Spectral geometry of tubes

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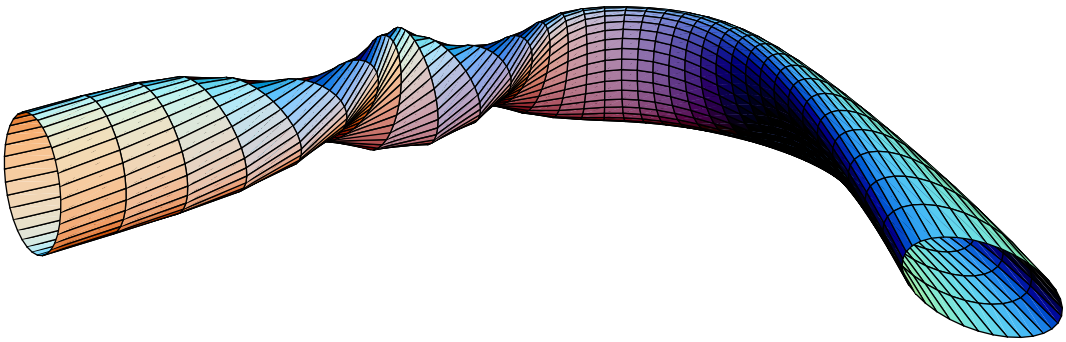


Abstract

The goal of this lecture is to give an account on spectral properties of tubular neighbourhoods of Riemannian manifolds. We focus on physical realisations when the base manifold is a curve in the three-dimensional Euclidean space, intensively studied during the last three decades because of physical motivations in quantum waveguides.

These lecture notes consists of two parts:

1. General framework of arbitrary manifolds and dimensions: Light presentation and state of the art.
2. Special Euclidean tubes about curves: Detailed proofs and occasionally new results.

Contents

1 General framework

1.1 Why spectrum?

Most processes in Nature can be under first approximation described by one of the typical linear second-order differential equations: the wave equation, the heat equation and the Schrödinger equation. Qualitative properties of the respective solutions are very different, which of course reflects the variety of the physical systems. However, the common denominator of these evolution equations is the Helmholtz equation

$$-\Delta\psi = \lambda\psi\,. \tag{1.1}$$

Indeed, the time-dependent problems can be solved by considering this stationary equation, or more precisely, the spectral problem for the Laplacian. Here λ is a point in the spectrum (not necessarily an eigenvalue) and ψ is its (generalised) eigenfunction.

More specifically, given any self-adjoint operator H in a Hilbert space $\mathcal{H}$, we decompose its spectrum as follows:

$$\sigma(H) = \sigma_{\rm disc}(H) \,\dot\cup\, \sigma_{\rm ess}(H)\,.$$

Here the *discrete spectrum* $\sigma_{\rm disc}(H)$ is composed of isolated eigenvalues of finite multiplicities (like in a finite-dimensional vector space), while the *essential spectrum* contains the rest: (isolated) eigenvalues of infinite multiplicities and accumulation points of the spectrum.

1.2 Why geometry?

The geometry arises naturally due to the manifold Ω, in which the Helmholtz equation (1.1) is considered. Unless the boundary $\partial\Omega$ is empty, it is necessary to assign boundary conditions. In this course, we restrict ourselves to Dirichlet boundary conditions

$$\psi = 0 \qquad \text{on} \qquad \partial\Omega\,. \tag{1.2}$$

More specifically, the spectral problem we consider is that of the unbounded self-adjoint operator in the Hilbert space $L^2(\Omega)$ defined by

$$-\Delta_D^\Omega\psi := -\Delta\psi\,, \qquad \psi \in \mathrm{dom}(-\Delta_D^\Omega) := \{\psi\in H^1_0(\Omega)\,:\,\Delta\psi\in L^2(\Omega)\}\,. \tag{1.3}$$

The action of the operator is considered in the sense of distribution and the Dirichlet boundary conditions are introduced in a weak sense via the Sobolev space

$$H^1_0(\Omega) := \overline{C^\infty_0(\Omega)}^{|||\cdot|||}\,,$$

where $|||\psi||| := \sqrt{\|\nabla\psi\|^2+\|\psi\|^2}$ with $\|\cdot\|$ denoting the norm of $L^2(\Omega)$. The operator is associated with the quadratic form

$$\delta_D^\Omega[\psi] := \|\nabla\psi\|^2\,, \qquad \psi\in\mathrm{dom}(\delta_D^\Omega) := H^1_0(\Omega)\,. \tag{1.4}$$

A feature of this lecture is that many spectral properties can be deduced by staying on the level of forms.

The goal of spectral geometry in the present context is to study the interplay between the spectrum of the Dirichlet Laplacian $-\Delta_D^\Omega$ and the geometry of the underlying manifold Ω.

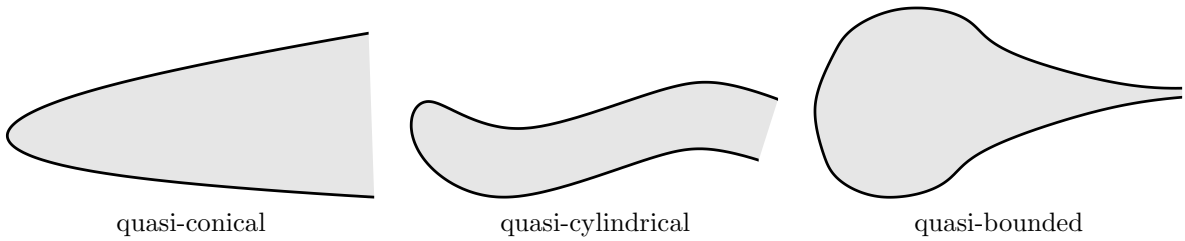


Figure 1: Examples of planar domains as regards the Glazman classification.

1.3 Why tubes?

There exists the following useful classification of Euclidean manifolds due to Glazman [22] (see also [16, Sec. X.6.1]). A non-empty open set $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ is said to be

(1) *quasi-conical* if it contains arbitrarily large balls;

(2) *quasi-cylindrical* if it is not quasi-conical but it contains infinitely many (pairwise) disjoint identical (*i.e.* of the same radius, congruent) balls;

(3) *quasi-bounded* if it is neither quasi-conical nor quasi-cylindrical.

ad (1)

Spectral-geometrically, the class of the quasi-conical manifolds is trivial because the spectrum is purely essential and, as a set, independent of the geometry:

$$\Omega \text{ is quasi-conical} \quad \implies \quad \sigma(-\Delta_D^\Omega) = \sigma_{\rm ess}(-\Delta_D^\Omega) = [0,\infty)\,. \tag{1.5}$$

However, even in the case of the entire Euclidean case, there are more subtle spectral properties:

$$-\Delta_D^{\mathbb{R}^d} \quad \text{is} \quad \begin{cases} \text{critical} & \text{if} \quad d = 1, 2\,, \\ \text{subcritical} & \text{if} \quad d \geq 3\,. \end{cases}$$

Here the *subcriticality* means the existence of a Hardy inequality, *e.g.* the classical one:

$$d \geq 3 \quad \implies \quad -\Delta_D^{\mathbb{R}^d} \geq \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2}$$

valid in the sense of quadratic forms in $L^2(\mathbb{R}^d)$. Note that, due to (1.5), there can be no positive constant bounding the Dirichlet Laplacian from below, but the existence of a positive function is admissible (and indeed true if $d \geq 3$). On the other hand, the *criticality* says that no such positive function exists:

$$d \geq 1, 2 \quad \implies \quad \forall V \in L^\infty(\mathbb{R}^d),\ V \underset{\neq}{\leq} 0, \quad \inf \sigma(-\Delta_D^\Omega + V) < 0\,.$$

ad (3)

The other extreme situation is that of bounded quasi-bounded sets:

$$\Omega \text{ is bounded} \quad \implies \quad \sigma_{\rm ess}(-\Delta_D^\Omega) = \varnothing\,. \tag{1.6}$$

Here the spectrum is purely discrete and the same extends to all quasi-bounded sets with sufficiently regular boundary. Of course, the dependence of the discrete eigenvalues on the underlying geometry is a fascinating subject of spectral geometry, but in this course we are interested in other aspects.

ad (2)

It is the remaining class of manifolds which is the most relevant from our perspective. Indeed, it is not difficult to see that the essential spectrum is never empty for quasi-cylindrical sets:

$$\Omega \text{ is quasi-cylindrical} \quad \implies \quad \sigma_{\rm ess}(-\Delta_D^\Omega) \neq \varnothing\,. \tag{1.7}$$

Then a non-trivial question is whether there is some discrete spectrum too. If there are eigenvalues below the essential spectrum of $-\Delta_D^\Omega$, we occasionally say that the Dirichlet Laplacian is *supercritical*. Even if the spectrum of a quasi-cylindrical manifold is purely essential, is $-\Delta_D^\Omega$ critical or subcritical? More specifically, is there a Hardy inequality for the shifted operator $-\Delta_D^\Omega - \inf \sigma_{\rm ess}(-\Delta_D^\Omega)$? This is exactly the type of questions we are interested in in this course.

However, the class of quasi-cylindrically manifolds is too broad to make any systematic study. The union of the infinitely many identical balls is one example. On the other hand, it is easy to construct examples of quasi-cylindrical sets Ω for which $\overline{\Omega} = \mathbb{R}^d$. That is why we restrict ourselves to a special class of quasi-cylindrical sets: tubes.

Moreover, there is a strong physical motivation for considering tubular geometries in the context of waveguides in electromagnetism and quantum mechanics. In the latter setting, the Dirichlet Laplacian represents the Hamiltonian of a quantum (quasi-)particle in a nano-structure of shape of wire or layer [49, 19].

1.4 Definition of tubes

We define a tube by moving a cross-section along a base manifold embedded in an ambient space.

- The **ambient space** $\mathcal{A}$ is a d-dimensional Riemannian manifold with $d \geq 2$ equipped with a metric G.
- The **base manifold** Σ is a d_1-dimensional oriented Riemannian submanifold of $\mathcal{A}$ with $1 \leq d_1 < d$ equipped with the pull-back metric $g := \iota^* G$, where $\iota : \Sigma \to \mathcal{A}$ is the immersion.
- The **cross-section** $\omega \subset \mathbb{R}^{d_2}$ with $1 \leq d_2 < d$ such that $d_1 + d_2 = d$ is a bounded open connected set. The farthest point of $\overline{\omega}$ to the origin of $\mathbb{R}^{d_2}$ will be denoted by

$$a := \sup_{t \in \Omega} |t|$$

and called the *tube radius*.

Since we are interested in unbounded tubes, we assume that both $\mathcal{A}$ and Σ are complete and non-compact. The usual C^∞-smoothness assumption can be reduced to C^2-smoothness.

The **motion** of ω along Σ is defined via

- an orthonormal frame $(n_1, \dots, n_{d_2}) \subset N\Sigma$.

Here the usual C^∞-smoothness assumption can be reduced to C^1-smoothness.

Then the **tube** Ω about Σ in $\mathcal{A}$ is defined by

$$\Omega := \Big\{ \underbrace{\exp_{\iota(s)} \big(t^1\, n_1(s) + \dots + t^{d_2}\, n_{d_2}(s)\big)}_{\mathscr{L}(s,t)} \ : \ (s,t) \in \Sigma \times \omega \Big\}.$$

Recall that the exponential map $\exp_p : T_p\mathcal{A} \to \mathcal{A}$ assigns to any vector $V \in T_p\mathcal{A}$ the unit-speed geoesic $\gamma_V(1)$ satisfying $\gamma_V(0) = p$ and $\gamma_V'(0) = V$. Of course, one has to impose certain smallness condition on a in order to ensure that $\mathscr{L}$ is an immersion, see Figure 2 and below.

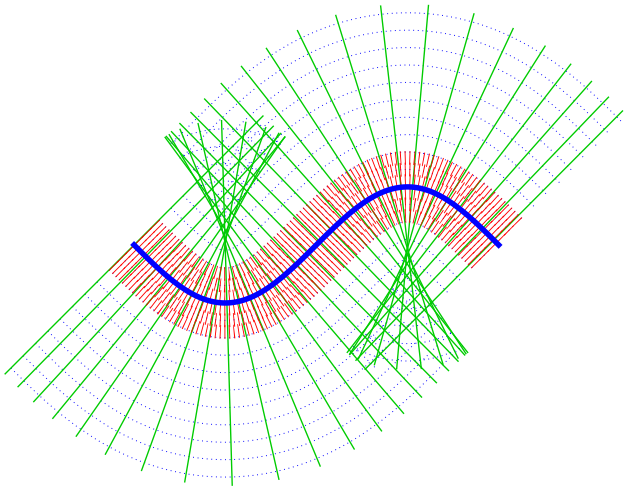


Figure 2: A Eucliean two-dimensional tube (red) constructed by means of transverse geodesics (green) emanating from the base curve (blue). A smallness restrition on the tube radius a must be imposed, in order to avoid "local self-intersections".

The Dirichlet Laplacian $-\Delta_D^\Omega$ associated with Ω is the Friedrichs extension of the Laplace–Beltrami operator

$$-|G|^{-1/2} \partial_A |G|^{1/2} G^{AB} \partial_B \qquad \text{in} \qquad L^2(\Omega)$$

intitially defined on the domain $C_0^\infty(\Omega)$. Here, as usual, $|G| := \det(G_{AB})$ and $(G^{AB}) := (G_{AB})^{-1}$, where (G_{AB}) is the matrix representation of G with respect to local coordinates $(x^1, \dots, x^d)$ of $\mathcal{A}$, and the Einstein summation convention is adopted, the range of capital Latin indices being $A \in \{1, \dots, d\}$. That is, the precise meaning of (1.4) is

$$\delta_D^\Omega[\psi] = \int_\Omega |\nabla_G \psi|^2 \, \mathrm{d}v_G$$

where $|\nabla_G\psi|^2 = \partial_A\bar\psi\, G^{AB}\,\partial_B\psi$ and $\mathrm{d}v_G = |G|^{1/2}\,\mathrm{d}x$ in the coordinates.

The natural strategy to study $-\Delta_D^\Omega$ is to pass to the *Fermi coordinates* $(s,t) := (s^1,\dots,s^{d_1},t^1,\dots,t^{d_2})$, where $(s^1,\dots,x^{d_1})$ are local coordinates of Σ. We shall use the convention of indexing the local coordinates $(s^1,\dots,x^{d_1})$ in Σ and and the Cartesian coordinates $(t^1,\dots,t^{d_2})$ in ω by small Latin and Greek letters, respectively, the range of them being $j\in\{1,\dots,d_1\}$ and $\alpha\in\{1,\dots,d_2\}$. Then one has the natural identification

$$\Omega\cong(\Sigma\times\omega,G)\,.$$

Unfortunately for this course (but fortunately from other respects), our life is finite. Let us therefore discuss only two special situations in more detail. Namely, those of the ambient space $\mathcal{A}$ being either the Euclidean space $\mathbb{R}^d$ or a two-dimensional manifold (in the latter case Σ is necessarily a curve).

