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# A note on the wave equation controlled with a dynamic saturating boundary control

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**Abstract:** This paper studies the nonlinear systems obtained by considering a wave equation in closed loop with a nonlinear dynamical boundary controller. The controller is subject to a magnitude limitation and modeled by a linear ordinary differential equation with a saturation map in the input. The well-posedness of the obtained infinite-dimensional system is first studied and then two stability results are given. These two stability results apply for two cascade cases and give sufficient conditions for the asymptotic stability of the equilibrium. The well-posedness is proven by using nonlinear semigroups techniques, whereas the global asymptotic stability results are obtained by Lyapunov-based arguments in infinite-dimensional state space.

*Keywords:* Infinite-dimensional systems, saturation, well-posedness, stability

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## 1. INTRODUCTION

Nonlinear control design is now very well developed for finite-dimensional systems, in particular for the stabilization of equilibrium by means of input with limited amplitude. Such a limitation on the control input can reduce the performance or even destabilize open-loop unstable systems (see e.g., Zaccarian and Teel (2011); Tarbouriech et al. (2011)). Lyapunov methods are fruitful techniques to analyze the stability of linear control systems in closed loop with saturating controls (see e.g., Dai et al. (2009)). Such methods are also instrumental to derive numerical techniques for the estimation of convergence speed or of the performance (as studied in Gomes da Silva Jr and Tarbouriech (2005)). In the context of infinite-dimensional systems, less results are available, except for specific partial differential equations as elastic systems (as considered in Lasiecka and Seidman (2003)), and some abstract systems as considered in Slemrod (1989). More recently estimation of the basin of attraction for reaction-diffusion models in closed loop with saturating controllers has been given in Mironchenko et al. (2021) (see also Lhachemi and Prieur (2023)).

The goal of this paper is to consider another class of partial differential equations, more specifically the wave equation with a saturating boundary control. The dynamical systems under consideration in this paper may result from a wave equation in closed loop with a dynamical boundary controller. The stability of such infinite-dimensional model has been already studied in Prieur et al. (2016) when closing the loop with saturating static boundary controller. The goal of this paper is to generalize this paper by considering dynamic feedback law. It asks to introduce specific

Hilbert state and to apply suitable abstract control theory providing a well-posedness result, that is the existence and the unicity of the solution to the nonlinear Cauchy problem. To analyze the stability, Lyapunov approach is developed and a LaSalle invariance principle is used in the infinite-dimensional state space. To handle the nonlinearity due to the use of saturating controller, sector conditions are used. Note that, in contrast to reaction-diffusion equations studied in e.g., Sakawa (1983), the wave equation is marginally stable in open loop, so that global asymptotic stability of the equilibrium is derived (and not only a local property). Such a coupling between infinite-dimensional systems and ordinary differential equations is quite usual in the linear case (see e.g., Baudouin et al. (2019); Bajodek et al. (2022); Karafyllis and Krstic (2019); Marx et al. (2021)), but not so well studied with nonlinearities in between (except in particular by the papers Ramirez et al. (2017); Schmid and Zwart (2021)).

The well-posedness is derived for a wave equation coupled at the boundary with an ordinary differential equation with a saturation in between, under a natural assumption. The asymptotic stability is given in this paper for two particular cases of cascade systems, that is when the wave equation is in cascade with the ODE, or vice-versa. Related works for hyperbolic systems controlled by saturating control inputs include the works Dus et al. (2020) and Shreim et al. (2022).

This paper is organized as follows. First in Section 2 the model and the problem statement are presented, together with the well-posedness results. The stability of the cascade systems are given in Section 3. Concluding remarks are collected in Section 4.

## 2. PROBLEM STATEMENT AND WELL-POSEDNESS RESULT

We are interested in a partial differential equation coupled at the boundary with an ordinary differential equation. To be more specific, the model that is considered in this paper is written as, for all  $0 < x < 1$  and for all  $t \geq 0$ ,

$$z_{tt}(x, t) = z_{xx}(x, t), \quad (1)$$

$$\dot{w} = Aw + Bz_t(1, t), \quad (2)$$

$$z(0, t) = 0, \quad (3)$$

$$z_x(1, t) + \text{sat}(Dz_t(1, t) + Cw(t)) = 0, \quad (4)$$

where  $z(x, t)$  is the amplitude of the wave dynamics with respect to the rest position, at point  $x$  in  $[0, 1]$  and at time  $t \geq 0$ ,  $w(t)$  is a dynamical state (in  $\mathbb{R}^n$ ) solving a linear finite-dimensional differential equation, and  $A$ ,  $B$  and  $C$  are matrices of appropriate dimensions. In (4),  $\text{sat}$  stands for the usual saturation function defined, for all  $s$  in  $\mathbb{R}$ , by

$$\begin{aligned} \text{sat}(s) &= s & \text{if } |s| \leq u_0 \\ &= u_0 & \text{otherwise} \end{aligned}$$

where  $u_0 > 0$  is a given scalar modeling the saturation level.

