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► **To cite this version:**

Cyril Letrouit, Simon Machado. Maximal multiplicity of Laplacian eigenvalues in negatively curved surfaces. 2023. hal-04160247v1

HAL Id: hal-04160247

<https://hal.science/hal-04160247v1>

Preprint submitted on 12 Jul 2023 (v1), last revised 22 May 2024 (v3)

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Maximal multiplicity of Laplacian eigenvalues in negatively curved surfaces

Cyril Letrouit* and Simon Machado†

July 12, 2023

Abstract

We obtain a general sublinear upper bound on the multiplicity of Laplacian eigenvalues for negatively curved surfaces. In particular, this yields progress on a longstanding conjecture by Colin de Verdière [Col86]. Our proof relies on a trace argument for the heat kernel, and on the idea of leveraging an r -net in the surface to control this trace. This last idea was introduced in [JTYZZ21, Theorem 2.2] for similar spectral purposes in the context of graphs of bounded degree. Our method is robust enough to also yield an upper bound on the “approximate multiplicity” of eigenvalues, i.e., the number of eigenvalues in windows of size $1/\log^\kappa(g)$, $\kappa > 0$.

1 Introduction

1.1 Main results

Let M be a closed, connected Riemannian manifold, and let Δ denote the Laplace-Beltrami operator on M , which is self-adjoint and non-positive. The operator $-\Delta$ has a discrete spectrum

$$0 = \lambda_1(M) < \lambda_2(M) \leq \dots \rightarrow +\infty, \quad (1)$$

where the $\lambda_i(M)$ are repeated according to their multiplicity.

In the present paper we consider the case where M is a closed negatively curved surface. We denote by \mathcal{T} the set of triples

$$\mathcal{T} = \{(a, b, \rho) \in \mathbb{R}^3 \mid b \leq a < 0, \rho > 0\}.$$

For any $(a, b, \rho) \in \mathcal{T}$, let $\mathcal{M}_g^{(a,b,\rho)}$ be the set of closed connected surfaces of genus g , with injectivity radius $\geq \rho$, and with Gaussian curvature bounded below by b and above by a . An important example is obtained by taking $(a, b, \rho) = (-1, -1, \rho)$, in which case $\mathcal{M}_g^{(a,b,\rho)}$ is the set of hyperbolic surfaces (i.e., with constant curvature -1) of injectivity radius $\geq \rho$.

In this paper, we obtain general sublinear upper bounds on the maximal multiplicity of $\lambda_2(M)$ for negatively curved surfaces. Our first main result is the following:

Theorem 1.1 (Maximal multiplicity of λ_2). *For any $(a, b, \rho) \in \mathcal{T}$, there exists $C_0 > 0$ such that for any $g \geq 2$ and any $M \in \mathcal{M}_g^{(a,b,\rho)}$, the multiplicity of $\lambda_2(M)$ is at most $C_0 \frac{g}{\log(\log(g))}$.*

For any $(a, b, \rho) \in \mathcal{T}$ and $\delta > 0$, there exist $C_0, \alpha > 0$ such that for any $g \geq 2$ and any $M \in \mathcal{M}_g^{(a,b,\rho)}$ with spectral gap $\lambda_2(M) \geq \delta$, the multiplicity of $\lambda_2(M)$ is at most $C_0 \frac{g}{\log(g)^\alpha}$.

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Our strategy of proof partly relies on a geometric idea which takes its source in [JTYZZ21]. This last work proves the same sublinear bound as ours, for the adjacency matrix of combinatorial graphs with a uniform bound on the degree.

Our next statement is stronger than Theorem 1.1, in the sense that it accommodates for “approximate multiplicity” in a window of size $O(1/\log^\kappa(g))$, $\kappa > 0$ (see Remark 4.2 for comments on the size of this window). This result parallels a similar statement [HSZZ22, Theorem 1.6] for graphs with a uniform bound on the degree of each vertex.

Theorem 1.2. *For any $j \in \mathbb{N}_{\geq 2}$, any $(a, b, \rho) \in \mathcal{T}$, and any $\kappa, K > 0$, there exists $C_0 > 0$ such that the number of eigenvalues in $[\lambda_j(M), (1 + \frac{K}{\log^\kappa(g)})\lambda_j(M)]$ is at most $C_0 \frac{g}{\log(\log(g))}$ for any $g \geq 2$ and any $M \in \mathcal{M}_g^{(a,b,\rho)}$.*

Following an analogous result on regular graphs [MRS21, Proposition 5.3], we also provide a construction of closed *hyperbolic* surfaces with high approximate multiplicity. This result shows that only little improvement is possible over Theorem 1.2 for bounds on approximate multiplicity, at least if the injectivity radius is allowed to tend to 0 (indeed, we warn the reader that the family of closed hyperbolic surfaces constructed in Proposition 1.3 has injectivity radius tending to 0 as $g \rightarrow +\infty$ at a rate which we do not control).

Proposition 1.3. *There exist $C_0, K > 0$ and a family of connected closed hyperbolic surfaces M of genus $g \rightarrow +\infty$, with at least $C_0 \frac{g}{\log^{3/2}(g)}$ eigenvalues in $[\lambda_2(M), (1 + K \frac{\log(\log(g))}{\log(g)})\lambda_2(M)]$.*

1.2 Bibliographical comments

The maximal multiplicity of Laplacian eigenvalues has been studied at least since the 1970’s and a seminal paper of Cheng [Che76]. We review the literature, focusing on the case of surfaces since on any closed manifold M of dimension $n \geq 3$, it is possible to construct a sequence of metrics whose first (non-trivial) eigenvalue multiplicity tends to $+\infty$ (see [Col86]). For M a closed surface of genus g , let $m_i(M)$ denote the maximal multiplicity of the i -th eigenvalue of a Riemannian Laplacian on M (with the convention (1) on indexing of eigenvalues).

Linear bounds. Cheng proved in [Che76] that $m_i(M) \leq \frac{1}{2}(2g+i)(2g+i+1)$. This result has been improved by Besson in [Bes80] who sharpened the bound down to $4g+2i-1$. Both papers proceed by bounding the order of vanishing of eigenfunctions and obtaining a contradiction if an eigenspace is too large (see also [SY94, Section III.6]). Then, Sévenec [Sev02] proved that in negative Euler characteristic, $m_2(M) \leq 5 - \chi(M)$; in particular, if M is orientable of genus $g \geq 2$, then $m_2(M) \leq 2g+3$. This bound has been improved to $2g-1$ for closed hyperbolic surfaces of sufficiently high genus in [FP23, Theorem 9.5].

Colin de Verdière’s conjecture. Colin de Verdière conjectures in [Col86, Section V] a much stronger bound of order \sqrt{g} for the maximal multiplicity. More precisely, he conjectures that

$$m_2(M) = \text{chr}(M) - 1 \tag{2}$$

where $\text{chr}(M)$ is the chromatic number of M , defined as the largest n such that the complete graph on n vertices embeds in M . By a result of Ringel and Youngs [RY68],

$$\text{chr}(M) = \left\lfloor \frac{1}{2} \left(7 + \sqrt{49 - 24\chi(M)} \right) \right\rfloor,$$

and since $\chi(M) = 2 - 2g$ for closed orientable surfaces, $m_2(g)$ would be of order $\sqrt{12g}$. The conjecture (2) has been verified for the sphere [Che76], the torus [Bes80], the projective plane

[Bes80], the Klein bottle [Col87], [Nad87]. The work [FP21] shows that the Klein quartic maximizes the multiplicity of λ_2 among all closed hyperbolic surfaces of genus 3, with multiplicity equal to 8, which also matches the conjecture (2). It is also proved in [Col87, Théorème 1.5] that $\overline{m}_2(M) \geq \text{chr}(M) - 1$ if $\overline{m}_2(M)$ denotes the maximal multiplicity of λ_2 of a Schrödinger operator on M for which $\lambda_1 = 0$.

Colbois and Colin de Verdière constructed in [CC88] for any $g \geq 3$ a closed hyperbolic surface of genus g such that the multiplicity of λ_2 is $\left\lfloor \frac{1+\sqrt{8g+1}}{2} \right\rfloor$, which has the same order of growth as the conjectured upper bound (2).

Sublinear bounds. Despite the difficulty to prove Colin de Verdière’s conjecture in full generality, sublinear bounds have been obtained on the multiplicity of eigenvalues on surfaces. However these bounds work only for *hyperbolic* surfaces and assume some control over the number of closed geodesics of length $\leq L$ for any large L independent of g , or for $L = c \log(g)$ where $c > 0$ is a small constant. This control is related to the notion of *Benjamini-Schramm convergence* (see [ABBGNS17]).

A quantitative control over the number of short closed geodesics, and thus over the asymptotics of multiplicities, holds with high probability for sequences of hyperbolic surfaces drawn with respect to Weil–Petersson probability measures. Monk [Mon22] constructs a set \mathcal{A}_g of closed hyperbolic surfaces M of high probability for which she obtains through Selberg’s trace formula a Weyl-type estimate and a sublinear upper bound of size $O(g/\sqrt{\log g})$ on the multiplicity of any eigenvalue. The set \mathcal{A}_g contains surfaces which converge in the sense of Benjamini-Schramm to the hyperbolic plane \mathbb{H}^2 . Another probabilistic sublinear upper bound has been proved in [GLST21, Corollary 1.7], under assumptions which are different but related to the ones of [Mon22]. We also mention [FP23, Proposition 9.3] which proves a sublinear bound when $\lambda_2(M) - \frac{1}{4}$ is of order $1/\log(g)^2$.

The above works are based either on the Selberg transform or on the Selberg trace formula which relates, in closed hyperbolic surfaces, the spectrum of the Laplacian to the set of lengths of closed geodesics. Our method, which works for general negatively curved surfaces and does not assume a control over the number of closed geodesics of length $\leq L$ for large L , is totally different. It relies mainly on heat kernel estimates, which correspond geometrically to random walks and not to closed geodesics. Heat kernel estimates by themselves are not sufficient; we need for our proof another ingredient inspired from the work [JTYZZ21] pertaining to multiplicities in graphs (see Section 1.3).

The work [ABBGNS17] shows sublinear bounds for the related problem of limit multiplicities under Benjamini–Schramm convergence. Namely, under Benjamini–Schramm convergence of a sequence of surfaces $(M_n)_{n \geq 0}$ to \mathbb{H}^2 with injectivity radius bounded below, and for a fixed eigenvalue $\lambda < \frac{1}{4}$, there holds $\frac{m(M_n, \lambda)}{\text{vol}(M_n)} \rightarrow m(\mathbb{H}^2, \lambda) = 0$. Under more restrictive arithmetic assumptions, precise rates of convergence - with power saving - can be established, see [DW78], [SX91], [Gam02].

We mention the fact that our results yield a sublinear bound on multiplicity when restricting to the set of Riemannian covers of a fixed negatively curved manifold. Also, when $a = b = -1$ and ρ is small, we see that the set of hyperbolic surfaces considered in Theorem 1.1 covers most of the moduli space of closed hyperbolic surfaces of genus $g \geq 2$ since the event of having injectivity radius $\geq \rho$ has probability roughly $1 - \rho^2$ for the Weil-Petersson probability measure (see [MP19, Theorem 4.1]).

Literature on graphs. As already mentioned in Section 1.1, our inspiration comes from the following result proved in [JTYZZ21, Theorem 2.2]:

Theorem 1.4 ([JTYZZ21]). *For every j and every d , there is a constant $C = C(d, j)$ so that the adjacency matrix of every connected n -vertex graph with maximum degree at most d has j -th eigenvalue multiplicity at most $Cn/\log \log n$.*

The main motivation of the authors of [JTYZZ21] is the equiangular problem, namely the computation of the maximal number of lines in \mathbb{R}^d which are pairwise separated by the same angle. This problem shows up for instance through tight frames in coding theory. In [JTYZZ21, Theorem 1.2], the equiangular problem for a fixed angle α between the lines is solved by showing that it may be reduced to Theorem 1.4.

We also mention the work [MRS21], in which an improvement of [JTYZZ21, Theorem 2.2] is proven for regular graphs. This improvement does not seem easy to transfer to (negatively curved) surfaces. The work [MRS21] also gives an explicit construction of regular graphs with low degree for which the first non-trivial eigenvalue has large approximate multiplicity. We use this construction to prove Proposition 1.3. Examples with even higher approximate multiplicity have been constructed in [HSZZ22], but for the adjacency matrix of irregular graphs and not for the Laplacian on graphs.

1.3 Strategy of proof

1.3.1 Warm-up: proof in the graph case

Our strategy to prove Theorems 1.1 and 1.2 is partly inspired by the proof of Theorem 1.4 worked out in [JTYZZ21]. We provide here a summary of this proof.

Let $d > 0$ and let G be a graph with degree $\leq d$, whose adjacency matrix is denoted by A_G . The authors of [JTYZZ21] introduce a subgraph $H \subset G$ whose complement $G \setminus H$ is an r_1 -net: it means that any vertex of G is at distance at most r_1 from $G \setminus H$. The parameter r_1 is chosen as $r_1 = \lfloor c \log \log(n) \rfloor$, where n is the number of vertices of G and $c > 0$ is a small constant.

The first step is to find an upper bound for the trace of $A_H^{2r_1}$, where A_H is the adjacency matrix of H . For this, the authors of [JTYZZ21] leverage the usual technique of expressing a trace as a number of closed paths. The trace $\text{Tr}(A_H^{2r_1})$ is bounded above by the number of paths of length $2r_1$ in G , which start from a given vertex $x \in H$ and do not belong to the r_1 -net $G \setminus H$ at time r_1 . It follows from the definition of an r_1 -net that this number is smaller by at least 1 than the total number of paths of length $2r_1$ in G which start from x and end at x : we call this the “gain of 1”.

This gain of 1 is transformed into a larger gain by considering the trace of $A_H^{2r_2}$ with $r_2 = \lfloor c \log(n) \rfloor \gg r_1$, instead of the trace of $A_H^{2r_1}$. The argument to get this larger gain relies on the Perron-Frobenius theorem and the min-max principle applied locally in balls of radius r_2 . The large gain which is obtained provides a strong bound on $\text{Tr}(A_H^{2r_2})$, and thus on the number of eigenvalues of A_H close to the largest non-trivial one $\lambda_2(A_H) < 1$ (largest after $\lambda_1(A_H) = 1$).