1.5 Euclidean tubes

Let $\mathcal{A}:=\mathbb{R}^d$ within this subsection. Since the geodesics are straight lines in this case, the exponential map is an identity and computations become rather elementary. In the Fermi coordinates, the metric tensor G of $\mathcal{A}$ acquires the block form

$$G\cong\begin{pmatrix}(I-t^\alpha S_\alpha)^2 g+CC^T & C\\ C^T & I\end{pmatrix},\qquad \begin{aligned}(S_\alpha)_{ij}&:=-\langle\nabla^\top_{\partial_i}n_\alpha,\partial_j\rangle\,,\\ C_{i\alpha}&:=t^\beta\langle\nabla^\perp_{\partial_i}n_\beta,n_\alpha\rangle\,,\end{aligned}\tag{1.8}$$

where ∇ is the Riemannian connection on $\mathcal{A}$ (previously denoted by ∇_G) with $\nabla^\top$ and $\nabla^\perp$ denoting the tangent and normal connections, respectively, with respect to the orthogonal decomposition $T_p\mathcal{A}=T_p\Sigma\oplus N_p\Sigma$. Since $\det G=\det(I-t^\alpha S_\alpha)^2\,\det g$, the right-hand side is a well-defined Riemannian manifold under the standing hypothesis

$$a\,\|S\|_\infty<1\,,\tag{1.9}$$

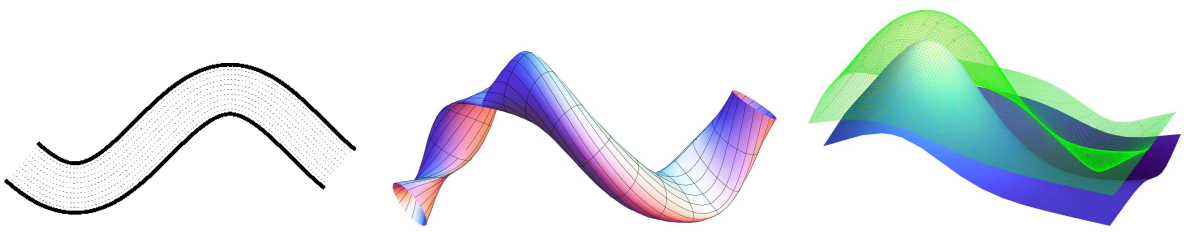
where $S:=\max_{\alpha=1,\dots,d_2}|S_\alpha|$ and $|S_\alpha|(x)$ denotes the norm of $S_\alpha:T_x\Sigma\to T_x\Sigma$. In other words, $\mathscr{L}$ is an immersion under (1.9). Since S_α is the shape operator associated with the vector field n_α and $\langle S_\alpha X,Y\rangle=\langle n_\alpha,I\!I(X,Y)\rangle$ for every $X,Y\in T\Sigma$ by the Weingarten equation, where $I\!I$ denotes the second fundamental form, hypothesis (1.9) represents a condition on the smallness of the tube radius in terms of curvatures of Σ.

From the structure of the metric (1.8), it is evident that there are two independent geometric deformations in the tubes:

- Ω is *bent* $:\Longleftrightarrow\ S\neq0$.
- Ω is *twisted* $:\Longleftrightarrow\ C\neq0$.

The condition $S\neq0$ means that the base manifold Σ is not flat (relative to $\mathcal{A}$). The condition $C\neq0$ means that the orthonormal frame $(n_1,\dots,n_{d_2})$ is not relatively parallel.

The situations which can be visualised (*i.e.* $d\leq3$) are depicted in Figure 3.



strips $d_1=1,\,d_2=1$ | tubes $d_1=1,\,d_2=2$ | layers $d_1=2,\,d_2=1$

Figure 3: Euclidean tubes in low dimensions.

Because of the codimension being one, there is no twisting for the strips and layers. From this perspective, the most interesting situation is that of tubes, to which we shall restrict on later.

1.6 Strips on surfaces

The simplest situation of a tube embedded in a non-trivially curved manifold is that of Σ being a curve of curvature k on a two-dimensional surface $\mathcal{A}$ of Gauss curvature K, see Figure 4. In this case, the metric tensor G in the Fermi coordinates reads

$$G \cong \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{cases} \partial_2^2 f + K f = 0\,, \\ f(\cdot,0) = 1\,, \\ \partial_2 f(\cdot,0) = -k\,. \end{cases} \tag{1.10}$$

More specifically, here k is the curvature of Σ as a curve on $\mathcal{A}$ computed with respect to a given normal vector field n along Σ (k is the geodesic curvature if $\mathcal{A}$ is embedded in $\mathbb{R}^3$). Now the basic hypothesis (1.9) is replaced by a suitable smallness of a with respect to $\|k\|_\infty$ and $\|K\|_\infty$.

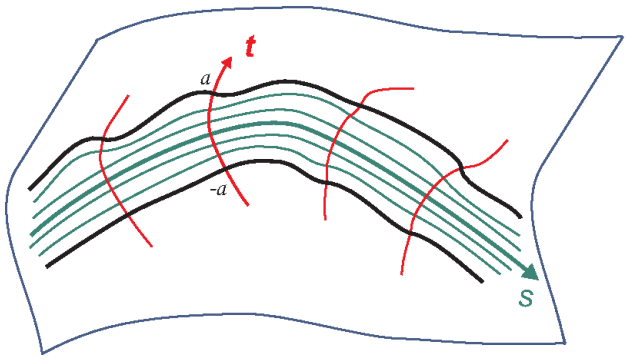


Figure 4: Strips on surfaces ($d_1 = 1 = d_2$).

If $\mathcal{A} = \mathbb{R}^2$, we are in the situation of the Euclidean strips above; since $K = 0$, the solution of the Jacobi equation in (1.10) yields the familiar formula $f(s,t) = 1 - k(s)t$. In general, there is an interesting interplay between the ambient (intrinsic) curvature K and the extrinsic curvature k.

There is one particular situation for which an interlink with the bending and twisting in the Euclidean case can be made: $\mathcal{A}$ being a *ruled surface* by itself embedded in $\mathbb{R}^3$. More specifically, let the Gauss curvature be given by

$$K(s,t) := -\frac{\theta'(s)^2}{f(s,t)^4} \qquad \text{with} \qquad f(s,t) := \sqrt{\big(1 - k(s)\,t\big)^2 + \theta'(s)^2\,t^2}\,,$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is C^1-smooth. Then it is easy to verify that (1.10) is satisfied. The same metric is obtained by considering the immersion

$$\tilde{\mathscr{L}} : \mathbb{R} \times (-a,a) \to \mathbb{R}^3 : \Big\{(s,t) \mapsto \Gamma(s) + \big[N_1(s)\cos\theta(s) - N_2(s)\sin\theta(s)\big]\,t\Big\},$$

where $\Gamma : \mathbb{R} \to \mathbb{R}^3$ is a unit-speed C^2-smooth curve and (N_1, N_2) is its relatively parallel adapted frame associated with the curvature vector (k_1, k_2) (see (2.1) below) after the identification $k := k_1 \cos\theta - k_2 \sin\theta$. Hence, the geometrical meaning of Ω in this case is that of a twisted strip about Γ in $\mathbb{R}^3$, see Figure 5. The terminology is not ideal because, in analogy with the Euclidean classification made above, there are two independent geometric deformations in the twisted strips:

- Ω is *bent* $:\Longleftrightarrow$ $k \neq 0$.
- Ω is *twisted* $:\Longleftrightarrow$ $\theta' \neq 0$.

However, it is important to emphasise that, contrary to the Euclidean strips, the twisted strip Ω is not an open subset of $\mathbb{R}^3$.

1.7 Straight tubes

The simplest tube is the straight tube $\Omega_0 := \Sigma \times \omega$ equipped with the corresponding metric $G_0 \cong \mathrm{diag}(g, 1)$, see Figure 6. Then

$$\sigma(-\Delta_D^{\Omega_0}) = \sigma(-\Delta^\Sigma) + \sigma(-\Delta_D^\omega)\,.$$

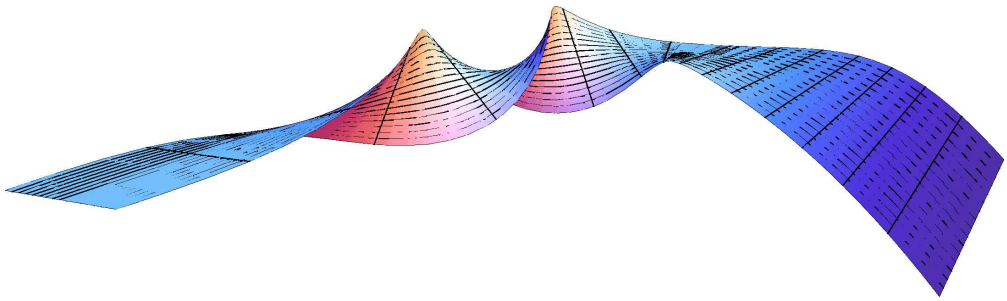


Figure 5: Strips on ruled surfaces as twisted strips.

Here we write $-\Delta^\Sigma$ instead of $-\Delta^\Sigma_D$, because Σ is complete and non-compact. If the Ricci curvature of Σ vanishes at infinity, then Lu and Zhou's result [51] implies

$$\sigma(-\Delta^\Sigma) = \sigma_{\rm ess}(-\Delta^\Sigma) = [0,\infty)\,. \tag{1.11}$$

On the other hand, since ω is bounded,

$$\sigma(-\Delta^\omega_D) = \sigma_{\rm disc}(-\Delta^\omega_D) =: \{E_1 < E_2 \leq E_3 \leq \dots \to \infty\}\,,$$

where E_1 is positive and simple, while the other eigenvalues are repeated according to their multiplicities. Consequently,

$$\sigma(-\Delta^{\Omega_0}_D) = \sigma_{\rm ess}(-\Delta^{\Omega_0}_D) = [E_1,\infty)\,.$$

Moreover, criticality properties of $-\Delta^{\Omega_0}_D$ are determined by the criticality properties of $-\Delta^\Sigma$, and vice versa.

Proposition 1.1. *$-\Delta^\Sigma$ is subcritical* $\iff$ *$-\Delta^{\Omega_0}_D - E_1$ is subcritical.*

Proof. Let $-\Delta^\Sigma$ be subcritical (also known as Σ being *non-parabolic* or *transient*). Then any Hardy inequality $-\Delta^\Sigma \geq \rho$ with a positive function ρ implies the Hardy inequality $-\Delta^{\Omega_0}_D - E_1 \geq \rho$.

Conversely, let Σ be critical (also known as Σ being *parabolic* or *recurrent*). This is equivalent to the analytic fact that any compact subset of Σ has zero capacity (see [53], [26] or [45]). From this, one can deduce that there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\Sigma)$ such that $0 \leq \varphi_n \leq 1$ for every $n \in \mathbb{N}$, $\varphi_n \to 1$ pointwise and $\int_\Sigma |\nabla_g \varphi_n|^2 \to 0$ as $n \to \infty$. Define $\psi_n(s,t) := \varphi_n(s)\mathcal{J}_1(t)$, where $\mathcal{J}_1$ is the positive eigenfunction of $-\Delta^\omega_D$ corresponding to E_1 normalised to 1 in $L^2(\omega)$. Then, given any non-trivial $V \in L^\infty(\Omega_0)$ such that $V \leq 0$, one has

$$\begin{aligned}
&\int_{\Omega_0} |\nabla_{G_0}\psi_n|^2 - E_1 \int_{\Omega_0} |\psi_n|^2 + \int_{\Omega_0} V\,|\psi_n|^2 \\
&= \int_\Sigma |\nabla_g \varphi_n|^2 \int_\omega |\mathcal{J}_1|^2 + \int_\Sigma |\varphi_n|^2 \int_\omega |\nabla \mathcal{J}_1|^2 - E_1 \int_\Sigma |\varphi_n|^2 \int_\omega |\mathcal{J}_1|^2 + \int_{\Omega_0} V\,|\psi_n|^2 \\
&= \int_\Sigma |\nabla_g \varphi_n|^2 \int_\omega |\mathcal{J}_1|^2 + \int_{\Omega_0} V\,|\psi_n|^2 \\
&\xrightarrow[n\to\infty]{} \int_{\Omega_0} V\,|1\otimes\mathcal{J}_1|^2 < 0\,,
\end{aligned}$$

where the last integral can be $-\infty$. By the minimax principle, $\inf\sigma(-\Delta^{\Omega_0}_D - E_1) < 0$. □

1.8 The essential spectrum

Since the essential spectrum is determined by the behaviour of geometry at infinity only, it is expected that

$$\Omega \text{ is asymptotically straight} \quad\Longrightarrow\quad \sigma_{\rm ess}(-\Delta^\Omega_D) = [E_1,\infty)\,. \tag{1.12}$$

Here the sufficient condition means that the bending and twisting vanish at infinity. More specifically, for the Euclidean tubes this means the vanishing of the tensors S and C at infinity (the former implies

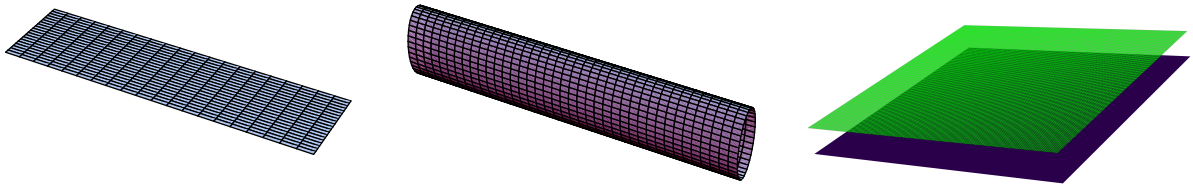


Figure 6: Straight strips, tubes and layers, respectively.

vanishing of the Ricci curvature of Σ). For strips on surfaces, one needs to assume the vanishing of both the curvatures K and k at infinity.

Despite that (1.12) is very natural, it took a considerable time to achieve it under the minimal hypotheses. The main obstacle was the usage of the classical Weyl criterion, which required to impose additional hypotheses on the decay of derivatives of the metric G at infinity. The first proof of (1.12) without these technical assumptions was obtained in [36] for the Euclidean strips with help of the Weyl criterion modified to quadratic forms (inspired by [13]). The Euclidean tubes about curves without twisting were covered in [8]. The proof of (1.12) for general Euclidean tubes is due to [37]. This ultimate result was delayed not only because of the ignorance of the Weyl criterion modified to quadratic forms before [36], but also because of the unavailability of the result (1.11) before [51].

The proofs of (1.12) for strips on surfaces about geodesics and twisted strips are due to [29] and [41], respectively. It is not difficult to extend these approaches to arbitrary strips on surfaces.

The result (1.12) is optimal in the sense that there are examples of Euclidean tubes for which $\inf\sigma_{\rm ess}(-\Delta_D^\Omega) < E_1$ or $\inf\sigma_{\rm ess}(-\Delta_D^\Omega) > E_1$ if Ω is not twisted but periodically bent or not bent but periodically twisted, respectively (see Sections 2.10 and 2.10, respectively). What is more, the spectrum can be purely discrete for tubes with asymptotically diverging twisting (see Section 2.12). The same situation for strips on surfaces has been recently analysed in [35].

1.9 The discrete spectrum

The spectral geometry of tubes was initiated in 1989 by Exner and Šeba's celebrated paper [20] on the existence of discrete spectra in thin curved Euclidean strips. A couple of years later, the superfluous restriction to thin strips was removed by physicists Goldstone and Jaffe [23]. Moreover, they extended the result to three-dimensional Euclidean tubes without twisting. However, the most significant contribution of [23] was to devise a robust variational approach to the proof of the discrete spectra. Despite the lack of mathematical rigour in their proof, the main idea is correct and the approach has been using in many other situations since. The proof is mathematically rectified in another celebrated paper of Duclos and Exner [14]. However, the proof [14] of requires en extra smoothness of the base curves, which is not needed in reality. The extension to higher-dimensional Euclidean tubes about curves without twisting is due to [8]. Even if the last mentioned paper represents the variational proof under minimal hypotheses, it still requires the existence of the Frenet frame, which is unnecessarily restrictive. The usage of the parallel adapted frame instead of the Frenet frame is due to [40]. This last extension enables us, in this course, to present the variational proof under indeed minimal hypotheses.

