Global Output Feedback Stabilization of Semilinear Reaction-Diffusion PDEs

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Abstract: This paper addresses the topic of global output feedback stabilization of semilinear reaction-diffusion PDEs. The semilinearity is assumed to be confined into a sector condition. We consider two different types of actuation configurations, namely: bounded control operator and right Robin boundary control. The measurement is selected as a left Dirichlet trace. The control strategy is finite dimensional and is designed based on a linear version of the plant. We derive a set of sufficient conditions ensuring the global exponential stabilization of the semilinear reaction-diffusion PDE. These conditions are shown to be feasible provided the order of the controller is large enough and the size of the sector condition in which the semilinearity is confined into is small enough.

Keywords: Semilinear reaction-diffusion PDEs, output feedback stabilization, finite-dimensional control.

1. INTRODUCTION

The topic of feedback stabilization of linear reaction-diffusion PDEs has been intensively studied in the literature (Boskovic et al., 2001; Liu, 2003) using different approaches such as backstepping (Krstic and Smyshlyaev, 2008) and spectral reduction methods (Russell, 1978; Coron and Trélat, 2004). The extension of these approaches to the stabilization of semilinear reaction-diffusion PDEs remains challenging. Among the reported contributions, one can find the study of stability by means of strict Lyapunov functionals (Mazenc and Prieur, 2011), control using quasi-static deformations (Coron and Trélat, 2004), state-feedback (Karafyllis and Krstic, 2019a,b; Karafyllis, 2021) or network control (Wu et al., 2019).

This paper addresses the topic of output feedback stabilization of 1-D semilinear reaction-diffusion PDEs. The case of a state-feedback was studied in (Karafyllis and Krstic, 2019a,b; Karafyllis, 2021). Using spectral reduction methods and small gain arguments, the authors derived in these works sufficient conditions on the size of the sector condition (in which the nonlinearity is confined into) so that the proposed control strategy achieves the global exponential stabilization of the plant. However, the case of the output feedback, as considered in this work, remains challenging. In this context, we consider in this paper the global output feedback stabilization of 1-D semilinear reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary conditions. Two different configurations for the actuation scheme are investigated: bounded control operator and right Robin boundary control. The measurement is selected as the left Dirichlet trace. The reported output feedback control strategy takes advantage of recent developments regarding the finite-dimensional control of parabolic PDEs (Curtain, 1982; Balas, 1988; Katz and Fridman, 2020) by leveraging control architectures introduced in (Sakawa, 1983). In particular, we adopt the enhanced procedures reported in (Lhachemi and Prieur, 2022b,e) that allow the design in a generic and systematic manner of finite-dimensional observer-based control strategies for general 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurements; these systematic procedures have been successfully extended to delayed boundary control (Lhachemi and Prieur, 2022a,f) and local stabilization in the presence of a saturation (Lhachemi and Prieur, 2022c,d). Assuming that the nonlinearity satisfies a sector condition, we derive a set of sufficient LMI conditions ensuring the global output feedback stabilization of the plant. We show that the derived stability conditions are always feasible when selecting the order of the observer large enough and for a size of the sector condition small enough.

The rest of the paper is organized as follows. Notations and preliminary properties are summarized in Section 2. The control design for semilinear reaction-diffusion PDEs in the case of a bounded input operator and left Dirichlet measurement is addressed in Section 3. Then, Section 4 reports the case of right Robin boundary control and left Dirichlet measurement. Finally, concluding remarks are formulated in Section 6.

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2. NOTATION AND PROPERTIES

2.1 Notation

Real spaces $\mathbb{R}^n$ are equipped with the usual Euclidean norm denoted by $\|\cdot\|$. The associated induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors X and Y, $\mathrm{col}(X,Y)$ represents the vector $[X^\top, Y^\top]^\top$. The space of square integrable functions on $(0,1)$ is denoted by $L^2(0,1)$ and is endowed with the inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$. The associated norm is denoted by $\|\cdot\|_{L^2}$. For an integer $m \geq 1$, $H^m(0,1)$ stands for the m-order Sobolev space and is endowed with its usual norm $\|\cdot\|_{H^m}$. For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that P is positive semi-definite (resp. positive definite).

2.2 Properties of Sturm-Liouville operators

Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^1([0,1])$, and $q \in \mathcal{C}^0([0,1])$ with $p > 0$ and $q \geq 0$. The Sturm-Liouville operator (Renardy and Rogers, 2006) is defined by $\mathcal{A} = -(pf')' + qf$ on the domain $D(\mathcal{A}) = \{f \in H^2(0,1) : \cos(\theta_1)f(0) - \sin(\theta_1)f'(0) = 0, \cos(\theta_2)f(1) + \sin(\theta_2)f'(1) = 0\}$. The eigenvalues $(\lambda_n)_{n\geq 1}$ of $\mathcal{A}$ are simple, non-negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. The associated unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. Owing to these eigenstructures, $\mathcal{A}$ is a Riesz spectral operator (Curtain and Zwart, 2012; Delattre et al., 2003): the domain of the operator $\mathcal{A}$ is characterized by $D(\mathcal{A}) = \left\{f \in L^2(0,1) : \sum_{n\geq 1} |\lambda_n|^2 |\langle f, \phi_n\rangle|^2 < +\infty\right\}$ while $\mathcal{A}f = \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle \phi_n$ for all $f \in D(\mathcal{A})$.

