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On the exact boundary controllability of semilinear wave equations

Sue Claret * Jérôme Lemoine † Arnaud Münch ‡

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Abstract

We address the exact boundary controllability of the semilinear wave equation $y_{tt} - \Delta y + f(y) = 0$ posed over a bounded domain Ω of \mathbb{R}^d . Assuming that f is continuous and satisfies the condition $\limsup_{|r| \rightarrow \infty} |f(r)|/(|r| \ln^p |r|) \leq \beta$ for some β small enough and some $p \in [0, 3/2)$, we apply the Schauder fixed point theorem to prove the uniform controllability for initial data in $L^2(\Omega) \times H^{-1}(\Omega)$. Then, assuming that f is in $C^1(\mathbb{R})$ and satisfies the condition $\limsup_{|r| \rightarrow \infty} |f'(r)|/\ln^p |r| \leq \beta$, we apply the Banach fixed point theorem and exhibit a strongly convergent sequence to a state-control pair for the semilinear equation.

AMS Classifications: 35L71, 93B05.

Keywords: Semilinear wave equation, Exact boundary controllability, Carleman estimates, Fixed point.

1 Introduction and main results

Let Ω be a bounded domain in \mathbb{R}^d of class C^2 and let $T > 0$. We set $Q := \Omega \times (0, T)$ and $\Sigma := \partial\Omega \times (0, T)$. We consider the semilinear problem in $y = y(x, t)$

$$Ly + f(y) = 0 \text{ in } Q, \quad y = v1_{\Gamma_0} \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega, \quad (1)$$

where $L := \partial_t^2 - \Delta$ denotes the wave operator, $(u_0, u_1) \in \mathbf{H} := L^2(\Omega) \times H^{-1}(\Omega)$ is a given initial state, $v \in L^2(\Sigma)$ is a control function and f a continuous function over \mathbb{R} . Γ_0 denotes a non empty subset of $\partial\Omega$.

The exact boundary controllability problem associated to (1) states as follows: given $T > 0$, $\Gamma_0 \subset \partial\Omega$ and $(u_0, u_1), (z_0, z_1) \in \mathbf{H}$, find a control function $v \in L^2(\Sigma)$ and $y \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ solution of (1) and such that $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$ in Ω .

The linear problem (1) with $f \equiv 0$ is exactly controllable provided that $T > 0$ and Γ_0 are sufficiently large (see [16, Theorem 6.1, p. 60] and [1, Theorem 4.9, p. 1058]). In the nonlinear case, a first exact boundary controllability result has been given in [21, Theorem 2.1] assuming f globally Lipschitz and initial data in $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ for $\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$ leading to Dirichlet control in $H_0^\gamma(0, T; L^2(\Gamma_0))$. A Schauder fixed point argument is used coupled with the HUM method developed in [16]. Still assuming $f' \in L^\infty(\mathbb{R})$, [14, Theorem 1.1] covers the case $\gamma = 0$ and generalizes the result to semilinear abstract systems by using a global inversion theorem. We also mention [7] where a boundary controllability result is proved in the one-dimensional case for a specific class of initial and final data and T large enough by a quasi-static deformation approach.

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1 Assuming $(u_0, u_1) \in \mathbf{V} := H_0^1(\Omega) \times L^2(\Omega)$, the boundary controllability may also be obtained indirectly
2 with the domain extension from interior controllability results. In this respect, we mention [22, Theorem
3 1] assuming $\Omega = (0, 1)$, $T > 2$ and that $f \in C^1(\mathbb{R})$ does not grow faster at infinity than $\beta|r| \ln^2(|r|)$
4 for some $\beta > 0$ small enough, for a global controllability result in \mathbf{V} . This result has been extended to
5 any spatial dimension, first in [15, Theorem 3.1] and then in [11, Theorem 4.5, page 116] assuming that
6 $f \in C^1(\mathbb{R})$ does not grow faster at infinity than $\beta|r| \ln^p(|r|)$ for some $\beta > 0$ small enough for $p = 1/2$ and
7 any $0 < p < 3/2$ respectively. The above results are based on the Schauder theorem together with an
8 estimate of the cost of control for linear wave equations with potential derived using Carleman estimates
9 (we refer to [9, Theorem 2.2, page 8]). Eventually, we mention [8] dealing with subcritical nonlinearities
10 satisfying the sign condition $rf(r) \geq 0$ for all $r \in \mathbb{R}$ (weakened later in [12] to an asymptotic sign condition
11 leading to a semi-global controllability result, in the sense that the final data (z_0, z_1) must be prescribed
12 in a precise subset of \mathbf{V}).

13 In this work, we directly address the exact boundary controllability for (1) under the usual conditions
14 on $(u_0, u_1), T$ and Γ_0 encountered in the linear case but with respect to [14], by replacing the condition
15 $f' \in L^\infty(\mathbb{R})$ by the slightly super-linear condition used in [11]. Our result is as follows.

Theorem 1. *For any $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$, let $\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}$ and $\Gamma_0 \subset \partial\Omega$ such that $\text{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0$ and let $T > 2 \max_{x \in \bar{\Omega}} |x - x_0|$.*

- Assume that $f \in C^0(\mathbb{R})$ and that there exists $0 \leq p < 3/2$ such that f satisfies

$$(\mathbf{H}_p) \quad \exists \alpha_1, \alpha_2, \beta^* > 0, \quad |f(r)| \leq \alpha_1 + |r| (\alpha_2 + \beta^* \ln_+^p(r)), \quad \forall r \in \mathbb{R}.$$

If β^ is small enough, then for any initial state (u_0, u_1) and final state (z_0, z_1) in \mathbf{H} , system (1) is exactly controllable with controls in $L^2(\Sigma)$.*

- Assume that $f \in C^1(\mathbb{R})$ and that there exists $0 \leq p < 3/2$ such that f satisfies

$$(\mathbf{H}'_p) \quad \exists \alpha, \beta^* > 0, \quad |f'(r)| \leq \alpha + \beta^* \ln_+^p(r), \quad \forall r \in \mathbb{R}.$$

If β^ is small enough, then for any initial state (u_0, u_1) and final state (z_0, z_1) in \mathbf{H} , one can construct a non trivial sequence $(y_k, v_k)_{k \in \mathbb{N}^*}$ that converges strongly to a controlled pair (y, v) in $(C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))) \times L^2(0, T; L^2(\partial\Omega))$ for system (1). Moreover, the convergence of $(y_k, v_k)_{k \in \mathbb{N}^*}$ holds at least with a linear rate for the norm $\|\rho \cdot\|_{L^2(Q)} + \|\rho \cdot\|_{L^2(\Sigma)}$ where $\rho = \rho(x, t, s)$ is defined in (7) and s is chosen sufficiently large depending on $\|(u_0, u_1)\|_{\mathbf{H}}$ and $\|(z_0, z_1)\|_{\mathbf{H}}$.*

- Assume that $f \in C^0(\mathbb{R})$ and satisfies $(\mathbf{H}_{3/2})$, i.e. (\mathbf{H}_p) with $p = 3/2$. If β^* is small enough, then for any initial state (u_0, u_1) and final state (z_0, z_1) in \mathbf{V} , system (1) is exactly controllable with controls in $H^1(0, T; L^2(\partial\Omega)) \cap C^0([0, T]; H^{1/2}(\partial\Omega))$.

17 Theorem 1 extends and generalizes to any dimension [3] devoted to $d = 1$. It relies on fixed point
18 approaches in a functional class $\mathcal{C}(s)$ dependent of a Carleman parameter s large enough. For any
19 $\hat{y} \in \mathcal{C}(s) \subset L^\infty(0, T; L^2(\Omega))$, the fixed point operator Λ_s is such that $y := \Lambda_s(\hat{y}) \in \mathcal{C}(s)$ is a controlled
20 solution of

$$Ly = B \text{ in } Q, \quad y = v \mathbf{1}_{\Gamma_0} \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega, \quad (2)$$

21 with $B = -f(\hat{y})$ satisfying $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$ through the boundary function v ; the pair (y, v) is
22 chosen as the minimizer of a quadratic functional (see (14)) involving Carleman weights and cut-off time
23 and space functions. While $\mathcal{C}(s)$ is a subset of $L^\infty(Q)$ in [3], the class here is a subset of $L^\infty(0, T; L^2(\Omega))$
24 allowing to consider any spatial dimension with a simpler proof.

25 We emphasize that we get the exact controllability for (1) under the conditions encountered in the
26 linear situation: the controllability time $T > T(x_0) := 2 \max_{x \in \bar{\Omega}} |x - x_0|$ and support Γ_0 satisfy the
27 usual geometric conditions introduced in [16] while the initial data is assumed in \mathbf{H} . This is in contrast

1 with [11, Theorem 4.5] devoted to distributed controllability where the initial data is taken in \mathbf{V} with
 2 a controllability time greater than $\max(2T(x_0), cd^{3/2})$ for some $c > 0$ (we refer to [11, Remark 4.7 page
 3 118]). Moreover, we consider nonlinearities f in $\mathcal{C}^0(\mathbb{R})$ while [11, Theorem 4.5] assumes f in $\mathcal{C}^1(\mathbb{R})$; this
 4 is due to the fact that the linearization (2), where the nonlinear term is seen as a right hand side, does
 5 not involve any derivative of f . As a matter of fact, the first item in Theorem 1 can not be obtained
 6 from the extension domain method and controllability results for the distributed case.

7 Remark also that the third item includes the value $p = 3/2$ in the exponent of the logarithm term (see
 8 (\mathbf{H}_p)) contrary to [11, Theorem 4.5]. Last, the second item, with a growth condition on the derivative
 9 of f , provides a constructive way to approximate control-state pair for (1), which is fundamental for
 10 applications. The regularity is used to estimate some $L^2(L^q)$ norm of $f(y_1) - f(y_2)$ for any elements
 11 y_1, y_2 in $\mathcal{C}(s)$ (see Proposition 6). It relaxes the Hölder assumption on f' used in [17, Theorem 2.3]
 12 based on a Newton type linearization. To our knowledge, this is the first result leading to a convergent
 13 approximation of boundary controls for superlinear nonlinearities without smallness assumption notably
 14 on the initial condition and target (in contrast to the recent works [4, 18]).

15 As in [3], the crucial technical point in the analysis is a regularity property of the state-control
 16 trajectories (y, v) for (2). We show and use that if the initial condition belongs to \mathbf{V} and if the right hand
 17 side B belongs to $L^2(Q)$, then the controlled trajectory y solution of (2) so that (y, v) is the minimizer
 18 of the quadratic functional J_s (see Remark 2) belongs to $\mathcal{C}^0([0, T], H^1(\Omega))$.

19 **Outline** - Section 2 discusses the exact null controllability of (2) and provides precise estimates of the
 20 control-state pair (y, v) in term of the Carleman parameter s according to the regularity of the data
 21 (u_0, u_1) and of the right hand side B : Proposition 2 for $(u_0, u_1, B) \in \mathbf{H} \times L^2(0, T; H^{-r}(\Omega))$, $r \in [0, 1]$,
 22 $r \neq \frac{1}{2}$, and Proposition 3 for $(u_0, u_1, B) \in \mathbf{V} \times L^2(Q)$. Section 3 is devoted to the proof of Theorem 1:
 23 in Section 3.1, we employ the Schauder fixed point theorem to the operator Λ_s (see (20)) defined on the
 24 class $\mathcal{C}(s) \subset L^\infty(0, T; L^2(\Omega))$ and prove the first item. In Section 3.2, we prove that, if the nonlinearity
 25 f satisfies (\mathbf{H}'_p) , then the operator $\Lambda(s)$ is contracting leading to the second item. Eventually, in Section
 26 3.3 assuming the initial data in \mathbf{V} , we define the operator $\Lambda(s)$ on a class $\tilde{\mathcal{C}}(s)$ subset of $H^1(Q)$ and reach
 27 the limit case $p = 3/2$ in the logarithmic exponent, as announced in the third item.

28 In the sequel, C denotes a generic constant which may changes from line to line, but depends only on
 29 Ω and T .

30 2 Controllability results for the linear wave equation

31 Existence of $L^2(\Sigma)$ controls for (2) with initial data in \mathbf{H} and right hand side in $L^2(0, T; H^{-1}(\Omega))$ is
 32 well-known (we refer to [16, chapter 2]); corresponding controlled solution belongs to $\mathcal{C}^0([0, T]; L^2(\Omega)) \cap$
 33 $\mathcal{C}^1([0, T]; H^{-1}(\Omega))$. For any initial data $(u_0, u_1) \in \mathbf{V}$, right hand side B in $L^2(Q)$ and $T > 0$ large enough,
 34 we analyze the existence of a control function $v \in H^1(0, T; L^2(\partial\Omega)) \cap \mathcal{C}^0([0, T]; H^{1/2}(\partial\Omega))$ such that the
 35 solution y of (2) satisfies $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$ and is bounded in $L^\infty(0, T; L^2(\Omega))$. Moreover, we
 36 aim to get precise weighted estimates of a particular state-control pair in term of the data, which will be
 37 crucial to handle the nonlinear system (1). As in [3], we employ a global Carleman estimates from [2] as
 38 a fundamental tool.

