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Nicolas Marie

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ON A CALCULABLE SKOROKHOD'S INTEGRAL BASED PROJECTION ESTIMATOR OF THE DRIFT FUNCTION IN FRACTIONAL SDE

NICOLAS MARIE[†]

ABSTRACT. This paper deals with a Skorokhod's integral based projection type estimator \widehat{b}_m of the drift function b_0 computed from $N \in \mathbb{N}^*$ independent copies X^1, \dots, X^N of the solution X of $dX_t = b_0(X_t)dt + \sigma dB_t$, where B is a fractional Brownian motion of Hurst index $H \in (1/2, 1)$. Skorokhod's integral based estimators cannot be calculated directly from X^1, \dots, X^N , but in this paper an \mathbb{L}^2 -error bound is established on a calculable approximation of \widehat{b}_m .

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1. INTRODUCTION

Consider the differential equation

$$(1) \quad X_t = X_0 + \int_0^t b_0(X_s)ds + \sigma B_t ; t \in [0, T],$$

where $T > 0$, $B = (B_t)_{t \in \mathbb{R}}$ is the two-sided fractional Brownian motion of Hurst index $H \in (1/2, 1)$ constructed in Hairer and Ohashi [6] (Section 4.1), X_0 is a square integrable and $\sigma((B_t)_{t \in \mathbb{R}_-})$ -measurable random variable, $b_0 \in C^1(\mathbb{R})$ and $\sigma \in \mathbb{R}^*$. Moreover, b'_0 is bounded, and then Equation (1) has a unique solution X .

The oldest kind of estimators of the drift function b_0 is based on the long-time behavior of the solution of Equation (1). In the parametric estimation framework, the reader may refer to the monograph [10] written by K. Kubilius, Y. Mishura and K. Ralchenko, but also to Kleptsyna and Le Breton [9], Tudor and Viens [17], Hu and Nualart [7], Neuenkirch and Tindel [14], Hu et al. [8], Marie and Raynaud de Fitte [13], etc. In the nonparametric estimation framework, the reader may refer to Saussereau [16] and Comte and Marie [2].

Recently, a new kind of estimators of b_0 have been investigated: those computed from N independent

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copies X^1, \dots, X^N of X observed on $[0, T]$ with T fixed but $N \rightarrow \infty$. The reader may refer to Marie [12] in the parametric framework, but also to Comte and Marie [3] dealing with an extension to Equation (1) of the projection least squares estimator computed from independent copies of a diffusion process in Comte and Genon-Catalot [1].

Note that copies based estimators are well-adapted to some situations difficult to manage with long-time behavior based estimators. For instance, in pharmacokinetics, X may model the elimination process of a drug administered to one people and then, in a clinical-trial involving N patients, X^i may model the elimination process of the same drug for the i -th patient.

The stochastic integral involved in the definition of the estimators studied in [7], [8], [13], [2], [3] and [12] is taken in the sense of Skorokhod. To be not directly calculable from one observation of X is the major drawback of the Skorokhod integral with respect to X . To bypass this difficulty by extending the fixed point strategy of Marie [12] to the nonparametric estimation framework is the main purpose of this paper.

Consider the copies X^1, \dots, X^N of X such that

$$X^i := \mathcal{I}_T(X_0^i, B^i) ; \forall i \in \{1, \dots, N\},$$

where \mathcal{I}_T is the solution map for Equation (1) and B^1, \dots, B^N (resp. X_0^1, \dots, X_0^N) are independent copies of B (resp. X_0). Moreover, assume that the probability distribution of X_t , $t \in (0, T]$, has a density f_t with respect to Lebesgue's measure such that $t \mapsto f_t(x)$ belongs to $\mathbb{L}^1([0, T])$ for every $x \in \mathbb{R}$. This legitimates to consider the density function f defined by

$$f(x) := \frac{1}{T} \int_0^T f_s(x) ds ; \forall x \in \mathbb{R}.$$

Consider the interval I of \mathbb{R} on which $x \mapsto b_0(x)$ will be estimated, and let $(\varphi_1, \dots, \varphi_m)$ be an I -supported orthonormal family of $\mathbb{L}^2(\mathbb{R}, dx)$. For instance, if I is compact, one can take the trigonometric basis, and if $I = \mathbb{R}$, one can take the Hermite basis. A suitable nonparametric estimator of b_0 could be

$$\hat{b}_m(x) := \frac{1}{f(x)} \sum_{j=1}^m \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \delta X_s^i \right) \varphi_j(x) ; x \in I,$$

but it is not directly calculable from X^1, \dots, X^N because the stochastic integral is taken in the sense of Skorokhod. However, by assuming that $\varphi_1, \dots, \varphi_m$ are differentiable on I ,

$$(2) \quad \hat{b}_m = \Phi_m(b_0) \approx \Phi_m(\hat{b}_m)$$

thanks to Nualart [15], Proposition 5.2.3, where Φ_m is the map defined on $C^1(\mathbb{R})$ by

$$\Phi_m(\varphi)(x) := \frac{1}{f(x)} \sum_{j=1}^m \left[\frac{1}{NT} \sum_{i=1}^N \left[I_{i,j} - \mathfrak{a} \int_0^T \int_0^t \varphi_j'(X_t^i) \exp \left(\int_s^t \varphi'(X_u^i) du \right) |t-s|^{2H-2} ds dt \right] \right] \varphi_j(x)$$

for every $\varphi \in C^1(\mathbb{R})$ and $x \in \mathbb{R}$, $\mathfrak{a} := \sigma^2 H(2H-1)$ and

$$I_{i,j} := \int_0^T \varphi_j(X_s^i) dX_s^i$$

for any $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, m\}$. In the definition of $I_{i,j}$, the stochastic integral is taken pathwise (in the sense of Young).

So, Section 3 deals with an \mathbb{L}^2 -error bound on the auxiliary (pseudo-)estimator \widehat{b}_m , and then Section 4 with the existence, the uniqueness and an \mathbb{L}^2 -error bound on the nonparametric estimator \widetilde{b}_m of b_0 defined as the fixed point of Φ_m in

$$\mathcal{S}_{m,\mathfrak{c}} := \{\varphi \in \mathcal{S}_m : \|\varphi'\|_\infty \leq \mathfrak{c}\} \quad \text{with} \quad \mathfrak{c} > \|b'_0\|_\infty \quad \text{and} \quad \mathcal{S}_m = \text{span}\{\varphi_1, \dots, \varphi_m\}.$$

For instance, if X is the stationary solution of Equation (1), which exists and is unique under an appropriate dissipativity condition on b_0 (see (6)), and if $(\varphi_1, \dots, \varphi_m)$ is the I -supported trigonometric basis with $I = [\ell, \mathfrak{r}]$ ($\ell, \mathfrak{r} \in \mathbb{R}$ such that $\ell < \mathfrak{r}$), then the main result of our paper (see Theorem 4.7) says that

$$(3) \quad \mathbb{E} \left(\int_{-\infty}^{\infty} (\widetilde{b}_m(x) \mathbf{1}_{\Delta_m} - b_0(x))^2 f_0(x) dx \right) \lesssim m^{-2} + m^3 (N^{-\frac{1}{2}} T^{-1} + T^{2H-1}),$$

where f_0 is the (stationary) density of X_0 and Δ_m is an event defined later. By Inequality (3), there is a double compromise to manage with. First, T needs to be of order $N^{-1/(4H)}$, and then m needs to be of order $N^{(2H-1)/(20H)}$ to get a rate of order $N^{-(2H-1)/(10H)}$.

