_0\int_{\Omega_T} e^{2s\varphi}\left(\frac{1}{s^2}\,|g(x,t)|^2 + \sum_k |g_k|^2\right)dxdt \tag{2.6}$$

*for all $s \geq s_0$.*

For the proof, we can refer to [6] or [9].

We deduce now a Carleman estimate for Maxwell system (2.2) which is one of the main ingredients in our proof.

**Lemma 2.2.** *Let $\mathbb{D}, \mathbb{B} \in H_0^1(\Omega_T)^3$ be solutions of system* (2.1)*. We assume that $(\varepsilon, \mu, \sigma) \in \mathcal{A}$. Then there exist $s_1 > 0$ and $C_1 > 0$ such that*

$$s \int_{\Omega_T} (|\mathbb{D}|^2 + |\mathbb{B}|^2) e^{2s\varphi} dxdt \leq C_1 \int_{\Omega_T} (|R_1|^2 + |R_2|^2) e^{2s\varphi} dxdt$$

*for all $s > s_1$.*

We will apply the Carleman estimate Lemma 2.2 to the system (2.2) for a sufficiently large $s$.
We also recall a Carleman estimate for a first order differential equation (see [6]) :

**Lemma 2.3.** *Let $\varphi$ be given by* (2.4)*. Then there exist $s_1 > 0$ and $K_1 > 0$ such that*

$$s \int_\Omega |w|^2 e^{2s\varphi(\cdot,0)} dx \leq K_1 \sum_{k=1}^{3} \int_\Omega |\partial_k w|^2 e^{2s\varphi(\cdot,0)} dx$$

*for all $s > s_1$ and $w \in \mathcal{C}_0^1(\Omega)$.*

We need now two technical lemma involving initial conditions and pointwise observations (see [10] and [14]).

**Lemma 2.4.** *We assume that $(\varepsilon, \mu, \sigma) \in \mathcal{A}$. Let $\mathbb{D}, \mathbb{B} \in H_0^1(\Omega_T)^3$ satisfy (2.1). Then there exist $s_2 > 0$ and $C_2 > 0$ such that*

$$\int_\Omega (|\mathbb{D}(\cdot,0)|^2 + |\mathbb{B}(\cdot,0)|^2) e^{2s\varphi(\cdot,0)} dx \leq C_2 \int_{\Omega_T} (|R_1|^2 + |R_2|^2) e^{2s\varphi} dxdt$$

*for all $s > s_2$.*

**Lemma 2.5.** *We assume that $(\varepsilon, \mu, \sigma) \in \mathcal{A}$. Let $\mathbb{D}, \mathbb{B}$ satisfy (2.1). Then there exists a constant $C_3 > 0$ such that*

$$\int_\Omega (|\mathbb{D}(\cdot,t_2)|^2 + |\mathbb{B}(\cdot,t_2)|^2) dx \leq C_3 \left( \int_\Omega (|\mathbb{D}(\cdot,t_1)|^2 + |\mathbb{B}(\cdot,t_1)|^2) dx + \int_{\Omega_T^*} (|R_1|^2 + |R_2|^2) dxdt \right)$$

*for $-T \leq t_1, t_2 \leq T$. Here $\Omega_T^* = \Omega \times (\min(t_1,t_2),\, \max(t_1,t_2)) \subset \Omega_T$.*

## 3. Main Stability Result

In the following, we consider a new system obtained by differentiating the system (2.1) with respect to $t$. Setting $\mathbb{D}_1 = \partial_t \mathbb{D}$ and $\mathbb{B}_1 = \partial_t \mathbb{B}$, we deduce

$$\begin{cases} \partial_t \mathbb{D}_1 - \text{curl}\,(\dfrac{1}{\mu}\mathbb{B}_1) + \dfrac{\sigma}{\varepsilon}\mathbb{D}_1 = R_{1,1}, & \text{in } \Omega_T, \\ \partial_t \mathbb{B}_1 + \text{curl}\,(\dfrac{1}{\varepsilon}\mathbb{D}_1) = R_{2,1}, & \text{in } \Omega_T, \\ \text{div}\,\mathbb{D}_1 = 0, \;\; \text{div}\,\mathbb{B}_1 = 0, & \text{in } \Omega_T, \\ \mathbb{D}_1 \times \nu = 0, & \text{on } \Sigma, \\ \mathbb{B}_1(x,0) = R_2(x,0) = -\text{curl}\,(d_{dp} B_0(x)), & \text{in } \Omega, \\ \mathbb{D}_1(x,0) = R_1(x,0) = \text{curl}\,(d_{mp} B_0(x)) - d_c D_0(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