Introducing constants $p_*, p^*, q^* \in \mathbb{R}$ such that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, the eigenvalues λ_n satisfy the estimates $0 \leq \pi^2(n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*$ for all $n \geq 1$ (Orlov, 2017). If we further assume that $q > 0$, there exist constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_{H^1}^2 \leq \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle^2 = \langle \mathcal{A}f, f\rangle \leq C_2\|f\|_{H^1}^2 \quad (1)$$

for all $f \in D(\mathcal{A})$. We infer, in particular, from the latter inequalities and the Riesz spectral nature of $\mathcal{A}$ that $f(0) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n(0)$ for all $f \in D(\mathcal{A})$. Finally, if we further assume that $p \in \mathcal{C}^2([0,1])$, we have for any $x \in [0,1]$ that $\phi_n(x) = O(1)$ and $\phi_n'(x) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ (Orlov, 2017). For an arbitrarily given integer $N \geq 1$, we define $\mathcal{R}_N f = \sum_{n\geq N+1} \langle f, \phi_n\rangle \phi_n$.

3. DISTRIBUTED COMMAND AND DIRICHLET BOUNDARY MEASUREMENT

3.1 Problem setting and spectral reduction

Consider the reaction-diffusion PDE described by

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}_0(x)z(t,x) + f(t,x,z(t,x)) + b(x)u(t) \quad (2a)$$

$$\cos(\theta_1)z(t,0) - \sin(\theta_1)z_x(t,0) = 0 \quad (2b)$$

$$\cos(\theta_2)z(t,1) + \sin(\theta_2)z_x(t,1) = 0 \quad (2c)$$

$$z(0,x) = z_0(x). \quad (2d)$$

for $t > 0$ and $x \in (0,1)$. Here we have $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q}_0 \in \mathcal{C}^0([0,1])$. The distributed control input is $u(t) \in \mathbb{R}$ and acts on the system via the shape function $b \in L^2(0,1)$. The state of the reaction-diffusion PDE is $z(t,\cdot) \in L^2(0,1)$ while $z_0 \in L^2(0,1)$ is the initial condition. The function $f : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is assumed to be globally Lipchitz continuous in z, uniformly in (t,x), so that $f(\cdot,\cdot,0) = 0$. Let $\tilde{q}_f \in \mathcal{C}^0([0,1])$ and $k_f > 0$ be such that

$$|f(t,x,z) - \tilde{q}_f(x)z| \leq k_f|z|, \quad \forall t \geq 0,\ \forall x \in [0,1],\ \forall z \in \mathbb{R}. \quad (3)$$

Inequality (3) is referred to as a *sector type condition*. Hence (2a) can be written as

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}(x)z(t,x) + g(t,x,z(t,x)) + b(x)u(t)$$

where $\tilde{q} = \tilde{q}_0 - \tilde{q}_f$ and $g(t,x,z) = f(t,x,z) - \tilde{q}_f(x)z$ with

$$|g(t,x,z)| \leq k_f|z|, \quad \forall t \geq 0, \quad \forall x \in [0,1], \quad \forall z \in \mathbb{R}. \quad (4)$$

The system output is selected as:

$$y(t) = z(t,0). \quad (5)$$

In perspective of control design, recalling that $\tilde{q} = \tilde{q}_0 - \tilde{q}_f$, and without loss of generality, we pick a function $q \in \mathcal{C}^0([0,1])$ and a constant $q_c \in \mathbb{R}$ such that

$$\tilde{q} = q - q_c, \quad q > 0. \quad (6)$$

The projection of (2) into the Hilbert basis $(\phi_n)_{n\geq 1}$ gives

$$\dot{z}_n(t) = (-\lambda_n + q_c)z_n(t) + b_n u(t) + g_n(t) \quad (7)$$

where $z_n(t) = \langle z(t,\cdot), \phi_n\rangle$, $b_n = \langle b, \phi_n\rangle$, and $g_n(t) = \langle g(t,\cdot,z(t,\cdot)), \phi_n\rangle$. Moreover, considering classical solutions, the system output (5) is expressed as:

$$y(t) = \sum_{n\geq 1} \phi_n(0)z_n(t). \quad (8)$$

3.2 Control design and truncated model

Let $\delta > 0$ be the desired exponential decay rate for the closed-loop system trajectories. Let an integer $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. We introduce an arbitrary integer $N \geq N_0 + 1$ that will be specified later. The control strategy is described by:

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + b_n u(t) - l_n\left\{\sum_{k=1}^{N} \phi_k(0)\hat{z}_k(t) - y(t)\right\}, \quad 1 \leq n \leq N_0 \quad (9a)$$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + b_n u(t), \quad N_0 + 1 \leq n \leq N \quad (9b)$$

$$u(t) = \sum_{n=1}^{N_0} k_n \hat{z}_n(t) \quad (9c)$$

where $l_n, k_n \in \mathbb{R}$ are the observer and feedback gains.