Some preliminary results on Skorokhod's integral are given in Section 2.

2. BASICS ON THE SKOROKHOD INTEGRAL

Let $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ be the inner product defined by

$$\langle h, \eta \rangle_{\mathcal{H}} := \alpha_H \int_0^T \int_0^T h(s) \eta(t) |t - s|^{2H-2} ds dt \quad \text{with} \quad \alpha_H = H(2H-1),$$

and consider the reproducing kernel Hilbert space $\mathcal{H} = \{h : \|h\|_{\mathcal{H}} < \infty\}$ of $B^T = (B_t)_{t \in [0, T]}$, where $\|\cdot\|_{\mathcal{H}}$ is the norm associated to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Consider also the isonormal Gaussian process $(\mathbf{B}(h))_{h \in \mathcal{H}}$ defined by

$$\mathbf{B}(h) := \int_0^T h(s) dB_s ; \quad h \in \mathcal{H},$$

which is the Wiener integral of h with respect to B on $[0, T]$.

Definition 2.1. *The Malliavin derivative of a smooth functional*

$$F = \varphi(\mathbf{B}(h_1), \dots, \mathbf{B}(h_n))$$

where $n \in \mathbb{N}^*$, $\varphi \in C_p^\infty(\mathbb{R}^n; \mathbb{R})$ (the space of all the smooth functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that φ and all its partial derivatives have polynomial growth) and $h_1, \dots, h_n \in \mathcal{H}$, is the \mathcal{H} -valued random variable

$$\mathbf{D}F := \sum_{k=1}^n \partial_k \varphi(\mathbf{B}(h_1), \dots, \mathbf{B}(h_n)) h_k.$$

Proposition 2.2. *The map \mathbf{D} is closable from $\mathbb{L}^2(\Omega; \mathbb{R})$ into $\mathbb{L}^2(\Omega; \mathcal{H})$. Its domain in $\mathbb{L}^2(\Omega; \mathbb{R})$, denoted by $\mathbb{D}^{1,2}$, is the closure of the smooth functionals space for the norm $\|\cdot\|_{1,2}$ defined by*

$$\|F\|_{1,2}^2 := \mathbb{E}(F^2) + \mathbb{E}(\|\mathbf{D}F\|_{\mathcal{H}}^2).$$

The Malliavin derivative of $F \in \mathbb{D}^{1,2}$ at time $s \in [0, T]$ is denoted by $\mathbf{D}_s F$.

See Nualart [15], Proposition 1.2.1 for a proof.

Definition 2.3. *The adjoint δ of the Malliavin derivative \mathbf{D} is the divergence operator. The domain of δ is denoted by $\text{dom}(\delta)$, and $Y \in \text{dom}(\delta)$ if and only if there exists a deterministic constant $\mathfrak{c}_Y > 0$ such that for every $F \in \mathbb{D}^{1,2}$,*

$$|\mathbb{E}(\langle \mathbf{D}F, Y \rangle_{\mathcal{H}})| \leq \mathfrak{c}_Y \mathbb{E}(F^2)^{\frac{1}{2}}.$$

For any process $Y = (Y_s)_{s \in [0, T]}$ and every $t \in (0, T]$, if $Y \mathbf{1}_{[0, t]} \in \text{dom}(\delta)$, then its Skorokhod integral with respect to B^T is defined on $[0, t]$ by

$$\int_0^t Y_s \delta B_s := \delta(Y \mathbf{1}_{[0, t]}),$$

and its Skorokhod integral with respect to X is defined by

$$\int_0^t Y_s \delta X_s := \int_0^t Y_s b_0(X_s) ds + \sigma \int_0^t Y_s \delta B_s.$$

Note that since δ is the adjoint of the Malliavin derivative \mathbf{D} , the Skorokhod integral of Y with respect to B^T on $[0, t]$ is a centered random variable. Indeed,

$$(4) \quad \mathbb{E} \left(\int_0^t Y_s \delta B_s \right) = \mathbb{E}(1 \cdot \delta(Y \mathbf{1}_{[0, t]})) = \mathbb{E}(\langle \mathbf{D}(1), Y \mathbf{1}_{[0, t]} \rangle_{\mathcal{H}}) = 0.$$

Let \mathcal{S} be the space of the smooth functionals presented in Definition 2.1 and consider $\mathbb{D}^{1,2}(\mathcal{H})$, the closure of

$$\mathcal{S}_{\mathcal{H}} := \left\{ \sum_{j=1}^n F_j h_j ; h_1, \dots, h_n \in \mathcal{H}, F_1, \dots, F_n \in \mathcal{S} \right\}$$

for the norm $\|\cdot\|_{1,2,\mathcal{H}}$ defined by

$$\|Y\|_{1,2,\mathcal{H}}^2 := \mathbb{E}(\|Y\|_{\mathcal{H}}^2) + \mathbb{E}(\|\mathbf{D}Y\|_{\mathcal{H} \otimes \mathcal{H}}^2).$$

Consider also the norm $\|\cdot\|_{\mathfrak{H}}$ defined by

$$\|h\|_{\mathfrak{H}} := \left(\alpha_H \int_0^T \int_0^T |h(s)| \cdot |h(t)| \cdot |t-s|^{2H-2} ds dt \right)^{\frac{1}{2}},$$

the Banach space $\mathfrak{H} := \{h : \|h\|_{\mathfrak{H}} < \infty\}$ and

$$\mathbb{D}^{1,2}(\mathfrak{H}) := \{Y \in \mathbb{D}^{1,2}(\mathcal{H}) : \mathbb{E}(\|Y\|_{\mathfrak{H}}^2) + \mathbb{E}(\|\mathbf{D}Y\|_{\mathfrak{H} \otimes \mathfrak{H}}^2) < \infty\}.$$

By Nualart [15], Proposition 1.3.1,

$$\mathbb{D}^{1,2}(\mathfrak{H}) \subset \mathbb{D}^{1,2}(\mathcal{H}) \subset \text{dom}(\delta).$$

The two following propositions are crucial in Sections 3 and 4.

Proposition 2.4. *For every $\varphi \in C^1(\mathbb{R})$ of bounded derivative, $(\varphi(X_t))_{t \in [0, T]}$ belongs to $\mathbb{D}^{1,2}(\mathfrak{H})$ and*

$$\int_0^T \varphi(X_s) \delta X_s = \int_0^T \varphi(X_s) dX_s - \alpha_H \sigma^2 \int_0^T \int_0^t \varphi'(X_t) \exp \left(\int_s^t b'_0(X_u) du \right) |t-s|^{2H-2} ds dt.$$

Proof. For any $s, t \in [0, T]$ such that $s < t$, since

$$\mathbf{D}_s B_t = \mathbf{D}_s (\mathbf{B}(\mathbf{1}_{[0, t]})) = \mathbf{1}_{[0, t]}(s),$$

since X_0 is a $\sigma((B_u)_{u \in \mathbb{R}_-})$ -measurable random variable, and by Equation (1) and Nualart [15], Proposition 1.2.3,

$$\mathbf{D}_s X_t = \int_0^t b'_0(X_u) \mathbf{D}_s X_u du + \sigma \mathbf{1}_{[0,t]}(s).$$