The main result of the aforementioned studies can be stated as the following supercriticality:

$$\left.\begin{array}{l}\mathcal{A}=\mathbb{R}^d \quad \& \quad \Sigma \text{ is a curve} \\ \Omega \text{ is bent but not twisted}\end{array}\right\} \quad \Longrightarrow \quad \inf\sigma(-\Delta_D^\Omega) < E_1\,. \tag{1.13}$$

The condition precisely means that Σ is not a straight line and that the orthonormal frame $(n_1,\ldots,n_{d_2})$ is relatively parallel (this is always the case if $d=2$). Here, in principle, we do not assume that Ω is asymptotically straight. If it is the case, however, (1.13) and (1.12) imply that $-\Delta_D^\Omega$ possesses at least one discrete eigenvalue below E_1.

If $\mathcal{A}$ is a two-dimensional manifold, the situation is more complex because of the presence of the ambient curvature K. In [30], it was shown that a sufficient condition to guarantee (1.13) in the case of strips on

surfaces is that K is non-negative:

$$\left.\begin{array}{l}\mathcal{A} \text{ is a surface} \quad \& \quad \Sigma \text{ is a curve} \\ k \neq 0 \quad \& \quad K \geq 0\end{array}\right\} \quad \Longrightarrow \quad \inf \sigma(-\Delta_D^\Omega) < E_1 \,. \tag{1.14}$$

This means $K = 0$ for the twisted strips.

The case of layers, *i.e.* the Euclidean tubes about surfaces, were first investigated in [15]. The expected analogue of (1.13) is as follows:

$$\left.\begin{array}{l}\mathcal{A} = \mathbb{R}^d \quad \& \quad \Sigma \text{ is a parabolic surface} \\ \Omega \text{ is bent but not twisted}\end{array}\right\} \quad \stackrel{?}{\Longrightarrow} \quad \inf \sigma(-\Delta_D^\Omega) < E_1 \,. \tag{1.15}$$

The question mark correctly suggests that this situation is by no means satisfactorily answered. Indeed, the pioneering work [15], focused on three-dimensional tubes (in which case there is no twisting), establishes (1.15) under extra hypotheses only, namely

$$K \in L^1(\Sigma). \tag{1.16}$$

together with any of the following conditions:

(a) $\int_\Sigma K \leq 0$.

(b) a is small enough.

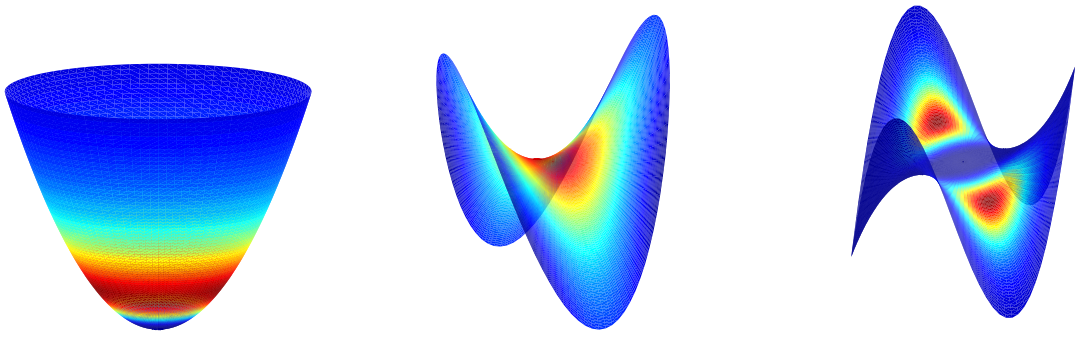
(c) $\int_\Sigma M^2 = \infty$ & $\nabla_g M \in L^2(\Sigma)$.

(d) Σ contains a rotationally symmetric end with $\int_\Sigma K > 0$.

Moreover, it is assumed in [15] that Σ possesses a pole, which is a very restrictive hypothesis. This technical condition was later removed in [7].

An extension of the results of [15, 7] to higher-dimensional layers was performed by Lin and Lu in [47, 46]. For other sufficient conditions in $\mathbb{R}^3$, see [48, 50]. Apart from these partial progresses, the complete validity of (1.15) remains a mystery.

It is interesting to notice that the requirement (1.16) implies that Σ is parabolic (see [24, 25] and [56] for independent discoveries). The extension of the results of [15, 7] to higher dimensions require to assume that the base manifold Σ is parabolic.



elliptic paraboloid
$\int_\Sigma K = 2\pi$

hyperbolic paraboloid
$\int_\Sigma K = -2\pi$

monkey saddle
$\int_\Sigma K = -4\pi$

Figure 7: Examples of surfaces and their total Gauss curvatures.

Condition (a) is closely related to topological properties of Σ. Indeed, Cohn–Vossen's inequality (see [9] and [28] for classical references and [54] for a recent survey) states that

$$\int_\Sigma K \leq 2\pi\, \chi(\Sigma)\,,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Recall that $\chi(\Sigma) = 2-2h-e$, where h is the genus of Σ, *i.e.* the number of handles, and e is the number of ends. In particular, condition (a) is always satisfied whenever the surface Σ is not simply connected. If it is simply connected, "negatively curved surfaces are preferred", see Figure 7.

Under condition (d), if Ω is asymptotically straight, there are actually an infinite number of discrete eigenvalues below E_1. Their asymptotics were studied in [10].

1.10 Hardy inequalities

It was already mentioned that non-parabolicity of Σ implies a Hardy inequality for $-\Delta_D^{\Omega_0} - E_1$. By certain hypothesis about the smallness of $G - G_0$, it is then possible to deduce Hardy inequalities for $-\Delta_D^{\Omega} - E_1$.

It is more interesting that there are Hardy inequalities even if Σ is parabolic. What is more, the manifold Σ can be straight:

$$\left.\begin{array}{l} \mathcal{A} = \mathbb{R}^3 \quad \& \quad \Sigma \text{ is a curve} \\ \Omega \text{ is twisted but not bent} \\ \omega \text{ is not circular} \end{array}\right\} \quad \Longrightarrow \quad -\Delta_D^{\Omega} - E_1 \text{ is subcritical}\,. \tag{1.17}$$

Since Ω is not bent, Σ is necessarily a straight line here. The hypothesis that the cross-section ω is not circular precisely means that it is not a disk or annulus centred at the origin. This requirement is necessary, because twisting a circular cross-section obviously leads to a straight tube.

Under extra hypotheses, result (1.17) is originally due to [17]. An alternative, more robust approach, which yields the subcriticality under the present minimal hypotheses, was devised in [32] and later in [42].

Hardy inequalities for strips on surfaces were originally derived in [31] and later also in [29, 41]. Here the subcriticality is generated by negatively curved ambient spaces or by twisting if the ambient space is the ruled surface (*i.e.* in the twisted-strip realisation).

To derive Hardy inequalities for tubes about parabolic two-dimensional manifolds Σ in higher-dimensional spaces (there is no twisting for the Euclidean layers in $\mathbb{R}^3$) constitutes an interesting direction of research to be performed.

2 Euclidean tubes about curves

After the general panorama, let us focus on the geometric realisation where the submanifold is a curve and the ambient manifold is the three-dimensional Euclidean space. The tube is constructed by translating a given cross-section $\omega \subset \mathbb{R}^2$ along the curve in $\mathbb{R}^3$ with respect to an arbitrary moving frame.

2.1 The base curve

Let $\Gamma : \mathbb{R} \to \mathbb{R}^3$ be a C^2-smooth curve which is (without loss of generality) parameterised by its arc-length (*i.e.* $|\Gamma'(s)| = 1$ for all $s \in \mathbb{R}$). By the regularity hypothesis, the tangent vector field $T := \Gamma'$ is C^1-smooth and the *curvature* $\kappa := |\Gamma''|$ is continuous.

2.2 The moving frame

We need a suitable moving frame along Γ. The most famous is the *Frenet frame* formed by the orthonormal triad (T, N, B), where $N := \Gamma''/|\Gamma''|$ is the *principal normal* and $B := T \times N$ is the *binormal*. It is characterised by the Frenet–Serret equations

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\tau := \det(\Gamma', \Gamma'', \Gamma''')/\kappa^2$ is the *torsion* of Γ.

A shortcoming of the Frenet frame is the necessity to assume that Γ is C^3-smooth. What is worse, the non-degeneracy condition $\kappa > 0$ is inevitable. In principle, the latter excludes straight lines! A straight line can be equipped with a constant frame, it is true, but there exist infinitely smooth curves for which no Frenet frame can be introduced, see Figure 8.

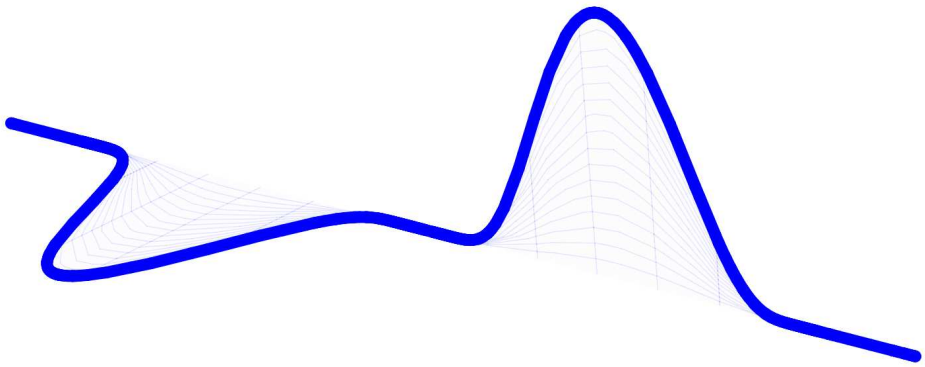


Figure 8: A curve without the Frenet frame.

Fortunately, any C^2-smooth curve can always be equipped with another moving frame, so-called *relatively parallel adapted frame* (T, N_1, N_2) satisfying

$$\begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}' = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}. \tag{2.1}$$

Here k_1, k_2 are continuous functions satisfying

$$k_1^2 + k_2^2 = \kappa^2. \tag{2.2}$$

This frame is *adapted* because N_1, N_2 are normal to Γ (the Frenet frame is adapted too). It is *relatively parallel* because the derivative of the normal vector fields N_1, N_2 is tangential (the Frenet frame is relatively parallel if, and only if, $\tau = 0$, *i.e.* the curve Γ is planar). The normal vectors thus rotate along the curve only whatever amount is necessary to remain normal. More specifically, each normal vector is translated along the curve as close to a parallel transport as possible without losing normality.

A relatively parallel adapted frame is constructed as follows. First, it is easily shown that any relatively parallel adapted frame is *uniquely* determined given initial conditions $N_j(0) \in \mathbb{R}^3$ with $j = 1, 2$. Second,

one constructs a relatively parallel adapted frame *locally* by suitably rotating an auxiliary adapted frame (the latter always exists locally because κ is locally bounded). Finally, to get the *global* existence, one patches together the local ones. Smoothness at the points where they link together is a consequence of the uniqueness part. We refer to [2] for a popular account on relatively parallel adapted frames, to [40] for an extension to merely $C^{1,1}$-smooth curves and to [41] for a generalisation to higher dimensions.

The general moving frame $(T, N_1^\theta, N_2^\theta)$ is then introduced by rotating the relatively parallel adapted frame (T, N_1, N_2) with respect to a given C^1-smooth rotation function $\theta : \mathbb{R} \to \mathbb{R}$:

$$\begin{pmatrix} N_1^\theta \\ N_2^\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} .$$

Using (2.1), it is straightforward to check that the new frame evolves along the curve via

$$\begin{pmatrix} T \\ N_1^\theta \\ N_2^\theta \end{pmatrix}' = \begin{pmatrix} 0 & k_1^\theta & k_2^\theta \\ -k_1^\theta & 0 & -\theta' \\ -k_2^\theta & \theta' & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1^\theta \\ N_2^\theta \end{pmatrix} , \tag{2.3}$$

where

$$\begin{pmatrix} k_1^\theta \\ k_2^\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} .$$

The Frenet frame corresponds to the special choice $\theta' = -\tau$ together with $k_1 = \kappa\cos\theta$ and $k_2 = -\kappa\sin\theta$.

2.3 The cross-section

Let ω be a bounded open connected set in $\mathbb{R}^2$. We do not assume any regularity hypotheses about the boundary $\partial\omega$ (it can be fractal). It is convenient to introduce the quantity

$$a := \sup_{t\in\omega} |t| , \tag{2.4}$$

which measures the distance of the farthest point of $\overline{\omega}$ to the origin.

We say that ω is *rotationally invariant with respect to the origin* if

$$\forall \vartheta \in (0, 2\pi) , \qquad \omega_\vartheta := \left\{ \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \ : \ (t_1, t_1) \in \omega \right\} = \omega ,$$

with the natural convention that we identify ω and ω_ϑ (and other open sets) provided that they differ at most on a set of zero capacity (in particular, removing a point is harmless). Hence, modulus a set of zero capacity, ω is rotationally symmetric with respect to the origin if, and only if, it is a disk or an annulus centred at the origin of $\mathbb{R}^2$.

2.4 The tube, bending and twisting

A *tube* Ω is defined by moving the cross-section ω along the base curve Γ together with the generally rotated frame $(T, N_1^\theta, N_2^\theta)$. More specifically, we set

$$\Omega := \Big\{ \underbrace{\Gamma(s) + t_1\, N_1^\theta(s) + t_2\, N_2^\theta(s)}_{\mathscr{L}(s,t)} \ : \ (s,t) \in \mathbb{R}\times\omega \Big\} . \tag{2.5}$$

In this way, Ω can be clearly understood as a deformation of the *straight tube* $\Omega_0 := \mathbb{R}\times\omega$.

It is clear from the equations of motion of the general moving frame (2.3) that there are two independent geometric effects in curved tubes:

- Ω is *bent* $:\Longleftrightarrow$ $\kappa \neq 0$ (*i.e.* Γ is not a straight line).
- Ω is *twisted* $:\Longleftrightarrow$ $\theta' \neq 0$ (*i.e.* ω is not translated parallelly along Γ).

The definition of bending is visually clear. On the other hand, the definition of twisting may be confusing. Indeed, even if $\theta' \neq 0$ (and $\kappa = 0$), it is possible that Ω is congruent to the straight tube $\mathbb{R}\times\omega$. For this reason, to have a non-trivially twisted tube, it is also important to assume that the cross-section ω is not

rotationally invariant with respect to the origin. It will be clear from the statements (and proofs) of the related theorems below.

Of course, Ω can be simultaneously bent and twisted (as in the figure in the cover page). Another example is illustrated in Figure 9, where the tube on the left is simultaneously bent and twisted, while the tube on the right is bent but untwisted (because $\theta' = 0$).

