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HILBERT 17TH PROPERTY AND CENTRAL ORDERINGS

GOULWEN FICHOU, JEAN-PHILIPPE MONNIER AND RONAN QUAREZ

ABSTRACT. This paper is dedicated to the study of the converse implication in Hilbert 17th problem for a general commutative ring. In this direction, we introduce the notions of central and precentral orderings which generalize the notion of central points of irreducible real algebraic varieties. We study these two families of orderings which both live in the real spectrum of the ring and allow to state new Positivstellensätze and to obtain an equivalence in Hilbert 17th problem.

1. INTRODUCTION

Let R be a real closed field. The famous Hilbert 17th problem, solved by Artin in 1926 [3], states that a nonnegative polynomial $f \in R[x_1, \dots, x_n]$ on R^n is a sum of squares $f = \sum_{i=1}^r f_i^2$ of rational functions $f_i \in R(x_1, \dots, x_n)$. Since a non-empty Zariski open subset of R^n is dense for the Euclidean topology, it is easy to check that the converse implication is true.

Real algebra was developed in order to solve Hilbert 17th problem and allows to formulate general Positivstellensätze for polynomials nonnegative on a given closed semialgebraic subset $W = \{f_1 \geq 0, \dots, f_r \geq 0\}$ of R^n . Among these Positivstellensätze, one notably recovers a real Nullstellensatz.

It has been possible to generalize these Positivstellensätze for functions in the coordinate ring $R[V]$ of an irreducible affine algebraic variety V over R (see [6, Cor. 4.4.3]). In particular, we get an Hilbert 17th property : given $f \in R[V]$,

$$f(V(R)) \geq 0 \Rightarrow f \in \sum \mathcal{K}(V)^2,$$

with $\sum \mathcal{K}(V)^2$ denoting the set of sums of squares in the field of rational functions $K(V)$ on V . But, as noted in [6, Example 6.1.8], a crucial difference with the original Hilbert 17th problem is that a polynomial function $f \in R[V]$ which can be written as a sum of squares in $\mathcal{K}(V)$ may not be nonnegative on the whole set of real closed points $V(R)$ of V . It appears that such a Positivstellensatz certifies positivity only on the central locus $\text{Cent } V(R)$ of V , which consists in the Euclidean closure of the nonsingular real closed points. More precisely, it provides an equivalence in the statement of Hilbert 17th property [6, Thm. 6.1.9], in the sense that given $f \in R[V]$:

$$f(\text{Cent } V(R)) \geq 0 \iff f \in \sum \mathcal{K}(V)^2.$$

The origin of the adjective "central", the key concept in this paper, comes from [8]. This notion appears several times in [6] where the link with the positivity of sums of squares of rational functions is noted. A theory of seminormalization of real algebraic varieties adapted to the central locus is developed in [11] and continued in [17] where the definition of central ideal is introduced.

Let us consider an abstract framework. Let A be a commutative domain with fraction field $\mathcal{K}(A)$. Let $\text{Spec}_r A$ denote the real spectrum of A as introduced in [6], namely the set of all orderings of A or equivalently the set of (classes of) morphisms from A to a real closed field. By the abstract

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Positivstellensatz, for $f \in A$ the implication

$$f(\text{Spec}_r A) \geq 0 \Rightarrow f \in A \cap \sum \mathcal{K}(A)^2$$

holds, and similarly to the geometric case, it is not always an equivalence.

The principal goal of this paper is to characterize the closed subsets $E \subset \text{Spec}_r A$ giving rise to an equivalence in Hilbert 17th property, namely, for any $f \in A$:

$$f(E) \geq 0 \iff f \in \sum \mathcal{K}(A)^2.$$

Artin's solution may be understood as the origin of real algebra. Here we lay the foundations of central algebra to study the problem stated above.

The paper is organized as follows. We start in section 2 with a reminder on real algebra and in particular the real spectrum of a ring. We are particularly interested in the support mapping from the set of cones of A to the set of ideals convex with respect to the cone of sums of squares.

In section 3, we study central ideals. This subcategory of real ideals has recently been used in [11] to develop the theory of central seminormalization. The motivation was the property that central ideals behave much better than real ideals when we consider integral extensions of rings. Similarly to Dubois notion of central point of a real algebraic variety, we consider the notion of central orderings introduced in [4], as the elements of $\text{Spec}_r A$ which are in the closure of $\text{Spec}_r \mathcal{K}(A)$ for the topology of the real spectrum. The set of central orderings, denoted by $\text{Spec}_c A$, is a closed subset of $\text{Spec}_r A$ and the supports of central orderings are exactly the central prime ideals of A . We study these central orderings whose definition of topological nature is not easy to handle in order to prove algebraic statements as Positivstellensätze. This motivates us to introduce another sort of orderings which we call precentral and are defined by a simple and natural algebraic condition. The precentral orderings are those orderings which contain the cone $A \cap \sum \mathcal{K}(A)^2$ and hence are a sup-class of central orderings. The set of precentral orderings, denoted by $\text{Spec}_{pc} A$, is also a closed subset of $\text{Spec}_r A$ and the supports of precentral orderings are again exactly the central prime ideals of A .

In section 4, we study the differences between central and precentral orderings, giving characterizations of these two kinds of orderings. Although these orderings are distinct in general, it appears that they coincide for real algebraic varieties of dimension less than or equal to two.

In section 5, we give some precentral Positivstellensätze which come naturally from the algebraic nature of precentral orderings. One of the main results of the paper is as follows.

Theorem A. *Let f, f_1, \dots, f_r in A . Denote by $P \subset A$ the cone $(A \cap \sum \mathcal{K}(A)^2)[f_1, \dots, f_r]$ and by $\Lambda \subset \text{Spec}_{pc} A$ the set $\{\alpha \in \text{Spec}_{pc} A \mid f_1(\alpha) \geq 0, \dots, f_r(\alpha) \geq 0\}$. Then*

- (1) $f \geq 0$ on Λ if and only if $f q = p + f^{2m}$ for some p, q in P and $m \in \mathbb{N}$.
- (2) $f > 0$ on Λ if and only if $f q = 1 + p$ for some p, q in P .
- (3) $f = 0$ on Λ if and only if $f^{2m} + p = 0$ for some p in P and $m \in \mathbb{N}$.