Therefore,

$$\mathbf{D}_s X_t = \sigma \mathbf{1}_{[0,t]}(s) \exp \left(\int_s^t b'_0(X_u) du \right)$$

and, by Nualart [15], Propositions 1.2.3 and 5.2.3,

$$\begin{aligned} \int_0^T \varphi(X_s) \delta X_s &= \int_0^T \varphi(X_s) dX_s - \alpha_H \sigma \int_0^T \int_0^T \mathbf{D}_s(\varphi(X_t)) |t-s|^{2H-2} ds dt \\ &= \int_0^T \varphi(X_s) dX_s \\ &\quad - \alpha_H \sigma^2 \int_0^T \int_0^t \varphi'(X_t) \exp \left(\int_s^t b'_0(X_u) du \right) |t-s|^{2H-2} ds dt. \end{aligned}$$

□

Proposition 2.5. *Consider*

$$M := \sup_{x \in \mathbb{R}} b'_0(x).$$

There exists a constant $\mathfrak{c}_{2.5} > 0$, only depending on H and σ , such that for every $\varphi \in C^1(\mathbb{R})$ of bounded derivative,

$$\mathbb{E} \left[\left(\int_0^T \varphi(X_s) \delta B_s \right)^2 \right] \leq \mathfrak{c}_{2.5} \bar{\mathfrak{m}}_T T^{2H-1} \left(\int_0^T \mathbb{E}(\varphi(X_s)^2) ds + \int_0^T \mathbb{E}(\varphi'(X_s)^2) ds \right)$$

with $\bar{\mathfrak{m}}_T = 1 \vee \mathfrak{m}_T$ and

$$\mathfrak{m}_T = \left(-\frac{H}{M} \right)^{2H} \mathbf{1}_{M < 0} + T^{2H} \mathbf{1}_{M=0} + \left(\frac{H}{M} \right)^{2H} e^{2MT} \mathbf{1}_{M > 0}.$$

See Hu et al. [8], Proposition 4.4.(2) and Comte and Marie [3], Theorem 2.9 for a proof.

For details on the Malliavin calculus, the reader may refer to Decreusefond [4] and Nualart [15].

3. AN \mathbb{L}^2 -ERROR BOUND ON THE AUXILIARY ESTIMATOR \widehat{b}_m

First, the probability distribution of X_t , $t \in (0, T]$, and the function b_0 need to fulfill the following assumption.

Assumption 3.1. *For every $t \in (0, T]$, the probability distribution of X_t has a density f_t with respect to Lebesgue's measure such that:*

- (1) *The function $t \mapsto f_t(x)$ belongs to $\mathbb{L}^1([0, T])$ for every $x \in \mathbb{R}$.*
- (2) *The function $|b_0|^\alpha$ belongs to $\mathbb{L}^2(\mathbb{R}, f(x)dx)$ for every $\alpha \in \mathbb{R}_+$, and \mathcal{S}_m is a subset of $\mathbb{L}^2(\mathbb{R}, f(x)dx)$, where f is the density function defined by*

$$f(x) := \frac{1}{T} \int_0^T f_s(x) ds ; \forall x \in \mathbb{R}.$$

Example 3.2. *In the two following situations, Assumption 3.1 is fulfilled:*

(1) Assume that $X_0(\cdot) = x_0$ with $x_0 \in \mathbb{R}$. For every $t \in (0, T]$, the probability distribution of X_t has a density f_t with respect to Lebesgue's measure such that, for every $x \in \mathbb{R}$,

$$(5) \quad f_t(x) \leq \mathbf{c}_{H,T} t^{-H} \exp\left(-\mathbf{m}_{H,T} \frac{(x - x_0)^2}{t^{2H}}\right)$$

where $\mathbf{c}_{H,T}$ and $\mathbf{m}_{H,T}$ are positive constants depending on T but not on t and x (see Li et al. [11], Theorem 1.3). So, $t \mapsto f_t(x)$ belongs to $\mathbb{L}^1([0, T])$. Still by Inequality (5),

$$|b_0|^\alpha \in \mathbb{L}^2(\mathbb{R}, f(x)dx) ; \forall \alpha \in \mathbb{R}_+$$

because b'_0 is bounded, and $\mathcal{S}_m \subset \mathbb{L}^2(\mathbb{R}, f(x)dx)$ because the φ_j 's belong to $\mathbb{L}^2(\mathbb{R}, dx)$.

(2) Assume that b_0 satisfies the dissipativity condition

$$(6) \quad \exists \mathbf{m} > 0 : \forall x \in \mathbb{R}, b'_0(x) \leq -\mathbf{m}.$$

Then, Equation (1) has a unique stationary solution X with a $\sigma((B_t)_{t \in \mathbb{R}_-})$ -measurable initial condition X_0 , and the common probability distribution of the X_t 's has a density f_0 with respect to Lebesgue's measure such that, for every $x \in \mathbb{R}$,

$$f_0(x) \leq \mathbf{c}_H e^{-\mathbf{m}_H x^2},$$

where \mathbf{c}_H and \mathbf{m}_H are positive constants not depending on x (see Li et al. [11], Theorem 1.1). So, b_0 and $f = f_0$ fulfill Assumption 3.1.(2). Of course, Assumption 3.1.(1) is fulfilled with $f_t = f_0$ for every $t \in (0, T]$.

Notations:

- For every measurable function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\|\varphi\|^2 := \int_I \varphi(x)^2 dx, \quad \|\varphi\|_f^2 := \int_I \varphi(x)^2 f(x) dx \quad \text{and} \quad \|\varphi\|_{f^2}^2 := \int_I \varphi(x)^2 f(x)^2 dx.$$

- The orthonormal projection of $b_0 f$ on \mathcal{S}_m is denoted by $(bf)_m$ in the sequel:

$$(bf)_m := \sum_{j=1}^m \langle b_0 f, \varphi_j \rangle \varphi_j.$$

Moreover, $b_m := (bf)_m / f$.

The following proposition provides a suitable \mathbb{L}^2 -error bound on \widehat{b}_m .

Proposition 3.3. *Under Assumption 3.1,*

$$\mathbb{E}(\|\widehat{b}_m - b_0\|_{f^2}^2) \leq \|b_m - b_0\|_{f^2}^2 + \frac{2}{N} \left(\|b_0\|_f^2 \mathfrak{L}(m) + \mathbf{c}_{2.5} \sigma^2 \frac{\overline{\mathbf{m}}_T}{T^{2-2H}} (\mathfrak{L}(m) + \mathfrak{R}(m)) \right),$$

where

$$\mathfrak{L}(m) := \sum_{j=1}^m \|\varphi_j\|_\infty^2 \quad \text{and} \quad \mathfrak{R}(m) := \sum_{j=1}^m \|\varphi'_j\|_\infty^2.$$

Proof. First of all,

$$(7) \quad \begin{aligned} \mathbb{E}(\|\widehat{b}_m - b_0\|_{f^2}^2) &= \mathbb{E}(\|\widehat{bf}_m - b_0 f\|^2) \\ &= \|(bf)_m - b_0 f\|^2 + \mathbb{E}(\|\widehat{bf}_m - (bf)_m\|^2) \end{aligned}$$

because $\widehat{bf}_m(\omega)$ ($\omega \in \Omega$) and $(bf)_m$ belong to \mathcal{S}_m , and $(bf)_m$ is the orthogonal projection of b_0f on \mathcal{S}_m . Since X^1, \dots, X^N are independent copies of X , and since the Skorokhod integral with respect to B^T is centered, for every $j \in \{1, \dots, m\}$,

$$\begin{aligned} \mathbb{E} \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \delta X_s^i \right) &= \frac{1}{T} \mathbb{E} \left(\int_0^T \varphi_j(X_s) b_0(X_s) ds \right) \\ &= \frac{1}{T} \int_{-\infty}^{\infty} \varphi_j(x) b_0(x) \int_0^T f_s(x) ds dx = \langle b_0f, \varphi_j \rangle. \end{aligned}$$