39 We introduce the usual geometric condition (see [2, Condition (1.2)]): for any $x_0 \notin \bar{\Omega}$, we introduce
 40 $\Gamma_1, \Gamma_0 \subset \partial\Omega$ such that

$$\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}, \quad \text{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0. \quad (3)$$

41 Let $\Psi \in \mathcal{C}^2(\partial\Omega)$ be a cut-off function such that

$$0 \leq \Psi \leq 1 \quad \text{on } \partial\Omega, \quad \Psi = 1 \quad \text{on } \Gamma_1, \quad \Psi = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0. \quad (4)$$

1 We assume that

$$T > 2 \max_{\overline{\Omega}} |x - x_0| \quad (5)$$

and define, for any $\delta > 0$ such that $T - 2\delta > 2 \max_{\overline{\Omega}} |x - x_0|$, a cut-off function $\eta \in C_c^1(\mathbb{R})$ satisfying

$$0 \leq \eta(t) \leq 1 \text{ in } (\delta, T - \delta), \quad \eta(t) = 0 \text{ in } (-\infty, \delta] \cup [T - \delta, +\infty). \quad (6)$$

2 Then, for any $\beta \in (0, 1)$ and $\lambda > 0$, we define the functions $\psi(x, t) = |x - x_0|^2 - \beta(t - \frac{T}{2})^2 + M_0$,
 3 $\phi(x, t) = e^{\lambda\psi(x, t)}$ in Q , with $M_0 > 0$ large enough so that $\psi > 0$ in \overline{Q} . Then, for all $s \geq s_0$, we define the
 4 weight function

$$\rho(x, t) := e^{-s\phi(x, t)} \quad \forall (x, t) \in Q. \quad (7)$$

5 Remark that $e^{-cs} \leq \rho \leq e^{-s}$ in Q with $c := \|\phi\|_{L^\infty(Q)}$ and $\rho, \rho^{-1} \in C^\infty(\overline{Q})$.

6 Let then $P := \{w \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), Lw \in L^2(Q)\}$ and recall that $\partial_\nu w \in L^2(\Sigma)$ for
 7 every $w \in P$ (see [16, Theorem 4.1]). The global Carleman estimate mentioned earlier reads as follows.

Proposition 1. *For any $x_0 \notin \overline{\Omega}$, we assume (3) and (5). There exists $s_0 > 0$, $\lambda > 0$ and $C > 0$, such that for any $s \geq s_0$ and every $w \in P$*

$$\begin{aligned} & s \int_Q \rho^{-2} (|w_t|^2 + |\nabla w|^2) dx dt + s^3 \int_Q \rho^{-2} |w|^2 dx dt \\ & + s \int_\Omega \rho^{-2}(0) (|w_t(\cdot, 0)|^2 + |\nabla w(\cdot, 0)|^2) dx + s^3 \int_\Omega \rho^{-2}(0) |w(\cdot, 0)|^2 dx \\ & \leq C \left(\int_Q \rho^{-2} |Lw|^2 dx dt + s \int_\Sigma \eta^2(t) \Psi(x) \rho^{-2} |\partial_\nu w|^2 dx dt \right). \end{aligned} \quad (8)$$

9 *Proof.* We refer to [2, Theorem 2.5 and Remark 2.9]. □

10 2.1 Estimates for the state-control pair in $C^0([0, T]; L^2(\Omega)) \times L^2(\Sigma)$

In all the sequel, we suppose that $s_0 \geq 1$. Proposition 1 allows to deduce the controllability for (2) with estimates of the state-control pair in $L^2(Q) \times L^2(\Sigma)$. For any $s \geq s_0$, we define the bilinear form

$$(w, z)_{P, s} := \int_Q \rho^{-2} Lw Lz dx dt + s \int_\Sigma \eta^2(t) \Psi(x) \rho^{-2} \partial_\nu w \partial_\nu z dx dt, \quad \forall w, z \in P. \quad (9)$$

11 According to (8), (9) defines a scalar product in P and if P_s denotes P endowed with this scalar product,
 12 then P_s is an Hilbert space. We now state the controllability result for the system (2) (without loss of
 13 generality in the null controllability case, for which $(z_0, z_1) = (0, 0)$ in Ω).

Proposition 2. *For any $x_0 \notin \overline{\Omega}$, we assume (3) and (5). Let $\eta \in C_c^1(\mathbb{R})$ and $\Psi \in C^2(\partial\Omega)$ be cut-off functions satisfying (4) and (6) respectively. For $s \geq s_0$, $B \in L^2(0, T, H^{-r}(\Omega))$, $r \in [0, 1]$, $r \neq 1/2$ and $(u_0, u_1) \in \mathbf{H}$, there exists a unique $w \in P_s$ such that, for all $z \in P_s$,*

$$(w, z)_{P, s} = \langle u_1, z(\cdot, 0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega u_0 z_t(\cdot, 0) dx + \langle B, z \rangle_{L^2(0, T; H^{-1}(\Omega)), L^2(0, T; H_0^1(\Omega))}. \quad (10)$$

14 Then $v := s\eta^2\Psi\rho^{-2}\partial_\nu w$ is a control function for (2) and $y := \rho^{-2}Lw$ is the associated controlled trajectory, that is $y(\cdot, T) = y_t(\cdot, T) = 0$ in Ω .

Moreover, there exists a constant $C_r > 0$ independent of s such that

$$\begin{aligned} & \|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}} v \right\|_{L^2(\Sigma)} + s^{-2} \|\rho y\|_{L^\infty(0, T; L^2(\Omega))} + s^{-2} \|(\rho y)_t\|_{L^\infty(0, T; H^{-1}(\Omega))} \\ & \leq C_r \left(s^{r-3/2} \|\rho B\|_{L^2(0, T; H^{-r}(\Omega))} + s^{-1/2} \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1/2} \|\rho(0)u_1\|_{H^{-1}(\Omega)} \right). \end{aligned} \quad (11)$$

Remark 1. With no vanishing target $(z_0, z_1) \in \mathbf{H}$, the right hand side of (11) contains the term $s^{-1/2}\|\rho(0)z_0\|_{L^2(\Omega)} + s^{-1/2}\|\rho(0)z_1\|_{H^{-1}(\Omega)}$. We choose in the sequel $(z_0, z_1) = (0, 0)$ which will make the computations shorter.

Proof. Some parts of the proof are only sketched as they are very similar to [3, Theorem 6] devoted to the case $r = 0$ (and $d = 1$). For any $0 \leq r \leq 1$, $r \neq 1/2$ and $z \in P_s$, we have from (8) :

$$\begin{aligned} & \left| \langle B, z \rangle_{L^2(0,T;H^{-1}(\Omega)), L^2(0,T;H_0^1(\Omega))} \right| = \left| \langle B, z \rangle_{L^2(0,T;H^{-r}(\Omega)), L^2(0,T;H_0^r(\Omega))} \right| \\ & \leq \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \|\rho^{-1}z\|_{L^2(0,T;H^r(\Omega))} \\ & \leq C \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \|\rho^{-1}z\|_{L^2(Q)}^{1-r} \|\nabla(\rho^{-1}z)\|_{L^2(Q)}^r \\ & \leq C \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \left(s^r \|\rho^{-1}z\|_{L^2(Q)} + \|\rho^{-1}z\|_{L^2(Q)}^{1-r} \|\rho^{-1}\nabla z\|_{L^2(Q)}^r \right) \\ & \leq C s^{r-3/2} \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \\ & \quad \left(s^{3/2} \|\rho^{-1}z\|_{L^2(Q)} + (s^{3/2} \|\rho^{-1}z\|_{L^2(Q)})^{1-r} (s^{1/2} \|\rho^{-1}\nabla z\|_{L^2(Q)})^r \right) \\ & \leq C s^{r-3/2} \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \|z\|_{P_s} \end{aligned}$$

and conclude that the right hand side of (10) is a linear continuous form on P_s . The Riesz representation theorem implies the existence of a unique $w \in P_s$ satisfying the formulation (10) and

$$\|w\|_{P_s} \leq C \left(s^{r-3/2} \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{-1/2} \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1/2} \|\rho(0)u_1\|_{L^2(\Omega)} \right). \quad (12)$$

Then, set $y = \rho^{-2}Lw$ and $v = s\eta^2\Psi\rho^{-2}\partial_\nu w$. From (10), the pair (y, v) satisfies

$$\begin{aligned} & \int_Q yLz \, dxdt + \int_\Sigma v\partial_\nu z \, d\Sigma = \langle u_1, z(\cdot, 0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega u_0 z_t(\cdot, 0) \, dx \\ & \quad + \langle B, z \rangle_{L^2(0,T;H^{-1}(\Omega)), L^2(0,T;H_0^1(\Omega))} \quad \forall z \in P_s, \end{aligned} \quad (13)$$

meaning that $y \in L^2(Q)$ is a solution to (2) associated with the function $v \in L^2(\Sigma)$ in the sense of transposition. Eventually, using (12), we get that $\rho y = \rho^{-1}Lw \in L^2(Q)$ and $s^{-1/2}\rho v = s^{1/2}\eta^2\Psi\rho^{-1}\partial_\nu w \in L^2(\Sigma)$ and deduce (11) for the first and second left hand side terms.

To get (11) for the third and fourth left hand side terms, we remark that $\rho y \in L^\infty(0, T; L^2(\Omega))$: indeed, $y \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ (see [16, Theorem 4.2 p.46]).

Moreover, since $\rho^{-1} \in C^\infty(\overline{Q})$, for any $z \in P_s$, we get

$$\begin{cases} L(\rho^{-1}z) = \rho^{-1}Lz + \tilde{B}_z, & \tilde{B}_z := 2\rho_t^{-1}z_t + \rho_{tt}^{-1}z - 2\nabla\rho^{-1} \cdot \nabla z - \Delta\rho^{-1}z, \\ (\rho^{-1}z)|_{\partial\Omega} = 0 \end{cases}$$

so that (13) rewrites

$$\begin{aligned} & \int_Q \rho y L(\rho^{-1}z) \, dxdt + \int_\Sigma \rho v \partial_\nu(\rho^{-1}z) \, d\Sigma \\ & = \langle \rho(0)u_1, (\rho^{-1}z)(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} - \int_\Omega \rho(0)u_0 (\rho^{-1}z)_t(\cdot, 0) \, dx + \int_\Omega \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) \, dx \\ & \quad + \langle \rho B, \rho^{-1}z \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} + \int_Q \rho y \tilde{B}_z \, dxdt. \end{aligned}$$

Remark that $z \in P_s$ if and only if $\tilde{z} := \rho^{-1}z \in P_s$; therefore, for all $\tilde{z} \in P_s$ and $z = \rho\tilde{z}$,

$$\begin{aligned} & \int_Q \rho y L\tilde{z} \, dxdt + \int_\Sigma \rho v \partial_\nu \tilde{z} \, d\Sigma \\ & = \langle \rho(0)u_1, \tilde{z}(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} - \int_\Omega \rho(0)u_0 \tilde{z}_t(\cdot, 0) \, dx + \langle \rho B, \tilde{z} \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} \\ & \quad + \int_\Omega \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) \, dx + \int_Q \rho y \tilde{B}_z \, dxdt. \end{aligned}$$

Moreover, using that $|\nabla\rho^{-1}| \leq Cs\rho^{-1}$, $|(\rho^{-1})_t| \leq Cs\rho^{-1}$, $|(\rho^{-1})_{tt}| \leq Cs^2\rho^{-1}$ and $|\nabla^2\rho^{-1}| \leq Cs^2\rho^{-1}$, we get the estimates

$$\begin{aligned} \left| \int_{\Omega} \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) dx \right| &\leq Cs\|\rho(0)u_0\|_{L^2(\Omega)}\|\tilde{z}(\cdot, 0)\|_{L^2(\Omega)}, \\ \|\tilde{B}_z\|_{L^2(Q)} &\leq C(s^2\|\tilde{z}\|_{L^2(Q)} + s\|\tilde{z}_t\|_{L^2(Q)} + s\|\nabla\tilde{z}\|_{L^2(Q)}) \end{aligned}$$

and thus, since $s \geq 1$:

$$\left| \int_Q \rho y \tilde{B}_z dx dt \right| \leq Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{z}\|_{L^2(Q)} + \|\tilde{z}_t\|_{L^2(Q)} + \|\nabla\tilde{z}\|_{L^2(Q)}).$$

Then, for all $\tilde{z} \in P_s$

$$\begin{aligned} \left| \int_Q \rho y L\tilde{z} dx dt \right| &\leq \left| \int_{\Sigma} \rho v \partial_{\nu} \tilde{z} d\Sigma \right| + \left| \langle \rho(0)u_1, \tilde{z}(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \right| + \left| \int_{\Omega} \rho(0)u_0 \tilde{z}_t(\cdot, 0) dx \right| \\ &\quad + \left| \int_{\Omega} \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) dx \right| + \left| \langle \rho B, \tilde{z} \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} \right| + \left| \int_Q \rho y \tilde{B}_z dx dt \right| \\ &\leq \|\rho v\|_{L^2(\Sigma)}\|\partial_{\nu}\tilde{z}\|_{L^2(\Sigma)} + \|\rho(0)u_1\|_{H^{-1}(\Omega)}\|\tilde{z}(\cdot, 0)\|_{H_0^1(\Omega)} \\ &\quad + \|\rho(0)u_0\|_{L^2(\Omega)}\|\tilde{z}_t(\cdot, 0)\|_{L^2(\Omega)} + Cs\|\rho(0)u_0\|_{L^2(\Omega)}\|\tilde{z}(\cdot, 0)\|_{L^2(\Omega)} \\ &\quad + \|\rho B\|_{L^1(0,T;H^{-1}(\Omega))}\|\tilde{z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} + Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{z}\|_{L^2(Q)} + \|\tilde{z}_t\|_{L^2(Q)} + \|\nabla\tilde{z}\|_{L^2(Q)}). \end{aligned}$$