Finally, the Cauchy interlacing theorem (Theorem A.1) converts this bound into a similar bound on the eigenvalues of A_G . The bounds depend on d .

1.3.2 Main steps

The main steps of our proof of Theorem 1.1 mimic the above proof, with many additional difficulties and several new ideas. The proof of Theorem 1.2 is an extension of the arguments provided below.

1. We consider, instead of directly $\lambda_2 = \lambda_2(M)$, the maximal multiplicity of $e^{-t\lambda_2(M)}$ as an eigenvalue of $e^{t\Delta}$, thus reinterpreting the problem in terms of heat kernels (and random walks).

2. In analogy with the graph case, we set $r_1 = c \log \log(g)$ where g is the genus of M and $c > 0$ is a small constant. We choose an r_1 -net $\{x_1, \dots, x_\ell\} \subset M$: this means that any point in M is at distance at most r_1 from one of the x_k 's. Then we fix around each x_k a small open set V_k of measure ~ 1 . We define the operator $P : L^2(M, \nu) \rightarrow L^2(M, \nu)$ as the orthogonal projection to the space of functions which are L^2 -orthogonal to the (normalized) characteristic functions of the V_k 's.
3. We use a Cauchy interlacing theorem in Hilbert spaces (see Theorem A.1): we compare the multiplicity m of $e^{-r_1 \lambda_2}$ as an eigenvalue of $e^{r_1 \Delta}$ with the multiplicity m' of $e^{-r_1 \lambda_2}$ as an eigenvalue of $P e^{r_1 \Delta} P$.

The Cauchy interlacing theorem implies that

$$m \leq m' + \text{rank}(\text{Id} - P). \quad (3)$$

Our choice of P guarantees that $\text{rank}(\text{Id} - P) = O(g / \log(\log(g)))$, or even $\text{rank}(\text{Id} - P) = O(g / \log^\alpha(g))$ when we work under the assumption $\lambda_2(M) \geq \delta$ in the second part of Theorem 1.1. The next steps prove an upper bound on m' .

4. We choose $r_2 = c \log(g)$ and $n \approx \lfloor r_2 / r_1 \rfloor$ and we compute the trace of $(P e^{r_1 \Delta} P)^{2n}$ to bound above m' :

$$m' e^{-2nr_1 \lambda_2} \leq \text{Tr}((P e^{r_1 \Delta} P)^{2n}). \quad (4)$$

The trace in the right-hand side may in turn be written as an integral of the form

$$\text{Tr}((P e^{r_1 \Delta} P)^{2n}) = \int_M \|(P e^{r_1 \Delta} P)^n \delta_x\|^2 d\nu(x). \quad (5)$$

5. We leverage the averaging properties of the heat kernel to prove an inequality which roughly looks like¹

$$\|(P e^{r_1 \Delta} P)^{\lfloor r_2 / r_1 \rfloor} \delta_x\| \leq \left((1 - \varepsilon(g)) e^{-r_1 \lambda_2} \right)^{\lfloor r_2 / r_1 \rfloor} \quad (6)$$

for “most points” $x \in M$. The assumption on the injectivity radius in Theorem 1.1 comes from the proof of (6), but also from the construction of the r_1 -net in Step 2.

Combining (4), (5), (6) with the Gauss-Bonnet formula we obtain for some $C_0 > 0$

$$m' \leq C_0 g (1 - \varepsilon(g))^{2\lfloor r_2 / r_1 \rfloor} \leq C_0 g \exp(-2\varepsilon(g) \lfloor r_2 / r_1 \rfloor) \quad (7)$$

The quantity $\varepsilon(g) > 0$, which depends on the genus g , is the “gain”, and we prove it to be sufficiently large, so that Theorem 1.1 follows from (3), (7) and our choices of r_1, r_2, P .

The proof of Step 5 is the heart of our contribution, and a more detailed summary of this step is provided at the beginning of Section 3.3, before its actual proof. Whereas the “gain” is straightforward to obtain for graphs (see Section 1.3.1), we have to face in the case of surfaces several difficulties.

A first difficulty comes from the infinite speed of propagation of the heat kernel². This property a priori prevents us from using any local argument in the manifold; however, as mentioned in Section 1.3.1, we need to apply the min-max principle locally in balls of radius $\approx r_2$

¹Here we warn the reader that the sequence of inequalities we prove is actually much more subtle than (6).

²Although there exists a “random walk at speed 1” on manifolds (see for instance [LM10]), whose kernel is the most obvious analogue of the adjacency matrix A_G of Section 1.3.1, we use in this paper the heat kernel because it seems more natural to understand the Laplacian on manifolds, and because the bounds available on the kernel of the “random walk at speed 1” are not as good as the ones available on the heat kernel.

to obtain the quantitative gain $\varepsilon(g)$. To overcome this difficulty we introduce some cut-offs χ_x (approximately the characteristic function of a ball of center x and radius $C'r_2$ for some large C') commuting with P , and consider the compact operators $B_x = P\chi_x e^{r_1\Delta}\chi_x P$ instead of $Pe^{r_1\Delta}P$ in (6). The remainder terms which unavoidably appear when replacing $Pe^{r_1\Delta}P$ by B_x are handled through classical heat kernel estimates in the universal cover of M .

Another difficulty arises from the fact that the operator B_x , which somehow plays locally around x the role of $A_H^{r_1}$ in Section 1.3.1, has one main difference with $A_H^{r_1}$: its matrix elements are not necessarily non-negative (the condition $f, g \geq 0$ does not imply that $(B_x f, g) \geq 0$), and the Perron-Frobenius theorem therefore does not apply to B_x ; however, as mentioned in Section 1.3.1, we do need to apply a Perron-Frobenius-type argument in local balls. We overcome this difficulty by analyzing the interplay between the positive and the negative part of the top eigenvector φ_x of B_x . This allows us to recover a gain $\varepsilon(g)$ despite the lack of positivity (in the sense of matrix elements) of B_x .

Organization of the paper. The paper is organized as follows. We introduce useful notation in Section 2.1, and we prove elementary results regarding r -nets in Section 2.2. In Section 2.3 we state estimates on the heat kernel in M and its universal cover \tilde{M} . Section 3 gathers the key lemmas used in the proof of Theorems 1.1 and 1.2: in Section 3.1 we compare the trace $\text{Tr}((Pe^{r_1\Delta}P)^n)$ to an integral of local Rayleigh quotients and we estimate the error terms; in Section 3.2 we draw several consequences from the min-max principle used to bound the previously mentioned Rayleigh quotients; in Section 3.3, we prove the gain described in Step 5 above. In Section 4.1, we proceed with the proof of Theorem 1.1, and in Section 4.2 we explain how to modify this proof to obtain Theorem 1.2. In Section 4.3 we prove Proposition 1.3, relying on constructions in [CC88] and [MRS21]. In Appendix A.1, we gather several elementary results such as the Cauchy interlacing theorem in infinite dimensional Hilbert spaces and an upper bound for the first eigenvalues of closed negatively curved surfaces. Finally in Section A.2 we prove the heat kernel estimates stated in Section 2.3.

Acknowledgment. The authors are thankful to Laura Monk and Yufei Zhao for answering questions related to this work. Part of this work was done while C.L. was supported by the Simons Foundation Grant 601948, DJ. S.M. was supported by the National Science Foundation under Grant No. DMS-1926686.

2 Preliminaries

This section gathers notation and elementary results concerning Voronoi cells in closed negatively curved surfaces, r -nets and the heat kernel.

2.1 Notation

We fix $(a, b, \rho) \in \mathcal{T}$ and $M \in \mathcal{M}_g^{(a, b, \rho)}$ for some $g \geq 3$. The Riemannian distance in M is denoted by $d(\cdot, \cdot)$ and the open ball of center $x \in M$ and radius $r > 0$ is $B_d(x, r)$.

We denote by ν the Riemannian volume on M , by $\langle \cdot, \cdot \rangle$ the scalar product with respect to ν , and by $\|\cdot\|$ the associated norm. We introduce

$$r_1 = c \log(\log(g)), \quad r_2 = c \log(g) \tag{8}$$

where $c > 0$ is a small constant which will be fixed in Section 4.1. Both r_1 and r_2 are positive since we assumed $g \geq 3$.

Let $(\widetilde{M}, d_{\widetilde{M}})$ be the universal cover of M endowed with the lifted Riemannian metric. By definition, \widetilde{M} is a Cartan-Hadamard manifold. Let $\text{Vol}_{\widetilde{M}}$ be the Riemannian volume on \widetilde{M} .

We recall that the volume of any ball of radius r in \widetilde{M} satisfies

$$\text{Vol}_{\widetilde{M}}(B(x, r)) \leq \frac{4\pi}{|b|} \sinh^2 \left(\frac{\sqrt{|b|}}{2} r \right) \quad (9)$$

according to the Bishop-Gromov inequality.

Heat kernels. We denote by $k_t : \widetilde{M} \times \widetilde{M} \rightarrow \mathbb{R}$ the heat kernel in \widetilde{M} , so that for any $f \in L^2(\widetilde{M}, d\text{Vol}_{\widetilde{M}})$, the solution $u : \mathbb{R}^+ \times \widetilde{M} \rightarrow \mathbb{R}$ of $\partial_t u = \Delta u$ with initial datum $u(0, \cdot) = f$ is given by

$$u(t, x) = \int_{\widetilde{M}} k_t(x, y) f(y) d\text{Vol}_{\widetilde{M}}(y).$$

By a slight abuse, we use the same notation k_t (with only one argument) for the function $k_t : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $k_t(d_{\widetilde{M}}(x, y)) = k_t(x, y)$ for any $x, y \in \widetilde{M}$. This definition makes sense since $k_t(x, y)$ only depends on $d_{\widetilde{M}}(x, y)$.

The linear operator e^{Δ} is compact, self-adjoint in $L^2(M, \nu)$, with norm 1. We denote by $K_t(x, y)$ the heat kernel on M , i.e., it satisfies for any $t \geq 0$

$$e^{t\Delta} f(x) = \int_M K_t(x, y) f(y) \nu(dy).$$

We have $e^{t\Delta} \delta_x = K_t(x, \cdot) \in L^2(M, \nu)$ for any $t > 0$ and $x \in M$. Writing $M = \Gamma \backslash \widetilde{M}$, we have the formula

$$K_t(x, y) = \sum_{\gamma \in \Gamma} k_t(\bar{x}, \gamma \bar{y}) \quad (10)$$

where \bar{x}, \bar{y} are lifts of x, y in a fixed fundamental domain of M in \widetilde{M} .

By rescaling the Riemannian metric on \mathbb{H}^2 , we create a space \widetilde{M}_K of constant Gaussian curvature $K < 0$, which is a simply connected space form. The universal cover of a closed surface with constant Gaussian curvature K is isometric to \widetilde{M}_K .

The following result is proved in [DGM76, Théorème 1].

Lemma 2.1 (Comparison for the heat kernel). *Let \widetilde{M} be a complete and simply connected Riemannian manifold with associated distance $d_{\widetilde{M}}$ and heat kernel $k_t(\cdot, \cdot)$. Assume that its sectional curvature is bounded above by a and below by b . Then for any $x, x_0 \in \widetilde{M}$ and any $t > 0$,*

$$k_t^{(b)}(d_{\widetilde{M}}(x_0, x)) \leq k_t(x_0, x) \leq k_t^{(a)}(d_{\widetilde{M}}(x_0, x))$$

where $k_t^{(K)}(\cdot)$ denotes the heat kernel (with radial variable) on \widetilde{M}_K .

Constants. Throughout the paper we use the following conventions to denote constants:

- we keep the same notation for constants which may change from line to line.
- Constants with an integer subscript, namely C_0, C_1, \dots , depend on a, b and ρ only.
- $C > 0$ and $C' > 0$ denote two sufficiently large constants whose values are fixed in the proof of Theorem 1.1, in (45)-(47). C and C' are introduced respectively in Lemma 2.4 and at the beginning of Section 3.1.
- The constant $c > 0$ introduced in (8) is fixed in (48) (chosen sufficiently small).

2.2 r -nets and Voronoi cells

As explained in Section 1.3, we need for our proof to consider a subset V of relatively small measure such that any point of M is at distance $\leq r$ of V . In the context of graphs, e.g., in [JTYZZ21], such sets, called r -nets, are subsets of the sets of vertices. In the case of a closed negatively curved surface M , we cut M into Voronoi cells and select a (not too large) subset of cells well distributed over M . This section gathers the necessary definitions and results.

In the sequel, an r -separated set is a set of points $x_1, \dots, x_m \in M$ such that $d(x_i, x_j) \geq r$ for any distinct $i, j \in \{1, \dots, m\}$. An r -net is a set of points $x_1, \dots, x_m \in M$ such that for any $y \in M$ there exists $i \in \mathcal{N}$ such that $d(y, x_i) \leq r$.

The following two lemmas prove the existence of r -nets whose size is not too large.