Then, since $(\varphi_1, \dots, \varphi_m)$ is an orthonormal family in $\mathbb{L}^2(\mathbb{R}, dx)$,

$$\begin{aligned} \mathbb{E}(\|\widehat{bf}_m - (bf)_m\|^2) &= \mathbb{E} \left[\sum_{j=1}^m \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \delta X_s^i - \langle b_0f, \varphi_j \rangle \right)^2 \right] \\ &= \sum_{j=1}^m \text{var} \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) \delta X_s^i \right) \\ &\leq \frac{2}{N} \sum_{j=1}^m \left[\mathbb{E} \left[\left(\frac{1}{T} \int_0^T \varphi_j(X_s) b_0(X_s) ds \right)^2 \right] + \sigma^2 \mathbb{E} \left[\left(\frac{1}{T} \int_0^T \varphi_j(X_s) \delta B_s \right)^2 \right] \right]. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{j=1}^m \mathbb{E} \left[\left(\frac{1}{T} \int_0^T \varphi_j(X_s) b_0(X_s) ds \right)^2 \right] &\leq \frac{1}{T} \sum_{j=1}^m \int_0^T \mathbb{E}(\varphi_j(X_s)^2 b_0(X_s)^2) ds \\ &= \sum_{j=1}^m \int_{-\infty}^{\infty} \varphi_j(x)^2 b_0(x)^2 f(x) dx \leq \mathfrak{L}(m) \|b_0\|_f^2 \end{aligned}$$

and, by Proposition 2.5,

$$\begin{aligned} \sum_{j=1}^m \mathbb{E} \left[\left(\frac{1}{T} \int_0^T \varphi_j(X_s) \delta B_s \right)^2 \right] &\leq \mathfrak{c}_{2.5} \frac{\overline{\mathfrak{m}}_T}{T^{2-2H}} \sum_{j=1}^m \int_{-\infty}^{\infty} (\varphi_j(x)^2 + \varphi_j'(x)^2) f(x) dx \\ &\leq \mathfrak{c}_{2.5} \frac{\overline{\mathfrak{m}}_T}{T^{2-2H}} (\mathfrak{L}(m) + \mathfrak{R}(m)). \end{aligned}$$

Therefore, by Equality (7),

$$\mathbb{E}(\|\widehat{bf}_m - b_0f\|_{f^2}^2) \leq \|(bf)_m - b_0f\|^2 + \frac{2}{N} \left(\|b_0\|_f^2 \mathfrak{L}(m) + \mathfrak{c}_{2.5} \sigma^2 \frac{\overline{\mathfrak{m}}_T}{T^{2-2H}} (\mathfrak{L}(m) + \mathfrak{R}(m)) \right).$$

□

4. EXISTENCE, UNIQUENESS AND \mathbb{L}^2 -ERROR BOUND ON THE FIXED POINT ESTIMATOR \widetilde{b}_m

The Skorokhod integral, and then \widehat{b}_m are uncomputable. However, by Proposition 2.4,

$$\int_0^T \varphi_j(X_s^i) \delta X_s^i = \underbrace{\int_0^T \varphi_j(X_s^i) dX_s^i}_{=I_{i,j}} - \alpha_H \sigma^2 \int_0^T \int_0^t \varphi_j'(X_t^i) \exp \left(\int_s^t b_0'(X_u^i) du \right) |t-s|^{2H-2} ds dt$$

for every $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, m\}$. Then, $\widehat{b}_m = \Phi_m(b_0)$, where Φ_m has been defined in Section 1 by

$$\Phi_m(\varphi)(x) := \frac{1}{f(x)} \sum_{j=1}^m \left[\frac{1}{NT} \sum_{i=1}^N \left[I_{i,j} - \mathfrak{a} \int_0^T \int_0^t \varphi'_j(X_t^i) \exp \left(\int_s^t \varphi'(X_u^i) du \right) |t-s|^{2H-2} ds dt \right] \right] \varphi_j(x)$$

for every $\varphi \in C^1(\mathbb{R})$ and $x \in \mathbb{R}$. Since \widehat{b}_m is a converging estimator of b_0 as established in Section 3,

$$\widehat{b}_m = \Phi_m(b_0) \approx \Phi_m(\widehat{b}_m),$$

which legitimates to consider the fixed point \widetilde{b}_m of Φ_m as a computable estimator of b_0 . This section deals with the existence, the uniqueness (thanks to Picard's theorem) and an \mathbb{L}^2 -error bound on \widetilde{b}_m .

In the sequel, consider the map F_m from \mathbb{R}^m into itself such that

$$F_m(\theta)_j := \frac{\mathfrak{a}}{NT} \sum_{i=1}^N \int_0^T \int_0^t \varphi'_j(X_t^i) \exp \left(\sum_{\ell=1}^m \theta_\ell \int_s^t \varphi'_\ell(X_u^i) du \right) |t-s|^{2H-2} ds dt$$

for every $\theta \in \mathbb{R}^m$ and $j \in \{1, \dots, m\}$. Moreover, I is compact and the density function f fulfills the following additional assumption.

Assumption 4.1. *There exists $\mathfrak{m}_f > 0$ such that*

$$f(x) \geq \mathfrak{m}_f ; \forall x \in I.$$

Example 4.2. *In the two following situations, the density function f fulfills Assumption 4.1:*

- (1) *Assume that $X_0(\cdot) = x_0$ with $x_0 \in \mathbb{R}$. By Li et al. [11], Theorem 1.3, there exist two constants $\bar{\mathfrak{c}}_{H,T}, \bar{\mathfrak{m}}_{H,T} > 0$ such that, for every $t \in (0, T]$ and $x \in I$,*

$$\begin{aligned} f_t(x) &\geq \bar{\mathfrak{c}}_{H,T} t^{-H} \exp \left(-\bar{\mathfrak{m}}_{H,T} \frac{(x-x_0)^2}{t^{2H}} \right) \\ &\geq u_I(t) := \bar{\mathfrak{c}}_{H,T} t^{-H} \exp \left(-\bar{\mathfrak{m}}_{H,T} \frac{x_I^2}{t^{2H}} \right) > 0 \end{aligned}$$

with $x_I = |\min(I - \{x_0\})| \vee |\max(I - \{x_0\})|$. Then, f fulfills Assumption 4.1 with

$$\mathfrak{m}_f = \frac{1}{T} \int_0^T u_I(s) ds > 0.$$

- (2) *Assume that b_0 fulfills the dissipativity condition (6). By Li et al. [11], Theorem 1.1, there exist two constants $\bar{\mathfrak{c}}_H, \bar{\mathfrak{m}}_H > 0$ such that, for every $x \in I$,*

$$f_0(x) \geq \bar{\mathfrak{c}}_H e^{-\bar{\mathfrak{m}}_H x^2} \geq \bar{\mathfrak{c}}_H e^{-\bar{\mathfrak{m}}_H x_I^2} > 0$$

with $x_I = |\min(I)| \vee |\max(I)|$. Then, $f = f_0$ fulfills Assumption 4.1 with $\mathfrak{m}_f = \bar{\mathfrak{c}}_H e^{-\bar{\mathfrak{m}}_H x_I^2}$.

