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Quasi-rational solutions to the cylindrical nonlinear Schrödinger equation

⁺Pierre Gaillard, ⁺ Université de Bourgogne Franche Comté, Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,

Abstract

Quasi-rational solutions to the cylindrical nonlinear Schrödinger equation (CNLS) in terms of wronskians and Fredholm determinants of order $2N$ depending on $2N - 2$ real parameters are given. We get multi-parametric families of quasi-rational solutions to the CNLS equation and we construct explicitly solutions of order 1 to 5.

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Keywords : Fredholm determinants, cylindrical NLS equation, wronskians.

1 Introduction

We consider the one dimensional cylindrical nonlinear Schrödinger equation (CNLS) which can be written in the form

$$iu_{\tilde{t}} + u_{\tilde{x}\tilde{x}} + i\frac{u}{2\tilde{t}} - 2|u|^2u = 0, \quad (1)$$

where u is a function of \tilde{x} and \tilde{t} and the subscripts denote partial derivatives. This equation plays a role in the theory of light wave envelopes in dispersive media with nonlinear refractive index [1, 2]. In particular, it describes the cylindrically diverging quasi-plane envelope waves in a nonlinear medium.

Different approaches have been made to solve this equation. For example, Lax pair for this equation has been derived through the Painlevé analysis [3]. The one-soliton solution has been given via the Bäcklund transformation corresponding to the Lax pair.

It is known that there exist solutions of (1) which have a singularity in finite time and there has been much interest in the determination of the structure of this singularity: we can quote for example, the works of Ablowitz and Segur [4] in 1979 , Landman [5] in 1988, Smirnov and Fraiman [6] in 1991...

Optical solitons find application in distortionless signal transmission along optical fibres. Such applications occur in long-distance data transmission.

Here we present solutions to the cylindrical nonlinear Schrödinger equation (CNLS) of order N depending on $2N - 2$ real parameters in terms of wronskians and Fredholm determinants. Families of quasi-rational solutions to the CNLS equation are obtained. We construct explicitly solutions of order 1 to 5.

2 Different representations of quasi-rational solutions to the CNLS equation

2.1 Quasi-rational solutions of the CNLS equation in terms of Fredholm determinant

We use the following notations. The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of the parameters $\lambda_\nu, 1 \leq \nu \leq 2N$; they are defined by the formulas :

$$\begin{aligned} \kappa_\nu &= 2\sqrt{1 - \lambda_\nu^2}, & \delta_\nu &= \kappa_\nu \lambda_\nu, & \gamma_\nu &= \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}; \\ x_{r,\nu} &= (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, & r &= 1, 3. \end{aligned} \quad (2)$$

The parameters $-1 < \lambda_\nu < 1, \nu = 1, \dots, 2N$, are real numbers such that

$$\begin{aligned} -1 < \lambda_{N+1} < \lambda_{N+2} < \dots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 1 \\ \lambda_{N+j} &= -\lambda_j, \quad j = 1, \dots, N. \end{aligned} \quad (3)$$

Complex numbers $e_\nu, 1 \leq \nu \leq 2N$ are defined by :

$$\begin{aligned} e_j &= i \sum_{l=1}^{N-1} a_l(j\epsilon)^{2l+1} - \sum_{l=1}^{N-1} b_l(j\epsilon)^{2l+1}, \\ e_{j+N} &= i \sum_{l=1}^{N-1} a_l(j\epsilon)^{2l+1} + \sum_{l=1}^{N-1} b_l(j\epsilon)^{2l+1}, \\ 1 \leq j &\leq N - 1. \end{aligned} \quad (4)$$

$\epsilon, a_\nu, b_\nu, \nu = 1 \dots 2N$ are arbitrary real numbers.

Let I be the unit matrix, and

$$\epsilon_j = j \quad 1 \leq j \leq N, \quad \epsilon_j = N + j, \quad N + 1 \leq j \leq 2N. \quad (5)$$

We consider the following matrix $D_r = (d_{jk}^{(r)})_{1 \leq j, k \leq 2N}$ defined by :

$$d_{\nu\mu}^{(r)} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \quad (6)$$

With these notations, the solution to the CNLS equation takes the form

Theorem 2.1 *Let v be the following expression*

$$v(x, t) = \frac{\det(I + D_3(x, t))}{\det(I + D_1(x, t))} e^{2it - i\varphi}, \quad (7)$$

then the function u defined by

$$u(X, T) = \frac{1}{T} v\left(-\frac{X}{T}, -\frac{1}{T}\right) e^{\frac{iX^2}{4T}} \quad (8)$$

is a solution to the cylindrical CNLS equation (1) depending on $2N - 1$ real parameters $a_j, b_j, \epsilon, 1 \leq j \leq N - 1$ with the matrix $D_r = (d_{jk}^{(r)})_{1 \leq j, k \leq 2N}$ defined by

$$d_{\nu\mu}^{(r)} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

where $\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$ being defined in(2), (3) and (4).

2.2 Wronskian representation

We define the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N + 1 \leq \nu \leq 2N, \quad r = 1, 3, \quad (9)$$

with the arguments

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2, \quad 1 \leq \nu \leq 2N. \quad (10)$$

The functions $\phi_{r,\nu}$ are defined by

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N + 1 \leq \nu \leq 2N, \quad r = 1, 3, \quad (11)$$

We denote $W_r(y)$ the wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}]. \quad (12)$$

We get the following result :

Theorem 2.2 *Let v be the expression defined by*

$$v(x, t) = \frac{W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0)} e^{2it - i\varphi},$$

then the function u defined by

$$u(X, T) = \frac{1}{T} v\left(-\frac{X}{T}, -\frac{1}{T}\right) e^{\frac{iX^2}{4T}} \quad (13)$$

is a solution to the cylindrical CNLS equation (1) depending on $2N - 1$ real parameters $a_j, b_j, \epsilon, 1 \leq j \leq N - 1$ with ϕ_ν^r defined in (11)

$$\begin{aligned} \phi_{r,\nu} &= \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad 1 \leq \nu \leq N, \\ \phi_{r,\nu} &= \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu/2), \quad N + 1 \leq \nu \leq 2N, \quad r = 1, 3, \end{aligned}$$

$\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$ being defined in(2), (3) and (4).

Another representation of the solutions to the CNLS equation depending only on terms γ_ν , $1 \leq \nu \leq 2N$ can be also given. From the relations (2), we can express the terms κ_ν , δ_ν and $x_{r,\nu}$ in function of γ_ν , for $1 \leq \nu \leq 2N$ and we obtain :

$$\begin{aligned} \kappa_j &= \frac{4\gamma_j}{(1+\gamma_j^2)}, & \delta_j &= \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, & x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, & 1 \leq j \leq N, \\ \kappa_j &= \frac{4\gamma_j}{(1+\gamma_j^2)}, & \delta_j &= -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, & x_{r,j} &= (r-1) \ln \frac{\gamma_j + i}{\gamma_j - i}, & N+1 \leq j \leq 2N. \end{aligned} \quad (14)$$

We have the following result

Theorem 2.3 *Let v be the expression defined by*

$$v(x, t) = \frac{\det[(\partial_y^{\mu-1} \tilde{\phi}_{3,\nu}(0))_{\nu, \mu \in [1, \dots, 2N]}]}{\det[(\partial_y^{\mu-1} \tilde{\phi}_{1,\nu}(0))_{\nu, \mu \in [1, \dots, 2N]}]} e^{2it - i\varphi}, \quad (15)$$

then the function defined by

$$u(X, T) = \frac{1}{T} v\left(-\frac{X}{T}, -\frac{1}{T}\right) e^{\frac{iX^2}{4T}} \quad (16)$$

is a solution to the cylindrical CNLS equation (1) depending on $2N - 1$ real parameters a_j , b_j , ϵ , $1 \leq j \leq N - 1$. The functions $\tilde{\phi}_{r,\nu}$ are defined by

$$\begin{aligned} \tilde{\phi}_{r,j}(y) &= \sin \left(\frac{2\gamma_j}{(1+\gamma_j^2)} x + i \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2} t - i \frac{(r-1)}{2} \ln \frac{\gamma_j - i}{\gamma_j + i} + \gamma_j y - i e_j \right), \\ \tilde{\phi}_{r,N+j}(y) &= \cos \left(\frac{2\gamma_j}{(1+\gamma_j^2)} x - i \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2} t + i \frac{(r-1)}{2} \ln \frac{\gamma_j - i}{\gamma_j + i} + \frac{1}{\gamma_j} y - i e_{N+j} \right), \end{aligned} \quad (17)$$

where $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}$, $1 \leq j \leq N$.