In preparation for stability analysis, we need to build a truncated model capturing the N first modes of the PDE (2) as well as the dynamics of the controller (9). To do so, let us define the error of observation $e_n = z_n - \hat{z}_n$, the scaled error of observation $\tilde{e}_n = \sqrt{\lambda_n}e_n$, and the vectors $\hat{Z}^{N_0} = [\hat{z}_1\ \dots\ \hat{z}_{N_0}]^\top$, $E^{N_0} = [e_1\ \dots\ e_{N_0}]^\top$, $\hat{Z}^{N-N_0} = [\hat{z}_{N_0+1}\ \dots\ \hat{z}_N]^\top$, $\tilde{E}^{N-N_0} = [\tilde{e}_{N_0+1}\ \dots\ \tilde{e}_N]^\top$, $R_1 = [g_1\ \dots\ g_{N_0}]^\top$, $\tilde{R}_2 = \left[\sqrt{\lambda_{N_0+1}}g_{N_0+1}\ \dots\ \sqrt{\lambda_N}g_N\right]^\top$, and $R = \mathrm{col}(R_1, \tilde{R}_2)$. We also introduce the matrices defined by $A_0 = \mathrm{diag}(-\lambda_1 + q_c, \dots, -\lambda_{N_0} + q_c)$, $A_1 =$

$\mathrm{diag}(-\lambda_{N_0+1}+q_c,\dots,-\lambda_N+q_c)$, $B_0=[b_1\ \dots\ b_{N_0}]^\top$, $B_1=[b_{N_0+1}\ \dots\ b_N]^\top$, $C_0=[\phi_1(0)\ \dots\ \phi_{N_0}(0)]$, $\tilde{C}_1=\left[\frac{\phi_{N_0+1}(0)}{\sqrt{\lambda_{N_0+1}}}\ \dots\ \frac{\phi_N(0)}{\sqrt{\lambda_N}}\right]$, $K=[k_1\ \dots\ k_{N_0}]$, and $L=[l_1\ \dots\ l_{N_0}]^\top$. With $X=\mathrm{col}\left(\hat{Z}^{N_0},E^{N_0},\hat{Z}^{N-N_0},\tilde{E}^{N-N_0}\right)$, we obtain from (7-8) and (9) that $u=K=K\hat{Z}^{N_0}$ and

$$\dot{X}=FX+\mathcal{L}\zeta+GR \tag{10}$$

where $\zeta=\sum_{n\geq N+1}\phi_n(0)z_n$,

$$F=\begin{bmatrix}A_0+B_0K & LC_0 & 0 & L\tilde{C}_1\\ 0 & A_0-LC_0 & 0 & -L\tilde{C}_1\\ B_1K & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix},\quad G=\begin{bmatrix}0&0\\I&0\\0&0\\0&I\end{bmatrix}$$

and $\mathcal{L}=\mathrm{col}(L,-L,0,0)$. Defining $\tilde{K}=[K\ 0\ 0\ 0]$, we also have that

$$u=\tilde{K}X. \tag{11}$$

We finally define the matrices

$$\Lambda=\mathrm{diag}(\lambda_{N_0+1},\dots,\lambda_N),\quad \Omega=\begin{bmatrix}I&I&0&0\\I&I&0&0\\0&0&I&\Lambda^{-1/2}\\0&0&\Lambda^{-1/2}&\Lambda^{-1}\end{bmatrix}$$

and $\tilde{\Lambda}=\mathrm{diag}(I,\Lambda)$. In particular, we have that $\Omega\preceq 2\max(1,1/\lambda_{N_0+1})I$ and $\tilde{\Lambda}^{-1}\succeq\min(1,1/\lambda_N)I$.

3.3 Stability result

Theorem 1. Let $\theta_1\in(0,\pi/2]$, $\theta_2\in[0,\pi/2]$, $p\in\mathcal{C}^2([0,1])$ with $p>0$, $\tilde{q}_0\in\mathcal{C}^0([0,1])$, and $b\in L^2(0,1)$. Let $f:\mathbb{R}_+\times[0,1]\times\mathbb{R}\to\mathbb{R}$ be globally Lipchitz continuous in z, uniformly in (t,x), so that $f(\cdot,\cdot,0)=0$. Let $\tilde{q}_f\in\mathcal{C}^0([0,1])$ and $k_f>0$ be so that (3) holds. Let $q\in\mathcal{C}^0([0,1])$ and $q_c\in\mathbb{R}$ be such that (6) holds. Let $\delta>0$ and $N_0\geq 1$ be such that $-\lambda_n+q_c<-\delta$ for all $n\geq N_0+1$. Assume that[1] $b_n\neq 0$ for all $1\leq n\leq N_0$. Let $K\in\mathbb{R}^{1\times N_0}$ and $L\in\mathbb{R}^{N_0}$ be such that A_0+B_0K and A_0-LC_0 are Hurwitz with eigenvalues that have a real part strictly less than $-\delta<0$. For a given $N\geq N_0+1$, assume that there exist a symmetric positive definite $P\in\mathbb{R}^{2N\times 2N}$, positive real numbers $\alpha_1,\alpha_2>1$ and $\beta,\gamma>0$ such that

$$\Theta_1\preceq 0,\quad \Theta_2\leq 0 \tag{12}$$

where

$$\Theta_1=\begin{bmatrix}\Theta_{1,1,1}&P\mathcal{L}&PG\\ \mathcal{L}^\top P&-\beta&0\\ G^\top P&0&-\alpha_2\gamma\tilde{\Lambda}^{-1}\end{bmatrix}$$