Lemma 2.2. *For any $(a, b, \rho) \in \mathcal{T}$, there exists $C_0 > 0$ such that for any $g \geq 2$, any $M \in \mathcal{M}_g^{(a, b, \rho)}$ and any $r \geq 1$, there exists an r -net in M of cardinal at most $\max(1, C_0 g/r)$.*

Proof. Let $(a, b, \rho) \in \mathcal{T}$. If $\text{diam}(M) \leq r$, then there is an r -net of size 1. So suppose that $\text{diam}(M) > r$. Let $x_1, \dots, x_\ell \in M$ be an r -separated set of maximal cardinality in M . Then $X := \{x_1, \dots, x_\ell\}$ is an r -net of M and $B_d(x_i, r/2) \cap B_d(x_j, r/2) = \emptyset$ for all $i \neq j$. So

$$\ell \min_{i \in \{1, \dots, \ell\}} \text{vol}(B_d(x_i, r/2)) \leq \sum_{i=1}^{\ell} \text{vol}(B_d(x_i, r/2)) \leq \text{vol}(M) \leq C_0 g \quad (11)$$

where the last inequality comes from the Gauss-Bonnet formula. It thus suffices to show that for any $x \in M$, $B_d(x, r/2)$ has volume at least $C_1 r$ for some constant $C_1 > 0$ (depending on $(a, b, \rho) \in \mathcal{T}$ only). Fix $x \in M$. Take any $y \in M$ such that $d(x, y) \geq r/2$ - there must be at least one such y since $\text{diam}(M) > r$ - and let $\gamma : [0; r/2] \rightarrow M$ be a continuous path of minimal length from x to y . Then the balls $B_d(\gamma(n), 1/2)$ for $n \in \{0, \dots, \lfloor (r-1)/2 \rfloor\}$ are pairwise disjoint and contained in $B_d(x, r/2)$. There is in addition a constant $C_2 > 0$ such that $\text{vol}(B_d(z, 1/2)) \geq C_2$ for all $z \in M$ (due to [Cro80, Proposition 14]), and we use this for $z = \gamma(n)$, $n \in \{0, \dots, \lfloor (r-1)/2 \rfloor\}$. So

$$\text{vol}(B_d(x, r/2)) \geq \left(\left\lfloor \frac{r-1}{2} \right\rfloor + 1 \right) C_2. \quad (12)$$

This concludes the proof of the lemma. \square

Lemma 2.3. *For any $(a, b, \rho) \in \mathcal{T}$ and $\delta > 0$ there exist $C_0, \delta' > 0$ such that for any $g \geq 2$, any $M \in \mathcal{M}_g^{(a, b, \rho)}$ with spectral gap $\lambda_2(M) \geq \delta$, and any r such that $1 \leq r \leq \frac{1}{\sqrt{|b|}} \log(g/4)$, there exists an r -net in M of cardinal at most $\max(1, C_0 g/e^{\delta' r})$.*

Proof. By the Buser inequality [Bus82], the Cheeger constant $h(M)$ verifies $\delta \leq \lambda_2(M) \leq 2h(M)\sqrt{|b|} + 10h(M)^2$, hence $h(M) \geq \delta'$ where δ' depends only on b and δ . Besides, for any $x \in M$ and any r as in the statement, we have

$$\text{vol}(B_d(x, r)) \leq \text{Vol}_{\widetilde{M}}(B_{\widetilde{M}}(r)) \leq \frac{4\pi}{|b|} \sinh\left(\sqrt{|b|} \frac{r}{2}\right)^2 \leq \frac{\pi}{|b|} g \leq \text{vol}(M)/2$$

where the second inequality comes from (9), the third one from our assumption on r , and the last one from the Gauss-Bonnet formula. Hence by definition of the Cheeger constant,

$$\frac{d\text{vol}(B_d(x, r))}{dr} = \lim_{\varepsilon \rightarrow 0} \frac{\text{vol}(B_d(x, r + \varepsilon)) - \text{vol}(B_d(x, r))}{\varepsilon} = |\partial B_d(x, r)| \geq \delta' \text{vol}(B_d(x, r))$$

which implies that $\text{vol}(B_d(x, r)) \geq C_1 e^{\delta' r}$ (where C_1 depends on ρ). Replacing in (11) and (12) we find the result. \square

We fix a 1-separated set $\mathcal{V} = \{v_1, \dots, v_m\}$ of maximal cardinality m in M , and we consider the Voronoi cells

$$V_k = \{q \in M \mid \forall i \in \{1, \dots, m\}, d(q, v_k) \leq d(q, v_i)\}, \quad k = 1, \dots, m.$$

By our choice of \mathcal{V} , there holds $B_d(v_k, 1/2) \subset V_k \subset B_d(v_k, 1)$ for any $k \in \{1, \dots, m\}$. We notice that if \tilde{v}_k denotes a lift of v_k to a fundamental domain of M in \widetilde{M} ,

$$C_1 \geq \text{Vol}_{\widetilde{M}}(B_{\widetilde{M}}(\tilde{v}_k, 1)) \geq \text{vol}(V_k) \geq C_0 \quad (13)$$

for some $C_0, C_1 > 0$ which depend only on $(a, b, \rho) \in \mathcal{T}$, where the leftmost inequality comes from (9), and the rightmost inequality from [Cro80, Proposition 14].

We fix a subset $\mathcal{N} \subset \{1, \dots, m\}$ with the following properties:

- $\#\mathcal{N} \leq C_0 g/r$ (or $\#\mathcal{N} \leq C_0 g/e^{\delta' r}$ if $\lambda_2(M) \geq \delta$ is assumed).
- For any $k \in \mathcal{N}$ there exists $x_k \in V_k$ such that the points $(x_k)_{k \in \mathcal{N}}$ form an r_1 -net.

The set \mathcal{N} is constructed by first considering an r_1 -net $\{x_1, \dots, x_\ell\}$, and then for each $j \in [\ell]$, putting in \mathcal{N} the index of (one of) the Voronoi cell(s) to which x_j belongs.

For any $k \in \mathcal{N}$, we denote by ψ_k the normalized characteristic function of the interior $\overset{\circ}{V}_k$ of V_k , i.e., $\psi_k(x) = \frac{1}{\sqrt{\text{vol}(V_k)}} \mathbf{1}_{x \in \overset{\circ}{V}_k}$. It follows from (13) that $\|\psi_k\|_{L^\infty(M)} \leq C_2$.

We denote by P the orthogonal projection³ on the orthogonal of the ψ_k with respect to ν :

$$\forall f \in L^2(M, \nu), \quad Pf = f - \sum_{k \in \mathcal{N}} \langle f, \psi_k \rangle \psi_k.$$

If δ_x denotes the Dirac mass on a manifold (defined as $\delta_x(f) = f(x)$) then $P\delta_x$ is a distribution, defined as

$$P\delta_x = \delta_x - \sum_{k \in \mathcal{N}} \psi_k(x) \psi_k. \quad (14)$$

2.3 Heat kernel estimates

The following lemmas on the heat kernels in \widetilde{M} and M (see definitions in Section 2.1) will be instrumental in the proofs of Theorems 1.1 and 1.2. The proofs of these lemmas are postponed to Appendix A.2.

Estimate (15) reflects the fact that the mass of k_t is concentrated in a ball of radius $\lesssim t$; thus its mass is small outside a ball of radius Ct for C sufficiently large. Then, the estimate (16) shows that when restricting to the interior of a ball of radius Ct , the heat kernel varies not too wildly over scales $\leq 4t$ (any other constant than 4 would also work, but 4 is the right constant for Lemma 3.8).

Lemma 2.4 (Estimates on the heat kernel in \widetilde{M}). *The following estimates hold:*

- (L^1 norm outside a ball of radius Ct). *There exists $C_0 > 0$ such that for any $C \geq 4|a|\sqrt{|b|} + 2$, $t \geq 1$ and $x \in \widetilde{M}$,*

$$\|k_t(x, \cdot)\|_{L^1(\widetilde{M} \setminus B_{\widetilde{M}}(x, Ct))} \leq C_0 \exp\left(|ab|t - \frac{C^2 t}{16|a|}\right). \quad (15)$$

³The idea of considering these projections is inspired by the paper [Bus77], which shows that in a closed hyperbolic surface M of genus g , the number of eigenvalues below $1/4$ is bounded above by $4g - 2$. The proof goes by considering a triangulation of M into $4g - 2$ geodesic triangles, and showing that in each of these triangles, the smallest positive eigenvalue of the Neumann problem is at least $1/4$. The functions which are considered in the Rayleigh quotient minimization problem are orthogonal to the characteristic functions of the geodesic triangles.

- (Variations over larger scales inside a ball of radius Ct). *There exists $C_0 > 0$ such that for any $t \geq 1$, $C > 0$, $x, y, z \in \widetilde{M}$ with $d(x, y) \leq Ct$ and $|d(x, y) - d(x, z)| \leq 4t + 4$, there holds*

$$\frac{k_t(x, z)}{k_t(x, y)} \geq C_0 \exp\left(-\left(1 + |b| + C + \frac{C+1}{|b|}\right)4t\right). \quad (16)$$

The following lemma on the L^∞ norm of the heat kernel in M is not sharp, but it is sufficient for our purpose.

Lemma 2.5 (L^∞ norm of the heat kernel in M). *There exists $C_1 > 0$ such that for any $t \geq 1$ there holds*

$$\|K_t(\cdot, \cdot)\|_{L^\infty(M \times M)} \leq C_1 \exp(4|ab|t).$$

3 Key lemmas

This section is devoted to the proof of several lemmas which are key ingredients of the proof of Theorem 1.1 provided in Section 4.1.

3.1 Error term estimates

As explained in Section 1.3, our proof relies on finding an upper bound on the trace of the trace-class operator $(Pe^{r_1\Delta}P)^n$, and the first step is to write this trace as an integral

$$\mathrm{Tr}((Pe^{r_1\Delta}P)^{2n}) = \int_M \|(Pe^{r_1\Delta}P)^n \delta_x\|^2 d\nu(x) \quad (17)$$

(see Lemma A.3). We actually choose $n = \lfloor r_2/r_1 \rfloor + 1$.

It turns out that to leverage the gain of $\varepsilon(g)$ obtained in Section 3.3 (and described in Step 5 of Section 1.3), we compare the integrand in the right-hand side of (17) to a local Rayleigh quotient (or spectral radius) at each point $x \in M$, see (19) below. We work this out in the present section, and we provide estimates for the error terms which unavoidably appear along the way.

We denote by χ_x the indicator function of the subset

$$B_d(x, C'r_2) \cup \bigcup_{V_k \cap B_d(x, C'r_2) \neq \emptyset} V_k, \quad (18)$$

where C' will be fixed in Section 4.1. We have $[\chi_x, P] = 0$ for any $x \in M$.

Throughout Section 3, we assume that g is sufficiently large, so that $r_1 \geq 1$. The main result of this section is the following one.

Lemma 3.1. *There exists $C_0 > 0$ such that if $C' \geq \max(10|ab|^{\frac{1}{2}}, 6)$, then for any $x \in M$,*

$$\|(Pe^{r_1\Delta}P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x\| \leq C_0 \left(\sup_{\|\varphi\|=1} \|(P\chi_x e^{r_1\Delta} \chi_x P)^{\lfloor r_2/r_1 \rfloor} \varphi\| + \exp\left(-\frac{C'^2 r_2}{32}\right) \right). \quad (19)$$

The proof of Lemma 3.1 relies on the following intermediate result.

Lemma 3.2. *There exists $C_0 > 0$ such that if $C' \geq \max(10|ab|^{\frac{1}{2}}, 6)$, then for any $x \in M$,*

$$\|(Pe^{r_1\Delta}P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x - (P\chi_x e^{r_1\Delta} \chi_x P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x\| \leq C_0 \exp\left(-\frac{C'^2 r_2}{32}\right).$$

Remark 3.3. *The cut-offs χ_x are introduced to overcome the difficulty caused by the infinite speed of propagation of the heat kernel. We expect that most of the mass of $(Pe^{r_1\Delta}P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x$ is contained in a ball of radius $C'r_2$. Lemma 3.2 proves that the remainder term coming from the complement of $\text{supp}(\chi_x)$ is small.*

Proof of Lemma 3.2. Write $n := \lfloor r_2/r_1 \rfloor + 1$. We first notice that the difference $(Pe^{r_1\Delta}P)^n \delta_x - (P\chi_x e^{r_1\Delta} \chi_x P)^n \delta_x$ is equal to the telescopic sum

$$\sum_{l=0}^{n-1} (P\chi_x e^{r_1\Delta})^{n-l-1} P(1-\chi_x) (e^{r_1\Delta}P)^{l+1} \delta_x \quad (20)$$

(to show this, write $P(1-\chi_x) = P - P\chi_x$, recall that $[P, \chi_x] = 0$, and after removing telescoping terms, use $\chi_x \delta_x = \delta_x$).

Next, we estimate the norm of each term of (20) individually. For $n-1 \geq l \geq 0$, the quantity

$$\| (P\chi_x e^{r_1\Delta})^{n-l-1} P(1-\chi_x) (e^{r_1\Delta}P)^{l+1} \delta_x \|$$

is bounded above by

$$\| (1-\chi_x) (e^{r_1\Delta}P)^{l+1} \delta_x \| \quad (21)$$

because e^Δ , χ_x and P have operator norm 1. We start by estimating this last quantity when $l = 0$. We remark first of all that

$$\| (1-\chi_x) e^{r_1\Delta} P \delta_x \| \leq \| (1-\chi_x) e^{(r_1-1)\Delta} |e^\Delta P \delta_x| \| \quad (22)$$

according to the triangle inequality. In order to make sense of the right-hand side, notice that $e^\Delta P \delta_x$ is equal to the element ϕ of $L^2(M, \nu)$ defined by

$$\phi : y \mapsto K_1(y, x) - \psi_{k_x}(x) \int_M K_1(y, z) \psi_{k_x}(z) d\nu(z)$$

where k_x denotes the unique $k \in \mathcal{N}$ such that $\psi_k(x) \neq 0$ if it exists, and $k_x = 0$ otherwise. The notation $|e^\Delta P \delta_x|$ then corresponds to the absolute value of ϕ .

We provide now an estimate of the right-hand side of (22). As a general fact, we have for any $t > 0$,

$$\| (1-\chi_x) e^{t\Delta} |e^\Delta P \delta_x| \| = \| (1-\chi_x) e^{t\Delta} \phi \| \leq \| (1-\chi_x) e^{t\Delta} K_1(\cdot, x) \| + C_0 \| (1-\chi_x) e^{t\Delta} (e^\Delta \psi_{k_x}) \|$$

where we have used that $|\psi_{k_x}(x)| \leq C_0$ for any x (see Section 2.2). We have moreover according to (18)

$$\| (1-\chi_x) e^{t\Delta} K_1(\cdot, x) \| \leq \| K_{t+1}(\cdot, x) \|_{L^2(M \setminus B_d(x, C'r_2))}. \quad (23)$$

Now, since the ψ_k 's are bounded above by C_0 and supported on subsets of diameter bounded above by 1, using (10) and Lemma 2.1 we obtain that there is a constant C_1 such that $K_1(y, x) \geq k_1(d(x, y)) \geq k_1^{(b)}(d(x, y)) \geq C_1 \psi_{k_x}(y)$. Hence,

$$\| (1-\chi_x) e^{t\Delta} (e^\Delta \psi_{k_x}) \| \leq C_1 \| (1-\chi_x) e^{(t+1)\Delta} K_1(\cdot, x) \| \leq C_1 \| K_{t+2}(\cdot, x) \|_{L^2(M \setminus B_d(x, C'r_2))}$$

According to Lemma 2.5,

$$\begin{aligned} \| K_{t+2}(\cdot, x) \|_{L^2(M \setminus B_d(x, C'r_2))} &\leq C_2 \exp(2|ab|t) \| K_{t+2}(\cdot, x) \|_{L^1(M \setminus B_d(x, C'r_2))}^{1/2} \\ &\leq C_2 \exp(2|ab|t) \| k_{t+2}(\cdot, x) \|_{L^1(\mathbb{H}^2 \setminus B_{\mathbb{H}}^2(x, C'r_2))}^{1/2} \end{aligned}$$

where in the last line we used (10). The same argument applied to the right-hand side of (23) yields the same bound, at time $t+1$. From now on, we assume $t \in [0, 2r_2]$. Using the heat kernel estimate (15) applied with $C = C'r_2/(t+2) \geq 2$ (here we use $C' \geq 6$) and $C = C'r_2/(t+1) \geq 2$ we thus have

$$\|(1 - \chi_x)e^{t\Delta} |e^\Delta P \delta_x|\| \leq C_3 \exp\left(\frac{5}{2}|ab|t - \frac{C'^2 r_2^2}{8(t+2)}\right). \quad (24)$$

Applying to $t = r_1 - 1$ and combining with (22) we get an upper bound for (21) for $l = 0$.

