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| Robin Khanfir. Convergences of looptrees coded by excursions. 2022. hal-04157345

HAL Id: hal-04157345

<https://hal.science/hal-04157345>

Preprint submitted on 10 Jul 2023

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CONVERGENCES OF LOOPTREES CODED BY EXCURSIONS

Robin KHANFIR*

January 20, 2023

Abstract

In order to study convergences of looptrees, we construct continuum trees and looptrees from real-valued càdlàg functions without negative jumps called excursions. We then provide a toolbox to manipulate the two resulting codings of metric spaces by excursions and we formalize the principle that jumps correspond to loops and that continuous growths correspond to branches. Combining these codings creates new metric spaces from excursions that we call veneration trees. They consist of a collection of loops and trees glued along a tree structure so that they unify trees and looptrees. We also propose a topological definition for veneration trees, which yields what we argue to be the right space to study convergences of looptrees. However, those first codings lack some functional continuity, so we adjust them. We thus obtain several limit theorems. Finally, we present some probabilistic applications, such as proving an invariance principle for random discrete looptrees.

Keywords Looptree · Tree · Coding by real-valued functions · Limit theorem · Scaling limit · Random metric space · Geodesic space

Mathematics Subject Classification 60F17 · 54C30 · 54E70 · 05C05 · 54F50

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1 Introduction

Background Informally, a looptree consists of a collection of loops tangentially glued together along some genealogical structure. Perhaps more clearly, one can associate a looptree with a finite ordered tree by replacing each vertex with a circle of length proportional to its degree and each edge with a unique point of tangency between the corresponding circles, thereby preserving the ordered tree structure. An example will be given with Figure 3 in Section 6.3. Ever since their introduction by Curien and Kortchemski in [18], looptrees have generated a growing interest thanks to their natural appearances in some random geometric models and because they can provide useful tools or insights to study other objects. Let us mention some of them. To begin with, when a sequence of trees admits a scaling limit and if the maximum degree becomes negligible against the height, then the scaling limit is often a continuum tree whose all branch points are of infinite degree. In this situation, it is then difficult to recover the limiting joint distribution of the scaled degrees from that object. However, a scaling limit of the associated looptrees would not only give such information (merely via the lengths of the loops) but also the whole asymptotic degree structure. This is exactly what happens for large critical Galton-Watson trees whose offspring distribution belongs to the domain of attraction of an α -stable law, with $\alpha \in (1, 2)$. While Duquesne showed in [21] these random trees converge, after suitable scaling, towards the α -stable tree introduced in [32, 23], Curien and Kortchemski proved the rescaled associated looptrees converge towards a random compact metric space they called the α -stable looptree. There are even more striking cases where the associated looptrees admit a non-degenerate scaling limit but the trees themselves do not so they are an interesting means to give sense to scaling limits of highly dense trees. An instance of such a situation is studied in [16] by Curien, Duquesne, Kortchemski, and Manolescu with the model of random trees built by linear preferential attachment, which was introduced and popularized in [41, 7, 11]. The scaling limit for this model is called the Brownian looptree and is distinct from the stable looptrees.

In addition to their help to study trees, looptrees spontaneously appear as scaling limits of other random structures. For example, random Boltzmann dissections of a regular polygon, that were introduced in [29], have a similar geometry as looptrees in some regimes. Indeed, Curien and Kortchemski proved in [18] that their scaling limit as the number of sides of the polygon tends to infinity is the α -stable looptree, depending on the regime. Let us mention links between looptrees and dissections were also detected in [17]. Several instances of looptrees can be found within random planar maps too. The goal of this active area of research is to understand universal large-scale properties of graphs or their proper embeddings in the two-dimensional sphere. We refer to [1, 37] for surveys of this field. In [19], Curien and Kortchemski considered boundaries of percolation clusters on the uniform infinite planar triangulation introduced in [6]. These boundaries are almost looptrees and the authors showed that in the critical case of the percolation, their scaling limit is the $3/2$ -stable looptree. Similarly, in [30, 39], Kortchemski and Richier discussed asymptotics for the boundaries of Boltzmann planar maps conditioned on having a large perimeter. Once again, stable looptrees arise as scaling limits in some regimes.

Nevertheless, the most important application of looptrees may be their substantial connections with scaling limits of random planar maps. The bijections of Bouttier, Di Francesco, Guitter [12] and of Janson, Stefánsson [26] yield a one-to-one correspondence between planar maps and some labeled trees (where each vertex is equipped with an integer), which can be reformulated into a bijection between planar maps and some labeled looptrees. Looptrees seem to be the right objects to consider here because the scaling limits of random planar maps with large faces introduced by Le Gall and Miermont in [33] are implicitly constructed from a Gaussian field on stable looptrees. Furthermore, Marzouk obtained limit theorems for planar maps in [34] without a strong control on the corresponding trees, but by observing their Lukasiewicz walks instead. As we will see later, the Lukasiewicz walk directly codes the geometry of the associated looptree. For more detailed discussions on the relations between maps and looptrees, we refer to Marzouk [35].

Motivations and main results The purpose of the present work is to build a framework and to provide a toolbox to construct, manipulate, and demonstrate convergences involving looptrees under the most

general setting possible. A first issue is how to define compact continuum fractal looptrees that may appear as scaling limits. Although compact continuum fractal trees are well-known since the introduction of the Brownian Continuum Random Tree (CRT) by Aldous in [3], mimicking the discrete setting to define continuum looptrees associated with them is not so easy. On the one hand, it could be not clear how to choose the lengths of the loops while ensuring the compactness of the resulting space. On the other hand, when the branch points are dense in the tree, it might be possible that two different cycles in the associated looptree would never share a common point. It is indeed what happens for the α -stable looptree \mathcal{L}_α of Curien and Kortchemski, with any $\alpha \in (1, 2)$. They avoided the above difficulties [18] by building directly \mathcal{L}_α from the excursion $X^{\text{exc},(\alpha)}$ of an α -stable spectrally positive Lévy process. This process also codes the α -stable tree, and Curien and Kortchemski justified that \mathcal{L}_α can be interpreted as its associated looptree. The idea to encode metric spaces by real-valued functions is in line with the founding work of Le Gall [25] which studies and constructs compact real trees from continuous excursions. Namely, if $f : [0, 1] \rightarrow [0, \infty)$ is continuous with $f(0) = f(1) = 0$ then setting

$$d_f^{\text{clas}}(t, s) = d_f^{\text{clas}}(s, t) = f(s) + f(t) - 2 \inf_{[s,t]} f \quad (1)$$

for all $s, t \in [0, 1]$ with $s \leq t$ defines a pseudo-distance on $[0, 1]$, which in turn induces a compact real tree by quotient. See [25, 22] for an extensive study of this coding. The definition of real trees will be reminded in Section 5 and the quotient metric space induced by a pseudo-distance is defined in Section 2.3. Similarly, the α -stable looptree is induced by another pseudo-distance inherited from $X^{\text{exc},(\alpha)}$.

Definition 1. A càdlàg function $f : [0, 1] \rightarrow \mathbb{R}$ is a right-continuous function that has left limits everywhere. In that case, we write $f(0-) = 0$, $\Delta_0(f) = 0$, $f(t-) = \lim_{s \rightarrow t-} f(s)$ and we denote $\Delta_t(f) = f(t) - f(t-)$ for all $t \in (0, 1]$. If no confusion over the function is possible, we will simply write Δ_t . We say a non-negative càdlàg function $f : [0, 1] \rightarrow [0, \infty)$ is an *excursion* when $f(1) = 0$ and when all jumps of f are non-negative, namely $\Delta_t \geq 0$ for all $t \in [0, 1]$. We denote by \mathcal{H} the set of excursions.

We straightforwardly extend the work of Curien and Kortchemski in [18] to construct a pseudo-distance d_f^{L} from any excursion f . We denote by \mathcal{L}_f and we call the looptree coded by f the quotient metric space induced by d_f^{L} . In particular, we have $\mathcal{L}_\alpha = \mathcal{L}_{X^{\text{exc},(\alpha)}}$. Without immediately giving the definition of d_f^{L} , let us already state that each jump of f of height Δ corresponds to a circle of length Δ in \mathcal{L}_f , that those circles are dense in \mathcal{L}_f , and that if g is a "sub-excursion" of f then \mathcal{L}_g is a subset of \mathcal{L}_f . This method has many benefits. While it allows an automatic construction of a large diversity of complex looptrees, it is also fairly simple to find an excursion coding a given discrete looptree. This kind of behavior is especially useful to tackle problems involving transitions from discrete to continuous worlds.

However, d^{L} alone is not enough to understand all convergences of looptrees. Indeed, a looptree consisting of a chain of n loops of lengths $1/n$ put back to back is asymptotically close to a segment that has no loops. It is not an unusual situation, as [17], [18], [19], and [30] provide as many examples of looptrees that converge towards loopless compact real trees. In particular, it ensures d^{L} lacks the key property of being continuous with respect to the coding function, contrary to d^{clas} . Furthermore, the coding brought by d^{L} does not fully exploit the diversity of excursions because it only cares about the jumps: if f is a continuous excursion with $f(0) = 0$, then $d_f^{\text{L}} = 0$ and the looptree coded by f is reduced to a single point. In fact, d^{L} disregards continuous growth that would classically code a real tree with d^{clas} . In contrast, applying d^{clas} to a discontinuous excursion would still give a tree but without really distinguishing between jumps and continuous growth. These observations motivate us to define a new pseudo-distance d_f^{T} that induces a real tree, and thus codes additional limits of looptrees we were lacking, but that only harnesses the continuous growth of f . Informally, d_f^{T} is obtained by stripping f of its jumps then by taking d^{clas} . We denote by \mathcal{T}_f and we call the tree coded by f the quotient metric space induced by d_f^{T} . Plus, we clarify that it naturally holds $d_0^{\text{L}} = d_0^{\text{T}} = 0$. Our first main result describes the relations between trees and looptrees coded by excursions. It will be specified by Theorems 4 and 5.

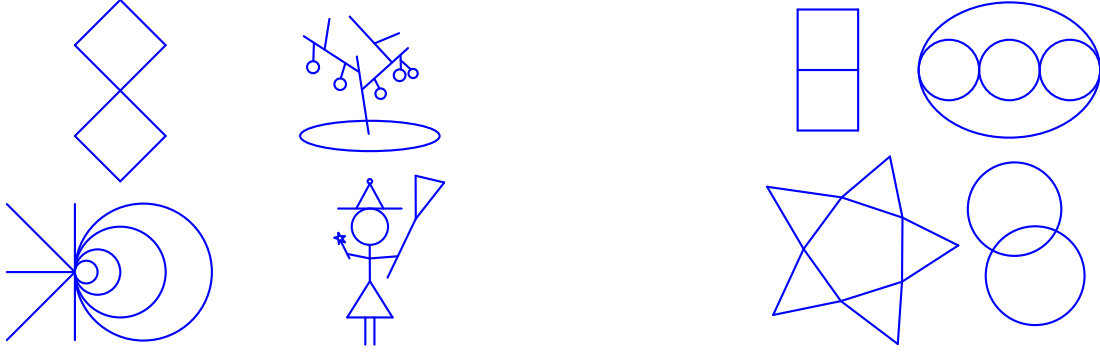


Figure 1: All eight figures have to be understood as intrinsic one-dimensional spaces. *Left*: Four examples of veneration trees. *Right*: Four non-examples of veneration trees.

Theorem 1. *Let us set $I : f \in \mathcal{H} \mapsto f \in \mathcal{H}$. There exists an operator $J : \mathcal{H} \rightarrow \mathcal{H}$ such that $J \circ J = J$ and $J \circ (I - J) = 0$. Moreover, it holds $d_f^l = d_{Jf}^l$ and $d_f^T = d_{f-Jf}^T = d_{f-Jf}^{\text{clas}}$ for all excursions f . Furthermore, f is a continuous excursion with $f(0) = 0$ if and only if $Jf = 0$.*

An excursion f such that $Jf = f$ or $Jf = 0$ will be respectively called a *pure jump growth (PJG) excursion* or a *continuous excursion*. These two classes of excursions and the operator J will be properly defined at the beginning of Section 3. A linear combination of d_f^T and d_f^l gives a pseudo-distance that induces a new hybrid metric space, denoted by \mathcal{V}_f , composed of loops and trees tangently glued together along some genealogical structure. More precisely, we choose $d_f^V = d_f^T + 2d_f^l$ because the diameter of a metric circle is half of its length. We call \mathcal{V}_f the *vernation tree* coded by f . The name veneration tree was chosen to distinguish this new object from the looptree \mathcal{L}_f . It refers to the arrangement of bud scales or young leaves in a leaf bud before it opens. Figure 1 presents some examples and non-examples of veneration trees. Let us already mention that independently, Blanc-Renaudie in [10] and Marzouk in [35] construct the same kind of spaces as part of their studies of scaling limits of models with prescribed degrees. While Blanc-Renaudie took the point of view of stick-breaking constructions and managed to mimic the association of a looptree with a discrete tree, Marzouk found the same coding by excursions as us and also identified the significance of PJG excursions. However, our methods to study convergences are completely different and we truly believe both should be useful according to context.

We argue veneration trees yield the right notion to study convergences of looptrees. Theorem 1 justifies this notion unifies the classical encoding of real trees and the encoding of looptrees introduced in [18] by Curien and Kortchemski. Moreover, veneration trees can naturally appear as limits of looptrees. Indeed, looptrees may converge to trees or other looptrees, and gluing two looptrees together still gives a looptree. Conversely, the limits of veneration trees are also veneration trees in some sense. To make this point rigorous, we will provide an intrinsic and topological definition for veneration trees in Section 5. Informally, a metric space is a veneration tree when any two points at distance r are joined by a path of length r and any two distinct loops do not intersect at more than one point. We let the reader compare this definition with Figure 1. Then, Theorems 9, 10, 11 imply the following result.

Theorem 2. *The space of compact veneration trees is the closure for the Gromov-Hausdorff topology of the set of veneration trees coded by excursions.*

The definition of the Gromov-Hausdorff topology, which formalizes convergence of (isometry classes of) compact metric spaces, is recalled in Section 4.3. Veneration trees reveal another interesting observation. Again in [18], Curien and Kortchemski have proved the α -stable looptrees converge in distribution to the Brownian CRT (up to a constant multiplicative factor) as α tends to 2, while the excursion $X^{\text{exc},(\alpha)}$ converge in distribution to a multiple of the Brownian excursion. However, we will see in Section 6.2 that the excursions $X^{\text{exc},(\alpha)}$ are PJG and the Brownian excursion is continuous. Plus, the multiplicative constants match. Hence, the veneration trees coded by the $X^{\text{exc},(\alpha)}$ converge to the veneration tree

coded by the limiting excursion. Furthermore, [18] and [30] exhibit several instances where rescaled looptrees associated with discrete trees converge to the looptree or the tree coded by the scaling limit of their Lukasiewicz walks. Thereby, it is natural to hope the coding $f \in \mathcal{H} \mapsto \mathcal{V}_f$ enjoys some kind of functional continuity. If it were true, understanding convergences of looptrees and veneration trees would merely be reduced to studying the convergences of their coding excursions. In general, it is a much simpler problem thanks to a lot of tools and methods that were developed over the years. In fact, one could argue the functional continuity of the classical coding $f \mapsto d_f^{\text{clas}}$ is the main reason for its success. Sadly, it does not hold for $d^{\mathcal{V}}$ or \mathcal{V} . We will discuss in more depth why this fails at the beginning of Section 4, but [17, Theorem 13] and [30, Theorem 2] already provide a counterexample where the limit differs from a strange multiplicative factor from what one could expect. In order to address this issue, we define another metric space $\tilde{\mathcal{V}}_f$, induced by a pseudo-distance $\tilde{d}_f^{\mathcal{V}}$, called the *shuffled veneration tree* coded by f . Informally, $\tilde{\mathcal{V}}_f$ is obtained by shuffling the positions on which the components of \mathcal{V}_f are glued on each loop while keeping the loops and the trees arranged in the same genealogical structure as \mathcal{V}_f . See Figure 2 for an example. Shuffling instructions are contained in an object Φ we will call *shuffle* and that one can choose with some freedom. The coding $f \mapsto \tilde{\mathcal{V}}_f$ is continuous in some sense, which justifies its definition. Let us denote by $D([0, 1])$ the space of càdlàg functions on $[0, 1]$ and let us endow it with the Skorokhod distance ρ defined by

$$\rho(f_1, f_2) = \inf_{\lambda \in \Lambda} \max(\|\lambda - \text{id}\|_{\infty}, \|f_1 - f_2 \circ \lambda\|_{\infty})$$

for all $f_1, f_2 \in D([0, 1])$, where Λ is the set of increasing bijections from $[0, 1]$ into itself. The distance ρ is not complete but induces a separable and completely metrizable topology on $\mathcal{D}([0, 1])$ called the Skorokhod topology, see [9, Chapter 3]. We set

$$\begin{aligned} \Theta : D([0, 1]) &\longrightarrow D([0, 1]) \\ f &\longmapsto \Theta f : t \mapsto \mathbf{1}_{2t \geq 1} f(2t - 1) \end{aligned}$$

and we write $f_n \rightarrow f$ for the *relaxed Skorokhod topology* when $\Theta f_n \rightarrow \Theta f$ for the Skorokhod topology on $D([0, 1])$. This induces a weaker separable and completely metrizable topology than the Skorokhod topology. It is clear that the set of excursions is closed for both of those topologies. We are now ready to state our limit theorem.

Theorem 3. *We denote by ∂ the metric space with a unique point. Let f be an excursion and let (f_n) be a sequence of excursions such that $f_n \rightarrow f$ for the relaxed Skorokhod topology.*

- (i) *If $B_f \cap \mathbb{B}(\Phi) = \emptyset$, then the convergence $\tilde{\mathcal{V}}_{f_n} \rightarrow \tilde{\mathcal{V}}_f$ holds for the pointed Gromov-Hausdorff-Prokhorov topology.*
- (ii) *If there is $N \geq 1$ such that all the f_n have at most N jumps, then the convergences $\mathcal{L}_{f_n} \rightarrow \mathcal{L}_f$, $\mathcal{T}_{f_n} \rightarrow \mathcal{T}_f$, and $\mathcal{V}_{f_n} \rightarrow \mathcal{V}_f$ hold for the pointed Gromov-Hausdorff-Prokhorov topology.*
- (iii) *If f is PJG, then the convergences $\mathcal{L}_{f_n} \rightarrow \mathcal{L}_f$, $\mathcal{T}_{f_n} \rightarrow \partial$, and $\mathcal{V}_{f_n} \rightarrow \mathcal{V}_f$ hold for the pointed Gromov-Hausdorff-Prokhorov topology.*

The pointed Gromov-Hausdorff-Prokhorov topology formalizes convergence of (equivalence classes of) compact metric spaces endowed with a distinguished point and a Borel probability measure. We will give a proper definition in Section 4.3. The assumption $B_f \cap \mathbb{B}(\Phi) = \emptyset$ is a minimal condition that is presented in Section 4.1. The price for the functional continuity of $f \mapsto \tilde{\mathcal{V}}_f$ is that there is no simple or canonical choice for the shuffle Φ . Moreover, it lacks some other useful properties that $f \mapsto \mathcal{V}_f$ shares with $f \mapsto d_f^{\text{clas}}$. That makes $\tilde{\mathcal{V}}_f$ inconvenient to use with explicit or discrete examples. Thus, we think it is best to first define a model or to express a given veneration tree by using (unshuffled) \mathcal{V}_f , and then to study the asymptotic behavior thanks to the point (ii) and (iii) of Theorem 3 or by comparing \mathcal{V}_f with $\tilde{\mathcal{V}}_f$. While this technical task will not be automatic, a nice choice of shuffle may ease it. For example, if

X is a random excursion, it may be possible to select Φ such that $\tilde{\mathcal{V}}_X$ and \mathcal{V}_X have the same distribution. Another advantage of the flexibility of the choice for the shuffle is that it is easy to ensure the condition $B_f \cap \mathbb{B}(\Phi) = \emptyset$ holds. We will give three concrete probabilistic applications of this method.

The first one reformulates in metric terms the asymptotics in terms of processes found by Aldous, Miermont, and Pitman in [5] for uniform random mappings. Indeed, such a mapping M can be represented by a graph on $\{1, 2, \dots, n\}$ with edges $(i, M(i))$ and each connected component of this graph is a veneration tree, as a collection of trees rooted on the same cycle. We give scaling limits for these components. The second application retrieves and completes [18, Theorem 1.2] of Curien and Kortchemski about the convergences of α -stable looptrees when α tends to 1 or 2 by showing that their whole family is continuous in distribution with respect to α . The strategy of the proof is to prove the coding excursions $X^{\text{exc},(\alpha)}$ are PJG and to find a shuffle such that $\tilde{\mathcal{V}}_{X^{\text{exc},(\alpha)}}$ is distributed as $\mathcal{V}_{X^{\text{exc},(\alpha)}}$. The last application provides two invariance principles for discrete looptrees associated with finite ordered trees. We first retrieve [18, Theorem 4.1] of Curien and Kortchemski under the PJG case. In the general case, we prove that if a sequence of *exchangeable* random ordered trees admits X as the scaling limit of their Lukasiewicz walks and if their heights become negligible against the scaling, then the scaling limit of their associated looptrees is $1/2 \cdot \mathcal{V}_X$, where $c \cdot E$ stands for the metric space obtained from E by multiplying all distances by $c > 0$. Of course, we will clarify what it means for a random ordered tree to be exchangeable, but informally, it means its distribution is invariant by changing the order.

Outline In Section 2, we construct the mentioned pseudo-distances and the metric spaces they induce by quotient. Section 3 gives some tools to better understand the relations between trees, looptrees, and excursions, and shows veneration trees generalize both trees and looptrees. Section 4 studies convergences of veneration trees by first stating the functional continuity of $f \mapsto \tilde{d}_f$ and then deducing other limit theorems under some particular cases. This results in a proof of Theorem 3. In Section 5, we introduce our topological notion of veneration trees and demonstrate Theorem 2. Section 6 is devoted to the three probabilistic applications of our work.

2 Tree, looptree, and veneration tree coded by an excursion

In all this Section, we fix an excursion denoted by f .

2.1 Genealogy associated with an excursion

In order to encode a looptree by an excursion f , we begin to construct a genealogy on $[0, 1]$ from f , which will give the tree structure of the loops in the associated looptree. If we take as a principle that f codes a looptree \mathcal{L} with root 0, then if $f(s-) \leq \inf_{[s,t]} f$, $s \leq t \leq r$, and $f(s-) = f(r)$, the excursion $u \mapsto f(s+u(r-s)) - f(u-)$ should encode a sub-looptree of \mathcal{L} , with root s , and containing t . Hence, the point s is an ancestor of t according to the genealogy of \mathcal{L} . This motivates the following definition, which is a direct extension of the genealogy defined by Curien and Kortchemski in [18] for the stable looptrees. We set for all $s, t \in [0, 1]$,

$$s \preceq_f t \text{ when both } s \leq t \text{ and } f(s-) \leq \inf_{[s,t]} f.$$

We also write $s \prec_f t$ when $s \preceq_f t$ and $s < t$. If the excursion f is obvious according to context, we would just write \preceq and \prec .

Proposition 1. *The relation \preceq enjoys the following properties.*

- (i) *The relation \preceq is a partial order on $[0, 1]$.*
- (ii) *The genealogy admits 0 as its root, namely $0 \preceq t$ for all $t \in [0, 1]$.*

(iii) *Two points have a most recent common ancestor. For any $s, t \in [0, 1]$, there is a unique $u \in [0, 1]$ such that for all $r \in [0, 1]$, $r \preceq u \iff r \preceq s, t$. We denote it by $s \wedge_f t$, or just by $s \wedge t$ if the excursion f is obvious according to context.*

(iv) *The ancestral lineages are totally ordered. For any $t \in [0, 1]$, the relation \preceq induces a total order on $\{s \in [0, 1] : s \preceq t\}$.*

Proof. Let $t, s, r \in [0, 1]$. Obviously, we have $t \leq t$ and $f(t-) \leq f(t) = \inf_{[t,t]} f$ because f is an excursion. So $t \preceq t$ and the relation \preceq is reflexive. If $s \preceq t$ and $t \preceq s$, then by definition $s \leq t \leq s$, so $t = s$, and \preceq is antisymmetric. To prove the transitivity, we suppose that $s \preceq t$ and $t \preceq r$ and we can assume $s < t < r$. We find $f(s-) \leq \inf_{[s,t]} f \leq f(t-) \leq \inf_{[t,r]} f$, which entails $s \preceq r$ and (i). The point (ii) is simply deduced from the fact that f is an excursion. Let us prove (iii). We set $u = \sup\{r \in [0, 1] : r \preceq s, t\}$. By definition, $u \leq s$ and $u \leq t$. We give ourselves a sequence $(r_n)_{n \geq 1}$ that converges towards u and such that $0 \leq r_n \leq r_{n+1} \leq u$ and $r_n \preceq s, t$ for all $n \geq 1$. We have $f(r_n-) \rightarrow f(u-)$ and thanks to the inequality

$$f(r_n-) \leq \inf_{[r_n,t]} f \leq \inf_{[u,t]} f,$$

we obtain $u \preceq t$. Likewise, $u \preceq s$. Conversely, if $r \preceq t$ and $r \preceq s$, then $r \leq u$ by definition and $f(r-) \leq \inf_{[r,t]} f \leq \inf_{[r,u]} f$, so $r \preceq u$. The uniqueness of $s \wedge t$ immediately follows from the antisymmetry of \preceq . Let us prove (iv). We suppose $s, r \preceq t$ and without loss of generality, we assume $r \leq s$. Then, $f(r-) \leq \inf_{[r,t]} f \leq \inf_{[r,s]} f$, so $r \preceq s$, which concludes the proof. \square

Lemma 1. *Let $s, t \in [0, 1]$. If $s \leq t$, then it holds that $\inf_{[s \wedge t, t]} f = \inf_{[s, t]} f$ and*

$$s \wedge t = \sup\{r \leq s : r \preceq t\} = \sup\left\{r \leq s : f(r-) \leq \inf_{[s, t]} f\right\}.$$

Proof. If $r \leq s$ and $r \preceq t$, then $f(r-) \leq \inf_{[r, t]} f \leq \inf_{[r, s]} f$, and so $r \preceq s$. Thus, $r \preceq s \wedge t$. Conversely, if $r \preceq s \wedge t$, then $r \leq s$ and $r \preceq t$. So, we have shown $s \wedge t = \sup\{r \leq s : r \preceq t\}$. In particular, if $s \wedge t < r \leq s$, then we cannot have $r \preceq t$. It follows that for all $r \in (s \wedge t, s]$, it holds

$$f(r) \geq f(r-) > \inf_{[r, t]} f$$

Since f is càdlàg, this leads to $\inf_{[r, s]} f > \inf_{[r, t]} f$. Hence, $\inf_{[r, t]} f = \inf_{[s, t]} f$ for all $r \in (s \wedge t, s]$ and we end the proof by making r tend towards $s \wedge t$. \square

As same as in [18], we define a quantity x_s^t which will represent the relative position of the ancestor of t on the loop containing s on the looptree associated with f , when $s \preceq t$. For all $s, t \in [0, 1]$, we set

$$x_s^t(f) = \mathbf{1}_{s \leq t} \max\left(\inf_{[s, t]} f - f(s-), 0\right).$$

When the excursion f is obvious according to context, we would just write x_s^t . Let us remark that $x_s^t \in [0, \Delta_s]$, $x_s^s = \Delta_s$, and $x_s^t > 0 \implies s \preceq t$. We finish this first step by providing two convenient lemmas about x_s^t .

Lemma 2. *Let $u, s, t \in [0, 1]$. If $u < t \wedge s$, then $x_u^t = x_u^s = x_u^{t \wedge s}$.*

Proof. We merely write $\inf_{[u, t \wedge s]} f \leq f(t \wedge s-) \leq \min(\inf_{[t \wedge s, t]} f, \inf_{[t \wedge s, s]} f)$. \square

Lemma 3. *Let $s, t \in [0, 1]$. If $s < t$, then*

$$\sum_{\substack{s < r \\ r < t}} x_r^t \leq f(t-) - \inf_{[s, t]} f. \quad (2)$$

Proof. Let $\{r_i : 1 \leq i \leq n\}$ be a finite subset of $\{r \in [0, 1] : r \prec t, s \prec r\}$ such that $r_i < r_{i+1}$ for any $1 \leq i \leq n-1$. Let us also set $r_{n+1} = t$. With these notations, we have $r_{i+1} \in (r_i, t]$ for any $1 \leq i \leq n$, so we can write

$$\sum_{i=1}^n x_{r_i}^t = \sum_{i=1}^n \left(\inf_{[r_i, t]} f - f(r_i-) \right) \leq \sum_{i=1}^n f(r_{i+1}-) - f(r_i-) = f(t-) - f(r_1-).$$

Eventually, $f(r_1-) \geq \inf_{[s, t]} f$ because $r_1 \in (s, t]$. The lemma follows. \square

2.2 Pseudo-distances coded by an excursion

We denote by (\mathcal{C}, δ) the metric circle of perimeter 1, seen as $[0, 1]$ endowed with the pseudo-distance δ given by $\delta(a, b) = \min(|a - b|, 1 - |a - b|)$ so that 0 and 1 are identified. We say that a function is càglàd when it is left-continuous with right limits (*continue à gauche, limite à droite* in French). We denote by $\overleftarrow{\mathcal{D}}([0, 1], \mathcal{C})$ the set of càglàd functions from $[0, 1]$ to \mathcal{C} and we endow it with the Skorokhod distance $\overleftarrow{\rho}$ given by

$$\overleftarrow{\rho}(\phi_1, \phi_2) = \inf_{\lambda \in \Lambda} \max \left(\|\lambda - \text{id}\|_\infty, \sup_{x \in [0, 1]} \delta(\phi_1(x), \phi_2 \circ \lambda(x)) \right)$$

for all $\phi_1, \phi_2 \in \overleftarrow{\mathcal{D}}([0, 1], \mathcal{C})$, where Λ is the set of increasing bijections from $[0, 1]$ into itself.

Definition 2. A *shuffle* is a Borel application $\Phi : \Delta > 0 \mapsto \phi_\Delta \in \overleftarrow{\mathcal{D}}([0, 1], \mathcal{C})$ such that for all $\Delta > 0$, ϕ_Δ is surjective,

$$\sup_{u \in (0, \Delta]} \sup_{x \in (0, 1]} \left| \frac{2}{x} \delta(\phi_u(0), \phi_u(x)) - 1 \right| < \infty, \text{ and } \sup_{x \in (0, 1]} \left| \frac{2}{x} \delta(\phi_\Delta(0), \phi_\Delta(x)) - 1 \right| \xrightarrow{\Delta \rightarrow 0+} 0. \quad (3)$$

Remark 1. Our requirements for the ϕ_Δ to be surjective with the convergence in (3) lead to the fact that the ϕ_Δ have to oscillate more and more quickly near $x = 0$ when Δ is small enough. Moreover, when Δ gets smaller, ϕ_Δ needs to jump an infinite number of times near 0. As a result, there is no simple or canonical choice for Φ . Nevertheless, it is not too difficult to construct various examples of shuffles. For instance, we could choose $\phi_\Delta(x) = x$ when $\Delta \geq 1$ and

$$\phi_{1-\Delta}(x) = \begin{cases} x - \frac{\Delta^k}{2} & \text{if } x \in \left(\frac{\Delta^k + \Delta^{k+1}}{2}, \Delta^k \right] \text{ with an integer } k \\ 1 - x + \frac{\Delta^{k+1}}{2} & \text{if } x \in \left(\Delta^{k+1}, \frac{\Delta^k + \Delta^{k+1}}{2} \right] \text{ with an integer } k \\ 0 & \text{if } x = 0 \end{cases}$$

when $\Delta \in (0, 1)$, for all $x \in [0, 1]$.

For the following, we fix a shuffle and let the dependence on it be implicit. As we said before, the loops of our space will match the jumps of f , and the lengths of the loops will be equal to the heights of the jumps. For $t \in [0, 1]$, we provide the pseudo-distances

$$\begin{aligned} \delta_t(a, b) &= \Delta_t \delta \left(\frac{a}{\Delta_t}, \frac{b}{\Delta_t} \right) = \min(|a - b|, \Delta_t - |a - b|), \\ \tilde{\delta}_t(a, b) &= \Delta_t \delta \left(\phi_{\Delta_t} \left(\frac{a}{\Delta_t} \right), \phi_{\Delta_t} \left(\frac{b}{\Delta_t} \right) \right) \end{aligned}$$

to the segment $[0, \Delta_t]$ when $\Delta_t > 0$. When $\Delta_t = 0$, we set $\delta_t(0, 0) = \tilde{\delta}_t(0, 0) = 0$. If needed, we would precise the dependence on f by writing $\delta_t^f(\cdot, \cdot)$ or $\tilde{\delta}_t^f(\cdot, \cdot)$. Let us notice that $([0, \Delta_t], \delta_t)$ and $([0, \Delta_t], \tilde{\delta}_t)$ are isometric to a circle of length Δ_t .

We are now ready to define our pseudo-distances of interest. For $s, t \in [0, 1]$ with $s \preceq t$, we set

$$d_f^\circ(s, t) = \sum_{s \prec r \preceq t} \delta_r(0, x_r^t),$$

$$\tilde{d}_f^\circ(s, t) = \sum_{s \prec r \preceq t} \tilde{\delta}_r(0, x_r^t).$$

It is the distance to run through between t and its ancestor on the loop of s . Indeed, each term of the sum corresponds to the length of the path getting through one of the loops between s and t . For \tilde{d}° , the points where two loops are glued were shuffled. We reach t from s by going through the loop of $t \wedge s$. Explicitly, we set

$$d_f^{\text{L}}(s, t) = \delta_{s \wedge t}(x_{s \wedge t}^s, x_{s \wedge t}^t) + d_f^\circ(s \wedge t, s) + d_f^\circ(s \wedge t, t),$$

$$\tilde{d}_f^{\text{L}}(s, t) = \tilde{\delta}_{s \wedge t}(x_{s \wedge t}^s, x_{s \wedge t}^t) + \tilde{d}_f^\circ(s \wedge t, s) + \tilde{d}_f^\circ(s \wedge t, t),$$

for all $s, t \in [0, 1]$. The pseudo-distance d^{L} is exactly the one constructed by Curien and Kortchemski in [18] for the stable looptrees. Now, we want to define a pseudo-distance from f that should induce a tree. For $s, t \in [0, 1]$ with $s \preceq t$, we set

$$d_f^{\text{T}}(s, t) = f(t) - \inf_{[s, t]} f - \sum_{s \prec r \preceq t} x_r^t.$$

While the term $f(t) - \inf_{[s, t]} f$ should correspond to the distance to travel if the metric space was a tree, the excursion has jumps we have to erase because we do not want to count the distance traveled on loops. Thus, we subtract the terms x_r^t that are already counted in the term $f(t) - \inf_{[s, t]} f$. More generally, if $s, t \in [0, 1]$, we set

$$d_f^{\text{T}}(t, s) = d_f^{\text{T}}(t \wedge s, t) + d_f^{\text{T}}(t \wedge s, s).$$

Observe the definition of d^{T} is consistent because $d^{\text{T}}(s, s) = f(s) - f(s) = 0$. In order to combine the paths on the loops and on the tree branches, we set

$$d_f^{\text{V}} = d_f^{\text{T}} + 2d_f^{\text{L}},$$

$$\tilde{d}_f^{\text{V}} = d_f^{\text{T}} + 2\tilde{d}_f^{\text{L}}.$$

The constant 2 before d^{L} (or \tilde{d}^{L}) could be replaced with another one but we will justify this choice by the fact that $f \mapsto \tilde{d}_f^{\text{V}}$ enjoys a functional continuity with the constant 2. It would be not the case with another constant. Nevertheless, Marzouk showed in [35] that metrics $d^{\text{L}} + ad^{\text{T}}$ with $a \neq 1/2$ can naturally appear as scaling limits of uniformly random discrete looptrees with prescribed degrees. When the function f is obvious according to context, we will just write $d^\circ, \tilde{d}^\circ, d^{\text{L}}, \tilde{d}^{\text{L}}, d^{\text{T}}, d^{\text{V}}$, and \tilde{d}^{V} . It is important to be aware that the shuffle Φ was used to construct \tilde{d}^{L} and \tilde{d}^{V} , but not for $d^{\text{L}}, d^{\text{T}}$, and d^{V} .