Let us use the following notation  $H_{(0)}^1(0, 1) = \{z \in H^1(0, 1), z(0) = 0\}$ ,  $\mathcal{H} = H_{(0)}^1(0, 1) \times L^2(0, 1)$  and  $\mathfrak{H} = \mathcal{H} \times \mathbb{R}^n$ . The linear system

$$z_{tt}(x, t) = z_{xx}(x, t), \quad (5)$$

$$\dot{w} = Aw + Bz_t(1, t), \quad (6)$$

$$z(0, t) = 0, \quad (7)$$

$$z_x(1, t) + Dz_t(1, t) + Cw(t) = 0, \quad (8)$$

is well-posed if and only if  $D \neq -1$ . The proof of this well-posedness result from the classical Lumer-Philips theorem (see e.g., (Pazy, 1983, Chapter 1), and is not given in this paper.

In this paper we focus on the nonlinear dynamics (1)-(4), and let us first prove the well-posedness result of the nonlinear dynamics (1)-(4). To do that, we need to introduce the following nonlinear operator  $\mathcal{A}$  defined on the domain

$$D(\mathcal{A}) = \left\{ (u, v, w) \in \mathfrak{H}, u \in H^2(0, 1), v \in H_{(0)}^1(0, 1), \right. \\ \left. u'(1) + \text{sat}(Dv(1) + Cw) = 0 \right\}$$

defined by, for all  $(u, v, w) \in D(\mathcal{A})$ ,

$$\mathcal{A}(u, v, w) = (v, u'', Aw + Bv(1)).$$

Given an initial condition  $(u_0, v_0, w_0)$  in  $D(\mathcal{A})$ , and  $T > 0$ , we say that a function  $t \mapsto (u, v, w)$  is a solution to the Cauchy problem

$$\begin{aligned} \frac{d}{dt}(u(t), v(t), w(t)) &= \mathcal{A}(u(t), v(t), w(t)) \\ (u(t=0), v(t=0), w(t=0)) &= (u_0, v_0, w_0) \end{aligned} \quad (9)$$

on  $[0, T]$ , if  $(u, v, w)$  is in  $\mathcal{C}^1([0, T]; \mathfrak{H})$ , and both lines of this equation make sense in the classical sense. The system (1)-(4) is said to be well-posed, if given any initial condition  $(u_0, v_0, w_0)$  in  $D(\mathcal{A})$ , there exists a unique solution to (9). The first main result is the following

**Theorem 1.** *If  $D > -1$ , then the system (1)-(4) is well-posed.*

To prove this result, we first prove the following

**Lemma 1.** *For all  $D$  in  $\mathbb{R}$ ,  $D(\mathcal{A})$  is dense in  $\mathfrak{H}$ , and for any sufficiently large  $\lambda$ , it holds  $\text{Ran}(\mathcal{A} - \lambda I) = \mathfrak{H}$ .*

**Proof of Lemma 1.** The density of  $D(\mathcal{A})$  follows from classical arguments.

Let  $\lambda > 0$  sufficiently large so that  $\lambda \notin \sigma(A)$ , and let  $(f, g, h)$  in  $\mathfrak{H}$ , and consider the map

$$\mathcal{F} : H_{(0)}^1(0, 1) \rightarrow H^2(0, 1) \cap H_{(0)}^1(0, 1)$$

defined by, for all  $y$  in  $H_{(0)}^1(0, 1)$ ,  $\mathcal{F}(y) = u$  where  $u$  is the unique solution to

$$\begin{cases} u'' - \lambda^2 u = g + \lambda f \\ u(0) = u'(1) \\ +\text{sat}(C(A - \lambda I_n)^{-1}f + F(\lambda)(\lambda y(1) + f(1))) = 0 \end{cases} \quad (10)$$

where  $F$  is defined, for all  $\lambda \notin \sigma(A)$ , by  $F(\lambda) = D + C(\lambda I_n - A)^{-1}B$ . It could be proven that  $u$  is explicit and given, for all  $x$  in  $(0, 1)$ , by

$$u(x) = \alpha \frac{\text{sh}(\lambda x)}{\lambda} + \int_0^x \frac{\text{sh}(\lambda(x-t))}{\lambda} (g(t) + \lambda f(t)) dt,$$

where

$$\begin{aligned} \alpha &= -\frac{1}{\text{ch}(\lambda)} \left( \int_0^1 \text{ch}(\lambda(1-t))(g(t) + \lambda f(t)) dt \right. \\ &\quad \left. + \text{sat}(C(A - \lambda I_p)^{-1}h + F(\lambda)(\lambda y(1) + f(1))) \right) \end{aligned}$$

The operator  $\mathcal{F}$  is continuous from  $H_{(0)}^1(0, 1)$  to  $H_{(0)}^1(0, 1)$ .