As a consequence, we obtain also some central Positivstellensätze when the positivity conditions on central and precentral orderings coincide. In particular, we get geometric central Positivstellensätze for algebraic varieties of dimension less than or equal to two.

The study done in section 4 shows that we cannot differentiate central and precentral orderings by the global positivity of a single function. It enables to state a general version of Hilbert 17th property.

Theorem B. *Let $f \in A$. The following properties are equivalent :*

- (1) $f \geq 0$ on $\text{Spec}_c A$.
- (2) $f \geq 0$ on $\text{Spec}_{pc} A$.
- (3) $f \in \sum \mathcal{K}(A)^2$.

Note that when A is the coordinate ring of an irreducible affine algebraic variety V over a real closed field R , the previous properties are equivalent to $f \geq 0$ on $\text{Cent } V(R)$.

In particular, we solve the problem of getting an equivalence in Hilbert 17th property. We show that the central and precentral spectra are respectively the smallest and the largest subset satisfying Hilbert 17th property. This result fully justifies the introduction of these two types of orderings.

Theorem C. *Let E be a closed subset of $\text{Spec}_r A$ containing $\text{Spec}_r \mathcal{K}(A)$. The following properties are equivalent :*

- (1) $\forall f \in A, f \geq 0 \text{ on } E \iff f \in \sum \mathcal{K}(A)^2$.
- (2) $\text{Spec}_c A \subset E \subset \text{Spec}_{pc} A$.

Using the abstract formalism developed here, we are able to extend our Positivstellensätze to other geometric settings than real algebraic varieties, namely the Nash and the real analytic settings.

The final section 6 deals with continuous rational functions. As previously recalled, one knows that a nonnegative $f \in R[x_1, \dots, x_n]$ on R^n is a sum of squares of rational functions. From [13] it appears that f is in fact a sum of squares of rational functions which can be extended continuously to the whole R^n . We study the question of adding a continuity property in the third property of Theorem B, and prove surprisingly that it is not always possible. Anyway, we establish a continuous central Hilbert 17th property when the non-negativity is assumed on the whole real spectrum.

In all the paper, R denotes a real closed field and all the rings are commutative and contain \mathbb{Q} .

2. PRELIMINARIES ON REAL ALGEBRA

In this section we revisit real algebra (introduced in [6] and [14]) from the angle of ideals convex with respect to the cone of sums of squares.

In this section A is a ring.

2.1. Preordering, convexity, convex and real ideals and the support mapping.

Definition 2.1. A cone of A is a subset P of A such that $P + P \subset P$, $P \cdot P \subset P$ and $A^2 \subset P$. A cone P is called proper if $-1 \notin P$.

Note that the set $\sum A^2$ of sums of squares is the smallest cone of A . In case $-1 \notin \sum A^2$, we say that A is a formally real ring, which means also that it admits a proper cone. Another example of major interest in the paper is, if A is an integral domain with fraction field $\mathcal{K}(A)$, the cone $A \cap \sum \mathcal{K}(A)^2$ of elements in A that are sum of squares of elements in $\mathcal{K}(A)$. This cone plays a crucial role in the paper, it will be denoted simply by $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$.

We will encounter the notion of cone generated by a subset. Let P be cone of A . If $S \subset A$ then $P[S] = \left\{ \sum_{i=1}^n t_i s_i \mid t_i \in P, s_i \in S \right\}$ is the smallest cone of A containing P and S . If $S = \{f_1, \dots, f_k\}$ then we also denote $P[S]$ by $P[f_1, \dots, f_k]$.

Recall that for a given cone P of A , the set $P \cap -P$ is called the support of P and is denoted by $\text{supp}(P)$.

Proposition 2.2. *We have a support map*

$$\text{supp} : \text{Cone}(A) \rightarrow \text{Ideal}(A), P \mapsto \text{supp}(P)$$

which preserves inclusions.

Let $A \rightarrow B$ be a ring morphism. The diagram

$$\begin{array}{ccc} \text{Cone}(B) & \xrightarrow{\text{supp}} & \text{Ideal}(B) \\ \downarrow & & \downarrow \\ \text{Cone}(A) & \xrightarrow{\text{supp}} & \text{Ideal}(A) \end{array}$$

is commutative, where the vertical arrows are the natural maps $\text{Ideal}(B) \rightarrow \text{Ideal}(A)$, $I \mapsto \varphi^{-1}(I)$, and $\text{Cone}(B) \rightarrow \text{Cone}(A)$, $P \mapsto \varphi^{-1}(P)$.

Proof. The fact that the support map is well-defined follows directly from the formula

$$xy = \frac{1}{4}x(y+1)^2 - \frac{1}{4}x(y-1)^2$$

for $x, y \in A$. The commutativity of the diagram is straightforward. \square

Note that the support map sends a proper cone on a proper ideal. We are interested more generally by characterizing the image of the support map. Note that this map is in general not surjective, for example the prime ideal $(x^2 + 1) \subset \mathbb{R}[x]$ is not the support of a cone otherwise this one would not be proper.

We recall to this aim the notion of convexity of an ideal related to a given cone [6].

Definition 2.3. Let P be a cone of A . An ideal I of A is called P -convex if

$$p_1 + p_2 \in I \text{ with } p_1 \in P \text{ and } p_2 \in P \Rightarrow p_1 \in I \text{ and } p_2 \in I.$$

The support of a cone P is always convex for this cone, and it is easy to check that it is even the smallest P -convex ideal.

We give an elementary property about convexity that will be useful in the sequel.

Lemma 2.4. *Let P and Q be cones of A with $P \subset Q$, and $I \subset A$ be a Q -convex ideal. Then I is P -convex.*

The following result is useful to study the image of the support map.

Lemma 2.5. *Let P be a cone and I be an ideal of A . There exists a cone Q of A such that $P \subset Q$ and $\text{supp}(Q) = I$ if and only if I is P -convex. In this situation, $I + P$ is the smallest cone containing P with support I and it satisfies $I + P = P[I]$.*

Proof. Assume that I is P -convex. The point is to prove that $\text{supp}(I + P) = I$. To prove the non-obvious inclusion, let $q = a + b \in \text{supp}(I + P)$ with $a \in I$ and $b \in P$. So $q = a + b \in -(I + P)$ and thus $b \in -(I + P)$. We have $b = -a' - b'$ with $a' \in I$ and $b' \in P$ and it follows that $b + b' \in I$. Since I is P -convex then $b \in I$ and thus $q \in I$. It proves I is the support of $I + P$.