We now show our quantities are finite. The functions $d^\circ, \tilde{d}^\circ, d^{\text{L}}$, and \tilde{d}^{L} obviously are non-negative because they are series of non-negative terms. Keeping only the term for $r = t$ yields that if $s \prec t$ then $d^{\text{T}}(s, t) \leq f(t-) - \inf_{[s, t]} f$. Moreover, the bound (2) implies that if $s \prec t$ then

$$0 \leq d^\circ(s, t) \leq f(t-) - \inf_{[s, t]} f, \tag{4}$$

$$0 \leq d^{\text{T}}(s, t) \leq f(t-) - \inf_{[s, t]} f, \tag{5}$$

because $\delta_t(0, x_t^t) = \delta_t(0, \Delta_t) = 0$ and $\delta(0, x) \leq x$ for all $x \in [0, 1]$. If we only have $s \preceq t$, the upper bound given by $f(t) - \inf_{[s, t]} f$ stays true. Now, we only assume $s < t$. We observe $t \wedge s \leq s < t$ and $\delta_{t \wedge s}(x_{s \wedge t}^s, x_{s \wedge t}^t) \leq x_{t \wedge s}^s - x_{t \wedge s}^t$. Recall we know $\inf_{[s, t]} f = \inf_{[t \wedge s, t]} f$ from Lemma 1, so if $s < t$ then

$$0 \leq d^{\text{L}}(s, t) \leq f(s) + f(t-) - 2 \inf_{[s, t]} f, \tag{6}$$

$$0 \leq d^{\text{T}}(s, t) \leq f(s) + f(t-) - 2 \inf_{[s, t]} f. \tag{7}$$

Remark we cannot find these bounds for \tilde{d}^\perp because $\tilde{\delta}_t(0, \Delta_t) \neq 0$ and $\tilde{\delta}_t(a, b) \not\leq |a - b|$. Nethertheless, the bound in (3) ensures there exists a constant $K \in (0, \infty)$ which only depends on $\|f\|_\infty$ and Φ such that $\tilde{\delta}_t(0, x) \leq Kx$ for any $t \in [0, 1]$ and $x \in [0, \Delta_t]$. In particular, if $s \preceq t$, then the bound (2) implies

$$0 \leq \tilde{d}^\circ(s, t) \leq K \left(f(t) - \inf_{[s, t]} f \right). \quad (8)$$

While the definitions of these quantities are intuitive and natural, the fact that $s \wedge t$ appears could make them a little troublesome to manipulate, from a computational point of view. To handle this issue, we verify that for all $s, t \in [0, 1]$ with $s \leq t$, it holds

$$d^\perp(s, t) = \sum_{r \in [0, 1]} \delta_r(x_r^s, x_r^t), \quad (9)$$

$$\tilde{d}^\perp(s, t) = \sum_{r \in [0, 1]} \tilde{\delta}_r(x_r^s, x_r^t), \quad (10)$$

$$d^\top(s, t) = f(s) + f(t) - 2 \inf_{[s, t]} f - \sum_{r \in [0, 1]} |x_r^s - x_r^t|. \quad (11)$$

Indeed, Lemma 2 ensures $x_r^s = x_r^t$ when $r \prec t \wedge s$, which yields

$$\begin{aligned} \delta_r(x_r^s, x_r^t) &= \delta_r(0, x_r^s) \mathbf{1}_{s \wedge t \prec r \preceq s} + \delta_r(0, x_r^t) \mathbf{1}_{s \wedge t \prec r \preceq t} + \delta_r(x_r^s, x_r^t) \mathbf{1}_{s \wedge t = r} \\ \tilde{\delta}_r(x_r^s, x_r^t) &= \tilde{\delta}_r(0, x_r^s) \mathbf{1}_{s \wedge t \prec r \preceq s} + \tilde{\delta}_r(0, x_r^t) \mathbf{1}_{s \wedge t \prec r \preceq t} + \tilde{\delta}_r(x_r^s, x_r^t) \mathbf{1}_{s \wedge t = r}, \\ |x_r^s - x_r^t| &= x_r^s \mathbf{1}_{s \wedge t \prec r \preceq s} + x_r^t \mathbf{1}_{s \wedge t \prec r \preceq t} + \left(\inf_{[r, s]} f - \inf_{[r, t]} f \right) \mathbf{1}_{s \wedge t = r}. \end{aligned}$$

Thus, (9) and (10) follow and Lemma 1 leads to (11). Let us show our quantities are pseudo-distances.

Proposition 2. *The functions d^\perp , \tilde{d}^\perp , and d^\top are pseudo-distances on $[0, 1]$. As linear combinations of pseudo-distances, d^\vee and \tilde{d}^\vee are also pseudo-distances on $[0, 1]$.*

Proof. We already proved these functions are well-defined and non-negative. The symmetries are obvious. Let $t \in [0, 1]$, it holds clearly $d^\perp(t, t) = \tilde{d}^\perp(t, t) = 0$ and we already saw $d^\top(t, t) = 0$. Let $s, r \in [0, 1]$. We want to prove the triangular inequalities on $d^\perp(t, r)$, $\tilde{d}^\perp(t, r)$, and $d^\top(t, r)$. The formula (9) makes it easy for d^\perp as a simple application of the triangular inequality of δ_u :

$$d^\perp(t, r) = \sum_{u \in [0, 1]} \delta_u(x_u^t, x_u^r) \leq \sum_{u \in [0, 1]} (\delta_u(x_u^t, x_u^s) + \delta_u(x_u^s, x_u^r)) = d^\perp(t, s) + d^\perp(s, r).$$

The proof for \tilde{d}^\perp is identical with (10) but the proof for d^\top requires some more care. We know either $s \wedge r \preceq t \wedge s$ or $t \wedge s \preceq s \wedge r$, because the ancestral lineage of s is totally ordered. By symmetry, we can assume $s \wedge r \preceq t \wedge s$ without loss of generality. Then, we also have $s \wedge r \preceq t \wedge r$. Thanks to the total order of the ancestral lineage of t , it holds either $t \wedge r \preceq t \wedge s$ or $t \wedge s \prec t \wedge r$. If $t \wedge r \preceq t \wedge s$, then $t \wedge r \preceq s \wedge r$, and by antisymmetry of \preceq , we have $t \wedge r = s \wedge r$. Likewise, if $t \wedge s \prec t \wedge r$ then $t \wedge s \preceq s \wedge r$ and $t \wedge s = s \wedge r$. From here, we treat the two cases separately. Recall Lemma 2 which we will use several times.

- If $t \wedge r = s \wedge r \preceq t \wedge s$: We can write $d^\top(t, r) = d^\top(t \wedge r, t) + d^\top(s \wedge r, r)$. Let us begin by bounding the distance between $t \wedge r$ and t . Because $t \wedge r \preceq t \wedge s \preceq t$, we can easily make the distance between $t \wedge s$ and t appear, which gives

$$d^\top(t \wedge r, t) = d^\top(t \wedge s, t) + \left(\inf_{[t \wedge s, t]} f - \inf_{[t \wedge r, t]} f \right) - \sum_{t \wedge r \prec u \preceq t \wedge s} x_u^t.$$

Now, we use Lemma 2 to transform the sum indexed by $s \wedge r = t \wedge r \prec u \prec t \wedge s$. We point out that the lemma does not allow us to transform the term for $u = t \wedge s$, thus we isolate it. Then, we recognize some terms of the distance between $s \wedge r$ and s , namely

$$\sum_{t \wedge r \prec u \preceq t \wedge s} x_u^t = \sum_{s \wedge r \prec u \prec t \wedge s} x_u^s + x_{t \wedge s}^t \mathbf{1}_{s \wedge r < t \wedge s}.$$

We make $d^\Gamma(s \wedge r, s)$ appear by adding and subtracting the x_u^s for $t \wedge s \preceq u \preceq s$, so that

$$- \sum_{t \wedge r \prec u \preceq t \wedge s} x_u^t = d^\Gamma(s \wedge r, s) + \inf_{[s \wedge r, s]} f - d^\Gamma(t \wedge s, s) - \inf_{[t \wedge s, s]} f - (x_{t \wedge s}^t - x_{t \wedge s}^s) \mathbf{1}_{s \wedge r < t \wedge s}.$$

Since $d^\Gamma(t \wedge s, s) \geq 0$, it follows

$$d^\Gamma(t \wedge r, t) \leq d^\Gamma(t \wedge s, t) + d^\Gamma(s \wedge r, s) - (x_{s \wedge r}^t - x_{s \wedge r}^s) + (x_{t \wedge s}^t - x_{t \wedge s}^s) \mathbf{1}_{s \wedge r = t \wedge s}.$$

Then, we remark that $-(x_{s \wedge r}^t - x_{s \wedge r}^s) + (x_{t \wedge s}^t - x_{t \wedge s}^s) \mathbf{1}_{s \wedge r = t \wedge s} = -(x_{s \wedge r}^t - x_{s \wedge r}^s) \mathbf{1}_{s \wedge r < t \wedge s}$, so Lemma 2 ensures $d^\Gamma(t \wedge r, t) \leq d^\Gamma(t \wedge s, t) + d^\Gamma(s \wedge r, s)$. Eventually, we find

$$d^\Gamma(t, r) \leq d^\Gamma(s \wedge r, r) + d^\Gamma(t \wedge s, t) + d^\Gamma(s \wedge r, s) \leq d^\Gamma(t, s) + d^\Gamma(s, r).$$

- If $t \wedge s = s \wedge r \prec t \wedge r$: In this case, the triangular inequality is rougher, so its proof is simpler. To bound $d^\Gamma(t \wedge r, t)$, we first make $d^\Gamma(t \wedge s, t)$ appear by writing

$$d^\Gamma(t \wedge r, t) = d^\Gamma(t \wedge s, t) + \inf_{[t \wedge s, t]} f - \inf_{[t \wedge r, t]} f + \sum_{t \wedge s \prec u \preceq t \wedge r} x_u^t.$$

We transform the sum thanks to Lemma 2 once again and we see $\inf_{[t \wedge r, t]} f = x_{t \wedge r}^t - x_{t \wedge r}^s + f(t \wedge r)$, so that we get

$$d^\Gamma(t \wedge r, t) = d^\Gamma(t \wedge s, t) + \inf_{[t \wedge s, t]} f - \inf_{[t \wedge s, t \wedge r]} f - d^\Gamma(t \wedge s, t \wedge r).$$

Lemma 2 also implies $\inf_{[t \wedge s, t]} f = \inf_{[t \wedge s, t \wedge r]} f$. Plus, $d^\Gamma(t \wedge s, t \wedge r)$ is non-negative, so it follows $d^\Gamma(t \wedge r, t) \leq d^\Gamma(t \wedge s, t)$. Thanks to the symmetry between t and r in this case, we find

$$d^\Gamma(t, r) \leq d^\Gamma(t \wedge s, t) + d^\Gamma(s \wedge r, r) \leq d^\Gamma(t, s) + d^\Gamma(s, r).$$

□

Remark 2. The triangular inequality for d^Γ will become clearer later. Indeed, we will prove with Theorem 4 that there exists a continuous excursion g with $g(0) = 0$ such that $d_f^\Gamma = d_g^{\text{clas}}$, where d^{clas} is the classical tree pseudo-distance defined by (1). In fact, Marzouk defines the same pseudo-distance d^Γ as such in [35].

2.3 Quotient metric spaces induced by those pseudo-distances

A *pointed metric space* is a triple (X, d, a) where (X, d) is a metric space equipped with a distinguished point, also called a root, $a \in X$. We say two pointed metric spaces (X_1, d_1, a_1) and (X_2, d_2, a_2) are *pointed-isometric* when there exists a bijective isometry λ from X_1 to X_2 such that $\lambda(a_1) = a_2$. A *pointed weighted metric space* is a quadruple (X, d, a, μ) where (X, d, a) is a pointed metric space also equipped with a Borel probability measure μ on (X, d) . We say two pointed weighted metric spaces (X_1, d_1, a_1, μ_1) and (X_2, d_2, a_2, μ_2) are *GHP-isometric* when there exists a bijective isometry λ from X_1 to X_2 such that $\lambda(a_1) = a_2$ and $\lambda_* \mu_1 = \mu_2$. When no confusion is possible, we will denote a metric space (possibly endowed with a root and/or a Borel probability measure) by its underlying set. Finally,

when $c > 0$, we write $c \cdot (X, d, a, \mu) = (X, cd, a, \mu)$ for the pointed weighted metric space obtained after multiplying all distances by c . As same, we write $c \cdot (X, d, a) = (X, cd, a)$ and $c \cdot (X, d) = (X, cd)$.

Let d be a pseudo-distance on a set X . We denote by $X/\{d = 0\}$ the quotient space obtained by identifying the points $x, y \in X$ such that $d(x, y) = 0$ and we endow it with the genuine distance (still denoted by d with a slight abuse of notation) induced by d . When $X = [0, 1]$, we equip the metric space $[0, 1]/\{d = 0\}$ with the canonical projection of 1 and with the image of the Lebesgue measure on $[0, 1]$, so that it is a pointed weighted metric space. We define the following pointed weighted metric spaces:

- the tree coded by f , denoted by \mathcal{T}_f , as $[0, 1]/\{d_f^\top = 0\}$,
- the looptree coded by f , denoted by \mathcal{L}_f , as $[0, 1]/\{d_f^L = 0\}$,
- the shuffled (with Φ) looptree coded by f , denoted by $\widetilde{\mathcal{L}}_f$, as $[0, 1]/\{\widetilde{d}_f^L = 0\}$,
- the veneration tree coded by f , denoted by \mathcal{V}_f , as $[0, 1]/\{d_f^V = 0\}$,
- the shuffled (with Φ) veneration tree coded by f , denoted by $\widetilde{\mathcal{V}}_f$, as $[0, 1]/\{\widetilde{d}_f^V = 0\}$.

Proposition 3. *The functions d^L, d^\top, d^V are continuous on $[0, 1]^2$. The metric spaces $\mathcal{L}_f, \mathcal{T}_f$, and \mathcal{V}_f are thus compact.*

Proof. Since d^L is a pseudo-distance, we only need to show that $d^L(t, t_n) \rightarrow 0$ when $t_n \rightarrow t-$ or $t_n \rightarrow t+$. We use the inequality (6) and the fact that f is an excursion. If $t_n < t < t_m$ then

$$\begin{aligned} d^L(t, t_n) &\leq f(t-) + f(t_n) - 2 \inf_{[t_n, t]} f \xrightarrow{t_n \rightarrow t-} f(t-) + f(t-) - 2f(t-) = 0, \\ d^L(t, t_m) &\leq f(t) + f(t_m-) - 2 \inf_{[t, t_m]} f \xrightarrow{t_m \rightarrow t+} f(t) + f(t) - 2f(t) = 0. \end{aligned}$$

We prove the continuity of d^\top in the same way with (7). The continuity of d^V follows. \square

The pseudo-distance \widetilde{d}^L (as well as \widetilde{d}^V) is not continuous in general because of the possible discontinuities of the ϕ_Δ . However, it still enjoys some regularity.

Definition 3. We denote by $\mathcal{D}([0, 1]^2)$ the set of functions $\psi : [0, 1]^2 \rightarrow \mathbb{R}$ such that for every monotonous sequences (s_n) and (t_n) of elements of $[0, 1]$, the sequence $(\psi(s_n, t_n))$ converges, and such that its limit is $\psi(\lim s_n, \lim t_n)$ when (s_n) and (t_n) are non-increasing. We provide a distance ρ_2 to $\mathcal{D}([0, 1]^2)$ by setting

$$\rho_2(\psi_1, \psi_2) = \inf_{\lambda, \mu \in \Lambda} \max(\|\psi_1 - \psi_2 \circ (\lambda, \mu)\|_\infty, \|\lambda - \text{id}\|_\infty, \|\mu - \text{id}\|_\infty)$$

for all $\psi_1, \psi_2 \in \mathcal{D}([0, 1]^2)$, where Λ is the set of increasing bijections from $[0, 1]$ into itself. The distance ρ_2 is not complete but induces a separable and completely metrizable topology on $\mathcal{D}([0, 1]^2)$, that we call the Skorokhod topology.

This is a generalization of the Skorokhod space $\mathcal{D}([0, 1])$ for bivariate functions, and is more precisely presented in [40]. Let us at least mention that if (ψ_n) converges to ψ for this topology and if ψ is continuous on $[0, 1]^2$, then the convergence also happens uniformly on $[0, 1]^2$. Indeed, such ψ would be continuous on a compact, so uniformly continuous, which is enough to conclude.

Proposition 4. *The metric spaces $\widetilde{\mathcal{L}}_f$ and $\widetilde{\mathcal{V}}_f$ are compact and the functions \widetilde{d}^L and \widetilde{d}^V are in $\mathcal{D}([0, 1]^2)$. Equivalently, the canonical projection maps from $[0, 1]$ to \mathcal{L}_f or \mathcal{V}_f are càdlàg.*

Proof. By triangular inequality, it is enough to show that if $s_n \uparrow s$ and $t_n \downarrow t$, then $\tilde{d}^{\downarrow}(t_n, t) \rightarrow 0$ and there is $\bar{s} \in [0, 1]$ such that $\tilde{d}^{\downarrow}(s_n, \bar{s}) \rightarrow 0$ and $d^{\uparrow}(s, \bar{s}) = 0$. We begin by proving $\tilde{d}^{\downarrow}(t, t_n) \rightarrow 0$. Thanks to the inequality (8) and Lemma 1, there exists a constant $K \in (0, \infty)$ such that

$$\tilde{d}^{\circ}(t \wedge t_n, t) + \tilde{d}^{\circ}(t \wedge t_n, t_n) \leq K \left(f(t) + f(t_n) - 2 \inf_{[t, t_n]} f \right),$$

but $f(t) + f(t_n) - 2 \inf_{[t, t_n]} f \rightarrow 0$ because f is càdlàg. Hence, we only need to show $\tilde{d}_{t \wedge t_n}(x_{t \wedge t_n}^t, x_{t \wedge t_n}^{t_n})$ tends to 0. Thanks to Lemma 1, we observe the sequence $(t \wedge t_n)$ is non-decreasing. Moreover, the set $\{r \in [0, 1] : \Delta_r \geq \varepsilon\}$ is finite for all $\varepsilon > 0$, so only two cases are possible: either $\Delta_{t \wedge t_n} \rightarrow 0$, or there exists $\tau \in [0, 1]$ with $\Delta_\tau > 0$ such that $t \wedge t_n = \tau$ for n large enough.

- In the first case: We have $\tilde{d}_{t \wedge t_n}(x_{t \wedge t_n}^t, x_{t \wedge t_n}^{t_n}) \leq \Delta_{t \wedge t_n} \rightarrow 0$.
- In the second case: When n is large enough, we have $\tilde{d}_{t \wedge t_n}(x_{t \wedge t_n}^t, x_{t \wedge t_n}^{t_n}) = \tilde{d}_\tau(x_\tau^t, x_\tau^{t_n})$. But, $x_\tau^{t_n} \uparrow x_\tau^t$ because f is càdlàg. The function ϕ_{Δ_τ} is càglàd, so $\tilde{d}_\tau(x_\tau^t, x_\tau^{t_n}) \rightarrow \hat{d}_\tau(x_\tau^t, x_\tau^t) = 0$.

This concludes the proof of $\tilde{d}^{\downarrow}(t, t_n) \rightarrow 0$.

We set $s' = \inf\{r \geq s : f(r) \leq f(s-)\}$, so that $\inf_{[s, s']} f = f(s-) = f(s') = f(s'-)$ because f is an excursion. Thus, $s \preceq s'$ and $d^{\uparrow}(s, s') = 0$ by the inequality (5). The sequence $(s_n \wedge s')$ is non-decreasing according to Lemma 1, so as before, either $\Delta_{s_n \wedge s'} \rightarrow 0$, or there exists $\sigma \in [0, 1]$ with $\Delta_\sigma > 0$ such that $s_n \wedge s' = \sigma$ for n large enough.

- In the first case: We set $\bar{s} = s'$. As f is càdlàg, it holds $f(\bar{s}) + f(s_n) - 2 \inf_{[s_n, \bar{s}]} f \rightarrow 0$ by definition of $s' = \bar{s}$. Plus, we have $\tilde{d}_{\bar{s} \wedge s_n}(x_{\bar{s} \wedge s_n}^{s_n}, x_{\bar{s} \wedge s_n}^{\bar{s}}) \leq \Delta_{s_n \wedge s'} \rightarrow 0$ by assumption. Then, recall the use of (8) and of Lemma 1 to conclude $\tilde{d}^{\downarrow}(s_n, \bar{s}) \rightarrow 0$. We already know $d^{\uparrow}(s, \bar{s}) = 0$.
- In the second case: When n is large enough, it holds $\sigma = s_n \wedge s' \preceq s'$ so $\inf_{[\sigma, s']} f = \inf_{[s_n, s']} f$ according to Lemma 1. But, we see $\inf_{[s_n, s']} f \rightarrow f(s')$ by definition of s' , so the inequality (5) yields $d^{\uparrow}(\sigma, s') = 0$. We already know $d^{\uparrow}(s, s') = 0$ so $d^{\uparrow}(\sigma, s) = 0$. Let us set

$$\bar{s} = \inf \left\{ r \geq \sigma : \phi_{\Delta_\sigma} \left(\frac{f(r) - f(\sigma-)}{\Delta_\sigma} \right) = \phi_{\Delta_\sigma} \left(\frac{x_\sigma^s}{\Delta_\sigma} + \right) \right\},$$

which is well-defined because ϕ_{Δ_σ} is surjective and càglàd, and because f is an excursion. Using properties of excursions, we find $\sigma \preceq \bar{s}$ and $f(\bar{s}) = \inf_{[\sigma, \bar{s}]} f$. As a result, $\sigma \preceq s_n \wedge \bar{s}$ when n is large enough and if $r \in (\sigma, \bar{s}]$ then $x_r^{\bar{s}} = 0$. It follows $d^{\uparrow}(\sigma, \bar{s}) = 0 = d^{\uparrow}(s, \bar{s})$ and

$$\tilde{d}^{\downarrow}(s_n, \bar{s}) \leq \tilde{d}^{\circ}(s_n \wedge s', s_n) + \tilde{d}_\sigma(x_\sigma^{s_n}, x_\sigma^{\bar{s}})$$

when n is large enough. As seen in the previous case, $f(s_n) - \inf_{[s_n, s']} f \rightarrow 0$ so (8) and Lemma 1 ensure $\tilde{d}^{\circ}(s_n \wedge s', s_n) \rightarrow 0$. Then, we check $\phi_{\Delta_\sigma}(x_\sigma^{\bar{s}}/\Delta_\sigma) = \phi_{\Delta_\sigma}(x_\sigma^s/\Delta_\sigma +)$ using the left-continuity of ϕ_{Δ_σ} . Since $f(s) \geq f(s-)$, $x_\sigma^{s_n}$ non-increasingly tends towards x_σ^s . Thus, $\tilde{d}_\sigma(x_\sigma^{s_n}, x_\sigma^{\bar{s}}) \rightarrow 0$ and the proof is completed. \square

2.4 First properties and an example

Here, we provide some basic properties of the codings $f \in \mathcal{H} \mapsto d_f^{\downarrow}, \tilde{d}_f^{\downarrow}, d_f^{\uparrow}, d_f^{\vee}, \tilde{d}_f^{\vee}$, that they share with the classical coding of a tree (1). These properties make the codings somewhat easy to manipulate.

Proposition 5. *The application $f \in \mathcal{H} \mapsto d_f^{\downarrow}$ (respectively $d_f^{\uparrow}, d_f^{\vee}$) is homogeneous. Namely, if $\alpha > 0$, then it holds $d_{\alpha f}^{\downarrow} = \alpha d_f^{\downarrow}$. Thus, the pointed weighted metric spaces $\mathcal{L}_{\alpha f}$ and $\alpha \cdot \mathcal{L}_f$ are GHP-isometric.*

The application $f \in \mathcal{H} \mapsto d_f^{\downarrow}$ (respectively $d_f^{\uparrow}, \tilde{d}_f^{\downarrow}, d_f^{\vee}, \tilde{d}_f^{\vee}$) enjoys the time-changing property. Namely, for any increasing bijection $\lambda : [0, 1] \rightarrow [0, 1]$, it holds $d_{f \circ \lambda}^{\downarrow}(s, t) = d_f^{\downarrow}(\lambda(s), \lambda(t))$ for all $s, t \in [0, 1]$. Thus, the pointed metric spaces $\mathcal{L}_{f \circ \lambda}$ and \mathcal{L}_f are pointed-isometric.

Proof. The first thing to see is that αf and $f \circ \lambda$ are indeed excursions. Then, it is obvious that $x_s^t(\alpha f) = \alpha x_s^t(f)$ and $x_s^t(f \circ \lambda) = x_{\lambda(s)}^{\lambda(t)}(f)$, for all $s, t \in [0, 1]$. In particular, $\Delta_s(\alpha f) = \alpha \Delta_s(f)$ and $\Delta_s(f \circ \lambda) = \Delta_{\lambda(s)}(f)$. By definition of δ_s and $\tilde{\delta}_s$ and thanks to the identities (9), (10), and (11), the proposition follows. Let us point out that we have used $\delta_s(\alpha a, \alpha b) = \alpha \delta_s(a, b)$ which does not hold for $\tilde{\delta}_s$, which explains why $f \mapsto \tilde{d}_f^{\text{L}}$ and $f \mapsto \tilde{d}_f^{\text{V}}$ are not homogeneous. \square

Another important property of the codings is the branching property. It tells that a "sub-excursion" codes a subspace of the looptree or the veneration tree. It allows observing directly the structure of the coded space on the graph of the excursion. We have already presented this idea when we defined the genealogy \preceq .

Definition 4. Let (X_0, d_0, a_0) and (X_1, d_1, a_1) be two pointed metric spaces and let $a \in X_0$. We write $a_0^* = a$ and $a_1^* = a_1$. The *gluing* of X_1 on X_0 at a is the pointed metric space denoted by $(X_0 \vee_a X_1, d, a_0)$ that is obtained from the quotient of the disjoint union $X_0 \sqcup X_1$ by the identification $a \sim a_1$, endowed with the distance d defined by setting $d(x, y) = d_i(x, y)$ if $x, y \in X_i$ with $i \in \{0, 1\}$, and by setting $d(x, y) = d_i(x, a_i^*) + d_{1-i}(a_{1-i}^*, y)$ if $x \in X_i$ and $y \in X_{1-i}$ with $i \in \{0, 1\}$. We see X_0 and X_1 as closed subsets of $X_0 \vee_a X_1$, so that $X_0 \cup X_1 = X_0 \vee_a X_1$, $X_0 \cap X_1 = \{a\} = \{a_1\}$, and the distinguished points of X_0 and $X_0 \vee_a X_1$ are the same.

Proposition 6. *The application $f \in \mathcal{H} \mapsto d_f^{\text{L}}$ (respectively $d_f^{\text{T}}, \tilde{d}_f^{\text{L}}, d_f^{\text{V}}, \tilde{d}_f^{\text{V}}$) enjoys the branching property. Namely, let $u, v \in [0, 1]$ be such that $u \prec v$ and $f(u-) = f(v-)$, and let us set*

$$g : t \in [0, 1] \mapsto \begin{cases} f(t) & \text{if } t \notin [u, v) \\ f(u-) & \text{if } t \in [u, v) \end{cases},$$

$$h : t \in [0, 1] \mapsto \begin{cases} f(u + t(v - u)) - f(u-) & \text{if } t < 1 \\ 0 & \text{if } t = 1 \end{cases}.$$

If $s, t \in [0, 1)$ and $a, b \in [0, 1] \setminus [u, v)$, then

$$d_f^{\text{L}}(a, b) = d_g^{\text{L}}(a, b),$$

$$d_f^{\text{L}}(u + s(v - u), u + t(v - u)) = d_h^{\text{L}}(s, t),$$

$$d_f^{\text{L}}(u + s(v - u), a) = d_g^{\text{L}}(a, u) + d_h^{\text{L}}(s, 1).$$

Thus, the pointed metric space \mathcal{L}_f is pointed-isometric to the gluing of \mathcal{L}_h on \mathcal{L}_g at the canonical projection of u .

Proof. First, we observe that g and h are indeed excursions. We only show the result for d^{L} and d^{T} , as the same proof holds for \tilde{d}^{L} and because the branching property is preserved by linear combination. Let $a, b \in [0, 1] \setminus [u, v)$ and $s, t \in [0, 1)$. We assume $a \leq b$ and $s \leq t$, and we set $w = u + s(v - u)$ to lighten the notations. We immediately observe that $x_{u+s(v-u)}^{u+t(v-u)}(f) = x_s^t(h)$, and if $[u, v) \cap [a, b] = \emptyset$ then $x_a^b(f) = x_a^b(g)$. If $[a, b] \subset [u, v)$ then we still have $x_a^b(f) = x_a^b(g)$ because $\inf_{[u, v)} f = f(u-) = \inf_{[u, v)} g$. Concerning x_a^w , if $w < a$ then $x_a^w(f) = x_a^u(g) = 0$ by definition, since $u \leq w$. If $a \leq w$ then $a < u \leq w < v$, so $x_a^w(f) = x_a^u(g)$ still holds, because

$$\inf_{[u, w]} f \geq \inf_{[u, v]} f \geq f(u-) = \inf_{[u, w]} g \geq \inf_{[a, u]} f = \inf_{[a, u]} g.$$

Concerning x_w^a , if $a < w$ then $x_w^a(f) = x_w^a(g) = 0$ by definition. If $w \leq a$ then $u \leq w < v \leq a$, so $x_w^a(f) = x_w^a(g) = 0$ still holds because

$$f(w-) \geq \min \left(f(u-), \inf_{[u, v]} f \right) \geq f(u-) = g(w-) = f(v-) = g(v-) \geq \max \left(\inf_{[w, a]} f, \inf_{[w, a]} g \right).$$

Moreover, $x_w^u(g) \leq \Delta_w(g) = 0$ and $x_s^1(h) \leq h(1) = 0$, so that $x_w^u(g) = x_s^1(h) = 0$. To sum up, we have

$$\begin{aligned}\Delta_a(f) &= \Delta_a(g) \text{ and } \Delta_{u+s(v-u)}(f) = \Delta_s(h), \\ x_a^b(f) &= x_a^b(g) \text{ and } x_{u+s(v-u)}^{u+t(v-u)}(f) = x_s^t(h), \\ x_a^w(f) &= x_a^u(g) \text{ and } x_w^a(f) = x_w^a(g) = x_w^u(g) = x_s^1(h) = 0.\end{aligned}$$

Combining these identities with the formula (9), we obtain the branching property for d^L . Some easy other observations together with the formula (11) and the above identities lead to the branching property for d^T . \square

Corollary 1. *Let $s, t \in [0, 1]$. If $s \preceq t$, then $d^T(t, 1) = d^T(s, 1) + d^T(s, t)$.*

Proof. Let $\bar{s} = \inf\{r \geq t : f(r) = f(s-)\}$, we have $t \leq \bar{s}$, $s \preceq \bar{s}$, and $f(s-) = f(\bar{s}) = f(\bar{s}-)$. Observe $f(\bar{s}) = \inf_{[t, \bar{s}]} f \leq \inf_{[s, t]} f$ so $d_f^T(s, \bar{s}) = 0$ thanks to (5). If $t = \bar{s}$ then the result becomes obvious. Let us assume that $t < \bar{s}$. We use the notations of Proposition 6 with $u = s$ and $v = \bar{s}$. The inequality (5) gives $d_f^T(0, 1) = d_h^T(0, 1) = d_f^T(s, \bar{s}) = d_g^T(s, \bar{s}) = 0$. Then, the branching property implies the desired result. \square

Let us discuss how one can intuitively understand the looptree or the veneration tree coded by an excursion f from a glance at its graph. On the one hand, if f does not have any jump, then (9) and (11) ensure that $d^L = 0$ and that d^T is equal to the classical tree pseudo-distance d^{clas} defined by (1). On the other hand, each jump of f indeed corresponds to a loop in the associated looptree or veneration tree. Let $s \in [0, 1]$ such that $\Delta_s > 0$ and let $x \in \mathcal{C}$, the times

$$\begin{aligned}\tau(x) &= \inf \left\{ t \geq s : \frac{f(t) - f(s-)}{\Delta_s} = x \right\}, \\ \tilde{\tau}(x) &= \inf \left\{ t \geq s : \phi_{\Delta_s} \left(\frac{f(t) - f(s-)}{\Delta_s} \right) = x \right\}\end{aligned}$$

are well-defined, with $x_s^{\tau(x)}/\Delta_s = x$ and $\phi_{\Delta_s}(x_s^{\tilde{\tau}(x)}/\Delta_s) = x$. Then, we let the reader check that $d^T(\tau(x), \tau(y)) = d^T(\tilde{\tau}(x), \tilde{\tau}(y)) = 0$ and

$$d^L(\tau(x), \tau(y)) = \delta_s(x_s^{\tau(x)}, x_s^{\tau(y)}) = \Delta_s \delta(x, y) = \tilde{\delta}_s(x_s^{\tilde{\tau}(x)}, x_s^{\tilde{\tau}(y)}) = \tilde{d}^L(\tilde{\tau}(x), \tilde{\tau}(y))$$

for any $x, y \in \mathcal{C}$. Hence, the metric subspaces of \mathcal{L}_f and $\tilde{\mathcal{L}}_f$ respectively induced by $\{\tau(x) : x \in \mathcal{C}\}$ and $\{\tilde{\tau}(x) : x \in \mathcal{C}\}$ are both isometric to $\Delta_s \cdot \mathcal{C}$, namely a metric circle of length Δ_s . The metric subspaces of \mathcal{V}_f and $\tilde{\mathcal{V}}_f$ induced by the same previous subsets of $[0, 1]$ are isometric to $2\Delta_s \cdot \mathcal{C}$, namely a metric circle of length $2\Delta_s$. These observations together with the homogeneity, the time-changing property, and the branching property are generally enough to accurately draw \mathcal{T}_f , \mathcal{L}_f , or \mathcal{V}_f from the graph of f when it has simple variations. We give an example with Figure 2. For $\tilde{\mathcal{L}}_f$ and $\tilde{\mathcal{V}}_f$, the lack of homogeneity and the complexity of the shuffle can make them hard to be precisely drawn. Nevertheless, we can understand them as transformed versions of \mathcal{L}_f and \mathcal{V}_f where the positions of the joint points between loops or trees were shuffled, see Figure 2 once again.