Note that under Assumption 4.1, the norms $\|\cdot\|$ and $\|\cdot\|_{f^2}$ are equivalent on $\mathcal{S}_{m,\mathfrak{c}}$. Indeed, for every $\varphi \in \mathcal{S}_{m,\mathfrak{c}}$,

$$\|\varphi\|_{f^2} \leq \|f\|_\infty^2 \|\varphi\| \quad \text{and} \quad \|\varphi\| \leq \frac{1}{\mathfrak{m}_f^2} \|\varphi\|_{f^2}.$$

Notations:

- For any $\varphi \in \mathcal{S}_m$, its coordinate vector in the basis $(\varphi_1, \dots, \varphi_m)$ is denoted by $\theta^\varphi = (\theta_1^\varphi, \dots, \theta_m^\varphi)$.

- The usual norm on $\mathcal{L}(\mathbb{R}^m)$ is denoted by $\|\cdot\|_{\text{op}}$.

First, an event on which the map Φ_m has a unique fixed point in $\mathcal{S}_{m,\mathfrak{c}}$ is provided in the following proposition.

Proposition 4.3. *Under Assumptions 3.1 and 4.1, consider $\mathfrak{l} \in (0, 1)$ and the event $\Delta_m := \Delta_{m,\mathfrak{c}} \cap \Delta_{m,\mathfrak{l}}$, where*

$$\Delta_{m,\mathfrak{c}} := \left\{ \sup_{\varphi \in \mathcal{S}_{m,\mathfrak{c}}} \|\Phi_m(\varphi)'\|_{\infty} \leq \mathfrak{c} \right\} \quad \text{and} \quad \Delta_{m,\mathfrak{l}} := \left\{ \frac{1}{\mathfrak{m}_f^2} \sup_{\varphi \in \mathcal{S}_{m,\mathfrak{c}}} \|DF_m(\theta^\varphi)\|_{\text{op}}^2 \leq \mathfrak{l}^2 \right\}.$$

On the event Δ_m , Φ_m is a contraction from $\mathcal{S}_{m,\mathfrak{c}}$ into itself, and then it has a unique fixed point \tilde{b}_m in $\mathcal{S}_{m,\mathfrak{c}}$.

Proof. First, by the definition of $\Delta_{m,\mathfrak{c}}$, $\Phi_m(\mathcal{S}_{m,\mathfrak{c}}) \subset \mathcal{S}_{m,\mathfrak{c}}$ on $\Delta_{m,\mathfrak{c}}$. For any $\varphi, \bar{\varphi} \in \mathcal{S}_{m,\mathfrak{c}}$, since $(\varphi_1, \dots, \varphi_m)$ is an orthonormal family of $\mathbb{L}^2(\mathbb{R}, dx)$,

$$\begin{aligned} \|\Phi_m(\varphi) - \Phi_m(\bar{\varphi})\|_{f^2}^2 &= \|F_m(\theta^\varphi) - F_m(\theta^{\bar{\varphi}})\|_m^2 \\ &\leq \left(\sup_{\psi \in \mathcal{S}_{m,\mathfrak{c}}} \|DF_m(\theta^\psi)\|_{\text{op}} \right)^2 \|\theta^\varphi - \theta^{\bar{\varphi}}\|_m^2 \leq \left(\frac{1}{\mathfrak{m}_f^2} \sup_{\psi \in \mathcal{S}_{m,\mathfrak{c}}} \|DF_m(\theta^\psi)\|_{\text{op}}^2 \right) \|\varphi - \bar{\varphi}\|_{f^2}^2. \end{aligned}$$

This leads to

$$\|\Phi_m(\varphi) - \Phi_m(\bar{\varphi})\|_{f^2} \leq \mathfrak{l} \|\varphi - \bar{\varphi}\|_{f^2} \quad \text{on} \quad \Delta_{m,\mathfrak{l}}.$$

Therefore, Φ_m is a contraction from $\mathcal{S}_{m,\mathfrak{c}}$ into itself on the event Δ_m . \square

Now, assume that b_0 fulfills the dissipativity condition (6). Then, as already mentioned in Section 3, Equation (1) has a unique stationary solution X , and $f = f_0$ is the density of the common probability distribution of the X_t 's. Assumptions 3.1 and 4.1 are fulfilled as established in Example 3.2.(2) and Example 4.2.(2) respectively. Moreover, consider

$$\mathfrak{I}(m) := \sum_{j=1}^m \|\bar{\varphi}_j\|_{\infty}^2$$

where, for every $j \in \{1, \dots, m\}$, $\bar{\varphi}_j$ is a primitive function of φ_j . In the sequel, X^1, \dots, X^N are independent copies of the stationary solution X of Equation (1), and the φ_j 's fulfill the following additional assumption.

Assumption 4.4. *There exists a constant $\mathfrak{c}_\varphi > 0$, not depending on m , such that*

$$\mathfrak{I}(m) \vee \mathfrak{L}(m) \leq \mathfrak{c}_\varphi \mathfrak{R}(m).$$

Example 4.5. *Assume that $I = [\ell, \mathfrak{r}]$ with $\ell, \mathfrak{r} \in \mathbb{R}$ satisfying $\ell < \mathfrak{r}$, and that $(\varphi_1, \dots, \varphi_m)$ is the trigonometric basis:*

$$\begin{aligned} \varphi_1(x) &:= \sqrt{\frac{1}{\mathfrak{r} - \ell}} \mathbf{1}_I(x), \\ \varphi_{2j+1}(x) &:= \sqrt{\frac{2}{\mathfrak{r} - \ell}} \sin\left(2\pi j \frac{x - \ell}{\mathfrak{r} - \ell}\right) \mathbf{1}_I(x) \quad \text{and} \\ \varphi_{2j}(x) &:= \sqrt{\frac{2}{\mathfrak{r} - \ell}} \cos\left(2\pi j \frac{x - \ell}{\mathfrak{r} - \ell}\right) \mathbf{1}_I(x) \end{aligned}$$

for every $x \in I$ and $j \in \mathbb{N}^*$ satisfying $2j + 1 \leq m$. Let us show that the trigonometric basis fulfills Assumption 4.4. On the one hand,

$$\mathfrak{R}(m) = \underbrace{\frac{4\pi^2}{(\mathfrak{x} - \ell)^3}}_{=: \mathfrak{c}_1} m^3 \geq \mathfrak{c}_1 m = \mathfrak{c}_1 \mathfrak{L}(m)$$

as established in Comte and Marie [3], Section 3.2.1. On the other hand, assume that for every $j \in \{1, \dots, m\}$, the primitive function $\bar{\varphi}_j$ of φ_j is chosen such that $\bar{\varphi}_j(x_j) = 0$ with $x_j \in I$. Then,

$$\begin{aligned} |\bar{\varphi}_j(\cdot)| &= \left| \int_{x_j}^{\cdot} \varphi_j(y) dy \right| \\ &\leq \lambda_I \|\varphi_j\|_{\infty} \quad \text{with} \quad \lambda_I = \max(I) - \min(I), \end{aligned}$$

leading to $\mathfrak{J}(m) \leq \lambda_I^2 \mathfrak{L}(m)$.

In order to provide an \mathbb{L}^2 -error bound on the truncated estimator

$$\tilde{b}_m^{\mathfrak{c}, \mathfrak{l}} := \tilde{b}_m \mathbf{1}_{\Delta_m}$$

in Theorem 4.7, a suitable control of $\mathbb{P}(\Delta_m^c)$ needs to be established first.

