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# On the exact boundary controllability of semilinear wave equations

Sue Claret <sup>\*</sup>      Jérôme Lemoine <sup>†</sup>      Arnaud Münch <sup>‡</sup>

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## Abstract

We address the exact boundary controllability of the semilinear wave equation  $y_{tt} - \Delta y + f(y) = 0$  posed over a bounded domain  $\Omega$  of  $\mathbb{R}^d$ . Assuming that  $f$  is continuous and satisfies the condition  $\limsup_{|r| \rightarrow \infty} |f(r)|/(|r| \ln^p |r|) \leq \beta$  for some  $\beta$  small enough and some  $p \in [0, 3/2)$ , we apply the Schauder fixed point theorem to prove the uniform controllability for initial data in  $L^2(\Omega) \times H^{-1}(\Omega)$ . Then, assuming that  $f$  is in  $C^1(\mathbb{R})$  and satisfies the condition  $\limsup_{|r| \rightarrow \infty} |f'(r)|/\ln^p |r| \leq \beta$ , we apply the Banach fixed point theorem and exhibit a strongly convergent sequence to a state-control pair for the semilinear equation.

**AMS Classifications:** 35L71, 93B05.

**Keywords:** Semilinear wave equation, Exact boundary controllability, Carleman estimates, Fixed point.

## 1 Introduction and main results

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^d$  of class  $C^2$  and let  $T > 0$ . We set  $Q := \Omega \times (0, T)$  and  $\Sigma := \partial\Omega \times (0, T)$ . We consider the semilinear problem in  $y = y(x, t)$

$$Ly + f(y) = 0 \text{ in } Q, \quad y = v1_{\Gamma_0} \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega, \quad (1)$$

where  $L := \partial_t^2 - \Delta$  denotes the wave operator,  $(u_0, u_1) \in \mathbf{H} := L^2(\Omega) \times H^{-1}(\Omega)$  is a given initial state,  $v \in L^2(\Sigma)$  is a control function and  $f$  a continuous function over  $\mathbb{R}$ .  $\Gamma_0$  denotes a non empty subset of  $\partial\Omega$ .

The exact boundary controllability problem associated to (1) states as follows: given  $T > 0$ ,  $\Gamma_0 \subset \partial\Omega$  and  $(u_0, u_1), (z_0, z_1) \in \mathbf{H}$ , find a control function  $v \in L^2(\Sigma)$  and  $y \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$  solution of (1) and such that  $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$  in  $\Omega$ .

The linear problem (1) with  $f \equiv 0$  is exactly controllable provided that  $T > 0$  and  $\Gamma_0$  are sufficiently large (see [16, Theorem 6.1, p. 60] and [1, Theorem 4.9, p. 1058]). In the nonlinear case, a first exact boundary controllability result has been given in [21, Theorem 2.1] assuming  $f$  globally Lipschitz and initial data in  $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$  for  $\gamma \in (0, 1)$ ,  $\gamma \neq \frac{1}{2}$  leading to Dirichlet control in  $H_0^\gamma(0, T; L^2(\Gamma_0))$ . A Schauder fixed point argument is used coupled with the HUM method developed in [16]. Still assuming  $f' \in L^\infty(\mathbb{R})$ , [14, Theorem 1.1] covers the case  $\gamma = 0$  and generalizes the result to semilinear abstract systems by using a global inversion theorem. We also mention [7] where a boundary controllability result is proved in the one-dimensional case for a specific class of initial and final data and  $T$  large enough by a quasi-static deformation approach.

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1 Assuming  $(u_0, u_1) \in \mathbf{V} := H_0^1(\Omega) \times L^2(\Omega)$ , the boundary controllability may also be obtained indirectly  
2 with the domain extension from interior controllability results. In this respect, we mention [22, Theorem  
3 1] assuming  $\Omega = (0, 1)$ ,  $T > 2$  and that  $f \in C^1(\mathbb{R})$  does not grow faster at infinity than  $\beta|r| \ln^2(|r|)$   
4 for some  $\beta > 0$  small enough, for a global controllability result in  $\mathbf{V}$ . This result has been extended to  
5 any spatial dimension, first in [15, Theorem 3.1] and then in [11, Theorem 4.5, page 116] assuming that  
6  $f \in C^1(\mathbb{R})$  does not grow faster at infinity than  $\beta|r| \ln^p(|r|)$  for some  $\beta > 0$  small enough for  $p = 1/2$  and  
7 any  $0 < p < 3/2$  respectively. The above results are based on the Schauder theorem together with an  
8 estimate of the cost of control for linear wave equations with potential derived using Carleman estimates  
9 (we refer to [9, Theorem 2.2, page 8]). Eventually, we mention [8] dealing with subcritical nonlinearities  
10 satisfying the sign condition  $rf(r) \geq 0$  for all  $r \in \mathbb{R}$  (weakened later in [12] to an asymptotic sign condition  
11 leading to a semi-global controllability result, in the sense that the final data  $(z_0, z_1)$  must be prescribed  
12 in a precise subset of  $\mathbf{V}$ ).

13 In this work, we directly address the exact boundary controllability for (1) under the usual conditions  
14 on  $(u_0, u_1), T$  and  $\Gamma_0$  encountered in the linear case but with respect to [14], by replacing the condition  
15  $f' \in L^\infty(\mathbb{R})$  by the slightly super-linear condition used in [11]. Our result is as follows.

**Theorem 1.** *For any  $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$ , let  $\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}$  and  $\Gamma_0 \subset \partial\Omega$  such that  $\text{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0$  and let  $T > 2 \max_{x \in \bar{\Omega}} |x - x_0|$ .*

- Assume that  $f \in C^0(\mathbb{R})$  and that there exists  $0 \leq p < 3/2$  such that  $f$  satisfies

$$(\mathbf{H}_p) \quad \exists \alpha_1, \alpha_2, \beta^* > 0, \quad |f(r)| \leq \alpha_1 + |r| (\alpha_2 + \beta^* \ln_+^p(r)), \quad \forall r \in \mathbb{R}.$$

*If  $\beta^*$  is small enough, then for any initial state  $(u_0, u_1)$  and final state  $(z_0, z_1)$  in  $\mathbf{H}$ , system (1) is exactly controllable with controls in  $L^2(\Sigma)$ .*

- Assume that  $f \in C^1(\mathbb{R})$  and that there exists  $0 \leq p < 3/2$  such that  $f$  satisfies

$$(\mathbf{H}'_p) \quad \exists \alpha, \beta^* > 0, \quad |f'(r)| \leq \alpha + \beta^* \ln_+^p(r), \quad \forall r \in \mathbb{R}.$$

*If  $\beta^*$  is small enough, then for any initial state  $(u_0, u_1)$  and final state  $(z_0, z_1)$  in  $\mathbf{H}$ , one can construct a non trivial sequence  $(y_k, v_k)_{k \in \mathbb{N}^*}$  that converges strongly to a controlled pair  $(y, v)$  in  $(C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))) \times L^2(0, T; L^2(\partial\Omega))$  for system (1). Moreover, the convergence of  $(y_k, v_k)_{k \in \mathbb{N}^*}$  holds at least with a linear rate for the norm  $\|\rho \cdot\|_{L^2(Q)} + \|\rho \cdot\|_{L^2(\Sigma)}$  where  $\rho = \rho(x, t, s)$  is defined in (7) and  $s$  is chosen sufficiently large depending on  $\|(u_0, u_1)\|_{\mathbf{H}}$  and  $\|(z_0, z_1)\|_{\mathbf{H}}$ .*

- Assume that  $f \in C^0(\mathbb{R})$  and satisfies  $(\mathbf{H}_{3/2})$ , i.e.  $(\mathbf{H}_p)$  with  $p = 3/2$ . If  $\beta^*$  is small enough, then for any initial state  $(u_0, u_1)$  and final state  $(z_0, z_1)$  in  $\mathbf{V}$ , system (1) is exactly controllable with controls in  $H^1(0, T; L^2(\partial\Omega)) \cap C^0([0, T]; H^{1/2}(\partial\Omega))$ .

17 Theorem 1 extends and generalizes to any dimension [3] devoted to  $d = 1$ . It relies on fixed point  
18 approaches in a functional class  $\mathcal{C}(s)$  dependent of a Carleman parameter  $s$  large enough. For any  
19  $\hat{y} \in \mathcal{C}(s) \subset L^\infty(0, T; L^2(\Omega))$ , the fixed point operator  $\Lambda_s$  is such that  $y := \Lambda_s(\hat{y}) \in \mathcal{C}(s)$  is a controlled  
20 solution of

$$Ly = B \text{ in } Q, \quad y = v \mathbf{1}_{\Gamma_0} \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega, \quad (2)$$

21 with  $B = -f(\hat{y})$  satisfying  $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$  through the boundary function  $v$ ; the pair  $(y, v)$  is  
22 chosen as the minimizer of a quadratic functional (see (14)) involving Carleman weights and cut-off time  
23 and space functions. While  $\mathcal{C}(s)$  is a subset of  $L^\infty(Q)$  in [3], the class here is a subset of  $L^\infty(0, T; L^2(\Omega))$   
24 allowing to consider any spatial dimension with a simpler proof.

25 We emphasize that we get the exact controllability for (1) under the conditions encountered in the  
26 linear situation: the controllability time  $T > T(x_0) := 2 \max_{x \in \bar{\Omega}} |x - x_0|$  and support  $\Gamma_0$  satisfy the  
27 usual geometric conditions introduced in [16] while the initial data is assumed in  $\mathbf{H}$ . This is in contrast

1 with [11, Theorem 4.5] devoted to distributed controllability where the initial data is taken in  $\mathbf{V}$  with  
2 a controllability time greater than  $\max(2T(x_0), cd^{3/2})$  for some  $c > 0$  (we refer to [11, Remark 4.7 page  
3 118]). Moreover, we consider nonlinearities  $f$  in  $\mathcal{C}^0(\mathbb{R})$  while [11, Theorem 4.5] assumes  $f$  in  $\mathcal{C}^1(\mathbb{R})$ ; this  
4 is due to the fact that the linearization (2), where the nonlinear term is seen as a right hand side, does  
5 not involve any derivative of  $f$ . As a matter of fact, the first item in Theorem 1 can not be obtained  
6 from the extension domain method and controllability results for the distributed case.

7 Remark also that the third item includes the value  $p = 3/2$  in the exponent of the logarithm term (see  
8  $(\mathbf{H}_p)$ ) contrary to [11, Theorem 4.5]. Last, the second item, with a growth condition on the derivative  
9 of  $f$ , provides a constructive way to approximate control-state pair for (1), which is fundamental for  
10 applications. The regularity is used to estimate some  $L^2(L^q)$  norm of  $f(y_1) - f(y_2)$  for any elements  
11  $y_1, y_2$  in  $\mathcal{C}(s)$  (see Proposition 6). It relaxes the Hölder assumption on  $f'$  used in [17, Theorem 2.3]  
12 based on a Newton type linearization. To our knowledge, this is the first result leading to a convergent  
13 approximation of boundary controls for superlinear nonlinearities without smallness assumption notably  
14 on the initial condition and target (in contrast to the recent works [4, 18]).

15 As in [3], the crucial technical point in the analysis is a regularity property of the state-control  
16 trajectories  $(y, v)$  for (2). We show and use that if the initial condition belongs to  $\mathbf{V}$  and if the right hand  
17 side  $B$  belongs to  $L^2(Q)$ , then the controlled trajectory  $y$  solution of (2) so that  $(y, v)$  is the minimizer  
18 of the quadratic functional  $J_s$  (see Remark 2) belongs to  $\mathcal{C}^0([0, T], H^1(\Omega))$ .

19 **Outline** - Section 2 discusses the exact null controllability of (2) and provides precise estimates of the  
20 control-state pair  $(y, v)$  in term of the Carleman parameter  $s$  according to the regularity of the data  
21  $(u_0, u_1)$  and of the right hand side  $B$ : Proposition 2 for  $(u_0, u_1, B) \in \mathbf{H} \times L^2(0, T; H^{-r}(\Omega))$ ,  $r \in [0, 1]$ ,  
22  $r \neq \frac{1}{2}$ , and Proposition 3 for  $(u_0, u_1, B) \in \mathbf{V} \times L^2(Q)$ . Section 3 is devoted to the proof of Theorem 1:  
23 in Section 3.1, we employ the Schauder fixed point theorem to the operator  $\Lambda_s$  (see (20)) defined on the  
24 class  $\mathcal{C}(s) \subset L^\infty(0, T; L^2(\Omega))$  and prove the first item. In Section 3.2, we prove that, if the nonlinearity  
25  $f$  satisfies  $(\mathbf{H}'_p)$ , then the operator  $\Lambda(s)$  is contracting leading to the second item. Eventually, in Section  
26 3.3 assuming the initial data in  $\mathbf{V}$ , we define the operator  $\Lambda(s)$  on a class  $\tilde{\mathcal{C}}(s)$  subset of  $H^1(Q)$  and reach  
27 the limit case  $p = 3/2$  in the logarithmic exponent, as announced in the third item.

28 In the sequel,  $C$  denotes a generic constant which may changes from line to line, but depends only on  
29  $\Omega$  and  $T$ .

## 30 2 Controllability results for the linear wave equation

31 Existence of  $L^2(\Sigma)$  controls for (2) with initial data in  $\mathbf{H}$  and right hand side in  $L^2(0, T; H^{-1}(\Omega))$  is  
32 well-known (we refer to [16, chapter 2]); corresponding controlled solution belongs to  $\mathcal{C}^0([0, T]; L^2(\Omega)) \cap$   
33  $\mathcal{C}^1([0, T]; H^{-1}(\Omega))$ . For any initial data  $(u_0, u_1) \in \mathbf{V}$ , right hand side  $B$  in  $L^2(Q)$  and  $T > 0$  large enough,  
34 we analyze the existence of a control function  $v \in H^1(0, T; L^2(\partial\Omega)) \cap \mathcal{C}^0([0, T]; H^{1/2}(\partial\Omega))$  such that the  
35 solution  $y$  of (2) satisfies  $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$  and is bounded in  $L^\infty(0, T; L^2(\Omega))$ . Moreover, we  
36 aim to get precise weighted estimates of a particular state-control pair in term of the data, which will be  
37 crucial to handle the nonlinear system (1). As in [3], we employ a global Carleman estimates from [2] as  
38 a fundamental tool.