3 Unification and relations between trees, looptrees, and veneration trees

In this section, we inspect more in-depth the intuitive principle that jumps code for loops and continuous growths code for branches in the associated veneration tree. The veneration distance d^V , or \tilde{d}^V , is the combinaison of two distances d^L , or \tilde{d}^L , and d^T . Similarly, we will see that we can always decompose an excursion into two others whose associated veneration pseudo-distances are respectively reduced to $2d^L$ and d^T . Let us begin with the definition of these two kinds of excursions.

Definition 5. Let f be an excursion. We say f is a *continuous excursion* when $\Delta_t = 0$ for all $t \in [0, 1]$. Notice that it do not only require f to be continuous on $[0, 1]$ but also that $f(0) = 0$. We say f is a *pure jump growth (PJG) excursion* when $f(t) = \sum_{r \preceq t} x_r^t$ for all $t \in [0, 1]$.

The following lemma is practical to both wrap one's head around the notion of PJG excursions and to identify them in simple cases.

Lemma 4. *Let f be an excursion. If there exists a partition $0 = r_0 < r_1 < \dots < r_n < r_{n+1} = 1$ such that f is non-increasing on $[r_i, r_{i+1})$ for all $0 \leq i \leq n$, then f is PJG.*

Proof. First, observe that f is PJG if and only if $d^\top(t, 1) = 0$ for all $t \in [0, 1]$. Indeed, $x_s^1 \leq f(1) = 0$ for all $s \in [0, 1]$ so the identity (11) becomes $d^\top(t, 1) = f(t) - \sum_{r \preceq t} x_r^t$. We prove the lemma by induction on n . If $n = 0$, then f is non-increasing on $[0, 1]$ so

$$d^\top(t, 1) \leq f(t) - x_0^t = f(t) - \inf_{[0, t]} f = 0,$$

and it follows that f is PJG. If $n \geq 1$, we set $\bar{r} = \inf\{r \geq r_n : f(r) = f(r_n-)\}$ so that $r_n \preceq \bar{r}$ and $f(r_n-) = f(\bar{r}) = f(\bar{r}-)$. We can assume $r_n < \bar{r}$ because otherwise, $f(r_n) = f(r_n-)$ which means that f is in fact non-increasing on $[r_{n-1}, 1]$ and so f is PJG by induction. We apply the branching property as given by Proposition 6 with $u = r_n$ and $v = \bar{r}$ and we keep the same notations. It is easy to see that h is non-increasing on $[0, 1]$ and that g is non-increasing on $[r_{n-1}, 1]$ and on $[r_i, r_{i+1})$ for all $0 \leq i \leq n-2$. By induction, g and h are PJG, so $d_g^\top(t, 1) = d_h^\top(t, 1) = 0$ for all $t \in [0, 1]$. The branching property then ensures $d_f^\top(t, 1) = 0$ for all $t \in [0, 1]$, so f is PJG. \square

For any $\varepsilon > 0$, we define the two following operators.

$$\begin{aligned} J : \mathcal{H} &\longrightarrow \mathcal{H} & J^\varepsilon : \mathcal{H} &\longrightarrow \mathcal{H} \\ f &\longmapsto Jf : t \mapsto \sum_{r \preceq t} x_r^t & f &\longmapsto J^\varepsilon f : t \mapsto \sum_{\substack{r \preceq t \\ \Delta_r \geq \varepsilon}} x_r^t \end{aligned}$$

The operator J is the same as in Theorem 1 and the operator J^ε will be necessary for some approximations later. We are now ready to formulate the decomposition of an excursion into a sum of a continuous excursion with a PJG excursion, and how this decomposition translates into the associated pseudo-distances.

Theorem 4. *Let f be an excursion and let $\varepsilon > 0$. The following points hold.*

- (i) *The function $f - Jf$ is a continuous excursion and the functions Jf and $J^\varepsilon f$ are PJG excursions. Moreover, for any $t \in [0, 1]$, $\Delta_t(J^\varepsilon f) > 0 \implies \Delta_t(J^\varepsilon f) \geq \varepsilon$.*
- (ii) *The excursion f is PJG if and only if $Jf = f$.*
- (iii) *The excursion f is a continuous excursion if and only if $Jf = 0$.*
- (iv) *We have $Jf = J^\varepsilon f$ if and only if for any $t \in [0, 1]$, $\Delta_t(f) > 0 \implies \Delta_t(f) \geq \varepsilon$.*
- (v) *The function $f - Jf + J^\varepsilon f$ is an excursion and $J(f - Jf + J^\varepsilon f) = J^\varepsilon f$.*

Proof. For any $t \in [0, 1]$, the quantities $Jf(t)$ and $J^\varepsilon f(t)$ are sums of non-negative terms, so $Jf(t) \geq 0$ and $J^\varepsilon f(t) \geq 0$. Plus, $x_t^1(f) \leq f(1) = 0$ so $Jf(1) = J^\varepsilon f(1) = 0 = f(1)$. It is clear $f(0) - J(0) = 0$. As seen in the proof of the above lemma, we have

$$d_f^\top(t, 1) = f(t) - Jf(t). \tag{12}$$

Since d_f^\top is a continuous pseudo-distance, this immediately gives that $f - Jf$ is continuous and non-negative on $[0, 1]$, that Jf is càdlàg, and that $\Delta_t(Jf) = \Delta_t(f)$ for all $t \in [0, 1]$. We remark the fact that f is an excursion implies that for any $r \in [0, 1]$, the function $t \in [0, 1] \mapsto x_r^t(f)$ is càdlàg with a unique jump at r , which has height $\Delta_r(f)$. Hence, the expression

$$J^\varepsilon f(t) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} x_r^t(f)$$

shows $J^\varepsilon f$ is càdlàg as a finite sum of càdlàg functions and $\Delta_t(J^\varepsilon f) = \Delta_t(f) \mathbf{1}_{\Delta_t(f) \geq \varepsilon}$. Let us show that Jf and $J^\varepsilon f$ are PJG to complete the proof of the point (i). The set $\{s \in [0, 1] : \Delta_s(f) \geq \varepsilon\}$ is finite, so it provides a partition $0 = s_0 < s_1 < \dots < s_{n-1} < s_n = 1$ such that $J^\varepsilon f$ is non-increasing on $[s_i, s_{i+1})$ for all $0 \leq i \leq n-1$, as a finite sum of non-increasing functions. Hence, $J^\varepsilon f$ is PJG according to Lemma 4. Let $s, t, u \in [0, 1]$ such that $s \leq u \leq t$. If $s \preceq_f t$ then $s \preceq_f u$ and

$$Jf(u) = Jf(s-) + f(u) - f(s-) - d_f^\top(s, u)$$

thanks to (12) and Corollary 1. The inequality (5) leads to

$$x_s^u \leq Jf(u) - Jf(s-) \leq f(u) - f(s-).$$

By taking the infimum over u , we can deduce that if $s \preceq_f t$ then $x_s^t(Jf) = x_s^t(f)$. Now, we set $\bar{s} = \inf\{r \geq s : f(r) = f(s-)\}$, so that $s \preceq_f \bar{s}$ but $x_{\bar{s}}^{\bar{s}}(f) = 0$. Then, if we do not have $s \preceq_f t$ then $\bar{s} < t$, and $x_s^t(Jf) \leq x_{\bar{s}}^t(Jf) = x_{\bar{s}}^t(f) = 0 = x_s^t(f)$. Thus, we have proven that for all $s, t \in [0, 1]$,

$$x_s^t(Jf) = x_s^t(f). \quad (13)$$

It follows that Jf is PJG because

$$Jf(t) = \sum_{r \in [0,1]} x_r^t(f) = \sum_{r \in [0,1]} x_r^t(Jf) = \sum_{r \preceq_{Jf} t} x_r^t(Jf).$$

The point (ii) is a rewording of the definition of PJG excursions. The point (iii) follows from the definition of J and from (12), by continuity of d_f^\top . Proving the point (iv) is easy by using the identities $\Delta_t(Jf) = \Delta_t(f)$ and $\Delta_t(J^\varepsilon f) = \Delta_t(f) \mathbf{1}_{\Delta_t(f) \geq \varepsilon}$.

It remains to prove the point (v). We set $g = f - Jf + J^\varepsilon f$. First, it is easy to see that g is indeed an excursion with the expression $g(t) = d_f^\top(t, 1) + J^\varepsilon f(t)$ inherited from (12). Plus, it also gives $\Delta_t(g) = \Delta_t(J^\varepsilon f) = \Delta_t(f) \mathbf{1}_{\Delta_t(f) \geq \varepsilon}$. Also recall $\Delta_t(Jf) = \Delta_t(f)$. Let $s, t \in [0, 1]$ with $s \preceq_f t$ and $u \in [s, t]$, so that $s \preceq_f u$. We obtain

$$Jf(u) - J^\varepsilon f(u) \geq (Jf(s) - \Delta_s(f)) - (J^\varepsilon f(s) - \Delta_s(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon}) \quad (14)$$

by expressing the left-hand-side as a series over $r \in [0, 1]$, then by cutting the series at s , and by applying Lemma 2. A similar trick together with Corollary 1 leads to

$$d_f^\top(u, 1) + J^\varepsilon f(u) \geq d_f^\top(s, 1) + (J^\varepsilon f(s) - \Delta_s(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon}) + x_s^u(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon}. \quad (15)$$

The expressions for $\Delta_s(Jf)$, $\Delta_s(J^\varepsilon f)$, and g and the bounds (14) and (15) eventually get us

$$x_s^u(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon} \leq g(u) - g(s-) \leq f(u) - f(s-),$$

which yields $x_s^t(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon} \leq x_s^t(g) \leq x_s^t(f)$ by taking the infimum over u . But as $x_s^t(g) \leq \Delta_s(g) = \Delta_t(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon}$, we deduce that if $s \preceq_f t$ then $x_s^t(g) = x_s^t(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon}$. Conversely, the same argument as for Jf gives that if we do not have $s \preceq_f t$ then $x_s^t(g) = x_s^t(f) = 0$. Thus, we have shown that for all $s, t \in [0, 1]$,

$$x_s^t(f - Jf + J^\varepsilon f) = x_s^t(f) \mathbf{1}_{\Delta_s(f) \geq \varepsilon}. \quad (16)$$

Finally, we end the proof by writing

$$J(f - Jf + J^\varepsilon f)(t) = \sum_{r \in [0,1]} x_r^t(f - Jf + J^\varepsilon f) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} x_r^t(f) = J^\varepsilon f(t).$$

□

Theorem 5. *Let f be an excursion. It holds $d_f^\top = d_{f-Jf}^\top = d_{f-Jf}^{\text{clas}}$, $d_f^{\text{L}} = d_{Jf}^{\text{L}}$, and $\tilde{d}_f^{\text{L}} = \tilde{d}_{Jf}^{\text{L}}$. Moreover,*

$$d_{J^\varepsilon f}^{\text{L}}(s, t) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} \delta_r^f(x_r^s(f), x_r^t(f)), \quad (17)$$

$$\tilde{d}_{J^\varepsilon f}^{\text{L}}(s, t) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} \tilde{\delta}_r^f(x_r^s(f), x_r^t(f)), \quad (18)$$

for all $s, t \in [0, 1]$.

Before proving the latter theorem, we sum up why veneration trees can be considered as a natural unification of trees and looptrees. On the one hand, if f is a continuous excursion then $d^{\text{L}} = 0$ and $d^{\text{V}} = d^\top = d^{\text{clas}}$, so f naturally codes a real tree. On the other hand, if f is a PJG excursion then $d^\top = 0$ and $d^{\text{V}} = 2d^{\text{L}}$, so f naturally codes a looptree. Conversely, each excursion has a natural continuous part, which codes its tree distance d^\top , and a natural PJG part, which codes its looptree distance d^{L} . Observe Theorem 1 is a direct consequence of Theorems 4 and 5.

Proof. Recall (1) for the definition of d^{clas} . Let $s, t \in [0, 1]$ with $s < t$. The identities (12) and (11) yield

$$d_{f-Jf}^\top(s, t) = d_{f-Jf}^{\text{clas}}(s, t) = d_f^\top(s, 1) + d_f^\top(t, 1) - 2 \inf_{u \in [s,t]} d_f^\top(u, 1)$$

because $f - Jf$ is a continuous excursion. If $u \in [s, t]$ then $s \wedge_f t \preceq_f u$ so $d_f^\top(u, 1) \geq d_f^\top(s \wedge_f t, 1)$ by Corollary 1. Moreover, Lemma 1 and the inequality (5) allow us to bound

$$\inf_{u \in [s,t]} d_f^\top(s \wedge_f t, u) \leq \inf_{u \in [s,t]} f(u) - \inf_{[s,t]} f = 0.$$

Hence, we conclude

$$d_{f-Jf}^\top(s, t) = d_f^\top(s, 1) + d_f^\top(t, 1) - 2d_f^\top(s \wedge_f t, 1) = d_f^\top(s \wedge_f t, s) + d_f^\top(s \wedge_f t, t) = d_f^\top(s, t)$$

with another application of Corollary 1. Thanks to (13), it holds $\Delta_r(Jf) = \Delta_r(f)$ and $x_r^t(Jf) = x_r^t(f)$ for all $r \in [0, 1]$. The formula (9) then implies $d_{Jf}^{\text{L}} = d_f^{\text{L}}$, and the formula (10) implies $\tilde{d}_{Jf}^{\text{L}} = \tilde{d}_f^{\text{L}}$. The same argument can be applied to $f - Jf + J^\varepsilon f$, so that $\Delta_r(J^\varepsilon f) = \Delta_r(f - Jf + J^\varepsilon f)$ and $x_r^t(J^\varepsilon f) = x_r^t(f - Jf + J^\varepsilon f)$ thanks to the point (v) of Theorem 4. Furthermore, the identity (16) ensures $\Delta_r(f - Jf + J^\varepsilon f) = \Delta_r(f) \mathbf{1}_{\Delta_r(f) \geq \varepsilon}$ and $x_r^t(f - Jf + J^\varepsilon f) = x_r^t(f) \mathbf{1}_{\Delta_r(f) \geq \varepsilon}$. Eventually, the formulas (9) and (10) lead to

$$\begin{aligned} d_{J^\varepsilon f}^{\text{L}}(s, t) &= \sum_{\substack{r \in [0,1] \\ \Delta_r(J^\varepsilon f) > 0}} \delta_r^{J^\varepsilon f}(x_r^s(J^\varepsilon f), x_r^t(J^\varepsilon f)) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} \delta_r^f(x_r^s(f), x_r^t(f)), \\ \tilde{d}_{J^\varepsilon f}^{\text{L}}(s, t) &= \sum_{\substack{r \in [0,1] \\ \Delta_r(J^\varepsilon f) > 0}} \tilde{\delta}_r^{J^\varepsilon f}(x_r^s(J^\varepsilon f), x_r^t(J^\varepsilon f)) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} \tilde{\delta}_r^f(x_r^s(f), x_r^t(f)). \end{aligned}$$

□

Corollary 2. For all excursion f and $\varepsilon \in (0, 1)$, it holds

$$\begin{aligned} \|d_f^{\perp} - d_{J^\varepsilon f}^{\perp}\|_{\infty} &\leq 2\|Jf - J^\varepsilon f\|_{\infty}, \\ \|\tilde{d}_f^{\perp} - \tilde{d}_{J^\varepsilon f}^{\perp}\|_{\infty} &\leq 2K\|Jf - J^\varepsilon f\|_{\infty}, \end{aligned}$$

where K is a finite constant which only depends on the shuffle Φ , as

$$K = \sup_{\Delta \in (0,1]} \sup_{x \in (0,1]} \frac{1}{x} \delta(\phi_{\Delta}(0), \phi_{\Delta}(x)).$$

Proof. From the definition of a shuffle, we know K is indeed finite. By definition, we have $\tilde{\delta}_r^f(0, x) \leq Kx$ for any $r \in [0, 1]$ such that $\Delta_r(f) \leq 1$ and for any $x \in [0, \Delta_r(f)]$. Plus, it is clear that $\delta_r^f(0, x) \leq x$. It follows

$$\begin{aligned} \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \varepsilon}} \delta_r^f(0, x_r^t(f)) &\leq \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \varepsilon}} x_r^t(f) = Jf(t) - J^\varepsilon f(t), \\ \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \varepsilon}} \tilde{\delta}_r^f(0, x_r^t(f)) &\leq \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \varepsilon}} Kx_r^t(f) = K(Jf(t) - J^\varepsilon f(t)). \end{aligned}$$

We conclude by applying the formulas (9), (10), (17), and (18), as well as the triangular inequalities of the δ_r^f and the $\tilde{\delta}_r^f$. \square

We finish this section by providing some tools to study convergences involving J or J^ε .

Proposition 7. Let f be an excursion, $\varepsilon > 0$, and let $\lambda : [0, 1] \rightarrow [0, 1]$ be an increasing bijection. The following points hold.

- (i) $J(\Theta f) = \Theta(Jf)$ and $J^\varepsilon(\Theta f) = \Theta(J^\varepsilon f)$,
- (ii) $J(f \circ \lambda) = (Jf) \circ \lambda$ and $J^\varepsilon(f \circ \lambda) = (J^\varepsilon f) \circ \lambda$,
- (iii) If $\varepsilon \notin \{\Delta_t(f) : t \in [0, 1]\}$, then J^ε is continuous at f with respect to the topology of uniform convergence,
- (iv) $\|Jf - J^\varepsilon f\|_{\infty} \xrightarrow{\varepsilon \rightarrow 0} 0$.

Proof. When $s, t \in [1/2, 1]$, recall $\Theta f(s) = f(2s-1)$ so $x_s^t(\Theta f) = x_{2s-1}^{2t-1}(f)$. Then, the point (i) comes from the fact that $\Delta_r(\Theta f) = 0$ when $r \in [0, 1/2)$. As for the proof of the time-changing property, we deduce (ii) from the strict monotony of λ and from the equality $x_s^t(f \circ \lambda) = x_{\lambda(s)}^{\lambda(t)}(f)$ for all $s, t \in [0, 1]$. Let us prove (iii). The excursion f is càdlàg so the set $\{t \in [0, 1] : \Delta_t(f) > \varepsilon/2\}$ is finite. Thus,

$$\begin{aligned} \eta_1 &= \max\{\Delta_t(f) : t \in [0, 1] \text{ such that } \varepsilon/2 < \Delta_t(f) < \varepsilon\} \cup \{\varepsilon/2\}, \\ \eta_2 &= \min\{\Delta_t(f) : t \in [0, 1] \text{ such that } \Delta_t(f) \geq \varepsilon\} \cup \{2\varepsilon\} \end{aligned}$$

are well-defined and $\eta_1 < \varepsilon < \eta_2$. Let g be an excursion such that $\|f - g\|_{\infty} < \min(\varepsilon - \eta_1, \eta_2 - \varepsilon)/2$. If $\Delta_r(g) \geq \varepsilon$ then $\Delta_r(f) > \eta_1$ and we obtain $\Delta_r(f) \geq \varepsilon$ by definition of η_1 . Likewise, if $\Delta_r(g) < \varepsilon$ then $\Delta_r(f) < \eta_2$ and we obtain $\Delta_r(f) < \varepsilon$ by definition of η_2 . Hence,

$$J^\varepsilon g(t) = \sum_{\substack{r \in [0,1] \\ \Delta_r(f) \geq \varepsilon}} x_r^t(g)$$

for all $t \in [0, 1]$, so $\|J^\varepsilon f - J^\varepsilon g\|_\infty \leq 2N(f) \times \|f - g\|_\infty$ where we denote by $N(f)$ the number of jumps of f of height at least ε . The point (iii) follows. To show the point (iv), we mimic the proof of Dini's theorem. We fix $\nu > 0$ and we set

$$U_n = \{t \in [0, 1] : Jf(t) - J^{1/n}f(t) < \nu \text{ and } Jf(t-) - J^{1/n}f(t-) < \nu\}$$

for all $n \geq 1$. We remark that $U_n \subset U_{n+1}$ because $J^{1/n}f \leq J^{1/(n+1)}f \leq Jf$. The functions Jf and $J^{1/n}f$ are càdlàg, so U_n is open for all $n \geq 1$. Furthermore, the monotone convergence theorem implies that $J^{1/n}f(t) \xrightarrow[n \rightarrow \infty]{} Jf(t)$ for all $t \in [0, 1]$. When n is large enough (depending on t), we have

$$\Delta_t(J^{1/n}f) = \Delta_t(f)\mathbf{1}_{\Delta_t(f) \geq 1/n} = \Delta_t(f) = \Delta_t(Jf)$$

which yields $Jf(t-) - J^{1/n}f(t-) = Jf(t) - J^{1/n}f(t)$. Therefore, the family $(U_n)_{n \geq 1}$ covers $[0, 1]$, so there exists $N \geq 1$ such that $U_N = [0, 1]$ by compactness. As a result, $\|Jf - J^{1/n}f\|_\infty \leq \nu$ when $n \geq N$. \square

4 Limit theorems

The main flaw of the pseudo-distances d^L , d^T , and d^V is that they do not enjoy any good property of functional continuity, unlike the classical pseudo-distance d^{clas} which induces a real tree from a continuous excursion f . We have already explained why a PJG excursion should code a looptree. However, it is possible to show every encoding of a pseudo-distance from an excursion cannot be continuous if it is equal to d_f^L whenever f is PJG. For example, we observe the two sequences of functions defined by

$$(f_n(t), g_n(t)) = \begin{cases} (k/2n + 1/n - (t - k/2n), (k + 1)/2n) & \text{if } \frac{k}{2n} \leq t < \frac{k+1}{2n} \text{ with } 0 \leq k \leq n - 1 \\ (1/2 - (t - 1/2), 1/2 - (t - 1/2)) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

are PJG thanks to Lemma 4. Both have the same asymptotic behavior because both uniformly converge to the same continuous excursion $f : t \in [0, 1] \mapsto 1/2 - |t - 1/2|$, while their associated looptrees have very different limits. Indeed, on the one hand, \mathcal{L}_{f_n} consists of a chain of n loops of length $1/n$ put back to back. Intuitively, this sequence converges to a segment of length $1/2$, which is also \mathcal{T}_f . On the other hand, \mathcal{L}_{g_n} consists of a bouquet of n loops of length $1/2n$ all glued at the same point. This sequence converges to a single point, which is also \mathcal{L}_f , as the sequence of the diameters tends to 0.

4.1 Continuity for the shuffled veneration tree pseudo-distance

We fix a shuffle Φ and we set $\mathbb{B}(\Phi) = \mathbb{B}^1(\Phi) \cup \mathbb{B}^2(\Phi)$, where

$$\begin{aligned} \mathbb{B}^1(\Phi) &= \{\Delta > 0 : \Delta \text{ discontinuity point of } \Phi \text{ for the Skorokhod topology}\}, \\ \mathbb{B}^2(\Phi) &= \{(\Delta, x) : \Delta > 0, x \in [0, 1], x \text{ discontinuity point of } \phi_\Delta\}. \end{aligned}$$

We are going to show that \tilde{d}^V enjoys a property of functional continuity, which justifies its construction. The set $\mathbb{B}(\Phi)$ will represent the imperfections of that continuity. Let us remark the definition of a shuffling implies $\mathbb{B}(\Phi) \cap ((0, \infty) \times \{0, 1\}) = \emptyset$. For any excursion f , we also define $B_f = B_f^1 \cup B_f^2$, where

$$\begin{aligned} B_f^1 &= \{\Delta_t : t \in [0, 1] \text{ such that } \Delta_t > 0\}, \\ B_f^2 &= \left\{ \left(\Delta_r, \frac{x_r^t}{\Delta_r} \right) : r, s, t \in [0, 1] \text{ such that } s \neq t, \Delta_r > 0, \text{ and } x_r^t = x_r^s > 0 \right\}, \end{aligned}$$

which will indicate if the functional $g \mapsto \tilde{d}_g^V$ is continuous at f . We remind that the set of excursions \mathcal{H} is a closed subset of the Skorokhod space $\mathcal{D}([0, 1])$, endowed with the Skorokhod distance ρ . Also recall from Definition 3 the Skorokhod topology on $\mathcal{D}([0, 1]^2)$ induced by the distance ρ_2 . Furthermore, we already know from Proposition 4 that $\tilde{d}_f^V \in \mathcal{D}([0, 1]^2)$ for any $f \in \mathcal{H}$.

Theorem 6. *Let f be an excursion. If $B_f \cap \mathbb{B}(\Phi) = \emptyset$, then the functional $g \in \mathcal{H} \mapsto \tilde{d}_g^{\mathcal{Y}} \in \mathcal{D}([0, 1]^2)$ is continuous at f for the Skorokhod topologies.*

Proof. The proof requires a lot of care and can be tedious sometimes. The main idea is to use the convergence in (3) to forget the small jumps of f . Then, the hypothesis $B_f \cap \mathbb{B}(\Phi) = \emptyset$ and the continuity of Φ allow us to control the variations generated by the big jumps.

Choice of some parameters. Let $\varepsilon \in (0, 1)$. We are able to choose some $\Delta \in (0, \varepsilon)$ such that

$$\text{if } u \in (0, 2\Delta) \text{ then } \sup_{x \in (0, 1]} \left| \frac{2}{x} \delta(\phi_u(0), \phi_u(x)) - 1 \right| \leq \varepsilon \quad (19)$$

thanks to (3). The set $\{s \in [0, 1] : \Delta_s \geq \Delta\}$ is finite because f is càdlàg. We denote by N its size and by $r_1 < r_2 < \dots < r_N$ its elements. The functions $\phi_{\Delta_{r_n}}$ are càglàd and continuous at 0. Hence, we have some $\nu > 0$ and some partitions of $[0, 1]$ denoted by $0 < y_{n,1} < y_{n,2} < \dots < y_{n,k_n} < 1$, where the $y_{n,i}$ are discontinuity points of $\phi_{\Delta_{r_n}}$ for any $1 \leq i \leq k_n$, such that for all $1 \leq n \leq N$,

$$\max_{1 \leq i \leq k_n} \sup_{\substack{y, y' \in (y_{n,i}, y_{n,i+1}] \\ |y - y'| \leq \nu}} \delta(\phi_{\Delta_{r_n}}(y), \phi_{\Delta_{r_n}}(y')) \leq \frac{\varepsilon}{N + 1}, \quad (20)$$

$$\sup_{\substack{y, y' \in [0, y_{n,1}] \\ |y - y'| \leq \nu}} \delta(\phi_{\Delta_{r_n}}(y), \phi_{\Delta_{r_n}}(y')) \leq \frac{\varepsilon}{N + 1}, \quad (21)$$

with $y_{n,k_n+1} = 1$ by convention. We now take a parameter $\eta > 0$ that respects the following inequalities:

$$\eta \leq \min\left(\frac{\Delta}{4}, \frac{\varepsilon}{N + 1}\right) \text{ and } \left(1 + \frac{4}{\Delta} + \frac{4}{\Delta^2} \|f\|_{\infty}\right) \eta \leq \nu. \quad (22)$$

The fact that f is an excursion ensures that for $1 \leq n \leq N$ and $1 \leq i \leq k_n$,

$$t_{n,i} = \inf\{t \geq r_n : f(t) - f(r_n-) = y_{n,i} \Delta_{r_n}\}$$

is well-defined, $f(t_{n,i}) = f(t_{n,i}-)$, and $x_{r_n}^{t_{n,i}} = y_{n,i} \Delta_{r_n} > 0$. In particular, the $t_{n,i}$, for $1 \leq n \leq N$ and $1 \leq i \leq k_n$, are continuity points of f , so they are distinct from the r_n , for $1 \leq n \leq N$. Moreover, if $t_{n,i} = t_{m,j}$ with $n < m$, $1 \leq i \leq k_n$ and $1 \leq j \leq k_m$, then $x_{r_n}^{t_{n,i}} = x_{r_m}^{t_{m,j}}$ because $f(r_m-) \leq \inf_{[r_m, t_{m,j}]} f$. Since $t_{n,i} \neq r_m$, we find $(\Delta_{r_n}, y_{n,i}) \in B_f$. By choice of $y_{n,i}$, we also have $(\Delta_{r_n}, y_{n,i}) \in \mathbb{B}(\Phi)$, which contradicts the assumption $B_f \cap \mathbb{B}(\Phi) = \emptyset$. If $t_{n,i} = t_{n,j}$, with $1 \leq i, j \leq k_n$, then $y_{n,i} \Delta_{r_n} = x_{r_n}^{t_{n,i}} = x_{r_n}^{t_{n,j}} = y_{n,j} \Delta_{r_n}$ so $i = j$. Thus, the $t_{n,i}$ are distinct.

Eventually, we obtain the existence of some $\gamma \in (0, \varepsilon)$ such that the intervals $[t_{n,i} - \gamma, t_{n,i} + \gamma]$, for $1 \leq n \leq N$ and $1 \leq i \leq k_n$, are included in $[0, 1]$, disjoint, do not contain any of the r_m , and such that

$$\text{if } t \in [0, 1] \text{ with } |t - t_{n,i}| \leq \gamma \text{ then } |f(t) - f(t_{n,i})| \leq \eta/3. \quad (23)$$

We set

$$\omega = \min_{\substack{1 \leq n \leq N \\ 1 \leq i \leq k_n}} \min(x_{r_n}^{t_{n,i} - \gamma} - x_{r_n}^{t_{n,i}}, x_{r_n}^{t_{n,i}} - x_{r_n}^{t_{n,i} + \gamma}),$$

and we know $\omega > 0$ thanks to the assumption $B_f \cap \mathbb{B}(\Phi) = \emptyset$. Then, we choose $\xi \in (0, \eta/3)$ such that

$$\left(\frac{8}{\Delta} + 1\right) \xi < \frac{\omega}{2 + \|f\|_{\infty}}. \quad (24)$$

Recall the definition of the Skorokhod distance \overleftarrow{p} for \mathcal{C} -valued càglàd functions given just before Definition 2. Because $\Phi : u > 0 \mapsto \phi_u$ is continuous at Δ_{r_n} for $1 \leq n \leq N$ thanks to the assumption $B_f \cap \mathbb{B}(\Phi) = \emptyset$, we can choose some $\nu' \in (0, \Delta)$ such that for all $1 \leq n \leq N$,

$$\sup_{|u - \Delta_{r_n}| \leq \nu'} \overleftarrow{p}(\phi_u, \phi_{\Delta_{r_n}}) \leq \xi/2. \quad (25)$$

Correspondence between big discontinuities. Let $g \in \mathcal{H}$ such that $\|f - g\|_\infty \leq \min(\xi, \nu'/2)$. We are going to construct an increasing bijection from $[0, 1]$ to itself which makes correspond the discontinuity points of \tilde{d}_f^\vee and \tilde{d}_g^\vee . If $s, t \in [0, 1]$ then $|x_s^t(f) - x_s^t(g)| \leq 2\|f - g\|_\infty$. In particular for $s = t$, we have $|\Delta_s(f) - \Delta_s(g)| \leq \nu'$, so (25) gives us some increasing bijections from $[0, 1]$ to itself, denoted by μ_n for $1 \leq n \leq N$, such that

$$\|\mu_n - \text{id}\|_\infty \leq \xi \text{ and } \sup_{x \in [0,1]} \delta(\phi_{\Delta_{r_n}(g)}(x), \phi_{\Delta_{r_n}(f)} \circ \mu_n(x)) \leq \xi. \quad (26)$$

Let λ be the continuous function which is affine on the intervals generated by the points

$$\{t_{n,i}, t_{n,i} + \gamma, t_{n,i} - \gamma : 1 \leq n \leq N \text{ and } 1 \leq i \leq k_n\}$$

such that $\lambda(0) = 0$, $\lambda(1) = 1$, $\lambda(t_{n,i} - \gamma) = t_{n,i} - \gamma$, $\lambda(t_{n,i} + \gamma) = t_{n,i} + \gamma$, and

$$\lambda(t_{n,i}) = \inf \left\{ t \geq r_n : \frac{x_{r_n}^t(g)}{\Delta_{r_n}} \leq \mu_n^{-1}(y_{n,i}) \right\}.$$

It is well-defined because g is an excursion. Observe that $x_{r_n}^{\lambda(t_{n,i})}(g) = \Delta_{r_n}(g) \times \mu_n^{-1}(y_{n,i})$. Let us prove that λ is an increasing function, close to the identity. Let $1 \leq n \leq N$ and $1 \leq i \leq k_n$. We use $\|f - g\|_\infty \leq \xi$ then we recall the definitions of $t_{n,i}$ and ω to find

$$\frac{x_{r_n}^{t_{n,i}-\gamma}(g)}{\Delta_{r_n}} \geq \frac{x_{r_n}^{t_{n,i}-\gamma}(f) - 2\xi}{\Delta_{r_n}(g)} \geq \frac{y_{n,i}\Delta_{r_n}(f) + \omega - 2\xi}{\Delta_{r_n}(g)}.$$

Plus, we found $\Delta_{r_n}(g) \geq \Delta_{r_n}(f) - 2\xi \geq \Delta - 2\eta$, so $\Delta_{r_n}(g) \geq \Delta/2$ according to (22). We also have $\Delta_{r_n}(g) \leq \Delta_{r_n}(f) + 2\xi \leq 2 + \|f\|_\infty$. It follows

$$\frac{x_{r_n}^{t_{n,i}-\gamma}(g)}{\Delta_{r_n}} \geq y_{n,i} + \frac{\omega}{2 + \|f\|_\infty} - \frac{8\xi}{\Delta},$$

then successive applications of (26) and (24) lead to

$$\frac{x_{r_n}^{t_{n,i}-\gamma}(g)}{\Delta_{r_n}} \geq \mu_n^{-1}(y_{n,i}) + \frac{\omega}{2 + \|f\|_\infty} - \left(\frac{8}{\Delta} + 1\right)\xi > \mu_n^{-1}(y_{n,i}).$$

We prove $x_{r_n}^{t_{n,i}+\gamma}(g)/\Delta_{r_n}(g) < \mu_n^{-1}(y_{n,i})$ in the same way. Thus, $|\lambda(t_{n,i}) - t_{n,i}| < \gamma$. We deduce that λ is strictly increasing and

$$\|\lambda - \text{id}\|_\infty \leq \gamma \leq \varepsilon. \quad (27)$$

Moreover, λ is a bijection from $[0, 1]$ to itself because $\lambda(0) = 0$, $\lambda(1) = 1$, and λ is continuous.