We turn now to the case $l \geq 1$. For any $f \in L^2(M, \nu)$, $t \geq 0$ and $y \in M$ we have

$$|Pe^{t\Delta} f(y)| \leq e^{t\Delta} |f|(y) + \sum_{k \in \mathcal{N}} \psi_k(y) \int_M \psi_k(w) K_t(w, z) |f(z)| d\nu(w) d\nu(z).$$

The ψ_k 's are bounded by C_0 and supported on pairwise disjoint subsets of diameter bounded above by 2. Therefore, there is a constant $C_4 > 0$ such that for any $y, w \in M$,

$$\sum_{k \in \mathcal{N}} \psi_k(y) \psi_k(w) \leq C_4 k_1^{(b)}(d(y, w)) \leq C_4 k_1(d(y, w)) \leq C_4 K_1(y, w)$$

(where we used Lemma 2.1 and (10)). This yields

$$\sum_{k \in \mathcal{N}} \psi_k(y) \int_M \psi_k(w) K_t(w, z) |f(z)| d\nu(w) d\nu(z) \leq C_4 e^{(t+1)\Delta} |f|(y).$$

Hence,

$$|Pe^{t\Delta} f(y)| \leq (e^{t\Delta} + C_4 e^{(t+1)\Delta}) |f|(y). \quad (25)$$

A simple induction then shows that,

$$|P(e^{t\Delta} P)^{l-1} f(y)| \leq (e^{t\Delta} + C_4 e^{(t+1)\Delta})^{l-1} |Pf|(y).$$

Applying the above with $t = r_1$ and $f = e^{r_1\Delta} P \delta_x$ yields

$$\|(1 - \chi_x) (e^{r_1\Delta} P)^{l+1} \delta_x\| \leq \|(1 - \chi_x) e^{r_1\Delta} (e^{r_1\Delta} + C_4 e^{(r_1+1)\Delta})^{l-1} |Pe^{r_1\Delta} P \delta_x|\|.$$

Applying (25) with $t = r_1 - 1$ and $f = e^\Delta P \delta_x$ we find

$$|Pe^{r_1\Delta} P \delta_x|(y) \leq (e^{(r_1-1)\Delta} + C_4 e^{r_1\Delta}) |e^\Delta P \delta_x|(y)$$

for all $y \in M$. So the triangle inequality yields

$$\|(1 - \chi_x) (e^{r_1\Delta} P)^{l+1} \delta_x\| \leq \sum_{j=0}^l \binom{l}{j} C_4^j \|(1 - \chi_x) e^{((l+1)r_1+j-1)\Delta} |e^\Delta P \delta_x|\|. \quad (26)$$

All in all, we have, using successively (26) and (24) (since $(l+1)r_1 + j + 1 \leq (l+1)(r_1 + 1) \leq 2r_2$)

$$\begin{aligned} & \|(P \chi_x e^{r_1\Delta})^{n-l-1} P (1 - \chi_x) (e^{r_1\Delta} P)^{l+1} \delta_x\| \\ & \leq \sum_{j=0}^l \binom{l}{j} C_4^j \|(1 - \chi_x) e^{((l+1)r_1+j-1)\Delta} |e^\Delta P \delta_x|\| \\ & \leq \sum_{j=0}^l \binom{l}{j} C_4^j C_3 \exp\left(\frac{5}{2}|ab|r_2 - \frac{C'^2 r_2^2}{8(l+1)(r_1+1)}\right) \\ & = (1 + C_4)^l C_3 \exp\left(\frac{5}{2}|ab|r_2 - \frac{C'^2 r_2^2}{8(l+1)(r_1+1)}\right). \end{aligned} \quad (27)$$

Finally, combining (20) with the estimate (27) and the estimate on the $l = 0$ term, we find

$$\begin{aligned}
& \| \left((Pe^{r_1\Delta}P)^n - (P\chi_x e^{r_1\Delta}\chi_x P)^n \right) \delta_x \| \\
& \leq C_3 \exp \left(\frac{5}{2}|ab|r_2 - \frac{C'^2 r_2^2}{8(r_1+1)} \right) + \sum_{l=1}^{n-1} (1+C_4)^l C_3 \exp \left(\frac{5}{2}|ab|r_2 - \frac{C'^2 r_2^2}{8(l+1)(r_1+1)} \right) \\
& \leq C_3 \exp \left(\frac{5}{2}|ab|r_2 - \frac{C'^2 r_2^2}{8(r_1+1)} \right) + C_3 n (1+C_4)^n \exp \left(\frac{5}{2}|ab|r_2 - \frac{C'^2 r_2^2}{8n(r_1+1)} \right) \\
& \leq C_5 \exp \left(-\frac{C'^2 r_2^2}{32} \right)
\end{aligned}$$

where we have estimated all terms of the sum by the maximal one in going from the second line to the third one, and we used that $n(r_1+1) \leq 2r_2$ and that $n(1+C_4)^n$ is negligible compared to $\exp(|ab|r_2/2)$ to go from the third to the last line (together with the fact that $C' \geq 10|ab|^{1/2}$). \square

The end of this section is devoted to the proof of Lemma 3.1.

Proof of Lemma 3.1. Notice that the operator $P\chi_x e^{r_1\Delta}\chi_x P$ is compact, as a composition of bounded (linear) operators with a compact operator e^Δ . Now we notice that

$$\| (Pe^{r_1\Delta}P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x \| \leq \| (P\chi_x e^{r_1\Delta}\chi_x P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x \| + C_0 \exp \left(-\frac{C'^2 r_2^2}{32} \right)$$

for some $C_0 > 0$ due to Lemma 3.2. Then we set $B_x = P\chi_x e^{r_1\Delta}\chi_x P$. We prove below that $\|B_x \delta_x\| \leq C_1$ for some $C_1 > 0$ (as always, depending only on $(a, b, \rho) \in \mathcal{T}$). Therefore,

$$\begin{aligned}
\| (Pe^{r_1\Delta}P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x \| & \leq \| B_x^{\lfloor r_2/r_1 \rfloor} B_x \delta_x \| + C_0 \exp \left(-\frac{C'^2 r_2^2}{32} \right) \\
& \leq C_1 \sup_{\|\varphi\|=1} \| B_x^{\lfloor r_2/r_1 \rfloor} \varphi \| + C_0 \exp \left(-\frac{C'^2 r_2^2}{32} \right),
\end{aligned}$$

which is exactly (19). There remains to justify that $\|B_x \delta_x\| \leq C_1$. We first have

$$\| B_x \delta_x \| \leq \| e^\Delta \chi_x P \delta_x \| \tag{28}$$

since the operator norms of P , χ_x and $e^{(r_1-1)\Delta}$ are all equal to 1. If we denote by k_x the only $k \in \mathcal{N}$ such that $x \in V_{k_x}$ when it exists (and $\psi_{k_x} = 0$ otherwise), then by definition of P we have

$$\| e^\Delta \chi_x P \delta_x \| = \| e^\Delta P \chi_x \delta_x \| \leq \| e^\Delta \delta_x \| + C_0 \| e^\Delta \psi_{k_x} \| \leq \| e^\Delta \delta_x \| + C_0 \tag{29}$$

where we used that $\|\psi_k\| = 1$ and that the operator norm of $e^\Delta : L^2(M, \nu) \rightarrow L^2(M, \nu)$ is equal to 1. Using (10), we get

$$\| e^\Delta \delta_x \|^2 = \| K_1(x, \cdot) \|^2 \leq \| K_1(x, \cdot) \|_{L^\infty} \| K_1(x, \cdot) \|_{L^1} = \| K_1(x, \cdot) \|_{L^\infty} \leq C_2 \tag{30}$$

where the last inequality is a consequence of Lemma 2.5. Combining (28), (29), (30), we get $\|B_x \delta_x\| \leq C_1$, which concludes the proof. \square

3.2 Applications of the min-max

Our proofs of Theorems 1.1 and 1.2 rely fundamentally on the Courant–Fischer min-max lemma (see e.g. [RS78, Theorem XIII.1]) through the two lemmas proved in this section. As in Section 3.1, the main quantity of interest in this section is, for any fixed $x \in M$,

$$\sup_{\|\varphi\|=1} \|P\chi_x e^{r_1\Delta}\chi_x P\varphi\|. \quad (31)$$

Lemma 3.4. *There holds*

$$\sup_{\|\varphi\|=1} \|(P\chi_x e^{r_1\Delta}\chi_x P)^{\lfloor r_2/r_1 \rfloor} \varphi\| = \sup_{\|\varphi\|=1} \|(P\chi_x e^{r_1\Delta}\chi_x P)\varphi\|^{\lfloor r_2/r_1 \rfloor} \quad (32)$$

Proof. The operator $P\chi_x e^{r_1\Delta}\chi_x P : L^2(M, \nu) \rightarrow L^2(M, \nu)$ is selfadjoint and compact (because e^Δ is compact). Hence by the min-max theorem, the left-hand side of (32) is equal to the top eigenvalue of $(P\chi_x e^{r_1\Delta}\chi_x P)^{2\lfloor r_2/r_1 \rfloor}$, which is also equal to the top eigenvalue of $(P\chi_x e^{r_1\Delta}\chi_x P)^2$, raised to the power $\lfloor r_2/r_1 \rfloor$. Using again the min-max theorem we obtain (32). \square

In all the sequel,

$$\mu_2 = e^{-\lambda_2(M)}$$

denotes the largest eigenvalue of e^Δ strictly smaller than 1.

We denote by $\varphi_x \in L^2(M, \nu)$ a function which attains the supremum (31), i.e.,

$$\|P\chi_x e^{r_1\Delta}\chi_x P\varphi_x\| = \sup_{\|\varphi\|=1} \|P\chi_x e^{r_1\Delta}\chi_x P\varphi\|, \quad \|\varphi_x\| = 1. \quad (33)$$

By the min-max principle φ_x is an eigenfunction of the compact, self-adjoint and non-negative operator $P\chi_x e^{r_1\Delta}\chi_x P$. Since $[\chi_x, P] = 0$, $\|P\|_{\text{op}} = 1$ and $\|\chi_x\|_{\text{op}} = 1$, the eigenfunction φ_x verifies

$$P\varphi_x = \varphi_x \quad \text{and} \quad \chi_x \varphi_x = \varphi_x. \quad (34)$$

The following result serves as a replacement for the bound on the set U in the proof of [JTYZZ21, Theorem 2.2].

Lemma 3.5. *There exists $C_0 > 0$ and a subset $S \subset M$ of area $\nu(S) \leq C_0 \exp(2C'r_2\sqrt{|b|})$ such that for any $x \notin S$,*

$$\|e^{r_1\Delta}|\varphi_x|\| \leq \mu_2^{r_1}. \quad (35)$$

Proof. Assume by contradiction that it is possible to find $x_1, x_2 \in M$ at distance $> 2C'r_2 + 4$ such that (35) does not hold. Then $\|e^{r_1\Delta}|\varphi_{x_1}|\| > \mu_2^{r_1}$ and $\|e^{r_1\Delta}|\varphi_{x_2}|\| > \mu_2^{r_1}$. Then $|\varphi_{x_1}|$ and $|\varphi_{x_2}|$ have disjoint supports since $\chi_{x_j}\varphi_{x_j} = \varphi_{x_j}$ for $j = 1, 2$ (recall (18) and the fact that the V_k have diameter at most 2). Thus they are orthogonal, and they contradict the min-max principle for $e^{2r_1\Delta}$ in $L^2(M, \nu)$. This proves (35) and shows that S has diameter $\leq 2C'r_2 + 4$. Finally, take $x \in S$ (if not empty) and notice that

$$\nu(S) \leq \nu(B_d(x, 2C'r_2 + 4)) \leq \frac{4\pi}{|b|} \exp((2C'r_2 + 4)\sqrt{|b|})$$

by (9), which concludes the proof. \square

3.3 Estimate of the gain $\varepsilon(g)$

The heart of our proof, the “gain of $\varepsilon(g)$ ”, corresponding to Step 5 in the strategy of proof in Section 1.3, is carried out in this section.

At the beginning of the proof of Theorem 1.1 in Section 4.1, we will prove using Section 3.2 that m' is controlled by the integral in x of the quantity $\|P\chi_x e^{r_1\Delta}\chi_x P\varphi_x\|$ where φ_x has been introduced in (33).

In the present section we show that, in turn, the quantity $\|P\chi_x e^{r_1\Delta}\chi_x P\varphi_x\| = \|P\chi_x e^{r_1\Delta}\varphi_x\|$ (see (34)) is bounded above by $(1 - \varepsilon(g))\|e^{r_1\Delta}|\varphi_x|\|$ up to remainders, where $\varepsilon(g)$ is “not too small”. This gain of a quantitative $\varepsilon(g)$, combined with an upper bound on $\|e^{r_1\Delta}|\varphi_x|\|$ obtained through Lemma 3.5, is sufficient to conclude the proof of Theorem 1.1 in Section 4.1.