Lemma 4.6. *If b_0 fulfills the dissipativity condition (6), and if X^1, \dots, X^N are independent copies of the stationary solution X of Equation (1), then there exists a constant $\mathfrak{c}_{4.6,1} > 0$, not depending on m , N and T , such that*

$$\mathbb{P}(\Delta_m^c) \leq \mathfrak{c}_{4.6,1} ((\mathfrak{L}(m)^{\frac{1}{2}} + \mathfrak{R}(m)^{\frac{1}{2}})(N^{-\frac{1}{2}} \mathfrak{J}(m)^{\frac{1}{2}} T^{-1} + \mathfrak{R}(m)^{\frac{1}{2}} e^{\mathfrak{c}T} T^{2H-1}) + \mathfrak{R}(m) e^{\mathfrak{c}T} T^{2H}).$$

Under Assumption 4.4, if $T \in (0, 1]$, then there exists a constant $\mathfrak{c}_{4.6,2} > 0$, not depending on m , N and T , such that

$$(8) \quad \mathbb{P}(\Delta_m^c) \leq \mathfrak{c}_{4.6,2} \mathfrak{R}(m) (N^{-\frac{1}{2}} T^{-1} + T^{2H-1}).$$

Inequality (8) suggests that, at the end, there will be a double compromise to manage with. First, the usual bias-variance tradeoff between the number N of observations and the dimension m of the projection space, but also a compromise between N and T . Indeed, the time horizon T should be chosen near to 0 in order to minimize T^{2H-1} , but not that much because $N^{-1/2} T^{-1}$ and the variance term in the \mathbb{L}^2 -error bound on the auxiliary estimator \hat{b}_m (see Proposition 3.3) explode when T goes to 0. See the final remarks for details. Finally, note that if X^1, \dots, X^N have been observed on $[0, T_{\max}]$ for a fixed $T_{\max} > 0$, then our estimator is calculable for $T = T(N)$ with $T(N) \downarrow 0$ when $N \rightarrow \infty$.

Proof. The proof of Lemma 4.6 is dissected in three steps. Step 1 deals with a bound on $\mathbb{P}(\Delta_{m,\mathfrak{c}}^c)$, Step 2 with a bound on $\mathbb{P}(\Delta_{m,\mathfrak{l}}^c)$, and the conclusion comes in Step 3.

Step 1. Consider $\varphi \in \mathcal{S}_{m,\mathfrak{c}}$. Since $f = f_0 \in C^1(\mathbb{R})$ and f' is bounded by Li et al. [11], Theorem

1.1, for every $x \in \mathbb{R}$,

$$\begin{aligned} |\Phi_m(\varphi)'(x)| &\leq \left| \frac{f'(x)}{f(x)^2} \right| \sum_{j=1}^m |\varphi_j(x)| U_j(\varphi) + \left| \frac{1}{f(x)} \right| \sum_{j=1}^m |\varphi_j'(x)| U_j(\varphi) \\ &\leq \underbrace{\left(\frac{\|f'\|_\infty}{\mathfrak{m}_f^2} \vee \frac{1}{\mathfrak{m}_f} \right)}_{=: \mathfrak{m}_{f,f'}} \sum_{j=1}^m (|\varphi_j(x)| + |\varphi_j'(x)|) U_j(\varphi) \end{aligned}$$

where, for every $j \in \{1, \dots, m\}$,

$$\begin{aligned} U_j(\varphi) &:= \frac{1}{NT} \left| \sum_{i=1}^N \left[I_{i,j} - \mathfrak{a} \int_0^T \int_0^t \varphi_j'(X_u^i) \exp \left(\int_s^t \varphi_j'(X_u^i) du \right) |t-s|^{2H-2} ds dt \right] \right| \\ &\leq \left| \frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) dX_s^i \right| + \frac{\mathfrak{a}}{NT} \|\varphi_j'\|_\infty e^{\epsilon T} \left(N \int_0^T \int_0^t |t-s|^{2H-2} ds dt \right) \\ &= \left| \frac{1}{NT} \sum_{i=1}^N (\bar{\varphi}_j(X_T^i) - \bar{\varphi}_j(X_0^i)) \right| + \frac{\mathfrak{a}}{2H(2H-1)} \|\varphi_j'\|_\infty e^{\epsilon T} T^{2H-1} \end{aligned}$$

by the change of variable formula for Young's integral. So,

$$\begin{aligned} \|\Phi_m(\varphi)'\|_\infty &\leq \mathfrak{m}_{f,f'} \sum_{j=1}^m (\|\varphi_j\|_\infty + \|\varphi_j'\|_\infty) \\ &\quad \times \left(\left| \frac{1}{NT} \sum_{i=1}^N (\bar{\varphi}_j(X_T^i) - \bar{\varphi}_j(X_0^i)) \right| + \frac{\mathfrak{a}}{2H(2H-1)} \|\varphi_j'\|_\infty e^{\epsilon T} T^{2H-1} \right) =: V_m. \end{aligned}$$

Moreover, since X is a stationary process, $f = f_0 = f_T$, and then

$$\begin{aligned} \mathbb{E} \left(\left| \frac{1}{NT} \sum_{i=1}^N (\bar{\varphi}_j(X_T^i) - \bar{\varphi}_j(X_0^i)) \right| \right) &\leq \text{var} \left(\frac{1}{NT} \sum_{i=1}^N (\bar{\varphi}_j(X_T^i) - \bar{\varphi}_j(X_0^i)) \right)^{\frac{1}{2}} \\ &\leq \frac{1}{N^{1/2}T} \mathbb{E}((\bar{\varphi}_j(X_T) - \bar{\varphi}_j(X_0))^2)^{\frac{1}{2}} \leq \frac{2\|\bar{\varphi}_j\|_\infty}{N^{1/2}T} \end{aligned}$$

for every $j \in \{1, \dots, m\}$. This leads to

$$\begin{aligned} \mathbb{E} \left(\sum_{j=1}^m (\|\varphi_j\|_\infty + \|\varphi_j'\|_\infty) \left| \frac{1}{NT} \sum_{i=1}^N (\bar{\varphi}_j(X_T^i) - \bar{\varphi}_j(X_0^i)) \right| \right) &\leq \frac{2}{N^{1/2}T} \sum_{j=1}^m (\|\varphi_j\|_\infty + \|\varphi_j'\|_\infty) \|\bar{\varphi}_j\|_\infty \\ &\leq \frac{2}{N^{1/2}T} (\mathfrak{L}(m)^{\frac{1}{2}} + \mathfrak{R}(m)^{\frac{1}{2}}) \mathfrak{J}(m)^{\frac{1}{2}}. \end{aligned}$$

Therefore, by Markov's inequality,

$$\begin{aligned} \mathbb{P}(\Delta_{m,\epsilon}^c) &\leq \frac{\mathbb{E}(|V_m|)}{\epsilon} \\ &\leq \frac{\mathfrak{m}_{f,f'}}{\epsilon} (\mathfrak{L}(m)^{\frac{1}{2}} + \mathfrak{R}(m)^{\frac{1}{2}}) \left(2N^{-\frac{1}{2}} \mathfrak{J}(m)^{\frac{1}{2}} T^{-1} + \frac{\mathfrak{a}}{2H(2H-1)} \mathfrak{R}(m)^{\frac{1}{2}} e^{\epsilon T} T^{2H-1} \right). \end{aligned}$$