Moreover, we can prove that there exists  $M$  such that, for all  $y$  in  $H_{(0)}^1(0, 1)$ ,

$$\begin{aligned} \|\mathcal{F}(y)\|_{L^2} &\leq M \\ \|\mathcal{F}(y)'\|_{L^2} &\leq M \\ \|\mathcal{F}(y)''\|_{L^2} &\leq M \end{aligned}$$

Therefore, there exists  $K$  (depending only on  $(f, g, h)$ ) such that  $\mathcal{F}(H_{(0)}^1(0, 1)) \subset C$  where  $C$  is the convex set defined by

$$C = \{f \in H_{(0)}^1(0, 1), f \in H^2(0, 1), \|f\|_{H^2} \leq K\}$$

The set  $C$  is compact in  $H_{(0)}^1(0, 1)$ . Therefore, with the Schauder fixed point theorem (see e.g., (Coron, 2007, Theorem B.17)), there exists  $u$  in  $H^2(0, 1) \cap H_{(0)}^1(0, 1)$  solution to (10). Denoting  $v = \lambda u + f$  and  $w = (A - \lambda I_p)^{-1}(h - f(1) + \lambda u(1))B$ , we have  $(u, v, w)$  in  $D(\mathcal{A})$  and  $(\mathcal{A} - \lambda I)(u, v, w) = (f, g, h)$ .

This concludes the proof of Lemma 1.  $\square$

Let us now state the following

**Lemma 2.** *If  $D > -1$ , then there exists a scalar product on  $\mathfrak{H}$  such that the operator  $\mathcal{A}$  is quasi-dissipative.*

**Proof of Lemma 2.** Recall that  $H_{(0)}^1(0, 1)$  is a Hilbert space with the inner product defined by, for all  $u_1$  and  $u_2$  in  $H_{(0)}^1(0, 1)$ ,

$$\langle u, \tilde{u} \rangle_{H_{(0)}^1(0, 1)} = \int_0^1 u_1'(x) u_2'(x) dx.$$

Given a symmetric positive definite matrix  $P$  in  $\mathbb{R}^{n \times n}$  (that will be determined later), let us consider the scalar product  $\langle \cdot, \cdot \rangle_{\mu, P}$  on  $\mathfrak{H} \times \mathfrak{H}$  defined, for all  $(u_1, v_1, w_1)$  and  $(u_2, v_2, w_2)$  in  $\mathfrak{H}$ , by

$$\begin{aligned}
& \langle (u_1, v_1, w_1), (u_2, v_2, w_2) \rangle_{\mu, P} = \\
& \int_0^1 e^{\mu x} (v_1 + u_1')(v_2 + u_2') dx \\
& + \int_0^1 e^{-\mu x} (v_1 - u_1')(v_2 - u_2') dx \\
& + w_1^\top P w_2
\end{aligned} \tag{11}$$

Denote by  $\mathcal{I}$  the identity operator in  $\mathfrak{H}$ .

Given a positive  $\alpha$  (that will be determined later) and  $(u_i, v_i, w_i)$  in  $\mathfrak{H}$ , for  $i = 1, 2$ . Let us denote  $(\delta u, \delta v, \delta w) = (u_1, v_1, w_1) - (u_2, v_2, w_2)$ . We compute

$$\begin{aligned}
& \langle (u_1, v_1, w_1) - (u_2, v_2, w_2), \\
& (\mathcal{A} - \alpha \mathcal{I})(u_1, v_1, w_1) - (\mathcal{A} - \alpha \mathcal{I})(u_2, v_2, w_2) \rangle_{\mu, P} \\
& = \langle (\delta u, \delta v, \delta w), \\
& (\mathcal{A} - \alpha \mathcal{I})(\delta u, \delta v, \delta w) \rangle_{\mu, P} \\
& = \int_0^1 e^{\mu x} (\delta v + \delta u') (\delta u'' - \alpha \delta v + \delta v' - \alpha \delta u') dx \\
& + \int_0^1 e^{-\mu x} (\delta v - \delta u') (\delta u'' - \alpha \delta v - \delta v' + \alpha \delta u') dx \\
& + \delta w^\top P (A - \alpha I_n) \delta w + \delta w^\top P \delta v(1) \\
& = \int_0^1 e^{\mu x} (\delta v + \delta u') (\delta u'' + \delta v') dx \\
& + \int_0^1 e^{-\mu x} (\delta v - \delta u') (\delta u'' - \delta v') dx \\
& - \alpha \int_0^1 e^{\mu x} (\delta v + \delta u')^2 dx - \alpha \int_0^1 e^{-\mu x} (\delta v - \delta u')^2 dx \\
& + \delta w^\top P (A - \alpha I_n) \delta w + \delta w^\top P B \delta v(1)
\end{aligned}$$