Control on the big loops. We easily see that if t is outside all the intervals $[t_{n,i} - \gamma, t_{n,i} + \gamma]$ then $\lambda(t) = t$. But if $t \in [t_{n,i} - \gamma, t_{n,i} + \gamma]$ then $\lambda(t) \in [t_{n,i} - \gamma, t_{n,i} + \gamma]$ too. Therefore, (23) allows us to deduce

$$\|f - g \circ \lambda\|_\infty \leq \|f - f \circ \lambda\|_\infty + \|f - g\|_\infty \leq 2\eta/3 + \eta/3 = \eta.$$

Now, we set $h = g \circ \lambda$ so that $\|f - h\|_\infty \leq \eta$. Our goal is to show that $\|\tilde{d}_f^\vee - \tilde{d}_h^\vee\|_\infty$ is small, but before that, we are going to bound the difference of the distances on the big loops. We begin by recalling that the point r_n is outside all the intervals $[t_{n,i} - \gamma, t_{n,i} + \gamma]$ so $\lambda(r_n) = r_n$, which implies $\Delta_{r_n}(h) = \Delta_{r_n}(g)$ and $x_{r_n}^t(h) = x_{r_n}^{\lambda(t)}(g)$. Let $t \geq r_n$ and $1 \leq i \leq k_n$. Recall $(\Delta_{r_n}(f), y_{n,i}) \notin B_f$ because $(\Delta_{r_n}(f), y_{n,i}) \in \mathbb{B}(\Phi)$, so

$$\frac{x_{r_n}^t(f)}{\Delta_{r_n}} \leq y_{n,i} \iff t_{n,i} \leq t.$$

Then, $\lambda(r_n) = r_n \leq \lambda(t)$ because λ is increasing. By definition of $\lambda(t_{n,i})$ together with the observation that $x_{r_n}^{\lambda(t_{n,i})}(g) = \Delta_{r_n}(g) \times \mu_n^{-1}(y_{n,i})$, we have $x_{r_n}^{\lambda(t)}(g)/\Delta_{r_n}(g) \leq \mu_n^{-1}(y_{n,i}) \iff \lambda(t_{n,i}) \leq \lambda(t)$. The functions λ and μ_n are strictly increasing so it follows

$$\frac{x_{r_n}^t(f)}{\Delta_{r_n}} \leq y_{n,i} \iff \mu_n \left(\frac{x_{r_n}^t(h)}{\Delta_{r_n}} \right) \leq y_{n,i}.$$

Hence, the points $x_{r_n}^t(f)/\Delta_{r_n}(f)$ and $\mu_n(x_{r_n}^t(h)/\Delta_{r_n}(h))$ are either both in the interval $[0, y_{n,1}]$ or both in the same interval $(y_{n,i}, y_{n,i+1}]$ with $1 \leq i \leq k_n$. Furthermore, we can bound

$$\left| \frac{x_{r_n}^t(f)}{\Delta_{r_n}} - \mu_n \left(\frac{x_{r_n}^t(h)}{\Delta_{r_n}} \right) \right| \leq \|\mu_n - \text{id}\|_\infty + \frac{2\|f-h\|_\infty}{\Delta_{r_n}(g)} + |x_{r_n}^t(f)| \frac{2\|f-h\|_\infty}{\Delta_{r_n}(f)\Delta_{r_n}(g)},$$

so that the inequalities $\Delta_{r_n}(g) \geq \Delta/2$, (26), and (22) give

$$\left| \frac{x_{r_n}^t(f)}{\Delta_{r_n}} - \mu_n \left(\frac{x_{r_n}^t(h)}{\Delta_{r_n}} \right) \right| \leq \left(1 + \frac{4}{\Delta} + \frac{4}{\Delta^2} \|f\|_\infty \right) \eta \leq \nu.$$

Therefore, it follows from (20) and (21) that

$$\delta \left(\phi_{\Delta_{r_n}(f)} \left(\frac{x_{r_n}^t(f)}{\Delta_{r_n}} \right), \phi_{\Delta_{r_n}(f)} \left(\mu_n \left(\frac{x_{r_n}^t(h)}{\Delta_{r_n}} \right) \right) \right) \leq \frac{\varepsilon}{N+1} \quad (28)$$

for all $t \in [0, 1]$, the case $t < r_n$ being obvious. Now, let $s, t \in [0, 1]$, the triangular inequality on δ implies

$$\begin{aligned} \left| \tilde{\delta}_{r_n}^f(x_{r_n}^s(f), x_{r_n}^t(f)) - \tilde{\delta}_{r_n}^h(x_{r_n}^s(h), x_{r_n}^t(h)) \right| &\leq |\Delta_{r_n}(f) - \Delta_{r_n}(h)| \\ &+ \Delta_{r_n}(f) \times \delta \left(\phi_{\Delta_{r_n}(f)} \left(\frac{x_{r_n}^t(f)}{\Delta_{r_n}} \right), \phi_{\Delta_{r_n}(h)} \left(\frac{x_{r_n}^t(h)}{\Delta_{r_n}} \right) \right) \\ &+ \Delta_{r_n}(f) \times \delta \left(\phi_{\Delta_{r_n}(f)} \left(\frac{x_{r_n}^s(f)}{\Delta_{r_n}} \right), \phi_{\Delta_{r_n}(h)} \left(\frac{x_{r_n}^s(h)}{\Delta_{r_n}} \right) \right). \end{aligned}$$

Recall that $\Delta_{r_n}(h) = \Delta_{r_n}(g)$, $\|f-h\|_\infty \leq \eta$, and $\xi \leq \eta$. We apply the inequalities (22), (26), and (28) into the previous bound to conclude that for all $s, t \in [0, 1]$,

$$\left| \tilde{\delta}_{r_n}^f(x_{r_n}^s(f), x_{r_n}^t(f)) - \tilde{\delta}_{r_n}^h(x_{r_n}^s(h), x_{r_n}^t(h)) \right| \leq (2 + 4\|f\|_\infty) \frac{\varepsilon}{N+1}. \quad (29)$$

Uniform control. We are now ready to bound $\|\tilde{d}_f^{\mathcal{V}} - \tilde{d}_h^{\mathcal{V}}\|_\infty$. Let $s, t \in [0, 1]$ with $s \leq t$. With the help of Lemma 1 and of the identities (10) and (11), we verify we can write

$$\begin{aligned} \left| \tilde{d}_f^{\mathcal{V}}(s, t) - \tilde{d}_h^{\mathcal{V}}(s, t) \right| &\leq |f(s) - h(s)| + |f(t) - h(t)| + 2 \left| \inf_{[s,t]} f - \inf_{[s,t]} h \right| \\ &+ \sum_{n=1}^N |x_{r_n}^s(f) - x_{r_n}^s(h)| + \sum_{n=1}^N |x_{r_n}^t(f) - x_{r_n}^t(h)| \\ &+ 2 \sum_{n=1}^N \left| \tilde{\delta}_{r_n}^f(x_{r_n}^s(f), x_{r_n}^t(f)) - \tilde{\delta}_{r_n}^h(x_{r_n}^s(h), x_{r_n}^t(h)) \right| \\ &+ 3\Delta_{s \wedge_f t}(f) \mathbf{1}_{\Delta_{s \wedge_f t}(f) < \Delta} + 3\Delta_{s \wedge_h t}(h) \mathbf{1}_{\Delta_{s \wedge_h t}(f) < \Delta} \\ &+ \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} \left| 2\tilde{\delta}_r^f(0, x_r^s(f)) - x_r^s(f) \right| + \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} \left| 2\tilde{\delta}_r^f(0, x_r^t(f)) - x_r^t(f) \right| \\ &+ \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} \left| 2\tilde{\delta}_r^h(0, x_r^s(h)) - x_r^s(h) \right| + \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} \left| 2\tilde{\delta}_r^h(0, x_r^t(h)) - x_r^t(h) \right|. \end{aligned}$$

The sum of all the terms in the first two rows of the right-hand side can be easily bounded by 8ε because $\|f - h\|_\infty \leq \eta \leq \varepsilon/(N + 1)$, thanks to (22). The term of the third row is smaller than $(4 + 8\|f\|_\infty)\varepsilon$ by a direct application of (29). Next, if $\Delta_r(f) < \Delta$ then $\Delta_r(h) < \Delta + 2\eta < 2\Delta$, since the inequality (22) justifies $\eta \leq \Delta/4$. As we chose Δ so that $\Delta < \varepsilon$, the sum of the terms in the fourth row is bounded by 9ε . Moreover, the fact (19) ensures that if $\Delta_r(f) < \Delta$, then it holds

$$\left| 2\tilde{\delta}_r^f(0, x) - x \right| \leq \varepsilon x \text{ and } \left| 2\tilde{\delta}_r^h(0, y) - y \right| \leq \varepsilon y$$

for all $x \in [0, \Delta_r(f)]$ and for all $y \in [0, \Delta_r(h)]$. It allows us to control the last four terms of the right-hand side. For instance,

$$\begin{aligned} \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} \left| 2\tilde{\delta}_r^f(0, x_r^s(f)) - x_r^s(f) \right| &\leq \varepsilon \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} x_r^s(f) \leq \varepsilon Jf(s) \leq \|f\|_\infty \varepsilon, \\ \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} \left| 2\tilde{\delta}_r^h(0, x_r^s(h)) - x_r^s(h) \right| &\leq \varepsilon \sum_{\substack{r \in [0,1] \\ \Delta_r(f) < \Delta}} x_r^s(h) \leq \varepsilon Jh(s) \leq (1 + \|f\|_\infty) \varepsilon, \end{aligned}$$

because we know from Theorem 4 that $f - Jf \geq 0$. Eventually, we find

$$\left| \tilde{d}_f^V(s, t) - \tilde{d}_h^V(s, t) \right| \leq (23 + 12\|f\|_\infty) \varepsilon.$$

Conclusion. To sum up, we have shown that for all $\varepsilon > 0$, there exists $\eta > 0$ such that if $\|f - g\|_\infty \leq \eta$ then there exists λ an increasing bijection from $[0, 1]$ to itself with $\|\tilde{d}_f^V - \tilde{d}_{g \circ \lambda}^V\|_\infty \leq \varepsilon$ and $\|\lambda - \text{id}\|_\infty \leq \varepsilon$. Indeed, recall (27). Now, if the Skorokhod distance between f and g is smaller than $\min(\eta, \varepsilon)/2$, then we have μ an increasing bijection from $[0, 1]$ to itself such that $\|\mu - \text{id}\|_\infty \leq \varepsilon$ and $\|f - g \circ \mu\|_\infty \leq \eta$. Hence, there exists λ an increasing bijection from $[0, 1]$ to itself such that $\|\mu \circ \lambda - \text{id}\|_\infty \leq 2\varepsilon$ and $\|\tilde{d}_f^V - \tilde{d}_{g \circ \mu \circ \lambda}^V\|_\infty \leq \varepsilon$. Eventually, thanks to the time-changing property, $\tilde{d}_{g \circ \mu \circ \lambda}^V = \tilde{d}_g^V \circ (\mu \circ \lambda, \mu \circ \lambda)$. Thereby, if $\rho(f, g) \leq \min(\eta, \varepsilon)/2$ then $\rho_2(\tilde{d}_f^V, \tilde{d}_g^V) \leq 2\varepsilon$, where we remind that ρ is the Skorokhod distance on \mathcal{H} and ρ_2 is the Skorokhod distance for bivariate functions. \square

4.2 Two particular cases for convergences of unshuffled pseudo-distances

Even if d^\perp , d^\top , and d^V do not enjoy a property of functional continuity, we are still able to provide two particular cases where it is possible to automatically deduce a convergence for $(d_{f_n}^\perp)$, $(d_{f_n}^\top)$, or $(d_{f_n}^V)$ from the convergence of (f_n) .

Theorem 7. *If $f_n \rightarrow f$ for the Skorokhod topology on \mathcal{H} and if there exists $N \geq 1$ such that all the f_n have at most N jumps, then $d_{f_n}^\perp \rightarrow d_f^\perp$ and $d_{f_n}^\top \rightarrow d_f^\top$ uniformly on $[0, 1]^2$. It follows $d_{f_n}^V \rightarrow d_f^V$ uniformly on $[0, 1]^2$.*

Proof. The limiting excursion f also has at most N jumps. Let $\varepsilon \in (0, 1)$ such that all the jumps of f are higher than 2ε . Hence, we get $J^\varepsilon f_n \rightarrow J^\varepsilon f = Jf$ for the Skorokhod topology thanks to Theorem 4 and to Proposition 7. Let $s \in [0, 1]$, we know there is $t \in [0, 1]$ such that $|\Delta_t(f) - \Delta_s(f_n)| \leq 2\rho(f_n, f)$. If $\rho(f_n, f) < \varepsilon/2$ and $\Delta_s(f_n) < \varepsilon$, then $\Delta_t(f) < 2\varepsilon$, so that $\Delta_t(f) = 0$ and $\Delta_s(f_n) \leq 2\rho(f_n, f)$. Hence, when n is large enough, we find

$$\|Jf_n - J^\varepsilon f_n\|_\infty \leq \sum_{s \in [0,1]} \Delta_s(f_n) \mathbf{1}_{0 < \Delta_s(f_n) < \varepsilon} \leq 2N\rho(f_n, f) \xrightarrow{n \rightarrow \infty} 0.$$

In particular, the sequence of excursions $g_n := f_n - Jf_n + J^\varepsilon f_n$ converges to f for the Skorokhod topology. Let us fix $\Phi : \Delta > 0 \mapsto \phi_\Delta$ such that $\phi_\Delta(x) = x$ for all $\Delta \geq \varepsilon$ and $\phi_{1-\Delta}(x)$ is expressed

as in Remark 1 for all $1 - \Delta \in (0, \varepsilon)$, for all $x \in [0, 1]$. It is straightforward to check that Φ is a shuffle. We observe $B_f \cap \mathbb{B}(\Phi) = \emptyset$ and $B_{Jf} \cap \mathbb{B}(\Phi) = \emptyset$, so Theorem 6 justifies

$$\tilde{d}_{J^\varepsilon f_n}^{\mathcal{V}} \xrightarrow{n \rightarrow \infty} \tilde{d}_{Jf}^{\mathcal{V}} \text{ and } \tilde{d}_{g_n}^{\mathcal{V}} \xrightarrow{n \rightarrow \infty} \tilde{d}_f^{\mathcal{V}}$$

for the Skorokhod topology on $\mathcal{D}([0, 1]^2)$. Then, for all $n \geq 1$, $\tilde{d}_{J^\varepsilon f_n}^{\mathcal{L}} = d_{J^\varepsilon f_n}^{\mathcal{L}}$ and $\tilde{d}_f^{\mathcal{L}} = d_f^{\mathcal{L}}$ because $\Delta_t(J^\varepsilon f_n)$ and $\Delta_t(f)$ are not in $(0, \varepsilon)$ for all $t \in [0, 1]$. Recall that $J^\varepsilon f_n$ and Jf are PJG, which yields that $d_{J^\varepsilon f_n}^{\mathcal{L}}$ converges to $d_f^{\mathcal{L}}$ for the Skorokhod topology, by applying Theorem 5. Since $d_f^{\mathcal{L}}$ is continuous on $[0, 1]^2$, this convergence also happens uniformly on $[0, 1]^2$. Similarly, Theorems 4 and 5 ensure that $\tilde{d}_{g_n}^{\mathcal{L}} = \tilde{d}_{Jg_n}^{\mathcal{L}} = \tilde{d}_{J^\varepsilon f_n}^{\mathcal{L}} = d_{J^\varepsilon f_n}^{\mathcal{L}}$ and $\tilde{d}_{g_n}^{\mathcal{T}} = \tilde{d}_{g_n - Jg_n}^{\mathcal{T}} = \tilde{d}_{f_n - Jf_n}^{\mathcal{T}} = d_{f_n}^{\mathcal{T}}$. Hence,

$$d_{f_n}^{\mathcal{T}} + 2d_{J^\varepsilon f_n}^{\mathcal{L}} \xrightarrow{n \rightarrow \infty} d_f^{\mathcal{T}} + 2d_f^{\mathcal{L}}$$

for the Skorokhod topology on $\mathcal{D}([0, 1]^2)$, which implies that $d_{f_n}^{\mathcal{T}}$ uniformly converges to $d_f^{\mathcal{T}}$ on $[0, 1]^2$, because $d_f^{\mathcal{T}}$ is continuous too. To conclude, recall that $\|Jf_n - J^\varepsilon f_n\|_\infty \xrightarrow{n \rightarrow \infty} 0$. We deduce the uniform convergence of $(d_{f_n}^{\mathcal{L}})$ to $d_f^{\mathcal{L}}$ on $[0, 1]^2$ with the help of Corollary 2. \square

Theorem 8. *If $f_n \rightarrow f$ for the Skorokhod topology on \mathcal{H} and if f is PJG, then $d_{f_n}^{\mathcal{L}} \rightarrow d_f^{\mathcal{L}}$ and $d_{f_n}^{\mathcal{T}} \rightarrow 0$ uniformly on $[0, 1]^2$. It follows $d_{f_n}^{\mathcal{V}} \rightarrow d_f^{\mathcal{V}} = 2d_f^{\mathcal{L}}$ uniformly on $[0, 1]^2$.*

Proof. Let $\varepsilon > 0$ such that $\varepsilon \notin \{\Delta_t(f) : t \in [0, 1]\}$. This condition ensures $J^\varepsilon f_n \rightarrow J^\varepsilon f$ for the Skorokhod topology on \mathcal{H} by Proposition 7, and even more, it holds $\|f_n - J^\varepsilon f_n\|_\infty \rightarrow \|f - J^\varepsilon f\|_\infty$. The triangular inequality of $d_{f_n}^{\mathcal{T}}$ and the formula (12) lead to $\|d_{f_n}^{\mathcal{T}}\|_\infty \leq 2\|f_n - J^\varepsilon f_n\|_\infty$, because $J^\varepsilon f_n \leq Jf_n \leq f_n$. Moreover, Corollary 2 implies

$$\|d_{f_n}^{\mathcal{L}} - d_f^{\mathcal{L}}\|_\infty \leq 2\|f_n - J^\varepsilon f_n\|_\infty + 2\|f - J^\varepsilon f\|_\infty + \|d_{J^\varepsilon f_n}^{\mathcal{L}} - d_{J^\varepsilon f}^{\mathcal{L}}\|_\infty.$$

All the jumps of the $J^\varepsilon f_n$ are higher or equal to ε and $J^\varepsilon f_n$ converges to $J^\varepsilon f$, thus there is $N \geq 1$ such that all the $J^\varepsilon f_n$ have at most N jumps. By the previous theorem, $\|d_{J^\varepsilon f_n}^{\mathcal{L}} - d_{J^\varepsilon f}^{\mathcal{L}}\|_\infty \rightarrow 0$, so we get

$$\limsup \|d_{f_n}^{\mathcal{T}}\|_\infty + \limsup \|d_{f_n}^{\mathcal{L}} - d_f^{\mathcal{L}}\|_\infty \leq 6\|f - J^\varepsilon f\|_\infty.$$

Since the excursion f is PJG, it holds $f = Jf$ and $\|f - J^\varepsilon f\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0$ thanks to Proposition 7. The set $\{\Delta_t(f) : t \in [0, 1]\}$ is countable, so we can make $\varepsilon \rightarrow 0$ within the above inequality. \square

4.3 From convergences of pseudo-distances to convergences of quotient metric spaces

Recall the notions presented at the start of Section 2.3. To see how Theorems 6, 7, and 8 can bring us some convergences of pointed weighted metric spaces under the form of Theorem 3, we have to present the definition of the pointed Gromov-Hausdorff-Prokhorov distance, which truly induces a topology on the set of the GHP-isometry classes of pointed weighted compact metric spaces. Recall that when no confusion is possible, we simply denote a metric space (possibly endowed with some additional structure) by its underlying set.

A *correspondence* between two compact metric spaces (X_1, d_1) and (X_2, d_2) is a subset \mathcal{R} of $X_1 \times X_2$ such that for all $x_1 \in X_1$ and $y_2 \in X_2$, there exists $x_2 \in X_2$ and $y_1 \in X_1$ such that $(x_1, x_2), (y_1, y_2) \in \mathcal{R}$. We say a correspondence between X_1 and X_2 is compact if it is compact in the product space $X_1 \times X_2$. The distortion of a correspondence \mathcal{R} is defined by

$$\text{dis}(\mathcal{R}) = \sup \{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

The Gromov-Hausdorff distance between X_1 and X_2 is then expressed by the formula

$$d_{\text{GH}}(X_1, X_2) = \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}),$$

where the infimum is taken over all correspondences \mathcal{R} between X_1 and X_2 . Similarly, the pointed Gromov-Hausdorff distance $d_{\text{GH}}^\bullet(X_1, X_2)$ between two pointed compact metric spaces (X_1, d_1, a_1) and (X_2, d_2, a_2) is expressed by the same formula up to the difference that the infimum is instead taken over all correspondences \mathcal{R} between X_1 and X_2 such that $(a_1, a_2) \in \mathcal{R}$. For both distances, we may restrict the infimum to compact correspondences without modifying the value. Indeed, a correspondence and its closure have the same distortion, and the latter is a compact correspondence because $X_1 \times X_2$ is compact. Let (X_1, d_1, a_1, μ_1) and (X_2, d_2, a_2, μ_2) be two pointed weighted compact metric spaces. A *coupling* between μ_1 and μ_2 is a Borel probability measure on $X_1 \times X_2$ whose marginals on X_1 and X_2 are μ_1 and μ_2 . The pointed Gromov-Hausdorff-Prokhorov distance between X_1 and X_2 is then defined by the formula

$$d_{\text{GHP}}^\bullet(X_1, X_2) = \inf_{\mathcal{R}, \nu} \max \left(\frac{1}{2} \text{dis}(\mathcal{R}), 1 - \nu(\mathcal{R}) \right)$$

where the infimum is taken over all couplings ν between μ_1 and μ_2 and all compact correspondences \mathcal{R} between X_1 and X_2 such that $(a_1, a_2) \in \mathcal{R}$.

The functions d_{GH} , d_{GH}^\bullet , and d_{GHP}^\bullet are only pseudo-distances but $d_{\text{GH}}(X_1, X_2) = 0$ if and only if X_1 and X_2 are isometric, $d_{\text{GH}}^\bullet(X_1, X_2) = 0$ if and only if X_1 and X_2 are pointed-isometric, and $d_{\text{GHP}}^\bullet(X_1, X_2) = 0$ if and only if X_1 and X_2 are GHP-isometric. Hence, d_{GH} defines a genuine distance on the space \mathbb{K} of isometry classes of compact metric spaces, d_{GH}^\bullet defines a genuine distance on the space \mathbb{K}^\bullet of pointed-isometry classes of pointed compact metric spaces, and d_{GHP}^\bullet defines a genuine distance on the space \mathbb{K}_w^\bullet of GHP-isometry classes of pointed weighted compact metric spaces. We call their respective topologies the Gromov-Hausdorff, the pointed Gromov-Hausdorff, and the pointed Gromov-Hausdorff-Prokhorov topologies. Furthermore, the metric spaces \mathbb{K} , \mathbb{K}^\bullet , and \mathbb{K}_w^\bullet are separable and complete. See [28] and [2] for proof of the stated facts and for more information. Now, recall from Section 2.3 the definition of quotient metric spaces and Definition 3.

Lemma 5. *For all $n \geq 1$, let $d_n, d \in \mathcal{D}([0, 1]^2)$ be two pseudo-distances on $[0, 1]$ such that the quotient pointed weighted metric spaces $X_n = [0, 1]/\{d_n = 0\}$ and $X = [0, 1]/\{d = 0\}$ are compact. It holds*

$$d_{\text{GH}}^\bullet(X_n, X) \leq \rho_2(d_n, d), \quad (30)$$

$$d_{\text{GHP}}^\bullet(X_n, X) \leq \|d_n - d\|_\infty. \quad (31)$$

Furthermore, if $d_n \rightarrow d$ for the Skorokhod topology on $\mathcal{D}([0, 1]^2)$, then $X_n \rightarrow X$ for the pointed Gromov-Hausdorff-Prokhorov topology.

Proof. Let us denote by $[s]_n$ and $[s]$ the canonical projections of $s \in [0, 1]$ respectively on X_n and X , and by μ_n and μ the push-forward measures of the Lebesgue measure on $[0, 1]$ by the respective canonical projections. Let $\lambda_n, \mu_n : [0, 1] \rightarrow [0, 1]$ be two increasing bijections. The triangular inequality yields $\|d_n - d \circ (\lambda_n, \lambda_n)\|_\infty \leq 2\|d_n - d \circ (\lambda_n, \mu_n)\|_\infty$. But the quantity $\|d_n - d \circ (\lambda_n, \lambda_n)\|_\infty$ is the distortion of the correspondence $\mathcal{R}_n = \{([s]_n, [\lambda_n(s)]) : s \in [0, 1]\}$. This proves (30) by taking the infimum over λ_n, μ_n . When $\lambda_n = \text{id}$, the correspondence \mathcal{R}_n has total mass for the coupling ν_n between μ_n and μ defined by

$$\int_{X_n \times X} g d\nu = \int_0^1 g([s]_n, [s]) ds$$

for any bounded measurable g . This shows (31). If $d_n \rightarrow d$ for the Skorokhod topology on $\mathcal{D}([0, 1]^2)$ then we can assume $\|\lambda_n - \text{id}\|_\infty + \|d_n - d \circ (\lambda_n, \lambda_n)\|_\infty \rightarrow 0$. For all $\varepsilon > 0$, we define another compact correspondence $\mathcal{R}_n^\varepsilon = \{([s]_n, x) : s \in [0, 1], x \in X \text{ such that } d([\lambda_n(s)], x) \leq \varepsilon\}$ between X_n and X . It holds $\limsup \text{dis}(\mathcal{R}_n^\varepsilon) \leq 2\varepsilon$. Remark the compactness of $[0, 1]/\{d = 0\}$ when $d \in \mathcal{D}([0, 1]^2)$ implies the canonical projection $[0, 1] \rightarrow X$ is càdlàg, so it is continuous almost everywhere. Thus, $[\lambda_n(s)] \rightarrow [s]$ almost everywhere and $\nu_n(\mathcal{R}_n^\varepsilon) \rightarrow 1$ according to the dominated convergence theorem. \square

Proof of Theorem 3. By definition of the relaxed Skorokhod topology, $\Theta f_n \rightarrow \Theta f$ for the (rigid) Skorokhod topology. Observe that if $t \in [0, 1/2)$ then $\tilde{d}_{\Theta f}^\vee(t, 1) = 0$, and that if $s, t \in [1/2, 1]$ then

$\tilde{d}_{\Theta f}^{\vee}(s, t) = \tilde{d}_f^{\vee}(2s - 1, 2t - 1)$ thanks to the branching property. It is then a small exercise to check the convergences $\tilde{\mathcal{V}}_{f_n} \rightarrow \tilde{\mathcal{V}}_f$ and $\tilde{\mathcal{V}}_{\Theta f_n} \rightarrow \tilde{\mathcal{V}}_{\Theta f}$ for the pointed Gromov-Hausdorff-Prokhorov topology are equivalent. Finally, the previous lemma and Theorem 6 implies the point (i) because $B_{\Theta f} = B_f$. Likewise, we show the two other points using Theorems 7 and 8. Indeed, f_n has at most N jumps if and only if Θf_n has at most N jumps. When f is PJG, $J(\Theta f) = \Theta f$ by Proposition 7, thus Θf is PJG. \square

5 Topological space of veneration trees

Here, we propose a topological notion of veneration trees that does not require coding by excursions. First, let us fix our framework by giving some definitions. Let (X, d) be a metric space and let $x, y \in X$. We say that γ is a *path* on X from x to y when $\gamma : [0, 1] \rightarrow X$ is continuous such that $\gamma(0) = x$ and $\gamma(1) = y$. In that case, we will denote by $\text{Im } \gamma = \gamma([0, 1])$ its image. If γ is also injective, we say that γ is an *arc* on X from x to y . By compactness of $[0, 1]$, an arc provides a topological embedding of the segment $[0, 1]$ into X . Moreover, we say that γ is a *geodesic* on X from x to y when $d(\gamma(t), \gamma(s)) = |t - s|d(x, y)$ for all $t, s \in [0, 1]$. A geodesic from x to y is an arc if and only if $x \neq y$. Finally, we say that γ is a *loop* on X based at x when $\gamma : [0, 1] \rightarrow X$ is continuous, injective on $[0, 1)$, and such that $\gamma(0) = \gamma(1) = x$. A loop provides a topological embedding of the circle \mathcal{C} into X .

Definition 6. A metric space (X, d) is *geodesic* when there exists a geodesic on X from x to y , for all $x, y \in X$. A metric space (X, d) is a *real tree* when it is geodesic and when all arcs on X from x to y have the same image, for all $x, y \in X$. We say a metric space (X, d) is a *veneration tree* when it is geodesic and when it satisfies that for any γ_1, γ_2 , two loops on X based at x , then either $\text{Im } \gamma_1 = \text{Im } \gamma_2$ or $\text{Im } \gamma_1 \cap \text{Im } \gamma_2 = \{x\}$, for all $x \in X$.

We point out that any geodesic space is arc-connected by definition. For our work, we exclusively are interested in compact metric spaces. When a metric space is indeed compact, it is geodesic if and only if it is a length space. See [20, Chapter 2] for a definition of length spaces, a proof of the just mentioned fact, and an extensive study of the notion. For a detailed study of real trees, we refer to Evans [24] or to Paulin [38]. Observe a geodesic space is locally path-connected. Indeed, for any $x \in X$ and $r > 0$, if $d(x, y) < r$ then a geodesic between x and y stays inside the ball of radius r and of center x . Moreover, it is well-known that a Hausdorff space, so a metric space, is pathwise connected if and only if it is arcwise connected, see [42, Chapter 31] or [13] for example. We deduce the following convenient lemma.

Lemma 6. *Let (X, d) be a compact geodesic space. If F is a finite subset of X , then the connected components of $X \setminus F$ are open and arcwise connected.*

Proposition 8. *When (X, d) is a compact geodesic space, the following statements are equivalent.*

- (i) *The space (X, d) is a real tree.*
- (ii) *For all $x, y \in X$, if $x \neq y$ then there exists $u \in X \setminus \{x, y\}$ such that x and y are in different connected components of $X \setminus \{u\}$.*
- (iii) *There are no loops on X .*

The implications (i) \implies (ii) \implies (iii) are obvious. The equivalence between (i) and (iii) is quite easy to show and can be found in [15, Proposition 2.3]. We observe that the point (iii) obviously ensures that a real tree is also a veneration tree, which might not be so clear by only looking at the definitions. The interest of the point (ii) is that it can be naturally adapted into an equivalent definition for veneration trees.

Proposition 9. *When (X, d) is a compact geodesic space, the following statements are equivalent.*

- (i) *The space (X, d) is a veneration tree.*

(ii) For all $x, y \in X$, if $x \neq y$ then there exists $u, v \in X \setminus \{x, y\}$ such that x and y are in different connected components of $X \setminus \{u, v\}$.

(iii) For all $x, y \in X$, if $x \neq y$ then there exists $u, v \in X \setminus \{x, y\}$ such that x and y are in different connected components of $X \setminus \{u, v\}$ and such that $2 \min(d(x, u), d(x, v), d(y, u), d(y, v)) \geq d(x, y)$.

Proof. The implication (iii) \implies (ii) is obvious. Let us assume that (ii) holds. Let γ_1, γ_2 be two loops on X based at x and let $s \in (0, 1)$ such that $\gamma_1(s) \notin \text{Im } \gamma_2$. We set

$$a_1 = \sup\{t \in [0, s] : \gamma_1(t) \in \text{Im } \gamma_2\} \text{ and } b_1 = \inf\{t \in [s, 1] : \gamma_1(t) \in \text{Im } \gamma_2\},$$

so that $\gamma_1(a_1), \gamma_2(b_1) \in \text{Im } \gamma_2$ because γ_1 is continuous and $\text{Im } \gamma_2$ is closed. We choose $a_2, b_2 \in [0, 1]$ such that $\gamma_2(a_2) = \gamma_1(a_1)$ and $\gamma_2(b_2) = \gamma_1(b_1)$. Even if it means reversing γ_2 , we can assume that $a_2 \leq b_2$ without loss of generality. We can define three paths on X from $\gamma_1(a_1)$ to $\gamma_1(b_1)$ by setting

$$\begin{aligned} \gamma'_1 : t \in [0, 1] &\longmapsto \gamma_1(a_1 + t(b_1 - a_1)), \\ \gamma'_2 : t \in [0, 1] &\longmapsto \gamma_2(a_2 + t(b_2 - a_2)), \\ \gamma'_3 : t \in [0, 1] &\longmapsto \begin{cases} \gamma_2(a_2 - t) & \text{if } t \leq a_2 \\ x & \text{if } a_2 \leq t \leq b_2 \\ \gamma_2(b_2 + 1 - t) & \text{if } b_2 \leq t \leq 1 \end{cases} \end{aligned}$$

By definition of a_1 and b_1 , it is clear that $\text{Im } \gamma'_1 \cap \text{Im } \gamma_2 = \{\gamma_1(a_1), \gamma_1(b_1)\}$, so it easily follows that $\text{Im } \gamma'_1 \cap \text{Im } \gamma'_2 = \text{Im } \gamma'_1 \cap \text{Im } \gamma'_3 = \{\gamma_1(a_1), \gamma_1(b_1)\}$. Moreover, $\text{Im } \gamma'_2 \cap \text{Im } \gamma'_3 = \{\gamma_1(a_1), \gamma_1(b_1)\}$ because γ_2 is injective on $[0, 1]$. In the case where $\gamma_1(a_1) \neq \gamma_1(b_1)$, there exists $u, v \in X \setminus \{\gamma_1(a_1), \gamma_1(b_1)\}$ such that $\gamma_1(a_1)$ and $\gamma_1(b_1)$ are in different pathwise connected components of $X \setminus \{u, v\}$ thanks to (ii). In particular, we would have $\text{Im } \gamma'_i \cap \{u, v\} \neq \emptyset$ for $i = 1, 2, 3$, which is impossible regarding the intersections between the $\text{Im } \gamma'_i$. Hence, we find that $\gamma_1(a_1) = \gamma_1(b_1)$ and since $a_1 < s < b_1$, we conclude that $a_1 = 0$ and $b_1 = 1$. This means $\text{Im } \gamma_1 \cap \text{Im } \gamma_2 = \{x\}$. By contraposition, if $\text{Im } \gamma_1 \cap \text{Im } \gamma_2 \neq \{x\}$ then $\text{Im } \gamma_1 \subset \text{Im } \gamma_2$ and $\text{Im } \gamma_1 = \text{Im } \gamma_2$ by symmetry, which ends the proof of (ii) \implies (i).

