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Uniform bound for the volume of the solutions of elliptic equations.

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Abstract

We consider variational problems with regular Hölderian weight or boundary singularity, and Dirichlet condition. We prove the boundedness of the volume of the solutions to these equations on analytic domains.

Keywords: Regular Hölderian weight, boundary singularity, a priori estimate, analytic domain, Lipschitz condition.

MSC: **35J60, 35B45.**

1 Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on analytic domain $\Omega \subset \subset \mathbb{R}^2$.

We consider the following boundary value problem:

$$(P) \begin{cases} -\Delta u = (1 + |x - x_0|^{2\beta})Ve^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here:

$$\beta \in (0, 1/2), \quad x_0 \in \partial\Omega,$$

and,

$$u \in W_0^{1,1}(\Omega), \quad e^u \in L^1(\Omega) \text{ and } 0 < a \leq V \leq b.$$

This is an equation with regular Hölderian weight not Lipschitz in x_0 but have a weak derivative.

This problem is defined in the sense of distributions as mentioned in [10]. It arises in different geometrical and physical situations, see for example [1, 5, 24, 28]. This type of problems was studied by many authors, with and without boundary conditions, in the subcritical case and the critical case, also for surfaces, see [1-28], we can find in the reference existence and compactness results. In [9] we have the following important Theorem,

Theorem A (Brezis-Merle [9]). *For u and V two functions relative to (P) with,*

$$0 < a \leq V \leq b < +\infty$$

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then it holds, for all compact set K of Ω :

$$\sup_K u \leq c,$$

with c depending on a, b, β, x_0, K and Ω .

We deduce from Theorem A and from the elliptic estimates that, u is uniformly bounded in $C_{loc}^2(\Omega)$.

In [12] we have the following important Theorem and local estimate near points $\partial\Omega \ni y \neq x_0$ obtained by mean of the method of moving-plane for C^1 functions V . For \mathcal{V} a neighborhood of $\partial\Omega$:

Theorem B (Chen-Li [12]). For u and V two functions relative to (P) with V a C^1 function satisfying,

$$0 < a \leq V \leq b < +\infty, \|\nabla V\|_{L^\infty} \leq A$$

then it holds, for all compact set K of $\mathcal{V} - \{x_0\}$:

$$\sup_K u \leq c',$$

with c' depending on $a, b, A, \beta, x_0, K, \mathcal{V}$ and Ω .

We deduce from Theorem B and from the elliptic estimates that, u is uniformly bounded in $C_{loc}^2(\bar{\Omega} - \{x_0\})$.

In this paper we try to prove that we have on all Ω the boundedness of the volume of the solutions of the boundary value problem (P) if we assume that V is uniformly Lipschitz and C^1 with the weight $(1 + |x - x_0|^{2\beta}), \beta \in (0, 1/2)$.

Our tools are the previous two theorems A and B and a local conformal map near x_0 of the analytic domain Ω and a Pohozaev type identity.

Here we have:

Theorem 1.1 Assume that u is a solution of (P) relative to V with the following conditions:

$$x_0 \in \partial\Omega, \beta \in (0, 1/2),$$

and V a C^1 function satisfying,

$$0 < a \leq V \leq b, \|\nabla V\|_{L^\infty} \leq A,$$

we have,

$$\int_{\Omega} e^u \leq C = C(a, b, A, \beta, x_0, \Omega).$$

A consequence of this theorem is a compactness result of the solutions to this Liouville type equation on analytic domains, see [4].

We have the same result if we consider the following boundary value problem on an analytic domain $\Omega \subset \subset \mathbb{R}^2$:

$$(P_\beta) \begin{cases} -\Delta u = |x - x_0|^{2\beta} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here:

$$\beta \in (-1/2, +\infty), x_0 \in \partial\Omega,$$

and,

$$u \in W_0^{1,1}(\Omega), |x - x_0|^{2\beta} e^u \in L^1(\Omega) \text{ and } 0 < a \leq V \leq b.$$

Theorems A and B are true for the boundary value problem (P_β) in a neighborhood of all $y \in \bar{\Omega} - \{x_0\}$, because their proofs are local (see [9] and [12]) and $x_0 \in \partial\Omega$.

In this paper we try to prove that we have on all Ω the boundedness of the volume of the solutions of the boundary value problem (P_β) if we assume that V is uniformly Lipschitz and C^1 and with the weight $|x - x_0|^{2\beta}$, $\beta \in (-1/2, +\infty)$.

Our tools are the previous two theorems A and B and a local conformal map near x_0 of the analytic domain Ω and a Pohozaev type identity.