λ_j is an arbitrary real parameter such that $0 < \lambda_j < 1$, $\lambda_{N+j} = -\lambda_j$, $1 \leq j \leq N$.

The terms e_ν are defined by (4),

where a_j and b_j are arbitrary real numbers, $1 \leq j \leq N - 1$.

2.3 Families of quasi-rational solutions of the CNLS equation in terms of a quotient of two determinants

We define the following notations :

$$X_\nu = \kappa_\nu x / 2 + i \delta_\nu t - i x_{3,\nu} / 2 - i e_\nu / 2,$$

$$Y_\nu = \kappa_\nu x / 2 + i \delta_\nu t - i x_{1,\nu} / 2 - i e_\nu / 2,$$

for $1 \leq \nu \leq 2N$, with κ_ν , δ_ν , $x_{r,\nu}$ defined in (2).

Parameters e_ν are defined by (4).

We define the following functions :

$$\begin{aligned}\varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k,\end{aligned}\quad (18)$$

for $1 \leq k \leq N$, and

$$\begin{aligned}\varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k},\end{aligned}\quad (19)$$

for $1 \leq k \leq N$.

We consider the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k .

$$\begin{aligned}\psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k,\end{aligned}\quad (20)$$

for $1 \leq k \leq N$, and

$$\begin{aligned}\psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k},\end{aligned}\quad (21)$$

for $1 \leq k \leq N$.

Then we have the following result

Theorem 2.4 *Let v be the expression defined by*

$$v(x, t) = \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} e^{2it - i\varphi}, \quad (22)$$

then the function defined by

$$u(X, T) = \frac{1}{T} v\left(-\frac{X}{T}, -\frac{1}{T}\right) e^{\frac{iX^2}{4T}} \quad (23)$$

is a quasi-rational solution of the cylindrical CNLS equation (1) depending on $2N - 2$ real parameters a_j, b_j , $1 \leq j \leq N - 1$, where

$$\begin{aligned}n_{j1} &= \varphi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= \varphi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= \psi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= \psi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ & & & 2 \leq k \leq N, \quad 1 \leq j \leq 2N\end{aligned}$$

The functions φ and ψ are defined in (18), (19), (20), (21).

3 Some examples of quasi-rational solutions to the CNLS equation

Within fluid physics-based models, one family of nonlinear equations is recurrent known as the cylindrical nonlinear Schrödinger equation. It describes very different kinds of processes and physical mechanisms.

The previous representations of the solutions to the CNLS equation give efficient methods to construct and study them. In the following, we give explicit solutions of order 1 until 5.

3.1 Solutions of order 1 to the CNLS equation

Example 3.1 *The function $v_k(x, t)$ defined by*

$$v_k(X, T) = \frac{(4X^2 - 3T^2 + 16iT + 16) e^{\frac{1/4 i(-s+x^2)}{T}}}{T(4X^2 + 16 + T^2)} \quad (24)$$

is a solution to the (CNLS) equation (1).

3.2 Solutions of order 2 to the CNLS equation

Example 3.2 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{N(X, T)}{D(X, T)}, \quad (25)$$

with

$$N(X, T) = -(-4096 - 384T^3b_1 + 180X^2T^4 + 144X^4T^2 + 5760X^2T^2 - 3072X^2 - 64X^6 - 768X^4 + 8448T^2 - 45T^6 + 1872T^4 - 6144iX^2T - 768iX^4T + 1152iX^2T^3 + 36ib_1T^6 - 576ib_1T^4 + 72b_1T^5 + 720iT^5 - 1536iT^3 - 12288iT - 9a_1^2T^6 - 9b_1^2T^6 - 48X^3a_1T^3 - 108Xa_1T^5 + 576Xa_1T^3 + 288X^2b_1T^3 + 576iXa_1T^4 + 144iX^2b_1T^4) e^{\frac{1/4 i(-s+x^2)}{T}}$$

and

$$D(X, T) = T(9T^6 + 1584T^4 + 108X^2T^4 + 4096 + 384T^3b_1 + 3072X^2 + 48X^3a_1T^3 + 9a_1^2T^6 + 768X^4 + 216b_1T^5 - 36Xa_1T^5 - 1152X^2T^2 - 576Xa_1T^3 - 288X^2b_1T^3 + 9b_1^2T^6 + 64X^6 + 6912T^2 + 48X^4T^2)$$

is a solution to the (CNLS) equation (1).

3.3 Solutions of order 3 to the CNLS equation

Example 3.3 *The function $v_k(x, t)$ defined by*

$$v_k(x, t) = \frac{N(X, T)}{D(X, T)} \quad (26)$$

with

$$N(X, T) = (67108864 - 64800Xa_1b_1b_2T^{11} + 345600ib_1a_1X^3T^9 - 129600ib_1Xa_2T^{11} +$$