The converse implication comes from Lemma 2.4. \square

We answer now to the question asked above concerning the image of the support map.

Theorem 2.6. *The image of the support map $\text{supp} : \text{Cone}(A) \rightarrow \text{Ideal}(A)$ is the set of $\sum A^2$ -convex ideals of A .*

Proof. For P a cone of A , $\text{supp}(P)$ is P -convex and thus $\sum A^2$ -convex since $\sum A^2 \subset P$.

Let I be a $\sum A^2$ -convex ideal of A . Then the cone $I + \sum A^2$ is a cone with support I by Lemma 2.5. \square

There exists a notion of radical ideal with respect to a cone which is no more than the convexity with respect to the cone plus the classical radicality.

Definition 2.7. Let P be a cone of A . An ideal I of A is called P -radical if

$$a^2 + p \in I \text{ with } a \in A \text{ and } p \in P \Rightarrow a \in I$$

It means equivalently that the ideal is radical and P -convex by [6, Prop. 4.2.5]. For instance a real ideal, which is by definition a $\sum A^2$ -radical ideal, is radical and $\sum A^2$ -convex. Our interest in the notion of $\sum A^2$ -convex ideals is motivated by the natural feeling that some non real ideals seem closer to be real (like the ideal (x^2) in $\mathbb{R}[x]$) than others (e.g the ideal $(x^2 + 1)$ in $\mathbb{R}[x]$). And indeed one may check that the ideal (x^2) is $\sum \mathbb{R}[x]^2$ -convex.

2.2. Orderings, real and Zariski spectra and the support mapping. We denote by $\text{Spec } A$ (resp. $\text{R-Spec } A$) the (resp. real) Zariski spectrum of A , i.e the set of all (resp. real) prime ideals of A . The set of maximal (resp. and real) ideals is denoted by $\text{Max } A$ (resp. $\text{R-Max } A$). We endow $\text{Spec } A$ with the Zariski topology (whose closed sets are) generated by the sets $\mathcal{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p}\}$ for $f \in A$. The subsets $\text{R-Spec } A$, $\text{Max } A$ and $\text{R-Max } A$ of $\text{Spec } A$ are endowed with the induced Zariski topology. We denote also $\mathcal{V}(I) = \{\mathfrak{p} \in \text{Spec } A \mid I \subset \mathfrak{p}\}$ for I an ideal of A .

For $\mathfrak{p} \in \text{Spec } A$, we denote by $k(\mathfrak{p})$ the residue field at \mathfrak{p} i.e the fraction field of A/\mathfrak{p} .

Definition 2.8. A proper cone P is called an ordering if it satisfies

$$ab \in P \Rightarrow a \in P \text{ or } -b \in P.$$

The set of orderings of A is denoted by $\text{Spec}_r A$.

We recall the principal properties of orderings.

Proposition 2.9. [6] *Let P be an ordering of A . We have*

- (1) $P \cup -P = A$.
- (2) $\text{supp}(P)$ is a real prime ideal of A .
- (3) $\overline{P} = \{\overline{a}/\overline{b} \in k(\text{supp}(P)) \mid ab \in P\}$ is an ordering of $k(\text{supp}(P))$ such that $P = \varphi^{-1}(\overline{P})$ with $\varphi : A \rightarrow k(\text{supp}(P))$ the canonical morphism and $\overline{a}, \overline{b}$ denote the classes of a and b in $A/\text{supp}(P)$.
- (4) There exists a morphism $\alpha : A \rightarrow R_\alpha$ such that R_α is a real closed field, $\ker \alpha = \text{supp}(P)$ and $P = \phi^{-1}((R_\alpha)_+)$.

Conversely, given $\alpha : A \rightarrow R_\alpha$ a morphism into a real closed field, $P_\alpha = \alpha^{-1}((R_\alpha)_+)$ is an ordering of A with support $\ker \alpha = \text{supp}(P_\alpha)$.

Thus one can see an ordering of A equivalently as a morphism into a real closed field. We will use this identification in all the paper.

By [6, Thm. 4.3.7], A is formally real if and only if $\text{Spec}_r A \neq \emptyset$ if and only if $\text{R-Spec } A \neq \emptyset$. Let $\alpha \in \text{Spec}_r A$. Let $a \in A$, we set $a(\alpha) \geq 0$ if $a \in P_\alpha$, $a(\alpha) > 0$ if $a \in P_\alpha \setminus \text{supp}(P_\alpha)$, $a(\alpha) = 0$ if $a \in \text{supp}(P_\alpha)$. A set of the form

$$\mathcal{S}(f_1, \dots, f_k) = \{\alpha \in \text{Spec}_r A \mid f_1(\alpha) > 0, \dots, f_k(\alpha) > 0\}$$

for f_1, \dots, f_k some elements of A , is called a basic open subset of $\text{Spec}_r A$. A basic open subset of the form $\mathcal{S}(f)$, for a $f \in A$, is called principal. The real spectrum $\text{Spec}_r A$ is a topological space for the topology (whose open sets are) generated by the basics open sets. In the sequel, if T is a subset of $\text{Spec}_r A$ then we will denote by \overline{T} the closure of T for the real spectrum topology. A constructible subset of $\text{Spec}_r A$ is a finite boolean combination of basic open sets.

Given two orderings α and β , one says that β specializes to α , and write $\beta \rightarrow \alpha$, when $P_\beta \subset P_\alpha$. An equivalent characterization from [6, Prop. and Defn. 7.1.18] is $(P_\alpha \setminus (-P_\alpha)) \subset (P_\beta \setminus (-P_\beta))$, and another is $\alpha \in \overline{\{\beta\}}$.

As a consequence closed subsets of the real spectrum are closed by specialization, and the converse is also true for constructible closed subsets [6, Prop. 7.1.21].

The restriction of the support map $\text{supp} : \text{Cone}(A) \rightarrow \text{Ideal}(A)$ to orderings gives a map $\text{supp} : \text{Spec}_r A \rightarrow \text{Spec } A$ whose image is contained in $\text{R-Spec } A$. We complete here the study of this support map initiated in Theorem 2.6 and Proposition 2.2.