where  $\delta v(1) = v_1(1) - v_2(1)$ ,  $\delta u' = u_1 - u_2'$  and similarly for  $\delta u''$  and  $\delta v'$ . To ease the reading, denote the previous expression by  $\Delta$ . Using an integration by parts in the second integral and using the definition of  $D(\mathcal{A})$ , we get

$$\begin{aligned}
\Delta & = -\left(\frac{\mu}{2} + \alpha\right) \int_0^1 e^{\mu x} (\delta v + \delta u')^2 dx \\
& - \left(\frac{\mu}{2} + \alpha\right) \int_0^1 e^{-\mu x} (\delta v - \delta u')^2 dx \\
& + \frac{1}{2} [e^{\mu x} (\delta v + \delta u')^2]_0^1 - \frac{1}{2} [e^{-\mu x} (\delta v - \delta u')^2]_0^1 \\
& + \delta w^\top P (A - \alpha I_n) \delta w + \delta w^\top P B \delta v(1) \\
& = -\left(\frac{\mu}{2} + \alpha\right) \int_0^1 e^{\mu x} (\delta v + \delta u')^2 dx \\
& - \left(\frac{\mu}{2} + \alpha\right) \int_0^1 e^{-\mu x} (\delta v - \delta u')^2 dx \\
& + \frac{1}{2} [e^{\mu x} (\delta v + \delta u')^2]_0^1 - \frac{1}{2} [e^{-\mu x} (\delta v - \delta u')^2]_0^1 \\
& + \delta w^\top P (A - \alpha I_n) \delta w + \delta w^\top P B \delta v(1) \\
& = -\left(\frac{\mu}{2} + \alpha\right) \int_0^1 e^{\mu x} (\delta v + \delta u')^2 dx \\
& - \left(\frac{\mu}{2} + \alpha\right) \int_0^1 e^{-\mu x} (\delta v - \delta u')^2 dx \\
& + \frac{1}{2} e^\mu (\delta v(1) + \delta u'(1))^2 \\
& - \frac{1}{2} e^{-\mu} (\delta v(1) - \delta u'(1))^2 \\
& + \delta w^\top P (A - \alpha I_n) \delta w + \delta w^\top P B \delta v(1)
\end{aligned}$$

where in the last equation  $\delta u'(1)$  is defined by

$$\delta u'(1) = -\text{sat}(Dv_1(1) + Cw_1) + \text{sat}(Dv_1(1) + Cw_1).$$

Denote the deadzone values  $\varphi_i = \text{sat}(Dv_i(1) + Cw_i) - (Dv_i(1) + Cw_i)$ , for  $i = 1, 2$ . By denoting  $\delta\varphi = \varphi_1 - \varphi_2$  it holds

$$\delta u'(1) = -\delta\varphi - (D\delta v(1) + C\delta w)$$

We will select  $\alpha > 0$  satisfying different conditions including the following inequality:

$$\alpha > -\frac{\mu}{2}. \tag{12}$$

We deduce

$$\begin{aligned}
\Delta & \leq \begin{pmatrix} \delta w \\ \delta v(1) \\ \delta\varphi \end{pmatrix}^\top \\
& \times \begin{pmatrix} (A - \alpha I_n)^\top P + P(A - \alpha I_n) + 2 \text{sh}(\mu) C^\top C \\ * \\ * \end{pmatrix} \\
& \begin{vmatrix} PB + 2(D \text{sh}(\mu) - \text{ch}(\mu)) C^\top & 2 \text{sh}(\mu) C^\top \\ (1 - D)^2 e^\mu - (1 + D)^2 e^{-\mu} & 2(\text{sh}(\mu) D - \text{ch}(\mu)) \\ * & 2 \text{sh}(\mu) \end{vmatrix} \\
& \times \begin{pmatrix} \delta w \\ \delta v(1) \\ \delta\varphi \end{pmatrix}
\end{aligned} \tag{13}$$

Recall that the deadzone values satisfy the so-called global sector condition (see (Tarbouriech et al., 2011, Lemma 1.4)), for all  $i = 1, 2$ :

$$\varphi_i(\varphi_i + Dv_i(1) + Cw_i) \leq 0.$$

In a similar way, denoting  $x_i = Dv_i(1) + Cw_i$ , for all  $i = 1, 2$ , it holds

$$\begin{aligned}
& \delta\varphi(\delta\varphi + D\delta v(1) + C\delta w) \\
& = (\varphi(x_1) - \varphi(x_2))(\varphi(x_1) - \varphi(x_2) + x_1 - x_2) \\
& \leq 0,
\end{aligned} \tag{14}$$

where to get this inequality we used that the deadline zone function  $\varphi$  is non-increasing and that the saturation map  $\text{sat}$  is non-decreasing.