Here we have:

Theorem 1.2 *Assume that u is a solution of (P_β) relative to V with the following conditions:*

$$x_0 \in \partial\Omega, \beta \in (-1/2, +\infty),$$

and V a C^1 function satisfying,

$$0 < a \leq V \leq b, \|\nabla V\|_{L^\infty} \leq A,$$

we have,

$$\int_{\Omega} |x - x_0|^{2\beta} e^u \leq C = C(a, b, A, \beta, x_0, \Omega).$$

A consequence of this theorem is a compactness result of the solutions to this Liouville type equation on analytic domains, see [3].

Remarks:

1) We can consider many singularities for the previous theorem 1.1, and take $V = W \cdot \prod_{k=1}^m (1 + a_k |x - x_k|^{2\beta})$, $x_k \in \partial\Omega$, $|a_k| \ll 1$, $\beta \in (0, 1/2)$, $m \in \mathbb{N}^*$, $W \in C^1$, W bounded in C^1 .

2) We can consider any number of singularities for the previous theorem 1.2. $\prod_{k=1}^m |x - x_k|^{2\beta}$, $x_k \in \partial\Omega$, $\beta \in (-1/2, +\infty)$.

3) We can consider any analytic domain, a disk or an annulus or other analytic domain.

2 Proof of the Theorems:

Proof of the theorems:

The proofs of theorems 1.1 and 1.2 are similar, we do the proof of theorem 1.1.

By corollary 1 of the paper of Brezis-Merle, we have: $e^{ku} \in L^1(\Omega)$ for all $k > 2$ and the elliptic estimates and the Sobolev embedding imply that: $u \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega})$, $\epsilon > 0$. By the maximum principle $u \geq 0$.

Step 1: *Brezis-Merle interior estimates*. By using the first eigenvalue and the first eigenfunction, with Dirichlet boundary condition, the volume is locally uniformly bounded, and thus the solutions are locally uniformly bounded by Brezis-Merle result. The solutions $u > 0$ are locally uniformly bounded in $C^{2,\epsilon}(\Omega)$ for ϵ small.

Step 2: *Chen-Li boundary estimates*. Let's consider $y \in \partial\Omega, y \neq x_0$. Applying the moving-plane method around y and the result of Chen-Li, the solutions are uniformly bounded in a neighborhood of $y \neq x_0$ in $C^{2,\epsilon}$ for ϵ small.

Step 3: *estimates around the singularity x_0* .

We use a conformal map $f = f_{x_0}$ around x_0 , which maps a neighborhood of x_0, D_{x_0} , to a unit half disk centered in 0, $B^+(0,1)$ with $f(x_0) = 0$ and $f(\partial\Omega \cap D_{x_0}) = \{z \in B^+(0,1), z_1 = 0\}$. The solution, $v = u \circ f^{-1}$ is bounded uniformly outside 0 in C^2 norm, and is a solution of:

$$-\Delta v = |g'(z)|^2(1 + |g(z) - x_0|)^{2\beta} W e^v, \text{ in } B^+(0, 1/2),$$

with $g = f^{-1}$ and $W = V \circ g$, and $|g'(0)| \neq 0$.

We use a Pohozaev type identity. We use the fact that v is uniformly bounded outside 0. We multiply the equation by $z \cdot \nabla v$ and we integrate by parts:

1) We have on a small half ball $B^+(0, \epsilon)$:

$$\int_{B^+(0,\epsilon)} (\Delta v)(z \cdot \nabla v) dz = \int_{B^+(0,\epsilon)} -|g'(z)|^2(1 + |g(z) - x_0|^{2\beta}) W [z \cdot \nabla(e^v)] dz,$$

Thus,

$$\begin{aligned} & \int_{\partial B^+(0,\epsilon)} (z \cdot \nabla v)(\nabla v \cdot \nu) - \frac{1}{2}(z \cdot \nu)|\nabla v|^2 = \\ & = \int_{B^+(0,\epsilon)} (2 + 2(\beta + 1 + O(\epsilon))|z|^{2\beta}) W e^v dz + \\ & + \int_{B^+(0,\epsilon)} (1 + |g(z) - x_0|^{2\beta}) [(z \cdot \nabla W)|g'(z)|^2 + (z \cdot \nabla |g'(z)|^2) W] e^v dz + \\ & - \int_{\partial B^+(0,\epsilon)} (1 + |g(z) - x_0|^{2\beta})(z \cdot \nu) W e^v d\sigma \end{aligned}$$