Proposition 2.10. *The support map $\text{supp} : \text{Spec}_r A \rightarrow \text{Spec } A$ is continuous and its image is $\text{R-Spec } A$.*

A morphism $\varphi : A \rightarrow B$ induces natural maps $\text{R-Spec}(B) \rightarrow \text{R-Spec}(A)$ and $\text{Spec}_r B \rightarrow \text{Spec}_r A$ and hence a commutative diagram:

$$\begin{array}{ccc} \text{Spec}_r B & \xrightarrow{\text{supp}} & \text{R-Spec } B \\ \downarrow & & \downarrow \\ \text{Spec}_r A & \xrightarrow{\text{supp}} & \text{R-Spec } A \end{array}$$

Proof. Let \mathfrak{p} be a real prime ideal. Since \mathfrak{p} is $\sum A^2$ -convex then it follows from [6, Prop. 4.3.8] that there exists an ordering with support equal to \mathfrak{p} . It proves $\text{supp}(\text{Spec}_r A) = \text{R-Spec } A$.

The continuity follows from [6, Prop. 7.1.8] or [14, Prop. 4.11]. \square

For the convenience of the reader, we recall from [6, Prop. 4.4.1] the formal Positivstellensatz, a key tool that we will use several times in the paper.

Theorem 2.11. *Let A a commutative ring. In A consider a subset H , a monoid M generated by the $(b_j)_{j \in L}$ and an ideal I generated by the $(c_k)_{k \in T}$.*

There is no $\alpha \in \text{Spec}_r A$ such that $H \subset P_\alpha$, $\forall j \in L$ $b_j \notin \text{supp}(\alpha)$, and $\forall k \in T$ $c_k \in \text{supp}(\alpha)$ if and only if we have an identity

$$p + b^2 + c = 0$$

where $p \in \sum A^2[H]$, $b \in M$, $c \in I$.

2.3. Real spectrum of geometric rings. Let us recall how the real points of a variety are related to the real spectrum of its coordinate ring. Assume $V = \text{Spec } R[V]$ is an affine algebraic variety over R with coordinate ring $R[V]$. We denote by $V(R)$ the set of real closed points of V i.e the subset of $\mathfrak{p} \in V$ such that $k(\mathfrak{p}) = R$. We have inclusions

$$V(R) \hookrightarrow \text{R-Spec } R[V] \hookrightarrow \text{Spec } R[V]$$

that makes $V(R)$ a topological space for the (induced) Zariski topology. The real zero sets $\mathcal{Z}(f) = \mathcal{V}(f) \cap V(R)$, for $f \in R[V]$, generate the closed subsets of $V(R)$ for the Zariski topology. If T is a subset of $V(R)$ then we will denote by \overline{T}^Z the closure of T for the Zariski topology. Since $R[V] = R[x_1, \dots, x_n]/I$ for a radical ideal $I \subset R[x_1, \dots, x_n]$ then we get an inclusion

$$V(R) \hookrightarrow R^n$$

that identifies $V(R)$ as a closed subset of R^n for the Zariski topology and also for the Euclidean topology. Recall that the unique ordering on R gives rise to an order topology on the affine spaces R^n called the Euclidean topology [6], in a similar way than the Euclidean topology on \mathbb{R}^n , even if the topological space R is not connected (except in the case $R = \mathbb{R}$) or the closed interval $[0, 1]$ is in general not compact. If T is a subset of $V(R)$ then we will denote by \overline{T}^E the closure of T for the Euclidean topology. An element of $V(R)$ can also be seen as a morphism from $R[V]$ to R . So we get a third inclusion

$$V(R) \hookrightarrow \text{Spec}_r R[V]$$

that identifies $V(R)$ as a closed (by specialization) subset of $\text{Spec}_r R[V]$. For $x \in V(R)$, we denote by $\alpha_x : R[V] \rightarrow R$, $f \mapsto f(x)$ the associated ordering of $R[V]$. A set of the form

$$S(f_1, \dots, f_k) = \{x \in V(R) \mid f_1(x) > 0, \dots, f_k(x) > 0\}$$

for f_1, \dots, f_k some elements of $R[V]$, is called a basic open subset of $V(R)$, it is an open subset of $V(R)$ for the Euclidean topology. A basic open subset of the form $S(f)$, for a $f \in R[V]$, is called principal. Clearly, the principal open subsets generates the Euclidean topology. A semialgebraic subset of $V(R)$ is a finite boolean combination of basic open sets. By [6, Prop. 7.2.2 and Thm. 7.2.3], the inclusion $V(R) \hookrightarrow \text{Spec}_r R[V]$ induces a one-to-one map between the semialgebraic subsets of $V(R)$ and the constructible subsets of $\text{Spec}_r R[V]$, this map sends the semialgebraic set S to the constructible \widetilde{S} described by the same inequalities than S . For f_1, \dots, f_k in $R[V]$ we have $S(\widetilde{f_1, \dots, f_k}) = S(f_1, \dots, f_k)$.

One important property of this map is the commutation with the closures for the Euclidean topology and the real spectrum topology [6, Thm. 7.2.3], namely for a semialgebraic subset S of $V(R)$ we have

$$\widetilde{(\widetilde{S}^E)} = \widetilde{(\widetilde{S})}.$$

2.4. Stability index. The material of this subsection will be used in section 4, hence the reader may momentarily skip it until reaching this section.

Definition 2.12. The stability index of A is the infimum of the numbers $k \in \mathbb{N}$ such that for any basic open subset S of $\text{Spec}_r A$ there exist f_1, \dots, f_k in A such that $S = \mathcal{S}(f_1, \dots, f_k)$.

Similarly, the stability index $s(U)$ of an open subset U of $\text{Spec}_r A$ is the infimum of the numbers $k \in \mathbb{N}$ such that for any basic open subset S of $\text{Spec}_r A$ such that $S \subset U$ there exist f_1, \dots, f_k in A such that $S = \mathcal{S}(f_1, \dots, f_k)$. If $U = \emptyset$ then we set $s(U) = 0$.