Let us assume that (i) holds. Let g be a geodesic on X from x to y with $x \neq y$, and let us set $u = g(1/2)$ so that $d(x, u) = d(y, u) = d(x, y)/2$. Let γ_1 be an arc on $X \setminus \{u\}$ from x to y . It is also an arc on X such that $u \notin \text{Im } \gamma_1$. We set

$$a_1 = \sup\{t \in [0, 1] : \gamma_1(t) \in g([0, 1/2])\} \text{ and } b_1 = \inf\{t \in [a_1, 1] : \gamma_1(t) \in g([1/2, 1])\},$$

which are well-defined because $\gamma_1(0) = g(0)$ and $\gamma_1(1) = g(1)$. Since γ_1 and g are continuous, there exists $\alpha_1, \beta_1 \in [0, 1]$ such that $\alpha_1 \leq 1/2 \leq \beta_1$, $g(\alpha_1) = \gamma_1(a_1)$, and $g(\beta_1) = \gamma_1(b_1)$. In fact, it even holds $\alpha_1 < 1/2 < \beta_1$ because $u \notin \text{Im } \gamma_1$, so $a_1 < b_1$ because g is injective. The real-valued function $t \in [a_1, b_1] \longmapsto d(x, \gamma_1(t))$ is continuous and

$$d(x, \gamma_1(a_1)) = \alpha_1 d(x, y) < d(x, y)/2 < \beta_1 d(x, y) = d(x, \gamma_1(b_1)),$$

so there exists $s \in (a_1, b_1)$ such that $d(x, \gamma_1(s)) = d(x, y)/2$. We set $v = \gamma_1(s)$, and we observe that $d(x, y) \leq d(x, y)/2 + d(y, v)$. We only need to verify that x and y are in different connected components of $X \setminus \{u, v\}$ in order to show (i) \implies (iii). Let γ_2 be an arc on $X \setminus \{u, v\}$ from x to y by contradiction with Lemma 6. It is also an arc on X such that $\text{Im } \gamma_2 \cap \{u, v\} = \emptyset$. Similarly as with γ_1 , we set

$$a_2 = \sup\{t \in [0, 1] : \gamma_2(t) \in g([0, 1/2])\} \text{ and } b_2 = \inf\{t \in [a_2, 1] : \gamma_2(t) \in g([1/2, 1])\}$$

and we fix $\alpha_2, \beta_2 \in [0, 1]$ such that $\alpha_2 < 1/2 < \beta_2$, $g(\alpha_2) = \gamma_2(a_2)$, and $g(\beta_2) = \gamma_2(b_2)$. Moreover, $a_2 < b_2$. Then, we define two paths γ'_1, γ'_2 on X from u to u by setting for all $t \in [0, 1]$,

$$\gamma'_1(t), \gamma'_2(t) = \begin{cases} g(1/2 + 3t(\beta_1 - 1/2)), g(1/2 + 3t(\beta_2 - 1/2)) & \text{if } t \leq 1/3 \\ \gamma_1(b_1 + 3(t - 1/3)(a_1 - b_1)), \gamma_2(b_2 + 3(t - 1/3)(a_2 - b_2)) & \text{if } 1/3 \leq t \leq 2/3 \\ g(\alpha_1 + 3(t - 2/3)(1/2 - \alpha_1)), g(\alpha_2 + 3(t - 2/3)(1/2 - \alpha_2)) & \text{if } 2/3 \leq t \end{cases}$$

It is clear that γ'_1 and γ'_2 are paths on X from u to u . By definition of a_1, b_1, a_2, b_2 , if $t_1 \in (a_1, b_1)$ and $t_2 \in (a_2, b_2)$, then $\gamma_1(t_1) \notin \text{Im } g$ and $\gamma_2(t_2) \notin \text{Im } g$. Since g, γ_1, γ_2 are injective, it follows that γ'_1 and γ'_2 are loops on X based at u . We observe that if $1/2 < t < \min(\beta_1, \beta_2)$ then $g(t) \neq u$ and $g(t) \in \text{Im } \gamma'_1 \cap \text{Im } \gamma'_2$, so it ensures that $\text{Im } \gamma'_1 = \text{Im } \gamma'_2$. However, $s \in (a_1, b_1)$ so $v = \gamma_1(s) \in \text{Im } \gamma'_1$, while $v \notin \text{Im } g$ and $v \notin \text{Im } \gamma_2$, so $v \notin \text{Im } \gamma'_2$. That is a contradiction, which concludes the proof. \square

When (X, d) is a compact metric space, we write $\text{diam}(X) = \sup_{x, y \in X} d(x, y)$. According to the next lemma, a compact veneration tree has only a countable number of images of loops.

Lemma 7. *If (X, d) is a compact veneration tree, then the set $\{\text{Im } \gamma : \gamma \text{ loop on } X, \text{diam}(\text{Im } \gamma) \geq \varepsilon\}$ is finite for all $\varepsilon > 0$.*

Proof. We assume by contradiction the set of interest is infinite. Then, the compactness of X yields there exists two loops γ_1, γ_2 on X such that $\text{Im } \gamma_1 \neq \text{Im } \gamma_2$, and there exists $x_1, y_1 \in \text{Im } \gamma_1$ and $x_2, y_2 \in \text{Im } \gamma_2$ such that $d(x_1, y_1) \geq \varepsilon$, $d(x_1, x_2) \leq \varepsilon/4$, and $d(y_1, y_2) \leq \varepsilon/4$. We give ourselves two geodesics g_x, g_y on X respectively from x_1 to x_2 and from y_1 to y_2 . Observe that $\text{Im } g_x \cap \text{Im } g_y = \emptyset$ because $d(\text{Im } g_x, \text{Im } g_y) \geq d(x_1, y_1) - d(x_1, x_2) - d(y_1, y_2) > 0$. Considering $a_z = \sup g_z^{-1}(\text{Im } \gamma_1)$ and $b_z = \inf g_z^{-1}(\text{Im } \gamma_2) \cap [a_z, 1]$ for $z \in \{x, y\}$ allows us to assume $\text{Im } g_x \cap \text{Im } \gamma_1 = \{x_1\}$, $\text{Im } g_x \cap \text{Im } \gamma_2 = \{x_2\}$, $\text{Im } g_y \cap \text{Im } \gamma_1 = \{y_1\}$, and $\text{Im } g_y \cap \text{Im } \gamma_2 = \{y_2\}$ at the cost of replacing the inequality $d(x_1, y_1) \geq \varepsilon$ with $d(x_1, y_1) \geq \varepsilon/2$. Since X is a veneration tree and $\text{Im } \gamma_1 \neq \text{Im } \gamma_2$, it holds $\# \text{Im } \gamma_1 \cap \text{Im } \gamma_2 \leq 1$, so choosing one of the two semi-loops of γ_1 gives us an arc γ'_1 on X from x_1 to y_1 such that $\text{Im } \gamma'_1 \subset \text{Im } \gamma_1$, and $\text{Im } \gamma'_1 \cap \text{Im } \gamma_2 = \{x_1, y_1\} \cap \{x_2, y_2\}$. Thus, we can construct a loop γ by following g_x from x_1 to x_2 , then following γ_2 from x_2 to y_2 , then following g_y from y_2 to y_1 , and finally following γ'_1 from y_1 to x_1 . Indeed, $d(x_1, y_2) \geq d(x_1, y_1) - d(y_1, y_2) > 0$ and $x_2 \neq y_1$ in the same way. We have found a contradiction because $x_1, y_1 \in \text{Im } \gamma_1 \cap \text{Im } \gamma$ but either $g_x(1/2)$ or $g_y(1/2)$ is in $\text{Im } \gamma$ but not in $\text{Im } \gamma_1$, because it would hold $x_1, y_1 \in \text{Im } \gamma_1 \cap \text{Im } \gamma_2$ otherwise. \square

Recall from Section 4.3 that \mathbb{K} is the space of compact metric spaces up to an isometry, endowed with the Gromov-Hausdorff distance d_{GH} , and that it is separable and complete. Let \mathbb{L} be the space of isometry classes of compact geodesic spaces and let \mathbb{T} be the space of isometry classes of compact real trees. They are both closed subsets of \mathbb{K} , see for example [24, Theorem 4.19] and [24, Lemma 4.22], so they are also separable and complete. We denote by \mathbb{V} the space of isometry classes of compact veneration trees. We have the chain of inclusions $\mathbb{T} \subset \mathbb{V} \subset \mathbb{L} \subset \mathbb{K}$. The next theorem justifies that \mathbb{V} is closed in \mathbb{K} , and so that it is separable and complete.

Theorem 9. *Let (X_n, d_n) be a compact veneration tree for all $n \geq 0$ and let (X, d) be a compact metric space. If $X_n \rightarrow X$ for the Gromov-Hausdorff topology then X is a compact veneration tree.*

Proof. First, the subspace \mathbb{L} is closed so X is a compact geodesic space. Let $x, y \in X$ with $x \neq y$, we only need to find $u, v \in X \setminus \{x, y\}$ such that x and y are in different connected components of $X \setminus \{u, v\}$. There exists a sequence (\mathcal{R}_n) of correspondences respectively between X_n and X such that $\text{dis}(\mathcal{R}_n) \rightarrow 0$. For any $n \geq 0$, there are $x_n, y_n \in X_n$ such that $(x_n, x) \in \mathcal{R}_n$ and $(y_n, y) \in \mathcal{R}_n$. In particular, $\lim d_n(x_n, y_n) = d(x, y) > 0$ so we can assume that $x_n \neq y_n$ for all $n \geq 0$ by only considering large enough n . Hence, there exists $u_n, v_n \in X_n$ such that in the point (iii) of Proposition 9. Then, we have $u'_n, v'_n \in X$ such that $(u_n, u'_n) \in \mathcal{R}_n$ and $(v_n, v'_n) \in \mathcal{R}_n$ for all $n \geq 0$. Since X is compact, we can assume that there exists $u, v \in X$ such that $u'_n \rightarrow u$ and $v'_n \rightarrow v$, by restricting ourselves on a subsequence. It follows that $2d(x, u) = \lim 2d_n(x_n, u_n) \geq \lim d_n(x_n, y_n) = d(x, y)$, so $x \neq u$. In the same way, we show that $u, v \in X \setminus \{x, y\}$. We are going to show that x and y are in different connected components of $X \setminus \{u, v\}$.

Let $\gamma : [0, 1] \rightarrow X$ be a path on X from x to y . For all $n \geq 1$ and $1 \leq i \leq n - 1$, there exists $w_{n,i} \in X_n$ such that $(w_{n,i}, \gamma(i/n)) \in \mathcal{R}_n$. We also write $w_{n,0} = x_n$ and $w_{n,n} = y_n$. We denote by $\gamma_n : [0, 1] \rightarrow X_n$ a path on X_n such that the restriction of γ_n on $[i/n, (i+1)/n]$ corresponds to a geodesic from $w_{n,i}$ to $w_{n,i+1}$ for all $0 \leq i \leq n - 1$. Since γ_n is a path on X_n from x_n to y_n , we

have $\{u_n, v_n\} \cap \text{Im } \gamma_n \neq \emptyset$, so by restricting ourselves on another subsequence, we can assume that $u_n \in \text{Im } \gamma_n$ for all $n \geq 1$ without loss of generality. We choose $t_n \in [0, 1]$ such that $\gamma_n(t_n) = u_n$, and we can assume that there exists $t \in [0, 1]$ such that $t_n \rightarrow t$. Let us prove that $\gamma(t) = u$. Observe that $d_n(w_{n, \lfloor nt_n \rfloor}, u_n) \leq d_n(w_{n, \lfloor nt_n \rfloor}, w_{n, \lfloor nt_n \rfloor + 1})$ by definition of γ_n . By definition of the $w_{n,i}$, we find

$$d_n(w_{n, \lfloor nt_n \rfloor}, u_n) \leq 2\text{dis}(\mathcal{R}_n) + \sup_{\substack{s, r \in [0, 1] \\ |s-r| \leq 1/n}} |\gamma(s) - \gamma(r)|,$$

so $\lim d_n(w_{n, \lfloor nt_n \rfloor}, u_n) = 0$ thanks to the continuity of γ . It follows $\lim d(\gamma(\lfloor nt_n \rfloor/n), u'_n) = 0$ too. Moreover, we have $t_n \rightarrow t$, so $\gamma(\lfloor nt_n \rfloor/n) \rightarrow \gamma(t)$. Recall that $u'_n \rightarrow u$, thus $d(\gamma(t), u) = 0$. \square

Besides the comparison between Proposition 8 and 9, we motivate our topological definition of veneration trees with the following theorem.

Theorem 10. *If f is an excursion, then the spaces \mathcal{V}_f and $\tilde{\mathcal{V}}_f$ are compact veneration trees.*

Observe the spaces \mathcal{T}_f , \mathcal{L}_f , and $\tilde{\mathcal{L}}_f$ are also veneration trees, thanks to Theorems 4 and 5. That result together with the fact that \mathbb{V} is closed in the space \mathbb{K} argues in favor of the existence of a result such as Theorem 3. In order to demonstrate Theorem 10, recall the gluing of pointed metric spaces presented in Definition 4 and let us give ourselves the following lemma.

Lemma 8. *Let X_0 and X_1 be two pointed compact veneration trees, and let $a \in X_0$. The gluing of X_1 on X_0 at a is also a pointed compact veneration tree.*

Proof. The space $X := X_0 \vee_a X_1$ is compact as a union of two compact spaces. Let $x, y \in X_i$ with $i \in \{0, 1\}$, there is a geodesic on X_i from x to y , so seen as an X -valued function, it is also a geodesic on X from x to y . Let $x \in X_i$ and $y \in X_{1-i}$ with $i \in \{0, 1\}$, we check the concatenation of a geodesic on X_i from x to a with a geodesic on X_{1-i} from a to y induces a geodesic on X from x to y . Hence, the metric space X is geodesic. Remark that $\partial X_0 = \partial X_1 = \{a\}$, so any path γ on X from a point of X_0 to a point of X_1 must verify $a \in \text{Im } \gamma$ by connectedness. Thus, if γ is a loop on X then $\text{Im } \gamma \subset X_0$ or $\text{Im } \gamma \subset X_1$. Indeed, if $\gamma(s_0) \in X_0 \setminus \{a\}$ and $\gamma(s_1) \in X_1 \setminus \{a\}$ with $s_0 < s_1$ for example, then there exists $t \in (s_0, s_1)$ such that $\gamma(t) = a$ but there also exists $t' \in [0, 1] \setminus (s_0, s_1)$ such that $\gamma(t') = a$. This contradicts the injectivity of γ on $[0, 1]$. Let γ, γ' be two loops on X based at the same point x . If $\text{Im } \gamma \subset X_i$ and $\text{Im } \gamma' \subset X_i$ with the same $i \in \{0, 1\}$, then they can be seen as loops on the veneration tree X_i based at the same point. Otherwise, it holds $\{x\} \subset \text{Im } \gamma \cap \text{Im } \gamma' \subset X_0 \cap X_1 = \{a\}$. Eventually, the metric space $X_1 \vee_a X_2$ is a veneration tree. \square

Proof of Theorem 10. Recall we already know the spaces \mathcal{V}_f and $\tilde{\mathcal{V}}_f$ are compact with Propositions 3 and 4. We begin by assuming f is a continuous excursion. In that case, $\mathcal{V}_f = \tilde{\mathcal{V}}_f = \mathcal{T}_f$ and $d_f^\top = d_f^{\text{clas}}$, as defined by (1). It is well-known that \mathcal{T}_f is a real tree, see [25] for example, so it is a veneration tree. Now, we suppose that the only jump of f is at 0, namely $\Delta_0 = f(0) > 0$ and $\Delta_t = 0$ for all $t \in (0, 1]$. The set $B(f) = \{x_0^t(f) : t \in [0, 1] \text{ such that } x_0^t(f) < f(t)\}$ is countable because f is càdlàg. If $B(f) = \emptyset$ then f is non-increasing on $[0, 1]$, and we can check that $d^{\mathbb{V}}(s, t) = 2\delta_0(f(s), f(t))$ and $\tilde{d}^{\mathbb{V}}(s, t) = 2\tilde{\delta}_0(f(s), f(t))$ for all $s, t \in [0, 1]$. It follows that both \mathcal{V}_f and $\tilde{\mathcal{V}}_f$ are metric circles of perimeter $2\Delta_0$. Using Proposition 9, it is clear that a circle is a veneration tree. If $x \in B(f)$, a judicious application of the branching property, as stated by Proposition 6, shows there exists a continuous excursion h and an excursion g with its only jump at 0 such that $B(g) = B(f) \setminus \{x\}$, such that \mathcal{V}_f (respectively $\tilde{\mathcal{V}}_f$) is isometric to a gluing of \mathcal{V}_h (respectively $\tilde{\mathcal{V}}_h$) on \mathcal{V}_g (respectively $\tilde{\mathcal{V}}_g$). By induction on $\#B(f)$ and thanks to the previous lemma, we deduce that if f has its only jump at 0 and if $B(f)$ is finite, then \mathcal{V}_f and $\tilde{\mathcal{V}}_f$ are veneration trees. In the case where $B(f)$ is infinite, we set

$$f_n(t) = \begin{cases} f(t) & \text{if there exists } s \in [0, 1] \text{ such that } x_0^s(f) = x_0^t(f) \leq f(s) - 1/n \\ x_0^t(f) & \text{otherwise} \end{cases}$$

for all $t \in [0, 1]$ and $n \geq 1$. It is straightforward to check that f_n is an excursion with a unique jump at 0, that $|f_n(t) - f(t)| \leq 1/n$, and that $x_0^t(f_n) = x_0^t(f)$ for all $t \in [0, 1]$. In particular, $d_{f_n}^L = d_f^L$ and $\tilde{d}_{f_n}^L = \tilde{d}_f^L$, and we find $d_{f_n}^T \rightarrow d_f^T$ uniformly on $[0, 1]^2$ by applying Theorem 7. Hence, $\mathcal{V}_{f_n} \rightarrow \mathcal{V}_f$ and $\tilde{\mathcal{V}}_{f_n} \rightarrow \tilde{\mathcal{V}}_f$ for the Gromov-Hausdorff topology thanks to Lemma 5. Moreover, we observe the identity $B(f_n) = \{x_0^t(f) : t \in [0, 1] \text{ such that } x_0^t(f) \leq f(t) - 1/n\}$ holds, so $B(f_n)$ is finite because f is càdlàg, thus \mathcal{V}_{f_n} and $\tilde{\mathcal{V}}_{f_n}$ are veneration trees. Theorem 9 then ensures this is also the case for \mathcal{V}_f and $\tilde{\mathcal{V}}_f$.

We just proved that if f has at most one jump at 0, then \mathcal{V}_f and $\tilde{\mathcal{V}}_f$ are veneration trees. Induction over the number of jumps of f shows the result is still valid with the weaker assumption of having a finite number of jumps. Indeed, using the branching property at the last jump of f allows writing \mathcal{V}_f as a gluing of \mathcal{V}_h on \mathcal{V}_g , where g is an excursion with one less jump than f and where h is an excursion with a unique jump at 0. The same argument holds for $\tilde{\mathcal{V}}_f$. Finally, we consider a general excursion f . We set $g_n = f - Jf + J^{1/n}f$ for all $n \geq 1$. According to Theorem 4, all the jumps of g_n are not lower than $1/n$ so there is only a finite number of them, which implies \mathcal{V}_{g_n} and $\tilde{\mathcal{V}}_{g_n}$ are veneration trees. Furthermore, it holds $Jg_n = J^{1/n}f$, so Theorem 5 yields $d_{g_n}^T = d_f^T$, $d_{g_n}^L = d_{J^{1/n}f}^L$, and $\tilde{d}_{g_n}^L = \tilde{d}_{J^{1/n}f}^L$. Corollary 2 and the point (iv) of Proposition 7 lead to $\mathcal{V}_{g_n} \rightarrow \mathcal{V}_f$ and $\tilde{\mathcal{V}}_{g_n} \rightarrow \tilde{\mathcal{V}}_f$ for the Gromov-Hausdorff topology. This concludes the proof, once again thanks to Theorem 9. \square

Theorem 11. *If X is a pointed compact veneration tree, then there exists two sequences of excursions (f_n) and (g_n) such that $\mathcal{V}_{f_n} \rightarrow X$ and $\tilde{\mathcal{V}}_{g_n} \rightarrow X$ for the pointed Gromov-Hausdorff topology.*

This is a weak converse for Theorem 10. For the proof of that theorem, we denote by $\overline{\mathcal{V}_{\mathcal{H}}}$ the set of isometry classes of compact metric spaces that are both limits for the pointed Gromov-Hausdorff topology of spaces of the form \mathcal{V}_f and of the form $\tilde{\mathcal{V}}_f$, with f an excursion, for any choice of their distinguished point. With this notation, Theorem 11 simply states $\mathbb{V} = \overline{\mathcal{V}_{\mathcal{H}}}$. Let us already observe $\overline{\mathcal{V}_{\mathcal{H}}}$ is closed for the Gromov-Hausdorff topology. Moreover, we set $\ell(X) = \{\text{Im } \gamma : \gamma \text{ loop on } X\}$ for any compact metric space X .

Lemma 9. *Let (X, d) be a compact veneration tree. If the set $\ell(X)$ is finite, then $X \in \overline{\mathcal{V}_{\mathcal{H}}}$.*

Proof. First, if $\ell(X) = \emptyset$ then X is a real tree according to the point (iii) of Proposition 8. But as a matter of fact, Duquesne has already shown in [22, Lemma 4.2] that any pointed compact real tree is coded by a continuous non-negative function F via the classical pseudo-distance d^{clas} given by (1). We point out that the coding function F in [22] satisfies $F(0) = 0$ but is defined on $[0, M]$ with an arbitrary $M > 0$ without its last value having to be 0. Nevertheless, the function $f : [0, 1] \rightarrow [0, \infty)$ defined by $f(t) = F(2tM)$ if $t \leq 1/2$ and by $f(t) = 2F(M)(1 - t)$ if $t \geq 1/2$ is a genuine continuous excursion in our sense, and it is then straightforward to check that \mathcal{T}_f is pointed-isometric to the pointed tree coded by F via d^{clas} . Hence, for all $a \in X$, there exists $\underline{f} \in \mathcal{H}$ such that (X, d, a) and $\mathcal{V}_{\underline{f}} = \tilde{\mathcal{V}}_{\underline{f}} = \mathcal{T}_{\underline{f}}$ are pointed-isometric. Thus, if $\ell(X) = \emptyset$ then $X \in \overline{\mathcal{V}_{\mathcal{H}}}$. Now, let us assume that $\#\ell(X) \geq 1$ and that $Y \in \overline{\mathcal{V}_{\mathcal{H}}}$ whenever Y is a compact veneration tree Y with $\#\ell(Y) \leq \#\ell(X) - 1$.

Let γ be a loop on X . The function $d(\cdot, \text{Im } \gamma)$ is well-defined and continuous on X because $\text{Im } \gamma$ is compact. For $u \in \text{Im } \gamma$, the subset $X_u = \{x \in X : d(x, \text{Im } \gamma) = d(x, u)\}$ is closed in X so it is compact. Clearly, $X_u \cap \text{Im } \gamma = \{u\}$. Moreover, if g is a geodesic on X from $x \in X_u$ to u , then $\text{Im } g \subset X_u$. Indeed, for any $y \in \text{Im } g$, we can write $d(y, u) = d(x, \text{Im } \gamma) - d(x, y) \leq d(y, \text{Im } \gamma)$. Now, let $u, v \in \text{Im } \gamma$ with $u \neq v$ and let $x \in X_u \cap X_v$. We fix two geodesics g_u, g_v from x to u, v respectively and we set $a = \sup g_u^{-1}(\text{Im } g_v)$, so that $g_u(a) \in \text{Im } g_u \cap \text{Im } g_v$. Following g_u from u to $g_u(a)$, then following g_v from $g_u(a)$ to v , and following one of the two semi-loops of γ from v to u describes another loop γ' . However, $u, v \in \text{Im } \gamma \cap \text{Im } \gamma'$ and $g_u(a) \in \text{Im } \gamma' \setminus \text{Im } \gamma$ because $g_u(a) \in X_u \cap X_v$, so this contradicts the fact X is a veneration tree. Hence, $\{X_u : u \in \text{Im } \gamma\}$ is a partition of X . A similar argument yields any geodesic from some point of X_u to some point of X_v has to hit $\text{Im } \gamma$. Such geodesic would then hit the subset $X_u \cap \text{Im } \gamma = \{u\}$. Thereby, any geodesic from some point of X_u to some point in X_v hits u then hits v . It follows that for all $x \in X_u$ and $y \in X_v$,

$$d(x, y) = d(x, u) + d(u, v) + d(v, y). \quad (32)$$

Furthermore, a geodesic between two points of X_u stays in X_u , because it would have to hit u at least two times otherwise. Thus, the subspace (X_u, d) is compact and geodesic for all $u \in \text{Im } \gamma$. The same holds for the subspace $(\text{Im } \gamma, d)$. Hence, these subspaces are compact veneration trees because a loop on one of them is also a loop on X . Plus, $\text{Im } \gamma$ is not included in X_u , so $\#\ell(X_u) \leq \#\ell(X) - 1$ and $X_u \in \overline{\mathcal{V}_{\mathcal{H}}}$ by induction hypothesis. As for $\text{Im } \gamma$, this geodesic space is homeomorphic to the metric circle \mathcal{C} , and this implies that $\text{Im } \gamma$ is isometric to a metric circle $2\Delta \cdot \mathcal{C}$, with some $\Delta > 0$. One can indeed show this by parametrizing γ by arc length. Alternatively, one can fix $u, v \in \text{Im } \gamma$ such that $d(u, v) = \text{diam}(\text{Im } \gamma)$ and a geodesic g from u to v , and then check that $\text{Im } \gamma \setminus g([0, 1])$ and $g([0, 1])$ are isometric via $w \mapsto g(d(v, w)/d(v, u))$. In particular, $\text{Im } \gamma \in \overline{\mathcal{V}_{\mathcal{H}}}$ because the (shuffled) veneration tree coded by the excursion $f : t \in [0, 1] \mapsto \Delta(1 - t)$ is pointed-isometric to $2\Delta \cdot \mathcal{C}$, for any choice of the root. Then, (32) and the compactness of X imply there is only a finite number of $u \in \text{Im } \gamma$ such that $\text{diam}(X_u) \geq 1/n$, for all $n \geq 1$. It follows the subset

$$X_n = \text{Im } \gamma \cup \left(\bigcup_{\substack{u \in \text{Im } \gamma \\ \text{diam}(X_u) \geq 1/n}} X_u \right)$$

of X can be constructed with a finite number of consecutive gluings only involving elements of $\overline{\mathcal{V}_{\mathcal{H}}}$, thanks to (32). With the help of Proposition 6 and of the bounds (6), (7), and (8), it is not hard to show that if g, h are excursions, then the gluing of \mathcal{V}_h on \mathcal{V}_g at any point of \mathcal{V}_g is pointed-isometric to another veneration tree coded by some excursion f . The same result also holds with shuffled veneration trees. Then, we observe that if (Y_n) and (Z_n) respectively converge to Y and Z for the pointed Gromov-Hausdorff topology, then for all $a \in Y$, there is a sequence of points $a_n \in Y_n$ such that $(Y_n \vee_{a_n} Z_n)$ converges to $Y \vee_a Z$ for the pointed Gromov-Hausdorff topology. Hence, $\overline{\mathcal{V}_{\mathcal{H}}}$ is stable by gluing, so it follows that $X_n \in \overline{\mathcal{V}_{\mathcal{H}}}$ for all $n \geq 1$. To conclude, it is clear that $d_{\text{GH}}(X, X_n) \leq 1/n$, so $X \in \overline{\mathcal{V}_{\mathcal{H}}}$ because $\overline{\mathcal{V}_{\mathcal{H}}}$ is closed for the Gromov-Hausdorff topology. \square

Remark 3. Along the lines of the previous proof, we have shown that if γ is a loop on a compact veneration tree (X, d) then $(\text{Im } \gamma, d)$ is isometric to the metric circle $2\text{diam}(\text{Im } \gamma) \cdot \mathcal{C}$.

Proof of Theorem 11. We denote by $\text{Int } S$ the interior of a subset S of $(0, 1)$, and we denote by Leb the Lebesgue measure on $(0, 1)$. Let $x, y \in X$ and let g be a geodesic on X from x to y . For all $\varepsilon > 0$ and $c \in [0, 1]$, we set

$$d_\varepsilon^c(x, y) = d(x, y) - c d(x, y) \text{Leb}(\{t \in (0, 1) : \exists L \in \ell(X), t \in \text{Int } g^{-1}(L), \text{diam}(L) < \varepsilon\}). \quad (33)$$

We point out the subset measured by Leb in (33) is open and for any $t \in (0, 1)$, if there exists $L \in \ell(X)$ such that $t \in \text{Int } g^{-1}(L)$ then L is unique because X is a veneration tree. Clearly, $(1 - c)d(x, y) \leq d_\varepsilon^c(x, y) \leq d(x, y)$. Let us verify that $d_\varepsilon^c(x, y)$ does not depend on the choice of the geodesic g . Let g_1, g_2 be two geodesics on X from x to y , and let $t \in (0, 1)$ such that there is $\eta > 0$ and $L \in \ell(X)$ with $(t - \eta, t + \eta) \subset g_1^{-1}(L)$ and $\text{diam}(L) < \varepsilon$. Let $s \in (t - \eta, t + \eta)$, if $g_2(s) \in \text{Im } g_1$ then $g_2(s) = g_1(s')$ with $s' \in [0, 1]$, but $s = s'$ because g_1, g_2 are geodesics, so $g_2(s) = g_1(s) \in L$. If $g_2(s) \notin \text{Im } g_1$ then we set $a = \sup g_2^{-1}(\text{Im } g_1) \cap [0, s]$ and $b = \inf g_2^{-1}(\text{Im } g_1) \cap [s, 1]$, so that $a < s < b$. The same argument as just before ensures $g_1(a) = g_2(a)$ and $g_1(b) = g_2(b)$. Following g_1 from $g_1(a)$ to $g_1(b)$ then following g_2 from $g_1(b)$ to $g_1(a)$ defines a loop γ with $g_1(s), g_2(s) \in \text{Im } \gamma$. Both $g_1^{-1}(\text{Im } \gamma)$ and $g_1^{-1}(L)$ contain a neighborhood of s , thus $\text{Im } \gamma = L$. Whatever the case, we have shown $g_2(s) \in L$, which gives $(t - \eta, t + \eta) \subset g_2^{-1}(L)$. It follows d_ε^c is indeed well-defined. Plus, it is continuous on $X \times X$ because it is Lipschitz. Informally, it corresponds to the metric obtained by contracting all small loops of X .

Let $x, y, z \in X$. A geodesic g from x to y gives a geodesic from y to x after reversing the time, so d_ε^c is symmetric. Moreover, if $z \in \text{Im } g$ then $d(x, y) = d(x, z) + d(z, y)$ and g induces a geodesic from x to z and a geodesic from z to y , thus it becomes easy to compute $d_\varepsilon^c(x, y) = d_\varepsilon^c(x, z) + d_\varepsilon^c(z, y)$. Now we suppose $z \notin \text{Im } g$ and we fix a geodesic g_1 from x to z and a geodesic g_2 from y to z . We set $a =$

$\sup g^{-1}(\text{Im } g_1)$, $u = g(a)$, $v = g(\inf g^{-1}(\text{Im } g_2) \cap [a, 1])$, and $w = g_1(\inf g_1^{-1}(\text{Im } g_2) \cap [g_1^{-1}(u), 1])$. The positions of u, v, w on the geodesics yields

$$\begin{aligned} d_\varepsilon^c(x, y) &= d_\varepsilon^c(x, u) + d_\varepsilon^c(u, v) + d_\varepsilon^c(v, y), \\ d_\varepsilon^c(x, z) &= d_\varepsilon^c(x, u) + d_\varepsilon^c(u, w) + d_\varepsilon^c(w, z), \\ d_\varepsilon^c(y, z) &= d_\varepsilon^c(y, v) + d_\varepsilon^c(v, w) + d_\varepsilon^c(w, z). \end{aligned}$$

Either $u = v = w$ or u, v, w are distinct and we can construct a loop γ that follows g from u to v , then follows g_2 from v to w , and finally follows g_1 from w to u . In the latter case, we choose the geodesic g_* from u to v induced by g to write (33), so that $(0, 1) \subset g_*^{-1}(\text{Im } \gamma)$. This implies $d_\varepsilon^c(u, v) = (1 - c\mathbf{1}_{\text{diam}(\text{Im } \gamma) < \varepsilon})d(u, v)$ by uniqueness of L . Whatever the case we are in, it eventually holds $d_\varepsilon^c(u, v) \leq d_\varepsilon^c(u, w) + d_\varepsilon^c(v, w)$, and the triangular inequality of d_ε^c follows. Eventually, d_ε^1 is a pseudo-distance on X and d_ε^c is a distance on X which generates the same topology as d , for all $c \in [0, 1)$ and $\varepsilon > 0$. If $c \in [0, 1)$ and $x \neq y$ then the function $t \in [0, 1] \mapsto d_\varepsilon^c(x, g(t))/d_\varepsilon^c(x, y) \in [0, 1]$ is a continuous increasing bijection, so its inverse function λ is continuous and increasing. Using the identity $d_\varepsilon^c(x, y) = d_\varepsilon^c(x, z) + d_\varepsilon^c(z, y)$ when $z \in \text{Im } g$ gives that $g \circ \lambda$ is a geodesic on (X, d_ε^c) from x to y . Thus, (X, d_ε^c) is a compact veneration tree when $c \in [0, 1)$. We denote by $Y_\varepsilon = X/\{d_\varepsilon^1 = 0\}$ the quotient metric space induced by the pseudo-distance d_ε^1 , as defined in Section 2.3. It is clear that $(d_\varepsilon^c)_{0 \leq c < 1}$ non-increasingly converges pointwise to d_ε^1 when c goes to 1, and that $(d_\varepsilon^1)_{\varepsilon > 0}$ non-decreasingly converges pointwise to d when ε goes to 0. These functions are continuous on the compact space $X \times X$, so the Dini's theorem implies $(X, d_\varepsilon^c) \xrightarrow{c \rightarrow 1} Y_\varepsilon$ and $Y_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} X$ for the Gromov-Hausdorff topology. In particular, Y_ε is a compact veneration tree for all $\varepsilon > 0$, according to Theorem 9. Since $\overline{\mathcal{V}_\mathcal{H}}$ is closed for the Gromov-Hausdorff topology, we only need to show $Y_\varepsilon \in \overline{\mathcal{V}_\mathcal{H}}$ for all $\varepsilon > 0$ to complete the proof. To do this, we will examine the diameters of loops on Y_ε .