Let  $\eta$  be a positive value that will be selected later. From (13) and (14), it follows

$$\begin{aligned}
\Delta & \leq \begin{pmatrix} \delta w \\ \delta v(1) \\ \delta\varphi \end{pmatrix}^\top \\
& \times \begin{pmatrix} (A - \alpha I_n)^\top P + P(A - \alpha I_n) + 2 \text{sh}(\mu) C^\top C \\ * \\ * \end{pmatrix} \\
& \begin{vmatrix} PB + 2(D \text{sh}(\mu) - \text{ch}(\mu)) C^\top & (2 \text{sh}(\mu) - \eta) C^\top \\ (1 - D)^2 e^\mu - (1 + D)^2 e^{-\mu} & 2(\text{sh}(\mu) D - \text{ch}(\mu)) - \eta D \\ * & 2(\text{sh}(\mu) - \eta) \end{vmatrix} \\
& \times \begin{pmatrix} \delta w \\ \delta v(1) \\ \delta\varphi \end{pmatrix}
\end{aligned}$$

Let us prove that there exist a symmetric definite matrix  $P$ , positive values  $\alpha, \eta$ , and a value  $\mu$  such that (12) holds and such that the matrix

$$\begin{pmatrix} (A - \alpha I_n)^\top P + P(A - \alpha I_n) + 2 \text{sh}(\mu) C^\top C \\ * \\ * \end{pmatrix} \begin{vmatrix} PB + 2(D \text{sh}(\mu) - \text{ch}(\mu)) C^\top & (2 \text{sh}(\mu) - \eta) C^\top \\ (1 - D)^2 e^\mu - (1 + D)^2 e^{-\mu} & 2(\text{sh}(\mu) D - \text{ch}(\mu)) - \eta D \\ * & 2(\text{sh}(\mu) - \eta) \end{vmatrix} \tag{15}$$

is symmetric definite negative.

To do that we use the change of variables  $P = e^{-\mu}Q$  and  $\eta = e^{-\mu}\tilde{\eta}$  where  $Q$  and  $\tilde{\eta}$  are to be determined independently of  $\mu$ . When  $\mu$  goes to  $-\infty$ , the matrix in (15) is symmetric definite negative, if the matrix

$$e^{-\mu}N(\alpha, Q)$$

is symmetric definite negative where

$$N(\alpha, Q) = \begin{pmatrix} (A - \alpha I_n)^\top Q + Q(A - \alpha I_n) - C^\top C & * \\ QB + (D+1)C^\top & (1+\tilde{\eta})C^\top \\ -(1+D)^2 & -D-1-\tilde{\eta}D \\ * & -(1+2\tilde{\eta}) \end{pmatrix}$$

(by noting that  $2\text{sh}(\mu) \sim -e^{-\mu}$  and  $2\text{ch}(\mu) \sim e^{-\mu}$  as  $\mu$  goes to  $-\infty$ ).

Let us prove the inequality

$$\begin{pmatrix} (1+D)^2 & D+1+\tilde{\eta}D \\ * & 1+2\tilde{\eta} \end{pmatrix} > 0. \quad (16)$$

To prove this inequality, it is sufficient to select  $\tilde{\eta} > 0$  and to ensure that the determinant of the matrix is positive, that is  $(1+D)^2(1+2\tilde{\eta}) - (D+1+\tilde{\eta}D)^2 > 0$ . This inequality is equivalent to  $(1+D)^2(1+2\tilde{\eta}) - (1+\frac{\tilde{\eta}D}{D+1})^2 > 0$ . Now selecting  $\tilde{\eta}$  positive and close to 0, we get that this inequality holds as soon as  $2\tilde{\eta}(1+D)^2(1-\frac{D}{D+1}) > 0$ . Using the assumption  $D > -1$ , this later inequality is equivalent to  $\tilde{\eta} > 0$ . Therefore if  $\tilde{\eta}$  is selected positive and close to 0, then the inequality (16) holds.