$$\Theta_{1,1,1}=F^\top P+PF+2\delta P+\alpha_1\gamma\|\mathcal{R}_N b\|_{L^2}^2\tilde{K}^\top\tilde{K}+\alpha_2\gamma k_f^2\Omega$$

$$\Theta_2=2\gamma\left\{-\left[1-\frac{1}{2}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\right]\lambda_{N+1}+q_c+\delta\right\}+\beta M_\phi+\frac{\alpha_2\gamma k_f^2}{\lambda_{N+1}}$$

with $M_\phi=\sum_{n\geq N+1}\frac{\phi_n(0)^2}{\lambda_n}$. Then, considering the closed-loop system composed of the plant (2) with the system output (5) and the controller (9), there exists $M>0$ such that for any initial conditions $z_0\in H^2(0,1)$ and $\hat{z}_n(0)\in\mathbb{R}$ so that $\cos(\theta_1)z_0(0)-\sin(\theta_1)z_0'(0)=0$ and $\cos(\theta_2)z_0(1)+\sin(\theta_2)z_0'(1)=0$, the system trajectory satisfies

$$\|z(t,\cdot)\|_{H^1}^2+\sum_{n=1}^N\hat{z}_n(t)^2\leq Me^{-2\delta t}\left(\|z_0\|_{H^1}^2+\sum_{n=1}^N\hat{z}_n(0)^2\right)$$

for all $t\geq 0$. Moreover, when selecting N to be sufficiently large, there exists $k_f>0$ (small enough) so that the constraints (12) are feasible.

Proof. Consider the Lyapunov functional defined for $X\in\mathbb{R}^{2N}$ and $z\in D(\mathcal{A})$ by $V(X,z)=X^\top PX+\gamma\sum_{n\geq N+1}\lambda_n z_n^2$. The computation of the time derivative of V along the system trajectories (7) and (10) reads

$$\dot{V}+2\delta V=X^\top\{F^\top P+PF+2\delta P\}X+2X^\top P\mathcal{L}\zeta+2X^\top PGR$$
$$+2\gamma\sum_{n\geq N+1}\lambda_n(-\lambda_n+q_c+\delta)z_n^2+2\gamma\sum_{n\geq N+1}\lambda_n(b_nu+g_n)z_n.$$

We define $\tilde{X}=\mathrm{col}(X,\zeta,R)$. As $\zeta^2\leq M_\phi\sum_{n\geq N+1}\lambda_n z_n^2$ and using Young's inequality and (11), we deduce that

$$\dot{V}+2\delta V\leq$$
$$\tilde{X}^\top\begin{bmatrix}F^\top P+PF+2\delta P+\alpha_1\gamma\|\mathcal{R}_N b\|_{L^2}^2\tilde{K}^\top\tilde{K}&P\mathcal{L}&PG\\ \mathcal{L}^\top P&-\beta&0\\ G^\top P&0&0\end{bmatrix}\tilde{X}$$
$$+2\gamma\sum_{n\geq N+1}\lambda_n\left\{-\lambda_n+q_c+\delta+\frac{1}{2}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\lambda_n\right\}z_n^2$$
$$+\beta M_\phi\sum_{n\geq N+1}\lambda_n z_n^2+\alpha_2\gamma\|\mathcal{R}_N g(t,\cdot,z)\|_{L^2}^2$$

for any $\alpha_1,\alpha_2>0$. Owing to the sector condition (4), we infer that

$$\|\mathcal{R}_N g(t,\cdot,z)\|_{L^2}^2=\|g(t,\cdot,z)\|_{L^2}^2-\sum_{n=1}^N g_n^2$$
$$\leq k_f^2\|z\|_{L^2}^2-\|R_1\|^2-\sum_{n=N_0+1}^N\frac{1}{\lambda_n}\left(\sqrt{\lambda_n}g_n\right)^2$$
$$\leq k_f^2\sum_{n=1}^N z_n^2+k_f^2\sum_{n\geq N+1}z_n^2-R^\top\tilde{\Lambda}^{-1}R. \tag{13}$$

Recalling that $e_n=z_n-\hat{z}_n$ and $\tilde{e}_n=\sqrt{\lambda}_n e_n$, we deduce that $\sum_{n=1}^{N_0}z_n^2=\sum_{n=1}^{N_0}\left(\hat{z}_n^2+2\hat{z}_n e_n+e_n^2\right)$ and $\sum_{n=N_0+1}^N z_n^2=\sum_{n=N_0+1}^N\left(\hat{z}_n^2+\frac{2}{\sqrt{\lambda_n}}\hat{z}_n\tilde{e}_n+\frac{1}{\lambda_n}\tilde{e}_n^2\right)$, hence $\sum_{n=1}^N z_n^2=X^\top\Omega X$. Putting everything together, this implies that $\dot{V}+2\delta V\leq\tilde{X}^\top\Theta_1\tilde{X}+\sum_{n\geq N+1}\lambda_n\Gamma_n z_n^2$ where $\Gamma_n=2\gamma\left\{-\left[1-\frac{1}{2}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\right]\lambda_n+q_c+\delta\right\}+\beta M_\phi+\frac{\alpha_2\gamma k_f^2}{\lambda_n}\leq\Theta_2$ for all $n\geq N+1$; we have used that $1-\frac{1}{2}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)>0$ because $\alpha_1,\alpha_2>1$. Owing to (12), we infer that $\dot{V}+2\delta V\leq 0$. Using the definition of V along with (1), we deduce that the claimed exponential stability estimate holds true.