When $V = \text{Spec } R[V]$ is an affine algebraic variety over R with coordinate ring $R[V]$, from the properties of the map $S \mapsto \widetilde{S}$ exposed previously then, it is clear that the stability index of $R[V]$ is also the infimum of the numbers $k \in \mathbb{N}$ such that for any basic open subset S of $V(R)$ there exist f_1, \dots, f_k in $R[V]$ such that $S = \mathcal{S}(f_1, \dots, f_k)$. In that case, the stability index of $R[V]$ is also called the stability index of $V(R)$.

We recall the famous theorem of Bröcker [7] and Scheiderer [19]:

Theorem 2.13. (*Bröcker-Scheiderer*)

Let $V = \text{Spec } R[V]$ be an affine algebraic variety over R with coordinate ring $R[V]$. Then, the stability index of $R[V]$ coincides with the stability index of $V(R)$ and is equal to the dimension of $V(R)$ (as a semialgebraic set).

Note that in the case of finitely generated algebras over a non real closed field, the formula is not as simple. Concerning the stability index of abstract rings, we refer to [2].

3. CENTRAL ALGEBRA

From now on, the ring A is assumed to be a domain with fraction field $\mathcal{K}(A)$. Classical real algebra is developed around the structural cone $\sum A^2$. It is a fruitful tool to make a link between algebra and the geometry of the real points of a variety. In this section, we develop a notion of central algebra in order to take into account the central points of a real variety, i.e. the Euclidean closure of the nonsingular real closed points. The central locus of a real algebraic variety has been defined in [6] inspired by the work of Dubois [8]. This central algebra is built around the cone $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$ of the sums of squares of elements of the fraction field that belong to the ring A .

Note that the word *central* already appeared in the literature in an algebraic context : a notion of central ideal is introduced in [17], a definition of central ordering is given in [4]. Our goal is to show that the central algebra unifies these notions, and is a good framework to state abstract central Positivstellensätze.

3.1. Cones and orderings with support the null ideal. In this section we are interested by describing the inverse image of the null ideal by the support maps $\text{Cone}(A) \rightarrow \text{Ideal}(A)$ and $\text{Spec}_r A \rightarrow \text{Spec } A$.

Remark that the cones (resp. orderings) of $\mathcal{K}(A)$ with support the null ideal are exactly the proper cones (resp. the orderings) of $\mathcal{K}(A)$. Now we aim to compare the proper cones of $\mathcal{K}(A)$ with the cones of A with support the null ideal.

Proposition 3.1. *The map $P \mapsto P \cap A$ sends injectively the proper cones (resp. the orderings) of $\mathcal{K}(A)$ into the cones (resp. the orderings) of the ring A with support the null ideal.*

The map between $\text{Spec}_r \mathcal{K}(A)$ and the set of orderings of A with support (0) given by $P \mapsto P \cap A$ is bijective and the inverse map is given by

$$Q \mapsto Q_{\mathcal{K}(A)} := \{a/b \in \mathcal{K}(A) \mid ab \in Q\}.$$

Proof. The first point comes from the property of the support map, cf. Proposition 2.2 and Proposition 2.10. The second point is a consequence of (3) of Proposition 2.9. \square

In the sequel, we identify the proper cones of $\mathcal{K}(A)$ with a subset of $\text{Cone}(A)$ with support the null ideal and $\text{Spec}_r \mathcal{K}(A)$ with the subset of $\text{Spec}_r A$ with support the null ideal. We illustrate in the following example the non-surjectivity in the case of cones.

Example 3.2. Let $A = \mathbb{R}[V]$ be the coordinate ring of the cubic curve V with an isolated point, namely $A = \mathbb{R}[x, y]/(y^2 - x^2(x - 1))$. It is easy to see that the cone $\sum A^2$ has support the null ideal, however it does not come from a cone of $\mathcal{K}(A)$. Indeed, assume that $\sum A^2 = P \cap A$ for a cone P of $\mathcal{K}(A)$. We have $x - 1 = (y/x)^2 \in \mathcal{K}(A)^2$ and thus $x - 1 \in P \cap A = \sum A^2$. It follows that $x - 1$ must be nonnegative on $V(\mathbb{R})$ and evaluating at the isolated point we get a contradiction.

3.2. \mathcal{C} -convex and central ideals. We also recall the definition of central ideal introduced in [17].

Definition 3.3. Let I be an ideal of A . Then I is called central if I is \mathcal{C} -radical. The central radical of I is defined as

$$\sqrt[\mathcal{C}]{I} = \{a \in A \mid \exists m \in \mathbb{N} \exists b \in \mathcal{C} \text{ such that } a^{2m} + b \in I\}.$$

We denote by $\text{C-Spec } A$ (resp. $\text{C-Max } A$) the subset of $\text{Spec } A$ of central prime (resp. maximal) ideals.

From [6, Prop. 4.2.5], we know that an ideal is central if and only if it is radical and \mathcal{C} -convex. A \mathcal{C} -convex (resp. central) ideal is $\sum A^2$ -convex (resp. real) but the converse is not true as illustrated by the ideal $I = (x, y)$ in Example 3.2. Indeed we have $1 + (y/x)^2 = x \in I$, $1 \in \mathcal{C}$ and $(y/x)^2 \in \mathcal{C}$ but $1 \notin I$. So I is a real ideal which is not \mathcal{C} -convex.

By [17, Prop. 3.14], $\sqrt[\mathcal{C}]{I}$ is the intersection of the central prime ideals containing I and moreover I is central if and only if $I = \sqrt[\mathcal{C}]{I}$.

We give several characterizations of the existence of a central ideal in a domain.

Proposition 3.4. *The following properties are equivalent:*

- (1) $\mathcal{K}(A)$ is a formally real field.
- (2) $\text{C-Spec } A \neq \emptyset$.
- (3) A has a proper \mathcal{C} -convex ideal.
- (4) A has a proper central ideal.
- (5) (0) is a central ideal of A .