For any $g \in \mathcal{D}(Q)$, let $\tilde{z} \in P_s$ be solution of $L\tilde{z} = g$ and $(\tilde{z}(0), \tilde{z}_t(0)) = (0, 0)$ so that

$$\|\tilde{z}_t\|_{L^{\infty}(0,T;L^2(\Omega))} + \|\tilde{z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} + \|\partial_{\nu}\tilde{z}\|_{L^2(\Sigma)} \leq C\|g\|_{L^1(0,T;L^2(\Omega))}$$

and thus, using (11), for all $0 \leq r \leq 1$, $r \neq 1/2$, since $s \geq 1$:

$$\begin{aligned} \left| \int_Q \rho y g dx dt \right| &\leq \|\rho v\|_{L^2(\Sigma)}\|\partial_{\nu}\tilde{z}\|_{L^2(\Sigma)} \\ &\quad + \|\rho B\|_{L^1(0,T;H^{-1}(\Omega))}\|\tilde{z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} + Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{z}\|_{L^2(Q)} + \|\tilde{z}_t\|_{L^2(Q)} + \|\nabla\tilde{z}\|_{L^2(Q)}) \\ &\leq C\left(s^{r+1/2}\|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right)\|g\|_{L^1(0,T;L^2(\Omega))}. \end{aligned}$$

Therefore, $\rho y \in (L^1(0, T; L^2(\Omega)))' = L^{\infty}(0, T; L^2(\Omega))$ and

$$\|\rho y\|_{L^{\infty}(0,T;L^2(\Omega))} \leq C\left(s^{r+1/2}\|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right).$$

Similarly, for any $g \in \mathcal{D}(Q)$, let $\tilde{Z} \in P_s$ satisfying $L\tilde{Z} = g_t$ and $(\tilde{Z}(0), \tilde{Z}_t(0)) = (0, 0)$. Then using [16, (4.19) p.51] we obtain :

$$\begin{aligned} \left| \int_Q \rho y g_t dx dt \right| &\leq \|\rho v\|_{L^2(\Sigma)}\|\partial_{\nu}\tilde{Z}\|_{L^2(\Sigma)} + \|\rho B\|_{L^1(0,T;H^{-1}(\Omega))}\|\tilde{Z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} \\ &\quad + Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{Z}\|_{L^2(Q)} + \|\tilde{Z}_t\|_{L^2(Q)} + \|\nabla\tilde{Z}\|_{L^2(Q)}) \\ &\leq C\left(s^{1/2+r}\|\rho(s)B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right)\|g\|_{L^1(0,T;H_0^1(\Omega))} \end{aligned}$$

and thus $(\rho y)_t \in L^{\infty}(0, T; H^{-1}(\Omega))$ and

$$\|(\rho y)_t\|_{L^{\infty}(0,T;H^{-1}(\Omega))} \leq C\left(s^{r+1/2}\|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right).$$

Remark 2. The state-control pair (y, v) introduced in Proposition 2 is the unique minimizer of the functional

$$J_s(z, u) := s \int_Q \rho^2 |z|^2 dxdt + \int_\delta^{T-\delta} \int_{\partial\Omega} \eta^{-2} \Psi^{-1} \rho^2 |u|^2 dxdt \quad (14)$$

over $\{(z, u) : z \in L^2(Q), \eta^{-1} \Psi^{-1/2} \rho u \in L^2(\delta, T - \delta; L^2(\partial\Omega))\}$ solution of (2) with $z(\cdot, T) = z_t(\cdot, T) = 0$ in Ω . We refer to [5, Section 2].

Remark 3. The controlled state $y = \rho^{-2} Lw$ satisfies

$$Ly = B \text{ in } Q, \quad y = s\eta^2 \Psi \rho^{-2} \partial_\nu w \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega,$$

so that $y \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))$. On the other hand, the function w satisfies

$$Lw = \rho^2 y \text{ in } Q, \quad w = 0 \text{ on } \Sigma$$

implying according to (8) that $(w(\cdot, 0), \partial_t w(\cdot, 0)) \in \mathbf{V}$ and $w \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$.

2.2 Estimates for the state-control pair with $(u_0, u_1, B) \in \mathbf{V} \times L^2(Q)$

The state-control pair (y, v) given by Proposition 2 enjoys additional regularity properties, under additional regularity assumption on the data and introduction of appropriate cut-off function in space and time. Such gain of regularity is known for the wave equation since [8, 10] and more recently [3]. The following first regularity result extends [3, Theorem 7] to the multi-dimensional case and is proven in Appendix A. It will be crucial for the analysis of the semilinear case discussed in Section 3.

Proposition 3. For any $x_0 \notin \bar{\Omega}$, we assume (3) and (5). Let $\eta \in \mathcal{C}_c^1(\mathbb{R})$ and $\Psi \in \mathcal{C}^2(\partial\Omega)$ be cut-off functions satisfying (4) and (6) respectively. Let any $(u_0, u_1) \in \mathbf{V}$ and $B \in L^2(Q)$ be given. For any $s \geq s_0$, the solution (y, v) of (2) defined in Proposition 2 satisfies

$$v \in \mathcal{C}^0([0, T]; H^{1/2}(\partial\Omega)), \quad y \in \mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)) \quad (15)$$

and the following estimate :

$$\begin{aligned} & \|(\rho y)_t\|_{L^2(Q)} + s^{-1/2} \left\| \frac{1}{\eta \Psi^{1/2}} (\rho v)_t \right\|_{L^2(\Sigma)} + s^{-1} \|\nabla(\rho y)\|_{L^2(Q)} + s^{-3/2} \|\rho v\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} \\ & + s^{-2} (\|\rho y\|_{\mathcal{C}^0([0, T]; H^1(\Omega))} + \|(\rho y)_t\|_{\mathcal{C}^0([0, T]; L^2(\Omega))}) \\ & \leq C s^{-1/2} (\|\rho B\|_{L^2(Q)} + \|\rho(0)u_1\|_{L^2(\Omega)} + s\|\rho(0)u_0\|_{L^2(\Omega)} + \|\rho(0)\nabla u_0\|_{L^2(\Omega)}). \end{aligned} \quad (16)$$

2.3 Estimates for the state-control pair with $(u_0, u_1, B) \in H_0^{1-r}(\Omega) \times H^{-r}(\Omega) \times L^2(0, T; H^{-r}(\Omega))$

Proposition 4. For any $x_0 \notin \bar{\Omega}$, we assume (3) and (5). Let $\eta \in \mathcal{C}_c^1(\mathbb{R})$ and $\Psi \in \mathcal{C}^2(\partial\Omega)$ be cut-off functions satisfying (4) and (6) respectively. For any $s \geq s_0$ and $r \in (0, 1)$, $r \neq 1/2$, there exists a constant $C > 0$ (depending on s and r) such that

$$\begin{aligned} & \|\rho y\|_{\mathcal{C}^1([0, T]; H^{-r}(\Omega))} + \|\rho y\|_{\mathcal{C}([0, T]; H^{1-r}(\Omega))} + \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{H^{1-r}(0, T; L^2(\partial\Omega))} \\ & \leq C (\|\rho B\|_{L^2(0, T; H^{-r}(\Omega))} + \|\rho(0)u_0\|_{H^{1-r}(\Omega)} + \|\rho(0)u_1\|_{H^{-r}(\Omega)}). \end{aligned} \quad (17)$$

Proof. Let $\Lambda_s^0 : (B, u_0, u_1) \mapsto (y, v)$ be the linear operator with (y, v) the control-state pair given by Proposition 2. Then, from Proposition 2 and Proposition 3,

$$\Lambda_s^0 : L^2(0, T; H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega) \rightarrow (\mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))) \times L^2(0, T; L^2(\partial\Omega))$$

and

$$\Lambda_s^0 : L^2(0, T; L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \rightarrow (\mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))) \times H^1(0, T; L^2(\partial\Omega))$$

are linear continuous. By interpolation, for all $0 < \theta < 1$, Λ_s^0 is linear continuous from $L^2(0, T; (H^{-1}(\Omega), L^2(\Omega))_\theta) \times (L^2(\Omega), H_0^1(\Omega))_\theta \times (H^{-1}(\Omega), L^2(\Omega))_\theta$ to $(\mathcal{C}^0([0, T]; (L^2(\Omega), H^1(\Omega))_\theta) \cap \mathcal{C}^1([0, T]; (H^{-1}(\Omega), L^2(\Omega))_\theta)) \times (L^2(0, T; L^2(\partial\Omega)), H^1(0, T; L^2(\partial\Omega)))_\theta$.

Thus, for $\theta = 1 - r$, $0 < r < 1$, $r \neq 1/2$, Λ_s^0 is linear, continuous from $L^2(0, T; H^{-r}(\Omega)) \times H_0^{1-r}(\Omega) \times H^{-r}(\Omega)$ to $(\mathcal{C}^0([0, T]; H^{1-r}(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-r}(\Omega))) \times H^{1-r}(0, T; L^2(\partial\Omega))$. \square

3 Proof of Theorem 1

In this section, we prove the controllability of the semilinear equation (1). In this respect, for all $s \geq s_0$ and for all \hat{y} in an appropriate subset $\mathcal{C}(s)$ of $L^\infty(0, T; L^2(\Omega))$, we consider the linearized boundary control problem

$$Ly = -f(\hat{y}) \text{ in } Q, \quad y = v 1_{\Gamma_0} \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega \quad (18)$$

such that $(y(\cdot, T), y_t(\cdot, T)) = (0, 0)$ in Ω . The existence of a controlled trajectory $y \in L^2(Q)$ is guaranteed by Proposition 2 with a source term $B = -f(\hat{y}) \in L^2(0, T; H^{-1}(\Omega))$.

3.1 First part of Theorem 1

Without restriction, we assume that $f \in C^0(\mathbb{R})$ satisfies (\mathbf{H}_p) for some $1 < p < 3/2$. For any $s \geq s_0 \geq 1$, we introduce the class $\mathcal{C}(s)$ defined as the closed convex subset of $L^\infty(0, T; L^2(\Omega))$

$$\mathcal{C}(s) := \left\{ y \in L^\infty(0, T; L^2(\Omega)) : \|\rho y\|_{L^2(Q)} \leq s, \|\rho y\|_{L^\infty(0, T; L^2(\Omega))} \leq s^3 \right\}. \quad (19)$$

We prove the existence of s and of a fixed point of the operator

$$\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s) \quad \hat{y} \mapsto y \quad (20)$$

where y is a solution of the null controllability problem (18) associated with the control v given by Proposition 2 for $B = -f(\hat{y})$. We employ the Schauder theorem and we prove that : **i**) for $\beta^* > 0$ small enough, there exists $s \geq s_0$ large enough such that $\mathcal{C}(s)$ is stable under the map Λ_s (Section 3.1.2); **ii**) $\Lambda_s(\mathcal{C}(s))$ is a relatively compact subset of $\mathcal{C}(s)$ for the norm $\|\cdot\|_{L^\infty(0, T; L^2(\Omega))}$ (Section 3.1.3); **iii**) Λ_s is a continuous map in $\mathcal{C}(s)$ with respect to the $L^\infty(0, T; L^2(\Omega))$ norm (Section 3.1.4). This ensures the existence of a fixed point for Λ_s , which is a controlled trajectory for (1).

3.1.1 Estimates of $\Lambda_s(\hat{y})$

Lemma 1. *Assume that $f \in C^0(\mathbb{R})$ satisfies (\mathbf{H}_p) . For any $s \geq s_0$ and $\hat{y} \in \mathcal{C}(s)$, there exists $C > 0$ such that*

$$\|\rho f(\hat{y})\|_{L^2(0, T; H^{p-3/2}(\Omega))} \leq C(\alpha_1 e^{-s} T^{1/2} |\Omega|^{1/2} + \alpha_2 s + \beta^* c^p s^{1+p})$$

with $c = \|\phi\|_{L^\infty(Q)}$.