It remains to show that the constraints (12) are feasible when N is large enough and $k_f>0$ is small enough. First, the application of the Lemma in appendix of Lhachemi and Prieur (2022b) to the matrix $F+\delta I$ shows that the unique solution $P\succ 0$ to the Lyapunov equation $F^\top P+PF+2\delta P=-I$ is such that $\|P\|=O(1)$ as $N\to+\infty$. We fix $\alpha_1>1$ and $\gamma>0$ arbitrarily. We also fix $\beta=N$,

[1] This implies that (A_0,B_0) satisfies the Kalman condition. Note that (A_0,C_0) satisfies the Kalman condition by arguments from Lhachemi and Prieur (2022e).

$\alpha_2 = N^3$, and $k_f = 1/N^2$. Hence it can be seen that $\Theta_2 \to -\infty$ as $N \to +\infty$. Moreover, since $\|\tilde{K}\| = \|K\|$, $\|\mathcal{L}\| = \sqrt{2}\|L\|$, and $\|G\| = 1$ are constants independent of N, $\|P\| = O(1)$ as $N \to +\infty$, and $\Omega \preceq 2\max(1, 1/\lambda_{N_0+1})I$ and $-\tilde{\Lambda}^{-1} \preceq -\min(1, 1/\lambda_N)I$, the Schur complement shows that $\Theta_1 \preceq 0$ for N selected to be large enough. This completes the proof.

Remark 2. For a fixed order $N \geq N_0+1$, let us arbitrarily fix the value of the decision variable $\gamma > 0$ (following the last part of the proof of Theorem 1, the obtained constraints remain feasible for N large enough and $k_f > 0$ small enough). Now $\Theta_1 \preceq 0$ takes the form of a LMI w.r.t. the decision variables $P, \alpha_1, \alpha_2, \beta$ while, using Schur complement, $\Theta_2 \leq 0$ is equivalent to the LMI formulation: $\begin{bmatrix} \mu & \sqrt{\gamma\lambda_{N+1}} & \sqrt{\gamma\lambda_{N+1}} \\ \sqrt{\gamma\lambda_{N+1}} & -\alpha_1 & 0 \\ \sqrt{\gamma\lambda_{N+1}} & 0 & -\alpha_2 \end{bmatrix} \preceq 0$ with $\mu = 2\gamma\{-\lambda_{N+1} + q_c + \delta\} + \beta M_\phi + \alpha_2\gamma k_f^2/\lambda_{N+1}$. A similar remark applies to the constraints of Theorem 3.

4. ROBIN BOUNDARY CONTROL AND DIRICHLET BOUNDARY MEASUREMENT

4.1 Problem setting and spectral reduction

We now consider the boundary stabilization of the reaction-diffusion PDE described by

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}_0(x)z(t,x) + f(t,x,z(t,x)) \tag{14a}$$
$$\cos(\theta_1)z(t,0) - \sin(\theta_1)z_x(t,0) = 0 \tag{14b}$$
$$\cos(\theta_2)z(t,1) + \sin(\theta_2)z_x(t,1) = u(t) \tag{14c}$$
$$z(0,x) = z_0(x). \tag{14d}$$

for $t > 0$ and $x \in (0,1)$ where $p \in \mathcal{C}^2([0,1])$ and $\theta_1 \in (0, \pi/2]$. The boundary measurement is selected as the left Dirichlet trace (5). We still assume that there exist $\tilde{q}_f \in \mathcal{C}^0([0,1])$ and $k_f > 0$ so that (3) holds. Hence we define $\tilde{q} = \tilde{q}_0 - \tilde{q}_f$ and $g(t,x,z) = f(t,x,z) - \tilde{q}_f(x)z$.

Consider the change of variable

$$w(t,x) = z(t,x) - \frac{x^2}{\cos\theta_2 + 2\sin\theta_2}u(t) \tag{15}$$

so that, introducing $v = \dot{u}$, the PDE (14) is equivalently rewritten under the following homogeneous representation:

$$\dot{u}(t) = v(t) \tag{16a}$$
$$w_t(t,x) = (p(x)w_x(t,x))_x - \tilde{q}(x)w(t,x) + g(t,x,z(t,x)) + a(x)u(t) + b(x)v(t) \tag{16b}$$
$$\cos(\theta_1)w(t,0) - \sin(\theta_1)w_x(t,0) = 0 \tag{16c}$$
$$\cos(\theta_2)w(t,1) + \sin(\theta_2)w_x(t,1) = 0 \tag{16d}$$
$$w(0,x) = w_0(x). \tag{16e}$$