To ensure that the matrix  $N(\alpha, Q)$  is symmetric definite negative, it is sufficient to note that, for all  $\beta > 0$ , there exists a positive value  $\alpha$  satisfying (12) and a symmetric definite matrix  $Q$  in  $\mathbb{R}^{n \times n}$  such that

$$(A - \alpha I_n)^\top Q + Q(A - \alpha I_n) < -\beta I_n.$$

For any  $\beta > 0$ , such positive value  $\alpha$  and matrix  $Q$  exist because by selecting  $\alpha$  large enough, the eigenvalues of  $A - \alpha I_n$  could have a real part a negative as desired.

This concludes the proof of Lemma 2.  $\square$

**Proof of Theorem 1.** Since  $A$  is dissipative (due to Lemma 2), it follows, from (Miyadera, 1992, Thm 4.2)), that  $\mathcal{A}$  generates a semigroup of contractions. Moreover due to (Miyadera, 1992, Thm 4.10), for all  $(u_0, v_0, w_0)$  in  $D(\mathcal{A})$ , there is a unique solution to the Cauchy Problem (9). This concludes the proof of Theorem 1.  $\square$

### 3. STABILITY RESULTS FOR THE CASCADE CASES

In this section we analyze the stability of the nonlinear system (1)-(4). Before doing that let us note that the linear system (5)-(8) is exponentially stable if and only if the spectrum of  $A$  is in the strict left part of the plane,  $\sigma(A) \subset \mathbb{C}_-$ , and if  $D > 0$ . This result could be proven by a spectral analysis of the linear operator describing (5)-(8). This motivates the introduction of the following.

**Assumption 1.** *It holds  $\sigma(A) \subset \mathbb{C}_-$  and  $D > 0$ .*

Now let us state the main results of this paper that are two stability results when the system (1)-(4) is in cascade form, that is either  $B = 0$  or  $C = 0$ .

#### 3.1 PDE-to-ODE case

Let us first consider the case where the PDE and the ODE are in cascade form in this order, that is when  $C = 0$ , namely:

$$z_{tt}(x, t) = z_{xx}(x, t), \quad (17)$$

$$\dot{w} = Aw + Bz_t(1, t), \quad (18)$$

$$z(0, t) = 0, \quad (19)$$

$$z_x(1, t) + \text{sat}(Dz_t(1, t)) = 0, \quad (20)$$

In that case the ODE dynamics do not have any impact on the PDE, and we can show that the necessary and sufficient condition for the asymptotic stability of the linear system (5)-(8) is also a sufficient condition for the asymptotic stability of the nonlinear system (17)-(20). To be more specific, we can state the following result:

**Theorem 2.** *Under Assumption 1 the system (17)-(20) is globally asymptotically stable, that is, there exists a symmetric definite positive matrix  $P$  in  $\mathbb{R}^{n \times n}$  such that for all  $(u, v, w)$  in  $D(\mathcal{A})$ , the following stability condition*

$$\begin{aligned} & \|z(\cdot, t)\|_{H_0^1(0,1)} + \|z_t(\cdot, t)\|_{L^2(0,1)} + w(t)^\top Pw(t) \\ & \leq \|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)} + w(0)^\top Pw(0), \quad \forall t \geq 0, \end{aligned} \quad (21)$$

holds, together with the attractivity property

$$\|z(\cdot, t)\|_{H_0^1(0,1)} + \|z_t(\cdot, t)\|_{L^2(0,1)} + \|w(t)\| \rightarrow_{t \rightarrow \infty} 0. \quad (22)$$

Due to space limitation of this conference paper, the proof of this theorem is omitted. It is based on the following Lyapunov function candidate

$$\begin{aligned} V_0(u, v, w) &= \frac{1}{2} \left( \int_0^1 e^{\mu x} (v + u')^2 dx + \int_0^1 e^{-\mu x} (v - u')^2 dx \right) + w^\top Pw \end{aligned}$$

with an appropriate choice of the positive scalar  $\mu$  and of the symmetric definite matrix  $P$ , being inspired by (Marx et al., 2017, Proof of Theorem 2.2)) and (Prieur et al., 2016, Proof of Theorem 2)).

#### 3.2 ODE-to-PDE case

Let us now consider the case where the ODE and the PDE are in cascade form in this order, that is when  $B = 0$ . In this context we have to consider the following system

$$z_{tt}(x, t) = z_{xx}(x, t), \quad (23)$$

$$\dot{w} = Aw, \quad (24)$$

$$z(0, t) = 0, \quad (25)$$

$$z_x(1, t) + \text{sat}(Dz_t(1, t) + Cw(t)) = 0, \quad (26)$$

for which we have the following result parallelling Theorem 2.