with $R_{1,1} = \partial_t R_1$ and $R_{2,1} = \partial_t R_2$.
We are now dealing with systems (2.1) and (3.1) and we consider a cut-off function $\chi_1(x)$ such that

$$\chi_1 \in \mathcal{C}_0^\infty(\Omega), 0 \leq \chi_1(x) < 1 \text{ on } \omega, \text{ and } \chi_1(x) = 1 \text{ for } x \in \overline{\Omega \backslash \omega}.$$

We set

$$\mathbb{D}_{1,1} = \chi_1 \mathbb{D}_1, \quad \mathbb{B}_{1,1} = \chi_1 \mathbb{B}_1 \in (W^{1,\infty}(Q))^3,$$

and we have

$$\left\{ \begin{array}{ll} \partial_t \mathbb{D}_{1,1} - \text{curl}\,(\frac{1}{\mu}\mathbb{B}_{1,1}) + \frac{\sigma}{\varepsilon}\mathbb{D}_{1,1}, & \\ = \chi_1 \text{curl}\,(d_{mp}\partial_t \widetilde{\mathrm{B}}) - \chi_1 d_c \partial_t \widetilde{\mathrm{D}} - (\nabla \chi_1) \times (\frac{1}{\mu}\mathbb{B}_1), & \text{in } \Omega_T, \\ \partial_t \mathbb{B}_{1,1} + \text{curl}\,(\frac{1}{\varepsilon}\mathbb{D}_{1,1}) = -\chi_1 \text{curl}\,(d_{dp}\partial_t \widetilde{\mathrm{D}}) + (\nabla \chi_1) \times (\frac{1}{\varepsilon}\mathbb{D}_1), & \text{in } \Omega_T, \\ \text{div}\,\mathbb{D}_{1,1} = \nabla \chi_1 \cdot \mathbb{D}_1, \qquad \text{div}\,\mathbb{B}_{1,1} = \nabla \chi_1 \cdot \mathbb{B}_1, & \text{in } \Omega_T, \\ \mathbb{D}_{1,1} = 0, \qquad \mathbb{B}_{1,1} = 0, & \text{on } \Sigma, \\ \mathbb{B}_{1,1}(x,0) = R_2(x,0) = -\text{curl}\,(d_{dp}B_0(x)), & \text{in } \Omega, \\ \mathbb{D}_{1,1}(x,0) = R_1(x,0) = \text{curl}\,(d_{mp}B_0(x)) - d_c D_0(x), & \text{in } \Omega. \end{array} \right. \tag{3.2}$$

Then, we use a cut-off function in time $\chi_2(t)$ satisfying

$$\chi_2 \in \mathcal{C}^\infty(\mathbb{R}), 0 \leq \chi_2(t) \leq 1 \text{ for } t \in \mathbb{R},$$

and

$$\chi_2(t) = \left\{ \begin{array}{ll} 0, & t \in [-T, -T+\delta] \cup [T-\delta, T], \\ 1, & t \in [-T+2\delta, T-2\delta]. \end{array} \right.$$

**Remark 3.1.** *We have*

$$\varphi(x,t) - \varphi(x,0) = e^{\lambda|x-x_0|^2}\left(e^{-\lambda\beta t^2} - 1\right) \leq e^{-\lambda\beta t^2} - 1 \leq 0 \quad \textit{for } x \in \overline{\Omega}.$$