39 We introduce the usual geometric condition (see [2, Condition (1.2)]): for any  $x_0 \notin \bar{\Omega}$ , we introduce  
40  $\Gamma_1, \Gamma_0 \subset \partial\Omega$  such that

$$\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}, \quad \text{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0. \quad (3)$$

41 Let  $\Psi \in \mathcal{C}^2(\partial\Omega)$  be a cut-off function such that

$$0 \leq \Psi \leq 1 \quad \text{on } \partial\Omega, \quad \Psi = 1 \quad \text{on } \Gamma_1, \quad \Psi = 0 \quad \text{on } \partial\Omega \setminus \Gamma_0. \quad (4)$$

1 We assume that

$$T > 2 \max_{\overline{\Omega}} |x - x_0| \quad (5)$$

and define, for any  $\delta > 0$  such that  $T - 2\delta > 2 \max_{\overline{\Omega}} |x - x_0|$ , a cut-off function  $\eta \in C_c^1(\mathbb{R})$  satisfying

$$0 \leq \eta(t) \leq 1 \text{ in } (\delta, T - \delta), \quad \eta(t) = 0 \text{ in } (-\infty, \delta] \cup [T - \delta, +\infty). \quad (6)$$

2 Then, for any  $\beta \in (0, 1)$  and  $\lambda > 0$ , we define the functions  $\psi(x, t) = |x - x_0|^2 - \beta(t - \frac{T}{2})^2 + M_0$ ,  
 3  $\phi(x, t) = e^{\lambda\psi(x, t)}$  in  $Q$ , with  $M_0 > 0$  large enough so that  $\psi > 0$  in  $\overline{Q}$ . Then, for all  $s \geq s_0$ , we define the  
 4 weight function

$$\rho(x, t) := e^{-s\phi(x, t)} \quad \forall (x, t) \in Q. \quad (7)$$

5 Remark that  $e^{-cs} \leq \rho \leq e^{-s}$  in  $Q$  with  $c := \|\phi\|_{L^\infty(Q)}$  and  $\rho, \rho^{-1} \in C^\infty(\overline{Q})$ .

6 Let then  $P := \{w \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)), Lw \in L^2(Q)\}$  and recall that  $\partial_\nu w \in L^2(\Sigma)$  for  
 7 every  $w \in P$  (see [16, Theorem 4.1]). The global Carleman estimate mentioned earlier reads as follows.

**Proposition 1.** *For any  $x_0 \notin \overline{\Omega}$ , we assume (3) and (5). There exists  $s_0 > 0$ ,  $\lambda > 0$  and  $C > 0$ , such that for any  $s \geq s_0$  and every  $w \in P$*

$$\begin{aligned} & s \int_Q \rho^{-2} (|w_t|^2 + |\nabla w|^2) dx dt + s^3 \int_Q \rho^{-2} |w|^2 dx dt \\ & + s \int_\Omega \rho^{-2}(0) (|w_t(\cdot, 0)|^2 + |\nabla w(\cdot, 0)|^2) dx + s^3 \int_\Omega \rho^{-2}(0) |w(\cdot, 0)|^2 dx \\ & \leq C \left( \int_Q \rho^{-2} |Lw|^2 dx dt + s \int_\Sigma \eta^2(t) \Psi(x) \rho^{-2} |\partial_\nu w|^2 dx dt \right). \quad (8) \end{aligned}$$

9 *Proof.* We refer to [2, Theorem 2.5 and Remark 2.9].  $\square$

## 10 2.1 Estimates for the state-control pair in $C^0([0, T]; L^2(\Omega)) \times L^2(\Sigma)$

In all the sequel, we suppose that  $s_0 \geq 1$ . Proposition 1 allows to deduce the controllability for (2) with estimates of the state-control pair in  $L^2(Q) \times L^2(\Sigma)$ . For any  $s \geq s_0$ , we define the bilinear form

$$(w, z)_{P, s} := \int_Q \rho^{-2} Lw Lz dx dt + s \int_\Sigma \eta^2(t) \Psi(x) \rho^{-2} \partial_\nu w \partial_\nu z dx dt, \quad \forall w, z \in P. \quad (9)$$

11 According to (8), (9) defines a scalar product in  $P$  and if  $P_s$  denotes  $P$  endowed with this scalar product,  
 12 then  $P_s$  is an Hilbert space. We now state the controllability result for the system (2) (without loss of  
 13 generality in the null controllability case, for which  $(z_0, z_1) = (0, 0)$  in  $\Omega$ ).

**Proposition 2.** *For any  $x_0 \notin \overline{\Omega}$ , we assume (3) and (5). Let  $\eta \in C_c^1(\mathbb{R})$  and  $\Psi \in C^2(\partial\Omega)$  be cut-off functions satisfying (4) and (6) respectively. For  $s \geq s_0$ ,  $B \in L^2(0, T, H^{-r}(\Omega))$ ,  $r \in [0, 1]$ ,  $r \neq 1/2$  and  $(u_0, u_1) \in \mathbf{H}$ , there exists a unique  $w \in P_s$  such that, for all  $z \in P_s$ ,*

$$(w, z)_{P, s} = \langle u_1, z(\cdot, 0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega u_0 z_t(\cdot, 0) dx + \langle B, z \rangle_{L^2(0, T; H^{-1}(\Omega)), L^2(0, T; H_0^1(\Omega))}. \quad (10)$$

14 Then  $v := s\eta^2\Psi\rho^{-2}\partial_\nu w$  is a control function for (2) and  $y := \rho^{-2}Lw$  is the associated controlled trajectory, that is  $y(\cdot, T) = y_t(\cdot, T) = 0$  in  $\Omega$ .

Moreover, there exists a constant  $C_r > 0$  independent of  $s$  such that

$$\begin{aligned} & \|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}} v \right\|_{L^2(\Sigma)} + s^{-2} \|\rho y\|_{L^\infty(0, T; L^2(\Omega))} + s^{-2} \|(\rho y)_t\|_{L^\infty(0, T; H^{-1}(\Omega))} \\ & \leq C_r \left( s^{r-3/2} \|\rho B\|_{L^2(0, T; H^{-r}(\Omega))} + s^{-1/2} \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1/2} \|\rho(0)u_1\|_{H^{-1}(\Omega)} \right). \quad (11) \end{aligned}$$

**Remark 1.** With no vanishing target  $(z_0, z_1) \in \mathbf{H}$ , the right hand side of (11) contains the term  $s^{-1/2}\|\rho(0)z_0\|_{L^2(\Omega)} + s^{-1/2}\|\rho(0)z_1\|_{H^{-1}(\Omega)}$ . We choose in the sequel  $(z_0, z_1) = (0, 0)$  which will make the computations shorter.

*Proof.* Some parts of the proof are only sketched as they are very similar to [3, Theorem 6] devoted to the case  $r = 0$  (and  $d = 1$ ). For any  $0 \leq r \leq 1$ ,  $r \neq 1/2$  and  $z \in P_s$ , we have from (8) :

$$\begin{aligned} & \left| \langle B, z \rangle_{L^2(0,T;H^{-1}(\Omega)), L^2(0,T;H_0^1(\Omega))} \right| = \left| \langle B, z \rangle_{L^2(0,T;H^{-r}(\Omega)), L^2(0,T;H_0^r(\Omega))} \right| \\ & \leq \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \|\rho^{-1}z\|_{L^2(0,T;H^r(\Omega))} \\ & \leq C \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \|\rho^{-1}z\|_{L^2(Q)}^{1-r} \|\nabla(\rho^{-1}z)\|_{L^2(Q)}^r \\ & \leq C \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \left( s^r \|\rho^{-1}z\|_{L^2(Q)} + \|\rho^{-1}z\|_{L^2(Q)}^{1-r} \|\rho^{-1}\nabla z\|_{L^2(Q)}^r \right) \\ & \leq C s^{r-3/2} \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \\ & \quad \left( s^{3/2} \|\rho^{-1}z\|_{L^2(Q)} + (s^{3/2} \|\rho^{-1}z\|_{L^2(Q)})^{1-r} (s^{1/2} \|\rho^{-1}\nabla z\|_{L^2(Q)})^r \right) \\ & \leq C s^{r-3/2} \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} \|z\|_{P_s} \end{aligned}$$

and conclude that the right hand side of (10) is a linear continuous form on  $P_s$ . The Riesz representation theorem implies the existence of a unique  $w \in P_s$  satisfying the formulation (10) and

$$\|w\|_{P_s} \leq C \left( s^{r-3/2} \|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{-1/2} \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1/2} \|\rho(0)u_1\|_{L^2(\Omega)} \right). \quad (12)$$

Then, set  $y = \rho^{-2}Lw$  and  $v = s\eta^2\Psi\rho^{-2}\partial_\nu w$ . From (10), the pair  $(y, v)$  satisfies

$$\begin{aligned} & \int_Q yLz \, dxdt + \int_\Sigma v\partial_\nu z \, d\Sigma = \langle u_1, z(\cdot, 0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega u_0 z_t(\cdot, 0) \, dx \\ & \quad + \langle B, z \rangle_{L^2(0,T;H^{-1}(\Omega)), L^2(0,T;H_0^1(\Omega))} \quad \forall z \in P_s, \end{aligned} \quad (13)$$

meaning that  $y \in L^2(Q)$  is a solution to (2) associated with the function  $v \in L^2(\Sigma)$  in the sense of transposition. Eventually, using (12), we get that  $\rho y = \rho^{-1}Lw \in L^2(Q)$  and  $s^{-1/2}\rho v = s^{1/2}\eta^2\Psi\rho^{-1}\partial_\nu w \in L^2(\Sigma)$  and deduce (11) for the first and second left hand side terms.

To get (11) for the third and fourth left hand side terms, we remark that  $\rho y \in L^\infty(0, T; L^2(\Omega))$ : indeed,  $y \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$  (see [16, Theorem 4.2 p.46]).

Moreover, since  $\rho^{-1} \in C^\infty(\overline{Q})$ , for any  $z \in P_s$ , we get

$$\begin{cases} L(\rho^{-1}z) = \rho^{-1}Lz + \tilde{B}_z, & \tilde{B}_z := 2\rho_t^{-1}z_t + \rho_{tt}^{-1}z - 2\nabla\rho^{-1} \cdot \nabla z - \Delta\rho^{-1}z, \\ (\rho^{-1}z)|_{\partial\Omega} = 0 \end{cases}$$

so that (13) rewrites

$$\begin{aligned} & \int_Q \rho y L(\rho^{-1}z) \, dxdt + \int_\Sigma \rho v \partial_\nu (\rho^{-1}z) \, d\Sigma \\ & = \langle \rho(0)u_1, (\rho^{-1}z)(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} - \int_\Omega \rho(0)u_0 (\rho^{-1}z)_t(\cdot, 0) \, dx + \int_\Omega \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) \, dx \\ & \quad + \langle \rho B, \rho^{-1}z \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} + \int_Q \rho y \tilde{B}_z \, dxdt. \end{aligned}$$

Remark that  $z \in P_s$  if and only if  $\tilde{z} := \rho^{-1}z \in P_s$ ; therefore, for all  $\tilde{z} \in P_s$  and  $z = \rho\tilde{z}$ ,

$$\begin{aligned} & \int_Q \rho y L\tilde{z} \, dxdt + \int_\Sigma \rho v \partial_\nu \tilde{z} \, d\Sigma \\ & = \langle \rho(0)u_1, \tilde{z}(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} - \int_\Omega \rho(0)u_0 \tilde{z}_t(\cdot, 0) \, dx + \langle \rho B, \tilde{z} \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} \\ & \quad + \int_\Omega \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) \, dx + \int_Q \rho y \tilde{B}_z \, dxdt. \end{aligned}$$

Moreover, using that  $|\nabla\rho^{-1}| \leq Cs\rho^{-1}$ ,  $|(\rho^{-1})_t| \leq Cs\rho^{-1}$ ,  $|(\rho^{-1})_{tt}| \leq Cs^2\rho^{-1}$  and  $|\nabla^2\rho^{-1}| \leq Cs^2\rho^{-1}$ , we get the estimates

$$\begin{aligned} \left| \int_{\Omega} \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) dx \right| &\leq Cs\|\rho(0)u_0\|_{L^2(\Omega)}\|\tilde{z}(\cdot, 0)\|_{L^2(\Omega)}, \\ \|\tilde{B}_z\|_{L^2(Q)} &\leq C(s^2\|\tilde{z}\|_{L^2(Q)} + s\|\tilde{z}_t\|_{L^2(Q)} + s\|\nabla\tilde{z}\|_{L^2(Q)}) \end{aligned}$$

and thus, since  $s \geq 1$  :

$$\left| \int_Q \rho y \tilde{B}_z dx dt \right| \leq Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{z}\|_{L^2(Q)} + \|\tilde{z}_t\|_{L^2(Q)} + \|\nabla\tilde{z}\|_{L^2(Q)}).$$

Then, for all  $\tilde{z} \in P_s$

$$\begin{aligned} \left| \int_Q \rho y L\tilde{z} dx dt \right| &\leq \left| \int_{\Sigma} \rho v \partial_{\nu} \tilde{z} d\Sigma \right| + \left| \langle \rho(0)u_1, \tilde{z}(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \right| + \left| \int_{\Omega} \rho(0)u_0 \tilde{z}_t(\cdot, 0) dx \right| \\ &\quad + \left| \int_{\Omega} \rho(0)u_0 \rho_t^{-1}(0)z(\cdot, 0) dx \right| + \left| \langle \rho B, \tilde{z} \rangle_{L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H_0^1(\Omega))} \right| + \left| \int_Q \rho y \tilde{B}_z dx dt \right| \\ &\leq \|\rho v\|_{L^2(\Sigma)}\|\partial_{\nu}\tilde{z}\|_{L^2(\Sigma)} + \|\rho(0)u_1\|_{H^{-1}(\Omega)}\|\tilde{z}(\cdot, 0)\|_{H_0^1(\Omega)} \\ &\quad + \|\rho(0)u_0\|_{L^2(\Omega)}\|\tilde{z}_t(\cdot, 0)\|_{L^2(\Omega)} + Cs\|\rho(0)u_0\|_{L^2(\Omega)}\|\tilde{z}(\cdot, 0)\|_{L^2(\Omega)} \\ &\quad + \|\rho B\|_{L^1(0,T;H^{-1}(\Omega))}\|\tilde{z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} + Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{z}\|_{L^2(Q)} + \|\tilde{z}_t\|_{L^2(Q)} + \|\nabla\tilde{z}\|_{L^2(Q)}). \end{aligned}$$