Let $L \in \ell(Y_\varepsilon)$, Remark 3 yields that L is isometric to $4\Delta \cdot \mathcal{C}$ where $\text{diam}(L) = 2\Delta > 0$. It follows there exists $c \in [0, 1)$ and $x_0, x_1, x_2, x_3 \in X$ such that $(1 - c)\varepsilon \leq \Delta/100$, $|d_\varepsilon^c(x_i, x_{i+1}) - \Delta| \leq \Delta/100$ and $|d_\varepsilon^c(x_i, x_{i+2}) - 2\Delta| \leq \Delta/100$ for all $i \in \mathbb{Z}/4\mathbb{Z}$. We choose a geodesic g on (X, d_ε^c) from x_0 to x_2 and we can assume $d_\varepsilon^c(g(1/2), x_1) \geq 9\Delta/10$ without loss of generality. Let us also fix a geodesic g_0 on (X, d_ε^c) from x_0 to x_1 and a geodesic g_2 on (X, d_ε^c) from x_2 to x_1 . Similarly as before, we set $a = \sup g^{-1}(\text{Im } g_0)$, $x'_0 = g(a)$, $x'_2 = g(\inf g^{-1}(\text{Im } g_2) \cap [a, 1])$, and $x'_1 = g_0(\inf g_0^{-1}(\text{Im } g_2) \cap [g_0^{-1}(x'_0), 1])$. Either $x'_0 = x'_1 = x'_2$ or x'_0, x'_1, x'_2 are distinct and we may construct a loop γ on X that follows g from x'_0 to x'_2 , then follows g_2 from x'_2 to x'_1 , and finally follows g_0 from x'_1 to x'_0 . Since $x'_0 \in \text{Im } g_0$, it holds $d_\varepsilon^c(x_0, x_1) = d_\varepsilon^c(x_0, x'_0) + d_\varepsilon^c(x'_0, x_1)$ so

$$d_\varepsilon^c(x_0, x'_0) \leq d_\varepsilon^c(x'_0, g(1/2)) + d_\varepsilon^c(x_0, x_1) - d_\varepsilon^c(g(1/2), x_1) < d_\varepsilon^c(x'_0, g(1/2)) + d_\varepsilon^c(x_0, g(1/2)).$$

Thereby, $d_\varepsilon^c(x_0, g(1/2)) = d_\varepsilon^c(x_0, x'_0) + d_\varepsilon^c(x'_0, g(1/2))$, because those three points are in $\text{Im } g$. Together with $d_\varepsilon^c(x_0, x_1) = d_\varepsilon^c(x_0, x'_0) + d_\varepsilon^c(x'_0, x_1)$, it then yields $d_\varepsilon^c(x_0, x'_0) \leq 3\Delta/5$. We find the same bound for $d_\varepsilon^c(x_2, x'_2)$, and it follows $d_\varepsilon^c(x'_0, x'_2) \geq 3\Delta/5$. The loop γ is thus well-defined and we compute

$$\frac{3\Delta}{5} \leq \sup_{x, y \in \text{Im } \gamma} d_\varepsilon^c(x, y) = (1 - c\mathbf{1}_{\text{diam}(\text{Im } \gamma) < \varepsilon})\text{diam}(\text{Im } \gamma) \leq d_\varepsilon^c(x_0, x_1) + d_\varepsilon^c(x_1, x_2) + d_\varepsilon^c(x_2, x_0)$$

using Remark 3 and the uniqueness of L in (33). As a result of $(1 - c)\varepsilon < 3\Delta/5$, it is necessary that $\varepsilon \leq \text{diam}(\text{Im } \gamma)$, thus $\varepsilon \leq 10\Delta$. The conclusion that $\text{diam}(L) \geq \varepsilon/5$ for all $L \in \ell(Y_\varepsilon)$ ensures $Y_\varepsilon \in \overline{\mathcal{V}_\mathcal{H}}$ thanks to Lemmas 7 and 9. \square

The reader might wonder if any compact veneration tree can be coded by an excursion. Let us informally explain why the answer is no for \mathcal{V} . For all $n \geq 1$ and for all $u \in \{+1, -1\}^n$, we give ourselves \mathcal{C}_u a pointed metric circle of length $2/n$, seen as $[-1/n, 1/n]$ with the identification $1/n = -1/n$ and whose root is 0. The metric space X is constructed by gluing each $\mathcal{C}_{(\varepsilon_1, \dots, \varepsilon_{n+1})}$ on $\mathcal{C}_{(\varepsilon_1, \dots, \varepsilon_n)}$ at the point $2\varepsilon_{n+1}/(n+1)^2$, and then by taking the closure. One can prove X is compact because

$\text{diam}(X) \leq 2 + 4 \sum 1/n^2 < \infty$, and that it is a veneration tree thanks to Lemma 8 and Theorem 9. However, if there was an excursion f such that \mathcal{V}_f was isometric to X , there would be a point $t \in [0, 1]$ whose projection would be on $\mathcal{C}_{(1)}$ or on $\mathcal{C}_{(-1)}$. The reader should now be able to convince oneself that there would be $t \in [0, 1]$ and distinct $r_1, \dots, r_n \in [0, 1]$ such that $x_{r_i}^t = 1/i - 1/(i+1)^2$ for all $1 \leq i \leq n$, for all $n \geq 1$. Recalling Theorem 4, this leads to $\infty = \sup Jf \leq \sup f$ and contradicts that f is càdlàg. The issue here is the same as what prevents d^V from enjoying a functional continuity, namely that high variations of excursions may code short distances on the veneration tree. This problem disappears for \tilde{d}^V thanks to (3), thus we believe any compact veneration tree is isometric to a shuffled veneration tree $\tilde{\mathcal{V}}_f$ coded by an excursion f . Nevertheless, we were not able to demonstrate it. We have recently realized that Blanc-Renaudie managed to construct a contour path continuously exploring an inhomogeneous continuum random looptree in finite time [10]. Adapting his construction and the coding of any real tree by a continuous excursion of Duquesne [22] may result in a proof of our conjecture.

6 Probabilistic applications

Words and plane trees Before presenting our three applications, we recall the formalism of plane trees, see [25] for example. Let $\mathbb{N}^* = \{1, 2, 3, \dots\}$ be the set of positive integers and let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. The set \mathcal{U} is totally ordered by the lexicographic order, denoted by \leq . An element of $\mathcal{U} \setminus \{\emptyset\}$ is a finite sequence of positive integers $u = (u_1, \dots, u_m)$, we set $|u| = m$ the generation or height of u , and $\overleftarrow{u} = (u_1, \dots, u_{m-1})$ the parent of u . We also set $|\emptyset| = 0$. If $u, v \in \mathcal{U}$, we write $u * v \in \mathcal{U}$ for the concatenation of u and v , we say u is an ancestor of $u * v$, and we write $u \preceq u * v$. If $j \in \mathbb{N}^*$, we say $u * (j)$ is a child of u and u is the parent of $u * (j)$.

A *plane tree* τ is a finite subset of \mathcal{U} such that:

- $\emptyset \in \tau$,
- if $v \in \tau$ and $v \neq \emptyset$, then $\overleftarrow{v} \in \tau$,
- for all $u \in \tau$, there exists a non-negative integer $k_u(\tau)$ (the number of children of u in τ) such that for every $j \in \mathbb{N}^*$, $u * (j) \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

We denote the total progeny of τ , which is the total number of vertices of τ , by $\#\tau$ and the height of τ , which is the maximal generation, by $|\tau| = \max_{u \in \tau} |u|$. If $u \in \tau$, we define the subtree $\theta_u \tau$ stemming from u by $\theta_u \tau = \{v \in \mathcal{U} : u * v \in \tau\}$, which is also a plane tree. Furthermore, we denote by $\emptyset = u(0) < u(1) < \dots < u(\#\tau - 1)$ the vertices of τ listed in lexicographic order. We call $(u(i))_{0 \leq i \leq \#\tau - 1}$ the *depth-first exploration* of τ . The *exploration by contour* $c = (c(i))_{0 \leq i \leq 2(\#\tau - 1)}$ gives another way to explore the tree. Informally, c starts at the root and continuously visits the whole tree from the left to the right. Maybe more precisely, $c(0) = \emptyset$, and $c(i+1)$ is the \leq -smallest child of $c(i)$ that has not been visited yet if one exists or $c(i+1)$ is the parent of $c(i)$ otherwise. This process crosses each edge two times, one time upwards and one time downwards, so the needed time to complete the exploration is indeed $2(\#\tau - 1)$ and $c(2\#\tau - 2) = \emptyset$. In order to link those two explorations, we define $\xi(i)$ for all $0 \leq i \leq 2\#\tau - 2$ as the largest integer l such that $u(l)$ has been visited by $(c(j))_{0 \leq j \leq i}$. Moreover, the exploration processes induce finite sequences of integers that characterize the plane tree. Namely, the *height process* $H(\tau) = (H_i(\tau))_{0 \leq i \leq \#\tau - 1}$ is defined by $H_i(\tau) = |u(i)|$, and the *contour process* $C(\tau) = (C_i(\tau))_{0 \leq i \leq 2\#\tau - 2}$ is defined by $C_i(\tau) = |c(i)|$. General arguments of [25, Section 1.6] draw relations between the height process and the contour process via the two bounds

$$\max_{0 \leq i \leq 2\#\tau - 2} \left| \xi(i) - \frac{i}{2} \right| \leq \frac{|\tau|}{2} + 1, \quad (34)$$

$$\max_{0 \leq i \leq 2\#\tau - 2} |C_i(\tau) - H_{\xi(i)}(\tau)| \leq 1 + \max_{0 \leq j \leq \#\tau - 1} |H_{i+1}(\tau) - H_i(\tau)|, \quad (35)$$

with the convention $H_{\#\tau}(\tau) = 0$. The process $C(\tau)$ is extended to the real interval $[0, 2\#\tau - 2]$ by specifying $C(\tau)$ is affine on each interval $[i, i + 1]$ for $0 \leq i \leq 2\#\tau - 3$. Then, $(C_{2t(\#\tau-1)}(\tau))_{t \in [0,1]}$ is a continuous excursion and it classically codes (via d^{clas}) the pointed weighted metric space spanned by τ . More precisely, the tree T coded by it is obtained by replacing each edge (\overleftarrow{u}, u) of τ with a line segment of length 1 and by endowing this space with the normalized sum of the Lebesgue measures of those segments. With a slight abuse of notations, τ can be seen as a finite subset of T , so that \emptyset is the distinguished point of T and that the graph distance of τ is inherited from the metric on T .

6.1 Metric asymptotics for uniform random mappings

We write $[n] = \{1, \dots, n\}$ for $n \geq 1$. We are interested in the asymptotic behavior of the patterns of random uniform mappings on $[n]$. We follow the presentation of Aldous, Miermont, and Pitman in [5]. A mapping $m : [n] \rightarrow [n]$ may be interpreted as a digraph with set of vertices $[n]$ and with oriented edges $i \rightarrow m(i)$, thus allowing edges of the form $i \rightarrow i$. It naturally induces a simple graph (that may have edges of the form (i, i)), that we will denote by $\mathcal{G}(m)$. We define $m^0(i) = i$ and $m^{k+1}(i) = m(m^k(i))$ the $(k + 1)$ -fold iteration of m on $i \in [n]$, for any $k \geq 0$. We say i is a cyclic point of m if $m^k(i) = i$ for some $k \geq 1$ and we denote by $\Gamma(m)$ the set of cyclic points of m . If $\gamma \in \Gamma(m)$, we say the finite set $\{m^k(\gamma) : k \geq 1\} \subset \Gamma(m)$ is a cycle of m and we write

$$\mathcal{T}_\gamma(m) = \{\gamma\} \cup \left\{ i \in [n] \setminus \Gamma(m) : \exists k \geq 0, m^k(i) = \gamma \right\}$$

for the tree component of the mapping graph with root γ . By independently putting each set of children of the vertices of $\mathcal{T}_\gamma(m)$ into uniform random order, the graph $\mathcal{T}_\gamma(m)$ naturally corresponds to a plane tree $T_\gamma(m)$. The tree components are bundled by the disjoint cycles $\Gamma_j(m)$, for $1 \leq j \leq k(m)$, to form the basins of attraction $\mathcal{B}_j(m)$ of m , that are the connected components of the graph $\mathcal{G}(m)$, namely

$$\mathcal{B}_j(m) = \bigsqcup_{\gamma \in \Gamma_j(m)} \mathcal{T}_\gamma(m).$$

While the $\Gamma_j(m)$ partition $\Gamma(m)$, the $\mathcal{B}_j(m)$ partition $[n]$. Let us explain how to index the cycles $\Gamma_j(m)$ while ordering $\Gamma(m)$. We consider a random sample $(X_k)_{k \geq 1}$ of independent uniform random points of $[n]$, independent from the random orders on the tree components. We index the basins of attraction $\mathcal{B}_j(m)$ for $1 \leq j \leq k(m)$ by the order of their first appearances in the sample. For example, $\mathcal{B}_1(m)$ is the basin that contains X_1 and $\mathcal{B}_2(m)$ is the basin which contains the first $X_k \notin \mathcal{B}_1(m)$ if it exists, and so on. When $\gamma_i \in \Gamma_i(m)$ and $\gamma_j \in \Gamma_j(m)$ with $1 \leq i < j \leq k(m)$, we write $\gamma_i \leq_m \gamma_j$. To extend \leq_m into a total order on $\Gamma(m)$, we only need to order each cycle $\Gamma_j(m)$. Let $\gamma_j^\bullet \in \Gamma_j(m)$ be the root of the tree component that contains the first $X_k \in \mathcal{B}_j(m)$. We then specify

$$m(\gamma_j^\bullet) \leq_m m^2(\gamma_j^\bullet) \leq_m \dots \leq_m m^{\#\Gamma_j(m)-1}(\gamma_j^\bullet) \leq_m \gamma_j^\bullet.$$

We point out that \leq_m may even be extended to $[n]$ using the lexicographic orders on the tree components. When the mapping M is random, that whole ordering procedure is done independently from M . Let us write $\gamma(1) <_m \gamma(2) <_m \dots <_m \gamma(\#\Gamma(m))$ for the elements of $\Gamma(m)$ listed in the \leq_m -increasing order. Finally, let us define the metric object associated with the mapping we are interested in. For all $1 \leq j \leq k(m)$, we denote by $B_j(m)$ the pointed weighted metric space $\mathcal{B}_j(m)$ endowed with its graph distance, with its uniform probability measure, and with γ_j^\bullet as its distinguished point. For all $j > k(m)$, we set $B_j(m) = \partial$ where ∂ is the pointed weighted metric space with a unique point. We define $G(m) = (\mathbf{1}_{j \leq k(m)} \# \mathcal{B}_j(m) / n, B_j(m))_{j \geq 1}$. One can observe $G(m)$ determines the mapping m up to the labels of the vertices and the orientations of the cycles. Hence, it is a good metric interpretation of the mapping pattern.

Given the random order on the graph, one can construct several processes associated with the mapping. We define the height process $H(m) = (H_i(m))_{0 \leq i \leq n}$ of the mapping as the concatenation of the

height processes of the tree components $(H_i(T_{\gamma(j)}(m)) ; 0 \leq i \leq \#T_{\gamma(j)}(m) - 1)$ for $1 \leq j \leq \#\Gamma(m)$, in that order, followed by a last zero term. Similarly, the contour process $C(m) = (C_i(m))_{0 \leq i \leq 2n}$ of the mapping is the concatenation of the sequences $(C_i(T_{\gamma(j)}(m)) ; 0 \leq i \leq 2\#T_{\gamma(j)}(m) - 1)$ for $1 \leq j \leq \#\Gamma(m)$, in that order and with the convention $C_{2\#T_{\gamma(j)}(m)-1}(T_{\gamma(j)}(m)) = 0$, followed by a last zero term. We point out we have inserted a zero term between the contour processes of successive tree components. Alternatively, $H(m)$ and $C(m)$ can be expressed as variations of the height and contour processes of a single plane tree. Indeed, let $T(m)$ be the unique plane tree such that $k_{\emptyset}(T(m)) = \#\Gamma(m)$ and $\theta_{(i)}T(m) = T_{\gamma(i)}(m)$ for all $1 \leq i \leq \#\Gamma(m)$, so that $\#T(m) = n + 1$. We let the reader observe that $H_n(m) = C_{2n}(m) = 0$,

$$H_i(m) = H_{i+1}(T(m)) - 1 \text{ and } C_{i'}(m) = \max(0, C_{i'+1}(T(m)) - 1)$$

for all $0 \leq i \leq n - 1$ and for all $0 \leq i' \leq 2n - 1$. We keep track of the number of cyclic points via two processes $\ell(m) = (\ell_i(m))_{0 \leq i \leq n}$ and $\ell'(m) = (\ell'_i(m))_{0 \leq i \leq 2n}$ defined by

$$\ell_i(m) = \#\{1 \leq j \leq i : H_i(m) = 0\} \text{ and } \ell'_{i'}(m) = \frac{1}{2} \#\{1 \leq j \leq i' : C_i(m) = C_{i-1}(m) = 0\}$$

for all $0 \leq i \leq n$ and for all $0 \leq i \leq 2n$. We extend the processes $C(m)$ and $\ell'(m)$ to the real interval $[0, 2n]$ by specifying they are affine on each interval $[i, i + 1]$ for $0 \leq i \leq 2n - 1$. Moreover, we set $Z_0(m) = 0$ and $Z_j(m) = Z_{j-1}(m) + \mathbf{1}_{j \leq k(m)} \#\mathcal{B}_j(m)$ for all $j \geq 1$. These marks delimit the intervals corresponding to the basins of attraction. Observe $C_{2Z_j(m)}(m) = 0$ and $\ell'_{2Z_j(m)}(m) - \ell'_{2Z_{j-1}(m)}(m) = \#\Gamma_j(m)$ for all $1 \leq j \leq k(m)$.

Now, let $B^{|\text{br}|}$ be the standard reflected Brownian bridge on $[0, 1]$ and let L be half its local time at 0, which is normalized to be the density of the occupation measure at 0 of the reflected Brownian bridge, so that for any $t \in [0, 1]$, it verifies the convergence in probability

$$\frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{B_s^{|\text{br}|} \leq \varepsilon\}} ds \xrightarrow[\varepsilon \rightarrow 0^+]{\mathbb{P}} L_t.$$

Let $(U_j)_{j \geq 1}$ be a sequence of independent uniform random variables on $[0, 1]$. We define a sequence of random points $(D_j)_{j \geq 0}$ by setting $D_0 = 0$ and $D_j = \inf\{t \geq D_{j-1} + U_j(1 - D_{j-1}) : B_t^{|\text{br}|} = 0\}$ for all $j \geq 1$. For all $n \geq 1$, let M_n be a uniform random mapping on $[n]$ and consider the associated processes $H(M_n)$, $C(M_n)$, $\ell(M_n)$, $\ell'(M_n)$, and $(Z_j(M_n))_{j \geq 1}$. Aldous, Miermont, and Pitman proved in [5, Theorem 1] that $2B^{|\text{br}|}$, L , and $(D_j)_{j \geq 1}$ are the respective scaling limits of $H(M_n)$, $\ell(M_n)$, and $(Z_j(M_n))_{j \geq 1}$ for the uniform topology. Since $B^{|\text{br}|}$ and L are almost surely continuous, the estimates (34) and (35) yield the following corollary.

Corollary 3. *The marks $(Z_j(M_n)/n)_{j \geq 1}$ converge in distribution to the sequence $(D_j)_{j \geq 1}$. Jointly with that convergence, the following convergence holds in distribution for the uniform topology on $[0, 1]^2$:*

$$\left(\frac{1}{\sqrt{n}} C_{2nt}(M_n), \frac{1}{\sqrt{n}} \ell'_{2nt}(M_n) \right)_{t \in [0,1]} \xrightarrow{d} \left(2B_t^{|\text{br}|}, L_t \right)_{t \in [0,1]}.$$

Recall from Section 4.3 the definition of the space \mathbb{K}_w^\bullet of GHP-isometry classes of pointed weighted compact metric spaces endowed with the pointed Gromov-Hausdorff-Prokhorov distance $\mathbf{d}_{\text{GHP}}^\bullet$. When (X, d) is a compact metric space, we write $\text{diam}(X) = \sup_{x,y \in X} d(x, y)$. The space

$$\mathbb{S} = \left\{ (\alpha_j, X_j)_{j \geq 1} : \alpha_j \geq 0, \sum_{j \geq 1} \alpha_j = 1, X_j \in \mathbb{K}_w^\bullet, \text{diam}(X_j) \rightarrow 0 \right\}$$

is made separable and complete by the uniform distance

$$\mathbf{d}_{\mathbb{S}}((\alpha_j, X_j)_{j \geq 1}, (\beta_j, Y_j)_{j \geq 1}) = \sup_{j \geq 1} \max(|\alpha_j - \beta_j|, \mathbf{d}_{\text{GHP}}^\bullet(X_j, Y_j)).$$

We write $\beta \cdot (\alpha_j, X_j)_{j \geq 1} = (\alpha_j, \beta \cdot X_j)_{j \geq 1}$ when $\beta > 0$. Up to the small abuse of identifying a pointed weighted compact metric space with its class in \mathbb{K}_w^\bullet , it holds $G(m) \in \mathbb{S}$ for any mapping m . Let us write $Z_j^n = Z_j(M_n)$ to lighten the notations. For $n, j \geq 1$ and $s \in [0, 1]$, we set

$$\begin{aligned} f_j^n(s) &= C_{2Z_{j-1}^n + 2s(Z_j^n - Z_{j-1}^n)}(M_n) + \frac{1}{2} \left(\ell'_{2Z_j^n}(M_n) - \ell'_{2Z_{j-1}^n + 2s(Z_j^n - Z_{j-1}^n)}(M_n) \right), \\ f_j(s) &= 2B_{D_{j-1} + s(D_j - D_{j-1})}^{|\text{br}|} + \frac{1}{2} \left(L_{D_j} - L_{D_{j-1} + s(D_j - D_{j-1})} \right). \end{aligned}$$

They are excursions with a unique jump at 0 and it holds $\Delta_0(f_j^n) = \#\Gamma_j(M_n)/2$. Recall the veneration tree coded by the contour process of a plane tree τ is the pointed weighted metric space spanned by τ , that we denote by $\bar{\tau}$ here. Let us remark $\ell'(M_n)$ only decreases (at speed 1/2) on the intervals where $C(M_n)$ is constant. Thanks to the branching property of d^V , we observe that $\mathcal{V}_{f_j^n}$ is obtained by regularly gluing the $\overline{T_\gamma(M_n)}$, for $\gamma \in \Gamma(j)$ and in the order prescribed by \leq_{M_n} , on a metric circle of length $\#\Gamma_j(M_n)$ endowed with its Lebesgue measure, and by normalizing the sum of the measures of each component. Thus, we find $d_{\text{GHP}}^\bullet(n^{-1/2} \cdot B_j(M_n), n^{-1/2} \cdot \mathcal{V}_{f_j^n}) \leq n^{-1/2}$ for all $j \geq 1$. The root of $\mathcal{V}_{f_j^n}$ corresponds to $M_n(\gamma_j^\bullet)$ instead of γ_j^\bullet , but they are at distance at most 1 anyway. Using the Skorokhod's representation theorem, we assume the convergences in distribution of Corollary 3 happen almost surely. It follows $n^{-1/2} f_j^n \rightarrow f_j$ uniformly on $[0, 1]$ for all $j \geq 1$ almost surely, by continuity of $B^{|\text{br}|}$ and L . Since d^V is homogeneous, Theorem 7 directly ensures $n^{-1/2} \cdot \mathcal{V}_{f_j^n} \rightarrow \mathcal{V}_{f_j}$ for the pointed Gromov-Hausdorff-Prokhorov topology for all $j \geq 1$ almost surely. Next, we point out that clearly $D_j \rightarrow 1$ almost surely, so for all $\varepsilon > 0$, there is $N \geq 1$ such that $\|f_j\|_\infty \leq \varepsilon$ and $\|f_j^n\|_\infty \leq \varepsilon\sqrt{n}$ for all $n \geq N$ and $j \geq N$. In that case, $\text{diam}(\mathcal{V}_{f_j}) \leq 6\varepsilon$ and $\text{diam}(n^{-1/2} \cdot \mathcal{V}_{f_j^n}) \leq 6\varepsilon$ according to the inequalities (6) and (7). Eventually, we deduce the desired scaling limit in distribution for $G(M_n)$.

Theorem 12. *The following convergence holds in distribution on the space $(\mathbb{S}, d_{\mathbb{S}})$:*

$$\frac{1}{\sqrt{n}} \cdot G(M_n) \xrightarrow{d} (D_j - D_{j-1}, \mathcal{V}_{f_j})_{j \geq 1}.$$

6.2 Continuity in distribution of random stable looptrees

We briefly present the stable Lévy processes as Curien and Kortchemski in [18]. Let us fix $\alpha \in (1, 2)$. Let $X^{(\alpha)}$ be the α -stable Lévy process, which is defined as the stable spectrally positive Lévy process such that $\mathbb{E} \left[\exp(-\lambda X_t^{(\alpha)}) \right] = \exp(t\lambda^\alpha)$ for all $t \geq 0$ and $\lambda > 0$. It is a random càdlàg process from $[0, \infty)$ to \mathbb{R} whose all jumps are positive, and it satisfies the important scaling property that $(c^{-1/\alpha} X_{ct}^{(\alpha)})_{t \geq 0}$ has the same distribution as $X^{(\alpha)}$ for all $c > 0$. As a Lévy process, it is also characterized by its Lévy measure

$$\Pi_\alpha(dr) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} r^{-\alpha-1} \mathbf{1}_{(0, \infty)}(r) dr.$$

Furthermore, $X^{(\alpha)}$ is a recurring process because $\limsup_{t \rightarrow +\infty} X_t^{(\alpha)} = +\infty$ and $\liminf_{t \rightarrow +\infty} X_t^{(\alpha)} = -\infty$ almost surely. We also know the local minima of $X^{(\alpha)}$ are distinct and reached almost surely, namely for all $t \geq 0$, if $X_{t-}^{(\alpha)} = \inf_{[0, t]} X^{(\alpha)}$ then $X_t^{(\alpha)} = X_{t-}^{(\alpha)}$. Plus, it holds

$$\mathbb{P} \left(X_t^{(\alpha)} = \inf_{[0, t]} X^{(\alpha)} \right) = 0 \tag{36}$$

for all $t > 0$. For more information about stable processes or the Lévy processes in general, and for the proof of the facts mentioned here, we refer to [8].

Now, we define the normalized excursion of X above its infimum straddling the time 1 by following Chaumont [14]. Its left and right boundaries are

$$\underline{g} = \sup \left\{ s \leq 1 : X_s^{(\alpha)} = \inf_{[0, s]} X^{(\alpha)} \right\} \text{ and } \underline{d} = \inf \left\{ s \geq 1 : X_s^{(\alpha)} = \inf_{[0, s]} X^{(\alpha)} \right\},$$

and its length is $\underline{\zeta} = \underline{d} - \underline{g}$. Almost surely, it holds $0 < \underline{g} < 1 < \underline{d}$ and

$$X_{\underline{g}}^{(\alpha)} = \inf_{[0, \underline{g}]} X^{(\alpha)} = \inf_{[0, 1]} X^{(\alpha)} = \inf_{[0, \underline{d}]} X^{(\alpha)} = X_{\underline{d}}^{(\alpha)}.$$

Moreover, the random variable $\underline{\zeta}$ admits a positive density on $(0, \infty)$. Then, we eventually set

$$X_t^{\text{exc}, (\alpha)} = \underline{\zeta}^{-1/\alpha} \left(X_{\underline{g} + \underline{\zeta}t}^{(\alpha)} - X_{\underline{g}-}^{(\alpha)} \right)$$

for all $t \in [0, 1]$. This process can be interpreted as an excursion of $X^{(\alpha)}$ above its infimum conditioned to have length 1. It is strictly positive on $(0, 1)$ because the minima of $X^{(\alpha)}$ are distinct, and it is an excursion in the sense that $X^{\text{exc}, (\alpha)} \in \mathcal{H}$. Furthermore, it is independent from $\underline{\zeta}$. To lighten the notations, we write $\mathcal{L}_\alpha = \mathcal{L}_{X^{\text{exc}, (\alpha)}}$, $\widetilde{\mathcal{L}}_\alpha = \widetilde{\mathcal{L}}_{X^{\text{exc}, (\alpha)}}$, $\mathcal{V}_\alpha = \mathcal{V}_{X^{\text{exc}, (\alpha)}}$, and $\widetilde{\mathcal{V}}_\alpha = \widetilde{\mathcal{V}}_{X^{\text{exc}, (\alpha)}}$. We point out that \mathcal{L}_α is exactly the random α -stable looptree constructed by Curien and Kortchemski in [18]. By analogy, one could call \mathcal{V}_α the random α -stable veneration tree. However, this name would not be so useful because the next result, presented in [18, Corollary 3.4], implies $\mathcal{V}_\alpha = 2 \cdot \mathcal{L}_\alpha$ almost surely.

Proposition 10. *For all $\alpha \in (1, 2)$, the excursion $X^{\text{exc}, (\alpha)}$ is PJG almost surely.*

Before proving the proposition, we straightforwardly extend the notations used for excursions. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a càdlàg function. We set $\Delta_t(f) = f(t) - f(t-)$ for $t > 0$ and $\Delta_0(f) = f(0)$, we write $s \preceq_f t$ when $s \leq t$ and $f(s-) \leq \inf_{[s, t]} f$, and we set $x_s^t(f) = \mathbf{1}_{s \leq t} \max(\inf_{[s, t]} f - f(s-), 0)$ for all $s, t \geq 0$. We also define $\underline{f}(t) = \inf_{[0, t]} f$ for all $t \geq 0$.

Proof. The proof can be essentially found in [18], so we only detail some tedious steps using tools we have developed earlier. We fix $\alpha \in (1, 2)$ and forget it in the notations. For $n \geq 0$ and $s \in [0, 1]$, let $T_n = \inf\{t \geq 0 : X_t = -n\}$ and $f_n(s) = X_{sT_n} + n$. By recurrence of X , T_n is finite almost surely, and f_n is an excursion. We see that $x_s^t(f_n) = x_{sT_n}^{tT_n}(X)$ for all $s, t \in (0, 1]$, and $x_0^t(f_n) = \underline{X}_{tT_n} + n$ for all $t \in [0, 1]$. We first prove that almost surely, f_n is PJG for all $n \geq 0$, which is equivalent to

$$X_t - \inf_{[0, t]} X = \sum_{s \preceq_X t} x_s^t(X) \quad (37)$$

for all $t \geq 0$ since $T_n \rightarrow +\infty$ almost surely. The proof that (37) holds almost surely for every fixed $t \geq 0$ is presented in [18]. The left-hand side of (37) is obviously càdlàg and it is clear that if $t \in [0, T_n]$ then the right-hand side is equal to $Jf_n(t/T_n) - x_0^{t/T_n}(f_n)$, and thus it is càdlàg by Theorem 4. Hence, (37) holds for all $t \geq 0$ almost surely. Now, recall with (12) that an excursion f is PJG if and only if $d_f^\top(t, 1) = 0$ for any $t \in [0, 1]$. Almost surely, there exists $N \geq 1$ such that $\underline{d} < T_N$. The excursion f_N is PJG so $d_{f_N}^\top(t, 1) = 0$ for any $t \in [0, 1]$, and it follows that $d_{X^{\text{exc}}}^\top(t, 1) = 0$ for any $t \in [0, 1]$ by an application of the homogeneity and of the branching property on the interval $(\underline{g}/T_N, \underline{d}/T_N)$ for d^\top . \square

We set $X_t^{\text{exc}, (1)} = 1 - t$ for all $t \in [0, 1]$, and we write $\mathcal{V}_1 = \mathcal{V}_{X^{\text{exc}, (1)}}$ and $\widetilde{\mathcal{V}}_1 = \widetilde{\mathcal{V}}_{X^{\text{exc}, (1)}}$. It is immediate that $X^{\text{exc}, (1)}$ is a PJG excursion and that $\mathcal{V}_1 = \widetilde{\mathcal{V}}_1 = 2 \cdot \mathcal{L}_{X^{\text{exc}, (1)}} = 2 \cdot \widetilde{\mathcal{L}}_{X^{\text{exc}, (1)}} = 2 \cdot \mathcal{C}$, where we remind that \mathcal{C} is the metric circle of length 1. Let \mathbf{e} be the standard Brownian excursion. We set $X^{\text{exc}, (2)} = \sqrt{2} \cdot \mathbf{e}$, and we write $\mathcal{V}_2 = \mathcal{V}_{X^{\text{exc}, (2)}}$ and $\widetilde{\mathcal{V}}_2 = \widetilde{\mathcal{V}}_{X^{\text{exc}, (2)}}$. It is immediate that $X^{\text{exc}, (2)}$ is a continuous excursion and $\mathcal{V}_2 = \widetilde{\mathcal{V}}_2 = \mathcal{T}_{X^{\text{exc}, (2)}} = \sqrt{2} \cdot \mathcal{T}_\mathbf{e}$, where $\mathcal{T}_\mathbf{e}$ is the Brownian Continuum Random Tree introduced by Aldous in [3] and [4].

Theorem 13. *The family $\alpha \in [1, 2] \mapsto \mathcal{V}_\alpha$ is continuous in distribution for the pointed Gromov-Hausdorff-Prokhorov topology. In other words, for all $\beta \in (1, 2)$, the following three convergences hold in distribution for the pointed Gromov-Hausdorff-Prokhorov topology:*

$$\mathcal{L}_\alpha \xrightarrow[\alpha \rightarrow 1+]{d} \mathcal{C}, \quad \mathcal{L}_\alpha \xrightarrow[\alpha \rightarrow \beta]{d} \mathcal{L}_\beta, \quad \mathcal{L}_\alpha \xrightarrow[\alpha \rightarrow 2-]{d} \frac{\sqrt{2}}{2} \mathcal{T}_\mathbf{e}.$$

The convergences when α tends toward 1 or 2 for the Gromov-Hausdorff topology have been shown by Curien and Kortchemski in [18]. Here, we retrieve and generalize them by using Theorem 3 and the lemma below.

Lemma 10. *The family $\alpha \in [1, 2] \mapsto X^{\text{exc},(\alpha)}$ is continuous in distribution for the relaxed Skorokhod topology.*

Proof of the lemma. The desired convergences when α tends towards 1 or 2 have already been proven in [18] with Propositions 3.5 and 3.6. However, we point out that when $\alpha \rightarrow 1$, the convergence as stated in [18] does not seem true even with the time-reversing, because it would entail $0 = X_0^{\text{exc},(\alpha)} \xrightarrow{d} 1$. The issue is that the Skorokhod topology is too rigid for the boundary values $f(0)$, $f(1)$, and $f(1-)$. The mistake is at the end of the proof of Proposition 3.6 with the application of the Vervaat transform because the time of the infimum of the bridge is close, but not equal, to the unique big jump of the bridge. Our use of the relaxed Skorokhod topology fixes the problem because it allows $f_n \rightarrow f$ even when $f_n(0) = 0$ and $f(0) = 1$.