$$\begin{aligned}
& 129600 ia_1 X b_2 T^{11} + 4147200 ib_1 X a_1 T^9 + 22118400 ib_1 X a_1 T^7 - 11059200 ib_1 a_1 X^3 T^7 - \\
& 2073600 ia_1 X b_2 T^9 + 2073600 ib_1 X a_2 T^9 + 518400 i X a_1 b_1^2 T^{10} + 276480 i X^5 a_1 b_1 T^7 - \\
& 172800 i X^3 a_1 b_2 T^9 + 172800 i X^3 a_2 b_1 T^9 + 129600 i X^2 a_1^2 b_1 T^{10} + 64800 ia_1 a_2 b_1 T^{12} - \\
& 946667520 iT^7 + 629145600 iT^3 - 1368391680 iT^5 + 402653184 iT + 1814400 iT^{11} - \\
& 15206400 iT^9 + 8100 a_1^4 T^{12} - 24300 b_2^2 T^{12} - 24300 a_2^2 T^{12} - 6075 b_1^2 T^{12} - 6075 a_1^2 T^{12} + \\
& 8100 b_1^4 T^{12} - 5529600 ia_1^2 X^2 T^7 + 16588800 ib_1^2 X^2 T^7 + 2764800 i X^4 b_2 T^6 + 907200 ia_1 a_2 T^{11} + \\
& 907200 ib_1 b_2 T^{11} - 2764800 ia_1 a_2 T^9 - 2764800 ib_1 b_2 T^9 - 518400 ia_1^2 b_1 T^{10} - 172800 i X^4 b_2 T^8 - \\
& 259200 i X^4 b_1 T^8 - 388800 i X^2 b_2 T^{10} - 184320 i X^8 b_1 T^4 + 16200 ia_1^2 b_1 T^{12} + 32400 ib_1^2 b_2 T^{12} - \\
& 32400 ia_1^2 b_2 T^{12} - 230400 i X^6 b_2 T^6 + 129600 i X^2 b_1^3 T^{10} + 712800 ib_1 X^2 T^{10} - \\
& 437760 ib_1 X^6 T^6 - 1658880 i X^5 a_2 T^6 + 2073600 i X^4 a_1^2 T^7 - 691200 i X^4 b_1^2 T^7 + \\
& 518400 i X a_1^3 T^{10} - 188743680 i X a_1 T^4 - 94371840 i X^3 a_1 T^4 - 29491200 i X^4 b_1 T^4 - \\
& 11796480 i X^5 a_1 T^4 - 5898240 ib_1 X^6 T^4 + 2332800 i X a_1 T^{10} - 14515200 ib_1 X^2 T^8 + \\
& 41472000 i X a_1 T^8 - 2592000 i X a_2 T^{10} + 7603200 i X^3 a_1 T^8 + 5529600 i X a_2 T^8 + \\
& 293068800 ib_1 X^2 T^6 + 1036800 ib_1^2 X^2 T^9 + 1036800 ia_1^2 X^2 T^9 + 314081280 i X a_1 T^6 - \\
& 70041600 i X^3 a_1 T^6 - 276480 i X^5 a_1 T^6 + 1382400 i X^3 a_2 T^8 - 4147200 i X^2 b_2 T^8 - \\
& 1382400 i X^4 b_1 T^6 + 44236800 i X a_2 T^6 + 33177600 i X^2 b_2 T^6 - 7372800 i X^3 a_2 T^6 - \\
& 1382400 b_1 b_2 T^8 - 1382400 a_1 a_2 T^8 + 2980800 a_1 a_2 T^{10} + 2980800 b_1 b_2 T^{10} + 259200 X^4 b_1^2 T^8 - \\
& 259200 X^4 a_1^2 T^8 + 234900 a_1^2 X^2 T^{10} + 234900 X^2 b_1^2 T^{10} + 32400 X^2 a_2^2 T^{10} + \\
& 32400 X^2 b_2^2 T^{10} + 34560 X^6 a_1^2 T^6 + 80640 X^6 b_1^2 T^6 + 8294400 X^2 b_1^2 T^6 - 2764800 a_1^2 X^2 T^6 + \\
& 2073600 X^4 a_1^2 T^6 - 691200 X^4 b_1^2 T^6 + 518400 a_1^3 X T^9 - 47185920 a_1 X T^3 - 31457280 a_1 X^3 T^3 - \\
& 5898240 a_1 X^5 T^3 - 30412800 X a_2 T^7 + 6912000 X^3 a_2 T^7 + 340623360 a_1 X T^5 + \\
& 23961600 a_1 X^3 T^5 + 8570880 a_1 X^5 T^5 - 6739200 X a_2 T^9 + 38016000 a_1 X^3 T^7 - \\
& 73267200 a_1 X T^7 - 12182400 a_1 X T^9 - 1658880 X^5 a_2 T^5 + 14745600 X a_2 T^5 - 3686400 X^3 a_2 T^5 - \\
& 3110400 X^2 b_1^2 T^8 + 5184000 a_1^2 X^2 T^8 - 64800 b_2 a_1^2 T^{11} - 345600 b_1 a_1^2 T^9 + 550800 b_1 a_1^2 T^{11} + \\
& 64800 b_2 b_1^2 T^{11} - 368640 X^8 b_1 T^3 - 24883200 X^2 b_2 T^7 - 345600 X^4 b_2 T^7 + 2810880 X^6 b_1 T^5 - \\
& 777600 X^2 b_2 T^9 - 109209600 b_1 X^2 T^7 + 2246400 X^4 b_1 T^7 + 7646400 b_1 X^2 T^9 + 259200 X^2 b_1^3 T^9 + \\
& 1843200 X^4 b_2 T^5 - 460800 X^6 b_2 T^5 + 46080 X^7 a_2 T^5 - 43200 X^3 a_1^3 T^9 + 61440 X^9 a_1 T^3 + \\
& 69120 X^7 a_1 T^5 + 311040 X^5 a_2 T^7 - 145800 a_1^3 X T^{11} + 812160 a_1 X^5 T^7 + 216000 X^3 a_2 T^9 - \\
& 712800 a_1 X^3 T^9 + 162000 X a_2 T^{11} - 437400 a_1 X T^{11} - 16588800 ib_1^2 T^9 - 11059200 ia_1^2 T^9 + \\
& 129600 ib_2^2 T^{11} + 129600 ia_2^2 T^{11} + 162000 ib_1^2 T^{11} + 5702400 ib_2 T^{10} - 208742400 ib_1 T^8 + \\
& 162000 ia_1^2 T^{11} - 154828800 i X^2 T^7 + 7257600 i X^2 T^9 - 8294400 i X^4 T^7 - 330301440 i X^4 T^3 - \\
& 55050240 i X^6 T^3 - 1474560 i X^8 T^3 - 377487360 i X^2 T^3 - 44236800 i X^4 T^5 + 672399360 i X^2 T^5 - \\
& 3686400 i X^6 T^5 + 503316480 i X^2 T + 251658240 i X^4 T + 62914560 i X^6 T + 7864320 i X^8 T + \\
& 393216 i X^{10} T - 20643840 ib_2 T^6 - 518400 ib_1^3 T^{10} + 141557760 ib_1 T^4 + 6635520 ib_1^2 T^7 + \\
& 15482880 ia_1^2 T^7 - 2851200 ib_1 T^{10} - 13824000 ib_2 T^8 - 13271040 ib_1 T^6 + 113400 ib_1 T^{12} - \\
& 32400 ib_2 T^{12} + 16200 ib_1^3 T^{12} - 19353600 a_1^2 T^8 + 745200 b_1 T^{11} + 550800 b_1^3 T^{11} + \\
& 20390400 b_2 T^9 - 64800 b_2 T^{11} - 345600 b_1^3 T^9 - 16588800 b_1^2 T^8 + 10530000 b_1^2 T^{10} + \\
& 4309200 a_1^2 T^{10} + 129600 a_2^2 T^{10} + 129600 b_2^2 T^{10} + 16200 a_1^2 b_1^2 T^{12} + 8100 b_1 b_2 T^{12} + \\
& 8100 a_1 a_2 T^{12} + 13271040 T^5 X^2 b_2 - 3932160 T^3 X^6 b_1 + 13824000 T^5 X^4 b_1 + 152616960 b_1 X^2 T^5 - \\
& 11796480 X^4 T^3 b_1 + 36864000 X^4 T^4 - 342835200 X^2 T^6 - 424673280 X^4 T^2 + 224133120 X^2 T^4 + \\
& 5160960 T^6 a_1^2 - 5898240 T^5 b_2 - 225054720 T^7 b_1 - 185794560 T^5 b_1 + 31457280 T^3 b_1 + \\
& 12165120 T^7 b_2 + 2211840 T^6 b_1^2 + 86572800 T^9 b_1 - 849346560 X^2 T^2 + 453600 T^{10} X^2 - \\
& 73728 T^2 X^{10} + 907200 T^8 X^4 - 691200 T^6 X^6 - 230400 T^4 X^8 + 73612800 X^2 T^8 + \\
& 2764800 X^4 T^6 - 7372800 X^8 T^2 + 8110080 X^6 T^4 - 90439680 X^6 T^2 + 3932160 X^8 + \\
& 20971520 X^6 - 578813952 T^2 - 2135162880 T^4 + 9072000 T^{10} + 335232000 T^8 - \\
& 610467840 T^6 - 56700 T^{12} + 62914560 X^4 + 100663296 X^2 + 16384 X^{12} + 393216 X^{10} -
\end{aligned}$$

$$\begin{aligned}
& 2073600 a_1 X b_1 T^{10} + 552960 X^5 a_1 b_1 T^6 + 129600 a_2 b_1 a_1 T^{11} - 43200 X^3 a_1 b_1^2 T^9 - \\
& 145800 a_1 X b_1^2 T^{11} + 32400 X a_2 b_1^2 T^{11} - 32400 X a_1^2 a_2 T^{11} + 97200 b_1 X^2 b_2 T^{10} + \\
& 97200 a_1 X^2 a_2 T^{10} + 86400 X^4 a_1 a_2 T^8 + 86400 X^4 b_1 b_2 T^8 + 518400 a_1 X b_1^2 T^9 + \\
& 259200 X^2 a_1^2 b_1 T^9 - 259200 X a_2 b_1 T^{10} + 259200 a_1 X b_2 T^{10} - 8294400 a_1 X b_1 T^8 + \\
& 3456000 a_1 X^3 b_1 T^8 + 1382400 X a_2 b_1 T^8 - 1382400 a_1 X b_2 T^8 - 345600 X^3 a_1 b_2 T^8 + \\
& 345600 X^3 a_2 b_1 T^8 + 8847360 a_1 X b_1 T^6 - 7372800 a_1 X^3 b_1 T^6) e^{\frac{1/4 i(-s+X^2)}{T}}
\end{aligned}$$