Proof. The equivalence between (1), (2), (4), (5) follows from [17, Prop. 3.16]. Clearly (4) implies (3). Let I be a proper \mathcal{C} -convex ideal. We claim that $\sqrt[\mathcal{C}]{I}$ is also proper and the proof will be done. Assume $1 \in \sqrt[\mathcal{C}]{I}$. There exists $b \in \mathcal{C}$ such that $1 + b \in I$ and since I is \mathcal{C} -convex then $1 \in I$, a contradiction. \square

In the case of the coordinate ring A of an irreducible affine algebraic variety V over R , the existence of a central ideal is equivalent to the existence of a so-called central point. From [6, Defn. 7.6.3], the central locus $\text{Cent } V(R)$ of $V(R)$ (or V) is defined to be the closure for the Euclidean topology of the nonsingular real closed points. In the sequel, we say that V is a central variety if $\text{Cent } V(R) = V(R)$. It follows from the definition that a nonsingular variety is central. On the contrary, the isolated point in the cubic exhibited in Example 3.2 is non-central. Note that $\text{Cent } V(R)$ is a closed semialgebraic set since $V_{\text{reg}}(R)$ is semialgebraic and the Euclidean closure of a semialgebraic set remains semialgebraic [6, Prop. 2.2.2].

The definition of central ideals gives a new formulation of the Central Nullstellensatz stated in [6, Cor. 7.6.6].

Theorem. (*Central Nullstellensatz*)

Let V be an irreducible affine algebraic variety over R . Then:

$$I \subset R[V] \text{ is a central ideal} \Leftrightarrow I = \mathcal{I}(\mathcal{Z}(I) \cap \text{Cent } V(R)) \Leftrightarrow I = \mathcal{I}(\mathcal{V}(I) \cap \text{Cent } V(R))$$

In particular, we have $\text{C-Max } R[V] = \text{Cent } V(R)$.

It furnishes a tool to decide geometrically whether an ideal is central.

Example 3.5. Let V be the Whitney umbrella given by the equation $y^2 = zx^2$. Then $\mathfrak{p} = (x, y) \subset \mathbb{R}[V]$ is a central prime ideal since the stick $\mathcal{Z}(\mathfrak{p})$ of the umbrella meets $\text{Cent } V(\mathbb{R})$ in maximal dimension.

Example 3.6. Let V be the Cartan umbrella given by the equation $x^3 = z(x^2 + y^2)$. Then $\mathfrak{p} = (x, y) \subset \mathbb{R}[V]$ is a real prime ideal but not a central ideal by the Central Nullstellensatz since the stick $\mathcal{Z}(\mathfrak{p})$ of the umbrella meets $\text{Cent } V(\mathbb{R})$ in a single point. Alternatively, one can show algebraically that \mathfrak{p} is not central by the identity

$$b = x^2 + y^2 - z^2 = x^2 + y^2 - \frac{x^6}{(x^2 + y^2)^2} = \frac{3x^4y^2 + 3x^2y^4 + y^6}{(x^2 + y^2)^2} \in \mathbb{R}[V] \cap \sum \mathcal{K}(V)^2.$$

Indeed $z^2 + b = x^2 + y^2 \in \mathfrak{p}$ but $z \notin \mathfrak{p}$.

One goal in the paper is to generalize this Central Nullstellensatz to Central Positivstellensätze in order to get algebraic certificates of positivity on subsets of the central locus.

In the same spirit, to show Positivstellensätze in the analytic settings, a variation of central radical (namely the Łojasiewicz radical) has been considered in [1].

3.3. Around Hilbert 17-th problem. Real geometers have always tried to give certificates of positivity for different types of functions. The most famous of these certificates is undoubtedly that of Hilbert 17th problem, attempting to characterize the polynomials of $A = \mathbb{R}[x_1, \dots, x_n]$ which are nonnegative on \mathbb{R}^n .

Artin answer to Hilbert 17th problem [3] says that a nonnegative polynomial is a sum of squares in $\mathcal{K}(A) = \mathbb{R}(x_1, \dots, x_n)$. In fact, the converse is also true as one may note using a continuity argument. Namely, one has the equivalence:

$$f(\mathbb{R}^n) \geq 0 \iff f \in \sum \mathbb{R}(x_1, \dots, x_n)^2.$$

Considering now varieties, the proof of Artin extends and one gets:

Proposition 3.7. *Let V be an irreducible affine algebraic variety over \mathbb{R} . Let $f \in \mathbb{R}[V]$. Then:*

$$f(V(\mathbb{R})) \geq 0 \Rightarrow f \in \sum \mathcal{K}(V)^2.$$

Contrary to Artin's theorem, the converse is no longer true. The following examples illustrate this phenomenon.

Example 3.8. Let $A = \mathbb{R}[V]$ be the coordinate ring of the cubic curve V with an isolated point considered in Example 3.2. Namely $A = \mathbb{R}[x, y]/(y^2 - x^2(x - 1))$. We have $x - 1 = (y/x)^2 \in \mathcal{K}(A)^2$ but $x - 1$ is negative at the isolated point of $V(\mathbb{R})$.

Example 3.9. Let V be the Cartan umbrella considered in Example 3.6. Namely $\mathbb{R}[V] = \mathbb{R}[x, y, z]/(z(x^2 + y^2) - x^3)$. We have already seen that $b = x^2 + y^2 - z^2 \in \mathbb{R}[V] \cap \sum \mathcal{K}(V)^2$ but b is negative on the stick without the origin.

We are interested by getting such Hilbert 17th properties with an equivalence for a general domain A , namely finding what kind of positivity is equivalent to the algebraic certificate of belonging to the cone $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$.

From the formal Positivstellensatz 2.11, one easily obtains an abstract version of Artin's solution to Hilbert 17th problem.

Proposition 3.10. *Let $f \in A$. Then,*

$$f(\text{Spec}_r A) \geq 0 \Rightarrow f \in \mathcal{C}.$$

We have already seen that the converse implication in the previous proposition is not always true. However, if A is a field then we clearly get an equivalence.

The above leads to the following definition.

Definition 3.11. Let E be a closed subset of $\text{Spec}_r A$. We say that E is a \mathcal{C} -subset of $\text{Spec}_r A$ if it satisfies the following property:

$$\forall f \in A, \quad f(E) \geq 0 \iff f \in \mathcal{C}.$$

Let V be an irreducible affine algebraic variety over R . We define similarly the \mathcal{C} -subsets of $V(R)$.