Proof. We infer, for $1/2 > r = 3/2 - p > 0$ and $1 \leq p^* = \frac{2d}{d+3-2p} < 2$ that (using that $\rho \leq e^{-s}$)

$$\begin{aligned} \|\rho f(\hat{y})\|_{L^2(0, T; H^{-r}(\Omega))} &\leq C\|\rho\alpha_1 + |\rho\hat{y}|(\alpha_2 + \beta^* \ln_+^p(\hat{y}))\|_{L^2(0, T; L^{p^*}(\Omega))} \\ &\leq C(\alpha_1 e^{-s} \sqrt{T} |\Omega|^{1/2} + \alpha_2 \|\rho\hat{y}\|_{L^2(Q)} + \beta^* \|\rho\hat{y} \ln_+^p(\hat{y})\|_{L^2(0, T; L^{p^*}(\Omega))}). \end{aligned}$$

Now, using that $\rho^{-1} \leq e^{cs}$ and $0 \leq \ln_+^p(\hat{y}) \leq C(\ln_+^p \rho^{-1} + \ln_+^p(\rho\hat{y})) \leq C((cs)^p + \ln_+^p(\rho\hat{y}))$, we get that

$$\|\rho\hat{y} \ln_+^p(\hat{y})\|_{L^2(0, T; L^{p^*}(\Omega))} \leq C((cs)^p \|\rho\hat{y}\|_{L^2(Q)} + \|\rho\hat{y} \ln_+^p(\rho\hat{y})\|_{L^2(0, T; L^{p^*}(\Omega))}).$$

1 But, for all $0 < \varepsilon < \frac{2-p^*}{p^*} = \frac{2r}{d} = \frac{3-2p}{d}$, there exists $C_\varepsilon > 0$ such that

$$\|\rho \widehat{y} \ln_+^p(\rho \widehat{y})\|_{L^2(0,T;L^{p^*}(\Omega))} \leq C_\varepsilon \|\rho \widehat{y}\|^{1+\varepsilon}_{L^2(0,T;L^{p^*}(\Omega))} \leq C_\varepsilon \|\rho \widehat{y}\|_{L^2(Q)} \|\rho \widehat{y}\|_{L^\infty(0,T;L^2(\Omega))}^\varepsilon. \quad (21)$$

2 Combining the above inequalities with $\varepsilon < \min(\frac{p}{3}, \frac{3-2p}{d})$ and using that $\widehat{y} \in \mathcal{C}(s)$ we get the result. \square

Proposition 5. For any $x_0 \notin \overline{\Omega}$, we assume (3) and (5). Assume $(u_0, u_1) \in \mathbf{H}$ and that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies (\mathbf{H}_p) . For $s \geq s_0$ and any $\widehat{y} \in \mathcal{C}(s)$, the solution $y = \Lambda_s(\widehat{y})$ to the linearized controlled system (18) with control v satisfies the following estimates:

$$\begin{aligned} & \|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\Psi^{1/2} \eta} v \right\|_{L^2(\Sigma)} + s^{-2} \|\rho y\|_{L^\infty(0,T;L^2(\Omega))} + s^{-2} \|(\rho y)_t\|_{L^\infty(0,T;H^{-1}(\Omega))} \\ & \leq C s \left(s^{-p} \alpha_2 + \beta^* c^p + e^{-s} (s^{-p-1} \alpha_1 T^{1/2} |\Omega|^{1/2} + s^{-3/2} (\|u_0\|_{L^2(\Omega)} + s \|u_1\|_{H^{-1}(\Omega)})) \right). \end{aligned} \quad (22)$$

4 *Proof.* The map $L : \mathcal{C}^1(\overline{\Omega}) \times H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$, $(\rho(0), u_1) \rightarrow \rho(0)u_1$ is bilinear continuous (see [20, Lemma 3, p. 1097]) so that $\|\rho(0)u_1\|_{H^{-1}(\Omega)} \leq C \|\rho(0)\|_{\mathcal{C}^1(\overline{\Omega})} \|u_1\|_{H^{-1}(\Omega)} \leq C s e^{-s} \|u_1\|_{H^{-1}(\Omega)}$. The estimates follow from Proposition 2 with $r = 3/2 - p$ and Proposition 3 with $B = -f(\widehat{y})$ and Lemma 1. \square

3.1.2 Stability of the class $\mathcal{C}(s)$

8 **Lemma 2.** Assume that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies (\mathbf{H}_p) with β^* small enough. Then, there exists $s \geq s_0$ large enough such that $\Lambda_s(\mathcal{C}(s)) \subset \mathcal{C}(s)$.

9 *Proof.* For any $s \geq s_0$ and $\widehat{y} \in \mathcal{C}(s)$, let $y := \Lambda_s(\widehat{y})$. From (22), we obtain that $\limsup_{s \rightarrow +\infty} s^{-1} \|\rho y\|_{L^2(Q)} \leq \beta^* C c^p$ and $\limsup_{s \rightarrow +\infty} s^{-3} \|\rho y\|_{L^\infty(0,T;L^2(\Omega))} \leq \beta^* C c^p$. Therefore, if $\beta^* > 0$ is small enough so that $\beta^* C c^p < 1$, then for any $s \geq s_0$ large enough, $y \in \mathcal{C}(s)$. \square

12 **Remark 4.** The lower bound for $s \geq s_0$ is related to the norm of the initial data $(u_0, u_1) \in \mathbf{H}$: in view of Proposition 5, the stability of Λ_s requires that $(C \|u_0\|_{L^2(\Omega)} + C s^{-1} (\alpha_1 T^{1/2} |\Omega|^{1/2} + \|u_1\|_{H^{-1}(\Omega)})) e^{-s} < s^{3/2}$. Therefore, the lower bound for s must be chosen as depending logarithmically on $\|u_0, u_1\|_{\mathbf{H}}$.

3.1.3 Relative compactness of the set $\Lambda_s(\mathcal{C}(s))$

14 **Lemma 3.** Under the assumptions of Lemma 2, $\Lambda_s(\mathcal{C}(s))$ is a relatively compact subset of $\mathcal{C}(s)$ for the $L^\infty(0, T; L^2(\Omega))$ norm.

Proof. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of $\Lambda_s(\mathcal{C}(s))$. Then, there exists $(\widehat{y}_n)_{n \in \mathbb{N}}$, sequence of $\mathcal{C}(s)$ such that, for all $n \in \mathbb{N}$, $y_n = \Lambda_s(\widehat{y}_n)$. Remark that the sequence $(y_n)_{n \in \mathbb{N}}$ does not enjoy enough regularity to use classical compactness results. However, for all $n \in \mathbb{N}$, $y_n - y_0$ is solution of (10) for $B = f(\widehat{y}_n) - f(\widehat{y}_0)$, $u_0 = u_1 = 0$ and thus, we deduce from Proposition 4, for $r = 3/2 - p$ that $\rho(y_n - y_0) \in \mathcal{C}^1(0, T; H^{-r}(\Omega)) \cap \mathcal{C}^0([0, T]; H^{1-r}(\Omega))$, $\rho(v_n - v_0) \in H^{1-r}(0, T; L^2(\partial\Omega))$ and

$$\begin{aligned} & \|\rho(y_n - y_0)\|_{\mathcal{C}^1(0,T;H^{-r}(\Omega))} + \|\rho(y_n - y_0)\|_{\mathcal{C}^0([0,T];H^{1-r}(\Omega))} + \left\| \frac{\rho}{\eta \Psi^{1/2}} (v_n - v_0) \right\|_{H^{1-r}(0,T;L^2(\partial\Omega))} \\ & \leq C(s) \|\rho(f(\widehat{y}_n) - f(\widehat{y}_0))\|_{L^2(0,T;H^{-r}(\Omega))}. \end{aligned}$$

15 This gives with Lemma 1 that $(\rho(y_n - y_0))_{n \in \mathbb{N}}$ is a bounded sequence of $\mathcal{C}^1(0, T; H^{-r}(\Omega)) \cap \mathcal{C}^0([0, T]; H^{1-r}(\Omega))$.
16 Thus, since $\mathcal{C}^1(0, T; H^{-r}(\Omega)) \cap \mathcal{C}^0([0, T]; H^{1-r}(\Omega)) \hookrightarrow L^\infty(0, T; L^2(\Omega))$ is compact (see [19], Corollary 5
17 p.86), there exist a subsequence $(\rho(y_{n_k} - y_0))_{n_k \in \mathbb{N}}$ and $z \in L^\infty(0, T; L^2(\Omega))$ such that $\rho(y_{n_k} - y_0) \rightarrow z$
18 in $L^\infty(0, T; L^2(\Omega))$. Therefore, $\rho y_{n_k} \rightarrow y = z + \rho y_0$ in $L^\infty(0, T; L^2(\Omega))$. Since $\rho y_{n_k} \in \mathcal{C}(s)$ for all n_k ,
19 $y = z + \rho y_0 \in \mathcal{C}(s)$. Thus $\Lambda_s(\mathcal{C}(s))$ is a relatively compact subset of $\mathcal{C}(s)$ for the $L^\infty(0, T; L^2(\Omega))$ norm. \square

3.1.4 Continuity of the map Λ_s in $\mathcal{C}(s)$

Lemma 4. *Assume that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies (\mathbf{H}_p) . Then, the map $\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s)$ is continuous for the $L^\infty(0, T; L^2(\Omega))$ norm.*

Proof. Let $(\hat{y}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}(s)$ such that $(\hat{y}_n)_{n \in \mathbb{N}}$ converges to $\hat{y} \in \mathcal{C}(s)$ with respect to the $L^\infty(0, T; L^2(\Omega))$ norm. Let $y_n := \Lambda_s(\hat{y}_n)$ for all $n \in \mathbb{N}$. Since $\hat{y}_n \rightarrow \hat{y}$ in $L^\infty(0, T; L^2(\Omega))$, $\hat{y}_n \rightarrow \hat{y}$ in $L^2(Q)$ and there exist a subsequence $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$ and $z \in L^2(Q)$ such that $\hat{y}_{n_k} \rightarrow \hat{y}$ a.e and $|\hat{y}_{n_k}| \leq z$ a.e for all n_k .

Thus, since $f \in \mathcal{C}^0(\mathbb{R})$, $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$ a.e. Let us choose $\varepsilon < 1$ small enough so $L^{\frac{2}{1+\varepsilon}}(\Omega) \hookrightarrow H^{-1}(\Omega)$. For all $n_k \in \mathbb{N}$, a.e:

$$\begin{aligned} |f(\hat{y}_{n_k})| &\leq \alpha_1 + |\hat{y}_{n_k}| (\alpha_2 + \beta^* \ln_+^p |\hat{y}_{n_k}|) \leq \alpha_1 + \alpha_2 |\hat{y}_{n_k}| + \beta^* \ln_+^p |\hat{y}_{n_k}| |\hat{y}_{n_k}| \\ &\leq \alpha_1 + \alpha_2 |z| + \beta^* \ln_+^{3/2} |z| |z| \leq \alpha_1 + \alpha_2 |z| + C_\varepsilon \beta^* |z|^{1+\varepsilon}. \end{aligned}$$

and $\alpha_1 + \alpha_2 |z| + C_\varepsilon \beta^* |z|^{1+\varepsilon} \in L^{\frac{2}{1+\varepsilon}}(Q)$, since $z \in L^2(Q)$ and thus, we deduce from the dominated convergence theorem that $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$ in $L^{\frac{2}{1+\varepsilon}}(Q)$.

In fact, $f(\hat{y}_n) \rightarrow f(\hat{y})$ in $L^{\frac{2}{1+\varepsilon}}(Q)$ (and thus $f(\hat{y}_n) \rightarrow f(\hat{y})$ in $L^q(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega))$ for all $1 \leq q < +\infty$ since $(f(\hat{y}_n))_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega))$). If not, there exists $\varepsilon_1 > 0$ and a subsequence $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$ of $(\hat{y}_n)_{n \in \mathbb{N}}$ such that $\|f(\hat{y}_{n_k}) - f(\hat{y})\|_{L^{\frac{2}{1+\varepsilon}}(Q)} \geq \varepsilon_1$. But, arguing as previously, there exist a subsequence $(\hat{y}_{n_{k'}})_{n_{k'} \in \mathbb{N}}$ of $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$ such that $f(\hat{y}_{n_{k'}}) \rightarrow f(\hat{y})$ in $L^{\frac{2}{1+\varepsilon}}(Q)$ which leads to a contradiction.

We denote by v_n and v the associated control to y_n and y respectively and we have, by definition of the operator Λ_s , that $(y_n, v_n) = \rho^{-2}(Lw_n, s\eta^2\Psi\partial_\nu w_n)$ and $(y, v) = \rho^{-2}(Lw, s\eta^2\Psi\partial_\nu w)$ with w_n and w solution of (10) associated with $(u_0, u_1, -f(\hat{y}_n))$ and $(u_0, u_1, -f(\hat{y}))$ respectively. In particular, $z_n := y_n - y = \Lambda_s(\hat{y}_n) - \Lambda_s(\hat{y})$ satisfies $z_n = \rho^{-2}L(w_n - w)$ with $w_n - w$ solution of (10) associated with data $(0, 0, f(\hat{y}) - f(\hat{y}_n))$. Thus, using that $L^2(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega)) \hookrightarrow L^2(0, T; H^{-1}(\Omega))$, estimate (11) with $B = f(\hat{y}_n) - f(\hat{y})$, $u_0 = u_1 = 0$ implies

$$\begin{aligned} \|\rho(y_n - y)\|_{L^\infty(0, T; L^2(\Omega))} &\leq Cs^{3/2} \|\rho(f(\hat{y}_n) - f(\hat{y}))\|_{L^2(0, T; H^{-1}(\Omega))} \\ &\leq Cs^{3/2} \|\rho(f(\hat{y}_n) - f(\hat{y}))\|_{L^2(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega))} \end{aligned} \quad (23)$$

and thus $y_n \rightarrow y$ as $n \rightarrow +\infty$ in $L^\infty(0, T; L^2(\Omega))$. \square

3.2 Second part of Theorem 1

We now assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies (\mathbf{H}'_p) with $0 \leq p < 3/2$; f then satisfies (\mathbf{H}_p) for $\alpha_1 = f(0)$ and $\alpha_2 = \alpha$ so that results of Section 3.1 remain true. We are going to show that $\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s)$ is a contracting mapping of the complete space $(\mathcal{C}(s), d)$ with $d : \mathcal{C}(s) \times \mathcal{C}(s) \rightarrow \mathbb{R}$ defined by $d(y, z) := \|\rho(y - z)\|_{L^2(Q)}$. The Banach fixed point theorem will ensure the existence of a unique fixed point of Λ_s which is a controlled trajectory for the semilinear problem (1).