We can write, ($v = 0$ and $z \cdot \nu = 0$ on $\{z \in B^+(0, \epsilon), z_1 = 0\}$) and v is uniformly bounded outside 0:

$$\begin{aligned} & \int_{\{z_1=0\}} \frac{1}{2}(z \cdot \nu)(\partial_\nu v)^2 d\sigma + O(1) = 0 + O(1) = \\ & = \int_{B^+(0,\epsilon)} (2 + 2(\beta + 1 + O(\epsilon))|g(z) - x_0|^{2\beta}) W e^v dz + \\ & + \int_{B^+(0,\epsilon)} (1 + |g(z) - x_0|^{2\beta}) [(z \cdot \nabla W)|g'(z)|^2 + (z \cdot \nabla |g'(z)|^2) W] e^v dz + O(1) \end{aligned}$$

thus, for ϵ small enough one can compare $z \cdot \nabla W$ and W ; $|z \cdot \nabla W| \leq \epsilon A \leq a \leq W$,

$$\int_{B^+(0,\epsilon)} |g'(z)|^2 (2 + 2(\beta + 1 + O(\epsilon)) |g(z) - x_0|^{2\beta}) W e^v dz = O(1),$$

Thus,

$$\int_{B^+(0,\epsilon)} |g'(z)|^2 (1 + |g(z) - x_0|^{2\beta}) W e^v dz = O(1),$$

and we have in a neighborhood D'_{x_0} of x_0 :

$$\int_{D'_{x_0}} (1 + |x - x_0|^{2\beta}) V e^u dx = O(1),$$

uniformly.

References

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.
- [2] Ambrosio. L, Fusco. N, Pallara, D. Functions of Bounded variations and Free discontinuity Problems, Oxford Press. 2000.
- [3] Bahoura.S.S. A uniform boundedness result for solutions to the Liouville type equation with boundary singularity. J. Math. Sci. Univ. Tokyo, 23, no 2, 487-497. 2016.
- [4] Bahoura.S.S. A compactness result for an equation with Holderian condition. Commun. Math. Anal. Vol 21, no 1, 23-34, 2018.
- [5] C. Bandle. Isoperimetric Inequalities and Applications. Pitman, 1980.
- [6] L. Boccardo, T. Gallouet. Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. 87 no 1, (1989), 149-169.
- [7] H. Brezis, YY. Li and I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [8] Brezis. H, Marcus. M, Ponce. A. C. Nonlinear elliptic equations with measures revisited. Mathematical aspects of nonlinear dispersive equations, 55-109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.
- [9] H. Brezis, F. Merle. Uniform estimates and Blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).
- [10] H. Brezis, W. A. Strauss. Semi-linear second-order elliptic equations in L1. J. Math. Soc. Japan 25 (1973), 565-590.

- [11] Chang, Sun-Yung A, Gursky, Matthew J, Yang, Paul C. Scalar curvature equation on 2- and 3-spheres. *Calc. Var. Partial Differential Equations* 1 (1993), no. 2, 205-229.
- [12] W. Chen, C. Li. A priori estimates for solutions to nonlinear elliptic equations. *Arch. Rational. Mech. Anal.* 122 (1993) 145-157.
- [13] C-C. Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . *Commun. Anal. Geom.* 6, No.1, 1-19 (1998).
- [14] D.G. De Figueiredo, P.L. Lions, R.D. Nussbaum, A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations, *J. Math. Pures et Appl.*, vol 61, 1982, pp.41-63.
- [15] Droniou. J. Quelques resultats sur les espaces de Sobolev. Hal 2001.
- [16] Ding.W, Jost. J, Li. J, Wang. G. The differential equation $\Delta u = 8\pi - 8\pi he^u$ on a compact Riemann surface. *Asian J. Math.* 1 (1997), no. 2, 230-248.
- [17] B. Gidas, W-M. Ni, L. Nirenberg. Symmetry and Related Properties via the Maximum Principle. *Commun. Math. Phys.* 68, 209-243 (1979).
- [18] D. Gilbarg, N. S, Trudinger. *Elliptic Partial Differential Equations of Second order*, Berlin Springer-Verlag.
- [19] Hofmann, S. Mitrea, M. Taylor, M. Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. *J. Geom. Anal.* 17 (2007), no. 4, 593-647.
- [20] Krantz, S. *Geometric functions theory*. Birkhauser.
- [21] YY. Li. Harnack Type Inequality: the method of moving planes. *Commun. Math. Phys.* 200,421-444 (1999).
- [22] YY. Li, I. Shafrir. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana. Math. J.* Vol 3, no 4. (1994). 1255-1270.
- [23] L. Ma, J-C. Wei. Convergence for a Liouville equation. *Comment. Math. Helv.* 76 (2001) 506-514.
- [24] Nagasaki, K, Suzuki,T. Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities. *Asymptotic Anal.* 3 (1990), no. 2, 173-188.
- [25] Necas, J. *Direct Methods in the Theory of Elliptic Equations*. Springer.
- [26] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. *C. R. Acad.Sci. Paris Sér. I Math.* 315 (1992), no. 2, 159-164.
- [27] Stoker, J. *Differential Geometry*.
- [28] Tarantello, G. Multiple condensate solutions for the Chern-Simons-Higgs theory. *J. Math. Phys.* 37 (1996), no. 8, 3769-3796.