and

$$\begin{aligned}
D(X, T) = & T(67108864 - 64800 X a_1 b_1 b_2 T^{11} + 8100 a_1^4 T^{12} + 8100 b_2^2 T^{12} + \\
& 8100 a_2^2 T^{12} + 34425 b_1^2 T^{12} + 34425 a_1^2 T^{12} + 8100 b_1^4 T^{12} - 1382400 b_1 b_2 T^8 - \\
& 1382400 a_1 a_2 T^8 + 907200 a_1 a_2 T^{10} + 907200 b_1 b_2 T^{10} + 86400 X^4 b_1^2 T^8 + 259200 X^4 a_1^2 T^8 - \\
& 24300 a_1^2 X^2 T^{10} - 24300 X^2 b_1^2 T^{10} + 32400 X^2 a_2^2 T^{10} + 32400 X^2 b_2^2 T^{10} + 34560 X^6 a_1^2 T^6 + \\
& 80640 X^6 b_1^2 T^6 + 8294400 X^2 b_1^2 T^6 - 2764800 a_1^2 X^2 T^6 + 2073600 X^4 a_1^2 T^6 - \\
& 691200 X^4 b_1^2 T^6 + 518400 a_1^3 X T^9 - 47185920 a_1 X T^3 - 31457280 a_1 X^3 T^3 - 5898240 a_1 X^5 T^3 + \\
& 24883200 X a_2 T^7 + 1382400 X^3 a_2 T^7 + 10321920 a_1 X T^5 - 94003200 a_1 X^3 T^5 - \\
& 276480 a_1 X^5 T^5 + 5702400 X a_2 T^9 - 14515200 a_1 X^3 T^7 + 53913600 a_1 X T^7 - 1814400 a_1 X T^9 - \\
& 1658880 X^5 a_2 T^5 + 14745600 X a_2 T^5 - 3686400 X^3 a_2 T^5 + 9331200 X^2 b_1^2 T^8 + \\
& 1036800 a_1^2 X^2 T^8 - 64800 b_2 a_1^2 T^{11} - 345600 b_1 a_1^2 T^9 + 291600 b_1 a_1^2 T^{11} + 64800 b_2 b_1^2 T^{11} - \\
& 368640 X^8 b_1 T^3 + 8294400 X^2 b_2 T^7 + 1036800 X^4 b_2 T^7 - 138240 X^6 b_1 T^5 + 1296000 X^2 b_2 T^9 + \\
& 95385600 b_1 X^2 T^7 + 1555200 X^4 b_1 T^7 - 1684800 b_1 X^2 T^9 + 259200 X^2 b_1^3 T^9 + 1843200 X^4 b_2 T^5 - \\
& 460800 X^6 b_2 T^5 + 46080 X^7 a_2 T^5 - 43200 X^3 a_1^3 T^9 + 61440 X^9 a_1 T^3 + 69120 X^7 a_1 T^5 - \\
& 103680 X^5 a_2 T^7 - 16200 a_1^3 X T^{11} - 362880 a_1 X^5 T^7 - 129600 X^3 a_2 T^9 + 151200 a_1 X^3 T^9 + \\
& 32400 X a_2 T^{11} - 113400 a_1 X T^{11} + 874800 b_1 T^{11} + 291600 b_1^3 T^{11} + 1036800 b_2 T^9 - \\
& 324000 b_2 T^{11} - 345600 b_1^3 T^9 - 8294400 b_1^2 T^8 + 4309200 b_1^2 T^{10} + 2235600 a_1^2 T^{10} + \\
& 129600 a_2^2 T^{10} + 129600 b_2^2 T^{10} + 16200 a_1^2 b_1^2 T^{12} - 24300 b_1 b_2 T^{12} - 24300 a_1 a_2 T^{12} + \\
& 13271040 T^5 X^2 b_2 - 3932160 T^3 X^6 b_1 - 15667200 T^5 X^4 b_1 + 152616960 b_1 X^2 T^5 - \\
& 11796480 X^4 T^3 b_1 - 22118400 X^4 T^4 + 320716800 X^2 T^6 + 15728640 X^4 T^2 + 884736000 X^2 T^4 + \\
& 5160960 T^6 a_1^2 - 5898240 T^5 b_2 - 32624640 T^7 b_1 + 97320960 T^5 b_1 + 31457280 T^3 b_1 - \\
& 18800640 T^7 b_2 + 2211840 T^6 b_1^2 + 35424000 T^9 b_1 + 283115520 X^2 T^2 + 194400 T^{10} X^2 + \\
& 24576 T^2 X^{10} + 216000 T^8 X^4 + 599040 T^6 X^6 + 138240 T^4 X^8 - 9331200 X^2 T^8 + \\
& 13824000 X^4 T^6 - 1474560 X^8 T^2 + 2211840 X^6 T^4 - 11796480 X^6 T^2 + 3932160 X^8 + \\
& 20971520 X^6 + 528482304 T^2 + 979107840 T^4 + 5961600 T^{10} + 147225600 T^8 + \\
& 250675200 T^6 + 8100 T^{12} + 62914560 X^4 + 100663296 X^2 + 16384 X^{12} + 393216 X^{10} + \\
& 552960 X^5 a_1 b_1 T^6 + 129600 a_2 b_1 a_1 T^{11} - 43200 X^3 a_1 b_1^2 T^9 - 16200 a_1 X b_1^2 T^{11} + \\
& 32400 X a_2 b_1^2 T^{11} - 32400 X a_1^2 a_2 T^{11} + 97200 b_1 X^2 b_2 T^{10} + 97200 a_1 X^2 a_2 T^{10} + \\
& 86400 X^4 a_1 a_2 T^8 + 86400 X^4 b_1 b_2 T^8 + 518400 a_1 X b_1^2 T^9 + 259200 X^2 a_1^2 b_1 T^9 + \\
& 777600 X a_2 b_1 T^{10} - 777600 a_1 X b_2 T^{10} + 13824000 a_1 X b_1 T^8 - 2073600 a_1 X^3 b_1 T^8 + \\
& 1382400 X a_2 b_1 T^8 - 1382400 a_1 X b_2 T^8 - 345600 X^3 a_1 b_2 T^8 + 345600 X^3 a_2 b_1 T^8 + \\
& 8847360 a_1 X b_1 T^6 - 7372800 a_1 X^3 b_1 T^6)
\end{aligned}$$

is a solution to the (CNLS) equation (1).

3.4 Solutions of order 4 to the CNLS equation

In what follows, we give only solutions without parameters.