Proposition 6. *Assume that there exists $0 \leq p < 3/2$ such that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies (\mathbf{H}'_p) with β^* and s as in Lemma 2. Then, Λ_s is a contraction mapping from $(\mathcal{C}(s), d)$ into itself.*

Proof. Without restriction, we assume that $p > 1$. Let $\hat{y}_1, \hat{y}_2 \in \mathcal{C}(s)$. From (11), we get that, for all $0 \leq r < 1/2$, $d(\Lambda_s(\hat{y}_2), \Lambda_s(\hat{y}_1)) \leq C_r s^{r-3/2} \|\rho(f(\hat{y}_2) - f(\hat{y}_1))\|_{L^2(0, T; H^{-r}(\Omega))}$. Let $r = 3/2 - p > 0$. There exists $1 \leq q < 2$ such that $L^q(\Omega) \hookrightarrow H^{-r}(\Omega)$. We then have

$$d(\Lambda_s(\hat{y}_2), \Lambda_s(\hat{y}_1)) \leq C_r s^{r-3/2} \|\rho(f(\hat{y}_2) - f(\hat{y}_1))\|_{L^2(0, T; L^q(\Omega))}.$$

But, for all $(m_1, m_2) \in \mathbb{R}^2$ there exists \bar{c} such that

$$|f(m_1) - f(m_2)| \leq |m_1 - m_2| |f'(\bar{c})| \leq |m_1 - m_2| (\alpha + \beta^* \ln_+^p |\bar{c}|) \leq |m_1 - m_2| (\alpha + \beta^* \ln_+^p (|m_1| + |m_2|))$$

1 and therefore, using that $0 \leq \ln_+^p \rho \leq c^p s^p$ and that $p = 3/2 - r$, we get

$$\begin{aligned} d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) &\leq C s^{-p} \|(\alpha + \beta^* \ln_+^p(|\widehat{y}_1| + |\widehat{y}_2|))\rho(\widehat{y}_2 - \widehat{y}_1)\|_{L^2(0,T;L^q(\Omega))} \\ &\leq C s^{-p} \|(\alpha + \beta^* \ln_+^p(|\widehat{y}_1| + |\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))} d(\widehat{y}_2, \widehat{y}_1) \\ &\leq C s^{-p} (\alpha + \beta^* c^p s^p + \beta^* \|\ln_+^p(\rho(|\widehat{y}_1| + |\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))}) d(\widehat{y}_2, \widehat{y}_1) \end{aligned} \quad (24)$$

with a such that $1/q = 1/2 + 1/a$. Now, using that, for $\varepsilon = \inf\{\frac{2}{a}, \frac{p}{3}\}$

$$\begin{aligned} \|\ln_+^p(\rho(|\widehat{y}_1| + |\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))} &\leq C (\|(\rho\widehat{y}_1)^\varepsilon\|_{L^\infty(0,T;L^a(\Omega))} + \|(\rho\widehat{y}_2)^\varepsilon\|_{L^\infty(0,T;L^a(\Omega))}) \\ &\leq C (\|\rho\widehat{y}_1\|_{L^\infty(0,T;L^2(\Omega))}^\varepsilon + \|\rho\widehat{y}_2\|_{L^\infty(0,T;L^2(\Omega))}^\varepsilon) \leq C s^p \end{aligned}$$

2 we infer that

$$d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) \leq C(s^{-p}\alpha + \beta^* c^p) d(\widehat{y}_2, \widehat{y}_1). \quad (25)$$

3 The contraction property follows for $s \geq s_0$ large enough and $\beta^* > 0$ small enough. \square

4 **Remark 5.** This also proves that $s^{-1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}}(v_2 - v_1) \right\|_{L^2(\Sigma)} \leq C(s^{-p}\alpha + \beta^* c^p) d(\widehat{y}_2, \widehat{y}_1)$, for all $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}(s)$ where v_1, v_2 are the associated controls.

5 As a corollary of the Banach fixed point theorem, the contraction property of the operator Λ_s for β^*
6 small enough and s large enough allows to define a convergent sequence $(y_k, v_k)_{k \in \mathbb{N}}$ to a controlled pair
7 for (1) and prove the following more precise version of the second item of Theorem 1.

8 **Theorem 2.** Let $(u_0, u_1) \in \mathbf{H}$. Assume that f is $\mathcal{C}^1(\mathbb{R})$ and satisfies (\mathbf{H}'_p) for some $0 \leq p < 3/2$ with
9 β^* small enough and s as in Lemma 2. Then, for any $y_0 \in \mathcal{C}(s)$, the sequence $(y_k)_{k \in \mathbb{N}^*} \subset \mathcal{C}(s)$ given
10 by $y_{k+1} = \Lambda_s(y_k)$, $k \geq 0$, (where Λ_s is defined by (20)) together with the associated sequence of controls
11 $(v_k)_{k \in \mathbb{N}^*} \subset L^2(\Sigma)$ strongly converges in $\mathcal{C}^0([0, T]; L^2(\Omega)) \times L^2(\Sigma)$ to a controlled solution for (1). The
convergence is at least linear with respect to the distance d .

9 *Proof.* The convergence of the sequence $(y_k)_{k \in \mathbb{N}}$ toward $y = \Lambda_s(y) \in \mathcal{C}(s)$ with linear rate follows from
10 the contraction property of Λ_s : $d(y, y_k) = d(\Lambda_s(y), \Lambda_s(y_{k-1})) \leq (C(s^{-p}\alpha + \beta^* c^p))^k d(y, y_0)$ for all $k \geq 0$
11 deduced from (25).

Let now $v \in H^1(0, T; L^2(\partial\Omega)) \cap \mathcal{C}^0([0, T]; H^{1/2}(\partial\Omega))$ be associated with y so that $y - y_k$ satisfies, for every $k \in \mathbb{N}^*$

$$\begin{cases} L(y - y_k) = -(f(y) - f(y_{k-1})) & \text{in } Q, \\ y - y_k = v - v_k, & \text{on } \Sigma, \\ ((y - y_k)(\cdot, 0), (y - y_k)_t(\cdot, 0)) = (0, 0) & \text{in } \Omega, \\ ((y - y_k)(\cdot, T), (y - y_k)_t(\cdot, T)) = (0, 0) & \text{in } \Omega. \end{cases}$$

12 From Remark 5, we deduce that $\left\| \frac{\rho}{\eta\Psi^{1/2}}(v - v_k) \right\|_{L^2(\Sigma)} \leq s^{1/2} C(s^{-p}\alpha + \beta^* c^p) d(y, y_{k-1})$ and the convergence
13 at a linear rate of the sequence $(v_k)_{k \in \mathbb{N}^*}$ toward a control v for (1).

Now, since $f \in \mathcal{C}^1(\mathbb{R})$ satisfies (\mathbf{H}'_p) , using (11) for $r = 3/2 - p > 0$ and (25), we get

$$\begin{aligned} \|(\rho(y - y_k))_t\|_{\mathcal{C}^0([0, T]; H^{-1}(\Omega))} + \|\rho(y - y_k)\|_{\mathcal{C}^0([0, T]; L^2(\Omega))} &\leq C s^{1/2+r} \|\rho(f(y) - f(y_{k-1}))\|_{L^2(0, T; H^{-r}(\Omega))} \\ &\leq C s^2 C(s^{-p}\alpha + \beta^* c^p) \|\rho(y - y_{k-1})\|_{L^2(Q)} \\ &\leq C s^2 (C(s^{-p}\alpha + \beta^* c^p))^k d(y, y_0). \end{aligned}$$

14 It follows that $y_k \rightarrow y$ in $\mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))$. \square

15 **Remark 6.** Assume that exists $0 \leq p < 3/2$ such that $\lim_{|r| \rightarrow +\infty} \frac{|f'(r)|}{\ln_+^p |r|} = 0$, i.e. that β^* is arbitrarily small
in (\mathbf{H}'_p) . Then, (25) shows that the constant of contraction of Λ_s behaves like s^{-p} .

Remark 7. Assume $d \leq 3$ and $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Then, one may prove (repeating one step more the arguments of the Appendix) that the optimal state y given in Proposition 2 belongs to $L^\infty(Q)$. This allows (following [3]) to reach the value $p = 3/2$ in (\mathbf{H}_p) . We refer to [6] together with numerical illustrations.

3.3 Third part of Theorem 1

Assuming the initial data in \mathbf{V} , we follow the arguments of Section 3.1 and get that the uniform controllability holds true under the condition (\mathbf{H}_p) with $p = 3/2$. For any $s \geq s_0$, we introduce the class $\tilde{\mathcal{C}}(s)$ defined as the closed convex subset of $L^2(Q)$

$$\tilde{\mathcal{C}}(s) := \left\{ y \in H^1(Q) : \|\rho y\|_{L^2(Q)} \leq s, \|(\rho y)_t\|_{L^2(Q)} \leq s^2, \|\nabla(\rho y)\|_{L^2(Q)} \leq s^3 \right\}. \quad (26)$$

3.3.1 Estimates of $\Lambda_s(\hat{y})$

Lemma 5. Assume that $f \in C^0(\mathbb{R})$ satisfies (\mathbf{H}_p) with $p = 3/2$. For any $s \geq s_0$ and $\hat{y} \in \tilde{\mathcal{C}}(s)$, there exists $C > 0$ such that

$$\|\rho f(\hat{y})\|_{L^2(Q)} \leq C \left(\alpha_1 e^{-sT^{1/2}} |\Omega|^{1/2} + \alpha_2 s + \beta^* (cs)^{3/2} s \right)$$

with $c = \|\phi\|_{L^\infty(Q)}$.

Proof. Using that $\rho \leq e^{-s}$, we infer that

$$\begin{aligned} \|\rho f(\hat{y})\|_{L^2(Q)} &\leq C \|\rho \alpha_1 + |\rho \hat{y}| \left(\alpha_2 + \beta^* \ln_+^{3/2}(\hat{y}) \right)\|_{L^2(Q)} \\ &\leq C (\alpha_1 e^{-sT^{1/2}} |\Omega|^{1/2} + \alpha_2 \|\rho \hat{y}\|_{L^2(Q)} + \beta^* \|\rho \hat{y} \ln_+^{3/2}(\hat{y})\|_{L^2(Q)}). \end{aligned}$$

Now, since, $0 \leq \ln_+^{3/2}(\hat{y}) \leq C(\ln_+^{3/2} \rho^{-1} + \ln_+^{3/2}(\rho \hat{y})) \leq C((cs)^{3/2} + \ln_+^{3/2}(\rho \hat{y}))$, we get that

$$\|\rho \hat{y} \ln_+^{3/2}(\hat{y})\|_{L^2(Q)} \leq C((cs)^{3/2} \|\rho \hat{y}\|_{L^2(Q)} + \|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)}).$$

But, for all $0 < \varepsilon \leq 4/d$, there exists $C_\varepsilon > 0$ such that

$$\|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)} \leq C_\varepsilon \|\rho \hat{y}\|_{L^2(Q)}^{1+\frac{\varepsilon}{2}} = C_\varepsilon \|\rho \hat{y}\|_{L^{2+\varepsilon}(Q)}^{1+\frac{\varepsilon}{2}} \leq C_\varepsilon \|\rho \hat{y}\|_{L^\infty(0,T;L^2(\Omega))}^{1-\frac{\varepsilon(d-2)}{4}} \|\nabla(\rho \hat{y})\|_{L^2(Q)}^{\frac{\varepsilon d}{4}} \quad (27)$$

and since $\hat{y} \in \tilde{\mathcal{C}}(s)$, we have $\|\rho \hat{y}\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{2} \|\rho \hat{y}\|_{L^2(Q)}^{1/2} \|(\rho \hat{y})_t\|_{L^2(Q)}^{1/2} \leq \sqrt{2} s^{3/2}$. Thus

$$\|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)} \leq C_\varepsilon s^{\frac{3}{2} - \frac{3\varepsilon(d-2)}{8}} s^{\frac{3\varepsilon d}{4}} = C_\varepsilon s^{\frac{3}{2} + \frac{3\varepsilon(d+2)}{8}}.$$

In particular for $\varepsilon = \frac{4}{3(d+2)} < \frac{4}{d}$ this gives $\|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)} \leq C s^2$. Combining the above inequalities, we get the result. \square

Proposition 7. For any $x_0 \notin \bar{\Omega}$, we assume (3) and (5). Assume $(u_0, u_1) \in \mathbf{V}$. Assume that $f \in C(\mathbb{R})$ satisfies (\mathbf{H}_p) with $p = 3/2$. For $s \geq s_0$ and for all $\hat{y} \in \tilde{\mathcal{C}}(s)$, the solution $y = \Lambda_s(\hat{y})$ to the linearized controlled system (18) with control v satisfies $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and the following estimates:

$$\begin{aligned} &\|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\Psi^{1/2} \eta} v \right\|_{L^2(\Sigma)} + s^{-1} \|(\rho y)_t\|_{L^2(Q)} + s^{-3/2} \left\| \frac{1}{\Psi^{1/2} \eta} (\rho v)_t \right\|_{L^2(\Sigma)} + s^{-2} \|\nabla(\rho y)\|_{L^2(Q)} \\ &\leq \left(C \alpha_2 s^{-3/2} + \beta^* C c^{3/2} \right) s + \left(C \|u_0\|_{L^2(\Omega)} + C s^{-1} \left(\alpha_1 T^{1/2} |\Omega|^{1/2} + \|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} \right) \right) s^{-1/2} e^{-s}. \end{aligned} \quad (28)$$

