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Samy Skander Bahoura. Uniform bound for the volume of the solutions of elliptic equations. 2023. hal-04159670v2

**HAL Id: hal-04159670**

**<https://hal.science/hal-04159670v2>**

Preprint submitted on 1 Aug 2023

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# Uniform bound for the volume of the solutions of elliptic equations.

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## Abstract

We consider variational problems with regular Hölderian weight or boundary singularity, and Dirichlet condition. We prove the boundedness of the volume of the solutions to these equations on analytic domains.

Keywords: Regular Hölderian weight, boundary singularity, a priori estimate, analytic domain, Lipschitz condition.

**MSC: 35J60, 35B45.**

## 1 Introduction and Main Results

We set  $\Delta = \partial_{11} + \partial_{22}$  on analytic domain  $\Omega \subset \subset \mathbb{R}^2$ .

We consider the following boundary value problem:

$$(P) \begin{cases} -\Delta u = (1 + |x - x_0|^{2\beta})Ve^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here:

$$\beta \in (0, 1/2), \quad x_0 \in \partial\Omega,$$

and,

$$u \in W_0^{1,1}(\Omega), \quad e^u \in L^1(\Omega) \text{ and } 0 < a \leq V \leq b.$$

This is an equation with regular Hölderian weight not Lipschitz in  $x_0$  but have a weak derivative.

This problem is defined in the sense of distributions as mentioned in [10]. It arises in different geometrical and physical situations, see for example [1, 5, 24, 28]. This type of problems was studied by many authors, with and without boundary conditions, in the subcritical case and the critical case, also for surfaces, see [1-28], we can find in the reference existence and compactness results. In [9] we have the following important Theorem,

**Theorem A** (Brezis-Merle [9]). *For  $u$  and  $V$  two functions relative to (P) with,*

$$0 < a \leq V \leq b < +\infty$$

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then it holds, for all compact set  $K$  of  $\Omega$ :

$$\sup_K u \leq c,$$

with  $c$  depending on  $a, b, \beta, x_0, K$  and  $\Omega$ .

We deduce from Theorem A and from the elliptic estimates that,  $u$  is uniformly bounded in  $C_{loc}^2(\Omega)$ .

In [12] we have the following important Theorem and local estimate near points  $\partial\Omega \ni y \neq x_0$  obtained by mean of the method of moving-plane for  $C^1$  functions  $V$ . For  $\mathcal{V}$  a neighborhood of  $\partial\Omega$ :

**Theorem B** (Chen-Li [12]). For  $u$  and  $V$  two functions relative to  $(P)$  with  $V$  a  $C^1$  function satisfying,

$$0 < a \leq V \leq b < +\infty, \quad \|\nabla V\|_{L^\infty} \leq A$$

then it holds, for all compact set  $K$  of  $\mathcal{V} - \{x_0\}$ :

$$\sup_K u \leq c',$$

with  $c'$  depending on  $a, b, A, \beta, x_0, K, \mathcal{V}$  and  $\Omega$ .

We deduce from Theorem B and from the elliptic estimates that,  $u$  is uniformly bounded in  $C_{loc}^2(\bar{\Omega} - \{x_0\})$ .

In this paper we try to prove that we have on all  $\Omega$  the boundedness of the volume of the solutions of the boundary value problem  $(P)$  if we assume that  $V$  is uniformly Lipschitz and  $C^1$  with the weight  $(1 + |x - x_0|^{2\beta})$ ,  $\beta \in (0, 1/2)$ .

Our tools are the previous two theorems A and B and a local conformal map near  $x_0$  of the analytic domain  $\Omega$  and a Pohozaev type identity.

Here we have:

**Theorem 1.1** Assume that  $u$  is a solution of  $(P)$  relative to  $V$  with the following conditions:

$$x_0 \in \partial\Omega, \quad \beta \in (0, 1/2),$$

and  $V$  a  $C^1$  function satisfying,

$$0 < a \leq V \leq b, \quad \|\nabla V\|_{L^\infty} \leq A,$$

we have,

$$\int_{\Omega} e^u \leq C = C(a, b, A, \beta, x_0, \Omega).$$

A consequence of this theorem is a compactness result of the solutions to this Liouville type equation on analytic domains, see [4].