Example 3.4 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{N(X, T)}{D(X, T)} \quad (27)$$

with

$$\begin{aligned} N(X, T) = & -(-1048576 X^{20} + 7864320 T^2 X^{18} - 41943040 iT X^{18} - 41943040 X^{18} - 1509949440 iT X^{16} + \\ & 44236800 T^4 X^{16} - 754974720 X^{16} + 1415577600 T^2 X^{16} + 283115520 iT^3 X^{16} + 37748736000 T^2 X^{14} + \\ & 25165824000 iT^3 X^{14} + 1415577600 iT^5 X^{14} - 4246732800 T^4 X^{14} - 24159191040 iT X^{14} - 8053063680 X^{14} + \\ & 324403200 T^6 X^{14} + 9083289600 iT^7 X^{12} - 56371445760 X^{12} - 10734796800 T^6 X^{12} - 225485783040 iT X^{12} - \\ & 13212057600 iT^5 X^{12} + 465064427520 T^2 X^{12} + 1354752000 T^8 X^{12} - 235615027200 T^4 X^{12} + \\ & 493250150400 iT^3 X^{12} + 4396972769280 iT^3 X^{10} + 9909043200 T^6 X^{10} - 3223742054400 T^4 X^{10} - \\ & 1352914698240 iT X^{10} - 1088673546240 iT^5 X^{10} + 32514048000 iT^9 X^{10} - 6096384000 T^{10} X^{10} - \\ & 270582939648 X^{10} + 138726604800 iT^7 X^{10} + 3297729576960 T^2 X^{10} - 74008166400 T^8 X^{10} - \\ & 1086898176000 T^8 X^8 - 9398592000 T^{12} X^8 - 121927680000 iT^{11} X^8 - 12049396531200 iT^5 X^8 - \\ & 5411658792960 iT X^8 + 10477161676800 T^6 X^8 + 14543833006080 T^2 X^8 + 5588700364800 iT^7 X^8 - \\ & 804722688000 T^{10} X^8 + 402554880000 iT^9 X^8 - 901943132160 X^8 + 20857434931200 iT^3 X^8 - \\ & 18232639488000 T^4 X^8 - 19920273408000 T^{10} X^6 + 52312701665280 iT^3 X^6 - 12472182374400 iT^7 X^6 - \\ & 14431090114560 iX^6 T + 16161649459200 T^8 X^6 + 47933344972800 T^6 X^6 - 11430720000 T^{14} X^6 - \\ & 48197586124800 iT^5 X^6 + 678730752000 T^{12} X^6 - 7153090560000 iT^{11} X^6 - 36923296972800 T^4 X^6 - \\ & 150377472000 iT^{13} X^6 - 2061584302080 X^6 + 40587440947200 T^2 X^6 + 19074908160000 iT^9 X^6 + \\ & 9644670000 T^{16} X^4 + 3292047360000 T^{12} X^4 + 291375415296000 T^8 X^4 + 98297708544000 iT^7 X^4 - \\ & 3092376453120 X^4 + 344240160768000 iT^9 X^4 - 207309570048000 T^{10} X^4 + 350542080000 T^{14} X^4 + \\ & 54116587929600 iT^3 X^4 + 52425444556800 T^4 X^4 + 69578470195200 T^2 X^4 - 37205154201600 iT^5 X^4 - \\ & 24739011624960 iX^4 T - 137168640000 iT^{15} X^4 - 70490456064000 iT^{11} X^4 + 273331047628800 T^6 X^4 - \\ & 3170119680000 iT^{13} X^4 + 1311675120000 T^{16} X^2 + 2679075000 T^{18} X^2 + 1290950148096000 iT^9 X^2 - \\ & 20615843020800 iX^2 T^3 + 6958006272000 T^{12} X^2 + 202937204736000 iT^5 X^2 - 2748779069440 X^2 + \\ & 77157360000 iT^{17} X^2 + 1181263645900800 T^6 X^2 + 66486093742080 T^2 X^2 - 1542243483648000 T^8 X^2 - \\ & 4511324160000 iT^{15} X^2 + 330497733427200 T^4 X^2 - 1756650405888000 T^{10} X^2 + 73156608000000 iT^{13} X^2 + \\ & 2693991392870400 iT^7 X^2 - 546681913344000 iT^{11} X^2 - 24739011624960 iX^2 T - 19203609600000 T^{14} X^2 - \\ & 10995116277760 iT - 53014155264000 iT^{15} + 806992478208000 iT^9 + 10716300000 iT^{19} + 1883933717299200 iT^7 + \\ & 369538986147840 iT^5 - 63909113364480 iT^3 - 240045120000 iT^{17} - 395779571712000 iT^{13} - \\ & 200930625 T^{20} + 569274531840000 iT^{11} + 26800595927040 T^2 + 796770120499200 T^6 + 379138238054400 T^4 - \\ & 523098390528000 T^{10} - 832835262873600 T^8 - 48039505920000 T^{14} - 963558457344000 T^{12} + \\ & 87873660000 T^{18} + 11341179360000 T^{16} - 1099511627776) e^{\frac{1/4 i(X^2 - 8)}{T}} \end{aligned}$$

and

$$\begin{aligned} D(X, T) = & T(1048576 X^{20} + 2621440 T^2 X^{18} + 41943040 X^{18} - 283115520 T^2 X^{16} + 26542080 T^4 X^{16} + \\ & 754974720 X^{16} + 471859200 T^4 X^{14} - 7549747200 T^2 X^{14} + 8053063680 X^{14} + 265420800 T^6 X^{14} + \\ & 7431782400 T^6 X^{12} + 56371445760 X^{12} + 1741824000 T^8 X^{12} + 6606028800 T^4 X^{12} - 70464307200 T^2 X^{12} + \\ & 1683763200 T^{10} X^{10} + 264241152000 T^4 X^{10} + 188271820800 T^6 X^{10} + 49235558400 T^8 X^{10} + \\ & 270582939648 X^{10} - 253671505920 T^2 X^{10} + 4280706662400 T^4 X^8 - 594542592000 T^6 X^8 + \\ & 1532805120000 T^8 X^8 - 128894976000 T^{10} X^8 + 901943132160 X^8 + 338228674560 T^2 X^8 + 4463424000 T^{12} X^8 + \\ & 134120448000 T^{12} X^6 + 68966940672000 T^6 X^6 + 42658430976000 T^8 X^6 - 2536095744000 T^{10} X^6 + \\ & 2061584302080 X^6 + 6313601925120 T^2 X^6 + 31962609745920 T^4 X^6 + 8382528000 T^{14} X^6 + \\ & 46039891968000 T^{10} X^4 + 332943851520000 T^6 X^4 + 23192823398400 T^2 X^4 - 365783040000 T^{12} X^4 + \\ & 320060160000 T^{14} X^4 + 1786050000 T^{16} X^4 + 123453466214400 T^4 X^4 + 3092376453120 X^4 + \\ & 161269678080000 T^8 X^4 + 15545779200000 T^{14} X^2 + 38654705664000 T^2 X^2 + 2748779069440 X^2 + \\ & 15893694720000 T^{12} X^2 - 122880240000 T^{16} X^2 - 417368899584000 T^8 X^2 + 241591910400000 T^4 X^2 + \\ & 296795661926400 T^6 X^2 - 58785398784000 T^{10} X^2 + 893025000 T^{18} X^2 + 46437300000 T^{18} + \\ & 1099511627776 + 17801441280000 T^{14} + 435233046528000 T^{12} + 22325625 T^{20} + 3594961440000 T^{16} + \\ & 25426206392320 T^2 + 592866548121600 T^6 + 190374425395200 T^4 + 727066135756800 T^{10} + \\ & 1053529473024000 T^8) \end{aligned}$$

is a solution to the (CNLS) equation (1).