Proof. This follows from Proposition 2 with $r = 0$, Proposition 3 and Lemma 5 with $B = -f(\hat{y})$. \square

3.3.2 Stability of the class $\tilde{\mathcal{C}}(s)$

Lemma 6. *Assume that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies (\mathbf{H}_p) with $p = 3/2$ and β^* small enough. Then, there exists $s \geq s_0$ large enough such that $\Lambda_s(\tilde{\mathcal{C}}(s)) \subset \tilde{\mathcal{C}}(s)$.*

Proof. For any $s \geq s_0$ and $\hat{y} \in \tilde{\mathcal{C}}(s)$, let $y := \Lambda_s(\hat{y})$. From the inequalities (28), we obtain that $\limsup_{s \rightarrow +\infty} s^{-1} \|\rho y\|_{L^2(Q)} \leq \beta^* C c^{3/2}$, $\limsup_{s \rightarrow +\infty} s^{-2} \|(\rho y)_t\|_{L^2(Q)} \leq \beta^* C c^{3/2}$ and $\limsup_{s \rightarrow +\infty} s^{-3} \|\nabla(\rho y)\|_{L^2(Q)} \leq \beta^* C c^{3/2}$. Therefore, if $\beta^* > 0$ is small enough so that $\beta^* C c^{3/2} < 1$, then for any $s \geq s_0$ large enough, $y \in \tilde{\mathcal{C}}(s)$. \square

3.3.3 Relative compactness of the set $\Lambda_s(\tilde{\mathcal{C}}(s))$

Lemma 7. *Under the assumptions of Lemma 2, $\Lambda_s(\tilde{\mathcal{C}}(s))$ is a relatively compact subset of $\tilde{\mathcal{C}}(s)$ for the $L^2(Q)$ norm.*

Proof. Since $\Lambda_s(\tilde{\mathcal{C}}(s)) \subset \tilde{\mathcal{C}}(s)$, this follows from the fact that $\tilde{\mathcal{C}}(s)$ is bounded in $H^1(Q)$. \square

Remark 8. *In fact $\tilde{\mathcal{C}}(s)$ is a compact subset of $L^p(Q)$ norm, $1 \leq p < 2 + \frac{4}{d}$ and thus $\Lambda_s(\tilde{\mathcal{C}}(s))$ is a relatively compact subset of $\tilde{\mathcal{C}}(s)$ for the $L^p(Q)$ norm.*

3.3.4 Continuity of the map Λ_s in $\tilde{\mathcal{C}}(s)$

Lemma 8. *Assume that $f \in \mathcal{C}^0(\mathbb{R})$ satisfies (\mathbf{H}_p) with $p = 3/2$. Then, the map $\Lambda_s : \tilde{\mathcal{C}}(s) \rightarrow \tilde{\mathcal{C}}(s)$ is continuous with respect to the $L^2(Q)$ norm.*

Proof. Let $(\hat{y}_n)_{n \in \mathbb{N}}$ be a sequence in $\tilde{\mathcal{C}}(s)$ such that $(\hat{y}_n)_{n \in \mathbb{N}}$ converges to $\hat{y} \in \tilde{\mathcal{C}}(s)$ with respect to the $L^2(Q)$ norm. Let $y_n := \Lambda_s(\hat{y}_n)$ for all $n \in \mathbb{N}$. Since $\hat{y}_n \rightarrow \hat{y}$ in $L^2(Q)$ then $\hat{y}_n \rightarrow \hat{y}$ in $L^{2+\frac{2}{d}}(Q)$ (since, see Remark 8, $\tilde{\mathcal{C}}(s)$ is a compact subset of $L^{2+\frac{2}{d}}(Q)$). Thus there exist a subsequence $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$ and $z \in L^{2+\frac{2}{d}}(Q)$ such that $\hat{y}_{n_k} \rightarrow \hat{y}$ a.e. and $|\hat{y}_{n_k}| \leq z$ a.e. for all n_k . It follows since $f \in \mathcal{C}^0(\mathbb{R})$ that $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$ a.e. But, for all $n_k \in \mathbb{N}$, a.e.

$$\begin{aligned} |f(\hat{y}_{n_k})| &\leq \alpha_1 + |\hat{y}_{n_k}| \left(\alpha_2 + \beta^* \ln_+^{3/2}(\hat{y}_{n_k}) \right) \leq \alpha_1 + \alpha_2 |\hat{y}_{n_k}| + \beta^* |\hat{y}_{n_k}| \ln_+^{3/2}(\hat{y}_{n_k}) \\ &\leq \alpha_1 + \alpha_2 |z| + \beta^* |z| \ln_+^{3/2}(z) \leq \alpha_1 + \alpha_2 |z| + C \beta^* |z|^{1+\frac{1}{d}} \end{aligned}$$

and $\alpha_1 + \alpha_2 |z| + C \beta^* |z|^{1+\frac{1}{d}} \in L^2(Q)$, since $z \in L^{2+\frac{2}{d}}(Q)$ and thus, from the dominated convergence theorem, $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$ in $L^2(Q)$. Indeed $f(\hat{y}_n) \rightarrow f(\hat{y})$ in $L^2(Q)$. Otherwise, there exists $\varepsilon > 0$ and a subsequence $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$ of $(\hat{y}_n)_{n \in \mathbb{N}}$ such that $\|f(\hat{y}_{n_k}) - f(\hat{y})\|_{L^2(Q)} \geq \varepsilon$. But, arguing as before, there exists a subsequence $(\hat{y}_{n_{k'}})_{n_{k'} \in \mathbb{N}}$ of $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$ such that $f(\hat{y}_{n_{k'}}) \rightarrow f(\hat{y})$ in $L^2(Q)$ which leads to a contradiction.

In particular, $z_n := y_n - y = \Lambda_s(\hat{y}_n) - \Lambda_s(\hat{y})$ satisfies $z_n = \rho^{-2} L(w_n - w)$ with $w_n - w$ solution of (10) associated with data $(0, 0, f(\hat{y}) - f(\hat{y}_n))$. Thus, using that $L^2(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega)) \hookrightarrow L^2(0, T; H^{-1}(\Omega))$, estimate (11) with $B = f(\hat{y}_n) - f(\hat{y})$, $u_0 = u_1 = 0$ implies

We denote by v_n and v the associated control to y_n and y respectively and we have, by definition of the operator Λ_s , that $(y_n, v_n) = \rho^{-2} (Lw_n, s\eta^2 \Psi \partial_\nu w_n)$ and $(y, v) = \rho^{-2} (Lw, s\eta^2 \Psi \partial_\nu w)$ with w_n and w solution of (10) associated with $(u_0, u_1, -f(\hat{y}_n))$ and $(u_0, u_1, -f(\hat{y}))$ respectively. In particular, $z := y_n - y = \Lambda_s(\hat{y}_n) - \Lambda_s(\hat{y})$ satisfies $z = \rho^{-2} L(w_n - w)$ with $w_n - w$ solution of (10) associated with data $(0, 0, f(\hat{y}) - f(\hat{y}_n))$. Then estimate (11) with $B = f(\hat{y}_n) - f(\hat{y})$, $u_0 = u_1 = 0$ and $r = 0$ implies $\|\rho(y_n - y)\|_{L^2(Q)} \leq C s^{-3/2} \|\rho(f(\hat{y}) - f(\hat{y}_n))\|_{L^2(Q)}$ so that $y_n \rightarrow y$ as $n \rightarrow +\infty$ in $L^2(Q)$. \square

A Appendix: Proof of Proposition 3

We check that the optimal pair (y, v) defined in Proposition 2 belongs to $C^0([0, T]; H^1(\Omega)) \times C^0(0, T; H^{1/2}(\partial\Omega))$ as soon as $B \in L^2(Q)$ and $(u_0, u_1) \in \mathbf{V}$. This property has been proved in [3] in the one dimensional case (by generalizing [10]). The occurrence of the weights and the coupling between the primal variable y and the dual one w (see Remark 3) make the arguments and the computations more involved with respect [10]. The proof is divided into four steps: the first and second steps are similar to [3, Appendix]; the third one is new with respect to [3, Appendix] as it provides an estimate of $\|\nabla(\rho y)\|_{L^2(Q)}$.

For all $f \in C^0(\mathbb{R}; E)$ (where E is a Banach space) and any $\tau \neq 0$, we define $\delta_\tau f := f(t + \frac{\tau}{2}) - f(t - \frac{\tau}{2})$, $\mathcal{T}_\tau f := \frac{1}{\tau} \delta_\tau \left(\frac{\delta_\tau f}{\tau} \right)$ and $\tilde{\delta}_\tau f(t) := \frac{f(t+\tau) - f(t)}{\tau}$.

Let now $w \in P_s$ and $y \in L^2(Q)$ be given by Proposition 2. Then $\mathcal{T}_\tau w$ belongs to P_s , where w as well as y can be extended uniquely on $(-\infty, 0)$ and $(T, +\infty)$. Indeed, in the time interval $(-\infty, 0)$ the solution y satisfies

$$Ly = 0 \text{ in } \Omega \times (-\infty, 0), \quad y = 0 \text{ on } \partial\Omega \times (-\infty, 0), \quad (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega,$$

where the source term $B \in L^2(Q)$ is assumed to be extendable by 0 outside $(0, T)$. Recall that the boundary condition $y(\cdot, t) = 0$ on Γ_0 holds outside $(0, T)$ since $\eta = 0$ (appearing in the formula of v) vanishes outside $(\delta, T - \delta)$, see (6).

Similarly, in $(T, +\infty)$ we can define the solution y uniquely, and $y(t) = 0$ for all $t \geq T$. It follows that the solution y satisfies $y \in C^0(\mathbb{R}; L^2(\Omega)) \cap C^1(\mathbb{R}; H^{-1}(\Omega))$ and $y \in C^0((-\infty, \delta]; H_0^1(\Omega)) \cap C^1((-\infty, \delta]; L^2(\Omega))$ and $y \in C^0([T - \delta, +\infty); H_0^1(\Omega)) \cap C^1([T - \delta, +\infty); L^2(\Omega))$ (see [13, Theorem 2.1, page 151]). We extend as well the weight ρ in $\Omega \times \mathbb{R}$. This ensures the extension of the solution w which satisfies the following set of equations in \mathbb{R}

$$Lw = \rho^2 y \text{ in } \Omega \times \mathbb{R}, \quad w = 0 \text{ on } \partial\Omega \times \mathbb{R}. \quad (29)$$

Moreover, it can be seen that $Lw = 0$ in $[T, +\infty)$.

Step 1. We assume that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $B \in \mathcal{D}(0, T; L^2(\Omega))$ and prove that $v \in H^1(0, T; L^2(\partial\Omega))$ and $y_t \in L^2(Q)$.

Since $w \in C^0(\mathbb{R}; H_0^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega))$ solves (29), $\mathcal{T}_\tau w \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and $\partial_\nu \mathcal{T}_\tau w \in L^2(\Sigma)$. With $z = \mathcal{T}_\tau w$ as test function in (10), we have

$$\begin{aligned} \int_Q \rho^{-2} Lw L \mathcal{T}_\tau w \, dxdt + s \int_\Sigma \eta^2(t) \Psi(x) \rho^{-2} \partial_\nu w \mathcal{T}_\tau \partial_\nu w \, d\Sigma \\ = \int_\Omega u_1 \mathcal{T}_\tau w(\cdot, 0) \, dx - \int_\Omega u_0 \mathcal{T}_\tau w_t(\cdot, 0) \, dx + \int_Q B \mathcal{T}_\tau w \, dxdt. \end{aligned} \quad (30)$$

Proceeding as [3, Appendix] and using that the smooth function η given by (6) satisfies $\eta = 0$ in $(-\infty, \delta] \cup [T - \delta, +\infty)$ (with $\delta > 0$ given in (6)), we get the following estimate (we refer to [3, sub-steps 1 and 2 pages 108-110])

$$\begin{aligned} \int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2} Lw)(t) \right|^2 \, dxdt + s \int_\Sigma \Psi \frac{\eta^2(t) \rho^{-2}(t) + \eta^2(t + \tau) \rho^{-2}(t + \tau)}{2} \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 \, d\Sigma \\ = \int_Q \rho^2(t) \tilde{\delta}_\tau(\rho^{-2} Lw)(t) \tilde{\delta}_\tau(\rho^{-2})(t) Lw(t + \tau) \, dxdt - \frac{1}{\tau} \int_{-\tau}^0 \int_\Omega \rho^{-2}(t + \tau) Lw(t + \tau) \tilde{\delta}_\tau Lw(t) \, dxdt \\ - \frac{1}{\tau} \int_T^{T-\tau} \int_\Omega \rho^{-2}(t + \tau) Lw(t + \tau) \tilde{\delta}_\tau Lw(t) \, dxdt \\ - s \int_\Sigma \Psi \tilde{\delta}_\tau(\eta^2 \rho^{-2})(t) \frac{\partial_\nu w(t) + \partial_\nu w(t + \tau)}{2} \tilde{\delta}_\tau(\partial_\nu w)(t) \, d\Sigma \\ - \int_Q B \mathcal{T}_\tau w(t) \, dxdt - \int_\Omega u_1 \mathcal{T}_\tau w(\cdot, 0) \, dx + \int_\Omega u_0 \mathcal{T}_\tau w_t(\cdot, 0) \, dx \end{aligned} \quad (31)$$

Then, following [3, step 1, sub-step 3 page 111-114], we get that each term of the right hand side of (31) are uniformly bounded with respect to $\tau \in [0, \delta]$. Then we can conclude, from (31), that the terms $\int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2}Lw)(t) \right|^2 dxdt$ and

$$\int_\Sigma \Psi \frac{\eta^2(t)\rho^{-2}(t) + \eta^2(t+\tau)\rho^{-2}(t+\tau)}{2} \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 d\Sigma$$

1 are bounded. Thus $\rho(\rho^{-2}Lw)_t \in L^2(Q)$, $\Psi^{1/2}\eta\rho^{-1}\partial_\nu w_t \in L^2(\Sigma)$, and $y \in H^1(0, T; L^2(\Omega))$, $v \in$
2 $H^1(0, T; L^2(\partial\Omega))$.