For any  $g \in \mathcal{D}(Q)$ , let  $\tilde{z} \in P_s$  be solution of  $L\tilde{z} = g$  and  $(\tilde{z}(0), \tilde{z}_t(0)) = (0, 0)$  so that

$$\|\tilde{z}_t\|_{L^{\infty}(0,T;L^2(\Omega))} + \|\tilde{z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} + \|\partial_{\nu}\tilde{z}\|_{L^2(\Sigma)} \leq C\|g\|_{L^1(0,T;L^2(\Omega))}$$

and thus, using (11), for all  $0 \leq r \leq 1$ ,  $r \neq 1/2$ , since  $s \geq 1$  :

$$\begin{aligned} \left| \int_Q \rho y g dx dt \right| &\leq \|\rho v\|_{L^2(\Sigma)}\|\partial_{\nu}\tilde{z}\|_{L^2(\Sigma)} \\ &\quad + \|\rho B\|_{L^1(0,T;H^{-1}(\Omega))}\|\tilde{z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} + Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{z}\|_{L^2(Q)} + \|\tilde{z}_t\|_{L^2(Q)} + \|\nabla\tilde{z}\|_{L^2(Q)}) \\ &\leq C\left(s^{r+1/2}\|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right)\|g\|_{L^1(0,T;L^2(\Omega))}. \end{aligned}$$

Therefore,  $\rho y \in (L^1(0, T; L^2(\Omega)))' = L^{\infty}(0, T; L^2(\Omega))$  and

$$\|\rho y\|_{L^{\infty}(0,T;L^2(\Omega))} \leq C\left(s^{r+1/2}\|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right).$$

Similarly, for any  $g \in \mathcal{D}(Q)$ , let  $\tilde{Z} \in P_s$  satisfying  $L\tilde{Z} = g_t$  and  $(\tilde{Z}(0), \tilde{Z}_t(0)) = (0, 0)$ . Then using [16, (4.19) p.51] we obtain :

$$\begin{aligned} \left| \int_Q \rho y g_t dx dt \right| &\leq \|\rho v\|_{L^2(\Sigma)}\|\partial_{\nu}\tilde{Z}\|_{L^2(\Sigma)} + \|\rho B\|_{L^1(0,T;H^{-1}(\Omega))}\|\tilde{Z}\|_{L^{\infty}(0,T;H_0^1(\Omega))} \\ &\quad + Cs^2\|\rho y\|_{L^2(Q)}(\|\tilde{Z}\|_{L^2(Q)} + \|\tilde{Z}_t\|_{L^2(Q)} + \|\nabla\tilde{Z}\|_{L^2(Q)}) \\ &\leq C\left(s^{1/2+r}\|\rho(s)B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right)\|g\|_{L^1(0,T;H_0^1(\Omega))} \end{aligned}$$

and thus  $(\rho y)_t \in L^{\infty}(0, T; H^{-1}(\Omega))$  and

$$\|(\rho y)_t\|_{L^{\infty}(0,T;H^{-1}(\Omega))} \leq C\left(s^{r+1/2}\|\rho B\|_{L^2(0,T;H^{-r}(\Omega))} + s^{3/2}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{3/2}\|\rho(0)u_1\|_{H^{-1}(\Omega)}\right).$$

**Remark 2.** The state-control pair  $(y, v)$  introduced in Proposition 2 is the unique minimizer of the functional

$$J_s(z, u) := s \int_Q \rho^2 |z|^2 \, dxdt + \int_\delta^{T-\delta} \int_{\partial\Omega} \eta^{-2} \Psi^{-1} \rho^2 |u|^2 \, dxdt \quad (14)$$

over  $\{(z, u) : z \in L^2(Q), \eta^{-1} \Psi^{-1/2} \rho u \in L^2(\delta, T - \delta; L^2(\partial\Omega))\}$  solution of (2) with  $z(\cdot, T) = z_t(\cdot, T) = 0$  in  $\Omega$ . We refer to [5, Section 2].

**Remark 3.** The controlled state  $y = \rho^{-2} Lw$  satisfies

$$Ly = B \text{ in } Q, \quad y = s\eta^2 \Psi \rho^{-2} \partial_\nu w \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega,$$

so that  $y \in \mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))$ . On the other hand, the function  $w$  satisfies

$$Lw = \rho^2 y \text{ in } Q, \quad w = 0 \text{ on } \Sigma$$

implying according to (8) that  $(w(\cdot, 0), \partial_t w(\cdot, 0)) \in \mathbf{V}$  and  $w \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$ .

## 2.2 Estimates for the state-control pair with $(u_0, u_1, B) \in \mathbf{V} \times L^2(Q)$

The state-control pair  $(y, v)$  given by Proposition 2 enjoys additional regularity properties, under additional regularity assumption on the data and introduction of appropriate cut-off function in space and time. Such gain of regularity is known for the wave equation since [8, 10] and more recently [3]. The following first regularity result extends [3, Theorem 7] to the multi-dimensional case and is proven in Appendix A. It will be crucial for the analysis of the semilinear case discussed in Section 3.

**Proposition 3.** For any  $x_0 \notin \bar{\Omega}$ , we assume (3) and (5). Let  $\eta \in C_c^1(\mathbb{R})$  and  $\Psi \in \mathcal{C}^2(\partial\Omega)$  be cut-off functions satisfying (4) and (6) respectively. Let any  $(u_0, u_1) \in \mathbf{V}$  and  $B \in L^2(Q)$  be given. For any  $s \geq s_0$ , the solution  $(y, v)$  of (2) defined in Proposition 2 satisfies

$$v \in \mathcal{C}^0([0, T]; H^{1/2}(\partial\Omega)), \quad y \in \mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)) \quad (15)$$

and the following estimate :

$$\begin{aligned} & \|(\rho y)_t\|_{L^2(Q)} + s^{-1/2} \left\| \frac{1}{\eta \Psi^{1/2}} (\rho v)_t \right\|_{L^2(\Sigma)} + s^{-1} \|\nabla(\rho y)\|_{L^2(Q)} + s^{-3/2} \|\rho v\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} \\ & + s^{-2} (\|\rho y\|_{\mathcal{C}^0([0, T]; H^1(\Omega))} + \|(\rho y)_t\|_{\mathcal{C}^0([0, T]; L^2(\Omega))}) \\ & \leq C s^{-1/2} (\|\rho B\|_{L^2(Q)} + \|\rho(0)u_1\|_{L^2(\Omega)} + s\|\rho(0)u_0\|_{L^2(\Omega)} + \|\rho(0)\nabla u_0\|_{L^2(\Omega)}). \end{aligned} \quad (16)$$

## 2.3 Estimates for the state-control pair with $(u_0, u_1, B) \in H_0^{1-r}(\Omega) \times H^{-r}(\Omega) \times L^2(0, T; H^{-r}(\Omega))$

**Proposition 4.** For any  $x_0 \notin \bar{\Omega}$ , we assume (3) and (5). Let  $\eta \in C_c^1(\mathbb{R})$  and  $\Psi \in \mathcal{C}^2(\partial\Omega)$  be cut-off functions satisfying (4) and (6) respectively. For any  $s \geq s_0$  and  $r \in (0, 1)$ ,  $r \neq 1/2$ , there exists a constant  $C > 0$  (depending on  $s$  and  $r$ ) such that

$$\begin{aligned} & \|\rho y\|_{\mathcal{C}^1([0, T]; H^{-r}(\Omega))} + \|\rho y\|_{\mathcal{C}([0, T]; H^{1-r}(\Omega))} + \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{H^{1-r}(0, T; L^2(\partial\Omega))} \\ & \leq C (\|\rho B\|_{L^2(0, T; H^{-r}(\Omega))} + \|\rho(0)u_0\|_{H^{1-r}(\Omega)} + \|\rho(0)u_1\|_{H^{-r}(\Omega)}). \end{aligned} \quad (17)$$

*Proof.* Let  $\Lambda_s^0 : (B, u_0, u_1) \mapsto (y, v)$  be the linear operator with  $(y, v)$  the control-state pair given by Proposition 2. Then, from Proposition 2 and Proposition 3,

$$\Lambda_s^0 : L^2(0, T; H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega) \rightarrow (\mathcal{C}^0([0, T]; L^2(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-1}(\Omega))) \times L^2(0, T; L^2(\partial\Omega))$$

and

$$\Lambda_s^0 : L^2(0, T; L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \rightarrow (\mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))) \times H^1(0, T; L^2(\partial\Omega))$$

are linear continuous. By interpolation, for all  $0 < \theta < 1$ ,  $\Lambda_s^0$  is linear continuous from  $L^2(0, T; (H^{-1}(\Omega), L^2(\Omega))_\theta) \times (L^2(\Omega), H_0^1(\Omega))_\theta \times (H^{-1}(\Omega), L^2(\Omega))_\theta$  to  $(\mathcal{C}^0([0, T]; (L^2(\Omega), H^1(\Omega))_\theta) \cap \mathcal{C}^1([0, T]; (H^{-1}(\Omega), L^2(\Omega))_\theta)) \times (L^2(0, T; L^2(\partial\Omega)), H^1(0, T; L^2(\partial\Omega)))_\theta$ .

Thus, for  $\theta = 1 - r$ ,  $0 < r < 1$ ,  $r \neq 1/2$ ,  $\Lambda_s^0$  is linear, continuous from  $L^2(0, T; H^{-r}(\Omega)) \times H_0^{1-r}(\Omega) \times H^{-r}(\Omega)$  to  $(\mathcal{C}^0([0, T]; H^{1-r}(\Omega)) \cap \mathcal{C}^1([0, T]; H^{-r}(\Omega))) \times H^{1-r}(0, T; L^2(\partial\Omega))$ .  $\square$

### 3 Proof of Theorem 1

In this section, we prove the controllability of the semilinear equation (1). In this respect, for all  $s \geq s_0$  and for all  $\hat{y}$  in an appropriate subset  $\mathcal{C}(s)$  of  $L^\infty(0, T; L^2(\Omega))$ , we consider the linearized boundary control problem

$$Ly = -f(\hat{y}) \text{ in } Q, \quad y = v 1_{\Gamma_0} \text{ on } \Sigma, \quad (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega \quad (18)$$

such that  $(y(\cdot, T), y_t(\cdot, T)) = (0, 0)$  in  $\Omega$ . The existence of a controlled trajectory  $y \in L^2(Q)$  is guaranteed by Proposition 2 with a source term  $B = -f(\hat{y}) \in L^2(0, T; H^{-1}(\Omega))$ .

#### 3.1 First part of Theorem 1

Without restriction, we assume that  $f \in C^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$  for some  $1 < p < 3/2$ . For any  $s \geq s_0 \geq 1$ , we introduce the class  $\mathcal{C}(s)$  defined as the closed convex subset of  $L^\infty(0, T; L^2(\Omega))$

$$\mathcal{C}(s) := \left\{ y \in L^\infty(0, T; L^2(\Omega)) : \|\rho y\|_{L^2(Q)} \leq s, \|\rho y\|_{L^\infty(0, T; L^2(\Omega))} \leq s^3 \right\}. \quad (19)$$

We prove the existence of  $s$  and of a fixed point of the operator

$$\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s) \quad \hat{y} \mapsto y \quad (20)$$

where  $y$  is a solution of the null controllability problem (18) associated with the control  $v$  given by Proposition 2 for  $B = -f(\hat{y})$ . We employ the Schauder theorem and we prove that : **i**) for  $\beta^* > 0$  small enough, there exists  $s \geq s_0$  large enough such that  $\mathcal{C}(s)$  is stable under the map  $\Lambda_s$  (Section 3.1.2); **ii**)  $\Lambda_s(\mathcal{C}(s))$  is a relatively compact subset of  $\mathcal{C}(s)$  for the norm  $\|\cdot\|_{L^\infty(0, T; L^2(\Omega))}$  (Section 3.1.3); **iii**)  $\Lambda_s$  is a continuous map in  $\mathcal{C}(s)$  with respect to the  $L^\infty(0, T; L^2(\Omega))$  norm (Section 3.1.4). This ensures the existence of a fixed point for  $\Lambda_s$ , which is a controlled trajectory for (1).

##### 3.1.1 Estimates of $\Lambda_s(\hat{y})$

**Lemma 1.** *Assume that  $f \in C^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$ . For any  $s \geq s_0$  and  $\hat{y} \in \mathcal{C}(s)$ , there exists  $C > 0$  such that*

$$\|\rho f(\hat{y})\|_{L^2(0, T; H^{p-3/2}(\Omega))} \leq C(\alpha_1 e^{-s} T^{1/2} |\Omega|^{1/2} + \alpha_2 s + \beta^* c^p s^{1+p})$$

with  $c = \|\phi\|_{L^\infty(Q)}$ .