The gain of $\varepsilon(g)$ is proved by writing the identity (full details are provided in the proof of Lemma 3.9)

$$\begin{aligned} \|e^{r_1\Delta}|\varphi_x|\|^2 - \|P\chi_x e^{r_1\Delta}\chi_x P\varphi_x\|^2 &= \underbrace{4\langle e^{r_1\Delta}\varphi_+, e^{r_1\Delta}\varphi_- \rangle}_{:=G_1} + \underbrace{\sum_{k \in \mathcal{N}_x} (\langle e^{r_1\Delta}\varphi_+, \psi_k \rangle - \langle e^{r_1\Delta}\varphi_-, \psi_k \rangle)^2}_{:=G_2} \\ &\quad + \|(1 - \chi_x)e^{r_1\Delta}\varphi_x\|^2 \end{aligned}$$

where $\varphi_{\pm} = \max(\pm\varphi_x, 0)$ and \mathcal{N}_x is introduced in (42). The key observation is that G_1 and G_2 play opposite roles. While G_1 quantifies the interaction (or lack thereof) between $e^{r_1\Delta}\varphi_+$ and $e^{r_1\Delta}\varphi_-$, G_2 measures the discrepancy between the mass left by $e^{r_1\Delta}\varphi_+$ and $e^{r_1\Delta}\varphi_-$ on the Voronoi cells V_k . Since the variations of $e^{r_1\Delta}\varphi_+$ and $e^{r_1\Delta}\varphi_-$ are very well controlled at scale $O(1)$ (see Lemma 3.6, which is a consequence of (16)), $e^{r_1\Delta}\varphi_+$ and $e^{r_1\Delta}\varphi_-$ must interact on the Voronoi cells V_k in order for G_2 to be small. In turn, this prevents G_1 from being small. In other words, $G_1 + G_2$ (hence $\varepsilon(g)$) cannot be small. This will be expressed in practice as a lower bound for $G_1 + G_2$ in terms of the L^2 -norm of $e^{r_1\Delta}|\varphi_x|$.

The following lemma illustrates the idea that the solutions of the heat equation at time r_1 do not vary too much over scales of size $\lesssim r_1$. We set

$$w(r_1) = C_0 \exp\left(-4\left(1 + |b| + C + \frac{C+1}{|b|}\right)r_1\right). \quad (36)$$

where C_0 is given in (16).

Lemma 3.6 (Small scale invariance). *There exists $C_0 > 0$ such that for any $C \geq \max(4|a|\sqrt{|b|} + 2, 16)$, any $r_1 \geq 1$, and any positive function f with $\|f\| = 1$ there exists $R \in L^2(M, \nu)$ with*

$$\|R\| \leq C_0 \exp\left(|ab|t - \frac{C^2 r_1}{64|a|}\right)$$

and the inequality

$$e^{r_1\Delta}f(x) \geq w(t)(e^{r_1\Delta}f(x') - R(x'))$$

holds for any $x, x' \in M$ at distance at most $4r_1 + 4$.

Proof. Recall that $e^{r_1\Delta}f(x) = \int_M K_{r_1}(x, y)f(y)dy$. For every $x, y \in M$ define

$$K_{r_1}^{\leq C}(x, y) := \sum_{\gamma \in \Gamma, d_{\bar{M}}(\bar{x}, \gamma\bar{y}) \leq Cr_1} k_{r_1}(\bar{x}, \gamma\bar{y})$$

and

$$K_{r_1}^{> C}(x, y) := \sum_{\gamma \in \Gamma, d_{\bar{M}}(\bar{x}, \gamma\bar{y}) > Cr_1} k_{r_1}(\bar{x}, \gamma\bar{y})$$

for any choice of lifts \bar{x}, \bar{y} of x, y to \widetilde{M} . We have $K_{r_1} = K_{r_1}^{>C} + K_{r_1}^{\leq C}$ according to (10).

For $x \in M$ we set

$$R_{>C}(x) := \int_M K_{r_1}^{>C}(x, y) f(y) d\nu(y) \quad (37)$$

Then it follows from the Cauchy-Schwarz inequality that

$$\|R_{>C}\|_{L^2} \leq \sup_{x \in M} \|K_{r_1}^{>C}(x, \cdot)\|_{L^1} \|f\|_{L^2}. \quad (38)$$

Take $x \in M$ and choose $\bar{x} \in \widetilde{M}$ a lift of x . We have

$$\|K_{r_1}^{>C}(x, \cdot)\|_{L^1} = \int_{\widetilde{M} \setminus B_{\widetilde{M}}(\bar{x}, Cr_1)} k_{r_1}(\bar{x}, \bar{y}) d\bar{y} \leq C_0 \exp\left(|ab|r_1 - \frac{C^2 r_1}{16|a|}\right) \quad (39)$$

according to (15). Besides, $K^{\leq C}$ has controlled variations in terms of C : we prove that for all $x, x', y \in M$ with $d(x, x') \leq 4r_1 + 4$ we have

$$K_{r_1}^{\leq C}(x, y) \geq w(r_1) K_{r_1}^{\leq C-1}(x', y). \quad (40)$$

For this, we fix lifts \bar{x}, \bar{x}' of x, x' at distance $\leq 4r_1 + 4$. Using that if $d_{\widetilde{M}}(\bar{x}', \gamma\bar{y}) \leq (C-8)r_1$, then $d_{\widetilde{M}}(\bar{x}, \gamma\bar{y}) \leq Cr_1$, together with (16) applied with $t = r_1$ we get

$$\begin{aligned} K_{r_1}^{\leq C}(x, y) &= \sum_{\gamma \in \Gamma, d_{\widetilde{M}}(\bar{x}, \gamma\bar{y}) \leq Cr_1} k_{r_1}(d_{\widetilde{M}}(\bar{x}, \gamma\bar{y})) \geq w(r_1) \sum_{\gamma \in \Gamma, d_{\widetilde{M}}(\bar{x}', \gamma\bar{y}) \leq Cr_1} k_{r_1}(d_{\widetilde{M}}(\bar{x}', \gamma\bar{y})) \\ &\geq w(r_1) \sum_{\gamma \in \Gamma, d_{\widetilde{M}}(\bar{x}', \gamma\bar{y}) \leq (C-8)r_1} k_{r_1}(d_{\widetilde{M}}(\bar{x}', \gamma\bar{y})) = w(r_1) K_{r_1}^{\leq C-8}(x', y). \end{aligned}$$

We deduce from (40)

$$e^{r_1 \Delta} f(x) \geq w(r_1) (e^{r_1 \Delta} f(x') - R_{>C-8}(x')).$$

Combining (38), (39) and the fact that $C-8 \geq C/2$ we obtain the lemma with $R = R_{>C-8}$. \square

For $f \in L^2(M, \nu)$ we set $f_{\pm} = \max(\pm f, 0)$. The next lemma leverages Lemma 3.6 to establish the fact that the interaction $\langle e^{r_1 \Delta} f_+, e^{r_1 \Delta} f_- \rangle$ between positive and negative parts can already be detected coarsely on the Voronoi cells V_k .

Lemma 3.7. *There exist $C_0, C_1 > 0$ such that for any $C \geq \max(4|a|\sqrt{|b|} + 2, 16)$, any $r_1 \geq 1$, and any $f \in L^2(M, \nu)$ with $\|f\| = 1$, there holds*

$$\langle e^{r_1 \Delta} f_+, e^{r_1 \Delta} f_- \rangle \geq C_0 w(r_1)^2 \sum_{k \in \mathcal{N}} m_k^+ m_k^- - C_1 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right)$$

where $m_k^{\pm} = \langle f_{\pm}, e^{r_1 \Delta} \psi_k \rangle$.

Proof. We denote by R^{\pm} the remainder corresponding to f^{\pm} in Lemma 3.6. We recall that $C_0 \leq \text{vol}(\text{supp}(\psi_k)) \leq C_1$ (see (13)).

Let $k \in \mathcal{N}$ and $x \in \text{supp}(\psi_k)$. Averaging over x' in the support of ψ_k and using that $4r_1 + 4 \geq 4 \geq \text{diam}(V_k)$ and $\|\psi_k\|_{L^\infty} \leq C_2$ (see Section 2.2), we deduce from Lemma 3.6 that

$$\begin{aligned} e^{r_1 \Delta} f_{\pm}(x) &\geq \frac{C_2^{-1} w(r_1)}{\text{vol}(\text{supp}(\psi_k))} (\langle e^{r_1 \Delta} f_{\pm}, \psi_k \rangle - \langle R^{\pm}, \psi_k \rangle) \\ &\geq C_3 w(r_1) \langle e^{r_1 \Delta} f_{\pm}, \psi_k \rangle - C_4 \|R^{\pm}\|_{L^2(\text{supp}(\psi_k))} \end{aligned} \quad (41)$$

where we used the Cauchy-Schwarz inequality for the second term. The remainder

$$R_{\pm}(x) = \begin{cases} \|R^{\pm}\|_{L^2(\text{supp}(\psi_{k_x}))} & \text{if } x \in \text{supp}(\psi_{k_x}) \text{ for some } k_x \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

verifies $\|R_{\pm}\| \leq C_1 \|R^{\pm}\|_{L^2(\bigcup_{k \in \mathcal{N}} \text{supp}(\psi_k))}$.

We notice that $m_k^{\pm} \leq \|A^{r_1} f_{\pm}\| \|\psi_k\| \leq 1$. Then using (41) and the fact that $w(r_1) \leq 1$ we obtain

$$\begin{aligned} \langle e^{r_1 \Delta} f_+, e^{r_1 \Delta} f_- \rangle &\geq \sum_{k \in \mathcal{N}} \int_{\text{supp}(\psi_k)} e^{r_1 \Delta} f_+(x) e^{r_1 \Delta} f_-(x) d\nu(x) \\ &\geq \sum_{k \in \mathcal{N}} \int_{\text{supp}(\psi_k)} (C_3 w(r_1) m_k^+ - C_4 R_+(x)) (C_3 w(r_1) m_k^- - C_4 R_-(x)) d\nu(x) \\ &\geq C_3 w(r_1)^2 \sum_{k \in \mathcal{N}} m_k^+ m_k^- - C_4 \exp\left(|ab|r_1 - \frac{C^2 r_1}{16|a|}\right) \end{aligned}$$

which concludes the proof. \square

We provide now what will serve as the lower bound for $G_1 + G_2$ alluded to at the beginning of this section.

Lemma 3.8. *For any $x \in M$ set*

$$\mathcal{N}_x = \{k \in \mathcal{N} \mid \text{supp}(\psi_k) \cap B_d(x, C'r_2) \neq \emptyset\}. \quad (42)$$

There exist constants $C_0, C_1 > 0$ such that for any $C \geq \max(4|a|\sqrt{|b|} + 2, 16)$, any $r_1 \geq 1$, any $f \in L^2(M, \nu)$ with $\|f\| = 1$, any $x \in M$,

$$\sum_{k \in \mathcal{N}_x} \langle e^{r_1 \Delta} |f|, \psi_k \rangle^2 \geq C_0 \exp(-2r_1 \sqrt{|b|}) w(r_1)^2 \|\chi_x e^{r_1 \Delta} |f|\|^2 - C_1 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right).$$

Proof. For any $k \in \mathcal{N}$, we denote by y_k a point where $e^{r_1 \Delta} |f|$ attains its minimum on $V_k = \text{supp}(\psi_k)$. We first show that

$$\text{supp}(\chi_x) \subset \bigcup_{k \in \mathcal{N}_x} B_d(y_k, 2r_1 + 2). \quad (43)$$

Let $z \in \text{supp}(\chi_x)$. If $z \in \text{supp}(\chi_x) \setminus B_d(x, C'r_2)$, then $z \in V_k$ for some $k \in \mathcal{N}$, and $d(z, y_k) \leq \text{diam}(V_k) \leq 2r_1 + 2$. Otherwise, $z \in B_d(x, C'r_2)$, and therefore there exists $z' \in B_d(x, C'r_2 - r_1)$ at distance at most r_1 from z . We have $B_d(z', r_1) \subset B(x, C'r_2)$, therefore there exists an element of the r_1 -net $x_k \in B(x, C'r_2)$ with $d(x_k, z') \leq r_1$. It follows that $d(y_k, z) \leq d(y_k, x_k) + d(x_k, z') + d(z', z) < 2r_1 + 2$.

Using Lemma 3.6 and noticing that $4r_1 + 4$ is the diameter of the ball $B_d(y_k, 2r_1 + 2)$, we have

$$\begin{aligned} \langle e^{r_1 \Delta} |f|, \psi_k \rangle^2 &\geq C_0 (e^{r_1 \Delta} |f|(y_k))^2 \\ &\geq C_0 \exp(-2r_1 \sqrt{|b|}) w(r_1)^2 \int_{B_d(y_k, 2r_1 + 2)} (e^{r_1 \Delta} |f|(x') - R(x'))^2 \nu(dx') \end{aligned}$$

where we used that $\text{vol}(B_d(y_k, 2r_1 + 2)) \leq C_1 \exp(2r_1 \sqrt{|b|})$ according to (9). Summing over $k \in \mathcal{N}_x$ and using (43), we obtain

$$\begin{aligned} \sum_{k \in \mathcal{N}_x} \langle e^{r_1 \Delta} |f|, \psi_k \rangle^2 &\geq C_0 \exp(-2r_1 \sqrt{|b|}) w(r_1)^2 \int_M \chi_x(x') (e^{r_1 \Delta} |f|(x') - R(x'))^2 \nu(dx') \\ &\geq C_0 \exp(-2r_1 \sqrt{|b|}) w(r_1)^2 \|\chi_x e^{r_1 \Delta} |f|\|^2 - C_2 \|R\| \end{aligned}$$

where in the last line we developed the square in the right-hand side, we used the Cauchy-Schwarz inequality in $L^2(M, \nu)$ and the bound $\|e^{r_1 \Delta} |f|\| \leq 1$. \square

In the next lemma, as in Section 3.2, φ_x denotes a function which attains the supremum in (31).