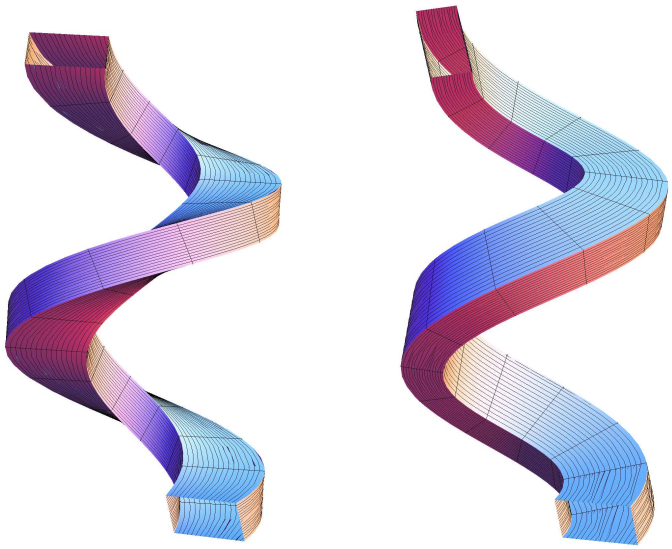


Figure 9: A tube of rectangular cross-section about a helix obtained by using the Frenet frame (left) or the relatively parallel frame (right).

2.5 The tube as a Riemannian manifold

Our strategy to deal with the curved geometry of the tube Ω is to use the identification

$$\Omega \cong (\mathbb{R} \times \omega, G)\,, \qquad \text{where} \qquad G := \nabla\mathscr{L} \cdot (\nabla\mathscr{L})^T \tag{2.6}$$

is the metric tensor induced by the mapping (*cf* (2.5))

$$\mathscr{L} : \mathbb{R} \times \omega \to \mathbb{R}^3\,. \tag{2.7}$$

In other words, we parameterise Ω globally by means of the "coordinates" (s,t) of (2.5). To this aim, we need to impose natural restrictions in order to ensure that $\mathscr{L} : \mathbb{R} \times \omega \to \Omega$ is a diffeomorphism.

Using (2.3), we find

$$G = \begin{pmatrix} f^2 + f_1^2 + f_2^2 & f_1 & f_2 \\ f_1 & 1 & 0 \\ f_2 & 0 & 1 \end{pmatrix}, \qquad \begin{aligned} f(s,t) &:= 1 - t_1\, k_1^\theta(s) - t_2\, k_2^\theta(s)\,, \\ f_1(s,t) &:= t_2\, \theta'(s)\,, \\ f_2(s,t) &:= -t_1\, \theta'(s)\,. \end{aligned} \tag{2.8}$$

Consequently,

$$|G| := \det(G) = f^2\,.$$

By virtue of the inverse function theorem, the mapping $\mathscr{L}$ induces a *local* C^1-diffeomorphism provided that the Jacobian f does not vanish on $\mathbb{R} \times \omega$. One has the uniform bounds

$$0 < 1 - a\,\|\kappa\|_\infty \;\leq\; f(s,t) \;\leq\; 1 + a\,\|\kappa\|_\infty < \infty \tag{2.9}$$

valid for every $(s,t) \in \mathbb{R} \times \omega$, where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R})}$. Consequently, the positivity of f is guaranteed by the hypothesis

$$\kappa \in L^\infty(\mathbb{R}) \qquad \text{and} \qquad a\,\|\kappa\|_\infty < 1\,. \tag{2.10}$$

The mapping $\mathscr{L} : \mathbb{R} \times \omega \to \Omega$ then becomes a *global* diffeomorphism if, in addition to (2.10), we assume that

$$\mathscr{L} \quad \text{is injective}\,. \tag{2.11}$$

In particular, Ω is an open set. For sufficient conditions ensuring (2.11) we refer to [17, App.].

Remark 2.1. In other words, hypothesis (2.10) ensures that $(\mathbb{R}\times\omega, G)$ is a Riemannian manifold and (2.7) represents its *immersion* in $\mathbb{R}^3$. Both hypotheses (2.10) and (2.11) then ensure that $(\mathbb{R}\times\omega, G)$ is an *embedded* submanifold of $\mathbb{R}^3$. Giving up the geometrical interpretation of Ω being a non-self-intersecting tube in $\mathbb{R}^3$, it is possible to work under the hypothesis (2.10) only.

2.6 The tube as a quantum Hamiltonian

The word "quantum" refers to that we consider the Hamiltonian of a free quantum particle constrained to Ω. As usual, we model the Hamiltonian by the Dirichlet Laplacian

$$-\Delta_D^\Omega \qquad \text{in} \qquad L^2(\Omega)\,.$$

It is introduced in the standard way as the self-adjoint operator associated with the quadratic form

$$\delta_D^\Omega[u] := \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x\,, \qquad u \in \operatorname{dom}\delta_D^\Omega := H_0^1(\Omega)\,. \tag{2.12}$$

Note that $-\Delta_D^\Omega$ is well defined under our minimal hypotheses (in particular, without any regularity assumptions about the cross-section ω), just because Ω is an open set due to (2.10) and (2.11).

Recalling the identification (2.6) induced by (2.7), we can identify $-\Delta_D^\Omega$ with an operator H acting in the Hilbert space

$$\mathcal{H} := L^2\big(\mathbb{R}\times\omega, f(s,t)\,\mathrm{d}s\,\mathrm{d}t\big)\,. \tag{2.13}$$

More specifically, $H := U(-\Delta_D^\Omega)U^{-1}$, where U is the unitary transform

$$U : L^2(\Omega) \to \mathcal{H} : \{u \mapsto \underbrace{u\circ\mathscr{L}}_{\psi}\}\,.$$

By definition, H is the operator associated with the quadratic form h in $\mathcal{H}$ defined by

$$h[\psi] := \delta_D^\Omega[U^{-1}\psi]\,, \qquad \psi \in \operatorname{dom} h := U\operatorname{dom}\delta_D^\Omega\,.$$

In other words, one has to pass to the curvilinear coordinates (s,t) in the integral (2.12). Using that $C_0^\infty(\Omega)$ is a core of δ_D^Ω and that $\mathscr{L} : \mathbb{R}\times\omega \to \Omega$ is a C^1-diffeomorphism, one has

$$h[\psi] = \big(\partial_i\psi, G^{ij}\partial_j\psi\big)\,, \qquad \psi \in \operatorname{dom} h = \overline{C_0^1(\mathbb{R}\times\omega)}^{|||\cdot|||}\,, \tag{2.14}$$

where $|||\psi||| := \sqrt{h[\psi] + \|\psi\|^2}$, G^{ij} stands for the coefficients of the inverse metric

$$G^{-1} = \frac{1}{f^2}\begin{pmatrix} 1 & -f_1 & -f_2 \\ -f_1 & f^2+f_1^2 & f_1f_2 \\ -f_2 & f_2f_1 & f^2+f_2^2 \end{pmatrix} \tag{2.15}$$

and the Einstein summation convention is adopted, with the range of indices being $1,2,3$. In the distributional sense, H acts as the Laplace–Beltrami operator

$$H = -|G|^{-1/2}\partial_i|G|^{1/2}G^{ij}\partial_j\,. \tag{2.16}$$

Remark 2.2. Defining H as the operator associated with (2.14), it is enough to assume (2.10).

In fact, (2.14) is a general formula for any curvilinear coordinates. Using the particular form of the inverse metric (2.15), we find

$$h[\psi] = \|f^{-1}(\partial_s\psi - \theta'\partial_\tau\psi)\|^2 + \|\nabla_t\psi\|^2\,, \tag{2.17}$$

where $\nabla_t := (\partial_{t_1}, \partial_{t_2})$ is the transverse gradient and $\partial_\tau := \tau\cdot\nabla_t$ with $\tau := (t_2, -t_1)$ is the transverse angular derivative (do not confuse the vector τ with the torsion that we shall never mention any more). Here we implicitly assume that $\psi = \psi(s,t)$. Moreover, with an abuse of notation, we denote by the same symbol θ' the function $(s,t)\mapsto\theta'(s)$.

Similarly, the general formula (2.16) in our case reads

$$H = -f^{-1}(\partial_s - \theta'\partial_\tau)f^{-1}(\partial_s - \theta'\partial_\tau) - f^{-1}\nabla_t\cdot f\nabla_t\,. \tag{2.18}$$

Proposition 2.3. *In addition to* (2.10)*, let us assume*

$$\theta' \in L^\infty(\mathbb{R})\,. \tag{2.19}$$

Then there exist positive constants $C_\pm$ depending on $\|\kappa\|_\infty$, $\|\theta'\|_\infty$ and a such that

$$\forall\psi\in C_0^1(\mathbb{R}\times\omega)\,,\qquad C_-\,\|\psi\|_{H^1(\mathbb{R}\times\omega)} \leq |||\psi||| \leq C_+\,\|\psi\|_{H^1(\mathbb{R}\times\omega)}\,. \tag{2.20}$$

In particular, $\operatorname{dom} h = H_0^1(\mathbb{R}\times\omega)$.

Proof. Let $\psi\in C_0^1(\mathbb{R}\times\omega)$. Recalling (2.9),

$$\frac{\|\partial_s\psi - \theta'\partial_\tau\psi\|^2_{\mathfrak{H}_0}}{1+a\,\|\kappa\|_\infty} + (1-a\,\|\kappa\|_\infty)\,\|\nabla_t\psi\|^2_{\mathfrak{H}_0} \leq h[\psi] \leq \frac{\|\partial_s\psi - \theta'\partial_\tau\psi\|^2_{\mathfrak{H}_0}}{1-a\,\|\kappa\|_\infty} + (1+a\,\|\kappa\|_\infty)\,\|\nabla_t\psi\|^2_{\mathfrak{H}_0}\,,$$

where $\mathcal{H}_0 := L^2(\mathbb{R}\times\omega)$. The upper bound in (2.20) is concluded by

$$\begin{aligned}\|\partial_s\psi - \theta'\partial_\tau\psi\|^2_{\mathfrak{H}_0} &\leq 2\,\|\partial_s\psi\|^2_{\mathfrak{H}_0} + 2\,\|\theta'\partial_\tau\psi\|^2_{\mathfrak{H}_0}\\ &\leq 2\,\|\partial_s\psi\|^2_{\mathfrak{H}_0} + 2\,\|\theta'\|^2_\infty\,a^2\,\|\nabla_t\psi\|^2_{\mathfrak{H}_0}\,,\end{aligned}$$

where we have used the pointwise bound $|\partial_\tau\psi| \leq |t||\nabla_t\psi|$ and (2.4). For the lower bound, we write

$$\begin{aligned}\|\partial_s\psi - \theta'\partial_\tau\psi\|^2_{\mathfrak{H}_0} &\geq \frac{\delta}{1+\delta}\,\|\partial_s\psi\|^2_{\mathfrak{H}_0} - \delta\,\|\theta'\partial_\tau\psi\|^2_{\mathfrak{H}_0}\\ &\geq \frac{\delta}{1+\delta}\,\|\partial_s\psi\|^2_{\mathfrak{H}_0} - \delta\,\|\theta'\|^2_\infty\,a^2\,\|\nabla_t\psi\|^2_{\mathfrak{H}_0}\end{aligned}$$

with any positive δ. It remains to choose δ sufficiently small, for instance

$$\delta := \frac{(1-a\,\|\kappa\|_\infty)(1+a\,\|\kappa\|_\infty)}{1+\|\theta'\|^2_\infty\,a^2}\,.$$

This establishes the lower bound in (2.20). From the equivalence of norms (2.20), the conclusion $\operatorname{dom} h = H_0^1(\mathbb{R}\times\omega)$ readily follows. □

From now on, we always assume (2.10) as a standing hypothesis. The additional hypothesis (2.19) will be often satisfied because we shall be primarily interested in asymptotically straight tubes satisfying (2.23). At the very end, however, we shall consider tubes which do not satisfy (2.19).

2.7 Straight tubes

If Ω is neither bent (*i.e.* $\kappa = 0$) nor twisted (*i.e.* $\theta' = 0$), the metric (2.8) reduces to the Euclidean metric. Consequently, Ω can be identified with the straight tube $\mathbb{R}\times\omega$, see Figure 10. In this case, one has the "separation of variables"

$$-\Delta_D^{\mathbb{R}\times\omega} \cong -\Delta_D^{\mathbb{R}}\otimes I^\omega + I^{\mathbb{R}}\otimes -\Delta_D^\omega \qquad\text{in}\qquad L^2(\mathbb{R}\times\omega)\cong L^2(\mathbb{R})\otimes L^2(\omega)\,.$$

Consequently, $\sigma(-\Delta_D^{\mathbb{R}\times\omega}) = \sigma(-\Delta_D^{\mathbb{R}}) + \sigma(-\Delta_D^{\omega})$. It is well known that

$$\sigma(-\Delta_D^{\mathbb{R}}) = \sigma_{\rm ess}(-\Delta_D^{\mathbb{R}}) = [0,\infty)\,,$$
$$\sigma(-\Delta_D^{\omega}) = \sigma_{\rm disc}(-\Delta_D^{\omega}) = \{E_1 < E_2 \leq E_3 \leq \dots \to \infty\}\,.$$

Therefore,

$$\sigma(-\Delta_D^{\mathbb{R}\times\omega}) = \sigma_{\rm ess}(-\Delta_D^{\mathbb{R}\times\omega}) = [E_1,\infty)\,.$$

In fact, the spectrum is purely absolutely continuous.

The eigenfunction corresponding to the lowest eigenvalue E_1 of $-\Delta_D^{\omega}$ will be denoted by $\mathcal{J}_1$. We choose it positive and normalised to 1 in $L^2(\omega)$. We shall frequently use the following Poincaré inequality

$$\forall \mathcal{J} \in H_0^1(\omega)\,, \qquad \|\nabla \mathcal{J}\|^2_{L^2(\omega)} \geq E_1 \, \|\mathcal{J}\|^2_{L^2(\omega)}\,, \tag{2.21}$$

which follows from the variational definition of E_1.

Despite the fact that the tubes we consider are three-dimensional, they are *quasi-one-dimensional* in the sense that there is only one "infinite direction". For straight tubes, using the criticality of $\mathbb{R}$, we get the following criticality of $\mathbb{R}\times\omega$.

Proposition 2.4 (Criticality of straight tubes)**.** *Let $V \in L^\infty(\mathbb{R}\times\omega)$ be such that $V \lneq 0$. Then*

$$\inf \sigma(-\Delta_D^{\mathbb{R}\times\omega} + V) < E_1\,.$$

Proof. The proof is based on the variational characterisation

$$\inf \sigma(-\Delta_D^{\mathbb{R}\times\omega} + V - E_1) = \inf_{\substack{\psi\in H_0^1(\mathbb{R}\times\omega)\\ \psi\neq 0}} \frac{Q[\psi]}{\|\psi\|^2}\,,$$

where $\|\psi\|^2 = \int_{\mathbb{R}\times\omega} |\psi(s,t)|^2 \,\mathrm{d}s\,\mathrm{d}t$ and

$$Q[\psi] := \int_{\mathbb{R}\times\omega} |\nabla\psi|^2 + \int_{\mathbb{R}\times\omega} V(s,t)\,|\psi(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t - E_1 \int_{\mathbb{R}\times\omega} |\psi(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t\,.$$

Hence it is enough to find a function $\psi \in H_0^1(\mathbb{R}\times\omega)$ such that $Q[\psi] < 0$. The main idea is that this can be achieved for a trial function built from $(s,t) \mapsto \mathcal{J}_1(t)$, a generalised eigenfunction of $-\Delta_D^{\mathbb{R}\times\omega}$ corresponding to E_1. More specifically, let $\psi_n(s,t) := \varphi_n(s)\,\mathcal{J}_1(t)$, where

$$\varphi_n(s) := \begin{cases} 1 & \text{if} \quad |s| \leq n\,, \\ 0 & \text{if} \quad |s| \geq 2n\,, \\ \dfrac{2n-|s|}{n} & \text{otherwise}\,. \end{cases} \tag{2.22}$$

Note that $\varphi_n \to 1$ pointwise as $n\to\infty$ and $\|\varphi_n'\|_{L^2(\mathbb{R})} \leq 2/n$. The latter makes the "longitudinal energy" disappear as $n\to\infty$:

$$\int_{\mathbb{R}\times\omega} |\partial_s\psi_n(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t = \|\varphi_n'\|^2_{L^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0\,.$$

On the other hand, the "transversal energy" is compensated by E_1:

$$\int_{\mathbb{R}\times\omega} |\nabla_t\psi_n(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t - E_1 \int_{\mathbb{R}\times\omega} |\psi_n(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t = 0\,.$$

It remains to notice that

$$\int_{\mathbb{R}\times\omega} V(s,t)\,|\psi_n(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t \xrightarrow[n\to\infty]{} \int_{\mathbb{R}\times\omega} V(s,t)\,|\mathcal{J}_1(t)|^2\,\mathrm{d}s\,\mathrm{d}t < 0$$

by the monotone convergence theorem (the limit can be $-\infty$). □

Of course, the most interesting situation is when V *vanishes at infinity* in the sense that

$$\lim_{s_0\to\infty} \operatorname*{ess\,sup}_{|s|>s_0,\,t\in\omega} |V(s,t)| = 0\,.$$

In this case,

$$\sigma_{\rm ess}(-\Delta_D^{\mathbb{R}\times\omega} + V) = [E_1,\infty)\,,$$

so the lowest point in the spectrum of $-\Delta_D^{\mathbb{R}\times\omega} + V$ corresponds to a discrete eigenvalue.