*Proof.* We infer, for  $1/2 > r = 3/2 - p > 0$  and  $1 \leq p^* = \frac{2d}{d+3-2p} < 2$  that (using that  $\rho \leq e^{-s}$ )

$$\begin{aligned} \|\rho f(\hat{y})\|_{L^2(0, T; H^{-r}(\Omega))} &\leq C\|\rho\alpha_1 + |\rho\hat{y}|(\alpha_2 + \beta^* \ln_+^p(\hat{y}))\|_{L^2(0, T; L^{p^*}(\Omega))} \\ &\leq C(\alpha_1 e^{-s} \sqrt{T} |\Omega|^{1/2} + \alpha_2 \|\rho\hat{y}\|_{L^2(Q)} + \beta^* \|\rho\hat{y} \ln_+^p(\hat{y})\|_{L^2(0, T; L^{p^*}(\Omega))}). \end{aligned}$$

Now, using that  $\rho^{-1} \leq e^{cs}$  and  $0 \leq \ln_+^p(\hat{y}) \leq C(\ln_+^p \rho^{-1} + \ln_+^p(\rho\hat{y})) \leq C((cs)^p + \ln_+^p(\rho\hat{y}))$ , we get that

$$\|\rho\hat{y} \ln_+^p(\hat{y})\|_{L^2(0, T; L^{p^*}(\Omega))} \leq C((cs)^p \|\rho\hat{y}\|_{L^2(Q)} + \|\rho\hat{y} \ln_+^p(\rho\hat{y})\|_{L^2(0, T; L^{p^*}(\Omega))}).$$

1 But, for all  $0 < \varepsilon < \frac{2-p^*}{p^*} = \frac{2r}{d} = \frac{3-2p}{d}$ , there exists  $C_\varepsilon > 0$  such that

$$\|\rho \widehat{y} \ln_+^p(\rho \widehat{y})\|_{L^2(0,T;L^{p^*}(\Omega))} \leq C_\varepsilon \|\rho \widehat{y}\|^{1+\varepsilon}_{L^2(0,T;L^{p^*}(\Omega))} \leq C_\varepsilon \|\rho \widehat{y}\|_{L^2(Q)} \|\rho \widehat{y}\|_{L^\infty(0,T;L^2(\Omega))}^\varepsilon. \quad (21)$$

2 Combining the above inequalities with  $\varepsilon < \min(\frac{p}{3}, \frac{3-2p}{d})$  and using that  $\widehat{y} \in \mathcal{C}(s)$  we get the result.  $\square$

**Proposition 5.** For any  $x_0 \notin \overline{\Omega}$ , we assume (3) and (5). Assume  $(u_0, u_1) \in \mathbf{H}$  and that  $f \in \mathcal{C}^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$ . For  $s \geq s_0$  and any  $\widehat{y} \in \mathcal{C}(s)$ , the solution  $y = \Lambda_s(\widehat{y})$  to the linearized controlled system (18) with control  $v$  satisfies the following estimates:

$$\begin{aligned} & \|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\Psi^{1/2} \eta} v \right\|_{L^2(\Sigma)} + s^{-2} \|\rho y\|_{L^\infty(0,T;L^2(\Omega))} + s^{-2} \|(\rho y)_t\|_{L^\infty(0,T;H^{-1}(\Omega))} \\ & \leq C s \left( s^{-p} \alpha_2 + \beta^* c^p + e^{-s} (s^{-p-1} \alpha_1 T^{1/2} |\Omega|^{1/2} + s^{-3/2} (\|u_0\|_{L^2(\Omega)} + s \|u_1\|_{H^{-1}(\Omega)})) \right). \end{aligned} \quad (22)$$

4 *Proof.* The map  $L : \mathcal{C}^1(\overline{\Omega}) \times H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$ ,  $(\rho(0), u_1) \rightarrow \rho(0)u_1$  is bilinear continuous (see [20, Lemma 3, p. 1097]) so that  $\|\rho(0)u_1\|_{H^{-1}(\Omega)} \leq C \|\rho(0)\|_{\mathcal{C}^1(\overline{\Omega})} \|u_1\|_{H^{-1}(\Omega)} \leq C s e^{-s} \|u_1\|_{H^{-1}(\Omega)}$ . The estimates follow from Proposition 2 with  $r = 3/2 - p$  and Proposition 3 with  $B = -f(\widehat{y})$  and Lemma 1.  $\square$

### 3.1.2 Stability of the class $\mathcal{C}(s)$

8 **Lemma 2.** Assume that  $f \in \mathcal{C}^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$  with  $\beta^*$  small enough. Then, there exists  $s \geq s_0$  large enough such that  $\Lambda_s(\mathcal{C}(s)) \subset \mathcal{C}(s)$ .

9 *Proof.* For any  $s \geq s_0$  and  $\widehat{y} \in \mathcal{C}(s)$ , let  $y := \Lambda_s(\widehat{y})$ . From (22), we obtain that  $\limsup_{s \rightarrow +\infty} s^{-1} \|\rho y\|_{L^2(Q)} \leq \beta^* C c^p$  and  $\limsup_{s \rightarrow +\infty} s^{-3} \|\rho y\|_{L^\infty(0,T;L^2(\Omega))} \leq \beta^* C c^p$ . Therefore, if  $\beta^* > 0$  is small enough so that  $\beta^* C c^p < 1$ , then for any  $s \geq s_0$  large enough,  $y \in \mathcal{C}(s)$ .  $\square$

12 **Remark 4.** The lower bound for  $s \geq s_0$  is related to the norm of the initial data  $(u_0, u_1) \in \mathbf{H}$ : in view of Proposition 5, the stability of  $\Lambda_s$  requires that  $(C \|u_0\|_{L^2(\Omega)} + C s^{-1} (\alpha_1 T^{1/2} |\Omega|^{1/2} + \|u_1\|_{H^{-1}(\Omega)})) e^{-s} < s^{3/2}$ . Therefore, the lower bound for  $s$  must be chosen as depending logarithmically on  $\|u_0, u_1\|_{\mathbf{H}}$ .

### 3.1.3 Relative compactness of the set $\Lambda_s(\mathcal{C}(s))$

14 **Lemma 3.** Under the assumptions of Lemma 2,  $\Lambda_s(\mathcal{C}(s))$  is a relatively compact subset of  $\mathcal{C}(s)$  for the  $L^\infty(0, T; L^2(\Omega))$  norm.

*Proof.* Let  $(y_n)_{n \in \mathbb{N}}$  be a sequence of  $\Lambda_s(\mathcal{C}(s))$ . Then, there exists  $(\widehat{y}_n)_{n \in \mathbb{N}}$ , sequence of  $\mathcal{C}(s)$  such that, for all  $n \in \mathbb{N}$ ,  $y_n = \Lambda_s(\widehat{y}_n)$ . Remark that the sequence  $(y_n)_{n \in \mathbb{N}}$  does not enjoy enough regularity to use classical compactness results. However, for all  $n \in \mathbb{N}$ ,  $y_n - y_0$  is solution of (10) for  $B = f(\widehat{y}_n) - f(\widehat{y}_0)$ ,  $u_0 = u_1 = 0$  and thus, we deduce from Proposition 4, for  $r = 3/2 - p$  that  $\rho(y_n - y_0) \in \mathcal{C}^1(0, T; H^{-r}(\Omega)) \cap \mathcal{C}^0([0, T]; H^{1-r}(\Omega))$ ,  $\rho(v_n - v_0) \in H^{1-r}(0, T; L^2(\partial\Omega))$  and

$$\begin{aligned} & \|\rho(y_n - y_0)\|_{\mathcal{C}^1(0,T;H^{-r}(\Omega))} + \|\rho(y_n - y_0)\|_{\mathcal{C}^0([0,T];H^{1-r}(\Omega))} + \left\| \frac{\rho}{\eta \Psi^{1/2}} (v_n - v_0) \right\|_{H^{1-r}(0,T;L^2(\partial\Omega))} \\ & \leq C(s) \|\rho(f(\widehat{y}_n) - f(\widehat{y}_0))\|_{L^2(0,T;H^{-r}(\Omega))}. \end{aligned}$$

15 This gives with Lemma 1 that  $(\rho(y_n - y_0))_{n \in \mathbb{N}}$  is a bounded sequence of  $\mathcal{C}^1(0, T; H^{-r}(\Omega)) \cap \mathcal{C}^0([0, T]; H^{1-r}(\Omega))$ .  
16 Thus, since  $\mathcal{C}^1(0, T; H^{-r}(\Omega)) \cap \mathcal{C}^0([0, T]; H^{1-r}(\Omega)) \hookrightarrow L^\infty(0, T; L^2(\Omega))$  is compact (see [19], Corollary 5  
17 p.86), there exist a subsequence  $(\rho(y_{n_k} - y_0))_{n_k \in \mathbb{N}}$  and  $z \in L^\infty(0, T; L^2(\Omega))$  such that  $\rho(y_{n_k} - y_0) \rightarrow z$   
18 in  $L^\infty(0, T; L^2(\Omega))$ . Therefore,  $\rho y_{n_k} \rightarrow y = z + \rho y_0$  in  $L^\infty(0, T; L^2(\Omega))$ . Since  $\rho y_{n_k} \in \mathcal{C}(s)$  for all  $n_k$ ,  
19  $y = z + \rho y_0 \in \mathcal{C}(s)$ . Thus  $\Lambda_s(\mathcal{C}(s))$  is a relatively compact subset of  $\mathcal{C}(s)$  for the  $L^\infty(0, T; L^2(\Omega))$  norm.  $\square$

### 1 3.1.4 Continuity of the map $\Lambda_s$ in $\mathcal{C}(s)$

2 **Lemma 4.** *Assume that  $f \in \mathcal{C}^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$ . Then, the map  $\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s)$  is continuous for the  $L^\infty(0, T; L^2(\Omega))$  norm.*

3 *Proof.* Let  $(\hat{y}_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{C}(s)$  such that  $(\hat{y}_n)_{n \in \mathbb{N}}$  converges to  $\hat{y} \in \mathcal{C}(s)$  with respect to the  
4  $L^\infty(0, T; L^2(\Omega))$  norm. Let  $y_n := \Lambda_s(\hat{y}_n)$  for all  $n \in \mathbb{N}$ . Since  $\hat{y}_n \rightarrow \hat{y}$  in  $L^\infty(0, T; L^2(\Omega))$ ,  $\hat{y}_n \rightarrow \hat{y}$  in  
5  $L^2(Q)$  and there exist a subsequence  $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$  and  $z \in L^2(Q)$  such that  $\hat{y}_{n_k} \rightarrow \hat{y}$  a.e and  $|\hat{y}_{n_k}| \leq z$  a.e  
6 for all  $n_k$ .

Thus, since  $f \in \mathcal{C}^0(\mathbb{R})$ ,  $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$  a.e. Let us choose  $\varepsilon < 1$  small enough so  $L^{\frac{2}{1+\varepsilon}}(\Omega) \hookrightarrow H^{-1}(\Omega)$ .  
For all  $n_k \in \mathbb{N}$ , a.e:

$$\begin{aligned} |f(\hat{y}_{n_k})| &\leq \alpha_1 + |\hat{y}_{n_k}| (\alpha_2 + \beta^* \ln_+^p |\hat{y}_{n_k}|) \leq \alpha_1 + \alpha_2 |\hat{y}_{n_k}| + \beta^* \ln_+^p |\hat{y}_{n_k}| \\ &\leq \alpha_1 + \alpha_2 |z| + \beta^* \ln_+^{3/2} |z| |z| \leq \alpha_1 + \alpha_2 |z| + C_\varepsilon \beta^* |z|^{1+\varepsilon}. \end{aligned}$$

7 and  $\alpha_1 + \alpha_2 |z| + C_\varepsilon \beta^* |z|^{1+\varepsilon} \in L^{\frac{2}{1+\varepsilon}}(Q)$ , since  $z \in L^2(Q)$  and thus, we deduce from the dominated  
8 convergence theorem that  $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$  in  $L^{\frac{2}{1+\varepsilon}}(Q)$ .

9 In fact,  $f(\hat{y}_n) \rightarrow f(\hat{y})$  in  $L^{\frac{2}{1+\varepsilon}}(Q)$  (and thus  $f(\hat{y}_n) \rightarrow f(\hat{y})$  in  $L^q(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega))$  for all  $1 \leq q < +\infty$   
10 since  $(f(\hat{y}_n))_{n \in \mathbb{N}}$  is bounded in  $L^\infty(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega))$ ). If not, there exists  $\varepsilon_1 > 0$  and a subsequence  
11  $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$  of  $(\hat{y}_n)_{n \in \mathbb{N}}$  such that  $\|f(\hat{y}_{n_k}) - f(\hat{y})\|_{L^{\frac{2}{1+\varepsilon}}(Q)} \geq \varepsilon_1$ . But, arguing as previously, there exist a  
12 subsequence  $(\hat{y}_{n_{k'}})_{n_{k'} \in \mathbb{N}}$  of  $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$  such that  $f(\hat{y}_{n_{k'}}) \rightarrow f(\hat{y})$  in  $L^{\frac{2}{1+\varepsilon}}(Q)$  which leads to a contradiction.

13 We denote by  $v_n$  and  $v$  the associated control to  $y_n$  and  $y$  respectively and we have, by definition  
14 of the operator  $\Lambda_s$ , that  $(y_n, v_n) = \rho^{-2}(Lw_n, s\eta^2\Psi\partial_\nu w_n)$  and  $(y, v) = \rho^{-2}(Lw, s\eta^2\Psi\partial_\nu w)$  with  $w_n$   
15 and  $w$  solution of (10) associated with  $(u_0, u_1, -f(\hat{y}_n))$  and  $(u_0, u_1, -f(\hat{y}))$  respectively. In particular,  
16  $z_n := y_n - y = \Lambda_s(\hat{y}_n) - \Lambda_s(\hat{y})$  satisfies  $z_n = \rho^{-2}L(w_n - w)$  with  $w_n - w$  solution of (10) associated with  
17 data  $(0, 0, f(\hat{y}) - f(\hat{y}_n))$ . Thus, using that  $L^2(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega)) \hookrightarrow L^2(0, T; H^{-1}(\Omega))$ , estimate (11) with  
18  $B = f(\hat{y}_n) - f(\hat{y})$ ,  $u_0 = u_1 = 0$  implies

$$\begin{aligned} \|\rho(y_n - y)\|_{L^\infty(0, T; L^2(\Omega))} &\leq Cs^{3/2} \|\rho(f(\hat{y}_n) - f(\hat{y}))\|_{L^2(0, T; H^{-1}(\Omega))} \\ &\leq Cs^{3/2} \|\rho(f(\hat{y}_n) - f(\hat{y}))\|_{L^2(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega))} \end{aligned} \quad (23)$$

19 and thus  $y_n \rightarrow y$  as  $n \rightarrow +\infty$  in  $L^\infty(0, T; L^2(\Omega))$ .  $\square$

## 20 3.2 Second part of Theorem 1

21 We now assume that  $f \in \mathcal{C}^1(\mathbb{R})$  satisfies  $(\mathbf{H}'_p)$  with  $0 \leq p < 3/2$ ;  $f$  then satisfies  $(\mathbf{H}_p)$  for  $\alpha_1 = f(0)$   
22 and  $\alpha_2 = \alpha$  so that results of Section 3.1 remain true. We are going to show that  $\Lambda_s : \mathcal{C}(s) \rightarrow \mathcal{C}(s)$  is  
23 a contracting mapping of the complete space  $(\mathcal{C}(s), d)$  with  $d : \mathcal{C}(s) \times \mathcal{C}(s) \rightarrow \mathbb{R}$  defined by  $d(y, z) :=$   
24  $\|\rho(y - z)\|_{L^2(Q)}$ . The Banach fixed point theorem will ensure the existence of a unique fixed point of  $\Lambda_s$   
25 which is a controlled trajectory for the semilinear problem (1).