*Then,*

$$\begin{array}{l} \displaystyle\int_{\Omega_T} \left(|d_{dp}|^2 + |\nabla d_{dp}|^2 + |d_{mp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2\right) e^{2s\varphi} dxdt \\ \displaystyle\leq C_9 \int_{\Omega_T} \left(|d_{dp}|^2 + |\nabla d_{dp}|^2 + |d_{mp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2\right) \\ \displaystyle\times e^{2s\varphi(\cdot,0)} \left[\int_{-T}^{T} e^{2s(\varphi(\cdot,t)-\varphi(\cdot,0)} dt\right] dx \\ \displaystyle\leq C_{10}\kappa_1(s) \int_\Omega \left(|d_{dp}|^2 + |\nabla d_{dp}|^2 + |d_{mp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2\right) e^{2s\varphi(0,x)} dx \end{array} \tag{3.3}$$

*where* $\kappa_1(s) = \displaystyle\int_{-T}^{T} e^{2s(e^{-\lambda\beta t^2}-1)} dt$.

*On the other hand, using the hypothesis and the definition of $\varphi$, we can write :*

$$\varphi(x,0) \geq \alpha \textit{ and } 0 < \varphi(x,-T) = \varphi(x,T) < \alpha := \exp(\lambda(\inf_{\overline{\Omega}} |x - x_0|^2 + \beta_0)). \tag{3.4}$$

*Thus, for* $\eta \in (0, \alpha - \sup_{\overline{\Omega}} \varphi(x,T))$ *we can choose* $\delta = \delta(\alpha)$ *such that :*

$$\varphi(x,t) \leq \alpha - \eta, \quad (x,t) \in \overline{\Omega} \times ([-T, -T+2\delta] \cup [T-2\delta, T]) \tag{3.5}$$

*and if we set* $\Phi = \sup_{\overline{\Omega_T}} \varphi$*, then* $\Phi \geq \alpha$*.*

Now we set

$\mathbb{D}_{1,2} = \chi_2 \mathbb{D}_{1,1}, \quad \mathbb{B}_{1,2} = \chi_2 \mathbb{B}_{1,1} \in (W^{2,\infty}(\Omega_T))^3.$

The system (3.2) becomes

$$\begin{cases} \partial_t \mathbb{D}_{1,2} - \text{curl}\,(\frac{1}{\mu}\mathbb{B}_{1,2}) + \frac{\sigma}{\varepsilon}\mathbb{D}_{1,2} \\ = \chi_2\chi_1 \text{curl}\,(d_{mp}\partial_t \widetilde{\text{B}}) - \chi_2\chi_1 d_c \partial_t \widetilde{\text{D}} - \chi_2(\nabla\chi_1) \times (\frac{1}{\mu}\mathbb{B}_2) + \partial_t\chi_2 \mathbb{D}_{1,1}, & \text{in } \Omega_T, \\ \partial_t \mathbb{B}_{1,2} + \text{curl}\,(\frac{1}{\varepsilon}\mathbb{D}_{1,2}) \\ = -\chi_2\chi_1 \text{curl}\,(d_{dp}\partial_t \widetilde{\text{D}}) + \chi_2(\nabla\chi_1) \times (\frac{1}{\varepsilon}\mathbb{D}_1) + \partial_t\chi_2 \mathbb{B}_{1,1}, & \text{in } \Omega_T, \\ \text{div}\,\mathbb{D}_{1,2} = \chi_2 \nabla\chi_1 \cdot \mathbb{D}_1, \qquad \text{div}\,\mathbb{B}_{1,2} = \chi_2 \nabla\chi_1 \cdot \mathbb{B}_1, & \text{in } \Omega_T, \\ \mathbb{D}_{1,2} = 0, \qquad \mathbb{B}_{1,2} = 0, & \text{on } \Sigma, \\ \mathbb{B}_{1,2}(x,0) = R_2(x,0) = -\text{curl}\,(d_{dp}B_0(x)), & \text{in } \Omega, \\ \mathbb{D}_{1,2}(x,0) = R_1(x,0) = \text{curl}\,(d_{mp}B_0(x)) - d_c D_0(x), & \text{in } \Omega. \end{cases} \tag{3.6}$$