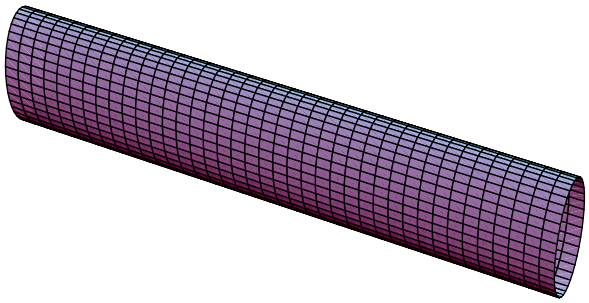


Figure 10: A straight tube.

2.8 Asymptotically straight tubes

The essential spectrum of the Laplacian in a manifold is determined by the behaviour of the metric at infinity (and possibly at the boundary) only. Inspecting the dependence of the coefficients of (2.8) on large "longitudinal distances" s, we see that the metric converges to the Euclidean metric provided that

$$\lim_{|s|\to\infty} \kappa(s) = 0 \qquad \text{and} \qquad \lim_{|s|\to\infty} \theta'(s) = 0\,. \tag{2.23}$$

Therefore the following stability result is not surprising.

Theorem 2.5 (Stability of the essential spectrum)**.** *Under the hypotheses* (2.23)*,*

$$\sigma_{\rm ess}(H) = [E_1,\infty)\,.$$

Proof. We divide the proof into two steps.

• $\boxed{\sigma_{\rm ess}(H) \subset [E_1,\infty)}$ (Neumann bracketing)

Given any arbitrary positive number s_0, we divide $\mathbb{R}\times\omega$ into an interior and an exterior part by considering:

$$I_{\rm int} := (-s_0, s_0) \qquad \text{and} \qquad I_{\rm ext} := (-\infty,-s_0)\cup(s_0,\infty)\,.$$

We impose the Neumann boundary condition on the interface $\Sigma_\pm := \{\pm s_0\}\times\omega$. On the level of forms, it leads to considering the quadratic form which acts as h in $\mathbb{R}\times\omega$ but satisfies no continuity (in the Sobolev setting) on $\Sigma_\pm$. More specifically, let us consider the quadratic form $h^N_{\rm int}$ in $\mathcal{H}_{\rm int} := L^2(I_{\rm int}\times\omega, f(s,t)\,{\rm d}s\,{\rm d}t)$ defined by

$$h^N_{\rm int}[\psi] := (\partial_i\psi, G^{ij}\partial_j\psi)_{\mathcal{H}_{\rm int}}\,,$$
$$\operatorname{dom} h^N_{\rm int} := \{\psi\restriction(I_{\rm int}\times\omega) :\ \psi\in\operatorname{dom} h\}\,.$$

We denote by $H^N_{\rm int}$ the operator associated with $h^N_{\rm int}$ in $\mathcal{H}_{\rm int}$. Similarly, we introduce a form $h^N_{\rm ext}$ and the associated operator $H^N_{\rm ext}$ in $\mathcal{H}_{\rm ext} := L^2(I_{\rm ext}\times\omega, f(s,t)\,{\rm d}s\,{\rm d}t)$. We set $H^N := H^N_{\rm int}\oplus H^N_{\rm ext}$. Since the form

domain H^N is larger than the form domain of H, while the forms act in the same way, one has $H \geq H^N$. By the minimax principle,

$$\begin{aligned} \inf \sigma_{\rm ess}(H) &\geq \inf \sigma_{\rm ess}(H^N) \\ &= \min\left\{\inf \sigma_{\rm ess}(H^N_{\rm int}), \inf \sigma_{\rm ess}(H^N_{\rm ext})\right\} \\ &= \inf \sigma_{\rm ess}(H^N_{\rm ext}) \\ &= \inf \sigma(H^N_{\rm ext})\,. \end{aligned} \tag{2.24}$$

Here the first equality holds because $H^N_{\rm int}$ is an operator with compact resolvent. To estimate the lowest point in the spectrum of $H^N_{\rm ext}$, we use the crude bound $G^{-1} \geq \mathrm{diag}(0,1,1)$. Then, for every $\psi \in \mathrm{dom}\, h^N_{\rm ext}$,

$$\begin{aligned} h^N_{\rm ext}[\psi] &\geq \|\nabla_t \psi\|^2_{\mathcal{H}_{\rm ext}} \\ &\geq \big(\inf_{I_{\rm ext}\times\omega} f\big)\, \|\nabla_t \psi\|^2_{L^2(I_{\rm ext}\times\omega)} \\ &\geq E_1\big(\inf_{I_{\rm ext}\times\omega} f\big)\, \|\psi\|^2_{L^2(I_{\rm ext}\times\omega)} \\ &\geq E_1\, \frac{\inf\limits_{I_{\rm ext}\times\omega} f}{\sup\limits_{I_{\rm ext}\times\omega} f}\, \|\psi\|^2_{\mathcal{H}_{\rm ext}}\,. \end{aligned}$$

Here the third inequality follows from (2.21) with help of Fubini's theorem. Using the first of the hypotheses (2.23), we see that the spectrum $H^N_{\rm ext}$ is estimated from below by E_1 times a function of s_0 tending to 1 as $s_0 \to \infty$. Since s_0 can be chosen arbitrarily large, it follows from (2.24) that $\inf \sigma_{\rm ess}(H) \geq E_1$.

• $\boxed{\sigma_{\rm ess}(H) \supset [E_1, \infty)}$ (Weyl criterion adapted to quadratic forms)

By the classical Weyl criterion, it suffices to construct, for each $k \in \mathbb{R}$, a sequence

$$\{\psi_n\}_{n=1}^\infty \subset \mathrm{dom}\, H \qquad \text{such that} \qquad \begin{aligned} &\text{(i)} \quad \liminf_{n\to\infty} \|\psi_n\|_{\mathcal{H}} > 0\,, \\ &\text{(ii)} \quad \|[H - (k^2 + E_1)]\psi_n\|_{\mathcal{H}} \xrightarrow[n\to\infty]{} 0\,. \end{aligned}$$

One is tempted to use a tensor product of plane waves "localised at infinity" and the first Dirichlet eigenfunction $\mathcal{J}_1$ in the cross-section. Namely, let us set

$$\psi_n(s,t) := \varphi_n(s)\, e^{iks}\, \mathcal{J}_1(t)\,,$$

where $\varphi_n(s) := n^{-1/2}\, \varphi(n^{-1}s - n)$ with $\varphi \in C^\infty_0(\mathbb{R})$ satisfying $\|\varphi\|_{L^2(\mathbb{R})} = 1$. Note that

$$\begin{aligned} &\|\varphi_n\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})} = 1\,, \\ &\|\varphi_n'\|_{L^2(\mathbb{R})} = n^{-1}\, \|\varphi'\|_{L^2(\mathbb{R})} \to 0\,, \qquad \text{and} \qquad \inf \mathrm{supp}\, \varphi_n \to \infty \\ &\|\varphi_n''\|_{L^2(\mathbb{R})} = n^{-2}\, \|\varphi''\|_{L^2(\mathbb{R})} \to 0\,, \end{aligned} \tag{2.25}$$

as $n \to \infty$. Then (i) follows at once by using (2.9) and the fact that $\|\psi_n\|_{\mathcal{H}_0} = 1$, where $\mathcal{H}_0 := L^2(\mathbb{R}\times\omega)$. Moreover, under additional assumptions about the decay of κ and θ' at infinity (involving derivatives), it is indeed possible to show that ψ_n is the desired sequence.

To avoid the additional assumptions, we use the Weyl criterion adapted to quadratic forms [37, App.] requiring to construct a sequence

$$\{\psi_n\}_{n=1}^\infty \subset \mathrm{dom}\, h \qquad \text{such that} \qquad \begin{aligned} &\text{(i)} \quad \liminf_{n\to\infty} \|\psi_n\|_{\mathcal{H}} > 0\,, \\ &\text{(ii')} \quad \|[H - (k^2 + E_1)]\psi_n\|_{\mathcal{H}_{-1}} \xrightarrow[n\to\infty]{} 0\,. \end{aligned}$$

The advantage is that the sequence is required to belong to the form domain instead of the operator domain. What is more, the weaker convergence in the dual space $\mathcal{H}_{-1} := \mathcal{H}_1^*$ is required, where $\mathcal{H}_1 := \mathrm{dom}\, h$ is equipped with the norm $|||\cdot|||$. Note that $\mathcal{H}_1 \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{H}_{-1}$ and

$$\|[H - (k^2 + E_1)]\psi_n\|_{\mathcal{H}_{-1}} = \sup_{\substack{\phi\in\mathcal{H}_1 \\ \phi\neq 0}} \frac{|h(\phi,\psi_n) - (k^2 + E_1)(\phi,\psi_n)_{\mathcal{H}}|}{\|\phi\|_{\mathcal{H}_1}}\,.$$

Therefore checking (ii') reduces to an analysis on the level of the form h.

Let $\phi \in C_0^1(\mathbb{R}\times\omega)$, a core of $\mathfrak{H}_1$. We write

$$|h(\phi,\psi_n) - (k^2+E_1)(\phi,\psi_n)_{\mathfrak{H}}| \le |h_1(\phi,\psi_n) - k^2\,(\phi,\psi_n)_{\mathfrak{H}}| + |h_2(\phi,\psi_n) - E_1\,(\phi,\psi_n)_{\mathfrak{H}}|\,,$$

where (with any $\psi\in\mathfrak{H}_1$)

$$h_1[\psi] := \|f^{-1}(\partial_s\psi - \theta'\partial_\tau\psi)\|^2_{\mathfrak{H}} \qquad \text{and} \qquad h_2[\psi] := \|\nabla_t\psi\|^2_{\mathfrak{H}}\,.$$

For the "transverse" form h_2, a repeated integration by parts using that $f(s,t)$ is linear in t yields

$$h_2(\phi,\psi_n) - E_1\,(\phi,\psi_n)_{\mathfrak{H}} = -(\phi,\nabla_t\psi_n\cdot\nabla_t f)_{\mathfrak{H}_0} = (\nabla_t\phi,\psi_n\nabla_t f)_{\mathfrak{H}_0} = \left(\nabla_t\phi,\psi_n\frac{\nabla_t f}{f}\right)_{\mathfrak{H}}.$$

Consequently,

$$\frac{|h_2(\phi,\psi_n) - E_1\,(\phi,\psi_n)_{\mathfrak{H}}|}{\|\phi\|_{\mathfrak{H}_1}} \le \frac{\|\nabla_t\phi\|_{\mathfrak{H}}}{\|\phi\|_{\mathfrak{H}_1}}\,\|\psi_n\|_{\mathfrak{H}}\left\|\frac{\nabla_t f}{f}\right\|_{\infty,n} \le \|f\|^{1/2}_{\infty,n}\left\|\frac{\nabla_t f}{f}\right\|_{\infty,n} \xrightarrow[n\to\infty]{} 0\,,$$

where $\|\cdot\|_{\infty,n} := \|\cdot\|_{L^\infty(\operatorname{supp}\varphi_n\times\omega)}$. The convergence holds because $\|\nabla_t f\|_{\infty,n}\to 0$ as $n\to\infty$ due to the first assumption of (2.23).

For the "longitudinal" form h_1, we further decompose

$$h_1(\phi,\psi_n) - k^2\,(\phi,\psi_n)_{\mathfrak{H}} = I_1 + I_2 + I_3 + I_4\,,$$

where

$$\begin{aligned} I_1 &:= (f^{-1}\partial_s\phi, f^{-1}\partial_s\psi_n)_{\mathfrak{H}} - k^2\,(\phi,\psi_n)_{\mathfrak{H}}\,,\\ I_2 &:= (f^{-1}\partial_s\phi, f^{-1}\theta'\partial_\tau\psi_n)_{\mathfrak{H}}\,,\\ I_3 &:= (f^{-1}\theta'\partial_\tau\phi, f^{-1}\partial_s\psi_n)_{\mathfrak{H}}\,,\\ I_4 &:= (f^{-1}\theta'\partial_\tau\phi, f^{-1}\theta'\partial_\tau\psi_n)_{\mathfrak{H}}\,. \end{aligned}$$

We start to estimate the last integral as follows:

$$\frac{|I_4|}{\|\phi\|_{\mathfrak{H}_1}} \le a^2\left\|\frac{\theta'}{f}\right\|^2_{\infty,n}\frac{\|\nabla_t\phi\|_{\mathfrak{H}}}{\|\phi\|_{\mathfrak{H}_1}}\,\|\nabla_t\psi_n\|_{\mathfrak{H}} \le a^2\left\|\frac{\theta'}{f}\right\|^2_{\infty,n}\|f\|^{1/2}_{\infty,n}\sqrt{E_1} \xrightarrow[n\to\infty]{} 0\,,$$

where the convergence holds due to the second assumption of (2.23). Similarly,

$$\frac{|I_3|}{\|\phi\|_{\mathfrak{H}_1}} \le a\left\|\frac{\theta'}{f^2}\right\|_{\infty,n}\frac{\|\nabla_t\phi\|_{\mathfrak{H}}}{\|\phi\|_{\mathfrak{H}_1}}\,\|\partial_s\psi_n\|_{\mathfrak{H}} \le a\left\|\frac{\theta'}{f}\right\|^2_{\infty,n}\|f\|^{1/2}_{\infty,n}\left(\|\varphi_n'\|_{L^2(\mathbb{R})} + |k|\,\|\varphi_n\|_{L^2(\mathbb{R})}\right) \xrightarrow[n\to\infty]{} 0$$

and

$$\frac{|I_2|}{\|\phi\|_{\mathfrak{H}_1}} \le a\left\|\frac{\theta'}{f^2}\right\|_{\infty,n}\frac{\|\partial_s\phi\|_{\mathfrak{H}}}{\|\phi\|_{\mathfrak{H}_1}}\,\|\nabla_t\psi_n\|_{\mathfrak{H}} \le a\left\|\frac{\theta'}{f}\right\|^2_{\infty,n}\|f\|_{\infty,n}\,C_-^{-1}\sqrt{E_1} \xrightarrow[n\to\infty]{} 0\,,$$

where C_- is the constant from Proposition 2.3. Finally, we write $I_1 = I_1^{(a)} + I_1^{(b)}$, where

$$\begin{aligned} I_1^{(a)} &:= (\partial_s\phi,\partial_s\psi_n)_{\mathfrak{H}_0} - k^2\,(\phi,\psi_n)_{\mathfrak{H}_0}\\ &= (\phi,-\partial_s^2\psi_n)_{\mathfrak{H}_0} - k^2\,(\phi,\psi_n)_{\mathfrak{H}_0}\\ &= -\left(\phi,(\varphi_n'' + 2ik\varphi_n')\,e^{iks}\,\mathfrak{J}_1\right)_{\mathfrak{H}_0} \end{aligned}$$

and

$$I_1^{(b)} := \left(\partial_s\phi,(f^{-1}-1)\partial_s\psi_n\right)_{\mathfrak{H}_0} - k^2\left(\phi,(f-1)\psi_n\right)_{\mathfrak{H}_0}.$$