**Theorem 3.** *Under Assumption 1 the system (23)-(26) is globally asymptotically stable.*

Before proving Theorem 3, let us note that Assumption 1 is necessary for the global asymptotic stability of the system (23)-(26). Indeed, the global asymptotic stability of the system (23)-(26) implies the global asymptotic stability of  $\dot{w} = Aw$ . Therefore Assumption 1 is a necessary condition for the conclusion of Theorem 3. Let us now prove that it is also a sufficient condition.

**Proof of Theorem 3.** Under Assumption 1, let  $Q$  a symmetric definite positive matrix in  $\mathbb{R}^{n \times n}$ , and  $P$  be a symmetric definite positive matrix in  $\mathbb{R}^{n \times n}$  such that

$$A^\top P + PA + \frac{D^{-1}}{2} C^\top C = -Q.$$

Inspired by Prieur et al. (2016), and using the norm associated to the scalar product defined in (11), let us consider the following function:  $(u, v, w) \mapsto V(u, v, w)$  defined by  $V(u, v, w) = \|(u, v, w)\|_{\mu, P}^2$  for all  $(u, v, w)$  in  $\mathfrak{H}$  as a candidate Lyapunov function. We compute along the solutions to (23)-(26) the following

$$\begin{aligned} \dot{V} &= \int_0^1 e^{\mu x} (z_t + z_x)(z_{tt} + z_{xt}) dx \\ &+ \int_0^1 e^{-\mu x} (z_t - z_x)(z_{tt} - z_{xt}) dx \\ &+ w^\top (A^\top P + PA)w \\ &= \int_0^1 e^{\mu x} (z_t + z_x)(z_{xx} + z_{xt}) dx \\ &- \int_0^1 e^{-\mu x} (z_t - z_x)(z_{xt} - z_{xx}) dx \\ &+ w^\top (A^\top P + PA)w \\ &= -\frac{\mu}{2} \int_0^1 e^{\mu x} (z_t + z_x)^2 dx + \frac{1}{2} [e^{\mu x} (z_t + z_x)^2]_{x=0}^{x=1} \\ &- \frac{\mu}{2} \int_0^1 e^{-\mu x} (z_t - z_x)^2 dx - \frac{1}{2} [e^{-\mu x} (z_t - z_x)^2]_{x=0}^{x=1} \\ &+ w^\top (A^\top P + PA)w \end{aligned}$$

where the partial differential equation (23) has been used in the first equality and two integrations by parts have been performed in the second equality.

Now, note that the boundary conditions (25) and (26) imply that, for all  $t \geq 0$ ,

$$\begin{aligned} [e^{\mu x} (z_t + z_x)^2](0, t) - [e^{-\mu x} (z_t - z_x)^2](0, t) \\ = z_x^2(0, t) - z_x^2(0, t) = 0 \end{aligned}$$

Therefore, with (26), it is deduced

$$\begin{aligned} \dot{V} &= -\mu V + \frac{e^\mu}{2} (z_t(1, t) + \text{sat}(Dz_t(1, t) + Cw(t)))^2 \\ &- \frac{e^{-\mu}}{2} (z_t(1, t) - \text{sat}(Dz_t(1, t) + Cw(t)))^2 \\ &+ w^\top (A^\top P + PA)w \end{aligned} \quad (27)$$

and thus

$$\begin{aligned} \dot{V} &= -\mu V + \frac{e^\mu}{2} (z_t(1, t) + Dz_t(1, t) + Cw(t) + \phi_2)^2 \\ &- \frac{e^{-\mu}}{2} (z_t(1, t) - Dz_t(1, t) - Cw(t) - \phi_2)^2 \\ &+ w^\top (A^\top P + PA)w \end{aligned} \quad (28)$$

where the notation  $\phi_2 = \text{sat}(Dz_t(1, t) + Cw(t)) - (Dz_t(1, t) + Cw(t))$  has been used. It follows

$$\begin{aligned} \dot{V} &= -\mu V + \begin{pmatrix} w \\ z_t(1, t) \\ \phi_2 \end{pmatrix}^\top \\ &\times \begin{pmatrix} A^\top P + PA + 2 \text{sh}(\mu) C^\top C \\ \star \\ \star \\ 2(D \text{sh}(\mu) - \text{ch}(\mu)) C^\top & 2 \text{sh}(\mu) C^\top \\ (1-D)^2 e^\mu - (1+D)^2 e^{-\mu} & 2(\text{sh}(\mu)D - \text{ch}(\mu)) \\ \star & 2 \text{sh}(\mu) \end{pmatrix} \\ &\times \begin{pmatrix} w \\ z_t(1, t) \\ \phi_2 \end{pmatrix} \end{aligned} \quad (29)$$