where $a(x) = \frac{1}{\cos\theta_2 + 2\sin\theta_2}\{2p(x) + 2xp'(x) - x^2\tilde{q}(x)\}$, $b(x) = -\frac{x^2}{\cos\theta_2+2\sin\theta_2}$, and $w_0(x) = z_0(x) - \frac{x^2}{\cos\theta_2+2\sin\theta_2}u(0)$. Note that we kept in (16b) the nonlinear term $g(t,x,z(t,x))$ expressed in original coordinate z. We introduce $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ so that (6) holds. Finally, we define the coefficients of projection $z_n(t) = \langle z(t,\cdot), \phi_n\rangle$, $w_n(t) = \langle w(t,\cdot), \phi_n\rangle$, $a_n = \langle a, \phi_n\rangle$, $b_n = \langle b, \phi_n\rangle$, and $g_n(t) = \langle g(t,\cdot,z(t,\cdot)), \phi_n\rangle$. From (15) we deduce that

$$w_n(t) = z_n(t) + b_n u(t), \quad n \geq 1. \tag{17}$$

Moreover, the projection of (16) into the Hilbert basis $(\phi_n)_{n\geq 1}$ gives $\dot{u} = v$ and

$$\dot{w}_n(t) = (-\lambda_n + q_c)w_n(t) + a_n u(t) + b_n v(t) + g_n(t) \tag{18}$$

with $w(t,\cdot) = \sum_{n\geq 1} w_n(t)\phi_n$. Inserting (17) into (18), the projection of (14) gives

$$\dot{z}_n(t) = (-\lambda_n + q_c)z_n(t) + \beta_n u(t) + g_n(t) \tag{19}$$

where $\beta_n = a_n + (-\lambda_n + q_c)b_n = p(1)\{-\cos(\theta_2)\phi_n'(1) + \sin(\theta_2)\phi_n(1)\} = O(\sqrt{\lambda_n})$. Finally, considering classical solutions, the system output (5) is expressed as:

$$y(t) = z(t,0) = w(t,0) = \sum_{n\geq 1}\phi_n(0)w_n(t). \tag{20}$$

4.2 Control design and truncated model

Let $\delta > 0$ be the desired exponential decay rate for the closed-loop system trajectories. Let an integer $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. For an arbitrary integer $N \geq N_0 + 1$, the control strategy reads:

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n u(t) \tag{21a}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t) - l_n\left\{\sum_{k=1}^{N}\phi_k(0)\hat{w}_k(t) - y(t)\right\}, \quad 1 \leq n \leq N_0 \tag{21b}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t), \quad N_0+1 \leq n \leq N \tag{21c}$$
$$u(t) = \sum_{n=1}^{N_0} k_n\hat{z}_n(t) \tag{21d}$$

where $l_n, k_n \in \mathbb{R}$ are the observer and feedback gains, respectively.

To build the truncated model, we adopt the notations of Subsection 3.2 except that we introduce the notations $\tilde{z}_n = \hat{z}_n/\lambda_n$ along with the vector $\tilde{Z}^{N-N_0} = [\tilde{z}_{N_0+1} \ \dots \ \tilde{z}_N]^\top$ and the matrices $\mathfrak{B}_0 = [\beta_1 \ \dots \ \beta_{N_0}]^\top$ and $\tilde{\mathfrak{B}}_1 = \begin{bmatrix}\frac{\beta_{N_0+1}}{\lambda_{N_0+1}} & \dots & \frac{\beta_N}{\lambda_N}\end{bmatrix}^\top$. Hence, introducing the state vector $X = \mathrm{col}\left(\hat{Z}^{N_0}, E^{N_0}, \tilde{Z}^{N-N_0}, \tilde{E}^{N-N_0}\right)$ we deduce similarly to the developments of Subsection 3.2 that the following truncated dynamics holds:

$$\dot{X} = FX + \mathcal{L}\zeta + GR \tag{22}$$

where

$$F = \begin{bmatrix} A_0 + \mathfrak{B}_0K & LC_0 & 0 & L\tilde{C}_1 \\ 0 & A_0 - LC_0 & 0 & -L\tilde{C}_1 \\ \tilde{\mathfrak{B}}_1K & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}$$

and $\mathcal{L} = \mathrm{col}(L, -L, 0, 0)$. We define the matrices

$$\Lambda = \mathrm{diag}(\lambda_{N_0+1}, \dots, \lambda_N), \quad \Omega = \begin{bmatrix} I & I & 0 & 0 \\ I & I & 0 & 0 \\ 0 & 0 & \Lambda^2 & \Lambda^{1/2} \\ 0 & 0 & \Lambda^{1/2} & \Lambda^{-1} \end{bmatrix}$$

and $\tilde{\Lambda} = \mathrm{diag}(I, \Lambda)$. These matrices which are such that $\Omega \preceq 2\max(1, 1/\lambda_{N_0+1}, \lambda_N^2)I$ and $\tilde{\Lambda}^{-1} \succeq \min(1, 1/\lambda_N)I$. Finally, introducing $\tilde{X} = \mathrm{col}(X, \zeta, R)$, we have

$$v = \dot{u} = K\dot{\hat{Z}}^{N_0} = E\tilde{X} \tag{23}$$

where $E = K\left[A_0 + \mathfrak{B}_0 K \;\; LC_0 \;\; 0 \;\; L\tilde{C}_1 \;\; L \;\; 0\right]$.