Now, we can apply Lemma 2.4 to the solutions of the previous system to obtain

$$\begin{aligned} &\int_\Omega \left(|\mathbb{D}_{1,2}(x,0)|^2 + |\mathbb{B}_{1,2}(x,0)|^2\right) e^{2s\varphi(0,x)} dx \\ &\leq C_{14} \int_{\omega_T} |\nabla\chi_1|^2 \left(|\mathbb{D}_1|^2 + |\mathbb{B}_1|^2\right) e^{2s\varphi} dxdt \\ &+C_{14} \int_\Omega \left(|d_{dp}|^2 + |\nabla d_{dp}|^2 + |d_{mp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2\right) e^{2s\varphi} dxdt \\ &+C_{14} \left(\int_{-T+\delta}^{-T+2\delta} + \int_{T-2\delta}^{T+\delta}\right) \int_\Omega |\partial_t\chi_2|^2 \left(|\mathbb{D}_{1,1}|^2 + |\mathbb{B}_{1,1}|^2\right) e^{2s\varphi} dxdt. \end{aligned} \tag{3.7}$$

Then we deal with the last integral and

$$\begin{aligned} &\left(\int_{-T+\delta}^{-T+2\delta} + \int_{T-2\delta}^{T+\delta}\right) \int_\Omega |\partial_t\chi_2|^2 \left(|\mathbb{D}_{1,1}|^2 + |\mathbb{B}_{1,1}|^2\right) e^{2s\varphi} dxdt \\ &\leq C_{15} e^{2s(\alpha-\eta)} \left(\int_{-T+\delta}^{-T+2\delta} + \int_{T-2\delta}^{T+\delta}\right) \int_\Omega \left(|\mathbb{D}_{1,1}|^2 + |\mathbb{B}_{1,1}|^2\right) dxdt \end{aligned} \tag{3.8}$$

It turns out by using Lemma 2.5 and (3.5) that

$$
\begin{aligned}
&C_{15}e^{2s(\alpha-\eta)}\left(\int_{-T+\delta}^{-T+2\delta}+\int_{T-2\delta}^{T+\delta}\right)\int_\Omega\left(|\mathbb{D}_{1,1}|^2+|\mathbb{B}_{1,1}|^2\right)dxdt\\
&\leq C_{16}\delta e^{2s(\alpha-\eta)}\left(\int_\Omega\left(|\mathbb{D}_{1,1}(x,0)|^2+|\mathbb{B}_{1,1}(x,0)|^2\right)dx+\int_{\omega_T}\left(|\mathbb{D}_1|^2+|\mathbb{B}_1|^2\right)dxdt\right.\\
&\left.+\int_\Omega\left(|d_{dp}|^2+|\nabla d_{dp}|^2+|d_{mp}|^2+|\nabla d_{mp}|^2+|d_c|^2\right)dx\right)\\
&\leq C_{17}\delta e^{-2s\eta}\left(\int_\Omega\left(|\mathbb{D}_{1,1}(x,0)|^2+|\mathbb{B}_{1,1}(x,0)|^2\right)e^{2s\varphi(x,0)}dx\right.\\
&\left.+\int_\Omega\left(|d_{dp}|\,2+|\nabla d_{dp}|^2+|d_{mp}|^2+|\nabla d_{mp}|^2+|d_c|^2\right)e^{2s\varphi(x,0)}dx\right)\\
&+C_{17}\delta e^{2s\Phi}\int_{\omega_T}\left(|\mathbb{D}_1|^2+|\mathbb{B}_1|^2\right)dxdt.
\end{aligned}
\tag{3.9}
$$

Since $\mathbb{D}_{1,1}(x,0)=\mathbb{D}_{1,2}(x,0)$ and $\mathbb{B}_{1,1}(x,0)=\mathbb{B}_{1,2}(x,0)$, gathering (3.3), (3.7), (3.9) and the definition of $\Phi$, we have