Due to the generalized sector condition (see e.g., (Tarbouriech et al., 2011, Lemma 1.5), for all  $1 > \ell > 0$ , and for all  $\eta > 0$ , for any initial condition  $(u_0, v_0, w_0)$  in  $D(\mathcal{A})$  such that

$$(1 - \ell) \|(u_0, v_0, w_0)\|_{\mu, P} \leq u_0, \quad (30)$$

it holds

$$\eta \phi_2 (\phi_2 + \ell D z_t(1, t) + \ell C w(t)) \leq 0.$$

Due to (29), in order to get an asymptotic stability, it is sufficient to ensure

$$\begin{aligned} &\begin{pmatrix} A^\top P + PA + 2 \text{sh}(\mu) C^\top C \\ \star \\ \star \\ 2(D \text{sh}(\mu) - \text{ch}(\mu)) C^\top & 2 \text{sh}(\mu) C^\top - \eta \ell C \\ (1-D)^2 e^\mu - (1+D)^2 e^{-\mu} & 2(\text{sh}(\mu)D - \text{ch}(\mu)) - \eta \ell D \\ \star & 2 \text{sh}(\mu) - 2\eta \end{pmatrix} \\ &\leq 0 \end{aligned} \quad (31)$$

Let us now prove that the matrix

$$\begin{pmatrix} (1-D)^2 e^\mu - (1+D)^2 e^{-\mu} & 2(\text{sh}(\mu)D - \text{ch}(\mu)) - \eta \ell D \\ \star & 2 \text{sh}(\mu) - 2\eta \end{pmatrix} \quad (32)$$

could be made definite negative by selecting suitable values for  $\mu, \ell$  and  $\eta$ . By continuity of the eigenvalues with respect to the matrix entries, and letting  $\mu = 0$  in the previous matrix, it is sufficient to prove that there exist  $0 < \ell < 1$  and  $0 < \eta$  such that

$$\begin{pmatrix} -4D & -2 - \eta \ell D \\ \star & -2\eta \end{pmatrix} < 0 \quad (33)$$

Since by Assumption 1,  $D > 0$ , it is sufficient to select  $0 < \ell < 1$  and  $0 < \eta$  such that

$$8\eta D - (2 + \eta \ell D)^2 > 0$$

that is

$$\sqrt{8\eta D} > 2 + \eta \ell D$$

A value  $\ell$  in  $(0, 1)$  satisfying the previous inequality exists if there exists  $\eta > 0$  such that

$$0 < \frac{\sqrt{8\eta D} - 2}{\eta D} < 1 \quad (34)$$

First note that for any  $\eta > 0$  satisfying  $\eta \neq \frac{1}{2D}$ , it holds  $\frac{\sqrt{8\eta D} - 2}{\eta D} < 1$ . Indeed this inequality is equivalent to  $8\eta D < \eta^2 D^2 + 4\eta D + 4$ , which is rewritten as  $(\eta D - 2)^2 > 0$ . Therefore to ensure (34), it is sufficient to select a positive  $\eta \neq \frac{1}{2D}$  such that

$$0 < \frac{\sqrt{8\eta D} - 2}{\eta D}$$

which is equivalent to  $1 < 2\eta D$ . Therefore selecting  $\eta > \frac{1}{2D}$  implies (34), and the existence of  $l$  in  $(0, 1)$  satisfying (33). It follows that there exists a positive  $\mu$  close to 0 such that the matrix in (32) is symmetric definite negative. By Schur complement, we can prove the existence of a symmetric definite positive matrix  $P$  in  $\mathbb{R}^{n \times n}$  such that (31) holds.

Therefore the system (23)-(26) is exponentially asymptotically stable for all initial condition satisfying (30). The remaining part of the proof is done as in the proof of Theorem 2.  $\square$

#### 4. CONCLUSION

Nonlinear systems obtained by considering a wave equation in closed loop with a nonlinear dynamical boundary controller have been considered in this paper. The controller was subject to a magnitude limitation and modeled by a linear ordinary differential equation in a saturation map. The well-posedness of the obtained infinite-dimensional system was first studied and then two stability results were given. These two stability results have been applied for two cascade cases and provided sufficient conditions for the asymptotic stability of the equilibrium. The well-posedness was proven by using nonlinear semigroups techniques, whereas the global asymptotic stability results were obtained by Lyapunov-based arguments in infinite-dimensional state space.

This work lets some questions open. In particular other performance criterion could be considered instead of the stability analysis, as the disturbance rejection as done in Shreim et al. (2022); Parada et al. (2022) for hyperbolic systems. In addition, other coupled ODE-PDE models could be considered as the beam equation controlled through an electric circuit (see Mattioni et al. (2022)). Finally the assumption  $B = 0$  or  $C = 0$ , that has been used in Section 3, could be removed in a future study.

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