4.3 Stability result

Theorem 3. Let $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q}_0 \in \mathcal{C}^0([0,1])$. Let $f : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ be globally Lipchitz continuous in z, uniformly in (t,x), so that $f(\cdot,\cdot,0) = 0$. Let $\tilde{q}_f \in \mathcal{C}^0([0,1])$ and $k_f > 0$ be so that (3) holds. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (6) holds. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0 + 1$. Let $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that [2] $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq N_0 + 1$, assume that there exist a symmetric positive definite $P \in \mathbb{R}^{2N\times 2N}$, positive real numbers $\alpha_1, \alpha_2, \alpha_3 > 3/2$ and $\beta, \gamma > 0$ such that

$$\Theta_1 \preceq 0, \quad \Theta_2 \leq 0 \tag{24}$$

where

$$\Theta_1 = \begin{bmatrix} \Theta_{1,1,1} & P\mathcal{L} & PG \\ \mathcal{L}^\top P & -\beta & 0 \\ G^\top P & 0 & -\alpha_3\gamma\tilde{\Lambda}^{-1} \end{bmatrix} + \alpha_2\gamma\|\mathcal{R}_N b\|_{L^2}^2 E^\top E$$

$$\Theta_{1,1,1} = F^\top P + PF + 2\delta P + \alpha_1\gamma\|\mathcal{R}_N a\|_{L^2}^2 \tilde{K}^\top\tilde{K} + 2\alpha_3\gamma k_f^2\|\mathcal{R}_N b\|_{L^2}^2\tilde{K}^\top\tilde{K} + \alpha_3\gamma k_f^2\Omega$$

$$\Theta_2 = 2\gamma\left\{-\left[1 - \frac{1}{2}\sum_{i=1}^{3}\frac{1}{\alpha_i}\right]\lambda_{N+1} + q_c + \delta\right\} + \beta M_\phi + \frac{2\alpha_3\gamma k_f^2}{\lambda_{N+1}}$$

with $M_\phi = \sum_{n\geq N+1}\frac{\phi_n(0)^2}{\lambda_n}$. Then, considering the closed-loop system composed of the plant (14) with the system output (5) and the controller (21), there exists $M > 0$ such that for any initial conditions $z_0 \in H^2(0,1)$ and $\hat{z}_n(0) \in \mathbb{R}$ so that $\cos(\theta_1)z_0(0) - \sin(\theta_1)z_0'(0) = 0$ and $\cos(\theta_2)z_0(1) + \sin(\theta_2)z_0'(1) = K\hat{Z}^{N_0}(0)$, the system trajectory satisfies

$$\|z(t,\cdot)\|_{H^1}^2 + \sum_{n=1}^{N}\hat{z}_n(t)^2 \leq Me^{-2\delta t}\left(\|z_0\|_{H^1}^2 + \sum_{n=1}^{N}\hat{z}_n(0)^2\right)$$

for all $t \geq 0$. Moreover, when selecting N to be sufficiently large, there exists $k_f > 0$ (small enough) so that the constraints (24) are feasible.

Proof. Consider the Lyapunov functional defined for $X \in \mathbb{R}^{2N}$ and $w \in D(\mathcal{A})$ by $V(X,w) = X^\top PX + \gamma\sum_{n\geq N+1}\lambda_n w_n^2$. Note here that, following (Lhachemi and Prieur, 2022e), the first term of the Lyapunov functional captures the N first modes of the PDE plant in original z coordinates (14) while the second term captures the residual modes (i.e., for $n \geq N+1$) in homogeneous w coordinates (16). The computation of the time derivative of V along the system trajectories (18) and (22) reads

$$\dot{V} + 2\delta V = X^\top\{F^\top P + PF + 2\delta P\}X + 2X^\top P\mathcal{L}\zeta + 2X^\top PGR + 2\gamma\sum_{n\geq N+1}\lambda_n(-\lambda_n + q_c + \delta)w_n^2 + 2\gamma\sum_{n\geq N+1}\lambda_n(a_n u + b_n v + g_n)w_n.$$

With $\tilde{X} = \mathrm{col}(X, \zeta, R)$, noting that $\zeta^2 \leq M_\phi\sum_{n\geq N+1}\lambda_n z_n^2$, and using Young's inequality, (11), and (23), we infer that

$$\dot{V} + 2\delta V \leq \tilde{X}^\top\begin{bmatrix} F^\top P + PF + 2\delta P + \alpha_1\gamma\|\mathcal{R}_N a\|_{L^2}^2\tilde{K}^\top\tilde{K} & P\mathcal{L} & PG \\ \mathcal{L}^\top P & -\beta & 0 \\ G^\top P & 0 & 0 \end{bmatrix}\tilde{X} + \alpha_2\gamma\|\mathcal{R}_N b\|_{L^2}^2\tilde{X}^\top E^\top E\tilde{X} + \beta M_\phi\sum_{n\geq N+1}\lambda_n w_n^2 + 2\gamma\sum_{n\geq N+1}\lambda_n\left\{-\lambda_n + q_c + \delta + \frac{1}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)\lambda_n\right\}w_n^2 + \alpha_3\gamma\|\mathcal{R}_N g(t,\cdot,z)\|_{L^2}^2.$$