Consequently,

$$\frac{|I_1^{(a)}|}{\|\phi\|_{\mathfrak{H}_1}} \le \frac{\|\phi\|_{\mathfrak{H}_0}}{\|\phi\|_{\mathfrak{H}_1}}\left(\|\varphi_n''\|_{L^2(\mathbb{R})} + 2\,|k|\,\|\varphi_n'\|_{L^2(\mathbb{R})}\right) \le \left\|\frac{1}{f}\right\|^{1/2}_{\infty,n}\left(\|\varphi_n''\|_{L^2(\mathbb{R})} + 2\,|k|\,\|\varphi_n'\|_{L^2(\mathbb{R})}\right) \xrightarrow[n\to\infty]{} 0\,,$$

where the convergence holds due to (2.25), and

$$\begin{aligned} \frac{|I_1^{(b)}|}{\|\phi\|_{\mathfrak{H}_1}} &\le \frac{\|\partial_s\phi\|_{\mathfrak{H}_0}}{\|\phi\|_{\mathfrak{H}_1}}\left\|\frac{1}{f}-1\right\|_{\infty,n}\|\partial_s\psi_n\|_{\mathfrak{H}_0} + k^2\,\frac{\|\phi\|_{\mathfrak{H}_0}}{\|\phi\|_{\mathfrak{H}_1}}\,\|\psi_n\|_{\mathfrak{H}_0}\\ &\le C_-^{-1}\left\|\frac{1}{f}-1\right\|_{\infty,n}\left(\|\varphi_n'\|_{L^2(\mathbb{R})} + |k|\,\|\varphi_n\|_{L^2(\mathbb{R})}\right) + k^2\,C_-^{-1} \xrightarrow[n\to\infty]{} 0\,, \end{aligned}$$

where the convergence holds due to the first assumption of (2.23). □

Remark 2.6. Note that the second of the hypotheses (2.23) was not used in the first part of the proof. Therefore $\kappa(s) \to 0$ as $|s| \to \infty$ is sufficient to ensure that $\inf \sigma_{\rm ess}(H) \geq E_1$.

At the same time, only vanishing of $\kappa(s)$ and $\theta'(s)$ as $s \to +\infty$ was sufficient to ensure that $\sigma_{\rm ess}(H) \subset [E_1, \infty)$ in the second part of the proof (the limit $s \to -\infty$ was not used).

Tubes without bending for which the twisting does not necessarily vanish at infinity were considered in [18], [6] and [4].

2.9 Thin tubes

As a foretaste what to expect when the tube is bent or twisted, let us look at the limit of shrinking cross-sections

$$\omega_\varepsilon := \{\varepsilon t : \ t \in \omega\}\,,$$

where ε is a small positive parameter.

• More specifically, let $H_{\varepsilon\varepsilon}$ be the operator in the Hilbert space

$$\mathcal{H}_{\varepsilon\varepsilon} := L^2\big(\mathbb{R} \times \omega_\varepsilon, f(s,t)\,\mathrm{d}s\,\mathrm{d}t\big)$$

associated with the quadratic form

$$h_{\varepsilon\varepsilon}[\psi] := \|f^{-1}(\partial_s\psi - \theta'\partial_\tau\psi)\|^2_{\mathcal{H}_{\varepsilon\varepsilon}} + \|\nabla_t\psi\|^2_{\mathcal{H}_{\varepsilon\varepsilon}}\,, \qquad \psi \in \mathrm{dom}\, h_{\varepsilon\varepsilon} = \overline{C_0^1(\mathbb{R}\times\omega_\varepsilon)}^{|||\cdot|||_{\varepsilon\varepsilon}}\,,$$

where $|||\psi|||_{\varepsilon\varepsilon} := \sqrt{h_{\varepsilon\varepsilon}[\psi] + \|\psi\|^2_{\mathcal{H}_{\varepsilon\varepsilon}}}$.

• As a first step, we introduce the unitary transform

$$U_{\varepsilon\varepsilon} : \mathcal{H}_{\varepsilon\varepsilon} \to \mathcal{H}_\varepsilon := L^2\big(\mathbb{R}\times\omega, f_\varepsilon(s,t)\,\mathrm{d}s\,\mathrm{d}t\big)$$

with $f_\varepsilon(s,t) := f(s,\varepsilon t)$ by rescaling the transverse variables:

$$(U_{\varepsilon\varepsilon}\psi)(s,t) := \varepsilon\,\psi(s,\varepsilon t)\,.$$

The unitarily equivalent operator $H_\varepsilon := U_{\varepsilon\varepsilon}H_{\varepsilon\varepsilon}{U_{\varepsilon\varepsilon}}^{-1}$ in $\mathcal{H}_\varepsilon$ is associated with the quadratic form

$$h_\varepsilon[\psi] := \|f_\varepsilon^{-1}(\partial_s\psi - \theta'\partial_\tau\psi)\|^2_{\mathcal{H}_\varepsilon} + \varepsilon^{-2}\,\|\nabla_t\psi\|^2_{\mathcal{H}_\varepsilon}\,, \qquad \psi \in \mathrm{dom}\, h_\varepsilon = \overline{C_0^1(\mathbb{R}\times\omega)}^{|||\cdot|||_\varepsilon}\,,$$

where $|||\psi|||_\varepsilon := \sqrt{h_\varepsilon[\psi] + \|\psi\|^2_{\mathcal{H}_\varepsilon}}$.

• In the second step, we introduce the unitary transform

$$U_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_0 := L^2\big(\mathbb{R}\times\omega\big) : \Big\{\psi \mapsto \sqrt{f_\varepsilon}\,\psi\Big\}\,.$$

The unitarily equivalent operator $\hat H_\varepsilon := U_\varepsilon H_\varepsilon U_\varepsilon^{-1}$ in $\mathcal{H}_0$ is associated with the quadratic form

$$\hat h_\varepsilon[\phi] := h_\varepsilon[U_\varepsilon^{-1}\phi]\,, \qquad \phi \in \mathrm{dom}\,\hat h_\varepsilon := U\,\mathrm{dom}\, h_\varepsilon\,.$$

To consider $\hat H_\varepsilon$, it is necessary to strengthen the regularity hypothesis. If one stays on the level of quadratic forms, it is enough to assume

$$k_1, k_2 \in C^1(\mathbb{R})\,. \tag{2.26}$$

• Given $\psi \in C_0^1(\mathbb{R}\times\omega)$, set $\phi := U_\varepsilon\psi$. Then

$$\|\nabla_t\psi\|^2_{\mathcal{H}_\varepsilon} = \|\nabla_t\phi\|^2_{\mathcal{H}_0} + \frac14\,\left\|\frac{|\nabla_t f_\varepsilon|}{f_\varepsilon}\,\phi\right\|^2_{\mathcal{H}_0} + \Re\left(\phi, \frac{\nabla_t f_\varepsilon}{f_\varepsilon}\cdot\nabla_t\phi\right)_{\mathcal{H}_0} = \|\nabla_t\phi\|^2_{\mathcal{H}_0} - \frac14\,\left\|\frac{|\nabla_t f_\varepsilon|}{f_\varepsilon}\,\phi\right\|^2_{\mathcal{H}_0}\,,$$

where the second equality follows by an integration by parts using that $f_\varepsilon(s,t)$ is linear in t. Consequently,

$$\varepsilon^{-2}\,\|\nabla_t\psi\|^2_{\mathcal{H}_\varepsilon} = \varepsilon^{-2}\,\|\nabla_t\phi\|^2_{\mathcal{H}_0} + (\phi, V_\varepsilon\phi)_{\mathcal{H}_0}$$

with

$$V_\varepsilon(s,t) := -\frac{|\nabla_t f_\varepsilon(s,t)|^2}{4\,\varepsilon^2 f_\varepsilon(s,t)^2} = -\frac{\kappa(s)^2}{4\,f_\varepsilon(s,t)^2} \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad -\frac{\kappa(s)^2}{4} =: V_0(s,t)\,.$$

Similarly (but without any integration by parts),

$$\|f_\varepsilon^{-1}(\partial_s\psi - \theta'\partial_\tau\psi)\|^2_{\mathcal{H}_\varepsilon} = \big\|f_\varepsilon^{-1}(\partial_s\phi - \theta'\partial_\tau\phi) - \tfrac{1}{2}f_\varepsilon^{-1}(\partial_s f_\varepsilon - \theta'\partial_\tau f_\varepsilon)\big\|^2_{\mathcal{H}_0} \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad \|\partial_s\phi - \theta'\partial_\tau\phi\|^2_{\mathcal{H}_0}\,.$$

This suggests that

$$\hat{h}_\varepsilon[\phi] \underset{\varepsilon\to 0}{\sim} \|\partial_s\phi - \theta'\partial_\tau\phi\|^2_{\mathcal{H}_0} + \varepsilon^{-2}\,\|\nabla_t\phi\|^2_{\mathcal{H}_0} + (\phi, V_0\phi)_{\mathcal{H}_0} =: \tilde{h}_\varepsilon[\phi]\,. \tag{2.27}$$

• One can do better. Since we consider Dirichlet boundary conditions, the spectrum of $\hat{H}_\varepsilon$ necessarily tends to $+\infty$ as $\varepsilon\to 0$. This is evident from the second term on the right-hand side of (2.27). Indeed, $\varepsilon^{-2}\,\|\nabla_t\phi\|^2_{\mathcal{H}_0} \geq \varepsilon^{-2}\,E_1\,\|\phi\|^2_{\mathcal{H}_0}$ due to the Poincaré inequality (2.21). What is more, if $\big(\mathcal{J}_1,\phi_\perp(s,\cdot)\big)_{L^2(\omega)} = 0$ for almost every $s\in\mathbb{R}$, then

$$\varepsilon^{-2}\left(\|\nabla_t\phi_\perp\|^2_{\mathcal{H}_0} - E_1\,\|\phi_\perp\|^2_{\mathcal{H}_0}\right) \geq \varepsilon^{-2}\,(E_2-E_1)\,\|\phi_\perp\|^2_{\mathcal{H}_0}\,, \tag{2.28}$$

where E_2 is the second eigenvalue of $-\Delta_D^\omega$. Since E_1 is simple, the fundamental gap E_2-E_1 is positive. Consequently, the right-hand side of (2.28) tends to $+\infty$ as $\varepsilon\to 0$ unless $\phi_\perp = 0$. Therefore, writing

$$\phi(s,t) = \underbrace{\varphi(s)\,\mathcal{J}_1(t)}_{\phi_1(s,t)} + \phi_\perp(s,t)\,,$$

it is ϕ_1 which should determine the leading order of $\tilde{h}_\varepsilon[\phi]$ as $\varepsilon\to 0$. That is

$$\begin{aligned}\hat{h}_\varepsilon[\phi] \underset{\varepsilon\to 0}{\sim} \tilde{h}_\varepsilon[\phi_1] &= \|\varphi'\mathcal{J}_1 - \theta'\varphi\,\partial_\tau\mathcal{J}_1\|^2_{\mathcal{H}_0} + \varepsilon^{-2}\,E_1\,\|\phi_1\|^2_{\mathcal{H}_0} + (\phi_1, V_0\phi_1)_{\mathcal{H}_0}\\ &= \|\varphi'\|^2_{L^2(\mathbb{R})} + \|\theta'\varphi\|^2_{L^2(\mathbb{R})}\,\|\partial_\tau\mathcal{J}_1\|^2_{L^2(\omega)} + \varepsilon^{-2}\,E_1\,\|\varphi\|^2_{L^2(\mathbb{R})} + (\varphi, V_0\varphi)_{L^2(\mathbb{R})}\,,\end{aligned}$$

where the second equality employs the normalisation of $\mathcal{J}_1$ and

$$(\varphi'\mathcal{J}_1, \theta'\varphi\,\partial_\tau\mathcal{J}_1)_{\mathcal{H}_0} = 0$$

due to an integration by parts in τ. In summary, it is expected that

$$-\Delta_D^{\Omega_\varepsilon} - \varepsilon^{-2}\,E_1 \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad -\frac{\mathrm{d}^2}{\mathrm{d}s^2} + C_\omega\,\theta'(s)^2 - \frac{\kappa(s)^2}{4}, \tag{2.29}$$

where $C_\omega := \|\partial_\tau\mathcal{J}_1\|^2_{L^2(\omega)}$ is a non-negative constant (it is zero if, and only if, ω is rotationally invariant with respect to the origin). The convergence (2.29) can be justified in a norm-resolvent sense by suitably identifying the different Hilbert spaces on which the operators act [40].

• The "effective Hamiltonian" on the right-hand side of (2.29) is a one-dimensional Schrödinger operator in $L^2(\mathbb{R})$. Its potential clearly encodes information on geometric properties of Ω. We can deduce from it several spectral consequences to be expected:

(i) bending acts as an attractive interaction (*i.e.* the curvature term in the potential is non-positive); the spectrum will be "pushed down" in bent tubes;

(ii) twisting acts as a repulsive interaction (*i.e.* the twisting term in the potential is non-negative); the spectrum will be "pushed up" in twisted tubes;

(iii) the spectrum will be purely discrete if $\lim_{|s|\to\infty}|\theta'(s)| = \infty$.

It will turn out that the expectations (i)–(ii) are correct, provided that the subtleties of asymptotically straight tubes are taken into account. The expectation (iii) is generally false.

Thin tubes has been extensively studied in the literature. For the Euclidean strips and tubes without twisting (*i.e.* $C_\omega = 0$ or $\theta' = 0$), the first result of the type (2.29) in a norm-resolvent sense goes back to [14]. Bounded twisted tubes were covered for the first time in [3] via Γ-convergence. An extension to unbounded tubes is due to [11] (see also [12]) and [40] by different approaches. Thin strips on ruled surfaces are considered in [57] and [41]. An analogous norm-resolvent convergence for thin Euclidean tubes about hypersurfaces is due to [39]. We refer to [38] for a unifying approach to this type of problems. Without the norm-resolvent convergence, thin tubes in general settings were considered in [55], [52], [21] and [58, 44, 43, 27].

2.10 The effect of bending

Now we restrict ourselves to tubes which are bent ($\kappa \neq 0$) but untwisted ($\theta' = 0$), see Figure 11.