Step 2. For every $\theta \in \mathbb{R}^m$,

$$\begin{aligned}\|DF_m(\theta)\|_{\text{op}}^2 &= \left(\sup_{z: \|z\|_m=1} \|DF_m(\theta) \cdot z\|_m \right)^2 \\ &= \sup_{z: \|z\|_m=1} \sum_{j=1}^m \left(\sum_{\ell=1}^m [\nabla F_m(\theta)_j]_{\ell} z_{\ell} \right)^2 \leq \sum_{j,\ell=1}^m [\nabla F_m(\theta)_j]_{\ell}^2.\end{aligned}$$

Then, for every $\varphi \in \mathcal{S}_{m,c}$,

$$\begin{aligned}\|DF_m(\theta^\varphi)\|_{\text{op}}^2 &\leq \sum_{j,\ell=1}^m \left[\frac{\mathfrak{a}}{NT} \sum_{i=1}^N \int_0^T \int_0^t \varphi'_j(X_t^i) \exp \left(\int_s^t \varphi'_j(X_u^i) du \right) \left(\int_s^t \varphi'_\ell(X_u^i) du \right) |t-s|^{2H-2} ds dt \right]^2 \\ &\leq \frac{\mathfrak{a}^2}{N^2 T^2} \mathfrak{R}(m)^2 e^{2\mathfrak{c}T} \left[N \int_0^T \int_0^t |t-s|^{2H-1} ds dt \right]^2 = \left(\frac{\mathfrak{a}}{2H(2H+1)} \right)^2 \mathfrak{R}(m)^2 e^{2\mathfrak{c}T} T^{4H}.\end{aligned}$$

So, by Markov's inequality,

$$\mathbb{P}(\Delta_{m,l}^c) \leq \frac{1}{\mathfrak{m}_f l} \mathbb{E} \left[\left(\sup_{\varphi \in \mathcal{S}_{m,c}} \|DF_m(\theta^\varphi)\|_{\text{op}} \right)^{\frac{1}{2}} \right] \leq \frac{1}{\mathfrak{m}_f l} \cdot \frac{\mathfrak{a}}{2H(2H+1)} \mathfrak{R}(m) e^{\mathfrak{c}T} T^{2H}.$$

Step 3 (conclusion). By the two previous steps, there exists a constant $\mathfrak{c}_1 > 0$, not depending on m , N and T (because $f = f_0$), such that

$$\begin{aligned}\mathbb{P}(\Delta_m^c) &\leq \mathbb{P}(\Delta_{m,c}^c) + \mathbb{P}(\Delta_{m,l}^c) \\ &\leq \mathfrak{c}_1 ((\mathfrak{L}(m)^{\frac{1}{2}} + \mathfrak{R}(m)^{\frac{1}{2}})(N^{-\frac{1}{2}} \mathfrak{J}(m)^{\frac{1}{2}} T^{-1} + \mathfrak{R}(m)^{\frac{1}{2}} e^{\mathfrak{c}T} T^{2H-1}) + \mathfrak{R}(m) e^{\mathfrak{c}T} T^{2H}).\end{aligned}$$

Moreover, by Assumption 4.4, if $T \in (0, 1]$, then there exists a constant $\mathfrak{c}_2 > 0$, not depending on m , N and T , such that

$$\mathbb{P}(\Delta_m^c) \leq \mathfrak{c}_2 \mathfrak{R}(m) (N^{-\frac{1}{2}} T^{-1} + T^{2H-1}).$$

□

Theorem 4.7. *Under Assumption 4.4, if $T \in (0, 1]$, b_0 fulfills the dissipativity condition (6), and if X^1, \dots, X^N are independent copies of the stationary solution X of Equation (1), then there exists a constant $\mathfrak{c}_{4.7} > 0$, not depending on m , N and T , such that*

$$\mathbb{E}(\|\tilde{b}_m^{\mathfrak{c},l} - b_0\|_{f^2}^2) \leq 2\|b_m - b_0\|_{f^2}^2 + \mathfrak{c}_{4.7} \mathfrak{R}(m) V(N, T),$$

where ($f = f_0$ doesn't depend on T , and)

$$V(N, T) := N^{-\frac{1}{2}} T^{-1} + T^{2H-1}.$$

Proof. The proof of Theorem 4.7 is dissected in two steps. The first one deals with the following regularity result on the map Φ_m :

$$(9) \quad \|\Phi_m(\varphi) - \Phi_m(\bar{\varphi})\|_{f^2} \leq \mathfrak{l}(m, T) \|\varphi' - \bar{\varphi}'\|_{\infty} ; \forall \varphi, \psi \in \mathcal{S}_{\mathfrak{c}}$$

with

$$\mathcal{S}_{\mathfrak{c}} = \{\varphi \in C^1(\mathbb{R}) : \|\varphi'\|_{\infty} \leq \mathfrak{c}\} \supset \mathcal{S}_{m,c} \quad \text{and} \quad \mathfrak{l}(m, T) = \frac{\mathfrak{a}}{2H(2H+1)} \mathfrak{R}(m)^{\frac{1}{2}} e^{\mathfrak{c}T} T^{2H}.$$

Then, Step 2 deals with the \mathbb{L}^2 -error bound on $\tilde{b}_m^{\mathfrak{c}, \mathfrak{l}}$.

Step 1. For any $\varphi, \bar{\varphi} \in \mathcal{S}_{\mathfrak{c}}$, since $(\varphi_1, \dots, \varphi_m)$ is an orthonormal family of $\mathbb{L}^2(\mathbb{R}, dx)$,

$$\|\Phi_m(\varphi) - \Phi_m(\bar{\varphi})\|_{f^2} = \|F(\varphi) - F(\bar{\varphi})\|_m,$$

where F is the map from $\mathcal{S}_{\mathfrak{c}}$ into \mathbb{R}^m defined by

$$F(\psi) := \left(\frac{\mathfrak{a}}{NT} \sum_{i=1}^N \int_0^T \int_0^t \varphi'_j(X_t^i) \exp \left(\int_s^t \psi'(X_u^i) du \right) |t-s|^{2H-2} ds dt \right)_{j \in \{1, \dots, m\}}$$

for every $\psi \in \mathcal{S}_{\mathfrak{c}}$. Moreover, for every $j \in \{1, \dots, m\}$,

$$\begin{aligned} |F(\varphi)_j - F(\bar{\varphi})_j| &\leq \frac{\mathfrak{a}}{NT} \sum_{i=1}^N \int_0^T \int_0^t |\varphi'_j(X_t^i)| \cdot |t-s|^{2H-2} \\ &\quad \times \left| \exp \left(\int_s^t \varphi'(X_u^i) du \right) - \exp \left(\int_s^t \bar{\varphi}'(X_u^i) du \right) \right| ds dt \\ &\leq \frac{\mathfrak{a}}{NT} \|\varphi'_j\|_{\infty} e^{\mathfrak{c}T} \left(N \int_0^T \int_0^t |t-s|^{2H-1} ds dt \right) \|\varphi' - \bar{\varphi}'\|_{\infty} \\ &= \frac{\mathfrak{a}}{2H(2H+1)} \|\varphi'_j\|_{\infty} e^{\mathfrak{c}T} T^{2H} \|\varphi' - \bar{\varphi}'\|_{\infty}. \end{aligned}$$