26 **Proposition 6.** *Assume that there exists  $0 \leq p < 3/2$  such that  $f \in \mathcal{C}^1(\mathbb{R})$  satisfies  $(\mathbf{H}'_p)$  with  $\beta^*$  and  $s$  as in Lemma 2. Then,  $\Lambda_s$  is a contraction mapping from  $(\mathcal{C}(s), d)$  into itself.*

*Proof.* Without restriction, we assume that  $p > 1$ . Let  $\hat{y}_1, \hat{y}_2 \in \mathcal{C}(s)$ . From (11), we get that, for all  
0  $\leq r < 1/2$ ,  $d(\Lambda_s(\hat{y}_2), \Lambda_s(\hat{y}_1)) \leq C_r s^{r-3/2} \|\rho(f(\hat{y}_2) - f(\hat{y}_1))\|_{L^2(0, T; H^{-r}(\Omega))}$ . Let  $r = 3/2 - p > 0$ . There  
exists  $1 \leq q < 2$  such that  $L^q(\Omega) \hookrightarrow H^{-r}(\Omega)$ . We then have

$$d(\Lambda_s(\hat{y}_2), \Lambda_s(\hat{y}_1)) \leq C_r s^{r-3/2} \|\rho(f(\hat{y}_2) - f(\hat{y}_1))\|_{L^2(0, T; L^q(\Omega))}.$$

But, for all  $(m_1, m_2) \in \mathbb{R}^2$  there exists  $\bar{c}$  such that

$$|f(m_1) - f(m_2)| \leq |m_1 - m_2| |f'(\bar{c})| \leq |m_1 - m_2| (\alpha + \beta^* \ln_+^p |\bar{c}|) \leq |m_1 - m_2| (\alpha + \beta^* \ln_+^p (|m_1| + |m_2|))$$

1 and therefore, using that  $0 \leq \ln_+^p \rho \leq c^p s^p$  and that  $p = 3/2 - r$ , we get

$$\begin{aligned} d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) &\leq C s^{-p} \|(\alpha + \beta^* \ln_+^p(|\widehat{y}_1| + |\widehat{y}_2|))\rho(\widehat{y}_2 - \widehat{y}_1)\|_{L^2(0,T;L^q(\Omega))} \\ &\leq C s^{-p} \|(\alpha + \beta^* \ln_+^p(|\widehat{y}_1| + |\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))} d(\widehat{y}_2, \widehat{y}_1) \\ &\leq C s^{-p} (\alpha + \beta^* c^p s^p + \beta^* \|\ln_+^p(\rho(|\widehat{y}_1| + |\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))}) d(\widehat{y}_2, \widehat{y}_1) \end{aligned} \quad (24)$$

with  $a$  such that  $1/q = 1/2 + 1/a$ . Now, using that, for  $\varepsilon = \inf\{\frac{2}{a}, \frac{p}{3}\}$

$$\begin{aligned} \|\ln_+^p(\rho(|\widehat{y}_1| + |\widehat{y}_2|))\|_{L^\infty(0,T;L^a(\Omega))} &\leq C (\|(\rho\widehat{y}_1)^\varepsilon\|_{L^\infty(0,T;L^a(\Omega))} + \|(\rho\widehat{y}_2)^\varepsilon\|_{L^\infty(0,T;L^a(\Omega))}) \\ &\leq C (\|\rho\widehat{y}_1\|_{L^\infty(0,T;L^2(\Omega))}^\varepsilon + \|\rho\widehat{y}_2\|_{L^\infty(0,T;L^2(\Omega))}^\varepsilon) \leq C s^p \end{aligned}$$

2 we infer that

$$d(\Lambda_s(\widehat{y}_2), \Lambda_s(\widehat{y}_1)) \leq C(s^{-p}\alpha + \beta^* c^p) d(\widehat{y}_2, \widehat{y}_1). \quad (25)$$

3 The contraction property follows for  $s \geq s_0$  large enough and  $\beta^* > 0$  small enough.  $\square$

4 **Remark 5.** This also proves that  $s^{-1/2} \left\| \frac{\rho}{\eta\Psi^{1/2}}(v_2 - v_1) \right\|_{L^2(\Sigma)} \leq C(s^{-p}\alpha + \beta^* c^p) d(\widehat{y}_2, \widehat{y}_1)$ , for all  $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}(s)$  where  $v_1, v_2$  are the associated controls.

5 As a corollary of the Banach fixed point theorem, the contraction property of the operator  $\Lambda_s$  for  $\beta^*$   
6 small enough and  $s$  large enough allows to define a convergent sequence  $(y_k, v_k)_{k \in \mathbb{N}}$  to a controlled pair  
7 for (1) and prove the following more precise version of the second item of Theorem 1.

8 **Theorem 2.** Let  $(u_0, u_1) \in \mathbf{H}$ . Assume that  $f$  is  $C^1(\mathbb{R})$  and satisfies  $(\mathbf{H}'_p)$  for some  $0 \leq p < 3/2$  with  
9  $\beta^*$  small enough and  $s$  as in Lemma 2. Then, for any  $y_0 \in \mathcal{C}(s)$ , the sequence  $(y_k)_{k \in \mathbb{N}^*} \subset \mathcal{C}(s)$  given  
10 by  $y_{k+1} = \Lambda_s(y_k)$ ,  $k \geq 0$ , (where  $\Lambda_s$  is defined by (20)) together with the associated sequence of controls  
11  $(v_k)_{k \in \mathbb{N}^*} \subset L^2(\Sigma)$  strongly converges in  $C^0([0, T]; L^2(\Omega)) \times L^2(\Sigma)$  to a controlled solution for (1). The  
convergence is at least linear with respect to the distance  $d$ .

9 *Proof.* The convergence of the sequence  $(y_k)_{k \in \mathbb{N}}$  toward  $y = \Lambda_s(y) \in \mathcal{C}(s)$  with linear rate follows from  
10 the contraction property of  $\Lambda_s$ :  $d(y, y_k) = d(\Lambda_s(y), \Lambda_s(y_{k-1})) \leq (C(s^{-p}\alpha + \beta^* c^p))^k d(y, y_0)$  for all  $k \geq 0$   
11 deduced from (25).

Let now  $v \in H^1(0, T; L^2(\partial\Omega)) \cap C^0([0, T]; H^{1/2}(\partial\Omega))$  be associated with  $y$  so that  $y - y_k$  satisfies, for every  $k \in \mathbb{N}^*$

$$\begin{cases} L(y - y_k) = -(f(y) - f(y_{k-1})) & \text{in } Q, \\ y - y_k = v - v_k, & \text{on } \Sigma, \\ ((y - y_k)(\cdot, 0), (y - y_k)_t(\cdot, 0)) = (0, 0) & \text{in } \Omega, \\ ((y - y_k)(\cdot, T), (y - y_k)_t(\cdot, T)) = (0, 0) & \text{in } \Omega. \end{cases}$$

12 From Remark 5, we deduce that  $\left\| \frac{\rho}{\eta\Psi^{1/2}}(v - v_k) \right\|_{L^2(\Sigma)} \leq s^{1/2} C(s^{-p}\alpha + \beta^* c^p) d(y, y_{k-1})$  and the convergence  
13 at a linear rate of the sequence  $(v_k)_{k \in \mathbb{N}^*}$  toward a control  $v$  for (1).

Now, since  $f \in C^1(\mathbb{R})$  satisfies  $(\mathbf{H}'_p)$ , using (11) for  $r = 3/2 - p > 0$  and (25), we get

$$\begin{aligned} \|(\rho(y - y_k))_t\|_{C^0([0,T];H^{-1}(\Omega))} + \|\rho(y - y_k)\|_{C^0([0,T];L^2(\Omega))} &\leq C s^{1/2+r} \|\rho(f(y) - f(y_{k-1}))\|_{L^2(0,T;H^{-r}(\Omega))} \\ &\leq C s^2 C(s^{-p}\alpha + \beta^* c^p) \|\rho(y - y_{k-1})\|_{L^2(Q)} \\ &\leq C s^2 (C(s^{-p}\alpha + \beta^* c^p))^k d(y, y_0). \end{aligned}$$

14 It follows that  $y_k \rightarrow y$  in  $C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ .  $\square$

15 **Remark 6.** Assume that exists  $0 \leq p < 3/2$  such that  $\lim_{|r| \rightarrow +\infty} \frac{|f'(r)|}{\ln_+^p |r|} = 0$ , i.e. that  $\beta^*$  is arbitrarily small  
in  $(\mathbf{H}'_p)$ . Then, (25) shows that the constant of contraction of  $\Lambda_s$  behaves like  $s^{-p}$ .

**Remark 7.** Assume  $d \leq 3$  and  $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ . Then, one may prove (repeating one step more the arguments of the Appendix) that the optimal state  $y$  given in Proposition 2 belongs to  $L^\infty(Q)$ . This allows (following [3]) to reach the value  $p = 3/2$  in  $(\mathbf{H}_p)$ . We refer to [6] together with numerical illustrations.

### 3.3 Third part of Theorem 1

Assuming the initial data in  $\mathbf{V}$ , we follow the arguments of Section 3.1 and get that the uniform controllability holds true under the condition  $(\mathbf{H}_p)$  with  $p = 3/2$ . For any  $s \geq s_0$ , we introduce the class  $\tilde{\mathcal{C}}(s)$  defined as the closed convex subset of  $L^2(Q)$

$$\tilde{\mathcal{C}}(s) := \left\{ y \in H^1(Q) : \|\rho y\|_{L^2(Q)} \leq s, \|(\rho y)_t\|_{L^2(Q)} \leq s^2, \|\nabla(\rho y)\|_{L^2(Q)} \leq s^3 \right\}. \quad (26)$$

#### 3.3.1 Estimates of $\Lambda_s(\hat{y})$

**Lemma 5.** Assume that  $f \in C^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$  with  $p = 3/2$ . For any  $s \geq s_0$  and  $\hat{y} \in \tilde{\mathcal{C}}(s)$ , there exists  $C > 0$  such that

$$\|\rho f(\hat{y})\|_{L^2(Q)} \leq C \left( \alpha_1 e^{-sT^{1/2}} |\Omega|^{1/2} + \alpha_2 s + \beta^* (cs)^{3/2} s \right)$$

with  $c = \|\phi\|_{L^\infty(Q)}$ .

*Proof.* Using that  $\rho \leq e^{-s}$ , we infer that

$$\begin{aligned} \|\rho f(\hat{y})\|_{L^2(Q)} &\leq C \|\rho \alpha_1 + |\rho \hat{y}| \left( \alpha_2 + \beta^* \ln_+^{3/2}(\hat{y}) \right)\|_{L^2(Q)} \\ &\leq C (\alpha_1 e^{-sT^{1/2}} |\Omega|^{1/2} + \alpha_2 \|\rho \hat{y}\|_{L^2(Q)} + \beta^* \|\rho \hat{y} \ln_+^{3/2}(\hat{y})\|_{L^2(Q)}). \end{aligned}$$

Now, since,  $0 \leq \ln_+^{3/2}(\hat{y}) \leq C(\ln_+^{3/2} \rho^{-1} + \ln_+^{3/2}(\rho \hat{y})) \leq C((cs)^{3/2} + \ln_+^{3/2}(\rho \hat{y}))$ , we get that

$$\|\rho \hat{y} \ln_+^{3/2}(\hat{y})\|_{L^2(Q)} \leq C((cs)^{3/2} \|\rho \hat{y}\|_{L^2(Q)} + \|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)}).$$

But, for all  $0 < \varepsilon \leq 4/d$ , there exists  $C_\varepsilon > 0$  such that

$$\|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)} \leq C_\varepsilon \|\rho \hat{y}\|^{1+\frac{\varepsilon}{2}}_{L^2(Q)} = C_\varepsilon \|\rho \hat{y}\|_{L^{2+\varepsilon}(Q)}^{1+\frac{\varepsilon}{2}} \leq C_\varepsilon \|\rho \hat{y}\|_{L^\infty(0,T;L^2(\Omega))}^{1-\frac{\varepsilon(d-2)}{4}} \|\nabla(\rho \hat{y})\|_{L^2(Q)}^{\frac{\varepsilon d}{4}} \quad (27)$$

and since  $\hat{y} \in \tilde{\mathcal{C}}(s)$ , we have  $\|\rho \hat{y}\|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{2} \|\rho \hat{y}\|_{L^2(Q)}^{1/2} \|(\rho \hat{y})_t\|_{L^2(Q)}^{1/2} \leq \sqrt{2} s^{3/2}$ . Thus

$$\|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)} \leq C_\varepsilon s^{\frac{3}{2} - \frac{3\varepsilon(d-2)}{8}} s^{\frac{3\varepsilon d}{4}} = C_\varepsilon s^{\frac{3}{2} + \frac{3\varepsilon(d+2)}{8}}.$$

In particular for  $\varepsilon = \frac{4}{3(d+2)} < \frac{4}{d}$  this gives  $\|\rho \hat{y} \ln_+^{3/2}(\rho \hat{y})\|_{L^2(Q)} \leq C s^2$ . Combining the above inequalities, we get the result.  $\square$

**Proposition 7.** For any  $x_0 \notin \bar{\Omega}$ , we assume (3) and (5). Assume  $(u_0, u_1) \in \mathbf{V}$ . Assume that  $f \in C(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$  with  $p = 3/2$ . For  $s \geq s_0$  and for all  $\hat{y} \in \tilde{\mathcal{C}}(s)$ , the solution  $y = \Lambda_s(\hat{y})$  to the linearized controlled system (18) with control  $v$  satisfies  $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  and the following estimates:

$$\begin{aligned} &\|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\Psi^{1/2} \eta} v \right\|_{L^2(\Sigma)} + s^{-1} \|(\rho y)_t\|_{L^2(Q)} + s^{-3/2} \left\| \frac{1}{\Psi^{1/2} \eta} (\rho v)_t \right\|_{L^2(\Sigma)} + s^{-2} \|\nabla(\rho y)\|_{L^2(Q)} \\ &\leq \left( C \alpha_2 s^{-3/2} + \beta^* C c^{3/2} \right) s + \left( C \|u_0\|_{L^2(\Omega)} + C s^{-1} \left( \alpha_1 T^{1/2} |\Omega|^{1/2} + \|\nabla u_0\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} \right) \right) s^{-1/2} e^{-s}. \end{aligned} \quad (28)$$

*Proof.* This follows from Proposition 2 with  $r = 0$ , Proposition 3 and Lemma 5 with  $B = -f(\hat{y})$ .  $\square$

### 3.3.2 Stability of the class $\tilde{\mathcal{C}}(s)$

**Lemma 6.** Assume that  $f \in \mathcal{C}^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$  with  $p = 3/2$  and  $\beta^*$  small enough. Then, there exists  $s \geq s_0$  large enough such that  $\Lambda_s(\tilde{\mathcal{C}}(s)) \subset \tilde{\mathcal{C}}(s)$ .