If $\beta \in (1, 2)$, we obtain the convergence in finite-dimensional distribution thanks to the Lévy property and thanks to the convergences $\mathbb{E} \left[\exp(-\lambda X_t^{(\alpha)}) \right] = \exp(t\lambda^\alpha) \xrightarrow{\alpha \rightarrow \beta} \exp(t\lambda^\beta) = \mathbb{E} \left[\exp(-\lambda X_t^{(\beta)}) \right]$ for any $t, \lambda \geq 0$. To show the tightness, we use the scaling property and bounds on fractional moments of stable distributions [31, 36] to prove there is $C(\beta) \in (0, \infty)$ that only depends on β such that

$$\mathbb{E} \left[|X_t^{(\alpha)} - X_s^{(\alpha)}|^\gamma |X_s^{(\alpha)} - X_r^{(\alpha)}|^\gamma \right] = \mathbb{E} \left[(X_1^{(\alpha)})^\gamma \right]^2 |t-s|^{\gamma/\alpha} |s-r|^{\gamma/\alpha} \leq C(\beta) |t-r|^{1+1/(2\beta+1)}$$

with $2\gamma = \beta + 1$, for all $\beta + 2 < 2\alpha < 2\beta + 1$ and $r \leq s \leq t \leq r + 1$. This implies the tightness by the Kolmogorov criterion for càdlàg processes, see [9, Theorem 13.5]. Hence, $X^{(\alpha)}$ converges in distribution to $X^{(\beta)}$ for the Skorokhod topology on $[0, T]$ when $\alpha \rightarrow \beta$, for all $T \geq 0$. Then, we can suppose that convergence happens almost surely on $[0, T]$ for all integers T by the Skorokhod's representation theorem. Since local minima of $X^{(\beta)}$ are distinct and reached almost surely, we get $\underline{g}(X^{(\alpha)}) \rightarrow \underline{g}(X^{(\beta)})$ almost surely. Moreover, either by applying the strong Markov property at $\underline{d}(X^{(\beta)})$ or by recalling that $X_{\underline{g}(X^{(\beta)})}^{(\beta)} = X_{\underline{d}(X^{(\beta)})}^{(\beta)}$ while the local local minima of $X^{(\beta)}$ are distinct, we check that $\underline{d}(X^{(\beta)})$ is not a local minima of $X^{(\beta)}$ almost surely, so $\underline{d}(X^{(\alpha)}) \rightarrow \underline{d}(X^{(\beta)})$ almost surely. See [8] for more details. Since $X^{(\beta)}$ is almost surely continuous at $\underline{g}(X^{(\beta)})$ and $\underline{d}(X^{(\beta)})$, it follows $X^{\text{exc},(\alpha)}$ converges in distribution to $X^{\text{exc},(\beta)}$ for the Skorokhod topology when $\alpha \rightarrow \beta$. \square

Because we know $X^{\text{exc},(\alpha)}$ is almost surely PJG for $\alpha \in [1, 2)$, Theorem 3 automatically shows the continuity in distribution of the family $\alpha \mapsto \mathcal{L}_\alpha$ on $[1, 2)$. However, it is not enough to prove the convergence for $\alpha \rightarrow 2$ because \mathbf{e} is a continuous excursion. Hence, we present a probabilistic method that will work for all $\alpha \in [1, 2]$.

Let us quickly present Itô's excursion theory applied to the stable Lévy process. We refer to [8, Chapter IV] for proof and details. Recall $\underline{X}_t = \inf\{X_s : s \in [0, t]\}$ is the running infimum process of X . Since the process $X - \underline{X}$ is strong Markov and 0 is regular for itself, one can use Itô's excursion theory and take $-\underline{X}$ as the local time of $X - \underline{X}$ at level 0. Let (g_j, d_j) , $j \in \mathcal{J}$, be the excursion intervals of $X - \underline{X}$ away from 0. Almost surely, we have $X_{g_j-} = X_{g_j} = \underline{X}_{g_j}$ for all $j \in \mathcal{J}$ and for all $t \geq 0$, if t is a jump time of X then there exists $j \in \mathcal{J}$ such that $t \in (g_j, d_j)$. For all $j \in \mathcal{J}$, we set $\omega_j(s) = X_{\min(g_j+s, d_j)} - X_{g_j}$ with $s \geq 0$, the excursion away from 0 indexed by j . The ω_j are elements of the space $\mathcal{E} = \{\omega : [0, \infty) \rightarrow [0, \infty) : \text{càdlàg such that } \exists M \geq 0, \forall t \geq M, \omega(t) = 0\}$. From Itô's excursion theory, the point measure

$$\mathcal{N}_X = \sum_{j \in \mathcal{J}} \delta_{(-X_{g_j}, \omega_j)}$$

is a Poisson point measure on $[0, \infty) \times \mathcal{E}$ with intensity $d\mathbf{n}(\underline{d}\omega)$, where \mathbf{n} is a σ -finite measure on \mathcal{E} . Moreover, X is measurable with respect to the measure \mathcal{N}_X because $X_t > \underline{X}_t$ almost everywhere almost surely thanks to (36) and because the local minima X_{g_j} are almost surely distinct.

Let us choose the shuffle we will work with. When $\Delta \in (0, 1/2]$, we define ϕ_Δ as in Remark 1, and when $\Delta \geq 1/2$, we set $\phi_\Delta = \phi_{1/2}$. It is straightforward to verify that $\Phi : \Delta > 0 \mapsto \phi_\Delta$ is a continuous shuffle. Notice all the ϕ_Δ have only a countable number of discontinuity points. From the strong Markov property and the distribution of \mathcal{N}_X , we see that $B_{X^{\text{exc},(\alpha)}} \cap \mathbb{B}(\Phi) = \emptyset$ almost surely for all $\alpha \in (1, 2)$. We also have $B_{X^{\text{exc},(1)}} = \{1\}$ and $B_{X^{\text{exc},(2)}} = \emptyset$ obviously. Thus, Lemma 10 and Theorem 3 ensure the family $\alpha \in [1, 2] \mapsto \mathcal{V}_\alpha$ is continuous in distribution for the pointed Gromov-Hausdorff-Prokhorov topology. The end of the proof of Theorem 13 is based on the observation that the Lebesgue measure on $[0, 1]$ is invariant under all the $x \in [0, 1] \mapsto \phi_\Delta(x) \in [0, 1]$. Informally, the shuffle under Φ of joint points between loops does not change the law of the stable looptree. Formally, the continuity in distribution of $\alpha \in [1, 2] \mapsto \mathcal{V}_\alpha$ will follow from the next theorem.

Theorem 14. *For all $\alpha \in (1, 2)$, the random pointed weighted compact metric spaces \mathcal{L}_α and $\widetilde{\mathcal{L}}_\alpha$ have the same distribution.*

Let $\mathbb{N}^* = \{1, 2, 3, \dots\}$ be the set of positive integers. We fix $\alpha \in (1, 2)$ and forget it in the notations. Let $\varepsilon > 0$. We define a sequence $(\tau_n(X))_{n \geq 0}$ of $\sigma(X)$ -measurable stopping times by setting $\tau_0(X) = 0$ and $\tau_n(X) = \inf \{t > \tau_{n-1}(X) : \Delta_t(X) \geq \varepsilon\}$ for $n \geq 1$. The $\tau_{n+1}(X) - \tau_n(X)$ are independent and have the same law as $\tau_1(X)$. Plus, $\tau_1(X) > 0$ almost surely because X is càdlàg with $X_0 = 0$. We postpone the proof of the following lemma.

Lemma 11. *There almost surely exists an α -stable Lévy process Y , a bijection $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^*$, and a function $\lambda : [0, \infty) \rightarrow [0, \infty)$ which avoids at most countably many points and preserves the Lebesgue measure on $[0, \infty)$ such that for all $t \geq 0$ and $k \in \mathbb{N}^*$, it holds $\underline{Y}_{\lambda(t)} = \underline{Y}_t = \underline{X}_t$, $\underline{Y}_{\tau_{\psi(k)}(Y)} = \underline{X}_{\tau_k(X)}$, $\Delta_{\tau_{\psi(k)}(Y)} = \Delta_{\tau_k(X)}$, and*

$$\frac{x_{\tau_{\psi(k)}(Y)}^{\lambda(t)}}{\Delta_{\tau_{\psi(k)}(Y)}} = \phi_{\Delta_{\tau_k(X)}} \left(\frac{x_{\tau_k(X)}^t}{\Delta_{\tau_k(X)}} \right) \text{ whenever } \tau_k(X) \leq \underline{d}(X).$$

The local minima of a stable Lévy process are almost surely distinct and $X_g = X_d = \inf_{[0, \underline{d}]} X$, so it holds $\underline{g}(X) = \inf \{t \leq 1 : \underline{X}_t = \underline{X}_1\}$ and $\underline{d}(X) = \sup \{t \geq 1 : \underline{X}_t = \underline{X}_1\}$. The identity $\underline{Y} = \underline{X}$ thus yields $\underline{g}(X) = \underline{g}(Y)$, $\underline{d}(X) = \underline{d}(Y)$, and $\underline{\zeta}(X) = \underline{\zeta}(Y)$. Furthermore, as $t \in [\underline{g}, \underline{d}]$ if and only if $\underline{X}_t = \underline{X}_1$, we deduce from the properties provided by Lemma 11 that

$$\begin{aligned} \tau_k(X) \in [\underline{g}, \underline{d}] &\iff \tau_{\psi(k)}(Y) \in [\underline{g}, \underline{d}], \\ t \in [\underline{g}, \underline{d}] &\iff \lambda(t) \in [\underline{g}, \underline{d}], \end{aligned}$$

for all $t \geq 0$ and $k \in \mathbb{N}^*$. Hence, we have that λ induces a function $\lambda^{\text{exc}} : [0, 1] \rightarrow [0, 1]$ such that $\lambda(\underline{g} + \zeta t) = \underline{g} + \zeta \lambda^{\text{exc}}(t)$ for all $t \in [0, 1]$. This function avoids at most countably many points and preserves the Lebesgue measure on $[0, 1]$. For all $s, t \in [0, 1]$, we compute

$$\begin{aligned} \widetilde{d}_{\underline{\zeta}^{1/\alpha} J^\varepsilon X^{\text{exc}}}(s, t) &= \sum_{\substack{k \in \mathbb{N}^* \\ \tau_k(X) \in [\underline{g}, \underline{d}]}} \widetilde{\delta}_{\tau_k(X)}^X \left(x_{\tau_k(X)}^{g+\zeta s}(X), x_{\tau_k(X)}^{g+\zeta t}(X) \right) \\ &= \sum_{\substack{k \in \mathbb{N}^* \\ \tau_k(Y) \in [\underline{g}, \underline{d}]}} \delta_{\tau_k(Y)}^Y \left(x_{\tau_k(Y)}^{\lambda(g+\zeta s)}(Y), x_{\tau_k(Y)}^{\lambda(g+\zeta t)}(Y) \right) = d_{\underline{\zeta}^{1/\alpha} J^\varepsilon Y^{\text{exc}}}^{\text{L}}(\lambda^{\text{exc}}(s), \lambda^{\text{exc}}(t)) \end{aligned}$$

by using the properties provided by Lemma 11 and the identities (17) and (18). The same kind of computation gives $d_{\underline{\zeta}^{1/\alpha} J^\varepsilon Y^{\text{exc}}}^{\text{L}}(\lambda^{\text{exc}}(1), 1) = 0$. It follows that the application λ^{exc} naturally induces an isometry λ^0 from $\widetilde{\mathcal{L}}_{\underline{\zeta}^{1/\alpha} J^\varepsilon X^{\text{exc}}}$ to $\mathcal{L}_{\underline{\zeta}^{1/\alpha} J^\varepsilon Y^{\text{exc}}}$ which avoids at most countably many points and maps the respective roots onto one another. In fact, we show λ^0 is genuinely surjective with Proposition 4. Let us denote by $\widetilde{\mu}$ and μ the respective measures of $\widetilde{\mathcal{L}}_{\underline{\zeta}^{1/\alpha} J^\varepsilon X^{\text{exc}}}$ and $\mathcal{L}_{\underline{\zeta}^{1/\alpha} J^\varepsilon Y^{\text{exc}}}$, which are the

images of the Lebesgue measure on $[0, 1]$ by the respective canonical projections. The invariance of the Lebesgue measure on $[0, 1]$ by λ^{exc} directly yields $\mu = \lambda_{\ast}^0 \tilde{\mu}$. Hence, the pointed weighted metric spaces $\widetilde{\mathcal{L}}_{\underline{\zeta}^{-1/\alpha}, J^\varepsilon X^{\text{exc}}}$ and $\mathcal{L}_{\underline{\zeta}^{-1/\alpha}, J^\varepsilon Y^{\text{exc}}}$ are GHP-isometric. Since X and Y have the same distribution, we get

$$\left(\underline{\zeta}(X), \widetilde{\mathcal{L}}_{\underline{\zeta}^{-1/\alpha}(X), J^\varepsilon X^{\text{exc}}} \right) \stackrel{d}{=} \left(\underline{\zeta}(X), \mathcal{L}_{\underline{\zeta}^{-1/\alpha}(X), J^\varepsilon X^{\text{exc}}} \right).$$

Proposition 7, Corollary 2, and (31) ensure the couples $\left(\underline{\zeta}, \widetilde{\mathcal{L}}_{\underline{\zeta}^{-1/\alpha} X^{\text{exc}}} \right)$ and $\left(\underline{\zeta}, \mathcal{L}_{\underline{\zeta}^{-1/\alpha} X^{\text{exc}}} \right)$ have the same distribution, by making $\varepsilon \rightarrow 0+$. We remind that d^{L} is homogeneous but not \tilde{d}^{L} . Nevertheless, recall that $\underline{\zeta}$ admits a positive density on $(0, \infty)$ and that $\underline{\zeta}$ and X^{exc} are independent. We can then write

$$\mathbb{E} [G(\mathcal{L}_{X^{\text{exc}}})] = \frac{1}{\mathbb{P}(|\underline{\zeta} - 1| < \varepsilon)} \mathbb{E} \left[\mathbf{1}_{\{|\underline{\zeta} - 1| < \varepsilon\}} G \left(\underline{\zeta}^{-1/\alpha} \cdot \widetilde{\mathcal{L}}_{\underline{\zeta}^{-1/\alpha} X^{\text{exc}}} \right) \right]$$

for any bounded continuous function G and for any $\varepsilon > 0$. With the help of the dominated convergence theorem and Theorem 6, we end the proof by making $\varepsilon \rightarrow 0+$.

Proof of Lemma 11. We set $X^0 = X$, $\psi_0 = \text{id}_{\mathbb{N}^*}$, and $\lambda_0 = \text{id}_{[0, \infty)}$. Let $n \geq 0$. By induction, let us assume we have constructed an α -stable Lévy process X^n , a bijection $\psi_n : \mathbb{N}^* \rightarrow \mathbb{N}^*$, and a function $\lambda_n : [0, \infty) \rightarrow [0, \infty)$ which avoids at most countably many points and preserves the Lebesgue measure on $[0, \infty)$ such that for all $t \geq 0$ and $k \in \mathbb{N}^*$, it holds $\underline{X}_{\lambda_n(t)}^n = \underline{X}_t^n = \underline{X}_t$, $\underline{X}_{\tau_{\psi_n(k)}(X^n)}^n = \underline{X}_{\tau_k(X)}$, $\Delta_{\tau_{\psi_n(k)}}(X^n) = \Delta_{\tau_k}(X)$, and

$$\frac{x_{\tau_{\psi_n(k)}}^{\lambda_n(t)}(X^n)}{\Delta_{\tau_{\psi_n(k)}}(X^n)} = \begin{cases} \phi_{\Delta_{\tau_k}(X)} \left(\frac{x_{\tau_k}^t(X)}{\Delta_{\tau_k}(X)} \right) & \text{if } \psi_n(k) \leq n \\ \frac{x_{\tau_k}^t(X)}{\Delta_{\tau_k}(X)} & \text{if } \psi_n(k) > n \end{cases}.$$

We write $\Delta = \Delta_{\tau_{n+1}(X^n)}(X^n)$ and $\tau = \tau_{n+1}(X^n)$ to lighten the notations. For any $t \geq 0$, we set $\mathbf{X}_t = X_{\tau+t}^n - X_\tau^n$. Thanks to the strong Markov property of an α -stable Lévy process, we know \mathbf{X} is an α -stable Lévy process independent from $\left(X_{\min(t, \tau)}^n \right)_{t \geq 0}$. Let us denote by (g_i, d_i) , $i \in \mathcal{I}$, the excursion intervals of $\mathbf{X} - \underline{\mathbf{X}}$ away from 0 and by ω_i its excursion away from 0 on (g_i, d_i) , so that the point measure

$$\mathcal{N}_{\mathbf{X}} = \sum_{i \in \mathcal{I}} \delta_{(-\mathbf{X}_{g_i}, \omega_i)}$$

is a Poisson point measure on $[0, \infty) \times \mathcal{E}$ with intensity $dt \underline{n}(d\omega)$. It is independent from $\left(X_{\min(t, \tau)}^n \right)_{t \geq 0}$, and so from Δ . Let us define two applications depending on Δ .

$$\begin{aligned} \varphi : [0, \infty) &\longrightarrow [0, \infty) & \Upsilon : [0, \infty) \times \mathcal{E} &\longrightarrow [0, \infty) \times \mathcal{E} \\ x &\longmapsto \begin{cases} \Delta - \Delta \phi_\Delta(1 - x/\Delta) & \text{if } x < \Delta \\ x & \text{if } x \geq \Delta \end{cases} & (x, \omega) &\longmapsto (\varphi(x), \omega) \end{aligned}$$

Recall the expression of ϕ_Δ with Remark 1. We check the Lebesgue measure on $[0, 1]$ is invariant by ϕ_Δ for all $\Delta > 0$. Indeed, $\phi_\Delta : [0, 1] \rightarrow [0, 1]$ is surjective and we can partition $(0, 1]$ into a countable number of intervals on whose ϕ_Δ is affine with slope ± 1 . Whatever the value of Δ is, it follows the product measure $dt \underline{n}(d\omega)$ is invariant by Υ , so conditionally given $\left(X_{\min(t, \tau)}^n \right)_{t \geq 0}$, the point measure $\Upsilon_* \mathcal{N}_{\mathbf{X}}$ is also a Poisson point measure with intensity $dt \underline{n}(d\omega)$. Therefore, there exists an α -stable Lévy process X^{n+1} that satisfies $\tau_{n+1}(X^{n+1}) = \tau$, $\left(X_{\min(t, \tau)}^{n+1} \right)_{t \geq 0} = \left(X_{\min(t, \tau)}^n \right)_{t \geq 0}$, and $\mathcal{N}_{\hat{\mathbf{X}}} = \Upsilon_* \mathcal{N}_{\mathbf{X}}$

almost surely, where $\hat{\mathbf{X}}_t = X_{\tau+t}^{n+1} - X_{\tau}^{n+1}$ for all $t \geq 0$. Analogously, we denote by (\hat{g}_j, \hat{d}_j) , $j \in \mathcal{J}$, the excursion intervals of $\hat{\mathbf{X}} - \underline{\hat{\mathbf{X}}}$ away from 0 and by $\hat{\omega}_j$ its excursion away from 0 on (\hat{g}_j, \hat{d}_j) , so that

$$\mathcal{N}_{\hat{\mathbf{X}}} = \sum_{j \in \mathcal{J}} \delta_{(-\hat{\mathbf{X}}_{\hat{g}_j}, \hat{\omega}_j)} = \sum_{i \in \mathcal{I}} \delta_{\Upsilon(-\mathbf{X}_{g_i}, \omega_i)}.$$

For all $x \geq 0$, we define $\sigma_x(\mathbf{X}) = \inf\{t \geq 0 : \underline{\mathbf{X}}_t \leq -x\}$ and $\sigma_x(\hat{\mathbf{X}})$ similarly. We also set $\zeta : \omega \in \mathcal{E} \mapsto \sup\{t \geq 0 : \omega(t) > 0\}$, so that $\zeta(\omega_i) = d_i - g_i$ is the lifetime of ω_i for any $i \in \mathcal{I}$. It follows from (36) that

$$\sigma_x(\mathbf{X}) = \sum_{i \in \mathcal{I}} \mathbf{1}_{-\mathbf{X}_{g_i} < x} \zeta(\omega_i) = \mathcal{N}_{\mathbf{X}} [\mathbf{1}_{[0, x)} \zeta],$$

almost surely, for all $x \geq 0$. But if $x \geq \Delta$, we notice that $(\mathbf{1}_{[0, x)} \zeta) \circ \Upsilon = \mathbf{1}_{[0, x)} \zeta$, which gives $\sigma_x(\mathbf{X}) = \sigma_x(\hat{\mathbf{X}})$ in that case. The processes \mathbf{X} and $\hat{\mathbf{X}}$ are continuous, so $\min(-\Delta, \underline{\mathbf{X}}) = \min(-\Delta, \underline{\hat{\mathbf{X}}})$ and $\sigma_{\Delta}(\mathbf{X}) = \sigma_{\Delta}(\hat{\mathbf{X}})$ almost surely. By definition of Δ , it is now clear that $\underline{X}^{n+1} = \underline{X}^n = \underline{X}$ almost surely. Next, recall the local minima of an α -stable Lévy process are distinct almost surely. In particular, the application Υ naturally induces a bijection $v : \mathcal{I} \rightarrow \mathcal{J}$, so that $(-\hat{\mathbf{X}}_{\hat{g}_{v(i)}}, \hat{\omega}_{v(i)}) = \Upsilon(-\mathbf{X}_{g_i}, \omega_i)$ for all $i \in \mathcal{I}$. When $\sigma_{\Delta}(\mathbf{X}) < g_i$ for some $i \in \mathcal{I}$, we can describe $[\hat{g}_{v(i)}, \hat{d}_{v(i)}]$ as the interval on which $\hat{\mathbf{X}} = \underline{\mathbf{X}}_{g_i} < -\Delta$, and this justifies $\hat{\mathbf{X}} = \mathbf{X}$ on the interval $[\hat{g}_{v(i)}, \hat{d}_{v(i)}] = [g_i, d_i]$. Eventually, we have found $X_t^{n+1} = X_t^n$ for all $t \in [0, \tau]$ or $t \geq \tau + \sigma_{\Delta}(\mathbf{X})$.

Next, we build a function $\lambda' : [0, \infty) \rightarrow [0, \infty)$. If $t \in [0, \tau)$, then we set $\lambda'(t) = t$. If $t \geq \tau$ and $t - \tau \in [g_i, d_i]$ with $i \in \mathcal{I}$, then we set $\lambda'(t) = \hat{g}_{v(i)} + t - g_i$. If $t \geq \tau$ but $t - \tau$ is not in any of the $[g_i, d_i]$, then we set $\lambda'(t) = \tau + \sigma_{\varphi(-\underline{\mathbf{X}}_{t-\tau})}(\hat{\mathbf{X}})$. Observe λ' gives bijections respectively from $[\tau + g_i, \tau + d_i]$ and $[0, \tau)$ to $[\tau + \hat{g}_{v(i)}, \tau + \hat{d}_{v(i)}]$ and $[0, \tau)$ that preserve their Lebesgue measures, for all $i \in \mathcal{I}$. Almost surely, each of these two countable families of intervals partitions a conull subset of $[0, \infty)$, according to (36), so the Lebesgue measure on $[0, \infty)$ is invariant by λ' . It is easy to check that if $s \neq \tau$ then

$$\underline{X}^{n+1}_{\lambda'(t)} = \underline{X}^n_t, \quad x_{\tau}^{\lambda'(t)}(X^{n+1}) = \Delta \phi_{\Delta} \left(\frac{x_{\tau}^t(X^n)}{\Delta} \right), \quad \text{and} \quad x_{\lambda'(s)}^{\lambda'(t)}(X^{n+1}) = x_s^t(X^n),$$

for all $s, t \geq 0$. If $t' - \tau > 0$ is not in any of the $[\hat{g}_{v(i)}, \hat{d}_{v(i)}]$ then $t' - \tau$ is the unique point where $\hat{\mathbf{X}}$ reaches $\underline{\hat{\mathbf{X}}}_{t'-\tau} < 0$, which means $t' = \tau + \sigma_{-\hat{\mathbf{X}}_{t'-\tau}}(\hat{\mathbf{X}})$. The function $\varphi : [0, \infty) \rightarrow (0, \infty)$ is surjective, so there exists $t \geq \tau$ such that $\lambda'(t) = t'$. Hence, the only point avoided by λ' is τ . Notice a jump has to be inside an excursion interval away from the infimum and that $\lambda'(\tau) = \tau + \sigma_{\Delta/2}(\hat{\mathbf{X}})$ is not a jump of X^{n+1} . Eventually, our reasoning justifies the existence of a unique bijection $\psi_{n+1} : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that $\psi_{n+1}(k) = n+1$ if $\psi_n(k) = n+1$ and $\tau_{\psi_{n+1}(k)}(X^{n+1}) = \lambda'(\tau_{\psi_n(k)}(X^n))$ otherwise, for all $k \in \mathbb{N}^*$. Furthermore, we set $\lambda_{n+1} = \lambda' \circ \lambda_n$. It is then straightforward to check that $(X^{n+1}, \psi_{n+1}, \lambda_{n+1})$ satisfies the properties desired for the induction.

Now, observe that $t \leq \underline{d}(X)$ if and only if $\underline{X}_t \geq \underline{X}_1$, for any $t \geq 0$. Hence, we find $\underline{d}(X^n) = \underline{d}(X)$, and that $\tau_k(X) \leq \underline{d}(X)$ if and only if $\tau_{\psi_n(k)}(X^n) \leq \underline{d}(X)$, for all $n \geq 0$ and $k \in \mathbb{N}^*$. This yields that X^n has the same number N of jumps of height at least ε in $[0, \underline{d}(X)]$ as X , which is finite because X is càdlàg. Thus, $\tau_k(X) \leq \underline{d}(X)$ if and only if $\psi_n(k) \leq N$. Finally, the choice $Y = X^N$, $\psi = \psi_N$, and $\lambda = \lambda_N$ verifies all the desired properties. In particular, X^N is indeed an α -stable Lévy process because N and \underline{d} are the same for all the X^n , so we can write for any non-negative measurable function G

$$\mathbb{E} [G(X^N)] = \sum_{n \geq 0} \mathbb{E} [\mathbf{1}_{\{\tau_n(X^n) \leq \underline{d}(X^n) < \tau_{n+1}(X^n)\}} G(X^n)] = \sum_{n \geq 0} \mathbb{E} [\mathbf{1}_{\{N=n\}} G(X)] = G(X).$$

□

Remark 4. Even if it was needed to replace the looptrees with veneration trees, we believe Theorem 14 should stay true for a larger class of random processes. The main argument was that the point measure

$\Upsilon_* \mathcal{N}_{\mathbf{X}}$ is distributed as $\mathcal{N}_{\mathbf{X}}$ conditionally given $\left(X_{\min(t, \tau)}^n\right)_{t \geq 0}$, which should be obtained with some kind of exchangeability. We think in particular of first passage bridges of processes with exchangeable increments occurring in the work [35] of Marzouk.

6.3 Invariance principle for random discrete looptrees

Let τ be a plane tree defined as at the start of Section 6. We associate a discrete looptree $\text{Loop}(\tau)$ with τ , defined as the graph on the set of vertices of τ where two vertices $u \leq v$ are joined by as many edges as verified conditions among the following:

- u and v are consecutive siblings of the same parent, which means $u = \overleftarrow{u} * (j)$ and $v = \overleftarrow{u} * (j+1)$ with some $j \geq 1$.
- v is the first child of u in τ , which means $v = u * (1)$,
- v is the last child of u in τ , which means $v = u * (k_u(\tau))$,
- $u = v$ and u does not have any children, which means $k_u(\tau) = 0$.

In particular, if v is the unique child of u in τ , then they are joined by exactly two edges. We endow this graph with the graph distance (every edge has unit length) and with the uniform probability measure on its vertices, and we distinguish \emptyset as its root so that $\text{Loop}(\tau)$ can be seen as a pointed weighted compact metric space. We can also inductively construct $\text{Loop}(\tau)$ by first arranging $\emptyset, (1), (2), \dots, (k_{\emptyset}(\tau))$ into a cycle of length $k_{\emptyset}(\tau) + 1$, then by gluing the root of $\text{Loop}(\theta_{(j)}\tau)$ respectively on (j) for each $1 \leq j \leq k_{\emptyset}(\tau)$. There are other ways to associate a discrete looptree with a tree. For example, Curien and Kortchemski make \emptyset correspond to a cycle of length $k_{\emptyset}(\tau)$ in [18]. For another version of the looptree, they erase the edges linking a leaf to itself. In [35], Marzouk contracts the last edge of each cycle, so that more than two cycles can share the same point.

Let $(u(i))_{0 \leq i \leq \#\tau - 1}$ be the depth-first exploration of τ and let $(c(i))_{0 \leq i \leq 2\#\tau - 2}$ be its exploration by contour. It is classical that the plane tree τ is coded by its Lukasiewicz walk $L(\tau) = (L_i(\tau))_{0 \leq i \leq \#\tau}$, which is defined by $L_0(\tau) = 0$ and $L_{i+1}(\tau) = L_i(\tau) + k_{u(i)}(\tau) - 1$ for all $0 \leq i \leq \#\tau - 1$. We introduce a variant of this process that is adapted with the exploration by contour instead of the depth-first exploration. Let $W(\tau) = (W_t(\tau))_{t \in [0, 2\#\tau - 1]}$ be the process which is affine with slope -1 on each of the intervals $[i, i+1)$ for $0 \leq i \leq 2\#\tau - 2$, and such that $W_0(\tau) = k_{\emptyset}(\tau) + 1$, $W_{2\#\tau - 1}(\tau) = 0$, and

$$W_i(\tau) = \begin{cases} W_{i-1}(\tau) + k_{c(i)}(\tau) & \text{if } |c(i)| = |c(i-1)| + 1 \\ W_{i-1}(\tau) - 1 & \text{if } |c(i)| = |c(i-1)| - 1 \end{cases}$$

for all $0 \leq i \leq 2\#\tau - 2$. Observe $W(\tau)$ can be expressed in terms of the Lukasiewicz walk and of the contour process by the identity

$$W_i(\tau) = L_{\xi(i)+1}(\tau) + C_i(\tau) + 2 \quad (38)$$

for all $0 \leq i \leq 2\#\tau - 2$, where we remind that $\xi(i)$ is the largest integer l such that $u(l)$ has been visited by $(c(j))_{0 \leq j \leq i}$. The Lukasiewicz walk is non-negative before its last value which is -1 , see [25, Proposition 1.1] for example, so the process $W(\tau)$ is positive on $[0, 2\#\tau - 1)$ and is continuous at $2\#\tau - 1$. Hence, the process $w(\tau) : t \in [0, 1] \mapsto w_t(\tau) = W_{(2\#\tau - 1)t}(\tau)$ is an excursion whose jumps are at the times $i/(2\#\tau - 1)$ with $i = 0$ or $|c(i)| = |c(i-1)| + 1$. Moreover, $w(\tau)$ is PJG according to Lemma 4. Let $1 \leq i \leq 2\#\tau - 2$ such that $|c(i)| = |c(i-1)| + 1$. We remark that $W_t(\theta_{c(i)}\tau) = W_{i+t}(\tau) - W_{i-}(\tau)$ for any $t \in [0, 2\#(\theta_{c(i)}\tau) - 1]$. This kind of branching property can be used with the help of Lemma 2 to compute that

$$x_{i_1/(2\#\tau - 1)}^{i_2/(2\#\tau - 1)}(w(\tau)) = \begin{cases} k_{c(i_1)}(\tau) + 1 & \text{if } c(i_1) = c(i_2) \\ k_{c(i_1)}(\tau) + 1 - j & \text{if } c(i_1) * (j) \preceq c(i_2) \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

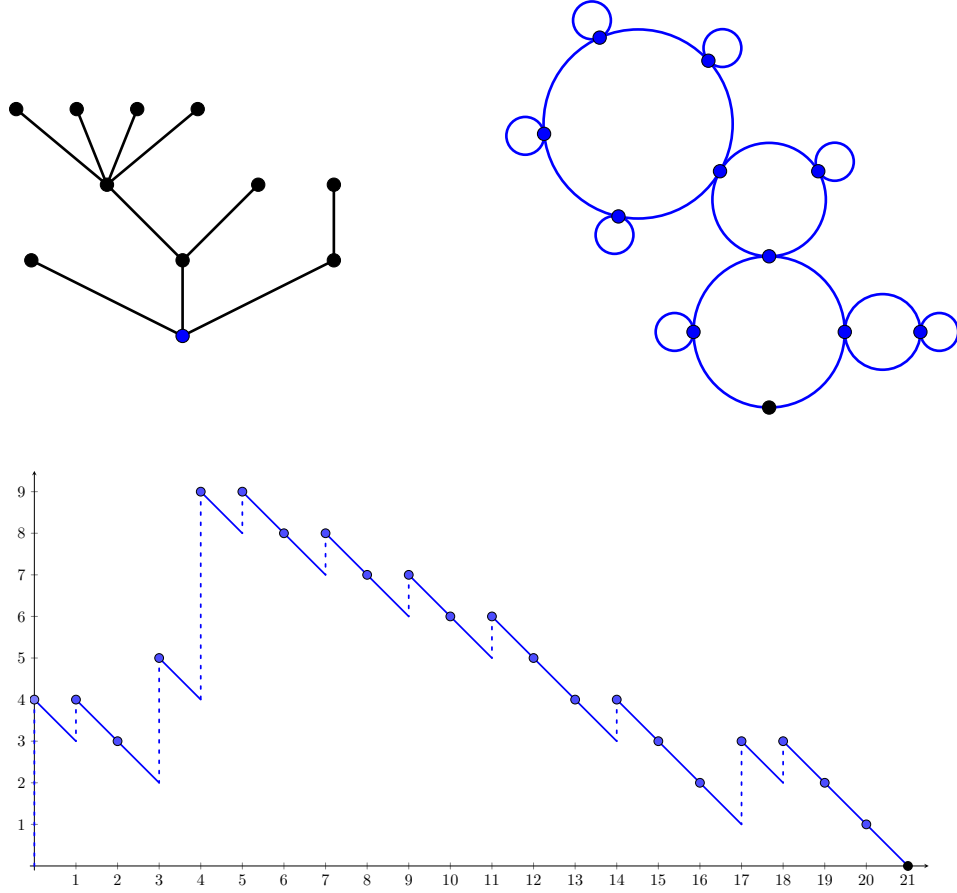


Figure 3: *Top Left:* A planar tree τ . *Top right:* Its looptree $\text{Loop}(\tau)$. *Bottom:* Its process $W(\tau)$.

when $0 \leq i_1, i_2 \leq 2\#\tau - 2$ are respectively either equal to 0 or such that $|c(i)| = |c(i - 1)| + 1$. Furthermore, applying Proposition 6 while recalling that $W(\tau)$ is affine of slope -1 by parts allows us to prove $\mathcal{L}_{w(\tau)}$ can be constructed by gluing the pointed weighted metric spaces $\mathcal{L}_{w(\theta_{(j)}\tau)}$ for $1 \leq j \leq k_\emptyset(\tau)$ respectively at the position j on a metric circle of length $k_\emptyset(\tau) + 1$ which is rooted at position 0 and endowed with its Lebesgue measure, and by normalizing the sum of the measures of the components. Hence, an easy induction on $\#\tau$ yields $\mathcal{L}_{w(\tau)}$ is the pointed weighted metric space spanned with the graph $\text{Loop}(\tau)$. Namely, it is obtained by replacing each edge with a line segment of length 1 and by endowing this space with the normalized sum of the Lebesgue measures of those segments. With a slight abuse of notations, we can also see $\text{Loop}(\tau)$ as a finite subset of $\mathcal{L}_{w(\tau)}$, so that they share the same root and that the graph distance of $\text{Loop}(\tau)$ is inherited from the metric on $\mathcal{L}_{w(\tau)}$. It is now clear that

$$d_{\text{GHP}}^\bullet \left(\mathcal{L}_{\frac{1}{a}w(\tau)}, \frac{1}{a} \cdot \text{Loop}(\tau) \right) \leq \frac{1}{a} + \frac{1}{\#\tau} \quad (40)$$

for any $a > 0$. Thus, the excursion $w(\tau)$ will be crucial for understanding the scaling limits of large discrete looptrees. The following theorem confirms this intuition in the PJG case.