$$
\begin{aligned}
&\int_\Omega\left(|\mathbb{D}_{1,1}(x,0)|^2+|\mathbb{B}_{1,1}(x,0)|^2\right)e^{2s\varphi(x,0)}dx\\
&=\int_\Omega\left(|\mathbb{D}_{1,2}(x,0)|^2+|\mathbb{B}_{1,2}(x,0)|^2\right)e^{2s\varphi(x,0)}dx\\
&\leq C_{18}\left(\delta e^{-2s\eta}\int_\Omega\left(|\mathbb{D}_{1,1}(x,0)|^2+|\mathbb{B}_{1,1}(x,0)|^2\right)e^{2s\varphi(x,0)}dx\right.\\
&+\kappa_2(s)\int_\Omega\left(|d_{dp}|^2+|\nabla d_{dp}|^2+|d_{mp}|^2+|\nabla d_{mp}|^2+|d_c|^2\right)e^{2s\varphi(x,0)}dx\\
&\left.+e^{2s\Phi}\int_{\omega_T}\left(|\mathbb{D}_1|^2+|\mathbb{B}_1|^2\right)dxdt\right).
\end{aligned}
\tag{3.10}
$$

Here $\kappa_2(s)=\kappa_1(s)+\delta e^{-2s\eta}$. Finally, for sufficiently large $s$, we obtain

$$
\begin{aligned}
&\int_\Omega\left(|\mathbb{D}_{1,1}(x,0)|^2+|\mathbb{B}_{1,1}(x,0)|^2\right)e^{2s\varphi(x,0)}dx\\
&\leq C_{19}\left(\kappa_2(s)\int_\Omega\left(|d_{dp}|^2+|\nabla d_{dp}|^2+|d_{mp}|^2+|\nabla d_{mp}|^2+|d_c|^2\right)e^{2s\varphi(x,0)}dx\right.\\
&\left.+e^{2s\Phi}\int_{\omega_T}\left(|\mathbb{D}_1|^2+|\mathbb{B}_1|^2\right)dxdt\right).
\end{aligned}
\tag{3.11}
$$

Now using the definition of the cut-off function in space $\chi_1(x)$, we have

$$\mathbb{D}_1(x,0)=\mathbb{D}_{1,1}(x,0)+(1-\chi_1(x))\mathbb{D}_1(x,0) \text{ and } \mathbb{B}_1(x,0)=\mathbb{B}_{1,1}(x,0)+(1-\chi_1(x))\mathbb{B}_1(x,0).$$

Since $1-\chi_1(x)=0$ for $x\in\Omega\setminus\omega$, we deduce

$$\begin{aligned}&\int_\Omega \left(|\mathbb{D}_1(x,0)|^2+|\mathbb{B}_1(x,0)|^2\right)e^{2s\varphi(x,0)}dx\\ &\leq C\left(\int_\Omega \left(|\mathbb{D}_{1,1}(x,0)|^2+|\mathbb{B}_{1,1}(x,0)|^2\right)e^{2s\varphi(x,0)}dx\right.\\ &\left.+\int_\omega (1-\chi_1(x))^2\left(|\mathbb{D}_1(x,0)|^2+|\mathbb{B}_1(x,0)|^2\right)e^{2s\varphi(x,0)}dx\right),\end{aligned}\tag{3.12}$$

and by the Sobolev imbedding we obtain

$$\begin{aligned}&\int_\omega (1-\chi_1(x))^2\left(|\mathbb{D}_1(x,0)|^2+|\mathbb{B}_1(x,0)|^2\right)e^{2s\varphi(x,0)}dx\\ &\leq C_{20}e^{2s\Phi}\int_{\omega_T}\left(|\mathbb{D}_1|^2+|\mathbb{B}_1|^2\right)dxdt.\end{aligned}\tag{3.13}$$

Finally, from (3.11), (3.12) and (3.13), we deduce

$$\begin{aligned}&\int_\Omega \left(|\mathbb{D}_1(x,0)|^2+|\mathbb{B}_1(x,0)|^2\right)e^{2s\varphi(x,0)}dx\\ &\leq C_{21}\left(\kappa_2(s)\int_\Omega\left(|d_{dp}|^2+|\nabla d_{dp}|^2+|d_{mp}|^2+|\nabla d_{mp}|^2+|d_c|^2\right)e^{2s\varphi(x,0)}dx\right.\\ &\left.+e^{2s\Phi}\int_{\omega_T}\left(|\mathbb{D}_1|^2+|\mathbb{B}_1|^2\right)dxdt\right).\end{aligned}\tag{3.14}$$