Lemma 3.9. *There exist $C_0, C_1 > 0$ such that for any $C \geq \max(4|a|\sqrt{|b|} + 2, 16)$, any $r_1 \geq 1$ and any $x \in M$,*

$$\|(P\chi_x e^{r_1 \Delta} \chi_x P)\varphi_x\|^2 \leq (1 - C_0 \exp(-2r_1 \sqrt{|b|}) w(r_1)^4) \|e^{r_1 \Delta} |\varphi_x|\|^2 + C_1 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right).$$

Proof. We fix $x \in M$. We compute $\varepsilon = \|e^{r_1 \Delta} \chi_x |\varphi_x|\|^2 - \|P\chi_x e^{r_1 \Delta} \chi_x P\varphi_x\|^2$ which can be simplified to

$$\varepsilon = \|e^{r_1 \Delta} |\varphi_x|\|^2 - \|P\chi_x e^{r_1 \Delta} \varphi_x\|^2$$

due to (34). First we compute without the absolute value on φ_x , and we use from line 1 to line 2 that $\chi_x \in \{0, 1\}$:

$$\begin{aligned} \|e^{r_1 \Delta} \varphi_x\|^2 - \|P\chi_x e^{r_1 \Delta} \varphi_x\|^2 &= \langle (\text{Id} - \chi_x P\chi_x) e^{r_1 \Delta} \varphi_x, e^{r_1 \Delta} \varphi_x \rangle \\ &= \|(1 - \chi_x) e^{r_1 \Delta} \varphi_x\|^2 + \|(\text{Id} - P)\chi_x e^{r_1 \Delta} \varphi_x\|^2 \\ &= \|(1 - \chi_x) e^{r_1 \Delta} \varphi_x\|^2 + \sum_{k \in \mathcal{N}_x} \langle e^{r_1 \Delta} \varphi_x, \psi_k \rangle^2 \end{aligned}$$

where \mathcal{N}_x has been introduced in (42).

Then we notice the following identity:

$$\|e^{r_1 \Delta} |\varphi_x|\|^2 - \|e^{r_1 \Delta} \varphi_x\|^2 = 4\langle e^{r_1 \Delta} \varphi_+, e^{r_1 \Delta} \varphi_- \rangle$$

where $\varphi_{\pm} = \max(\pm \varphi_x, 0)$. All in all,

$$\varepsilon = 4\langle e^{r_1 \Delta} \varphi_+, e^{r_1 \Delta} \varphi_- \rangle + \|(1 - \chi_x) e^{r_1 \Delta} \varphi_x\|^2 + \sum_{k \in \mathcal{N}_x} (\langle e^{r_1 \Delta} \varphi_+, \psi_k \rangle - \langle e^{r_1 \Delta} \varphi_-, \psi_k \rangle)^2$$

(we write the last term as a difference on purpose). Using Lemma 3.7, its notation and the fact that $m_k^+ + m_k^- = \langle e^{r_1 \Delta} |\varphi_x|, \psi_k \rangle$ we have

$$\begin{aligned} \varepsilon &\geq \sum_{k \in \mathcal{N}_x} (m_k^+ - m_k^-)^2 + 2 \left(C_0 w(r_1)^2 \sum_{k \in \mathcal{N}} m_k^+ m_k^- - C_1 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right) \right) \\ &\quad + 2\langle e^{r_1 \Delta} \varphi_+, e^{r_1 \Delta} \varphi_- \rangle + \|(1 - \chi_x) e^{r_1 \Delta} \varphi_x\|^2 \\ &\geq \min\left(1, \frac{C_0 w(r_1)^2}{2}\right) \sum_{k \in \mathcal{N}_x} \langle e^{r_1 \Delta} |\varphi_x|, \psi_k \rangle^2 - 2C_1 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right) + \frac{\|(1 - \chi_x) e^{r_1 \Delta} |\varphi_x|\|^2}{2}. \end{aligned}$$

Using Lemma 3.8 and its notation we obtain that there exist $C_2, C_3 > 0$ such that

$$\begin{aligned}\varepsilon &\geq C_2 \exp(-2r_1 \sqrt{|b|}) w(r_1)^4 \|\chi_x e^{r_1 \Delta} |\varphi_x|\|^2 + \frac{\|(1 - \chi_x) e^{r_1 \Delta} |\varphi_x|\|^2}{2} - C_3 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right) \\ &\geq C_2 \exp(-2r_1 \sqrt{|b|}) w(r_1)^4 \|e^{r_1 \Delta} |\varphi_x|\|^2 - C_3 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right),\end{aligned}$$

which concludes the proof. \square

4 Proof of the main results

4.1 Proof of Theorem 1.1

Building upon the results of Section 3, we proceed with the proof of Theorem 1.1.

We recall that $\mu_2 = e^{-\lambda_2(M)}$. We denote by m the multiplicity of $\lambda_2(M)$ as an eigenvalue of $-\Delta$. Then, m is also the multiplicity of $\mu_2^{r_1}$ as an eigenvalue of $e^{r_1 \Delta}$. We denote by m' the multiplicity of $\mu_2^{r_1}$ as an eigenvalue of $P e^{r_1 \Delta} P$, which is also compact, self-adjoint and non-negative.

In the sequel, we provide an upper bound on m' since such a bound is sufficient to bound m (by (53)).

Since e^Δ is a trace-class operator, $(P e^{r_1 \Delta} P)^{2\lfloor r_2/r_1 \rfloor + 2}$ is also trace-class. We have

$$\begin{aligned}m' \mu_2^{2r_1(\lfloor r_2/r_1 \rfloor + 1)} &\leq \text{Tr}((P e^{r_1 \Delta} P)^{2\lfloor r_2/r_1 \rfloor + 2}) \\ &= \int_M \|(P e^{r_1 \Delta} P)^{\lfloor r_2/r_1 \rfloor + 1} \delta_x\|^2 d\nu(x) \\ &\leq C_0 \left(\exp\left(-\frac{C'^2 r_2}{32}\right) g + \int_M \sup_{\|\varphi_x\|=1} \|(P \chi_x e^{r_1 \Delta} \chi_x P)^{\lfloor r_2/r_1 \rfloor} \varphi_x\|^2 d\nu(x) \right) \\ &= C_0 \left(\exp\left(-\frac{C'^2 r_2}{32}\right) g + \int_M \left(\sup_{\|\varphi_x\|=1} \|(P \chi_x e^{r_1 \Delta} \chi_x P) \varphi_x\|^2 \right)^{\lfloor r_2/r_1 \rfloor} d\nu(x) \right).\end{aligned}\tag{44}$$

The justification of the above computation is as follows. From first to second line we use Lemma A.3. From second to third line we use Lemma 3.1 and the Gauss-Bonnet theorem. From third to fourth line we use Lemma 3.4.

To continue, we need to fix the parameters C, C' and the parameter c introduced in (8). We denote by $C_2 \in (0, 1)$ a constant such that $\mu_2 \geq C_2$ for any M of curvature $\geq -b$ (thanks to Lemma A.2). We choose successively (in this order) C', C and $c > 0$ such that

$$\frac{C'^2}{32} \geq \max(10|ab|^{\frac{1}{2}}, 6, -4 \log C_2 + 1)\tag{45}$$

$$\frac{C^2}{64|a|} \geq |ab| + 16 \left(1 + \frac{\sqrt{|b|}}{8} + |b| + C + \frac{C+1}{|b|} \right) - 2 \log(C_2)\tag{46}$$

$$\text{and } C \geq \max(4|a|\sqrt{|b|} + 2, 16)\tag{47}$$

$$\frac{1}{4} \geq \left(2C' \sqrt{|b|} - 4 \log C_2 + 16 \left(1 + \frac{\sqrt{|b|}}{8} + |b| + C + \frac{C+1}{|b|} \right) \right) c.\tag{48}$$

We also assume g large enough so that $r_1 \geq 1$. We separate the integral in (44) into an integral over S and an integral over $M \setminus S$, where S is chosen as in Lemma 3.5. Due to Lemma 3.5, the integral over S is bounded above by $\nu(S) \leq C_1 \exp(2C'r_2\sqrt{|b|})$ since the operator norms of $e^{r_1\Delta}$, χ_x and P are equal to 1. The integral over $M \setminus S$ is bounded above by

$$\int_{M \setminus S} \left((1 - C_4 \exp(-hr_1))^2 \|e^{r_1\Delta}|\varphi_x|\|^2 + C_3 \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right) \right)^{\lfloor r_2/r_1 \rfloor} d\nu(x) \quad (49)$$

by Lemma 3.9 (which we can apply thanks to (47)), where

$$h = 16 \left(1 + \frac{\sqrt{|b|}}{8} + |b| + C + \frac{C+1}{|b|} \right).$$

For any $x \in M \setminus S$, we have by definition of S

$$(1 - C_4 \exp(-hr_1))^2 \|e^{r_1\Delta}|\varphi_x|\|^2 \leq (1 - C_4 \exp(-hr_1))^2 \mu_2^{2r_1}. \quad (50)$$

Thanks to (46), for sufficiently large g ,

$$\frac{C^2}{64|a|} r_1 \geq |ab|r_1 + hr_1 - 2 \log(C_2)r_1.$$

Therefore we get, again for g large enough,

$$\begin{aligned} \frac{C_4}{2} \exp\left(|ab|r_1 - \frac{C^2 r_1}{64|a|}\right) &\leq \frac{C_4}{2} \exp(-hr_1) \mu_2^{2r_1} \\ &\leq \mu_2^{2r_1} \left(1 - \frac{C_4}{2} \exp(-hr_1)\right)^2 - \mu_2^{2r_1} (1 - C_4 \exp(-hr_1))^2. \end{aligned}$$

Combining with (50), we obtain that (49) is bounded above by

$$C_0 \mu_2^{2r_1 \lfloor r_2/r_1 \rfloor} \int_{M \setminus S} \left(1 - \frac{C_4}{2} \exp(-hr_1)\right)^{2 \lfloor r_2/r_1 \rfloor} d\nu(x) \leq C_0 g \left(\mu_2^{r_1} \left(1 - \frac{C_4}{2} \exp(-hr_1)\right) \right)^{2 \lfloor r_2/r_1 \rfloor}$$

due to Lemma 3.5 and the Gauss-Bonnet theorem. Summarizing, we have obtained

$$\begin{aligned} m' \mu_2^{2r_1(\lfloor r_2/r_1 \rfloor + 1)} &\leq C_0 g \left(\mu_2^{r_1} \left(1 - \frac{C_4}{2} \exp(-hr_1)\right) \right)^{2 \lfloor r_2/r_1 \rfloor} \\ &\quad + C_0 \exp(2C'r_2\sqrt{|b|}) + C_0 \exp\left(-\frac{C'^2 r_2}{32}\right) g \end{aligned} \quad (51)$$

We divide by $\mu_2^{2r_1(\lfloor r_2/r_1 \rfloor + 1)}$ and use the inequality $1 - x \leq e^{-x}$ to deduce that

$$m' \leq C_0 \left(\frac{g}{\mu_2^{2r_1}} \exp(-C_5 \exp(-hr_1) \lfloor r_2/r_1 \rfloor) + \frac{\exp(2C'r_2\sqrt{|b|}) + \exp(-C'^2 r_2/32)g}{\mu_2^{4r_2}} \right)$$

Thanks to our choice of parameters (45) and (48) we get that

$$\frac{\exp(2C'r_2\sqrt{|b|}) + \exp(-C'^2 r_2/32)g}{\mu_2^{4r_2}} \leq g^{\frac{1}{2}} + g^{1-c}.$$

Thanks to (48) we have for g sufficiently large

$$\frac{g}{\mu_2^{2r_1}} \exp(-C_5 \exp(-hr_1) \lfloor r_2/r_1 \rfloor) \leq g \exp(-C_6 \log^{2/3} g).$$

All in all, we find that for g sufficiently large,

$$m' \leq C_0 g \exp(-C_6 \log^{2/3} g). \quad (52)$$

By the Cauchy interlacing theorem (Theorem A.1) there holds

$$m \leq m' + \text{rank}(\text{Id} - P). \quad (53)$$

Under the assumptions of the first part of Theorem 1.1, we can choose the r_1 -net in a way that $\text{rank}(\text{Id} - P) \leq C_0 g/r_1$ according to Lemma 2.2, which together with (52) and (53) concludes the proof in this first case.

Under the assumptions of the second part of Theorem 1.1, we can choose the r_1 -net in a way that $\text{rank}(\text{Id} - P) \leq C_0 g/e^{\delta' r_1}$ according to Lemma 2.3, which together with (52) and (53) concludes the proof in this second case.

4.2 Proof of Theorem 1.2

Theorem 1.2 follows from elementary modifications of Section 4.1. We need the following straightforward adaptation of Lemma 3.5.

Lemma 4.1. *For any $j \in \mathbb{N}_{\geq 1}$, there exists $C'_j > 0$ and a subset $S \subset M$ of area $\nu(S) \leq C'_j \exp(2C' r_2 \sqrt{|b|})$ such that for any $x \notin S$,*

$$\|e^{r_1 \Delta} |\varphi_x|\| \leq \mu_j^{r_1}$$

where φ_x has been introduced in Section 3.2.

Fix $j \in \mathbb{N}_{\geq 1}$ and $\kappa, K > 0$. We denote by m' the number of eigenvalues of $P e^{r_1 \Delta} P$ contained in $[\mu_j^{r_1} (1 - \delta), \mu_j^{r_1}]$ where $\delta = K \frac{\log \log(g)}{\log^\kappa(g)}$. Compared to (45)-(48), the constants C' and C are fixed using C_j (coming from Lemma A.2) instead of C_2 , and (48) is replaced by

$$\frac{\kappa}{4} \geq \left(2C' \sqrt{|b|} - 4 \log C_j + 16 \left(1 + \frac{\sqrt{|b|}}{8} + |b| + C + \frac{C+1}{|b|} \right) \right) c.$$

Instead of (51) we obtain using Lemma 4.1

$$\begin{aligned} & m' \mu_j^{2r_1(\lfloor r_2/r_1 \rfloor + 1)} (1 - \delta)^{2\lfloor r_2/r_1 \rfloor + 2} \\ & \leq C_0 g \left(\mu_j^{r_1} \left(1 - \frac{C_4}{2} \exp(-hr_1) \right) \right)^{2\lfloor r_2/r_1 \rfloor} + C_0 \exp(2C' r_2 \sqrt{|b|}) + C_0 \exp\left(-\frac{C'^2 r_2}{32}\right) g. \end{aligned}$$

Dividing by $\mu_j^{2r_1(\lfloor r_2/r_1 \rfloor + 1)} (1 - \delta)^{2\lfloor r_2/r_1 \rfloor + 2}$ and proceeding as in Section 4.1, we obtain instead of (52)

$$m' \leq C_0 g \exp\left(-\log^{1-\frac{\kappa}{2}} g\right) (1 - \delta)^{-2\lfloor r_2/r_1 \rfloor - 2}$$

and thanks to the definition of δ and the inequality $(1 - \delta)^n \leq e^{-n\delta}$, we finally get for sufficiently large g

$$m' \leq C_0 g \exp\left((4K \log^{1-\kappa} g) - (\log^{1-\frac{\kappa}{2}} g)\right) \leq C_0 \frac{g}{\log \log(g)}.$$

By the Cauchy interlacing theorem (Theorem A.1) we obtain that the number m of eigenvalues of $e^{r_1\Delta}$ in $[\mu_j^{r_1}(1-\delta), \mu_j^{r_1}]$ is bounded above by $C_0 \frac{g}{\log \log(g)}$. It implies the same bound for the number of eigenvalues of e^Δ in $[\mu_j(1 - \frac{K}{2c \log^k(g)}), \mu_j]$, and Theorem 1.2 follows.