We have the same result if we consider the following boundary value problem on an analytic domain  $\Omega \subset \subset \mathbb{R}^2$ :

$$(P_\beta) \begin{cases} -\Delta u = |x - x_0|^{2\beta} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here:

$$\beta \in (-1/2, +\infty), \quad x_0 \in \partial\Omega,$$

and,

$$u \in W_0^{1,1}(\Omega), \quad |x - x_0|^{2\beta} e^u \in L^1(\Omega) \text{ and } 0 < a \leq V \leq b.$$

Theorems A and B are true for the boundary value problem  $(P_\beta)$  in a neighborhood of all  $y \in \bar{\Omega} - \{x_0\}$ , because their proofs are local (see [9] and [12]) and  $x_0 \in \partial\Omega$ .

In this paper we try to prove that we have on all  $\Omega$  the boundedness of the volume of the solutions of the boundary value problem  $(P_\beta)$  if we assume that  $V$  is uniformly Lipschitz and  $C^1$  and with the weight  $|x - x_0|^{2\beta}$ ,  $\beta \in (-1/2, +\infty)$ .

Our tools are the previous two theorems A and B and a local conformal map near  $x_0$  of the analytic domain  $\Omega$  and a Pohozaev type identity.

Here we have:

**Theorem 1.2** *Assume that  $u$  is a solution of  $(P_\beta)$  relative to  $V$  with the following conditions:*

$$x_0 \in \partial\Omega, \quad \beta \in (-1/2, +\infty),$$

*and  $V$  a  $C^1$  function satisfying,*

$$0 < a \leq V \leq b, \quad \|\nabla V\|_{L^\infty} \leq A,$$

*we have,*

$$\int_{\Omega} |x - x_0|^{2\beta} e^u \leq C = C(a, b, A, \beta, x_0, \Omega).$$

A consequence of this theorem is a compactness result of the solutions to this Liouville type equation on analytic domains, see [3].

**Remarks:**

1) We can consider many singularities for the previous theorem 1.1, and take  $V = W \cdot \prod_{k=1}^m (1 + a_k |x - x_k|^{2\beta})$ ,  $x_k \in \partial\Omega$ ,  $|a_k| < 1$ ,  $\beta \in (0, 1/2)$ ,  $m \in \mathbb{N}^*$ ,  $W \in C^1$ ,  $W$  bounded in  $C^1$ .

2) We can consider any number of singularities for the previous theorem 1.2.  $\prod_{k=1}^m |x - x_k|^{2\beta}$ ,  $x_k \in \partial\Omega$ ,  $\beta \in (-1/2, +\infty)$ .

3) We can consider any analytic domain, a disk or an annulus or other analytic domain.

## 2 Proof of the Theorems:

*Proof of the theorems:*

The proofs of theorems 1.1 and 1.2 are similar, we do the proof of theorem 1.1.

By corollary 1 of the paper of Brezis-Merle, we have:  $e^{ku} \in L^1(\Omega)$  for all  $k > 2$  and the elliptic estimates and the Sobolev embedding imply that:  $u \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega})$ ,  $\epsilon > 0$ . By the maximum principle  $u \geq 0$ .

Step 1: *Brezis-Merle interior estimates*. By using the first eigenvalue and the first eigenfunction, with Dirichlet boundary condition, the volume is locally uniformly bounded, and thus the solutions are locally uniformly bounded by Brezis-Merle result. The solutions  $u > 0$  are locally uniformly bounded in  $C^{2,\epsilon}(\Omega)$  for  $\epsilon$  small.

Step 2: *Chen-Li boundary estimates*. Let's consider  $y \in \partial\Omega, y \neq x_0$ . Applying the moving-plane method around  $y$  and the result of Chen-Li, the solutions are uniformly bounded in a neighborhood of  $y \neq x_0$  in  $C^{2,\epsilon}$  for  $\epsilon$  small.