3 Eventually, following [3, step 1, sub-step 4 page 115], we prove that $v \in C^0([0, T]; H^{1/2}(\partial\Omega))$ and
4 $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.

Since $\tilde{\delta}_\tau w \in P$, Carleman estimates (8) gives

$$\begin{aligned} & s \int_Q \rho^{-2} \left(|\tilde{\delta}_\tau w_t|^2 + |\tilde{\delta}_\tau \nabla w|^2 \right) dxdt + s^3 \int_Q \rho^{-2} |\tilde{\delta}_\tau w|^2 dxdt \\ & + s \int_\Omega \rho^{-2}(0) \left(|\tilde{\delta}_\tau(w_t)(0)|^2 + |\tilde{\delta}_\tau(\nabla w)(0)|^2 \right) dx + s^3 \int_\Omega \rho^{-2}(0) |\tilde{\delta}_\tau w(0)|^2 dx \\ & \leq C \int_Q \rho^{-2} |L(\tilde{\delta}_\tau w)|^2 dxdt + Cs \int_\Sigma \eta^2 \Psi \rho^{-2} |\tilde{\delta}_\tau(\partial_\nu w)|^2 d\Sigma. \end{aligned} \quad (32)$$

Remark that $\int_\Sigma \Psi \eta^2 \rho^{-2} \left| \tilde{\delta}_\tau(\partial_\nu w) \right|^2 d\Sigma$ and $\int_Q \rho^{-2} |L(\tilde{\delta}_\tau w)|^2 dxdt$ are bounded since

$$\int_\Sigma \Psi \eta^2(t) \rho^{-2}(t) \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 d\Sigma \leq 2 \int_\Sigma \Psi \frac{\eta^2(t)\rho^{-2}(t) + \eta^2(t+\tau)\rho^{-2}(t+\tau)}{2} \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 d\Sigma$$

and

$$\int_Q \rho^{-2}(t) |L(\tilde{\delta}_\tau w)(t)|^2 dxdt \leq 2 \int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2}Lw)(t) \right|^2 dxdt + 2 \int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2})(t)Lw(t+\tau) \right|^2 dxdt$$

5 thus the right hand side in (32) is bounded. Therefore $\tau \mapsto \tilde{\delta}_\tau w_t$ and $\tau \mapsto \tilde{\delta}_\tau \nabla w$ are bounded in $L^2(Q)$
6 and thus $w_{tt} \in L^2(Q)$ and $w_t \in L^2(0, T; H_0^1(\Omega))$. Moreover, $\tau \mapsto \tilde{\delta}_\tau w(0)$ is bounded in $H_0^1(\Omega)$ thus
7 $w_t(0) \in H_0^1(\Omega)$. Eventually, since $\tau \mapsto L(\tilde{\delta}_\tau w)$ is bounded in $L^2(Q)$, we conclude that $Lw_t \in L^2(Q)$ and
8 then that $w_t \in P$; it follows that $\partial_\nu w_t \in L^2(\Sigma)$.

Remark 9. We then have $w \in C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$ and since $Lw \in C^0([0, T]; L^2(\Omega))$ we
9 deduce that $\Delta w (= w_{tt} - Lw) \in C^0([0, T]; L^2(\Omega))$ and thus $w \in C^0([0, T]; H^2(\Omega))$.

10 Since $w \in C^0([0, T]; H^2(\Omega))$ $v \in C^0([0, T]; H^{1/2}(\partial\Omega)) \cap H^1(0, T; L^2(\partial\Omega))$ and v satisfies the compatibility
11 conditions $v(0) = 0$, [13, Theorem 2.1, page 151] leads to $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap$
12 $C^2([0, T]; H^{-1}(\Omega))$.

Remark 10. As previously mentioned, y and w can be extended uniquely on $(-\infty; 0)$ and $(T, +\infty)$ so
13 that $w \in C^0(\mathbb{R}; H^2(\Omega)) \cap C^1(\mathbb{R}; H_0^1(\Omega)) \cap C^2(\mathbb{R}; L^2(\Omega))$, $\partial_\nu w \in C^0(\mathbb{R}; H^{1/2}(\partial\Omega))$, $\partial_\nu w_t \in L_{\text{loc}}^2(\mathbb{R}; L^2(\partial\Omega))$
and $y \in C^0(\mathbb{R}; H^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega))$.

Step 2. We prove estimate on v_t and y_t in (16). Carleman estimate (8) for $w_t \in P$ reads

$$\begin{aligned} & s \int_Q \rho^{-2} (|w_{tt}|^2 + |\nabla w_t|^2) dxdt + s^3 \int_Q \rho^{-2} |w_t|^2 dxdt \\ & + s \int_\Omega \rho^{-2}(0) (|w_{tt}(0)|^2 + |\nabla w_t(0)|^2) dx + s^3 \int_\Omega \rho^{-2}(0) |w_t(0)|^2 dx \\ & \leq C \int_Q \rho^{-2} |Lw_t|^2 dxdt + Cs \int_\Sigma \eta^2 \Psi \rho^{-2} |\partial_\nu w_t|^2 d\Sigma. \end{aligned} \quad (33)$$

- 1 Proceeding as in [3, step 2 sub-step 1, page 116], we pass to the limit when $\tau \rightarrow 0$ in the equality (31)
 2 and obtain

$$\begin{aligned}
 \int_Q \rho^2 |y_t|^2 dx dt + s \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma &= -4s\lambda\beta \int_Q (t - \frac{T}{2}) \phi \rho^2 y_t y dx dt - \int_\Omega y(0)(\rho^2 y)_t(0) dx \\
 - s \int_\Sigma \Psi (\eta^2 \rho^{-2})_t \partial_\nu w \partial_\nu w_t d\Sigma - \int_Q B w_{tt} dx dt - \int_\Omega w_{tt}(0) u_1 dx & \quad (34) \\
 - 2s\lambda\beta T \int_\Omega \phi(0) \rho^2(0) u_0^2 dx + \int_\Omega \rho^2(0) u_1 u_0 dx - \int_\Omega \nabla u_0 \nabla w_t(0) dx &
 \end{aligned}$$

and then estimate each term of the right side. In particular, for the third term, we write, using $(\rho^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho^{-1}$ that

$$\begin{aligned}
 |s \int_\Sigma \Psi (\eta^2 \rho^{-2})_t \partial_\nu w \partial_\nu w_t d\Sigma| &\leq 2 \left(s \int_\Sigma \Psi |(\eta\rho^{-1})_t|^2 |\partial_\nu w|^2 d\Sigma \right)^{1/2} \left(s \int_\Sigma \Psi \eta^2(t) \rho^{-2}(t) |\partial_\nu w_t|^2 d\Sigma \right)^{1/2} \\
 &\leq C \left(s^3 \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w|^2 d\Sigma + s \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma \right)^{1/2} \left(s \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma \right)^{1/2} \\
 &\leq C \left(s \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} v^2 d\Sigma + s \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma \right) + \frac{s}{8} \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma.
 \end{aligned}$$

We now estimate the term $\int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma$ appearing in the previous inequality: proceeding as in [16, Lemma 3.7] with $q(x, t) = h\rho^{-2}(x, t)$ where $h \in (C^1(\bar{\Omega}))^d$ satisfies $h(x) = \nu(x)$ on $\partial\Omega$, we get (with the notation $f_k g_k = \sum_{k=1}^d f_k g_k$)

$$\begin{aligned}
 \frac{1}{2} \int_\Sigma \rho^{-2} |\partial_\nu w|^2 d\Sigma &= \int_Q \rho^{-2} \nabla w \cdot \nabla h_k \frac{\partial w}{\partial x_k} + 2 \int_Q \rho^{-1} (\nabla w \cdot \nabla \rho^{-1} - (\rho^{-1})_t w_t) h \cdot \nabla w \\
 &\quad + \frac{1}{2} \int_Q \rho^{-2} (|w_t|^2 - |\nabla w|^2) \nabla \cdot h + \int_Q \rho^{-1} (|w_t|^2 - |\nabla w|^2) h \cdot \nabla \rho^{-1} \\
 &\quad + \int_\Omega [\rho^{-2} w_t h \cdot \nabla w]_0^T - \int_Q y h \cdot \nabla w.
 \end{aligned}$$

Writing that $|\nabla \rho^{-1}| \leq Cs\rho^{-1}$ and $|(\rho^{-1})_t| \leq Cs\rho^{-1}$, and using that $h \in (C^1(\bar{\Omega}))^d$ and $s \geq 1$ we obtain

$$\begin{aligned}
 \int_\Sigma \rho^{-2} |\partial_\nu w|^2 d\Sigma &\leq Cs \int_\Omega \left(\rho^{-2}(0)(|w_t|^2 + |\nabla w|^2)(0) + \rho^{-2}(T)(|w_t|^2 + |\nabla w|^2)(T) \right) dx \\
 &\quad + Cs \int_Q \rho^{-2} (|w_t|^2 + |\nabla w|^2) dx dt + Cs^{-1} \int_Q \rho^2 |y|^2 dx dt
 \end{aligned}$$

- 3 leading, using (8) and that $\Psi \in (C^1(\bar{\Omega}))^d$ to

$$s \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma \leq Cs \left(\|\rho y\|_{L^2(Q)}^2 + s^{-1} \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)}^2 \right). \quad (35)$$

Thus,

$$|s \int_\Sigma \Psi (\eta^2 \rho^{-2})_t \partial_\nu w \partial_\nu w_t d\Sigma| \leq Cs \left(\|\rho y\|_{L^2(Q)}^2 + \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)}^2 \right) + \frac{s}{8} \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma.$$

- 4 Proceeding as in [3, step 2 sub-step 2, page 117-120] for the other terms in the right hand side of (34),

1 we then get from (34)

$$\begin{aligned} & \int_Q \rho^2 |y_t|^2 \, dxdt + s \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 \, d\Sigma \\ & \leq C \left(s^2 \|\rho y\|_{L^2(Q)}^2 + s \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)}^2 + s^{-1} \|\rho B\|_{L^2(Q)}^2 \right. \\ & \quad \left. + s \|\rho(0) u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) \nabla u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) u_1\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (36)$$

2 We have, since $\eta \in C^1([0, T])$ and $v = s\eta^2 \Psi \rho^{-2} \partial_\nu w$

$$s^{-1} \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} |v_t|^2 \, d\Sigma \leq C s \left(\int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 \, d\Sigma + \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} v^2 \, d\Sigma + \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 \, d\Sigma \right) \quad (37)$$

thus, using that $\Psi \in (C^1(\bar{\Omega}))^d$ and (35), (36), implies for $s \geq s_0 \geq 1$ that

$$\begin{aligned} & \int_Q \rho^2 |y_t|^2 \, dxdt + s^{-1} \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} |v_t|^2 \, d\Sigma \\ & \leq C \left(s^{-1} \|\rho B\|_{L^2(Q)}^2 + s \|\rho(0) u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) \nabla u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) u_1\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

3 which is the announced estimates (16) for the first two terms in the case of regular data.