Remark 4.2. *In the present paper, we rely on the trace method to bound eigenvalue multiplicity. The natural time scale of the trace which we consider, namely $(Pe^{r_1\Delta}P)^{\lfloor r_2/r_1 \rfloor + 1} \approx e^{\lfloor r_2/r_1 \rfloor r_1 \Delta}$, is $O(r_1 \lfloor r_2/r_1 \rfloor) = O(c \log(g))$. With this time scale, it is impossible to distinguish eigenvalues that differ by $O(1/\log(g))$. However, as covered in Proposition 1.3, there exist closed hyperbolic surfaces such that there are $g/\log^{\frac{3}{2}}(g)$ eigenvalues in a range of (roughly) that size around the second eigenvalue. Therefore, as in the graph case (see [MRS21, Section 6]), our approach does not seem adequate to bound above the multiplicity of $\lambda_2(M)$ by a better bound than $g/\log^{\frac{3}{2}}(g)$ (which would be slightly better than our results).*

Analogously, the spectral bounds obtained in [Mon22, Theorems 4 and 5] do not give precise information in spectral windows of size $\ll 1/\sqrt{\log(g)}$.

4.3 Proof of Proposition 1.3

We recall that for every $d \geq 3$, there exists an infinite sequence of d -regular bipartite Ramanujan graphs, see [MSS15, Theorem 5.5]. The following result from [MRS21] shows that for these graphs the number of eigenvalues close to the second largest is at least $n/\log^{\frac{3}{2}}(n)$.

Proposition 4.3 (Proposition 5.3 in [MRS21]). *There exist $K, C_0 > 0$ such that for every fixed d , every bipartite d -regular Ramanujan graph G on n vertices satisfies*

$$m_{A_G} \left(\left[\left(1 - K \frac{\log \log(n)}{\log(n)}\right) \mu_2, \mu_2 \right] \right) \geq C_0 \frac{n}{\log^{\frac{3}{2}}(n)}$$

where $m_{A_G}([a, b])$ denotes the number of eigenvalues of the adjacency matrix A_G of G between a and b , and μ_2 is the second largest eigenvalue of A_G .

The above result is written in [MRS21] for the adjacency matrix A_G of G , but translates immediately to Laplacian eigenvalues, i.e., eigenvalues of $\tilde{A}_G = d\text{Id} - A_G$ (which is a non-negative matrix corresponding to the positive Laplacian on G): if $\lambda_2 = d - \mu_2 > 0$ denotes the smallest eigenvalue > 0 of \tilde{A}_G , we have

$$m_{\tilde{A}_G} \left(\left[\lambda_2, \left(1 + K' \frac{\log \log(n)}{\log(n)}\right) \lambda_2 \right] \right) \geq C_0 \frac{n}{\log^{\frac{3}{2}}(n)} \quad (54)$$

where K' may be computed in terms of K and d only.

The following lemma implies Proposition 1.3.

Lemma 4.4 (Extracted from [CC88]). *Let $G = (V, E)$ be a non-oriented finite graph, whose vertices have degrees $d_i \geq 3$ for any $i \in V$, and whose edge lengths are denoted by $(\theta_{i,j})_{\{i,j\} \in E}$. Then for any sufficiently small $\varepsilon > 0$ there exists a closed hyperbolic surface of genus $|E| - |V| + 1$ whose first $|V|$ eigenvalues of the positive Laplacian coincide (with corresponding multiplicities) with the eigenvalues of εq_θ where q_θ is the Dirichlet form*

$$q_\theta(x) = \frac{1}{\pi} \sum_{\{i,j\} \in E} \theta_{i,j} |x_i - x_j|^2, \quad x \in \mathbb{R}^V \quad (55)$$

on $L^2(V, \mu)$ and $\mu = 2\pi \sum_{i \in V} (d_i - 2)\delta_i$ with δ_i the Dirac mass on $i \in V$.

Sketch of proof of Lemma 4.4 extracted from [CC88]. For any $i \in V$, we denote by V_i the set of $j \in V$ such that $\{i, j\} \in E$. The degree of $i \in V$ is $d_i = |V_i| \geq 3$.

The authors of [CC88] first construct a closed hyperbolic surface X as follows: to the vertex $i \in V$ is associated X_i , a compact hyperbolic surface with d_i free geodesics $(\gamma_{i,j})_{j \in V_i}$ on its boundary, by gluing $d_i - 2$ pants (see [CC88, Section VI] and its figures for the case of the complete graph). We impose that $\ell(\gamma_{i,j}) = \ell(\gamma_{j,i}) = \theta_{i,j}$. To construct the surface X we glue the pieces X_i as indicated by the graph G : for $\{i, j\} \in E$, we glue X_i and X_j by identifying without twist $\gamma_{i,j}$ and $\gamma_{j,i}$.

In [CC88, Section II], the authors construct from X a family of closed hyperbolic surfaces X^ε ($0 < \varepsilon \leq 1$) as follows. The geodesics in the pant decomposition of X which do not belong to the boundary of one of the X_i , $i \in V$, remain of fixed length. For $\{i, j\} \in E$, the geodesic $\gamma_{i,j}$ of X is replaced in X^ε by a geodesic $\gamma_{i,j}^\varepsilon$ of length $\ell_{i,j}^\varepsilon = \varepsilon \theta_{i,j}$. Note that $\text{vol}(X_i^\varepsilon) = \text{vol}(X_i) = 2\pi(d_i - 2)$ for any $i \in V$, by the Gauss-Bonnet formula.

Then, in [CC88, Section V], they consider the measure $\mu = 2\pi \sum_{i \in V} (d_i - 2) \delta_i$ on G , and the quadratic form q_θ on $L^2(V, \mu)$ given by (55), which is the Dirichlet form on G endowed with edge lengths $\theta = (\theta_{i,j})_{\{i,j\} \in E}$. The key step in [CC88, Sections I and V] is to exhibit a quadratic form q_θ^ε on $L^2(V, \mu)$ (depending continuously on the geometric parameter θ) whose spectrum is the set of first $|V|$ non-trivial eigenvalues of X^ε and such that $\lim_{\varepsilon \rightarrow 0} \|(q_\theta^\varepsilon/\varepsilon) - q_\theta\| = 0$, uniformly in $\theta \in W$ for every compact $W \subseteq (\mathbb{R}_{>0})^E$. In particular it implies that the eigenvalues $(\lambda_i(\varepsilon))_{i \in V}$ of X^ε verify $\lambda_i(\varepsilon) \sim \varepsilon \lambda_i$ ($\varepsilon \rightarrow 0$) where the $(\lambda_i)_{i \in V}$ are the eigenvalues of q_θ on $L^2(V, \mu)$.

Finally, using a topological lemma in [CC88, Section VI] the authors conclude that for any $\theta_0 \in (\mathbb{R}_{>0})^{|E|}$ and any $\varepsilon > 0$ sufficiently small, there exists a choice of θ close to θ_0 such that $q_\theta^\varepsilon = \varepsilon q_{\theta_0}$. The spectrum of q_θ^ε consists of the $|V|$ first eigenvalues of X_θ^ε , and the genus of X_θ^ε is equal to $(p + 2)/2$ where $p = \sum_{i \in V} (d_i - 2) = 2|E| - 2|V|$ is the number of pants used in the decomposition. This concludes the proof of Lemma 4.4. \square

End of the proof of Proposition 1.3. We consider an infinite sequence $(G_k)_{k \in \mathbb{N}}$ of 3-regular bipartite Ramanujan graphs (see [MSS15, Theorem 5.5]). We use Lemma 4.4 for each $G_k = (V_k, E_k)$ and with $\theta_{i,j} = 1$ for every edge $\{i, j\} \in E_k$. In this case, the measure μ_k on V_k gives the same mass to each vertex, and q_θ given by (55) on $L^2(V_k, \mu_k)$ has the same spectrum as \tilde{A}_{G_k} by definition. Applying Proposition 4.3 (in the form (54)) to each $G_k = (V_k, E_k)$ for each $k \in \mathbb{N}$, we obtain a hyperbolic surface of genus $g_k = |E_k| - |V_k| + 1 = \frac{1}{2}|V_k| + 1$ satisfying the lower bound (54) with $n = |V_k|$ (and thus also with g_k replacing n , up to changing constants). This concludes the proof of Proposition 1.3. \square

A Appendix

We gather in this appendix several statements and proofs of elementary facts that are used throughout the proofs of Theorems 1.1 and 1.2.

A.1 Eigenvalues and trace

We start with the following infinite-dimensional version of the Cauchy interlacing theorem (see also [DD87, Theorem 2]).

Theorem A.1 (Cauchy interlacing theorem). *Let A be a compact self-adjoint operator on a Hilbert space H . Let $P = P^\top$ be an orthogonal projection onto a subspace of H of codimension $k \in \mathbb{N}$. We denote by $\alpha_1 \geq \dots \geq \alpha_m \geq \dots$ the eigenvalues of A , and by $\beta_1 \geq \dots \geq \beta_m \geq \dots$*

those of $B = PAP$. Then for any $j \in \mathbb{N}$,

$$\alpha_j \geq \beta_j \geq \alpha_{j+k}.$$

Proof of Theorem A.1. Since B is compact and self-adjoint, the spectral theorem provides a basis $(b_j)_{j \in \mathbb{N}}$ of normalized eigenvectors of B , with $Bb_j = \beta_j b_j$ for any $j \in \mathbb{N}$. We set $S_j = \text{Span}(b_1, \dots, b_j)$ and we notice that $S_j \subset \text{Im}(P)$. We compute

$$\beta_j = \min_{x \in S_j, \|x\|=1} (PAPx, x) = \min_{x \in S_j, \|x\|=1} (Ax, x) \leq \max_{V, \dim(V)=j} \min_{x \in V, \|x\|=1} (Ax, x) = \alpha_j.$$

Also, noticing that PS_{j-1}^\perp has codimension at most $k+j-1$ we obtain

$$\begin{aligned} \beta_j &= \max_{x \in S_{j-1}^\perp, \|x\|=1} (PAPx, x) \geq \max_{x \in PS_{j-1}^\perp, \|x\|=1} (PAPx, x) = \max_{x \in PS_{j-1}^\perp, \|x\|=1} (Ax, x) \\ &\geq \min_{V, \text{codim } V \leq k+j-1} \max_{x \in V, \|x\|=1} (Ax, x) = \alpha_{k+j} \end{aligned}$$

which concludes the proof. \square

We recall the following estimate:

Lemma A.2 (Upper bound on eigenvalues). *For any $b \in \mathbb{R}$ and any $j \in \mathbb{N}_{\geq 2}$, there exists $C_j > 0$ such that any closed surface M with curvature bounded below by b verifies $\lambda_j(M) \leq C_j$.*

Proof. The diameter d of a closed negatively curved surface M with curvature bounded below by b is bounded below since for any $x \in \widetilde{M}$,

$$\frac{4\pi}{|b|} \leq \text{vol}(M) \leq \text{Vol}_{\widetilde{M}}(B_{\widetilde{M}}(x, d)) \leq \frac{4\pi}{|b|} \sinh^2\left(\frac{d}{2}\sqrt{|b|}\right)$$

where the right-hand side comes from (9). Combining with [Che75, Corollary 2.3] we get the result. \square

Lemma A.3 (Computation of the trace). *For any $n \in \mathbb{N}_{\geq 1}$ and $t \geq 1$, there holds*

$$\text{Tr}((Pe^{t\Delta}P)^{2n}) = \int_M \|(Pe^{t\Delta}P)^n \delta_x\|^2 d\nu(x).$$

Proof. We set $Q = Pe^{mt\Delta}P$. Let $(u_j)_{j \in \mathbb{N}}$ denote an orthonormal basis of eigenfunctions of the compact and self-adjoint operator Q^n , with associated eigenvalues λ_j . For any $x \in M$ we set $u_x = \sum_{j \in \mathbb{N}} \lambda_j u_j(x) u_j$. We know that Q^{2n} is trace-class since e^Δ is trace-class, and

$$\text{Tr}(Q^{2n}) = \sum_{j \in \mathbb{N}} \lambda_j^2 = \int_M \left(\sum_{j \in \mathbb{N}} \lambda_j^2 u_j(x)^2 \right) \nu(dx) = \int_M \|u_x\|^2 \nu(dx). \quad (56)$$

In particular, $u_x \in L^2(M, \nu)$ for ν -almost every $x \in M$. For any such x and any $f \in C^\infty(M)$, written as $f = \sum_{j \in \mathbb{N}} a_j u_j$, we have

$$\begin{aligned} \langle Q^n \delta_x, f \rangle_{\mathcal{D}', \mathcal{D}} &= \langle P \delta_x, e^{t\Delta} P Q^{n-1} f \rangle_{\mathcal{D}', \mathcal{D}} = e^{t\Delta} P Q^{n-1} f(x) - \sum_{k \in \mathbb{N}} \langle e^{t\Delta} P Q^{n-1} f, \psi_k \rangle \psi_k(x) \\ &= Q^n f(x) = \sum_{j \in \mathbb{N}} a_j \lambda_j u_j(x) = \int_M u_x(y) f(y) \nu(dy) \end{aligned}$$

where the first equality comes from the fact that the transpose (in the sense of distributions) of the continuous linear map from smooth functions to smooth functions $e^{t\Delta} P Q^{n-1}$ is $Q^{n-1} P e^{t\Delta}$; and the second equality follows from (14). We deduce from this computation that $Q^n \delta_x$ coincides with the distribution $\langle u_x, \cdot \rangle_{L^2(M, \nu)}$, which is identified to $u_x \in L^2(M, \nu)$. Plugging into (56), this concludes the proof. \square

A.2 Heat kernel: comparison and estimates

We provide here the proofs of Lemma 2.4 and 2.5 on the heat kernel in \widetilde{M} and M .