We first prove there is a unique geometric \mathcal{C} -set by reformulating [6, Thm. 6.1.9]:

Theorem 3.12. *Let V be an irreducible affine algebraic variety over R . The central locus $\text{Cent } V(R)$ is the unique \mathcal{C} -subset of $V(R)$.*

Proof. The central locus $\text{Cent } V(R)$ is a \mathcal{C} -subset of $V(R)$ by [6, Thm. 6.1.9].

Now, let E be a \mathcal{C} -subset of $V(R)$.

Assume in addition that $\text{Cent } V(R) \not\subset E$. Let $x \in \text{Cent } V(R) \setminus E$. Considering a small open ball centred at x that does not intersect E , there exists $f \in R[V]$ such that $x \in S(f)$ and $f(E) \leq 0$. From the property satisfied by E then it follows that $-f \in \sum \mathcal{K}(V)^2$. Again, by [6, Thm. 6.1.9], one gets $-f \geq 0$ on $\text{Cent } V(R)$ which contradicts $f(x) > 0$.

Assume now $E \not\subset \text{Cent } V(R)$. Let $x \in E \setminus \text{Cent } V(R)$. Arguing similarly to the previous case, there exists $f \in R[V]$ such that $x \in S(f)$ and $f(\text{Cent } V(R)) \leq 0$. Since $\text{Cent } V(R)$ is a \mathcal{C} -subset of $V(R)$ then it follows that $-f \in \sum \mathcal{K}(V)^2$. Since $-f$ is negative at x , we contradict the property of E to be a \mathcal{C} -set. \square

One main purpose of the paper is to study the \mathcal{C} -subsets of $\text{Spec}_r A$ for a general domain A . However, for an abstract ring A , we cannot copy the previous argument since, in particular, principal open subsets of $\text{Spec}_r A$ may not form a basis of open neighbourhoods.

In the following, we will focus in the \mathcal{C} -subsets of $\text{Spec}_r A$ that contain $\text{Spec}_r \mathcal{K}(A)$. In order to do that, we introduce the families of central and precentral orderings in the following sections.

3.4. Central cones and precentral orderings. In this section we introduce the notion of central cones whose supports give the \mathcal{C} -convex ideals.

Definition 3.13. A cone $P \subset A$ is called central if there exists a cone Q of $\mathcal{K}(A)$ such that $(Q \cap A) \subset P$. We denote by $\text{Cone}_c(A)$ the subset of all central cones in $\text{Cone}(A)$.

An ordering which is a central cone is called a precentral ordering. We denote by $\text{Spec}_{pc} A$ the subset of all precentral orderings in $\text{Spec}_r A$. We say that A is precentral if $\text{Spec}_{pc} A = \text{Spec}_r A$.

Since any cone of $\mathcal{K}(A)$ contains $\sum \mathcal{K}(A)^2$, it follows from the definition that a cone P of A is central if and only if $\mathcal{C} \subset P$. This allows one to write

$$\text{Spec}_{pc} A = \bigcap_{f \in \mathcal{C}} \{\alpha \in \text{Spec}_r A \mid f(\alpha) \geq 0\}.$$

This shows that $\text{Spec}_{pc} A$ is a closed subset in $\text{Spec}_r A$ as an intersection of closed subsets. Beware that it is not necessarily a constructible set as it will be point in the sequel.

We now study the restriction of our support map to the set of central cones and show that it coincides with the set of all \mathcal{C} -convex ideals.

Proposition 3.14. *Let $\text{supp} : \text{Cone}(A) \rightarrow \text{Ideal}(A)$ be the support map. We have*

- (1) $\text{supp}(\text{Cone}_c(A))$ is the set of \mathcal{C} -convex ideals.
- (2) $\text{supp}(\text{Spec}_{pc} A) = \text{C-Spec } A$.

Proof. Let P be a central cone. Since $\mathcal{C} \subset P$ then $\text{supp}(P)$ is \mathcal{C} -convex by Lemma 2.4. Let I be a \mathcal{C} -convex ideal. Since I is \mathcal{C} -convex then it follows from Lemma 2.5 that $I + \mathcal{C}$ is a cone with support equal to I . Since $I + \mathcal{C}$ is clearly a central cone then we have proved (1).

From (1) then $\text{supp}(\text{Spec}_{pc} A) \subset \text{C-Spec } A$. To show the converse implication, assume \mathfrak{p} is a central prime ideal. Then \mathfrak{p} is \mathcal{C} -convex and we conclude by using [6, Prop. 4.3.8]. \square

Looking at the example 3.5 of the Whitney umbrella, it is easy to see that a cone and even an ordering with support a \mathcal{C} -convex ideal is not always central as a cone.

Proposition 3.15. *An ordering of A with support the null ideal is precentral i.e $\text{Spec}_r \mathcal{K}(A) \subset \text{Spec}_{pc} A$.*

Proof. Let $P \subset A$ be an ordering such that $\text{supp}(P) = (0)$. By Proposition 3.1 then there exists $Q \in \text{Spec}_r \mathcal{K}(A)$ such that $P = Q \cap A$. Since $\sum \mathcal{K}(A)^2 \subset Q$ then $\mathcal{C} \subset P$. \square

From the example 3.2 of the cubic, we know that a cone with support the null ideal is not always central i.e the statement of Proposition 3.15 cannot be relaxed to cones.

One may give equivalent conditions on the existence of a precentral ordering:

Proposition 3.16. *The following properties are equivalent:*

- (1) $\mathcal{K}(A)$ is a formally real field.
- (2) There is a proper cone in $\text{Cone}_c(A)$.
- (3) There exists a proper \mathcal{C} -convex ideal in A .
- (4) $\text{Spec}_{pc} A \neq \emptyset$.
- (5) $\text{C-Spec } A \neq \emptyset$.

Proof. One may use Propositions 3.4, 3.14 and 3.15. \square

Let us end this section by considering the ring $A = \mathbb{R}[x, y]$ which is clearly precentral, namely $\text{Spec}_{pc} A = \text{Spec}_r A$. However, $\sum A^2 \neq \mathcal{C}$ (for instance consider the Motzkin polynomial) and hence $\text{Cone}_c(A) \neq \text{Cone}(A)$.

3.5. Central orderings. We begin with recalling the definition of central ordering from [4], definition which has inspired the definition 3.13 of a central cone in the preceding section.