Step 3: *estimates around the singularity  $x_0$* .

We use a conformal map  $f = f_{x_0}$  around  $x_0$ , which maps a neighborhood of  $x_0, D_{x_0}$ , to a unit half disk centered in 0,  $B^+(0,1)$  with  $f(x_0) = 0$  and  $f(\partial\Omega \cap D_{x_0}) = \{z \in B^+(0,1), z_1 = 0\}$ . The solution,  $v = u \circ f^{-1}$  is bounded uniformly outside 0 in  $C^2$  norm, and is a solution of:

$$-\Delta v = |g'(z)|^2(1 + |g(z) - x_0|^{2\beta})W e^v, \text{ in } B^+(0, 1/2),$$

with  $g = f^{-1}$  and  $W = V \circ g$ , and  $|g'(0)| \neq 0$ .

We use a Pohozaev type identity. We use the fact that  $v$  is uniformly bounded outside 0. We multiply the equation by  $z \cdot \nabla v$  and we integrate by parts:

1) We have on a small half ball  $B^+(0, \epsilon)$ :

$$\int_{B^+(0, \epsilon)} (\Delta v)(z \cdot \nabla v) dz = \int_{B^+(0, \epsilon)} -|g'(z)|^2(1 + |g(z) - x_0|^{2\beta})W [z \cdot \nabla(e^v)] dz,$$

Thus,

$$\begin{aligned} & \int_{\partial B^+(0, \epsilon)} (z \cdot \nabla v)(\nabla v \cdot \nu) - \frac{1}{2}(z \cdot \nu)|\nabla v|^2 = \\ & = \int_{B^+(0, \epsilon)} (2 + 2(\beta + 1 + O(\epsilon))|z|^{2\beta})W e^v dz + \\ & + \int_{B^+(0, \epsilon)} (1 + |g(z) - x_0|^{2\beta})[(z \cdot \nabla W)|g'(z)|^2 + (z \cdot \nabla |g'(z)|^2)W]e^v dz + \\ & - \int_{\partial B^+(0, \epsilon)} (1 + |g(z) - x_0|^{2\beta})(z \cdot \nu)W e^v d\sigma \end{aligned}$$

We can write, ( $v = 0$  and  $z \cdot \nu = 0$  on  $\{z \in B^+(0, \epsilon), z_1 = 0\}$ ) and  $v$  is uniformly bounded outside 0:

$$\begin{aligned} & \int_{\{z_1=0\}} \frac{1}{2}(z \cdot \nu)(\partial_\nu v)^2 d\sigma + O(1) = 0 + O(1) = \\ & = \int_{B^+(0, \epsilon)} (2 + 2(\beta + 1 + O(\epsilon))|g(z) - x_0|^{2\beta})W e^v dz + \\ & + \int_{B^+(0, \epsilon)} (1 + |g(z) - x_0|^{2\beta})[(z \cdot \nabla W)|g'(z)|^2 + (z \cdot \nabla |g'(z)|^2)W]e^v dz + O(1) \end{aligned}$$

thus, for  $\epsilon$  small enough one can compare  $z \cdot \nabla W$  and  $W$ ;  $|z \cdot \nabla W| \leq \epsilon A \leq a \leq W$ ,

$$\int_{B^+(0,\epsilon)} |g'(z)|^2 (2 + 2(\beta + 1 + O(\epsilon)) |g(z) - x_0|^{2\beta}) W e^v dz = O(1),$$

Thus,

$$\int_{B^+(0,\epsilon)} |g'(z)|^2 (1 + |g(z) - x_0|^{2\beta}) W e^v dz = O(1),$$

and we have in a neighborhood  $D'_{x_0}$  of  $x_0$ :

$$\int_{D'_{x_0}} (1 + |x - x_0|^{2\beta}) V e^u dx = O(1),$$

uniformly.

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