This leads to Inequality (9):

$$\|\Phi_m(\varphi) - \Phi_m(\bar{\varphi})\|_{f^2} \leq \mathfrak{c}_1 \mathfrak{R}(m)^{\frac{1}{2}} e^{\mathfrak{c}T} T^{2H} \|\varphi' - \bar{\varphi}'\|_{\infty} \quad \text{with} \quad \mathfrak{c}_1 = \frac{\mathfrak{a}}{2H(2H+1)}.$$

Step 2. On Δ_m , by Equality (2) and Inequality (9),

$$\begin{aligned} \|\tilde{b}_m - b_0\|_{f^2} &\leq \|\Phi_m(\tilde{b}_m) - \Phi_m(b_0)\|_{f^2} + \|\Phi_m(b_0) - b_0\|_{f^2} \\ &\leq \mathfrak{l}(m, T) \underbrace{\|\tilde{b}'_m - b'_0\|_{\infty}}_{\leq 2\mathfrak{c}} + \|\tilde{b}_m - b_0\|_{f^2}. \end{aligned}$$

Moreover, by Proposition 3.3,

$$\begin{aligned} \mathbb{E}(\|\hat{b}_m - b_0\|_{f^2}^2) &\leq \|b_m - b_0\|_{f^2}^2 + \frac{2}{N} \left(\|b_0\|_f^2 \mathfrak{L}(m) + \mathfrak{c}_{2.5} \sigma^2 \frac{\overline{\mathfrak{m}}_T}{T^{2-2H}} (\mathfrak{L}(m) + \mathfrak{R}(m)) \right) \\ &\leq \|b_m - b_0\|_{f^2}^2 + \frac{\mathfrak{c}_2}{N} (\mathfrak{L}(m) + T^{2H-2} \mathfrak{R}(m)) \end{aligned}$$

with

$$\mathfrak{c}_2 = 2 \max \left\{ \|b_0\|_f^2 ; \mathfrak{c}_{2.5} \sigma^2 \left[1 \vee \left(-\frac{H}{M} \right)^{2H} \right] \right\}.$$

So,

$$\begin{aligned} \mathbb{E}(\|\tilde{b}_m^{\mathfrak{c}, \mathfrak{l}} - b_0\|_{f^2}^2) &= \mathbb{E}(\|\tilde{b}_m - b_0\|_{f^2}^2 \mathbf{1}_{\Delta_m}) + \|b_0\|_{f^2}^2 \mathbb{P}(\Delta_m^c) \\ &\leq 8\mathfrak{c}^2 \mathfrak{l}(m, T)^2 + 2\|b_m - b_0\|_{f^2}^2 + 2\mathfrak{c}_2 N^{-1} (\mathfrak{L}(m) + T^{2H-2} \mathfrak{R}(m)) \\ &\quad + \mathfrak{c}_{4.6,2} \|b_0\|_{f^2}^2 \mathfrak{R}(m) (N^{-\frac{1}{2}} T^{-1} + T^{2H-1}) \\ &\leq 2\|b_m - b_0\|_{f^2}^2 + \mathfrak{c}_3 \mathfrak{R}(m) (N^{-\frac{1}{2}} T^{-1} + T^{2H-1}), \end{aligned}$$

where \mathfrak{c}_3 is a positive constant not depending on m , N and T (because $f = f_0$). \square

Remarks:

- Consider $T(N) := N^{-1/(4H)}$. Then, for every $t \in (0, 1]$,

$$V(N, T(N)) = 2N^{-\frac{2H-1}{4H}} \leq V(N, t),$$

and by Theorem 4.7 with $T = T(N)$,

$$\mathbb{E}(\|\tilde{b}_m^{\mathfrak{c}, \mathfrak{l}} - b_0\|_{f^2}^2) \leq 2\|b_m - b_0\|_{f^2}^2 + 2\mathfrak{c}_{4.7}\Re(m)N^{-\frac{2H-1}{4H}}.$$

- Assume that $(\varphi_1, \dots, \varphi_m)$ is the trigonometric basis defined in Example 4.5, and that $(b_0 f)|_I$ belongs to the Fourier-Sobolev space

$$\mathbb{W}_2^\beta(I) := \left\{ \varphi : I \rightarrow \mathbb{R} \text{ } \beta \text{ times differentiable on } I : \int_I \varphi^{(\beta)}(x)^2 dx < \infty \right\}$$

with $\beta \in \mathbb{N}^*$. By DeVore and Lorentz [5], Corollary 2.4 p. 205, there exists a constant $\mathfrak{c}_{\beta, \ell, \mathbf{x}} > 0$, not depending on m , such that

$$\|(bf)_m - b_0 f\|^2 \leq \mathfrak{c}_{\beta, \ell, \mathbf{x}} m^{-2\beta}.$$

Then,

$$\mathbb{E}(\|\tilde{b}_m^{\mathfrak{c}, \mathfrak{l}} - b_0\|_{f^2}^2) \leq \bar{\mathfrak{c}}_{4.7}(m^{-2\beta} + m^3 N^{-\frac{2H-1}{4H}}),$$

where $\bar{\mathfrak{c}}_{4.7} > 0$ is a positive constant not depending on m and N . Therefore, the bias-variance tradeoff is reached for

$$m \asymp N^{\frac{2H-1}{4H(3+2\beta)}}.$$

- Thanks to a well-known consequence of Picard's fixed point theorem, for every $n \in \mathbb{N}^*$ and $\varphi \in \mathcal{S}_{m, \mathfrak{c}}$,

$$\| \underbrace{(\Phi_m \circ \dots \circ \Phi_m)}_{n \text{ times}}(\varphi) - \tilde{b}_m \|_{f^2} \leq \frac{\mathfrak{l}^n}{1 - \mathfrak{l}} \|\Phi_m(\varphi) - \varphi\|_{f^2} \quad \text{on } \Delta_m.$$

Then, the estimator $\tilde{b}_m^{\mathfrak{c}, \mathfrak{l}}$ can be approximated by $\tilde{b}_{m,n}^{\mathfrak{c}, \mathfrak{l}} := \tilde{b}_{m,n} \mathbf{1}_{\Delta_m}$, where the sequence $(\tilde{b}_{m,n})_{n \in \mathbb{N}}$ is defined by $\tilde{b}_{m,0} \in \mathcal{S}_{m, \mathfrak{c}}$ and

$$\tilde{b}_{m,n+1} = \Phi_m(\tilde{b}_{m,n}) ; n \in \mathbb{N}.$$

- In practice, the density function f is unknown. So, f needs to be estimated, for instance by the projection estimator

$$\hat{f}_m(x) := \sum_{j=1}^m \left(\frac{1}{NT} \sum_{i=1}^N \int_0^T \varphi_j(X_s^i) ds \right) \varphi_j(x) ; x \in I.$$

Then, for numerical purposes, $(\tilde{b}_{m,n})_{n \in \mathbb{N}}$ should be replaced by the sequence $(\bar{b}_{m,n})_{n \in \mathbb{N}}$ such that $\bar{b}_{m,0} \in \mathcal{S}_{m, \mathfrak{c}}$ and

$$\bar{b}_{m,n+1} = \hat{\Phi}_m(\bar{b}_{m,n}) ; n \in \mathbb{N},$$

where

$$\hat{\Phi}_m(.) := \frac{f}{\hat{f}_m} \Phi_m(.).$$

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[†]LABORATOIRE MODAL'X, UNIVERSITÉ PARIS NANTERRE, NANTERRE, FRANCE
Email address: nmarie@parisnanterre.fr