3.5 Solutions of order 5 to the CNLS equation

Example 3.5 The function $v_k(x, t)$ defined by

$$v_k(x, t) = \frac{N(X, T)}{D(X, T)} \quad (28)$$

with

$$\begin{aligned} N(X, T) = & (1073741824 X^{30} - 12079595520 T^2 X^{28} + 64424509440 X^{28} + 64424509440 iT X^{28} + \\ & 1803886264320 X^{26} + 3607772528640 iT X^{26} - 676457349120 iT^3 X^{26} - 105696460800 T^4 X^{26} - \\ & 3382286745600 T^2 X^{26} + 93802085744640 iT X^{24} - 5496215961600 iT^5 X^{24} + 18179791257600 T^4 X^{24} + \\ & 31267361914880 X^{24} - 1347629875200 T^6 X^{24} - 103723460198400 iT^3 X^{24} - 151075474636800 T^2 X^{24} - \\ & 64686234009600 iT^7 X^{22} + 1500833371914240 iT X^{22} - 3311935181291520 T^2 X^{22} + 375208342978560 X^{22} + \\ & 223230925209600 iT^5 X^{22} - 3788161155072000 iT^3 X^{22} + 2086870922035200 T^4 X^{22} - 14566293504000 T^8 X^{22} + \\ & 92590099660800 T^6 X^{22} - 69528992171950080 iT^3 X^{20} + 62031138914304000 T^4 X^{20} - 1686915514368000 T^6 X^{20} + \\ & 26536068891279360 iT^5 X^{20} - 390654119116800 iT^7 X^{20} - 45003354522255360 T^2 X^{20} - 62531017113600 T^{10} X^{20} + \\ & 930578064998400 T^8 X^{20} + 3301833418211328 X^{20} + 16509167091056640 iT X^{20} - 640916914176000 iT^9 X^{20} - \\ & 6814250827776000 iT^9 X^{18} - 53839240416460800 iT^7 X^{18} + 2201222788075520 X^{18} + 927044209528012800 T^4 X^{18} + \\ & 13263650684928000 T^{10} X^{18} - 2501240684544000 iT^{11} X^{18} + 662928202137600000 iT^5 X^{18} - \\ & 266639193302630400 T^6 X^{18} + 18789131354112000 T^8 X^{18} - 418443888961781760 T^2 X^{18} + 132073336728453120 iT X^{18} - \\ & 784185436825190400 iT^3 X^{18} + 924049244160000 T^{12} X^{18} - 5920441306038927360 iT^3 X^{16} - \\ & 2800208725829222400 T^2 X^{16} + 8305489215907430400 T^4 X^{16} + 33265772789760000 iT^{13} X^{16} + \\ & 110720477822976000 iT^{11} X^{16} + 8062613969279385600 iT^5 X^{16} + 141401773965312000 T^{12} X^{16} + \\ & 691271342279884800 T^8 X^{16} + 113205717195816960 X^{16} - 5599423059355238400 T^6 X^{16} + 390268459155456000 T^{10} X^{16} - \\ & 2527055248254566400 iT^7 X^{16} + 792440020370718720 iT X^{16} + 3041851760640000 T^{14} X^{16} - \\ & 295254607527936000 iT^9 X^{16} + 2797527418011648000 iT^{11} X^{14} - 480796932833280000 T^{14} X^{14} + \\ & 8459703734501376000 T^{10} X^{14} + 452822868783267840 X^{14} + 3622582950266142720 iT X^{14} - \\ & 55447639526316441600 T^6 X^{14} + 47385350256564633600 T^4 X^{14} + 3215987897794560000 iT^{13} X^{14} - \\ & 34921434190970880000 iT^7 X^{14} - 31088031568389734400 iT^3 X^{14} - 12783217463525376000 iT^9 X^{14} + \\ & 97339256340480000 iT^{15} X^{14} + 552039931183104000 T^8 X^{14} + 4169996886343680000 T^{12} X^{14} + \\ & 56637987994415923200 iT^5 X^{14} + 10439082178560000 T^{16} X^{14} - 13845930026257612800 T^2 X^{14} + \\ & 94609824263700480000 iT^{13} X^{12} - 10902446184333312000 iT^{11} X^{12} - 937719938211840000 T^{16} X^{12} + \\ & 170423092842489446400 T^4 X^{12} + 237544979183330918400 iT^5 X^{12} + 1408782258436833280 X^{12} - \\ & 328542671186008473600 T^6 X^{12} - 256437296495905996800 iT^7 X^{12} + 113598211414818816000 T^{12} X^{12} + \\ & 18678949135515648000 T^8 X^{12} - 33042253015941120000 T^{14} X^{12} + 292294300999680000 iT^{17} X^{12} - \\ & 51093297099177984000 iT^9 X^{12} + 73284670690689024000 T^{10} X^{12} + 152956750464000000 T^{18} X^{12} + \\ & 12679040325931499520 iT X^{12} - 114599018330534707200 iT^3 X^{12} - 51142859776233308160 T^2 X^{12} - \\ & 1275570322145280000 iT^{15} X^{12} - 568209551476654080000 T^{14} X^{10} + 33810774202483998720 iT X^{10} - \\ & 111496120466669568000 iT^{15} X^{10} - 835836308404096204800 T^{10} X^{10} + 1544635466429497344000 iT^{13} X^{10} - \\ & 3354471162040320000 T^{18} X^{10} - 1460544237383712768000 iT^{11} X^{10} - 1350240905863574323200 T^6 X^{10} - \\ & 6253924316774400000 T^{16} X^{10} + 4395752326103040000 iT^{17} X^{10} + 537540327664338862080 iT^5 X^{10} + \\ & 3381077420248399872 X^{10} - 291130272099273277440 iT^3 X^{10} + 335547089394912460800 T^4 X^{10} - \\ & 64851069709440000 T^{20} X^{10} - 140932409776700129280 T^2 X^{10} - 29559021493026816000 iT^9 X^{10} - \\ & 1496184251922933350400 iT^7 X^{10} + 367096201113600000 iT^{19} X^{10} + 181578307462181683200 T^8 X^{10} + \\ & 1995679229886332928000 T^{12} X^{10} - 8278185882032485171200 iT^7 X^8 + 35040900723266027520000 T^{12} X^8 - \\ & 25776719676221423616000 iT^{11} X^8 + 20037955353323765760000 iT^{13} X^8 + 46022478106145587200 T^4 X^8 - \\ & 4423140087741127065600 T^6 X^8 - 89719417680076800000 T^{18} X^8 + 1458152383840911360000 T^{16} X^8 - \end{aligned}$$