for any $\alpha_1, \alpha_2, \alpha_3 > 0$. Using the sector condition (4) we infer that (13) holds. Recalling that $e_n = z_n - \hat{z}_n$, $\tilde{e}_n = \sqrt{\lambda}_n e_n$, and $\tilde{z}_n = \hat{z}_n/\lambda_n$, we deduce that $\sum_{n=1}^{N_0} z_n^2 = \sum_{n=1}^{N_0}\left(\hat{z}_n^2 + 2\hat{z}_n e_n + e_n^2\right)$ and $\sum_{n=N_0+1}^{N} z_n^2 = \sum_{n=N_0+1}^{N}\left(\lambda_n^2\tilde{z}_n^2 + 2\sqrt{\lambda_n}\tilde{z}_n\tilde{e}_n + \frac{1}{\lambda_n}\tilde{e}_n^2\right)$, hence $\sum_{n=1}^{N} z_n^2 = X^\top\Omega X$. Moreover, we infer from (17) and using (11) that $\sum_{n\geq N+1} z_n^2 \leq 2\sum_{n\geq N+1} w_n^2 + 2\|\mathcal{R}_N b\|_{L^2}^2 X^\top\tilde{K}^\top\tilde{K}X$. This implies that $\dot{V} + 2\delta V \leq \tilde{X}^\top\Theta_1\tilde{X} + \sum_{n\geq N+1}\lambda_n\Gamma_n w_n^2$ with $\Gamma_n = 2\gamma\left\{-\left[1 - \frac{1}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right)\right]\lambda_n + q_c + \delta\right\} + \beta M_\phi + \frac{2\alpha_3\gamma k_f^2}{\lambda_n} \leq \Theta_2$ for all $n \geq N+1$ where we have used that $1 - \frac{1}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right) > 0$ because $\alpha_1, \alpha_2, \alpha_3 > 3/2$. Owing to (24), we infer that $\dot{V} + 2\delta V \leq 0$. Using the definition of V and (1), we deduce that the claimed exponential stability estimate holds true.

It remains to show that the constraints (24) are feasible when N is large enough and $k_f > 0$ is small enough. Since $\|\tilde{C}_1\| = O(1)$ and $\|\tilde{\mathfrak{B}}_1\| = O(1)$ as $N \to +\infty$, the same approach as in the proof of Theorem 1 shows that the unique solution $P \succ 0$ to the Lyapunov equation $F^\top P + PF + 2\delta P = -I$ is such that $\|P\| = O(1)$ as $N \to +\infty$. We fix $\alpha_1, \alpha_2 > 3/2$ and $\gamma > 0$ arbitrarily. We also fix $\beta = N$, $\alpha_3 = N^3$, and $k_f = 1/N^4$. Hence it can be seen that $\Theta_2 \to -\infty$ as $N \to +\infty$. Moreover, since $\|\tilde{K}\| = \|K\|$, $\|\mathcal{L}\| = \sqrt{2}\|L\|$, and $\|G\| = 1$ are constants independent of N, $\|P\| = O(1)$ and $\|E\| = O(1)$ as $N \to +\infty$, and $\Omega \preceq 2\max(1, 1/\lambda_{N_0+1}, \lambda_N^2)I$ and $-\tilde{\Lambda}^{-1} \preceq -\min(1, 1/\lambda_N)I$, the Schur complement shows that $\Theta_1 \preceq 0$ for N selected to be large enough. This completes the proof.

5. NUMERICAL ILLUSTRATION

For numerical illustration, we consider $p = 1$, $\tilde{q}_0 = -2$, $\tilde{q}_f = 3$, $\theta_1 = \pi/5$, and $\theta_2 = 0$. Hence the open-loop reaction diffusion PDEs (2) and (14) are unstable in the case of $k_f = 0$, i.e. for $f(t,x,z) = \tilde{q}_f z = 3z$. In the case of the reaction-diffusion PDE with bounded control input (2), we consider the shape function $b(x) = \cos(x)1|_{[1/10,3/10]}$.

The feedback gain K and the observer gain L are computed to achieve the same pole placement in the different studied settings. In the case of Theorem 1, corresponding to a bounded input operator and a left Dirichlet measurement, the gains are set as $K = -25.9768$ and $L = 5.9341$. In the case of Theorem 3, corresponding to a right boundary control and a left Dirichlet measurement, the gains are set as $K = -2.3307$ and $L = 5.9341$.

[2] Note that $(A_0, \mathfrak{B}_0)$ and (A_0, C_0) satisfy the Kalman condition by arguments from Lhachemi and Prieur (2022e).

Dimension of the observer	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
Theorem 1	$k_f = 1.99$	$k_f = 2.32$	$k_f = 2.45$	$k_f = 2.54$	$k_f = 2.59$
Theorem 3	$k_f = 1.93$	$k_f = 2.14$	$k_f = 2.21$	$k_f = 2.25$	$k_f = 2.27$

Table 1. Maximum value of k_f for the sector condition (3) obtained for different dimensions N of the observer.

The maximal value of $k_f > 0$ (corresponding to the size of the sector condition (3) in which the nonlinearity $f(t, x, z)$ is confined into) for which the stability of the closed-loop system is ensured by applying the theorems of this paper is detailed in Tab. 1 for the two studied configurations and for different values of the order of the observer.

6. CONCLUSION

This paper solved the problem of output feedback stabilization of semilinear reaction-diffusion PDEs for which the semilinearity is assumed to be confined into a sector condition. It is worth noting that even if the developments of this paper have been focused on a Dirichlet boundary measurement, the case of a Neumann boundary measurement can also be handled using the framework of this paper augmented with the procedure reported in Lhachemi and Prieur (2022b,e).

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