Now, in light of the two sets of initial conditions, we are going to get rid of the integrals of $|d_c|^2$, $|\nabla d_{dp}|^2$ and $|\nabla d_{mp}|^2$ in terms of integrals depending on $|\mathbb{D}_1(x,0)|^2$, $|\mathbb{B}_1(x,0)|^2$, $|d_{mp}|^2$ and $|d_{dp}|^2$. From system (3.1) we have

$$\begin{aligned}\mathbb{D}_1^k(x,0)=R_1^k(x,0) &= \text{curl}\,(d_{mp}B_0^k)-d_cD_0^k=d_{mp}\text{curl}\,(B_0^k)+\nabla d_{mp}\times B_0^k-d_cD_0^k\\ &= d_{mp}\text{curl}\,(B_0^k)+\textstyle\sum_{j=1}^3(\partial_j d_{mp})(e_j\times B_0^k)-d_cD_0^k,\\ \mathbb{B}_1^k(x,0)=R_2^k(x,0) &= -\text{curl}\,(d_{dp}D_0^k)=-d_{dp}\text{curl}\,(D_0^k)-\nabla d_{dp}\times D_0^k\\ &= -d_{dp}\text{curl}\,(D_0^k)-\textstyle\sum_{j=1}^3(\partial_j d_{dp})(e_j\times D_0^k),\end{aligned}$$

where the subscript $k=1,2$ corresponds to the choice of two sets of different initial data. We deduce the following linear systems

$$\begin{aligned}&\textstyle\sum_{j=1}^3(\partial_j d_{mp})(e_j\times B_0^k)-d_cD_0^k=\mathbb{D}_1^k(x,0)-d_{mp}\text{curl}\,(B_0^k),\\ &\textstyle\sum_{j=1}^3(\partial_j d_{dp})(e_j\times D_0^k)=-\mathbb{B}_1^k(x,0)-d_{dp}\text{curl}\,(D_0^k),\quad k=1,2.\end{aligned}$$

It can be rewritten in the form

$$\mathcal{G}(x)U=\begin{pmatrix}\mathbb{D}_1^1(x,0)-d_{mp}\text{curl}\,(B_0^1)\\ -\mathbb{B}_1^1(x,0)-d_{dp}\text{curl}\,(D_0^1)\\ \mathbb{D}_1^2(x,0)-d_{mp}\text{curl}\,(B_0^2)\\ -\mathbb{B}_1^2(x,0)-d_{dp}\text{curl}\,(D_0^2)\end{pmatrix}$$

with $U=(\partial_1 d_{mp},\partial_2 d_{mp},\partial_3 d_{mp},\partial_1 d_{dp},\partial_2 d_{dp},\partial_3 d_{dp},-d_c)^T$.

Thanks to the hypothesis (1.7), it follows that there exists a positive constant $C_{22}$ such that:

$$|\nabla d_{dp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2 \tag{3.15}$$
$$\leq C_{22} \sum_{k=1,2} \left( \left|\mathbb{D}_1^k(x,0)\right|^2 + \left|\mathbb{B}_1^k(x,0)\right|^2 + |d_{dp}|^2 + |d_{mp}|^2 \right) \quad \text{in } \Omega.$$

Now, from (3.15) we obtain

$$\int_\Omega \left( |d_{dp}|^2 + |\nabla d_{dp}|^2 + |d_{mp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2 \right) e^{2s\varphi(x,0)} dx \tag{3.16}$$
$$\leq C_{23} \int_\Omega \left[ \sum_{k=1,2} \left( \left|\mathbb{D}_1^k(x,0)\right|^2 + \left|\mathbb{B}_1^k(x,0)\right|^2 \right) + |d_{dp}|^2 + |d_{mp}|^2 \right] e^{2s\varphi(x,0)} dx$$

in $\Omega$ for sufficiently large $s > 0$. By Lemma 2.3 and (3.14) applied to $d_{dp}$ and $d_{mp}$, since $lim_{s\to+\infty}\kappa_2(s) = 0$, we obtain