$$\begin{aligned}
& 285766062730609950720 T^2 X^8 - 7233336603857387520000 T^{14} X^8 - 61134091048800000 T^{22} X^8 - \\
& 2881713935587737600000 iT^{15} X^8 - 16189547667976617984000 iT^9 X^8 + 220691774903943168000 iT^5 X^8 + \\
& 6147413491360727040 X^8 + 7563747035669594112000 T^{10} X^8 + 479895807293521920000 iT^{17} X^8 - \\
& 1297021394188800000 iT^{21} X^8 - 11440905685977600000 T^{20} X^8 + 67621548404967997440 iT X^8 - \\
& 402780022974185472000 T^8 X^8 - 5721446629785600000 iT^{19} X^8 - 477902289208187289600 iT^3 X^8 - \\
& 1602834625818368409600 T^4 X^6 - 467395040951193600000 T^{20} X^6 + 193399403798266380288000 T^{12} X^6 + \\
& 190438406808970199040000 iT^{13} X^6 - 410930947999420907520 iT^3 X^6 - 12203382359857928601600 T^6 X^6 + \\
& 8196551321814302720 X^6 - 413768215764664320000 T^2 X^6 + 753683451826352947200 T^8 X^6 + \\
& 38717817820412903424000 iT^{11} X^6 + 520477428164198400000 iT^{19} X^6 - 131637812201231745024000 iT^9 X^6 + \\
& 7260779169907200000 T^{22} X^6 + 206664200076398690304000 T^{10} X^6 - 111952588730572800000 iT^{21} X^6 - \\
& 34861071249996093849600 iT^7 X^6 - 2200212487328602521600 iT^5 X^6 + 98358615861771632640 iT X^6 - \\
& 978145456780800000 iT^{23} X^6 - 42107664753000000 T^{24} X^6 + 450403487459573760000 iT^{17} X^6 + \\
& 16019699552141967360000 T^{16} X^6 + 455829413581946880000 T^{18} X^6 - 46430361192297922560000 T^{14} X^6 - \\
& 3988777878568304640000 iT^{15} X^6 + 2788463132532000000 T^{24} X^4 + 85884451312317235200000 T^{14} X^4 + \\
& 6133269850478346240000 T^{16} X^4 + 44624483826048000000 T^{22} X^4 + 20299991297015808000000 T^{18} X^4 + \\
& 282458236080674045952000 iT^{11} X^4 - 6125838434395855257600 iT^5 X^4 + 7566047373982433280 X^4 + \\
& 28372677652434124800 iT^3 X^4 - 89129552752731488256000 iT^7 X^4 - 305831341933914488832000 iT^9 X^4 - \\
& 4240828662862263091200 T^4 X^4 + 19534263779250053120000 iT^{15} X^4 - 404310656547186278400 T^2 X^4 - \\
& 8337530399488819200000 T^{20} X^4 - 1665438726828441600000 iT^{21} X^4 + 613455147022380171264000 T^{10} X^4 - \\
& 31685379731712000000 iT^{23} X^4 + 20984485372113715200000 iT^{19} X^4 + 368327781629936271360000 T^{12} X^4 + \\
& 24562804439250000 T^{26} X^4 + 368827165444606525440000 iT^{13} X^4 + 43061589148277145600000 iT^{17} X^4 - \\
& 505291977036000000 iT^{25} X^4 - 25395929360528257843200 T^6 X^4 + 32912453273942753280000 T^8 X^4 + \\
& 98358615861771632640 iT X^4 - 218481740039438991360000 iT^9 X^2 + 5024847993858000000 T^{26} X^2 - \\
& 784859525671645347840000 iT^{13} X^2 - 117836606844530432409600 iT^7 X^2 + 961868885445039882240000 T^{16} X^2 - \\
& 423870752966285721600000 T^{18} X^2 + 4323455642275676160 X^2 - 681540543088485728256000 T^{12} X^2 + \\
& 389794236839887896576000 T^{10} X^2 + 4386215078437500 T^{28} X^2 - 25601460169824000000 iT^{25} X^2 - \\
& 238330492280446648320 T^2 X^2 + 72657779502379696128000 T^8 X^2 - 562865968465796136960000 iT^{15} X^2 + \\
& 1049210718538752000000 iT^{23} X^2 + 397217487134077747200 iT^3 X^2 - 26741593565813145600000 iT^{21} X^2 - \\
& 172534242340656000000 T^{24} X^2 + 366851799452045279232000 iT^{11} X^2 - 4959898712116140441600 T^4 X^2 + \\
& 1009453725613629112320000 iT^{17} X^2 - 297065860284211200000 T^{22} X^2 - 128132801085129523200000 T^{20} X^2 - \\
& 32407361079227921203200 T^6 X^2 + 125795650513010688000000 iT^{19} X^2 + 47163579464543109120000 T^{14} X^2 + \\
& 60528378991859466240 iT X^2 + 196502435514000000 iT^{27} X^2 - 6915840177780817920000 iT X^2 + 5 + \\
& 69112713747924000000 T^{26} - 219310753921875 T^{30} + 1152921504606846976 - 59407067127137225932800 iT^7 + \\
& 257245610715402731520 iT^3 - 2952177109735770685440 iT^5 - 273116254020814503936000 iT^9 - \\
& 7563452086736230809600 T^8 - 18131470989170363596800 T^6 - 2333652736912706764800 T^4 - \\
& 63770970723566223360 T^2 - 55384542847632015360000 T^{20} - 594531997780153466880000 T^{18} + \\
& 1004919187172480778240000 T^{16} + 1324780016039991705600000 T^{14} + 278603930601305564774400 T^{10} + \\
& 1251450861919898959872000 T^{12} + 214047295827750000 T^{28} - 572523598204992000000 T^{24} - \\
& 36951451474310553600000 T^{22} - 10390805225798860800000 iT^{23} - 860867812728000000 iT^{27} + \\
& 17544860313750000 iT^{29} - 483169375568678400000 iT^{25} + 1638427073853322690560000 iT^{17} + \\
& 892628700764861104128000 iT^{15} + 267820997302661677056000 iT^{13} - 536100687886946402304000 iT^{11} + \\
& 31199038332715008000000 iT^{21} + 125258042850567782400000 iT^{19} + 17293822569102704640 iT)e^{\frac{1/4 i(X^2-8)}{T}}
\end{aligned}$$

and

$$\begin{aligned}
D(X, T) = & T(1073741824 X^{30} + 64424509440 X^{28} + 4026531840 T^2 X^{28} + 1803886264320 X^{26} + \\
& 63417876480 T^4 X^{26} - 676457349120 T^2 X^{26} + 1268357529600 T^4 X^{24} - 33822867456000 T^2 X^{24} +
\end{aligned}$$

$$\begin{aligned}
& 31267361914880 X^{24} + 1083388723200 T^6 X^{24} + 375208342978560 X^{22} - 685476780441600 T^2 X^{22} + \\
& 111615462604800 T^4 X^{22} + 15557197824000 T^8 X^{22} + 36782368358400 T^6 X^{22} + 1083177330278400 T^6 X^{20} + \\
& 3301833418211328 X^{20} + 140148552499200 T^{10} X^{20} - 7857728567377920 T^2 X^{20} + 3693457126195200 T^4 X^{20} + \\
& 584316459417600 T^8 X^{20} + 292496375808000 T^{12} X^{18} + 4024458805248000 T^{10} X^{18} + 62694067116441600 T^4 X^{18} + \\
& 22012222788075520 X^{18} + 18611561299968000 T^8 X^{18} - 55242212958535680 T^2 X^{18} + 236760072192000 T^6 X^{18} + \\
& 129074007638016000 T^{10} X^{16} + 181121455226880000 T^6 X^{16} + 699207962685603840 T^4 X^{16} - \\
& 224778659624386560 T^2 X^{16} - 28686754381824000 T^{12} X^{16} + 1586913140736000 T^{14} X^{16} + 113205717195816960 X^{16} + \\
& 545095673708544000 T^8 X^{16} - 1340556800163840000 T^{12} X^{14} + 7594753259520000 T^{16} X^{14} + \\
& 7759669310049484800 T^6 X^{14} + 5782249187101900800 T^4 X^{14} + 452822868783267840 X^{14} + 5343349284274176000 T^{10} X^{14} - \\
& 261243962759577600 T^2 X^{14} + 50144465387520000 T^{14} X^{14} + 3116354450227200000 T^8 X^{14} + \\
& 1408782258436833280 X^{12} + 1874026490757120000 T^{14} X^{12} + 180895586118008832000 T^8 X^{12} + \\
& 3178581460992000000 T^{16} X^{12} + 101780313914474496000 T^6 X^{12} + 2743061608975564800 T^2 X^{12} + \\
& 171758275056893952000 T^{10} X^{12} - 19979128170086400000 T^{12} X^{12} + 36927427872345292800 T^4 X^{12} + \\
& 22861897228800000 T^{18} X^{12} + 180106931552963788800 T^4 X^{10} + 1934199855881021030400 T^8 X^{10} + \\
& 700945754557199155200 T^6 X^{10} + 15459612420833280000 T^{16} X^{10} + 812497040732160000 T^{18} X^{10} + \\
& 3381077420248399872 X^{10} + 196972179521273856000 T^{12} X^{10} + 15009061173120000 T^{20} X^{10} + \\
& 43749571213393920000 T^{14} X^{10} + 1596068259715193241600 T^{10} X^{10} + 19668767685098864640 T^2 X^{10} - \\
& 3566145079615684608000 T^{10} X^8 + 2990170937443221504000 T^6 X^8 + 23964304442400000 T^{22} X^8 - \\
& 1516864316889600000 T^{20} X^8 + 3552399109820252160000 T^{14} X^8 - 2308156689909547008000 T^{12} X^8 + \\
& 6753337805214056448000 T^8 X^8 + 648276893587891814400 T^4 X^8 + 69247066395472035840 T^2 X^8 + \\
& 47965854285619200000 T^{18} X^8 + 170631291853209600000 T^{16} X^8 + 6147413491360727040 X^8 + \\
& 8542697373447487488000 T^6 X^6 + 1646900940797305159680 T^4 X^6 + 8196551321814302720 X^6 + \\
& 20090180659518111744000 T^{14} X^6 + 3307433323810258944000 T^{12} X^6 - 23562324502936289280000 T^{10} X^6 - \\
& 51640272946483200000 T^{20} X^6 + 11414721564288000000 T^{22} X^6 + 263988120438000000 T^{24} X^6 + \\
& 151793825440522567680 T^2 X^6 + 13150445041365811200000 T^{16} X^6 + 10165439543767990272000 T^8 X^6 + \\
& 2770975783128268800000 T^{18} X^6 + 3721637036250000 T^{26} X^4 + 1509070651956000000 T^{24} X^4 + \\
& 101354235193825689600000 T^{16} X^4 + 30890484375811522560000 T^8 X^4 + 2774315886702074265600 T^4 X^4 + \\
& 5498665554816000000 T^{22} X^4 + 7566047373982433280 X^4 - 32618354647467294720000 T^{12} X^4 + \\
& 16716882429729570816000 T^6 X^4 + 14105232533274624000000 T^{18} X^4 + 1930323178144972800000 T^{20} X^4 + \\
& 51484021358466170880000 T^{14} X^4 + 756976879134572544000 T^{10} X^4 + 210430692588886425600 T^2 X^4 + \\
& 4323455642275676160 X^2 + 911313710011184578560000 T^{12} X^2 + 385121524902461964288000 T^{10} X^2 + \\
& 2771655948172158566400 T^4 X^2 - 5227964253337190400000 T^{20} X^2 - 375141013254000000 T^{26} X^2 + \\
& 120909394693584000000 T^{24} X^2 + 20881847429103196569600 T^6 X^2 + 170236065914604748800 T^2 X^2 + \\
& 861101664679514603520000 T^{14} X^2 + 96153349029441306624000 T^8 X^2 - 76443703652568268800000 T^{18} X^2 + \\
& 3165053670245376000000 T^{22} X^2 + 1196240475937500 T^{28} X^2 + 485926912220284846080000 T^{16} X^2 + \\
& 407046920571552006144000 T^{12} + 94104250773750000 T^{28} + 205201850696640000000 T^{24} + 19937341265625 T^{30} + \\
& 14746992739115212800000 T^{22} + 1152921504606846976 + 17358565796964000000 T^{26} + 71438052221024993280000 T^8 + \\
& 12509247583105214054400 T^6 + 1241304647293992960000 T^4 + 61609242902428385280 T^2 + \\
& 63522744757048442880000 T^{20} + 354454471673590579200000 T^{18} + 454189165926332497920000 T^{16} + \\
& 332695768397355417600000 T^{14} + 228185772462932439859200 T^{10}
\end{aligned}$$