Theorem 15. *Let (τ_n) be a sequence of plane trees, let f be an excursion, and let (a_n) be a sequence of positive real numbers which tends to ∞ such that $|\tau_n|/a_n \rightarrow 0$ and $(a_n^{-1}L_{\lfloor t\#\tau_n \rfloor}(\tau_n))_{t \in [0,1]} \rightarrow f$ for the relaxed Skorokhod topology. If f is a PJG excursion, then*

$$\frac{1}{a_n} \cdot \text{Loop}(\tau_n) \rightarrow \mathcal{L}_f$$

for the pointed Gromov-Hausdorff-Prokhorov topology.

Proof. If $\#\tau_n/a_n \rightarrow 0$ then $f = 0$ because $\sup L(\tau_n) \leq \#\tau_n$, and the desired result holds because the diameter of $\text{Loop}(\tau_n)$ is bounded by $\#\tau_n$. We can now assume $\#\tau_n/a_n \geq \varepsilon$ for some constant $\varepsilon > 0$, so that $\#\tau_n \rightarrow \infty$ and $|\tau_n|/\#\tau_n \rightarrow 0$ by the assumption $|\tau_n|/a_n \rightarrow 0$. The identity (38) and the inequality (34) ensure $a_n^{-1}w(\tau_n) \rightarrow f$ for the relaxed Skorokhod topology. Finally, we apply Theorem 3 and the inequality (40). \square

Recall the definitions of $X^{\text{exc},(\alpha)}$ and \mathcal{L}_α presented in the previous subsection, as well as that $X^{\text{exc},(\alpha)}$ is a PJG excursion. We thus retrieve the invariance principle of Curien and Kortchemski [18, Theorem 4.1] stating that if $|\tau_n|/a_n \xrightarrow{d} 0$ and $(a_n^{-1}L_{\lfloor t\#\tau_n \rfloor}(\tau_n))_{t \in [0,1]} \xrightarrow{d} X^{\text{exc},(\alpha)}$ for the Skorokhod topology, with some $\alpha \in (1, 2)$, then $a_n^{-1} \cdot \text{Loop}(\tau_n) \xrightarrow{d} \mathcal{L}_\alpha$ for the Gromov-Hausdorff topology. We point out again that our definition of $\text{Loop}(\tau)$ slightly differs from Curien and Kortchemski's, but the two objects are at Gromov-Hausdorff distance at most 10, so the result stays true anyway. Moreover, this result stills holds with $\alpha = 1$, when $X^{\text{exc},(1)} = 1 - t$ and $\mathcal{L}_1 = \mathcal{C}$. In the light of the asymptotic behavior when $\alpha \rightarrow 2$ studied in the previous subsection, it is tempting to state that if $|\tau_n|/a_n \xrightarrow{d} 0$ and $(a_n^{-1}L_{\lfloor t\#\tau_n \rfloor}(\tau_n))_{t \in [0,1]} \xrightarrow{d} \mathbf{e}$ for the relaxed Skorokhod topology then $2a_n^{-1} \cdot \text{Loop}(\tau_n) \xrightarrow{d} \mathcal{T}_\mathbf{e}$ for the Gromov-Hausdorff topology, where \mathbf{e} is the standard Brownian excursion. Sadly, the reader could convince oneself this is false by approximating \mathbf{e} by PJG excursions, in a similar fashion as the counterexample given at the beginning of Section 4. Here, the problem comes from a lack of exchangeability in the order of siblings inside the plane trees. Nevertheless, by adding a hypothesis that expresses such exchangeability, we will be able to state a general result for convergences of random discrete looptrees.

Let $\Psi = (\psi_k)_{k \geq 1}$ be a family of permutations ψ_k respectively of $\{1, \dots, k\}$. The family Ψ induces a bijection, also denoted by Ψ with a slight abuse of notations, from any plane tree τ to another plane tree $\Psi(\tau) = \{\Psi(u) : u \in \tau\}$ by inductively setting $\Psi(\emptyset) = \emptyset$ and $\Psi(u * (j)) = \Psi(u) * (\psi_{k_u(\tau)}(j))$ if $u \in \tau$ and $1 \leq j \leq k_u(\tau)$. Observe that if $u, v \in \tau$ then $|\Psi(u)| = |u|$, $k_{\Psi(u)}(\Psi(\tau)) = k_u(\tau)$, and $u \preceq v$ if and only if $\Psi(u) \preceq \Psi(v)$. Basically, $\Psi(\tau)$ is the plane tree recursively constructed by shuffling the order of siblings of the same parents, so $\text{Loop}(\Psi(\tau))$ can be retrieved by shuffling the joint points between the loops of $\text{Loop}(\tau)$. It is natural to try to compare the looptree shuffled by a family of permutations Ψ with the shuffled looptree $\widetilde{\mathcal{L}}_{w(\tau)}$ induced by a shuffle Φ .

Lemma 12. *We assume the shuffle Φ is such that the Lebesgue measure on $[0, 1]$ is invariant by ϕ_Δ and $\phi_\Delta(0) = 0$ for all $\Delta > 0$. Let τ be a plane tree, let Ψ be a family of permutations, and let $a > 0$. It holds*

$$\begin{aligned} & \mathbf{d}_{\text{GHP}}^\bullet \left(\widetilde{\mathcal{L}}_{\frac{1}{a}w(\tau)}, \mathcal{L}_{\frac{1}{a}w(\Psi(\tau))} \right) \\ & \leq \max_{v \in \tau} \sum_{\substack{u \in \tau, j \geq 1 \\ u*(j) \preceq v}} \frac{k_u(\tau) + 1}{a} \delta \left(\phi_{\frac{k_u(\tau)+1}{a}} \left(1 - \frac{j}{k_u(\tau) + 1} \right), 1 - \frac{\psi_{k_u(\tau)}(j)}{k_u(\tau) + 1} \right). \end{aligned}$$

Proof. We denote by c_1 the exploration by contour of τ and by c_2 the exploration by contour of $\Psi(\tau)$. For all $u \in \tau \setminus \{\emptyset\}$, we define $(2\#\tau - 1)r_1(u)$ as the unique integer i such that $c_1(i) = u$ and $|c_1(i)| = |c_1(i-1)| + 1$. Similarly, we define $(2\#\tau - 1)r_2(u)$ as the unique integer i such that $c_2(i) = \Psi(u)$ and $|c_2(i)| = |c_2(i-1)| + 1$. We also set $r_1(\emptyset) = r_2(\emptyset) = 0$. The $r_1(u)$ and the $r_2(u)$ for $u \in \tau$ are respectively all the jumps of $w(\tau)$ and $w(\Psi(\tau))$. For all $u \in \tau$ and $y \in [0, 1]$, the times

$$\begin{aligned} t_1(u, y) &= \inf \{ t \geq r_1(u) : w_t(\tau) = w_{r_1(u)-}(\tau) + y\Delta_{r_1(u)}(w(\tau)) \} \\ t_2(u, y) &= \inf \{ t \geq r_2(u) : w_t(\Psi(\tau)) = w_{r_2(u)-}(\Psi(\tau)) + y\Delta_{r_2(u)}(w(\Psi(\tau))) \} \end{aligned}$$

are well-defined and the applications $t_1, t_2 : \tau \times (0, 1] \rightarrow [0, 1]$ are bijective. It is a simple consequence of the fact $W(\tau)$ is strictly decreasing on the intervals $[i, i+1)$. We use the conventions $t_2(u, \phi_\Delta(x)) = t_2(u, 1)$ when $\phi_\Delta(x) = 0 = 1 \in \mathcal{C}$ and $t_2(u, \phi_\Delta(x)) = t_2(u, y)$ when $\phi_\Delta(x) = y \in (0, 1) \subset \mathcal{C}$. Let

us denote by $[s]_1$ and $[s]_2$ the canonical projections of $s \in [0, 1]$ respectively on $\widetilde{\mathcal{L}}_{\frac{1}{a}w(\tau)}$ and $\mathcal{L}_{\frac{1}{a}w(\Psi(\tau))}$. Since $\phi_\Delta : [0, 1] \rightarrow \mathcal{C}$ is always surjective, the set

$$\mathcal{R} = \{([1]_1, [1]_2)\} \cup \left\{ \left([t_1(u, y)]_1, [t_2(u, \phi_{\frac{k_u(\tau)+1}{a}}(y))]_2 \right) : u \in \tau, y \in [0, 1] \right\}$$

is a correspondence between $\widetilde{\mathcal{L}}_{\frac{1}{a}w(\tau)}$ and $\mathcal{L}_{\frac{1}{a}w(\Psi(\tau))}$. We check its distortion is bounded by two times the right-hand side of the desired inequality thanks to (39), Lemma 2, basic observations about Ψ , and the identities $\phi_\Delta(0) = 0$. We define a probability measure ν on $\widetilde{\mathcal{L}}_{\frac{1}{a}w(\tau)} \times \mathcal{L}_{\frac{1}{a}w(\Psi(\tau))}$ by setting

$$\int_{\widetilde{\mathcal{L}}_{\frac{1}{a}w(\tau)} \times \mathcal{L}_{\frac{1}{a}w(\Psi(\tau))}} g d\nu = \frac{1}{2\#\tau - 1} \sum_{u \in \tau} (k_u(\tau) + 1) \int_0^1 g \left([t_1(u, y)]_1, [t_2(u, \phi_{\frac{k_u(\tau)+1}{a}}(y))]_2 \right) dy$$

for any bounded measurable function g . Obviously, $\nu(\mathcal{R}) = 1$. The process $W(\tau)$ takes integer values at integer times and is affine of slope -1 on the intervals $[i, i + 1)$, so it follows that

$$t_1 \left(u, \frac{j - y}{k_u(\tau) + 1} \right) = t_1 \left(u, \frac{j}{k_u(\tau) + 1} \right) + \frac{y}{2\#\tau - 1}$$

for all $u \in \tau$, $y \in [0, 1)$, and $1 \leq j \leq k_u(\tau) + 1$. The application t_2 has an analogous behavior. Together with the invariance of the Lebesgue measure on $[0, 1]$ by ϕ_Δ , this allows us to verify that ν is a coupling between the probability measures of $\widetilde{\mathcal{L}}_{\frac{1}{a}w(\tau)}$ and $\mathcal{L}_{\frac{1}{a}w(\Psi(\tau))}$. \square

We say a random plane tree τ is called *exchangeable* if for any deterministic family Ψ of permutations, $\Psi(\tau)$ has the same distribution as τ . This will be the main assumption of our invariance principle for random discrete looptrees. For example, it is satisfied by uniform random trees or Galton-Watson trees conditioned on an event or a quantity that does not depend on the plane order: the number of vertices, the number of leaves, the height, the degree sequence, the generation size sequence, the existence of a chain with prescribed degrees...

Theorem 16. *Let (τ_n) be a sequence of random exchangeable plane trees, let X be a random excursion, and let (a_n) be a sequence of positive real numbers with $a_n \rightarrow \infty$. If the convergence $|\tau_n|/a_n \xrightarrow{d} 0$ holds in distribution and if the convergence $(a_n^{-1}L_{\lfloor t\#\tau_n \rfloor}(\tau_n))_{t \in [0,1]} \xrightarrow{d} X$ holds in distribution for the relaxed Skorokhod topology, then the convergence*

$$\frac{1}{a_n} \cdot \text{Loop}(\tau_n) \xrightarrow{d} \frac{1}{2} \cdot \mathcal{V}_X$$

holds in distribution for the pointed Gromov-Hausdorff-Prokhorov topology.

In particular, this theorem gives scaling limits for $\text{Loop}(\tau_n)$ when τ_n is a large critical Galton-Watson tree conditioned to have n vertices with offspring distribution in the domain of attraction of the Gaussian law but with infinite variance, because the scaling limit of the Lukasiewicz walk and $|\tau_n|$ are known in this case, see [21]. This was conjectured by Curien and Kortchemski in [18] and later proved by Kortchemski and Richier [30] with a spinal decomposition argument. As scaling limits of Lukasiewicz walks of uniform random trees with prescribed degree sequence are given by [27, Theorem 16.23], our theorem yields another proof for scaling limits of uniform random looptrees with prescribed cycle lengths obtained by Marzouk in [35], but only in the particular case where the height of the associated tree becomes negligible against a_n .

Proof. Thanks to the Skorokhod's representation theorem, we can assume all the convergences of the hypothesis happen almost surely. We can adapt the argument in the proof of Theorem 15 to prove that almost surely, $d_{\text{GHP}}^\bullet \left(a_n^{-1} \cdot \text{Loop}(\tau_n), \mathcal{L}_{a_n^{-1}w(\tau_n)} \right) \rightarrow 0$ and $a_n^{-1}w(\tau_n) \rightarrow X$ for the relaxed

Skorokhod topology. Thus, we only need to find the limit in distribution of $\mathcal{L}_{a_n^{-1}w(\tau_n)}$. The strategy of the proof is to construct a family of permutations Ψ and a shuffle Φ such that we can show $\mathcal{L}_{a_n^{-1}w(\Psi(\tau_n))}$ and $\widetilde{\mathcal{L}}_{a_n^{-1}w(\tau_n)}$ are close with Lemma 12. Then, as $\widetilde{\mathcal{V}}_{a_n^{-1}w(\tau_n)} = 2 \cdot \widetilde{\mathcal{L}}_{a_n^{-1}w(\tau_n)}$ because $w(\tau_n)$ is PJG, we would find the desired limit thanks to Theorem 3 and the exchangeability of τ_n . However, we have two technical difficulties to overcome. On the one hand, if $(k+1)/a_n = (l+1)/a_m = \Delta$, then ϕ_Δ would have to verify some relations with both ψ_k and ψ_l , which could complexify its construction. On the other hand, if the set of the $(k+1)/a_n$ is dense, it may be impossible for Φ to be continuous at some points, and so to verify the condition $B_X \cap \mathbb{B}(\Phi) = \emptyset$ almost surely.

By induction, we can choose a sequence $(b_n)_{n \geq 1}$ such that $b_1 = a_1$ and $|b_{n+1} - a_{n+1}| \leq 1$ but $b_{n+1} \notin \bigcup_{m=1}^n b_m \mathbb{Q}$ for all $n \geq 1$. Therefore, we can assume that $a_n/a_m \in \mathbb{Q} \implies n = m$ without loss of generality, which resolves the first difficulty. It is not difficult to express B_X as the set of values of a sequence of $\sigma(X)$ -measurable random variables because X is almost surely càdlàg, so that the set

$$E = \{x \in \mathbb{R} : \mathbb{P}(x \in B_X) + \mathbb{P}(\exists \Delta > 0, (\Delta, x) \in B_X) > 0\}$$

is countable. Let us fix $\varepsilon \in (0, \infty) \setminus E$ for the time being. For all $\eta \in (0, \varepsilon) \setminus E$, we are able to prove $\|J^\eta(a_n^{-1}w(\tau_n)) - J^\varepsilon(a_n^{-1}w(\tau_n))\|_\infty \longrightarrow \|J^\eta X - J^\varepsilon X\|_\infty$ almost surely with Proposition 7, since neither ε nor η are the height of some jump of X by definition of E . The set E is countable so we can give ourselves a deterministic strictly decreasing sequence $(\eta'_p)_{p \geq 1}$ of points of $(0, \varepsilon) \setminus E$ that tends to 0. Then, there exists a deterministic strictly increasing sequence $(m_p)_{p \geq 1}$ of integers such that

$$\mathbb{P}\left(\exists n \geq m_p, \left| \|J^{\eta'_p}(a_n^{-1}w(\tau_n)) - J^\varepsilon(a_n^{-1}w(\tau_n))\|_\infty - \|J^{\eta'_p} X - J^\varepsilon X\|_\infty \right| \geq \frac{1}{2^p} \right) \leq \frac{1}{2^p}$$

for all $p \geq 1$. We construct another deterministic sequence (η_n) by setting $\eta_n = \varepsilon$ if $n < m_1$ and $\eta_n = \eta'_p$ if $m_p \leq n < m_{p+1}$ with $p \geq 1$. This sequence tends to 0 so $\|JX - J^{\eta_n} X\|_\infty \longrightarrow 0$ almost surely according to Proposition 7, and the Borel-Cantelli lemma ensures that almost surely,

$$\|J^{\eta_n}(a_n^{-1}w(\tau_n)) - J^\varepsilon(a_n^{-1}w(\tau_n))\|_\infty \longrightarrow \|JX - J^\varepsilon X\|_\infty. \quad (41)$$

Plus, the only limit point of the set $D = \{(k+1)/a_n : k, n \geq 1 \text{ such that } k+1 < a_n \eta_n\}$ is 0 because $\eta_n \longrightarrow 0$, and that will resolve the second difficulty.

Let $k, n \geq 1$. We set

$$\psi_k(l) = \begin{cases} \frac{k+1-l+1}{2} & \text{if } k+1-l \text{ is odd} \\ k+1 - \frac{k+1-l}{2} & \text{if } k+1-l \text{ is even} \end{cases}$$

for all $1 \leq j \leq k$ and we easily check that ψ_k is a deterministic permutation of $\{1, \dots, k\}$. Since E is countable, it is possible to choose a deterministic $\alpha \in (1/2, 1)$ such that $\alpha^m(1+\alpha)/2 \notin E$ and $\alpha^m \notin E$ for all $m \geq 1$ and such that $\max(1-\alpha, 1/\alpha-1) \leq \min((k+1)/a_n, 1/(k+1))$. For all $m \geq 0$, we write $I_{2m} = (\alpha^m(1+\alpha)/2, \alpha^m]$ and $I_{2m+1} = (\alpha^{m+1}, \alpha^m(1+\alpha)/2]$. Plus, we also set

$$\varphi(x) = \begin{cases} x - \frac{\alpha^m}{2} & \text{if } x \in I_{2m} \text{ with an integer } m \\ 1 - x + \frac{\alpha^{m+1}}{2} & \text{if } x \in I_{2m+1} \text{ with an integer } m \end{cases}$$

for all $x \in (0, 1]$. Observe there is at most one $1 \leq j \leq k$ such that $j/(k+1) \in I_{2m} \cup I_{2m+1}$ because $1-\alpha \leq 1/(k+1)$. Let us denote by b the parity function defined on the set of integers by $b(2m) = 0$ and $b(2m+1) = 1$. Finally, we define $\phi_{\frac{k+1}{a_n}} : [0, 1] \longrightarrow \mathcal{C}$ by setting

$$\phi_{\frac{k+1}{a_n}}(x) = \begin{cases} 1 - \varphi(x) & \text{if } \exists j \in \{1, \dots, k\} \text{ such that } j/(k+1) \in I_{2m+1-b(j)} \\ \varphi(x) & \text{otherwise} \end{cases}$$

for all $x \in I_{2m} \cup I_{2m+1}$ with an integer m , and $\phi_{\frac{k+1}{a_n}}(0) = 0$. We emphasize the notation $\phi_{\frac{k+1}{a_n}}$ does not lead to any inconsistency. Indeed, if $(k+1)/a_n = (l+1)/a_m$ then $a_n/a_m \in \mathbb{Q}$, so $n = m$ and $k = l$. It is clear that $\phi_{\frac{k+1}{a_n}}$ is càglàd and surjective. The intervals $(I_m)_{m \geq 0}$ partition $(0, 1]$ and $\phi_{\frac{k+1}{a_n}}$ is affine with slope ± 1 on each one of them, so it preserves the Lebesgue measure on $[0, 1]$. We check that for all $k, n \geq 1$,

$$\begin{aligned} \sup_{x \in (0,1]} \left| \frac{2}{x} \delta \left(0, \phi_{\frac{k+1}{a_n}}(x) \right) - 1 \right| &\leq \min \left(\frac{k+1}{a_n}, 1 \right), \\ \sup_{1 \leq j \leq k} \delta \left(\phi_{\frac{k+1}{a_n}} \left(1 - \frac{j}{k+1} \right), 1 - \frac{\psi_k(j)}{k+1} \right) &\leq \frac{2}{k+1}. \end{aligned} \quad (42)$$

For all $n \geq 1$, we define a deterministic family of permutation $\Psi^n = (\psi_k^n)_{k \geq 1}$ by setting $\psi_k^n = \psi_k$ if $k+1 < a_n \eta_n$ and $\psi_k^n = \text{id}_{\{1, \dots, k\}}$ if $k+1 \geq a_n \eta_n$. Recall the only limit point of the set D is 0, D is bounded from above by ε , and E is countable. Hence, we can give ourselves a deterministic strictly decreasing sequence $(\beta_p)_{p \geq 1}$ of points of $(0, \infty) \setminus E$ which tends to 0, such that $\beta_1 = \varepsilon$, and such that there is exactly one element of D inside $[\beta_{p+1}, \beta_p)$ for all $p \geq 1$. Then, we define a deterministic shuffle $\Phi^\varepsilon : \Delta > 0 \mapsto \phi_\Delta^\varepsilon$ by setting $\phi_\Delta^\varepsilon(x) = x$ for all $x \in [0, 1]$ if $\Delta \geq \beta_1 = \varepsilon$, and by setting $\phi_\Delta^\varepsilon = \phi_{\frac{k+1}{a_n}}$ if $\Delta \in [\beta_{p+1}, \beta_p)$ with $p \geq 1$, where $(k+1)/a_n$ is the unique point of D also inside $[\beta_{p+1}, \beta_p)$. By definition of E and of the $\phi_{\frac{k+1}{a_n}}$, it holds $B_X \cap \mathbb{B}(\Phi^\varepsilon) = \emptyset$ almost surely. Theorem 3 ensures that almost surely, $2 \cdot \widetilde{\mathcal{L}}_{a_n^{-1}w(\tau_n)} \xrightarrow{\text{GHP}} \widetilde{\mathcal{V}}_f$ for the pointed Gromov-Hausdorff-Prokhorov topology, because $w(\tau_n)$ is PJG. Moreover, the Lebesgue measure on $[0, 1]$ is invariant by ϕ_Δ^ε and $\phi_\Delta^\varepsilon(0) = 0$ for all $\Delta > 0$, and

$$\sup_{\Delta > 0} \sup_{x \in [0,1]} \left| \frac{2}{x} \delta \left(0, \phi_\Delta^\varepsilon(x) \right) - 1 \right| \leq 1. \quad (43)$$

Thus, we can apply Lemma 12. Recall the constructions of Φ^ε and Ψ^n , and that $\eta_n \leq \varepsilon$. Let $u^*(j) \in \tau_n$ with $j \geq 1$, we want to bound the corresponding term in the right-hand-side of the inequality provided by Lemma 12. If $k_u(\tau_n) + 1 \geq a_n \varepsilon$, then this term is equal to 0 because $\phi_\Delta^\varepsilon(x) = x$ when $\Delta \geq \varepsilon$ and $\psi_k^n(j) = j$ when $k+1 \geq a_n \eta_n$. If $a_n \eta_n \leq k_u(\tau_n) + 1 < a_n \varepsilon$, then we control the term with the inequality (43). If $k_u(\tau_n) + 1 < a_n \eta_n$, then we use the inequality (42). Eventually, we obtain

$$\begin{aligned} d_{\text{GHP}}^\bullet \left(\mathcal{L}_{\frac{1}{a_n}w(\Psi^n(\tau_n))}, \widetilde{\mathcal{L}}_{\frac{1}{a_n}w(\tau_n)} \right) &\leq \max_{v \in \tau_n} \sum_{\substack{u \in \tau_n, j \geq 1 \\ u^*(j) \leq v}} \left(0 + 2 \mathbf{1}_{\{\eta_n \leq \frac{k_u(\tau_n)+1}{a_n} < \varepsilon\}} \frac{k_u(\tau_n) + 1 - j}{a_n} + \frac{2}{a_n} \right) \\ &\leq \frac{2|\tau_n|}{a_n} + 2 \sup_{t \in [0,1]} \sum_{\substack{s \in [0,1] \\ \eta_n \leq \Delta_s(a_n^{-1}w(\tau_n)) < \varepsilon}} x_s^t \left(\frac{1}{a_n} w(\tau_n) \right) \\ &\leq \frac{2|\tau_n|}{a_n} + 2 \|J^{\eta_n}(a_n^{-1}w(\tau_n)) - J^\varepsilon(a_n^{-1}w(\tau_n))\|_\infty. \end{aligned}$$

We point out the second inequality comes from the identity (39). Then, the almost sure convergences $|\tau_n|/a_n \rightarrow 0$, $2 \cdot \widetilde{\mathcal{L}}_{a_n^{-1}w(\tau_n)} \xrightarrow{\text{GHP}} \widetilde{\mathcal{V}}_X$, and (41) yield that almost surely,

$$\limsup_{n \rightarrow \infty} d_{\text{GHP}}^\bullet \left(\mathcal{L}_{\frac{1}{a_n}w(\Psi^n(\tau_n))}, \frac{1}{2} \cdot \widetilde{\mathcal{V}}_X \right) \leq 2 \|JX - J^\varepsilon X\|_\infty. \quad (44)$$

We remind that $\phi_\Delta^\varepsilon(x) = x$ for all $x \in [0, 1]$ whenever $\Delta \geq \varepsilon$. The identities (17) and (18) from Theorem 5 then imply that $d_{J^\varepsilon X}^L = \widetilde{d}_{J^\varepsilon X}^L$. Let us set $Y = X - JX + J^\varepsilon X$. We know $JY = J^\varepsilon Y$ from the point (v) of Theorem 4, thus $d_{Y-JY}^T = d_{X-JX}^T$ and $\widetilde{d}_{JY}^L = d_{JY}^L = d_{J^\varepsilon X}^L$. Moreover, Theorem 5 gives $d_Y^T = d_X^T$ and $d_Y^V = \widetilde{d}_Y^V$. Therefore, we find

$$\begin{aligned} \|\widetilde{d}_X^V - d_Y^V\|_\infty &= 2 \|\widetilde{d}_X^L - \widetilde{d}_{J^\varepsilon X}^L\|_\infty \leq 4 \|JX - J^\varepsilon X\|_\infty, \\ \|\widetilde{d}_X^V - d_Y^V\|_\infty &= 2 \|d_X^L - d_{J^\varepsilon X}^L\|_\infty \leq 4 \|JX - J^\varepsilon X\|_\infty, \end{aligned}$$

thanks to Corollary 2, that is properly used with the constant $K = 1$ according to (43). Eventually, we combine (44) and (31) to obtain $\limsup d_{\text{GHP}}^{\bullet} \left(\mathcal{L}_{\frac{1}{a_n} w(\Psi^n(\tau_n))}, \frac{1}{2} \cdot \mathcal{V}_X \right) \leq 10 \|JX - J^\varepsilon X\|_\infty$ almost surely. The random plane tree τ_n is exchangeable so it has the same distribution as $\Psi^n(\tau_n)$. However, neither $\mathcal{L}_{\frac{1}{a_n} w(\tau_n)}$ nor \mathcal{V}_X depend on ε . Plus, ε may be chosen arbitrarily small because E is countable. Hence, we conclude the convergence in distribution $2 \cdot \mathcal{L}_{\frac{1}{a_n} w(\tau_n)} \xrightarrow{d} \mathcal{V}_X$ holds for the pointed Gromov-Hausdorff-Prokhorov topology with the point (iv) of Proposition 7. \square

Remark 5. In Theorem 15, the assumption $|\tau_n|/a_n \rightarrow 0$ may be removed if one directly know the limit of $\frac{1}{a_n} w(\tau_n)$. However, doing the same for Theorem 16 is not possible because we also used the assumption $|\tau_n|/a_n \xrightarrow{d} 0$ to obtain the estimate (44). In fact, [30, Theorem 2] and [35, Theorem 7.5] provide examples which confirm the latter assumption is not superfluous. As shown by Marzouk, when $|\tau_n|/a_n$ is not negligible, an unbalance between d^L and d^T may occur so that the scaling limit of the looptrees is the quotient metric space induced by the pseudo-distance $ad^T + 2d^L$, with some $a > 1$.

References

- [1] Céline Abraham, Jérémie Bettinelli, Gwendal Collet, and Igor Kortchemski. Random maps. *ESAIM: Proceedings and Surveys*, 51:133–149, 2015.
- [2] Romain Abraham, Jean-François Delmas, and Patrick Hoscheit. A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electronic Journal of Probability*, 18:1 – 21, 2013.
- [3] David Aldous. The Continuum Random Tree I. *The Annals of Probability*, 19(1):1–28, 1991.
- [4] David Aldous. The Continuum Random Tree III. *The Annals of Probability*, 21(1):248–289, 1993.
- [5] David Aldous, Gregory Miermont, and Jim Pitman. Brownian Bridge Asymptotics for Random p -Mappings. *Electronic Journal of Probability*, 9:37 – 56, 2004.
- [6] Omer Angel and Oded Schramm. Uniform Infinite Planar Triangulations. *Communications in Mathematical Physics*, 241:191–213, 2003.
- [7] Albert-Laszlo Barabasi and Rita Albert. Emergence of Scaling in Random Networks. *Science (New York, N.Y.)*, 286:509–512, 1999.
- [8] Jean Bertoin. *Lévy Processes*. Volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, 1996.
- [9] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [10] Arthur Blanc-Renaudie. Looptree, Fennec, and Snake of ICRT. *Preprint available on arXiv*, 2022. arXiv:2203.10891.
- [11] Béla Bollobás, Oliver Riordan, Joel Spencer, and Gábor Tusnády. The degree sequence of a scale-free random graph process. *Random Structures Algorithms*, 18:279–290, 2001.
- [12] Jérémie Bouttier, Philippe Di Francesco, and Emmanuel Guitter. Planar maps as labeled mobiles. *Electronic Journal of Combinatorics*, 11(1):Research Paper 69, 27, 2004.
- [13] Reinhard Börger. How to make a path injective. *Recent developments of general topology and its applications : International Conference in Memory of Felix Hausdorff (1868-1942)*, pages 57–59, 1992.

- [14] Loïc Chaumont. Excursion normalisée, méandre et pont pour les processus de Lévy stables. *Bulletin des Sciences Mathématiques*, 121, pages 377–403, 1997.
- [15] Ian Chiswell. *Introduction To Lambda-Trees*. World Scientific Publishing Company, 2001.
- [16] Nicolas Curien, Thomas Duquesne, Igor Kortchemski, and Ioan Manolescu. Scaling limits and influence of the seed graph in preferential attachment trees. *Journal de l'École polytechnique — Mathématiques*, 2:1–34, 2015.
- [17] Nicolas Curien, Bénédicte Haas, and Igor Kortchemski. The CRT is the scaling limit of random dissections. *Random Structures & Algorithms*, 47(2):304–327, 2015.
- [18] Nicolas Curien and Igor Kortchemski. Random stable looptrees. *Electronic Journal of Probability*, 19:1–35, 2014.
- [19] Nicolas Curien and Igor Kortchemski. Percolation on random triangulations and stable looptrees. *Probability Theory and Related Fields*, 163:303–3037, 2015.
- [20] Sergei Ivanov Dmitri Burago, Yuri Burago. *A course in metric geometry*. Graduate studies in Mathematics. American Mathematical Society, Providence (R.I.), 2001.
- [21] Thomas Duquesne. A limit theorem for the contour process of conditioned Galton–Watson trees. *The Annals of Probability*, 31(2):996 – 1027, 2003.
- [22] Thomas Duquesne. The coding of compact real trees by real valued functions. *Preprint available on arXiv*, 2006. arXiv:math/0604106.
- [23] Thomas Duquesne and Jean-François Le Gall. *Random trees, Lévy processes and spatial branching processes*. Number 281 in Astérisque. Société mathématique de France, 2002.
- [24] Steven Evans. *Probability and Real Trees: École d'Été de Probabilités de Saint-Flour XXXV-2005*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2007.
- [25] Jean-François Le Gall. Random trees and applications. *Probability Surveys*, 2:245 – 311, 2005.
- [26] Svante Janson and Sigurdur Örn Stefánsson. Scaling limits of random planar maps with a unique large face. *The Annals of Probability*, 43(3):1045–1081, 2015.
- [27] Olav Kallenberg. *Foundations of Modern Probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
- [28] Ali Khezeli. Metrization of the Gromov–Hausdorff (-Prokhorov) topology for boundedly-compact metric spaces. *Stochastic Processes and their Applications*, 130(6):3842–3864, 2020.
- [29] Igor Kortchemski. Random stable laminations of the disk. *The Annals of Probability*, 42(2):725 – 759, 2014.
- [30] Igor Kortchemski and Loïc Richier. The boundary of random planar maps via looptrees. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 29(2):391–430, 2020.
- [31] Ercan Kuruoglu. Density parameter estimation of skewed alpha-stable distribution. *IEEE Transactions on Signal Processing*, 49:2192 – 2201, 2001.
- [32] Jean-François Le Gall and Yves Le Jan. Branching processes in Lévy processes: the exploration process. *The Annals of Probability*, 26(1):213–252, 1998.
- [33] Jean-François Le Gall and Grégory Miermont. Scaling limits of random planar maps with large faces. *The Annals of Probability*, 39(1):1 – 69, 2011.

- [34] Cyril Marzouk. On scaling limits of random trees and maps with a prescribed degree sequence. *Annales Henri Lebesgue*, 5:317–386, 2022.
- [35] Cyril Marzouk. Scaling limits of random looptrees and bipartite plane maps with prescribed large faces. *Preprint available on arXiv*, 2022. arXiv:2202.08666.
- [36] Muneya Matsui and Zbyněk Pawlas. Fractional absolute moments of heavy tailed distributions. *Brazilian Journal of Probability and Statistics*, 30(2):272–298, 2016.
- [37] Grégory Miermont. Aspects of random maps. Preliminary draft, 2014.
- [38] Frédéric Paulin. The Gromov topology on R-trees. *Topology and its Applications*, 32(3):197 – 221, 1989.
- [39] Loïc Richier. Limits of the boundary of random planar maps. *Probability Theory and Related Fields*, 172(3-4):789–827, 2018.
- [40] Miron L. Straf. Weak convergence of stochastic processes with several parameters. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory*, pages 187–221, Berkeley, California, 1972. University of California Press.
- [41] Jerzy Szymański. On a Nonuniform Random Recursive Tree. *North-Holland Mathematics Studies*, 144:297–306, 1987.
- [42] Stephen Willard. *General Topology*. Addison-Wesley series in mathematics. Dover Publications, 2004.