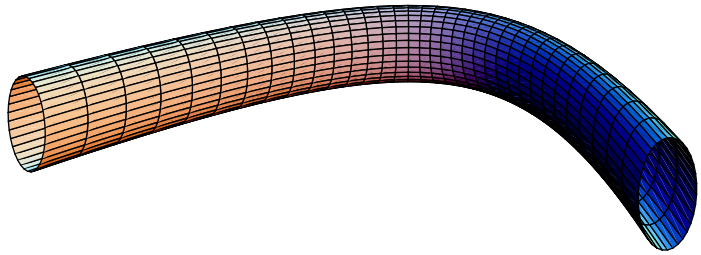


Figure 11: A purely bent tube.

It turns out that bending is supercritical in the sense that it gives rise to a spectrum below the energy E_1.

Theorem 2.7 (Supercriticality)**.** *Let $\kappa \neq 0$ and $\theta' = 0$. Then*

$$\inf \sigma(H) < E_1 \,.$$

Proof. As in the proof of Proposition 2.4, the proof is based on the variational characterisation

$$\inf \sigma(H - E_1) = \inf_{\substack{\psi \in \operatorname{dom} h \\ \psi \neq 0}} \frac{h[\psi] - E_1 \, \|\psi\|^2}{\|\psi\|^2} \,.$$

The main idea is that the Rayleigh quotient can be made negative for a trial function built from $(s,t) \mapsto \mathcal{J}_1(t)$, a generalised eigenfunction of $-\Delta_D^{\mathbb{R}\times\omega}$ corresponding to E_1.

• First, one verifies that

$$Q[\psi_n] := h[\psi_n] - E_1 \, \|\psi_n\|^2 \xrightarrow[n\to\infty]{} 0 \tag{2.30}$$

for the choice $\psi_n(s,t) := \varphi_n(s)\mathcal{J}_1(t)$ as in the proof of Proposition 2.4. The "longitudinal energy" of (2.17) disappear as $n \to \infty$:

$$\|\partial_s \psi_n\|^2 = \int_{\mathbb{R}\times\omega} \frac{|\varphi_n'(s)|^2 \, |\mathcal{J}_1(t)|^2}{f(s,t)} \, \mathrm{d}s \, \mathrm{d}t \leq \frac{\|\varphi_n'\|^2_{L^2(\mathbb{R})} \, \|\mathcal{J}_1\|^2_{L^2(\omega)}}{1 - a \, \|\kappa\|_\infty} \xrightarrow[n\to\infty]{} 0 \,.$$

The striking fact behind (2.30) is that the "transverse energy" is compensated by E_1, despite the presence of the Jacobian f in the Hilbert space (2.13). This follows by an explicit integration by parts and the observation that $f(s,t)$ is *linear* in t:

$$\begin{aligned}
\|\nabla_t \psi_n\|^2 - E_1 \, \|\psi_n\|^2 &= \int_{\mathbb{R}\times\omega} |\varphi_n(s)|^2 \, |\nabla_t \mathcal{J}_1(t)|^2 \, f(s,t) \, \mathrm{d}s \, \mathrm{d}t - E_1 \int_{\mathbb{R}\times\omega} |\varphi_n(s)|^2 \, |\mathcal{J}_1(t)|^2 \, f(s,t) \, \mathrm{d}s \, \mathrm{d}t \\
&= - \int_{\mathbb{R}\times\omega} |\varphi_n(s)|^2 \, \mathcal{J}_1(t) \, \nabla_t \mathcal{J}_1(t) \cdot \nabla_t f(s,t) \, \mathrm{d}s \, \mathrm{d}t \\
&= -\frac{1}{2} \int_{\mathbb{R}\times\omega} |\varphi_n(s)|^2 \, \nabla_t |\mathcal{J}_1(t)|^2 \cdot \nabla_t f(s,t) \, \mathrm{d}s \, \mathrm{d}t \\
&= \frac{1}{2} \int_{\mathbb{R}\times\omega} |\varphi_n(s)|^2 \, |\mathcal{J}_1(t)|^2 \cdot \Delta_t f(s,t) \, \mathrm{d}s \, \mathrm{d}t = 0 \,.
\end{aligned}$$

• In the second step, one considers a small perturbation

$$\psi_{n,\varepsilon}(s,t) := \psi_n(s,t) + \varepsilon \, \phi(s,t) \,,$$

where $\varepsilon \in \mathbb{R}$ and $\phi \in \operatorname{dom} h$ is real-valued. Then

$$Q[\psi_{n,\varepsilon}] = Q[\psi_n] + 2 \, \varepsilon \, Q(\psi_n, \phi) + \varepsilon^2 \, Q[\phi] \,. \tag{2.31}$$

Here the first term on the right-hand side is independent of ε and vanishes as $n \to \infty$. The last term is independent of n and can be made negligible with respect to the middle term by taking ε small. It remains to show that $Q(\psi_n, \phi)$ does not vanish as $n \to \infty$. In fact, by taking

$$\phi(s,t) := j(s)\,\xi(t)\,\mathcal{J}_1(t)\,,$$

where $j \in C_0^\infty(\mathbb{R})$ and ξ will be determined in a moment, $Q(\psi_n, \phi)$ is independent of n for all sufficiently large n. This follows from the fact $\varphi_n = 1$ on the support of j for all sufficiently large n. More specifically, we write $Q = Q_1 + Q_2$, where

$$Q_1(\psi_n, \phi) := \left(f^{-1}\partial_s\psi_n, f^{-1}\partial_s\phi\right) = \int_{\mathbb{R}\times\omega} \frac{\varphi_n'(s)\,\mathcal{J}_1(t)\,\partial_s\phi(s,t)}{f(s,t)}\,\mathrm{d}s\,\mathrm{d}t = 0$$

for all sufficiently large n (because $\varphi_n' = 0$ on the support of j) and

$$\begin{aligned}
Q_2(\psi_n, \phi) &:= (\nabla_t\psi_n, \nabla_t\phi) - E_1\,(\psi_n, \phi) \\
&= \int_{\mathbb{R}\times\omega} \varphi_n(s)\,\nabla_t\mathcal{J}_1(t)\cdot\nabla_t\phi(s,t)\,f(s,t)\,\mathrm{d}s\,\mathrm{d}t - E_1\int_{\mathbb{R}\times\omega} \varphi_n(s)\,\mathcal{J}_1(t)\,\phi(s,t)\,f(s,t)\,\mathrm{d}s\,\mathrm{d}t \\
&= -\int_{\mathbb{R}\times\omega} \varphi_n(s)\,\nabla_t\mathcal{J}_1(t)\,\phi(s,t)\cdot\nabla_t f(s,t)\,\mathrm{d}s\,\mathrm{d}t \\
&= -\frac{1}{2}\int_{\mathbb{R}\times\omega} \nabla_t|\mathcal{J}_1(t)|^2\,j(s)\,\xi(t)\cdot\nabla_t f(s,t)\,\mathrm{d}s\,\mathrm{d}t \\
&= -\frac{1}{2}\int_{\mathbb{R}\times\omega} |\mathcal{J}_1(t)|^2\,j(s)\,\nabla_t\xi(t)\cdot\nabla_t f(s,t)\,\mathrm{d}s\,\mathrm{d}t\,,
\end{aligned}$$

where the last but one equality holds for all sufficiently large n (because $\varphi_n = 1$ on the support of j). Choosing (note that θ is necessarily constant by the hypothesis $\theta' = 0$)

$$\xi(t) := \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}t_1 \\ t_2\end{pmatrix}$$

with any $\alpha_1, \alpha_2 \in \mathbb{R}$ and recalling that $\mathcal{J}_1$ is normalised to 1 in $L^2(\omega)$, we arrive at

$$Q_2(\psi_n, \phi) = \frac{1}{2}\int_{\mathbb{R}} j(s)\,[\alpha_1 k_1(s) + \alpha_2 k_2(s)]\,\mathrm{d}s$$

for all sufficiently large n. We claim that there exists a bounded interval $I \subset \mathbb{R}$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 k_1 + \alpha_2 k_2 < 0 \qquad \text{on} \qquad I\,.$$

Indeed, this follows by the hypothesis $\kappa \neq 0$, the relationship (2.2) and the continuity of κ. Choosing j non-negative and $\operatorname{supp} j := I$, we eventually get the desired result that

$$Q_2(\psi_n, \phi) \text{ is negative and independent of } n \text{ for all sufficiently large } n.$$

• In summary, the sum of the last two terms on the right-hand side of (2.31) are independent of n for all sufficiently large n and can be made negative by choosing ε positive and sufficiently small. Then we choose n so large that the sum with the first term on the right-hand side of (2.31) remains negative. □

For asymptotically straight tubes, Theorem 2.7 together with the stability of the essential spectrum (Theorem 2.5) implies the existence of discrete eigenvalues.

Corollary 2.8. *Let $\kappa \neq 0$ and $\theta' = 0$. Assume in addition $\lim_{|s|\to\infty} \kappa(s) = 0$. Then*

$$\sigma_{\mathrm{disc}}(H) \cap (0, E_1) \neq \varnothing\,.$$

The moral of this section is that

bending acts as an attractive interaction

in the sense that it diminishes the spectrum with respect to the spectrum of straight tubes (*cf* Theorem 2.7). If the bent tube is asymptotically straight, then the spectrum below E_1 is discrete (*cf* Corollary 2.8). For tubes which are not asymptotically straight, however, the effect of bending can be so strong that it pushes down the essential spectrum too. For instance, this always happens for periodically bent tubes (*i.e.* $\theta'=0$ and $\kappa\neq 0$ is periodic). This is clear from Theorem 2.7 and the fact that periodic systems admit no discrete eigenvalues.

2.11 The effect of twisting

Now we restrict ourselves to tubes which are twisted ($\theta'\neq 0$) but unbent ($\kappa=0$), see Figure 12.

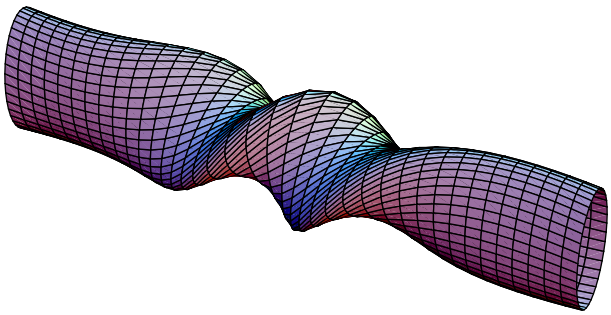


Figure 12: A purely twisted tube.

The effect of twisting is more subtle because of the following result.

Proposition 2.9. *Let $\kappa=0$. Assume in addition $\lim_{|s|\to\infty}\theta'(s)=0$. Then*

$$\sigma(H)=\sigma_{\rm ess}(H)=[E_1,\infty)\,.$$

Proof. By Theorem 2.5, $[E_1,\infty)\subset\sigma(H)$. The opposite inclusion follows from the fact that $f=1$ if $\kappa=0$:

$$h[\psi]\geq\|\nabla_t\psi\|^2=\int_{\mathbb{R}\times\omega}|\nabla_t\psi(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t\geq E_1\int_{\mathbb{R}\times\omega}|\psi(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t=E_1\,\|\psi\|^2 \tag{2.32}$$

for every $\psi\in\mathrm{dom}\,h$. Here the second inequality follows from the Poincaré inequality (2.21) with help of Fubini's theorem. □

That is, for asymptotically straight unbent tubes, the spectrum is purely essential and coincides with that of straight tubes. However, contrary to the case of straight tubes which are *critical* (*cf* Proposition 2.4), twisted tubes are *subcritical*. Indeed, there is always a Hardy-type inequality whenever the tube is non-trivially twisted.

Theorem 2.10 (Local Hardy inequality)**.** *Let $\kappa=0$. Assume that $\theta'\neq 0$ and ω is not rotationally invariant with respect to the origin. Let $I\subset\mathbb{R}$ be any interval on which $\theta'\neq 0$. Then there is a positive constant λ_1^I (depending on $\theta'\restriction I$ and ω) such that*

$$H-E_1\geq\lambda_1^I\;\chi_{I\times\omega}\,. \tag{2.33}$$

Proof. Given a bounded interval $I\subset\mathbb{R}$, let us consider the quadratic form in $L^2(I\times\omega)$ defined by

$$h^I[\psi]:=\|\partial_s\psi-\theta'\partial_\tau\psi\|^2_{L^2(I\times\omega)}+\|\nabla_t\psi\|^2_{L^2(I\times\omega)}\,,$$
$$\mathrm{dom}\,h^I:=\left\{\psi\restriction(I\times\omega):\ \psi\in H^1_0(\mathbb{R}\times\omega)\right\}.$$

The corresponding operator H^I acts as (2.18) and satisfies Neumann boundary conditions on $(\partial I)\times\omega$. Since I is bounded, H^I is an operator with compact resolvent. Consequently, the infimum

$$\lambda_1^I := \inf_{\substack{\psi\in\operatorname{dom} h^I\\ \psi\neq 0}} \frac{h^I[\psi]-E_1\,\|\psi\|^2_{L^2(I\times\omega)}}{\|\psi\|^2_{L^2(I\times\omega)}}$$

is achieved by a positive function $\psi_1\in\operatorname{dom} h^I$. One immediately has

$$h[\psi]-E_1\,\|\psi\|^2\geq h^I[\psi]-E_1\,\|\psi\|^2_{L^2(I\times\omega)}\geq\lambda_1^I\,\|\psi\|^2_{L^2(I\times\omega)}=\lambda_1^I\,\|\chi_I\,\psi\|^2$$

for every $\psi\in\operatorname{dom} h$, similarly as in (2.32) crucially employing that $f=1$ if the tube is unbent. It remains to ensure that λ_1^I is positive. To this purpose, we choose the interval I in such a way that $\theta'\neq 0$ on I (*i.e.* θ' is not identically equal to zero on I). By contradiction, assume that $\lambda_1^I=0$. Then

$$\|\partial_s\psi_1-\theta'\partial_\tau\psi_1\|^2_{L^2(I\times\omega)}=0\qquad\text{and}\qquad\|\nabla_t\psi_1\|^2_{L^2(I\times\omega)}-E_1\,\|\psi_1\|^2_{L^2(I\times\omega)}=0\,. \tag{2.34}$$

Writing $\psi_1(s,t)=\varphi(s)\,\mathcal{J}_1(t)+\phi(s,t)$, where $(\mathcal{J}_1,\phi(s,\cdot))_{L^2(\omega)}=0$ for almost every $s\in I$, we deduce from the second equality in (2.34) that $\phi=0$. The first identity in (2.34) is then equivalent to

$$\|\varphi'\|^2_{L^2(I)}\,\|\mathcal{J}_1\|^2_{L^2(\omega)}+\|\theta'\varphi\|^2_{L^2(I)}\,\|\partial_\tau\mathcal{J}_1\|^2_{L^2(\omega)}-2\,(\mathcal{J}_1,\partial_\tau\mathcal{J}_1)_{L^2(\omega)}\,(\varphi',\theta'\varphi)_{L^2(I)}=0\,.$$

Since $(\mathcal{J}_1,\partial_\tau\mathcal{J}_1)_{L^2(\omega)}=0$ by an integration by parts, it follows that φ must be constant and that

$$\|\theta'\|_{L^2(I)}=0\qquad\text{or}\qquad\|\partial_\tau\mathcal{J}_1\|_{L^2(\omega)}=0\,.$$

However, this is impossible under the stated assumptions because $\|\theta'\|_{L^2(I)}$ vanishes if, and only if, $\theta'=0$ almost everywhere in I, and $\partial_\tau\mathcal{J}_1=0$ identically in ω if, and only if, ω is rotationally invariant with respect to the origin. □

The inequality of Theorem 2.10 is particularly interesting in the setting of Proposition 2.9. Indeed, if the twist vanishes at infinity, the spectrum starts with E_1, so there can be no positive *constant* c such that $H-E_1\geq c$ (Poincaré inequality) holds. However, there can be a non-trivial non-negative *function* (Hardy weight) $\rho:\mathbb{R}\times\omega\to\mathbb{R}$ such that $H-E_1\geq\rho$ (Hardy inequality) holds. We call the Hardy inequality of Theorem 2.10 *local* because the Hardy weight is compactly supported there. However, there is always a way how to deduce a *global* Hardy inequality (*i.e.* with an everywhere positive Hardy weight) from the local one.