4 **Step 3.** We obtain the estimate on $\|\nabla(\rho y)\|_{L^2(Q)}$ and $\|\rho v\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))}$ in (16). With respect to [3],
5 this part is new. From the definition of v , since $Lw(T) = (\rho^2 y)(T) = 0$ and $s \geq 1$ we have

$$\begin{aligned} \|\rho v\|_{L^2(0, T; H^{1/2}(\partial\Omega))} &= s \|\Psi \rho^{-1} \partial_\nu w\|_{L^2(0, T; H^{1/2}(\partial\Omega))} = s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^2(0, T; H^{1/2}(\partial\Omega))} \\ &\leq C s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^2(0, T; H^1(\Omega))} \\ &\leq C (s \|\nabla(\rho^{-1} w)\|_{L^2(0, T; H^1(\Omega))} + s^2 \|\rho^{-1} w\|_{L^2(0, T; H^1(\Omega))}) \end{aligned} \quad (38)$$

6 and

$$\begin{aligned} \|\rho v\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} &= s \|\Psi \rho^{-1} \partial_\nu w\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} = s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} \\ &\leq C s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^\infty(0, T; H^1(\Omega))} \\ &\leq C (s \|\nabla(\rho^{-1} w)\|_{L^\infty(0, T; H^1(\Omega))} + s^2 \|\rho^{-1} w\|_{L^\infty(0, T; H^1(\Omega))}). \end{aligned} \quad (39)$$

Since $\nabla(\rho^{-1} w) = \nabla(\rho^{-1})w + \rho^{-1} \nabla w$ and $\Delta(\rho^{-1} w) = \Delta(\rho^{-1})w + 2\nabla(\rho^{-1}) \cdot \nabla w + \rho^{-1} \Delta w$ we have

$$\begin{aligned} \|\rho^{-1} w\|_{L^2(0, T; H^1(\Omega))} &\leq C \|\nabla(\rho^{-1} w)\|_{L^2(0, T; L^2(\Omega))} \\ &\leq C \left(s \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} \right), \end{aligned}$$

$$\begin{aligned} \|\nabla(\rho^{-1} w)\|_{L^2(0, T; H^1(\Omega))} &\leq C \|\Delta(\rho^{-1} w)\|_{L^2(0, T; L^2(\Omega))} \\ &\leq C \left(s^2 \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + s \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} + \|\rho^{-1} \Delta w\|_{L^2(0, T; L^2(\Omega))} \right) \\ &\leq C \left(s^2 \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + s \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} + \|\rho^{-1} w_{tt}\|_{L^2(0, T; L^2(\Omega))} \right. \\ & \quad \left. + \|\rho y\|_{L^2(0, T; L^2(\Omega))} \right) \end{aligned}$$

and thus

$$\begin{aligned} \|\rho v\|_{L^2(0, T; H^{1/2}(\partial\Omega))} &\leq C \left(s^3 \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + s^2 \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} + s \|\rho^{-1} w_{tt}\|_{L^2(0, T; L^2(\Omega))} \right. \\ & \quad \left. + s \|\rho y\|_{L^2(0, T; L^2(\Omega))} \right). \end{aligned}$$

1 Estimate (8) then rewrites :

$$\begin{aligned}
& s^{1/2} \|\rho^{-1}(0)w_t(0)\|_{L^2(\Omega)} + s^{1/2} \|\rho^{-1}(0)\nabla w(0)\|_{L^2(\Omega)} + s^{3/2} \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} \\
& + s^{1/2} \|\rho^{-1}w_t\|_{L^2(Q)} + s^{1/2} \|\rho^{-1}\nabla w\|_{L^2(Q)} + s^{3/2} \|\rho^{-1}w\|_{L^2(Q)} \\
& \leq C \left(\|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}} v \right\|_{L^2(\Sigma)} \right) \\
& \leq C \left(s^{-3/2} \|\rho B\|_{L^2(Q)} + s^{-1/2} \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-3/2} \|\rho(0)u_1\|_{L^2(\Omega)} \right)
\end{aligned} \tag{40}$$

2 while estimate (33) rewrites (using estimates (36)-(37))

$$\begin{aligned}
& s^{1/2} \|\rho^{-1}(0)w_{tt}(0)\|_{L^2(\Omega)} + s^{1/2} \|\rho^{-1}(0)\nabla w_t(0)\|_{L^2(\Omega)} + s^{3/2} \|\rho^{-1}(0)w_t(0)\|_{L^2(\Omega)} \\
& + s^{1/2} \|\rho^{-1}w_{tt}\|_{L^2(Q)} + s^{1/2} \|\rho^{-1}\nabla w_t\|_{L^2(Q)} + s^{3/2} \|\rho^{-1}w_t\|_{L^2(Q)} \\
& \leq C \left(s \|\rho y\|_{L^2(Q)} + \|\rho y_t\|_{L^2(Q)} + s^{1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}} v \right\|_{L^2(\Sigma)} + s^{-1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}} v_t \right\|_{L^2(\Sigma)} \right) \\
& \leq C s^{-1/2} \mathcal{A}(u_0, u_1, B).
\end{aligned}$$

with

$$\mathcal{A}(u_0, u_1, B) := \|\rho B\|_{L^2(Q)} + s \|\rho(0)u_0\|_{L^2(\Omega)} + \|\rho(0)u_1\|_{L^2(\Omega)} + \|\rho(0)\nabla u_0\|_{L^2(\Omega)}.$$

Since $s \geq 1$, $|(\rho^{-1})_t| = Cs|\rho^{-1}|$, $|(\rho^{-1})_{tt}| = Cs^2|\rho^{-1}|$, $|\nabla(\rho^{-1})| = Cs|\rho^{-1}|$, $|\Delta(\rho^{-1})| = Cs^2|\rho^{-1}|$ and $\rho^{-1}(0)\Delta w(0) = \rho^{-1}w_{tt}(0) - \rho(0)u_0$, we deduce

$$\begin{aligned}
& \|\Delta(\rho^{-1}w)(0)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} \\
& \leq C \left(\|\rho^{-1}(0)w_{tt}(0)\|_{L^2(\Omega)} + \|\rho(0)u_0\|_{L^2(\Omega)} + s \|\rho^{-1}(0)\nabla w(0)\|_{L^2(\Omega)} + s^2 \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} \right. \\
& \quad \left. + s \|\rho^{-1}(0)w_t(0)\|_{L^2(\Omega)} + \|\rho^{-1}(0)\nabla w_t(0)\|_{L^2(\Omega)} \right) \\
& \leq C s^{-1} \mathcal{A}(u_0, u_1, B)
\end{aligned}$$

and thus

$$\|\rho v\|_{L^2(0,T;H^{1/2}(\partial\Omega))} \leq C \mathcal{A}(u_0, u_1, B).$$

Since $w_{tt} - \Delta w = Lw$, $w|_{\partial\Omega} = 0$, we infer, using $\rho^{-1} \in C^\infty(\bar{Q})$ and $y = \rho^{-2}Lw$, that

$$\begin{cases} (\rho^{-1}w)_{tt} - \Delta(\rho^{-1}w) = \tilde{B}_w := \rho y + 2\rho_t^{-1}w_t + \rho_{tt}^{-1}w - 2\nabla\rho^{-1} \cdot \nabla w - \Delta\rho^{-1}w, \\ (\rho^{-1}w)|_{\partial\Omega} = 0. \end{cases}$$

for which the standard estimates

$$\|(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} \leq C \left(\|\tilde{B}_w\|_{L^2(Q)} + \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} + \|(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} \right)$$

and

$$\begin{aligned}
& \|(\rho^{-1}w)_{tt}(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\Delta(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} \\
& \leq C \left(\|\tilde{B}_w\|_{H^1(0,T;L^2(\Omega))} + \|\Delta(\rho^{-1}w)(0)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} \right)
\end{aligned}$$

3 hold true, for all $t \geq 0$.

Using again that $s \geq 1$, $|(\rho^{-1})_t| = Cs|\rho^{-1}|$, $|(\rho^{-1})_{tt}| = Cs^2|\rho^{-1}|$, $|\nabla(\rho^{-1})| = Cs|\rho^{-1}|$ and $|\Delta(\rho^{-1})| = Cs^2|\rho^{-1}|$ we have the following estimates :

$$\begin{aligned}
\|\tilde{B}_w\|_{L^2(Q)} & \leq C \left(\|\rho y\|_{L^2(Q)} + s \|\rho^{-1}w_t\|_{L^2(Q)} + s^2 \|\rho^{-1}w\|_{L^2(Q)} + s \|\rho^{-1}\nabla w\|_{L^2(Q)} \right) \\
& \leq C \left(s^{-1} \|\rho B\|_{L^2(Q)} + \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1} \|\rho(0)u_1\|_{L^2(\Omega)} \right),
\end{aligned}$$

$$\begin{aligned} \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} + \|(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} &\leq C\left(s\|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} + \|\rho^{-1}w_t(0)\|_{L^2(\Omega)}\right) \\ &\leq C\left(s^{-2}\|\rho B\|_{L^2(Q)} + s^{-1}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{-2}\|\rho(0)u_1\|_{L^2(\Omega)}\right) \end{aligned}$$

1 and therefore

$$\|(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)(t)\|_{L^2(\Omega)} \leq C\left(s^{-1}\|\rho B\|_{L^2(Q)} + \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1}\|\rho(0)u_1\|_{L^2(\Omega)}\right).$$

We also have

$$(\tilde{B}_w)_t = \rho_t y + \rho y_t + 3\rho_{tt}^{-1}w_t + 2\rho_t^{-1}w_{tt} + \rho_{ttt}^{-1}w - 2\nabla\rho_t^{-1} \cdot \nabla w - 2\nabla\rho^{-1} \cdot \nabla w_t - \Delta\rho_t^{-1}w - \Delta\rho^{-1}w_t$$

2 and thus

$$\begin{aligned} \|(\tilde{B}_w)_t\|_{L^2(Q)} &\leq C\left(s\|\rho y\|_{L^2(Q)} + \|\rho y_t\|_{L^2(Q)} + s\|\rho^{-1}w_{tt}\|_{L^2(Q)} + s^2\|\rho^{-1}w_t\|_{L^2(Q)}\right. \\ &\quad \left.+ s^3\|\rho^{-1}w\|_{L^2(Q)} + s\|\rho^{-1}\nabla w_t\|_{L^2(Q)} + s^2\|\rho^{-1}\nabla w\|_{L^2(Q)}\right) \\ &\leq C\mathcal{A}(u_0, u_1, B) \end{aligned}$$

3 and then

$$\|(\rho^{-1}w)_{tt}(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\Delta(\rho^{-1}w)(t)\|_{L^2(\Omega)} \leq C\mathcal{A}(u_0, u_1, B).$$

4 (39) then reads $\|\rho v\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))} \leq Cs\mathcal{A}(u_0, u_1, B)$ which is a part of (16).

5 Eventually, from (2), we get that

$$\begin{cases} (\rho y)_{tt} - \Delta(\rho y) = \tilde{B}_y := \rho B + 2\rho_t y_t + \rho_{tt} y - 2\nabla\rho \cdot \nabla y - \Delta\rho y & \text{in } Q, \\ \rho y|_\Sigma = \rho v \\ ((\rho y)(\cdot, 0), (\rho y)_t(\cdot, 0)) = (\rho(0)u_0, \rho(0)u_1 + \rho_t(0)u_0) & \text{in } \Omega, \end{cases}$$

for which we have the standard estimate

$$\begin{aligned} \|(\rho y)_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\rho y)\|_{L^\infty(0,T;L^2(\Omega))} &\leq C\left(\|\tilde{B}_y\|_{L^2(Q)} + \|(\rho v)_t\|_{L^2(\Sigma)} + \|\rho v\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))}\right. \\ &\quad \left.+ \|\rho(0)u_0\|_{H^1(\Omega)} + \|\rho(0)u_1 + \rho_t(0)u_0\|_{L^2(\Omega)}\right). \end{aligned}$$

But

$$\begin{aligned} \|\tilde{B}_y\|_{L^2(Q)} &\leq C\left(\|\rho B\|_{L^2(Q)} + s^2\|\rho y\|_{L^2(Q)} + s\|\rho y_t\|_{L^2(Q)} + s\|\nabla(\rho y)\|_{L^2(Q)}\right) \\ &\leq Cs^{1/2}\left(\mathcal{A}(u_0, u_1, B) + s^{1/2}\|\nabla(\rho y)\|_{L^2(Q)}\right), \end{aligned}$$

$$\begin{aligned} \|\nabla(\rho y)\|_{L^2(Q)} &\leq C\left(\|\rho(0)u_0\|_{L^2(\Omega)} + \|\rho(0)u_1 + \rho_t(0)u_0\|_{L^2(\Omega)} + \|(\rho y)_t\|_{L^2(Q)} + \|(\rho v)_t\|_{L^2(\Sigma)}\right. \\ &\quad \left.+ \|\rho v\|_{L^2(0,T;H^{1/2}(\partial\Omega))} + \|\tilde{B}_y\|_{L^2(0,T;H^{-1}(\Omega))}\right) \end{aligned}$$

and

$$\begin{aligned} \|\tilde{B}_y\|_{L^2(0,T;H^{-1}(\Omega))} &\leq C\left(\|\rho B\|_{L^2(Q)} + s^2\|\rho y\|_{L^2(Q)} + s\|\rho y_t\|_{L^2(Q)}\right) \\ &\leq Cs^{1/2}\mathcal{A}(u_0, u_1, B) \end{aligned}$$

leading to $\|\nabla(\rho y)\|_{L^2(Q)} \leq Cs^{1/2}\mathcal{A}(u_0, u_1, B)$ that is the estimate of the third term in (16) and $\|\tilde{B}_y\|_{L^2(Q)} \leq Cs^{3/2}\mathcal{A}(u_0, u_1, B)$. We then deduce that

$$\|(\rho y)_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\rho y)\|_{L^\infty(0,T;L^2(\Omega))} \leq Cs^{3/2}\mathcal{A}(u_0, u_1, B)$$

6 that is, the estimate of the last two terms in (16).

7 **Step 4.** Case where $B \in L^2(Q)$ and $(u_0, u_1) \in \mathbf{V}$. We proceed by density as in [3, Step 3, page 171].

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