is a solution to the (CNLS) equation (1).

We could go on and present more explicit quasi-rational solutions, but they become very complicated. It will be relevant to study in detail the structure of these solutions.

4 Conclusion

Different representations of quasi-rational solutions to the cylindrical nonlinear Schrödinger equation have been given. First quasi rational solutions in terms of wronskians of order $2N$ depending on $2N - 2$ real parameters have been presented. Another representation in terms of Fredholm determinants are given depending on $2N - 2$ real parameters. So we get a hierarchy of families of quasi-rational solutions of order N to the CNLS equation depending on $2N - 2$ real parameters.

References

- [1] R. Radhakrishnan, M. Lakshmanan, Bright and Dark Soliton Solutions to Coupled Nonlinear Schrödinger Equations, *J. Phys. A*, V. 28, 2683-2692, 1995
- [2] A. Hasegawa, Optical solitons in fibers, *Tract In Mod. Phys.*, V. 116, 2nd Ed., Springer, 1990
- [3] P.A. Clarkson, S. Hood, Symmetry reductions of a generalized, cylindrical nonlinear Schrödinger equation, *J. Phys. A : Math. Gen.*, V. 26, 133-150, 1993.
- [4] M.J. Ablowitz, H. Segur, On the evolution of packets of water waves, *J. Of Fluid Mech.*, V. 92, 691-715, 1979
- [5] M.J. Landman, G.C. Papanicolaou, C. Sulem, P.L. Sulem, Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension, *Phys. Rev. A Gen. Phys.*, 3837-3843, 1988
- [6] A.I. Smirnov, G.M. Fraiman, The interaction representation in the self-focusing theory, *Phys. D: Nonln. Phen.*, V. 52, N. 1, 2-15, 1991
- [7] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. 161, 2002
- [8] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. 68, 77-90, 2004
- [9] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. 89, 1-12, 2009
- [10] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. 42, 404409-1-16, 2009
- [11] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. 185, 247-258, 2010

- [12] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [13] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [14] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [15] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013
- [16] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013
- [17] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013
- [18] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [19] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [20] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [21] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014
- [22] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [23] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [24] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014
- [25] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014

- [26] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [27] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [28] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [29] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015
- [30] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015
- [31] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016
- [32] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [33] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [34] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [35] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016
- [36] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [37] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [38] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. **3**, 130-139, 2017
- [39] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. **3**, 60-70, 2017

- [40] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017
- [41] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 2, 92-98, 2017
- [42] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 9, 19-37, 2017
- [43] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 11, 24-30, 2017
- [44] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. 1, 14-19, 2018
- [45] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [46] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [47] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [48] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [49] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019
- [50] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [51] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [52] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021
- [53] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021

- [54] P. Gaillard, 2N parameter solutions to the Burgers equation, *Jour. Of Appl. Nonlin. Dyn.*, V. **1**, N. **1**, 1-6, 2021
- [55] P. Gaillard, Multi-Parametric rational Solutions to the KdV equation, *Asi. Jour. Of Res. and Rev. In Phys.*, V. **3**, N. 3, 14-21, 2021
- [56] P. Gaillard, Multiparametric solutions to the Gardner equation and the degenerate rational case, *Jour. Of Appl. Ana. And Appl.*, V. **11**, N. 4, 2102-2113, 2021
- [57] P. Gaillard, Other families of rational solutions to the KPI equation, *Asi. Res. Jour Of Math.*, V. **17**, N. 6, 27-34, 2021
- [58] P. Gaillard, Rational solutions to the mKdV equation associated to particular polynomials, *Wave Motion*, V. **107**, 102824-1-11, 2021
- [59] P. Gaillard, Multiparametric Rational Solutions of Order N to the KPI Equation and the Explicit Case of Order 3, *Arch. Of cur. Res. Int.*, V. **21**, N . 6, 58-71, 2021
- [60] P. Gaillard, Rational solutions to the KPI equation from particular polynomials, *Wave Motion*, V. **108**, 102828-1-9, 2022
- [61] P. Gaillard, Rogue Waves of the Lakshmanan Porsezian Daniel Equation Depending on Multi-parameters, *As. Jour. Of Adv. Res. And Rep.*, V 16, N. 3, 32-40, 2022
- [62] P. Gaillard, Quasi-rational and rational solutions to the defocusing NLS equation, *Jour. Of Math. Sci. And Mod. Res. And Rep.*, V 5, N. 1, 24-34, 2022
- [63] P. Gaillard, Rational solutions to the KdV equation in terms of particular polynomials, *Jour. Of Nonlin. Mod. and Ana.*, V. 4, N. 4, 615-626, 2022
- [64] P. Gaillard, Rational solutions to the Painlevé II equation from particular polynomials, *Int. J. Of Appl. Nonlin. Sci.*, V. 3, N. 3, 189-196, 2022
- [65] P. Gaillard, Multi-lump solutions to the KPI equation with a zero degree of derivation, *Jour. Of Theor. And Appl. Phys.*, V. 16, N. 4, 1-8, 2022
- [66] P. Gaillard, Families of solutions to the KPI equation given by an extended Darboux transformation, *Part. Diff. Equ. And Appl.*, V. 3, N. 80, 1-14, 2022
- [67] P. Gaillard, Rogue waves of the Hirota equation in terms of quasi-rational solutions depending on multi-parameters, *WSEAS Trans. On Math.*, DOI: 10.37394/23206.2023.22.24, V. 22, 190-203, 2023