*Proof.* For any  $s \geq s_0$  and  $\hat{y} \in \tilde{\mathcal{C}}(s)$ , let  $y := \Lambda_s(\hat{y})$ . From the inequalities (28), we obtain that  $\limsup_{s \rightarrow +\infty} s^{-1} \|\rho y\|_{L^2(Q)} \leq \beta^* C c^{3/2}$ ,  $\limsup_{s \rightarrow +\infty} s^{-2} \|(\rho y)_t\|_{L^2(Q)} \leq \beta^* C c^{3/2}$  and  $\limsup_{s \rightarrow +\infty} s^{-3} \|\nabla(\rho y)\|_{L^2(Q)} \leq \beta^* C c^{3/2}$ . Therefore, if  $\beta^* > 0$  is small enough so that  $\beta^* C c^{3/2} < 1$ , then for any  $s \geq s_0$  large enough,  $y \in \tilde{\mathcal{C}}(s)$ .  $\square$

### 3.3.3 Relative compactness of the set $\Lambda_s(\tilde{\mathcal{C}}(s))$

**Lemma 7.** Under the assumptions of Lemma 2,  $\Lambda_s(\tilde{\mathcal{C}}(s))$  is a relatively compact subset of  $\tilde{\mathcal{C}}(s)$  for the  $L^2(Q)$  norm.

*Proof.* Since  $\Lambda_s(\tilde{\mathcal{C}}(s)) \subset \tilde{\mathcal{C}}(s)$ , this follows from the fact that  $\tilde{\mathcal{C}}(s)$  is bounded in  $H^1(Q)$ .  $\square$

**Remark 8.** In fact  $\tilde{\mathcal{C}}(s)$  is a compact subset of  $L^p(Q)$  norm,  $1 \leq p < 2 + \frac{4}{d}$  and thus  $\Lambda_s(\tilde{\mathcal{C}}(s))$  is a relatively compact subset of  $\tilde{\mathcal{C}}(s)$  for the  $L^p(Q)$  norm.

### 3.3.4 Continuity of the map $\Lambda_s$ in $\tilde{\mathcal{C}}(s)$

**Lemma 8.** Assume that  $f \in \mathcal{C}^0(\mathbb{R})$  satisfies  $(\mathbf{H}_p)$  with  $p = 3/2$ . Then, the map  $\Lambda_s : \tilde{\mathcal{C}}(s) \rightarrow \tilde{\mathcal{C}}(s)$  is continuous with respect to the  $L^2(Q)$  norm.

*Proof.* Let  $(\hat{y}_n)_{n \in \mathbb{N}}$  be a sequence in  $\tilde{\mathcal{C}}(s)$  such that  $(\hat{y}_n)_{n \in \mathbb{N}}$  converges to  $\hat{y} \in \tilde{\mathcal{C}}(s)$  with respect to the  $L^2(Q)$  norm. Let  $y_n := \Lambda_s(\hat{y}_n)$  for all  $n \in \mathbb{N}$ . Since  $\hat{y}_n \rightarrow \hat{y}$  in  $L^2(Q)$  then  $\hat{y}_n \rightarrow \hat{y}$  in  $L^{2+\frac{2}{d}}(Q)$  (since, see Remark 8,  $\tilde{\mathcal{C}}(s)$  is a compact subset of  $L^{2+\frac{2}{d}}(Q)$ ). Thus there exist a subsequence  $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$  and  $z \in L^{2+\frac{2}{d}}(Q)$  such that  $\hat{y}_{n_k} \rightarrow \hat{y}$  a.e. and  $|\hat{y}_{n_k}| \leq z$  a.e. for all  $n_k$ . It follows since  $f \in \mathcal{C}^0(\mathbb{R})$  that  $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$  a.e. But, for all  $n_k \in \mathbb{N}$ , a.e.

$$\begin{aligned} |f(\hat{y}_{n_k})| &\leq \alpha_1 + |\hat{y}_{n_k}| \left( \alpha_2 + \beta^* \ln_+^{3/2}(\hat{y}_{n_k}) \right) \leq \alpha_1 + \alpha_2 |\hat{y}_{n_k}| + \beta^* |\hat{y}_{n_k}| \ln_+^{3/2}(\hat{y}_{n_k}) \\ &\leq \alpha_1 + \alpha_2 |z| + \beta^* |z| \ln_+^{3/2}(z) \leq \alpha_1 + \alpha_2 |z| + C \beta^* |z|^{1+\frac{1}{d}} \end{aligned}$$

and  $\alpha_1 + \alpha_2 |z| + C \beta^* |z|^{1+\frac{1}{d}} \in L^2(Q)$ , since  $z \in L^{2+\frac{2}{d}}(Q)$  and thus, from the dominated convergence theorem,  $f(\hat{y}_{n_k}) \rightarrow f(\hat{y})$  in  $L^2(Q)$ . Indeed  $f(\hat{y}_n) \rightarrow f(\hat{y})$  in  $L^2(Q)$ . Otherwise, there exists  $\varepsilon > 0$  and a subsequence  $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$  of  $(\hat{y}_n)_{n \in \mathbb{N}}$  such that  $\|f(\hat{y}_{n_k}) - f(\hat{y})\|_{L^2(Q)} \geq \varepsilon$ . But, arguing as before, there exists a subsequence  $(\hat{y}_{n_{k'}})_{n_{k'} \in \mathbb{N}}$  of  $(\hat{y}_{n_k})_{n_k \in \mathbb{N}}$  such that  $f(\hat{y}_{n_{k'}}) \rightarrow f(\hat{y})$  in  $L^2(Q)$  which leads to a contradiction.

In particular,  $z_n := y_n - y = \Lambda_s(\hat{y}_n) - \Lambda_s(\hat{y})$  satisfies  $z_n = \rho^{-2} L(w_n - w)$  with  $w_n - w$  solution of (10) associated with data  $(0, 0, f(\hat{y}) - f(\hat{y}_n))$ . Thus, using that  $L^2(0, T; L^{\frac{2}{1+\varepsilon}}(\Omega)) \hookrightarrow L^2(0, T; H^{-1}(\Omega))$ , estimate (11) with  $B = f(\hat{y}_n) - f(\hat{y})$ ,  $u_0 = u_1 = 0$  implies

We denote by  $v_n$  and  $v$  the associated control to  $y_n$  and  $y$  respectively and we have, by definition of the operator  $\Lambda_s$ , that  $(y_n, v_n) = \rho^{-2} (Lw_n, s\eta^2 \Psi \partial_\nu w_n)$  and  $(y, v) = \rho^{-2} (Lw, s\eta^2 \Psi \partial_\nu w)$  with  $w_n$  and  $w$  solution of (10) associated with  $(u_0, u_1, -f(\hat{y}_n))$  and  $(u_0, u_1, -f(\hat{y}))$  respectively. In particular,  $z := y_n - y = \Lambda_s(\hat{y}_n) - \Lambda_s(\hat{y})$  satisfies  $z = \rho^{-2} L(w_n - w)$  with  $w_n - w$  solution of (10) associated with data  $(0, 0, f(\hat{y}) - f(\hat{y}_n))$ . Then estimate (11) with  $B = f(\hat{y}_n) - f(\hat{y})$ ,  $u_0 = u_1 = 0$  and  $r = 0$  implies  $\|\rho(y_n - y)\|_{L^2(Q)} \leq C s^{-3/2} \|\rho(f(\hat{y}) - f(\hat{y}_n))\|_{L^2(Q)}$  so that  $y_n \rightarrow y$  as  $n \rightarrow +\infty$  in  $L^2(Q)$ .  $\square$

## A Appendix: Proof of Proposition 3

We check that the optimal pair  $(y, v)$  defined in Proposition 2 belongs to  $C^0([0, T]; H^1(\Omega)) \times C^0(0, T; H^{1/2}(\partial\Omega))$  as soon as  $B \in L^2(Q)$  and  $(u_0, u_1) \in \mathbf{V}$ . This property has been proved in [3] in the one dimensional case (by generalizing [10]). The occurrence of the weights and the coupling between the primal variable  $y$  and the dual one  $w$  (see Remark 3) make the arguments and the computations more involved with respect [10]. The proof is divided into four steps: the first and second steps are similar to [3, Appendix]; the third one is new with respect to [3, Appendix] as it provides an estimate of  $\|\nabla(\rho y)\|_{L^2(Q)}$ .

For all  $f \in C^0(\mathbb{R}; E)$  (where  $E$  is a Banach space) and any  $\tau \neq 0$ , we define  $\delta_\tau f := f(t + \frac{\tau}{2}) - f(t - \frac{\tau}{2})$ ,  $\mathcal{T}_\tau f := \frac{1}{\tau} \delta_\tau \left( \frac{\delta_\tau f}{\tau} \right)$  and  $\tilde{\delta}_\tau f(t) := \frac{f(t+\tau) - f(t)}{\tau}$ .

Let now  $w \in P_s$  and  $y \in L^2(Q)$  be given by Proposition 2. Then  $\mathcal{T}_\tau w$  belongs to  $P_s$ , where  $w$  as well as  $y$  can be extended uniquely on  $(-\infty, 0)$  and  $(T, +\infty)$ . Indeed, in the time interval  $(-\infty, 0)$  the solution  $y$  satisfies

$$Ly = 0 \text{ in } \Omega \times (-\infty, 0), \quad y = 0 \text{ on } \partial\Omega \times (-\infty, 0), \quad (y(\cdot, 0), \partial_t y(\cdot, 0)) = (u_0, u_1) \text{ in } \Omega,$$

where the source term  $B \in L^2(Q)$  is assumed to be extendable by 0 outside  $(0, T)$ . Recall that the boundary condition  $y(\cdot, t) = 0$  on  $\Gamma_0$  holds outside  $(0, T)$  since  $\eta = 0$  (appearing in the formula of  $v$ ) vanishes outside  $(\delta, T - \delta)$ , see (6).

Similarly, in  $(T, +\infty)$  we can define the solution  $y$  uniquely, and  $y(t) = 0$  for all  $t \geq T$ . It follows that the solution  $y$  satisfies  $y \in C^0(\mathbb{R}; L^2(\Omega)) \cap C^1(\mathbb{R}; H^{-1}(\Omega))$  and  $y \in C^0((-\infty, \delta]; H_0^1(\Omega)) \cap C^1((-\infty, \delta]; L^2(\Omega))$  and  $y \in C^0([T - \delta, +\infty); H_0^1(\Omega)) \cap C^1([T - \delta, +\infty); L^2(\Omega))$  (see [13, Theorem 2.1, page 151]). We extend as well the weight  $\rho$  in  $\Omega \times \mathbb{R}$ . This ensures the extension of the solution  $w$  which satisfies the following set of equations in  $\mathbb{R}$

$$Lw = \rho^2 y \text{ in } \Omega \times \mathbb{R}, \quad w = 0 \text{ on } \partial\Omega \times \mathbb{R}. \quad (29)$$

Moreover, it can be seen that  $Lw = 0$  in  $[T, +\infty)$ .

**Step 1.** We assume that  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $u_1 \in H_0^1(\Omega)$  and  $B \in \mathcal{D}(0, T; L^2(\Omega))$  and prove that  $v \in H^1(0, T; L^2(\partial\Omega))$  and  $y_t \in L^2(Q)$ .

Since  $w \in C^0(\mathbb{R}; H_0^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega))$  solves (29),  $\mathcal{T}_\tau w \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  and  $\partial_\nu \mathcal{T}_\tau w \in L^2(\Sigma)$ . With  $z = \mathcal{T}_\tau w$  as test function in (10), we have

$$\begin{aligned} \int_Q \rho^{-2} Lw L \mathcal{T}_\tau w \, dx dt + s \int_\Sigma \eta^2(t) \Psi(x) \rho^{-2} \partial_\nu w \mathcal{T}_\tau \partial_\nu w \, d\Sigma \\ = \int_\Omega u_1 \mathcal{T}_\tau w(\cdot, 0) \, dx - \int_\Omega u_0 \mathcal{T}_\tau w_t(\cdot, 0) \, dx + \int_Q B \mathcal{T}_\tau w \, dx dt. \end{aligned} \quad (30)$$

Proceeding as [3, Appendix] and using that the smooth function  $\eta$  given by (6) satisfies  $\eta = 0$  in  $(-\infty, \delta] \cup [T - \delta, +\infty)$  (with  $\delta > 0$  given in (6)), we get the following estimate (we refer to [3, sub-steps 1 and 2 pages 108-110])

$$\begin{aligned} \int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2} Lw)(t) \right|^2 \, dx dt + s \int_\Sigma \Psi \frac{\eta^2(t) \rho^{-2}(t) + \eta^2(t + \tau) \rho^{-2}(t + \tau)}{2} \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 \, d\Sigma \\ = \int_Q \rho^2(t) \tilde{\delta}_\tau(\rho^{-2} Lw)(t) \tilde{\delta}_\tau(\rho^{-2})(t) Lw(t + \tau) \, dx dt - \frac{1}{\tau} \int_{-\tau}^0 \int_\Omega \rho^{-2}(t + \tau) Lw(t + \tau) \tilde{\delta}_\tau Lw(t) \, dx dt \\ - \frac{1}{\tau} \int_T^{T-\tau} \int_\Omega \rho^{-2}(t + \tau) Lw(t + \tau) \tilde{\delta}_\tau Lw(t) \, dx dt \\ - s \int_\Sigma \Psi \tilde{\delta}_\tau(\eta^2 \rho^{-2})(t) \frac{\partial_\nu w(t) + \partial_\nu w(t + \tau)}{2} \tilde{\delta}_\tau(\partial_\nu w)(t) \, d\Sigma \\ - \int_Q B \mathcal{T}_\tau w(t) \, dx dt - \int_\Omega u_1 \mathcal{T}_\tau w(\cdot, 0) \, dx + \int_\Omega u_0 \mathcal{T}_\tau w_t(\cdot, 0) \, dx \end{aligned} \quad (31)$$

Then, following [3, step 1, sub-step 3 page 111-114], we get that each term of the right hand side of (31) are uniformly bounded with respect to  $\tau \in [0, \delta]$ . Then we can conclude, from (31), that the terms  $\int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2}Lw)(t) \right|^2 dxdt$  and

$$\int_\Sigma \Psi \frac{\eta^2(t)\rho^{-2}(t) + \eta^2(t+\tau)\rho^{-2}(t+\tau)}{2} \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 d\Sigma$$

1 are bounded. Thus  $\rho(\rho^{-2}Lw)_t \in L^2(Q)$ ,  $\Psi^{1/2}\eta\rho^{-1}\partial_\nu w_t \in L^2(\Sigma)$ , and  $y \in H^1(0, T; L^2(\Omega))$ ,  $v \in$   
2  $H^1(0, T; L^2(\partial\Omega))$ .