Theorem 2.11 (Global Hardy inequality)**.** *Let $\kappa=0$. If $\theta'\neq 0$ and ω is not rotationally invariant with respect to the origin, there is a positive constant c depending on θ' and ω such that*

$$H-E_1\geq c\,\rho\,, \tag{2.35}$$

where $\rho(s,t):=(1+s^2)^{-1}$.

Proof. Let $\psi\in C_0^1(\mathbb{R}\times\omega)$, a core of $\operatorname{dom} h$.

• By Theorem 2.10,

$$Q[\psi]:=h[\psi]-E_1\,\|\psi\|^2\geq\lambda_1^I\,\|\chi_I\psi\|^2\,, \tag{2.36}$$

where λ_1^I is positive and $I\subset\mathbb{R}$ is any bounded interval on which θ' is not identically equal to zero. Here we abbreviate $\chi_I:=\chi_{I\times\omega}$. Let us also recall the orthogonality relation

$$\Re\,(\psi,\partial_\tau\psi)_{L^2(\omega)}=0\,, \tag{2.37}$$

which was essentially used in the proof of Theorem 2.10 and follows by an integration by parts.

• Let us write $I=(s_0-R,s_0+R)$, so that $s_0\in\mathbb{R}$ is the centre of the interval I and $R>0$ its half-width. Let us define an auxiliary cut-off function $\eta\in C_0^\infty(\mathbb{R}\setminus\{s_0\})$ such that $0\leq\eta\leq 1$ and $\eta(s):=1$ if $|s-s_0|>R$. We denote by the same symbol η the function $(s,t)\mapsto\eta(s)$ on $\mathbb{R}\times\omega$, and similarly for its derivatives. We write

$$\psi=\eta\psi+(1-\eta)\psi\,.$$

• Applying this decomposition, one has the identity

$$\|\partial_s\psi-\theta'\partial_\tau\psi\|^2=\|\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi)\|^2+\|\partial_s((1-\eta)\psi)-\theta'\partial_\tau((1-\eta)\psi)\|^2+I_1+I_2+I_3+I_4\,,$$

where

$$\begin{aligned}I_1:=2\,\Re\big(\partial_s(\eta\psi),\partial_s((1-\eta)\psi)\big)&=2\,\big(\partial_s\psi,\eta(1-\eta)\partial_s\psi\big)+2\,\Re\big(\psi,[\eta(1-\eta)]'\partial_s\psi\big)+2\,\big(\psi,\eta'(1-\eta)'\psi\big)\\&=2\,\big(\partial_s\psi,\eta(1-\eta)\partial_s\psi\big)-\big(\psi,[\eta(1-\eta)]''\psi\big)-2\,\big(\psi,\eta'^2\psi\big)\\&\geq 2\,\underbrace{\big(\partial_s\psi,\eta(1-\eta)\partial_s\psi\big)}_{A\geq0}-\underbrace{\|[\eta(1-\eta)]''+2\eta'^2\|_\infty}_{C>0}\,\|\chi_I\psi\|^2\,,\end{aligned}$$

$$I_2:=2\,\Re\big(\theta'\partial_\tau(\eta\psi),\theta'\partial_\tau((1-\eta)\psi)\big)=2\,\underbrace{\big(\theta'\partial_\tau\psi,\eta(1-\eta)\theta'\partial_\tau\psi\big)}_{B\geq0}\,,$$

(using (2.37))

$$\begin{aligned}I_3:=-2\,\Re\big(\theta'\partial_\tau(\eta\psi),\partial_s((1-\eta)\psi)\big)&=-2\,\Re\big(\theta'\eta\partial_\tau\psi,(1-\eta)\partial_s\psi\big)\\&\geq-2\,\sqrt{A}\,\sqrt{B}\end{aligned}$$

and, similarly,

$$\begin{aligned}I_4:=-2\,\Re\big(\partial_s(\eta\psi),\theta'\partial_\tau((1-\eta)\psi)\big)&=-2\,\Re\big(\eta\partial_s\psi,\theta'(1-\eta)\partial_\tau\psi\big)=I_3\\&\geq-2\,\sqrt{A}\,\sqrt{B}\,.\end{aligned}$$

We therefore arrive at

$$\|\partial_s\psi-\theta'\partial_\tau\psi\|^2\geq\|\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi)\|^2+\|\partial_s((1-\eta)\psi)-\theta'\partial_\tau((1-\eta)\psi)\|^2-C\,\|\chi_I\psi\|^2\,.$$

Neglecting the second term on the right-hand side and using (2.21), we eventually get

$$Q[\psi]\geq\|\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi)\|^2-C\,\|\chi_I\psi\|^2\,. \tag{2.38}$$

• Since $\eta\psi$ vanishes at $s=s_0$, one has a classical version of the Hardy inequality for the first term on the right-hand side of (2.38):

$$\|\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi)\|^2\geq\frac14\,\left\|\frac{\eta\psi}{s-s_0}\right\|^2\,, \tag{2.39}$$

where s denotes the function $(s,t)\mapsto s$. Indeed, for every $\alpha\in\mathbb{R}$, one has

$$\begin{aligned}&\left\|\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi)-\alpha\frac{\eta\psi}{s-s_0}\right\|^2\\&\qquad=\|\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi)\|^2+\alpha^2\left\|\frac{\eta\psi}{s-s_0}\right\|^2+2\,\alpha\,\Re\left(\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi),\frac{\eta\psi}{s-s_0}\right)\,.\end{aligned}$$

Recalling (2.37), one has

$$2\,\Re\left(\partial_s(\eta\psi)-\theta'\partial_\tau(\eta\psi),\frac{\eta\psi}{s-s_0}\right)=2\,\Re\left(\partial_s(\eta\psi),\frac{\eta\psi}{s-s_0}\right)=\int_{\mathbb{R}\times\omega}\frac{\partial_s|\eta\psi|^2}{s-s_0}=\int_{\mathbb{R}\times\omega}\frac{|\eta\psi|^2}{(s-s_0)^2}\,,$$

where the last equality follows by an integration by parts. Choosing $\alpha:=-1/2$, we get the desired claim. In summary, plugging (2.39) into (2.38), we arrive at

$$Q[\psi]\geq\frac14\,\left\|\frac{\eta\psi}{s-s_0}\right\|^2-C\,\|\chi_I\psi\|^2\geq\frac14\,\|\tilde\rho\,\eta\psi\|^2-C\,\|\chi_I\psi\|^2\,, \tag{2.40}$$

where $\tilde\rho(s,t):=[1+(s-s_0)^2]^{-1/2}$.

• The negative term on the right-hand side of (2.40) can be controlled by (2.36). In more detail, given any $\beta\in\mathbb{R}$, let us interpolate between the estimates (2.36) and (2.40) as follows:

$$Q[\psi]=(1-\beta)\,Q[\psi]+\beta\,Q[\psi]\geq[(1-\beta)\,\lambda_1^I-C\,\beta]\,\|\chi_I\psi\|^2+\frac{\beta}{4}\,\|\tilde\rho\,\eta\psi\|^2\,.$$

Choosing β sufficiently small, for instance

$$\beta := \frac{4\lambda_1^I}{1+4(\lambda_1^I+C)}$$

which makes the constants standing in front of $\|\chi_I\psi\|^2$ and $\|\tilde\rho\,\eta\psi\|^2$ equal, we get

$$Q[\psi] \geq \frac{\beta}{4}\left(\|\chi_I\psi\|^2+\|\tilde\rho\,\eta\psi\|^2\right) \geq \frac{\beta}{4}\left(\|\chi_I\tilde\rho\,\psi\|^2+\|\tilde\rho\,\eta\psi\|^2\right) \geq \frac{\beta}{4}\,\|\tilde\rho\psi\|^2\,.$$

• Finally, the ultimate inequality (2.35) follows by the choice

$$c := \frac{\beta}{4}\,\sup_{s\in\mathbb{R}}\frac{1+s^2}{1+(s-s_0)^2}\,.$$

This concludes the proof of Theorem 2.11. □

The moral of this section is that

twisting acts as a repulsive interaction

in the sense that it has the tendency to raise the spectrum with respect to the spectrum of straight tubes. This is subtle for asymptotically straight twisted unbent tubes, for which the spectrum as a set coincides with the spectrum of straight tubes. In this case, the repulsiveness is quantified by the existence of Hardy-type inequalities at the threshold E_1 of the spectrum (*cf* Theorems 2.10 and 2.11). For tubes which are not asymptotically straight, however, the effect of twisting can be so strong that it pushes up the essential spectrum too. For instance, this always happens for periodically twisted tubes (*i.e.* $\kappa = 0$, ω is not rotationally invariant with respect to the origin and $\theta' \neq 0$ is periodic). This is clear from the proof of Theorem 2.10 yielding the Poincaré inequality $H - E_1 \geq \lambda_1^I > 0$, where I is the period of twisting.

2.12 Diverging twisting

For twisting diverging at infinity, *i.e.*,

$$\lim_{|s|\to\infty}|\theta'(s)| = \infty\,, \tag{2.41}$$

the situation can be even more drastic. We collect two results from [33].

Theorem 2.12 (Quasi-bounded realisation)**.** *Let $\kappa = 0$. If θ satisfies* (2.41) *and $\omega \subset (0,\infty)\times\mathbb{R}$, then*

$$\sigma_{\rm ess}(-\Delta_D^\Omega) = \varnothing\,.$$

It follows from Theorem 2.12 that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and the spectrum of $-\Delta_D^\Omega$ is purely discrete. Since the quasi-boundedness is a necessary condition for the compactness, we see that Ω is actually not quasi-cylindrical but quasi-bounded under the hypotheses of Theorem 2.12.

A geometric realisation of Theorem 2.12 is illustrated in Figure 13.

It turns out that it is crucial that the origin of $\mathbb{R}^2 \supset \omega$ is located outside the cross-section ω (the hypothesis that ω lies in the half-plane ensures this). Indeed, if $0 \in \omega$, then the cylinder $\mathbb{R}\times B_r(0)$ with $r := \mathrm{dist}(0,\partial\omega)$ clearly lies inside Ω. The spectrum of the cylinder coincides with $[\lambda_1,\infty)$, where λ_1 denotes the first eigenvalue of the Dirichlet Laplacian in the disk $B_r(0)$. Applying the Weyl criterion for $-\Delta_D^\Omega$ with a singular sequence for the cylinder, we easily end up with the following result.

Theorem 2.13 (Quasi-cylindrical realisation)**.** *Let $\kappa = 0$. If θ satisfies* (2.41) *and $\omega \ni 0$, then*

$$\sigma(-\Delta_D^\Omega) \supset [\lambda_1,\infty)\,.$$

A geometric realisation of Theorem 2.13 is illustrated in Figure 14. The embedded cylindrical channel responsible for the existence of the essential spectrum is clearly visible.

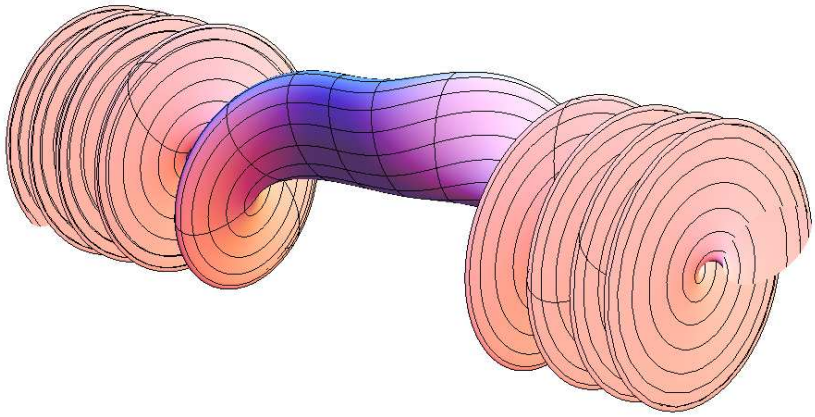


Figure 13: A tube with diverging twisting with purely discrete spectrum (Theorem 2.12).

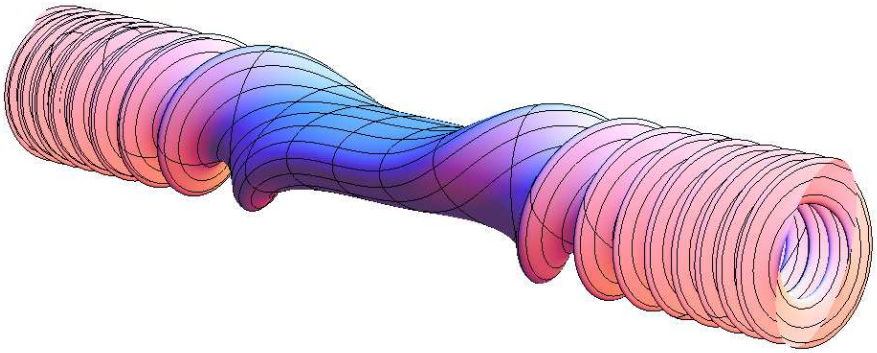


Figure 14: A tube with diverging twisting with a non-empty essential spectrum (Theorem 2.13).

2.13 Open problems

• How to get rid of the technical assumption (2.26) to establish the effective Hamiltonian for thin tubes (2.29) in a norm-resolvent sense? We expect the *optimal hypotheses* to be the standing ones (*i.e.* (2.10)). However, even the mild approach of [40] requires to assume slightly more than $\kappa \in L^\infty(\mathbb{R})$.

• Establish a quantitative *lower bound* to the constant λ_1^I appearing in Theorem 2.10. This is particularly relevant for the study of the large-time behaviour of the heat semigroup in twisted tubes [42].

• The results for diverging twisting are established in [33] by considering $-\Delta_D^\Omega$ in $L^2(\Omega)$, without the strategy of passing to the curvilinear coordinates (s,t). (This is the reason why we intentionally write $-\Delta_D^\Omega$ instead of H in Theorems 2.12 and 2.13.) To deduce the results *directly* from the expression for H acting in $\mathcal{H}$ constitutes an open problem.

• In the geometric realisation of Theorem 2.12, the spectrum is purely discrete. What are the *Weyl-type asymptotics* for the accumulation of the eigenvalues at infinity? The Weyl law must be necessarily non-standard because $|\Omega| = \infty$. In this direction, a Berezin-type upper bound for the eigenvalue moments has been recently established in [1].

• Find sufficient conditions which ensure the *absence of embedded eigenvalues*. The Mourre theory (which by itself does disprove the possibility of embedded eigenvalues) was developed in [34] and [5] for bent and twisted tubes, respectively.

• Extend the present results to the tubes about curves in *higher-dimensional* Euclidean spaces. Just bent tubes have been considered previously [8].

• Extend the present results to the tubes about curves in *curved ambient manifolds*. Just strips on two-dimensional surfaces have been considered previously [30, 31, 29, 35, 41].

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