Proof of Lemma 2.4. We recall from [Dav89, Theorem 5.7.2] that there exist constants $c_1, c_2 > 0$ such that for any $\eta, t > 0$,

$$c_1 g_1(t, \eta) \leq k_t^{\mathbb{H}^2}(\eta) \leq c_2 g_1(t, \eta)$$

where $k_t^{\mathbb{H}^2}$ denotes the heat kernel in the hyperbolic plane (equal to $k^{(-1)}$ with the notation of Lemma 2.1) and

$$g_1(t, \eta) = \frac{1}{t} \frac{1 + \eta}{(1 + \eta + t)^{\frac{1}{2}}} \exp\left(-\frac{t}{4} - \frac{\eta}{2} - \frac{\eta^2}{4t}\right).$$

For $K < 0$ we consider

$$g_{|K|}(t, \eta) = \frac{1}{|K|t} \frac{1 + \eta}{(1 + \eta + |K|t)^{\frac{1}{2}}} \exp\left(-\frac{|K|t}{4} - \frac{\eta}{2} - \frac{\eta^2}{4|K|t}\right),$$

which is the analogue of g_1 on the space form \widetilde{M}_K introduced in Section 2.1. Using Lemma 2.1 we obtain for the heat kernel $k_t(\cdot, \cdot)$ in \widetilde{M} that

$$C_1 g_{|b|}(t, d_{\widetilde{M}}(x, y)) \leq k_t(x, y) \leq C_2 g_{|a|}(t, d_{\widetilde{M}}(x, y)) \quad (57)$$

for any $x, y \in \widetilde{M}$ and any $t > 0$, where C_1 and C_2 depend on $(a, b, \rho) \in \mathcal{T}$ only.

For (15), we set for $n \in \mathbb{N}$

$$A_n = \left\{ y \in \widetilde{M} \mid Ct + n \leq d_{\widetilde{M}}(x, y) < Ct + n + 1 \right\} \subset \widetilde{M}.$$

Then $\text{Vol}_{\widetilde{M}}(A_n) \leq \text{Vol}_{\widetilde{M}}(B(x, Ct + n + 1)) \leq C_0 \exp((Ct + n)\sqrt{|b|})$ according to (9). We write $\widetilde{M} \setminus B_{\widetilde{M}}(x, Ct) = \bigcup_{n=0}^{\infty} A_n$, and then using (57) and the fact that $t \geq 1$, we obtain

$$\begin{aligned} \|k_t(x, \cdot)\|_{L^1(\widetilde{M} \setminus B_{\widetilde{M}}(x, Ct))} &\leq C_3 \sum_{n=0}^{\infty} \frac{(Ct + n)}{t} \exp\left(-\frac{|a|t}{4} - \frac{(Ct + n)^2}{4|a|t}\right) \text{Vol}_{\widetilde{M}}(A_n) \\ &\leq C_3 \int_{Ct-1}^{\infty} \frac{\eta}{t} \exp\left(-\frac{|a|t}{4} - \frac{\eta^2}{4|a|t}\right) \exp(\eta\sqrt{|b|}) d\eta \\ &\leq C_3 \exp(|ab|t) \int_{Ct-1}^{\infty} \frac{\eta}{t} \exp\left(-\frac{(\eta - 2|a|t\sqrt{|b|})^2}{4|a|t}\right) d\eta \end{aligned} \quad (58)$$

We make the change of variables $\eta' = \eta - 2|a|t\sqrt{|b|}$ and we use that $C - \frac{1}{t} - 2|a|\sqrt{|b|} \geq C/2$ and $\eta + 2|a|t\sqrt{|b|} \leq 2\eta$ for $\eta \geq Ct/2$ to obtain that (58) is bounded above by

$$C_3 \exp(|ab|t) \int_{Ct/2}^{\infty} \frac{\eta}{t} \exp\left(-\frac{\eta^2}{4|a|t}\right) d\eta.$$

Computing explicitly this last integral gives the result.

For (16), we set $\eta = d_{\widetilde{M}}(x, y)$ and $\alpha = d_{\widetilde{M}}(x, z) - d_{\widetilde{M}}(x, y)$. We have

$$\frac{k_t(x, z)}{k_t(x, y)} \geq C_0 \frac{g_{|b|}(t, \eta + \alpha)}{g_{|a|}(t, \eta)} = C_0 \frac{|a|}{|b|} \left(\frac{1 + \eta + |a|t}{1 + \eta + \alpha + |b|t} \right)^{\frac{1}{2}} h(\alpha, \eta) \quad (59)$$

where

$$\begin{aligned}
h(\alpha, \eta) &= \frac{1 + \eta + \alpha}{1 + \eta} \exp\left(-\frac{\alpha}{2} - \frac{|b|t}{4} + \frac{|a|t}{4} - \frac{(\eta + \alpha)^2}{4|b|t} + \frac{\eta^2}{4|a|t}\right) \\
&\geq C_0(1 + Ct)^{-1} \exp\left(-2t - \frac{|b|t}{4} - \frac{2C(t+1)}{|b|} - \frac{4t}{|b|}\right) \\
&\geq C_0 \exp\left(-\left(1 + |b| + C + \frac{C+1}{|b|}\right)4t\right)
\end{aligned} \tag{60}$$

where we used in the second line $\eta \leq Ct$ and $|\alpha| \leq 4t + 4$. We also notice that $\frac{1+\eta+|a|t}{1+\eta+\alpha+|b|t} \geq \frac{|a|}{|a|+|b|+8}$ when $|\alpha| \leq 4t + 4$. Combining (59) and (60) we get (16). \square

Proof of Lemma 2.5. We write $M = \Gamma \backslash \widetilde{M}$. We prove that there exists $C_0 > 0$ (depending only on $(a, b, \rho) \in \mathcal{T}$) such that for any $\eta \geq 0$ and any $\bar{x} \in \widetilde{M}$, the number of elements $\gamma \in \Gamma$ such that $d_{\widetilde{M}}(\bar{x}, \gamma\bar{x}) < \eta + 1$ is at most $C_0 e^{\eta\sqrt{|b|}}$. By definition of the injectivity radius ρ , the open balls B_γ of center $\gamma\bar{x}$ and radius $\rho/2$, for $\gamma \in \Gamma$, are disjoint. If $\gamma \in \Gamma$ is such that $d_{\widetilde{M}}(\bar{x}, \gamma\bar{x}) < \eta + 1$, then B_γ is included in the ball of center \bar{x} and radius $\eta + 1 + \rho/2$. According to (9), the volume of a ball of radius $\eta + 1 + \rho/2$ in M is at most $\frac{4\pi}{|b|} \sinh^2\left(\frac{1}{2}(\eta + 1 + \rho/2)\sqrt{|b|}\right)$, and according to [Cro80, Proposition 14], the volume of a ball of radius $\rho/2$ is at least $C_1 > 0$. Therefore, the number of $\gamma \in \Gamma$ such that $d_{\widetilde{M}}(\bar{x}, \gamma\bar{x}) < \eta + 1$ is smaller than

$$C_0 \sinh^2\left(\frac{1}{2}\left(\eta + 1 + \frac{\rho}{2}\right)\sqrt{|b|}\right)$$

which in turn is bounded above by $C_0 e^{\eta\sqrt{|b|}}$.

As a consequence, for any $\bar{x}, \bar{y} \in \widetilde{M}$ and $\eta \geq 0$,

$$\#\{\gamma \in \Gamma \mid \eta \leq d(\bar{x}, \gamma\bar{y}) < \eta + 1\} \leq C_0 e^{2\eta\sqrt{|b|}}. \tag{61}$$

Below, \bar{x}, \bar{y} are lifts of given $x, y \in M$ to a fundamental domain of M in \widetilde{M} . For any $y \in M$ we have, using (57) in the first line, (61) in the second line, and then a series-integral comparison for the last inequality (cutting the sum at $\eta = 4t|a|\sqrt{|b|}$)

$$\begin{aligned}
\sum_{\gamma \in \Gamma} k_t(\bar{x}, \gamma\bar{y}) &\leq C_0 \sum_{\eta=0}^{\infty} (\#\{\gamma \in \Gamma \mid \eta \leq d(\bar{x}, \gamma\bar{y}) < \eta + 1\}) \exp\left(-\frac{|a|t}{4} - \frac{\eta^2}{4|a|t}\right) \\
&\leq C_0 \exp\left(-\frac{|a|t}{4}\right) \sum_{\eta=0}^{\infty} \exp\left(2\eta\sqrt{|b|} - \frac{\eta^2}{4|a|t}\right) \leq C_0 \exp(4|ab|t).
\end{aligned}$$

Using (10) we get the result. \square

References

[ABBGNS17] Miklos Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault and Iddo Samet. *On the growth of L^2 -invariants for sequences of lattices in Lie groups*. Annals of Mathematics 185.3 (2017): 711-790.

[Bes80] Gérard Besson. *Sur la multiplicité de la première valeur propre des surfaces Riemanniennes*. Annales de l'institut Fourier 30.1 (1980): 109-128.

- [Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Springer Science & Business Media, 2010.
- [Bus77] Peter Buser. *Riemannsche Flächen mit Eigenwerten in $(0, 1/4)$* . Commentarii Mathematici Helvetici 52 (1977): 25-34.
- [Bus82] Peter Buser. *A note on the isoperimetric constant*. Annales scientifiques de l'École normale supérieure 15.2 (1982): 213-230.
- [Che75] Shiu-Yuen Cheng. *Eigenvalue comparison theorems and its geometric applications*. Mathematische Zeitschrift 143 (1975): 289-297.
- [Che76] Shiu-Yuen Cheng. *Eigenfunctions and nodal sets*. Commentarii Mathematici Helvetici 51 (1976): 43-55.
- [CC88] Bruno Colbois and Yves Colin de Verdière. *Sur la multiplicité de la première valeur propre d'une surface de Riemann à courbure constante*. Commentarii Mathematici Helvetici 63 (1988): 194-208.
- [Col86] Yves Colin de Verdière. *Sur la multiplicité de la première valeur propre non nulle du Laplacien*. Commentarii Mathematici Helvetici 61.1 (1986): 254-270.
- [Col87] Yves Colin de Verdière. *Construction de laplaciens dont une partie finie du spectre est donnée*. Annales scientifiques de l'École normale supérieure 20.4 (1987): 599-615.
- [Cro80] Christopher B. Croke. *Some isoperimetric inequalities and eigenvalue estimates*. Annales scientifiques de l'École normale supérieure. 13.4 (1980): 419-435.
- [DD87] Jerome Dancis and Horace C. Davis. *An interlacing theorem for eigenvalues of self-adjoint operators*. Linear Algebra and its Applications 88 (1987): 117-122.
- [Dav89] Edward B. Davies. *Heat kernels and spectral theory*. Cambridge University Press, 1989.
- [DGM76] Amédée Debiard, Bernard Gaveau and Edmond Mazet. *Théorèmes de comparaison en géométrie Riemannienne*. Publications of the Research Institute for Mathematical Sciences 12.2 (1976): 391-425.
- [DW78] David L. DeGeorge and Nolan R. Wallach. *Limit formulas for multiplicities in $L^2(\Gamma \backslash G)$* . Annals of Mathematics 107.2 (1978): 133-150.
- [FP21] Maxime Fortier Bourque and Bram Petri. *The Klein quartic maximizes the multiplicity of the first positive eigenvalue of the Laplacian*. arXiv:2111.14699 (2021). To appear in Journal of Differential Geometry.
- [FP23] Maxime Fortier Bourque and Bram Petri. *Linear programming bounds for hyperbolic surfaces*. arXiv:2302.02540 (2023).
- [Gam02] Alex Gamburd. *On the spectral gap for infinite index "congruence" subgroups of $SL_2(\mathbb{Z})$* . Israel Journal of Mathematics 127.1 (2002): 157-200.
- [GLST21] Clifford Gilmore, Etienne Le Masson, Tuomas Sahlsten and Joe Thomas. *Short geodesic loops and L^p norms of eigenfunctions on large genus random surfaces*. Geometric and Functional Analysis 31.1 (2021): 62-110.

- [HSZZ22] Milan Haiman, Carl Schildkraut, Shengtong Zhang and Yufei Zhao. *Graphs with high second eigenvalue multiplicity*. Bulletin of the London Mathematical Society 54.5 (2022): 1630-1652.
- [JTYZZ21] Zilin Jiang, Jonathan Tidor, Yuan Yao, Shengtong Zhang and Yufei Zhao. *Equian-gular lines with a fixed angle*. Annals of Mathematics 194.3 (2021): 729-743.
- [LM10] Gilles Lebeau and Laurent Michel. *Semi-classical analysis of a random walk on a man-ifold*. The Annals of Probability 38.1 (2010): 277-315.
- [MSS15] Adam W. Marcus, Daniel A. Spielman and Nikhil Srivastava. *Interlacing families I: Bipartite Ramanujan graphs of all degrees*. Annals of Mathematics 182.1 (2015): 307-325.
- [MRS21] Theo McKenzie, Peter Michael Reichstein Rasmussen and Nikhil Srivastava. *Support of closed walks and second eigenvalue multiplicity of graphs*. Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing. 2021.
- [MP19] Maryam Mirzakhani and Bram Petri. *Lengths of closed geodesics on random surfaces of large genus*. Commentarii Mathematici Helvetici 94.4 (2019): 869-889.
- [Mon22] Laura Monk. *Benjamini–Schramm convergence and spectra of random hyperbolic sur-faces of high genus*. Analysis & PDE 15.3 (2022): 727-752.
- [Nad87] Nikolai S. Nadirashvili. *Multiple eigenvalues of the Laplace operator*. Matematicheskii Sbornik 175.2 (1987): 223-237.
- [RS78] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press, New York-London, 1978.
- [RY68] Gerhard Ringel and John WT Youngs. *Solution of the Heawood map-coloring problem*. Proceedings of the National Academy of Sciences 60.2 (1968): 438-445.
- [SX91] Peter Sarnak and Xiaoxi Xue. *Bounds for multiplicities of automorphic representations*. Duke Mathematical Journal 64.1 (1991): 207-227.
- [SY94] Richard M. Schoen and Shing-Tung Yau. *Lectures on differential geometry*. Vol. 2. Cam-bridge, MA: International press, 1994.
- [Sev02] Bruno Sévenec. *Multiplicity of the second Schrödinger eigenvalue on closed surfaces*. Mathematische Annalen 324.1 (2002): 195-211.