3 Eventually, following [3, step 1, sub-step 4 page 115], we prove that  $v \in C^0([0, T]; H^{1/2}(\partial\Omega))$  and  
4  $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ .

Since  $\tilde{\delta}_\tau w \in P$ , Carleman estimates (8) gives

$$\begin{aligned} & s \int_Q \rho^{-2} \left( |\tilde{\delta}_\tau w_t|^2 + |\tilde{\delta}_\tau \nabla w|^2 \right) dxdt + s^3 \int_Q \rho^{-2} |\tilde{\delta}_\tau w|^2 dxdt \\ & + s \int_\Omega \rho^{-2}(0) \left( |\tilde{\delta}_\tau(w_t)(0)|^2 + |\tilde{\delta}_\tau(\nabla w)(0)|^2 \right) dx + s^3 \int_\Omega \rho^{-2}(0) |\tilde{\delta}_\tau w(0)|^2 dx \\ & \leq C \int_Q \rho^{-2} |L(\tilde{\delta}_\tau w)|^2 dxdt + Cs \int_\Sigma \eta^2 \Psi \rho^{-2} |\tilde{\delta}_\tau(\partial_\nu w)|^2 d\Sigma. \end{aligned} \quad (32)$$

Remark that  $\int_\Sigma \Psi \eta^2 \rho^{-2} \left| \tilde{\delta}_\tau(\partial_\nu w) \right|^2 d\Sigma$  and  $\int_Q \rho^{-2} |L(\tilde{\delta}_\tau w)|^2 dxdt$  are bounded since

$$\int_\Sigma \Psi \eta^2(t) \rho^{-2}(t) \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 d\Sigma \leq 2 \int_\Sigma \Psi \frac{\eta^2(t)\rho^{-2}(t) + \eta^2(t+\tau)\rho^{-2}(t+\tau)}{2} \left| \tilde{\delta}_\tau(\partial_\nu w)(t) \right|^2 d\Sigma$$

and

$$\int_Q \rho^{-2}(t) |L(\tilde{\delta}_\tau w)(t)|^2 dxdt \leq 2 \int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2}Lw)(t) \right|^2 dxdt + 2 \int_Q \rho^2(t) \left| \tilde{\delta}_\tau(\rho^{-2})(t)Lw(t+\tau) \right|^2 dxdt$$

5 thus the right hand side in (32) is bounded. Therefore  $\tau \mapsto \tilde{\delta}_\tau w_t$  and  $\tau \mapsto \tilde{\delta}_\tau \nabla w$  are bounded in  $L^2(Q)$   
6 and thus  $w_{tt} \in L^2(Q)$  and  $w_t \in L^2(0, T; H_0^1(\Omega))$ . Moreover,  $\tau \mapsto \tilde{\delta}_\tau w(0)$  is bounded in  $H_0^1(\Omega)$  thus  
7  $w_t(0) \in H_0^1(\Omega)$ . Eventually, since  $\tau \mapsto L(\tilde{\delta}_\tau w)$  is bounded in  $L^2(Q)$ , we conclude that  $Lw_t \in L^2(Q)$  and  
8 then that  $w_t \in P$ ; it follows that  $\partial_\nu w_t \in L^2(\Sigma)$ .

**Remark 9.** We then have  $w \in C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$  and since  $Lw \in C^0([0, T]; L^2(\Omega))$  we deduce that  $\Delta w (= w_{tt} - Lw) \in C^0([0, T]; L^2(\Omega))$  and thus  $w \in C^0([0, T]; H^2(\Omega))$ .

10 Since  $w \in C^0([0, T]; H^2(\Omega))$   $v \in C^0([0, T]; H^{1/2}(\partial\Omega)) \cap H^1(0, T; L^2(\partial\Omega))$  and  $v$  satisfies the compatibility  
11 conditions  $v(0) = 0$ , [13, Theorem 2.1, page 151] leads to  $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap$   
12  $C^2([0, T]; H^{-1}(\Omega))$ .

**Remark 10.** As previously mentioned,  $y$  and  $w$  can be extended uniquely on  $(-\infty; 0)$  and  $(T, +\infty)$  so that  $w \in C^0(\mathbb{R}; H^2(\Omega)) \cap C^1(\mathbb{R}; H_0^1(\Omega)) \cap C^2(\mathbb{R}; L^2(\Omega))$ ,  $\partial_\nu w \in C^0(\mathbb{R}; H^{1/2}(\partial\Omega))$ ,  $\partial_\nu w_t \in L_{\text{loc}}^2(\mathbb{R}; L^2(\partial\Omega))$  and  $y \in C^0(\mathbb{R}; H^1(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega))$ .

**Step 2.** We prove estimate on  $v_t$  and  $y_t$  in (16). Carleman estimate (8) for  $w_t \in P$  reads

$$\begin{aligned} & s \int_Q \rho^{-2} (|w_{tt}|^2 + |\nabla w_t|^2) dxdt + s^3 \int_Q \rho^{-2} |w_t|^2 dxdt \\ & + s \int_\Omega \rho^{-2}(0) (|w_{tt}(0)|^2 + |\nabla w_t(0)|^2) dx + s^3 \int_\Omega \rho^{-2}(0) |w_t(0)|^2 dx \\ & \leq C \int_Q \rho^{-2} |Lw_t|^2 dxdt + Cs \int_\Sigma \eta^2 \Psi \rho^{-2} |\partial_\nu w_t|^2 d\Sigma. \end{aligned} \quad (33)$$

- 1 Proceeding as in [3, step 2 sub-step 1, page 116], we pass to the limit when  $\tau \rightarrow 0$  in the equality (31)  
 2 and obtain

$$\begin{aligned}
 \int_Q \rho^2 |y_t|^2 dx dt + s \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma &= -4s\lambda\beta \int_Q (t - \frac{T}{2}) \phi \rho^2 y_t y dx dt - \int_\Omega y(0)(\rho^2 y)_t(0) dx \\
 - s \int_\Sigma \Psi (\eta^2 \rho^{-2})_t \partial_\nu w \partial_\nu w_t d\Sigma - \int_Q B w_{tt} dx dt - \int_\Omega w_{tt}(0) u_1 dx & \quad (34) \\
 - 2s\lambda\beta T \int_\Omega \phi(0) \rho^2(0) u_0^2 dx + \int_\Omega \rho^2(0) u_1 u_0 dx - \int_\Omega \nabla u_0 \nabla w_t(0) dx &
 \end{aligned}$$

and then estimate each term of the right side. In particular, for the third term, we write, using  $(\rho^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho^{-1}$  that

$$\begin{aligned}
 |s \int_\Sigma \Psi (\eta^2 \rho^{-2})_t \partial_\nu w \partial_\nu w_t d\Sigma| &\leq 2 \left( s \int_\Sigma \Psi |(\eta\rho^{-1})_t|^2 |\partial_\nu w|^2 d\Sigma \right)^{1/2} \left( s \int_\Sigma \Psi \eta^2(t) \rho^{-2}(t) |\partial_\nu w_t|^2 d\Sigma \right)^{1/2} \\
 &\leq C \left( s^3 \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w|^2 d\Sigma + s \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma \right)^{1/2} \left( s \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma \right)^{1/2} \\
 &\leq C \left( s \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} v^2 d\Sigma + s \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma \right) + \frac{s}{8} \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma.
 \end{aligned}$$

We now estimate the term  $\int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma$  appearing in the previous inequality: proceeding as in [16, Lemma 3.7] with  $q(x, t) = h\rho^{-2}(x, t)$  where  $h \in (C^1(\bar{\Omega}))^d$  satisfies  $h(x) = \nu(x)$  on  $\partial\Omega$ , we get (with the notation  $f_k g_k = \sum_{k=1}^d f_k g_k$ )

$$\begin{aligned}
 \frac{1}{2} \int_\Sigma \rho^{-2} |\partial_\nu w|^2 d\Sigma &= \int_Q \rho^{-2} \nabla w \cdot \nabla h_k \frac{\partial w}{\partial x_k} + 2 \int_Q \rho^{-1} (\nabla w \cdot \nabla \rho^{-1} - (\rho^{-1})_t w_t) h \cdot \nabla w \\
 &\quad + \frac{1}{2} \int_Q \rho^{-2} (|w_t|^2 - |\nabla w|^2) \nabla \cdot h + \int_Q \rho^{-1} (|w_t|^2 - |\nabla w|^2) h \cdot \nabla \rho^{-1} \\
 &\quad + \int_\Omega [\rho^{-2} w_t h \cdot \nabla w]_0^T - \int_Q y h \cdot \nabla w.
 \end{aligned}$$

Writing that  $|\nabla \rho^{-1}| \leq Cs\rho^{-1}$  and  $|(\rho^{-1})_t| \leq Cs\rho^{-1}$ , and using that  $h \in (C^1(\bar{\Omega}))^d$  and  $s \geq 1$  we obtain

$$\begin{aligned}
 \int_\Sigma \rho^{-2} |\partial_\nu w|^2 d\Sigma &\leq Cs \int_\Omega \left( \rho^{-2}(0) (|w_t|^2 + |\nabla w|^2)(0) + \rho^{-2}(T) (|w_t|^2 + |\nabla w|^2)(T) \right) dx \\
 &\quad + Cs \int_Q \rho^{-2} (|w_t|^2 + |\nabla w|^2) dx dt + Cs^{-1} \int_Q \rho^2 |y|^2 dx dt
 \end{aligned}$$

- 3 leading, using (8) and that  $\Psi \in (C^1(\bar{\Omega}))^d$  to

$$s \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 d\Sigma \leq Cs \left( \|\rho y\|_{L^2(Q)}^2 + s^{-1} \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)}^2 \right). \quad (35)$$

Thus,

$$|s \int_\Sigma \Psi (\eta^2 \rho^{-2})_t \partial_\nu w \partial_\nu w_t d\Sigma| \leq Cs \left( \|\rho y\|_{L^2(Q)}^2 + \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)}^2 \right) + \frac{s}{8} \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 d\Sigma.$$

- 4 Proceeding as in [3, step 2 sub-step 2, page 117-120] for the other terms in the right hand side of (34),

1 we then get from (34)

$$\begin{aligned} & \int_Q \rho^2 |y_t|^2 \, dxdt + s \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 \, d\Sigma \\ & \leq C \left( s^2 \|\rho y\|_{L^2(Q)}^2 + s \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)}^2 + s^{-1} \|\rho B\|_{L^2(Q)}^2 \right. \\ & \quad \left. + s \|\rho(0) u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) \nabla u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) u_1\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (36)$$

2 We have, since  $\eta \in C^1([0, T])$  and  $v = s\eta^2 \Psi \rho^{-2} \partial_\nu w$

$$s^{-1} \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} |v_t|^2 \, d\Sigma \leq C s \left( \int_\Sigma \Psi \eta^2 \rho^{-2} |\partial_\nu w_t|^2 \, d\Sigma + \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} v^2 \, d\Sigma + \int_\Sigma \Psi \rho^{-2} |\partial_\nu w|^2 \, d\Sigma \right) \quad (37)$$

thus, using that  $\Psi \in (C^1(\bar{\Omega}))^d$  and (35), (36), implies for  $s \geq s_0 \geq 1$  that

$$\begin{aligned} & \int_Q \rho^2 |y_t|^2 \, dxdt + s^{-1} \int_\Sigma \frac{\rho^2}{\eta^2 \Psi} |v_t|^2 \, d\Sigma \\ & \leq C \left( s^{-1} \|\rho B\|_{L^2(Q)}^2 + s \|\rho(0) u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) \nabla u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0) u_1\|_{L^2(\Omega)}^2 \right), \end{aligned}$$

3 which is the announced estimates (16) for the first two terms in the case of regular data.