$$\begin{aligned} &\int_\Omega \left( |d_{dp}|^2 + |\nabla d_{dp}|^2 + |d_{mp}|^2 + |\nabla d_{mp}|^2 + |d_c|^2 \right) dx \\ &\leq C_{24} e^{2s\Phi} \sum_{k=1,2} \int_{\omega_T} \left( |\mathbb{D}_1^k|^2 + |\mathbb{B}_1^k|^2 \right) dxdt \end{aligned} \tag{3.17}$$

for sufficiently large $s > 0$. Here $C_{24} > 0$ depends on $\Omega, \omega, T, C_0, \widehat{\varepsilon}, \alpha, M_0$ and $M_1$.

On the other hand, from the definition of $d_c$, we can write

$$\sigma - \widetilde{\sigma} = \varepsilon(d_c - \widetilde{\sigma} d_{dp}).$$

Then, it is easy to verify

$$\|\sigma - \widetilde{\sigma}\|_{H^1(\Omega)} \leq C(\|d_c\|_{H^1(\Omega)} + \|d_{dp}\|_{H^1(\Omega)}),$$

where $C > 0$ is a constant depending on $M_0$.

Now, we are going to improve the observation term in (3.17) and get rid of the observation by the component $\mathrm{B}$.

In a neighborhood $\omega$ of the boundary $\partial\Omega$ such that $\omega \subset \widetilde{\omega}$, we can write

$$\partial_t \mathrm{B}^k = -\mathrm{curl}\,(\varepsilon^{-1}\mathrm{D}^k), \text{ in } \omega \times (-T,T)$$

and

$$\partial_t \widetilde{\mathrm{B}}^k = -\mathrm{curl}\,(\widetilde{\varepsilon}^{-1}\widetilde{\mathrm{D}}^k), \text{ in } \omega \times (-T,T)$$

for $k = 1, 2$. Since $\varepsilon = \widetilde{\varepsilon} = \widehat{\varepsilon}$ in $\omega$, we can have

$$\partial_t(\mathrm{B}^k - \widetilde{\mathrm{B}}^k) = -\mathrm{curl}\,\frac{1}{\widehat{\varepsilon}}(\mathrm{D}^k - \widetilde{\mathrm{D}}^k) \text{ in } \omega \times (-T,T) \text{ for } k = 1,2.$$

Moreover we readily obtain

$$\partial_t(\mathrm{B}^k - \widetilde{\mathrm{B}}^k) = -\frac{1}{\widehat{\varepsilon}}\mathrm{curl}\,(\mathrm{D}^k - \widetilde{\mathrm{D}}^k) - \nabla\frac{1}{\widehat{\varepsilon}} \times (\mathrm{D}^k - \widetilde{\mathrm{D}}^k) \text{ in } \omega \times (-T,T) \text{ for } k = 1,2$$

and we deduce the estimate

$$\sum_{k=1,2} \int_{\omega_T} |\mathbb{B}_1^k|^2 dxdt \leq C_{25} \sum_{k=1,2} \left\| \mathrm{D}^k - \widetilde{\mathrm{D}}^k \right\|^2_{H^1(-T,T;L^2(\omega))},$$

where $C_{25} > 0$ depends on $M_0$.
Finally, we get rid of the observation by the first component $\mathrm{D}_1$ of $\mathrm{D}$.
Actually the geometry of $\partial\Omega$ allows us to choose the component $D_i$ that we are going to recover from the two others $D_j$ for $j \neq i$. For simplicity, we will give the proof of this result in the simplest case where the direction $(x_1, x_2)$ is normal to the $x_3$-direction and this direction corresponds to the boundary $\partial\omega$ of $\omega$ which intersects $\partial\Omega$. Since $\rho$ is done, and $\mathrm{D}_2$, $\mathrm{D}_3$ are known on $\omega_T$, from the relation $\operatorname{div} \mathrm{D} = \rho$, we obtain $\partial_1 \mathrm{D}_1$ on $\omega_T$. Then, the boundary condition on $D_1$ gives $\mathrm{D}_1$ on $\Sigma$, and then we see $\mathrm{D}_1$ on $\omega_T$.
This ends the proof of Theorem 1.1.

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