4 **Step 3.** We obtain the estimate on  $\|\nabla(\rho y)\|_{L^2(Q)}$  and  $\|\rho v\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))}$  in (16). With respect to [3],  
5 this part is new. From the definition of  $v$ , since  $Lw(T) = (\rho^2 y)(T) = 0$  and  $s \geq 1$  we have

$$\begin{aligned} \|\rho v\|_{L^2(0, T; H^{1/2}(\partial\Omega))} &= s \|\Psi \rho^{-1} \partial_\nu w\|_{L^2(0, T; H^{1/2}(\partial\Omega))} = s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^2(0, T; H^{1/2}(\partial\Omega))} \\ &\leq C s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^2(0, T; H^1(\Omega))} \\ &\leq C (s \|\nabla(\rho^{-1} w)\|_{L^2(0, T; H^1(\Omega))} + s^2 \|\rho^{-1} w\|_{L^2(0, T; H^1(\Omega))}) \end{aligned} \quad (38)$$

6 and

$$\begin{aligned} \|\rho v\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} &= s \|\Psi \rho^{-1} \partial_\nu w\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} = s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^\infty(0, T; H^{1/2}(\partial\Omega))} \\ &\leq C s \|\Psi \rho^{-1} h \cdot \nabla w\|_{L^\infty(0, T; H^1(\Omega))} \\ &\leq C (s \|\nabla(\rho^{-1} w)\|_{L^\infty(0, T; H^1(\Omega))} + s^2 \|\rho^{-1} w\|_{L^\infty(0, T; H^1(\Omega))}). \end{aligned} \quad (39)$$

Since  $\nabla(\rho^{-1} w) = \nabla(\rho^{-1})w + \rho^{-1} \nabla w$  and  $\Delta(\rho^{-1} w) = \Delta(\rho^{-1})w + 2\nabla(\rho^{-1}) \cdot \nabla w + \rho^{-1} \Delta w$  we have

$$\begin{aligned} \|\rho^{-1} w\|_{L^2(0, T; H^1(\Omega))} &\leq C \|\nabla(\rho^{-1} w)\|_{L^2(0, T; L^2(\Omega))} \\ &\leq C \left( s \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} \right), \end{aligned}$$

$$\begin{aligned} \|\nabla(\rho^{-1} w)\|_{L^2(0, T; H^1(\Omega))} &\leq C \|\Delta(\rho^{-1} w)\|_{L^2(0, T; L^2(\Omega))} \\ &\leq C \left( s^2 \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + s \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} + \|\rho^{-1} \Delta w\|_{L^2(0, T; L^2(\Omega))} \right) \\ &\leq C \left( s^2 \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + s \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} + \|\rho^{-1} w_{tt}\|_{L^2(0, T; L^2(\Omega))} \right. \\ & \quad \left. + \|\rho y\|_{L^2(0, T; L^2(\Omega))} \right) \end{aligned}$$

and thus

$$\begin{aligned} \|\rho v\|_{L^2(0, T; H^{1/2}(\partial\Omega))} &\leq C \left( s^3 \|\rho^{-1} w\|_{L^2(0, T; L^2(\Omega))} + s^2 \|\rho^{-1} \nabla w\|_{L^2(0, T; L^2(\Omega))} + s \|\rho^{-1} w_{tt}\|_{L^2(0, T; L^2(\Omega))} \right. \\ & \quad \left. + s \|\rho y\|_{L^2(0, T; L^2(\Omega))} \right). \end{aligned}$$

1 Estimate (8) then rewrites :

$$\begin{aligned}
& s^{1/2} \|\rho^{-1}(0)w_t(0)\|_{L^2(\Omega)} + s^{1/2} \|\rho^{-1}(0)\nabla w(0)\|_{L^2(\Omega)} + s^{3/2} \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} \\
& + s^{1/2} \|\rho^{-1}w_t\|_{L^2(Q)} + s^{1/2} \|\rho^{-1}\nabla w\|_{L^2(Q)} + s^{3/2} \|\rho^{-1}w\|_{L^2(Q)} \\
& \leq C \left( \|\rho y\|_{L^2(Q)} + s^{-1/2} \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)} \right) \\
& \leq C \left( s^{-3/2} \|\rho B\|_{L^2(Q)} + s^{-1/2} \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-3/2} \|\rho(0)u_1\|_{L^2(\Omega)} \right)
\end{aligned} \tag{40}$$

2 while estimate (33) rewrites (using estimates (36)-(37))

$$\begin{aligned}
& s^{1/2} \|\rho^{-1}(0)w_{tt}(0)\|_{L^2(\Omega)} + s^{1/2} \|\rho^{-1}(0)\nabla w_t(0)\|_{L^2(\Omega)} + s^{3/2} \|\rho^{-1}(0)w_t(0)\|_{L^2(\Omega)} \\
& + s^{1/2} \|\rho^{-1}w_{tt}\|_{L^2(Q)} + s^{1/2} \|\rho^{-1}\nabla w_t\|_{L^2(Q)} + s^{3/2} \|\rho^{-1}w_t\|_{L^2(Q)} \\
& \leq C \left( s \|\rho y\|_{L^2(Q)} + \|\rho y_t\|_{L^2(Q)} + s^{1/2} \left\| \frac{\rho}{\eta \Psi^{1/2}} v \right\|_{L^2(\Sigma)} + s^{-1/2} \left\| \frac{\rho}{\eta \Psi^{1/2}} v_t \right\|_{L^2(\Sigma)} \right) \\
& \leq C s^{-1/2} \mathcal{A}(u_0, u_1, B).
\end{aligned}$$

with

$$\mathcal{A}(u_0, u_1, B) := \|\rho B\|_{L^2(Q)} + s \|\rho(0)u_0\|_{L^2(\Omega)} + \|\rho(0)u_1\|_{L^2(\Omega)} + \|\rho(0)\nabla u_0\|_{L^2(\Omega)}.$$

Since  $s \geq 1$ ,  $|(\rho^{-1})_t| = Cs|\rho^{-1}|$ ,  $|(\rho^{-1})_{tt}| = Cs^2|\rho^{-1}|$ ,  $|\nabla(\rho^{-1})| = Cs|\rho^{-1}|$ ,  $|\Delta(\rho^{-1})| = Cs^2|\rho^{-1}|$  and  $\rho^{-1}(0)\Delta w(0) = \rho^{-1}w_{tt}(0) - \rho(0)u_0$ , we deduce

$$\begin{aligned}
& \|\Delta(\rho^{-1}w)(0)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} \\
& \leq C \left( \|\rho^{-1}(0)w_{tt}(0)\|_{L^2(\Omega)} + \|\rho(0)u_0\|_{L^2(\Omega)} + s \|\rho^{-1}(0)\nabla w(0)\|_{L^2(\Omega)} + s^2 \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} \right. \\
& \quad \left. + s \|\rho^{-1}(0)w_t(0)\|_{L^2(\Omega)} + \|\rho^{-1}(0)\nabla w_t(0)\|_{L^2(\Omega)} \right) \\
& \leq C s^{-1} \mathcal{A}(u_0, u_1, B)
\end{aligned}$$

and thus

$$\|\rho v\|_{L^2(0,T;H^{1/2}(\partial\Omega))} \leq C \mathcal{A}(u_0, u_1, B).$$

Since  $w_{tt} - \Delta w = Lw$ ,  $w|_{\partial\Omega} = 0$ , we infer, using  $\rho^{-1} \in C^\infty(\bar{Q})$  and  $y = \rho^{-2}Lw$ , that

$$\begin{cases} (\rho^{-1}w)_{tt} - \Delta(\rho^{-1}w) = \tilde{B}_w := \rho y + 2\rho_t^{-1}w_t + \rho_{tt}^{-1}w - 2\nabla\rho^{-1} \cdot \nabla w - \Delta\rho^{-1}w, \\ (\rho^{-1}w)|_{\partial\Omega} = 0. \end{cases}$$

for which the standard estimates

$$\|(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} \leq C \left( \|\tilde{B}_w\|_{L^2(Q)} + \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} + \|(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} \right)$$

and

$$\begin{aligned}
& \|(\rho^{-1}w)_{tt}(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\Delta(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} \\
& \leq C \left( \|\tilde{B}_w\|_{H^1(0,T;L^2(\Omega))} + \|\Delta(\rho^{-1}w)(0)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} \right)
\end{aligned}$$

3 hold true, for all  $t \geq 0$ .

Using again that  $s \geq 1$ ,  $|(\rho^{-1})_t| = Cs|\rho^{-1}|$ ,  $|(\rho^{-1})_{tt}| = Cs^2|\rho^{-1}|$ ,  $|\nabla(\rho^{-1})| = Cs|\rho^{-1}|$  and  $|\Delta(\rho^{-1})| = Cs^2|\rho^{-1}|$  we have the following estimates :

$$\begin{aligned}
& \|\tilde{B}_w\|_{L^2(Q)} \leq C \left( \|\rho y\|_{L^2(Q)} + s \|\rho^{-1}w_t\|_{L^2(Q)} + s^2 \|\rho^{-1}w\|_{L^2(Q)} + s \|\rho^{-1}\nabla w\|_{L^2(Q)} \right) \\
& \leq C \left( s^{-1} \|\rho B\|_{L^2(Q)} + \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1} \|\rho(0)u_1\|_{L^2(\Omega)} \right),
\end{aligned}$$

$$\begin{aligned} \|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} + \|(\rho^{-1}w)_t(0)\|_{L^2(\Omega)} &\leq C\left(s\|\rho^{-1}(0)w(0)\|_{L^2(\Omega)} + \|\rho^{-1}w_t(0)\|_{L^2(\Omega)}\right) \\ &\leq C\left(s^{-2}\|\rho B\|_{L^2(Q)} + s^{-1}\|\rho(0)u_0\|_{L^2(\Omega)} + s^{-2}\|\rho(0)u_1\|_{L^2(\Omega)}\right) \end{aligned}$$

1 and therefore

$$\|(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)(t)\|_{L^2(\Omega)} \leq C\left(s^{-1}\|\rho B\|_{L^2(Q)} + \|\rho(0)u_0\|_{L^2(\Omega)} + s^{-1}\|\rho(0)u_1\|_{L^2(\Omega)}\right).$$

We also have

$$(\tilde{B}_w)_t = \rho_t y + \rho y_t + 3\rho_{tt}^{-1}w_t + 2\rho_t^{-1}w_{tt} + \rho_{ttt}^{-1}w - 2\nabla\rho_t^{-1} \cdot \nabla w - 2\nabla\rho^{-1} \cdot \nabla w_t - \Delta\rho_t^{-1}w - \Delta\rho^{-1}w_t$$

2 and thus

$$\begin{aligned} \|(\tilde{B}_w)_t\|_{L^2(Q)} &\leq C\left(s\|\rho y\|_{L^2(Q)} + \|\rho y_t\|_{L^2(Q)} + s\|\rho^{-1}w_{tt}\|_{L^2(Q)} + s^2\|\rho^{-1}w_t\|_{L^2(Q)}\right. \\ &\quad \left.+ s^3\|\rho^{-1}w\|_{L^2(Q)} + s\|\rho^{-1}\nabla w_t\|_{L^2(Q)} + s^2\|\rho^{-1}\nabla w\|_{L^2(Q)}\right) \\ &\leq C\mathcal{A}(u_0, u_1, B) \end{aligned}$$

3 and then

$$\|(\rho^{-1}w)_{tt}(t)\|_{L^2(\Omega)} + \|\nabla(\rho^{-1}w)_t(t)\|_{L^2(\Omega)} + \|\Delta(\rho^{-1}w)(t)\|_{L^2(\Omega)} \leq C\mathcal{A}(u_0, u_1, B).$$

4 (39) then reads  $\|\rho v\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))} \leq Cs\mathcal{A}(u_0, u_1, B)$  which is a part of (16).

5 Eventually, from (2), we get that

$$\begin{cases} (\rho y)_{tt} - \Delta(\rho y) = \tilde{B}_y := \rho B + 2\rho_t y_t + \rho_{tt} y - 2\nabla\rho \cdot \nabla y - \Delta\rho y & \text{in } Q, \\ \rho y|_\Sigma = \rho v \\ ((\rho y)(\cdot, 0), (\rho y)_t(\cdot, 0)) = (\rho(0)u_0, \rho(0)u_1 + \rho_t(0)u_0) & \text{in } \Omega, \end{cases}$$

for which we have the standard estimate

$$\begin{aligned} \|(\rho y)_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\rho y)\|_{L^\infty(0,T;L^2(\Omega))} &\leq C\left(\|\tilde{B}_y\|_{L^2(Q)} + \|(\rho v)_t\|_{L^2(\Sigma)} + \|\rho v\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))}\right. \\ &\quad \left.+ \|\rho(0)u_0\|_{H^1(\Omega)} + \|\rho(0)u_1 + \rho_t(0)u_0\|_{L^2(\Omega)}\right). \end{aligned}$$

But

$$\begin{aligned} \|\tilde{B}_y\|_{L^2(Q)} &\leq C\left(\|\rho B\|_{L^2(Q)} + s^2\|\rho y\|_{L^2(Q)} + s\|\rho y_t\|_{L^2(Q)} + s\|\nabla(\rho y)\|_{L^2(Q)}\right) \\ &\leq Cs^{1/2}\left(\mathcal{A}(u_0, u_1, B) + s^{1/2}\|\nabla(\rho y)\|_{L^2(Q)}\right), \end{aligned}$$

$$\begin{aligned} \|\nabla(\rho y)\|_{L^2(Q)} &\leq C\left(\|\rho(0)u_0\|_{L^2(\Omega)} + \|\rho(0)u_1 + \rho_t(0)u_0\|_{L^2(\Omega)} + \|(\rho y)_t\|_{L^2(Q)} + \|(\rho v)_t\|_{L^2(\Sigma)}\right. \\ &\quad \left.+ \|\rho v\|_{L^2(0,T;H^{1/2}(\partial\Omega))} + \|\tilde{B}_y\|_{L^2(0,T;H^{-1}(\Omega))}\right) \end{aligned}$$

and

$$\begin{aligned} \|\tilde{B}_y\|_{L^2(0,T;H^{-1}(\Omega))} &\leq C\left(\|\rho B\|_{L^2(Q)} + s^2\|\rho y\|_{L^2(Q)} + s\|\rho y_t\|_{L^2(Q)}\right) \\ &\leq Cs^{1/2}\mathcal{A}(u_0, u_1, B) \end{aligned}$$

leading to  $\|\nabla(\rho y)\|_{L^2(Q)} \leq Cs^{1/2}\mathcal{A}(u_0, u_1, B)$  that is the estimate of the third term in (16) and  $\|\tilde{B}_y\|_{L^2(Q)} \leq Cs^{3/2}\mathcal{A}(u_0, u_1, B)$ . We then deduce that

$$\|(\rho y)_t\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(\rho y)\|_{L^\infty(0,T;L^2(\Omega))} \leq Cs^{3/2}\mathcal{A}(u_0, u_1, B)$$

6 that is, the estimate of the last two terms in (16).

7 **Step 4.** Case where  $B \in L^2(Q)$  and  $(u_0, u_1) \in \mathbf{V}$ . We proceed by density as in [3, Step 3, page 171].

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