Definition 3.17. A cone $P \in \text{Spec}_r A$ is called central if there exists an ordering $Q \in \text{Spec}_r \mathcal{K}(A)$ such that $(Q \cap A) \rightarrow P$. We denote by $\text{Spec}_c A$ the subset of $\text{Spec}_r A$ of central orderings. We say that A is central if $\text{Spec}_c A = \text{Spec}_r A$.

The notion of central ordering is a priori different from that of precentral ordering introduced in the preceding section, and studying the difference is a crucial issue in this paper.

In the geometric setting, let us see that the central spectrum is compatible with the notion of central points previously recalled. Again, this comes from results in [6]:

Proposition 3.18. *Let V be an irreducible affine algebraic variety over R . Then,*

$$\text{Spec}_c R[V] = \widetilde{\text{Cent } V(R)}.$$

Moreover, V is central if and only if $R[V]$ is central.

Proof. To show the first statement, one knows from [6, Prop. 7.6.2] that if $x \in \text{Cent } V(R)$, then α_x is the specialization of an ordering in $\text{Spec}_r \mathcal{K}(V)$, in other word, $\widetilde{\text{Cent } V(R)} \subset \overline{\text{Spec}_r \mathcal{K}(V)}$. The converse inclusion comes from [6, Prop. 7.6.4] which we recall the argument since we will need it in an abstract setting after.

For dimensional reasons, $\text{Spec}_r \mathcal{K}(V) \subset \widetilde{V_{\text{reg}}(R)}$ and thus, using that the tilde map commutes with the closures for the Euclidean topology and the real spectrum topology, we get

$$\text{Spec}_c R[V] = \overline{\text{Spec}_r \mathcal{K}(V)} \subset \overline{\widetilde{V_{\text{reg}}(R)}} = \overline{V_{\text{reg}}(R)^E} = \widetilde{\text{Cent } V(R)}.$$

It follows that $\text{Spec}_c R[V] = \widetilde{\text{Cent } V(R)}$.

Now, let us deduce the second statement. Assuming that $R[V]$ is central, then $V(R) = \text{Spec}_r R[V] \cap V(R) = \text{Spec}_c R[V] \cap V(R) = \text{Cent } V(R)$. Conversely, assuming that V is central, namely $V(R) = \text{Cent } V(R)$, one gets $(\text{Spec}_r R[V] \setminus \text{Spec}_c R[V]) = (\widetilde{V(R)} \setminus \widetilde{\text{Cent } V(R)}) = (V(R) \setminus \widetilde{\text{Cent } V(R)}) = \emptyset$ and hence $R[V]$ is a central domain. \square

An alternative way of saying that an ordering P is central is to say that there is $Q \in \text{Spec}_r A$ such that $\text{supp}(Q) = (0)$ and $Q \rightarrow P$. Of course, any central ordering is central as a cone and thus it is a precentral ordering. Moreover, $\text{Spec}_c A = \overline{\text{Spec}_r \mathcal{K}(A)}$ is naturally a closed subset of $\text{Spec}_r A$.

Let us see now how to mimic the geometric argument motivating our definition in the abstract case. As usual, if A is Noetherian then set $\text{Reg } A$ to be the set of all prime ideal \mathfrak{p} in A such that $A_{\mathfrak{p}}$ is a regular local ring. The complementary $\text{Sing } A$ of $\text{Reg } A$ in $\text{Spec } A$ is a non-empty closed subset for the Zariski topology whenever the ring A satisfies the so-called property (J1) [15, §32 B]. Note that excellent rings satisfy this condition.

If A is excellent then $\text{Reg } A$ and $\text{Sing } A$ are Zariski constructible subsets of $\text{Spec } A$. One may derive the associated constructible $\widetilde{\text{Reg } A}$ and $\widetilde{\text{Sing } A}$ in $\text{Spec}_r A$. Namely $\widetilde{\text{Sing } A} = \{\alpha \in \text{Spec}_r A \mid \text{supp}(\alpha) \in \text{Sing } A\}$ and $\widetilde{\text{Reg } A} = \text{Spec}_r A \setminus \widetilde{\text{Sing } A}$.

We give an abstract version of [6, Prop. 7.6.4].

Proposition 3.19. *Let A be an excellent domain. Considering the closure in $\text{Spec}_r A$, one has*

$$\overline{\text{Spec}_r \mathcal{K}(A)} = \overline{\widetilde{\text{Reg}(A)}}.$$

Proof. We start by showing that $\text{Spec}_r \mathcal{K}(A) \subset \widetilde{\text{Reg}(A)}$.

Let us assume that $\alpha \in \text{Spec}_r A \setminus \widetilde{\text{Reg}(A)}$. Then, $\text{supp}(\alpha) \in \text{Sing}(A)$ and thus $I \subset \text{supp}(\alpha)$, where $\text{Sing } A = \mathcal{V}(I)$. This is impossible if $\text{supp}(\alpha) = 0$.

It remains to show that

$$\widetilde{\text{Reg}(A)} \subset \overline{\text{Spec}_r \mathcal{K}(A)}.$$

For this, we use an analogous of (i) \implies (iii) from [6, Prop. 7.6.2].

Let α be an ordering in A whose support \mathfrak{p} is in $\text{Reg}(A)$. Since A has finite dimension and $A_{\mathfrak{p}}$ has same fraction field as A , we deduce the existence of $\beta \in \text{Spec}_r \mathcal{K}(A)$ which specializes to α , by using [2, Lem. 3.4] which says that, given any regular local ring A of dimension d , of residue field k and fraction field K , any ordering on k admits 2^d generalizations in $\text{Spec}_r A$ which are orderings in K . \square

With this framework in an abstract setting we recover the usual geometric properties.

The end of the section will consist in studying the restriction of the support mapping to central orderings. Let us start with the following:

Lemma 3.20. *Let $\beta \in \text{Spec}_c A$ with support the null ideal. Let \mathfrak{q} be a prime ideal of A which is P_{β} -convex. Then*

$$P_{\alpha} = \mathfrak{q} + P_{\beta}$$

is a central ordering of A with support \mathfrak{q} which is a specialization of P_{β} .

Proof. From Lemma 2.5 then P_{α} is a cone with support \mathfrak{q} and clearly $P_{\beta} \subset P_{\alpha}$. It follows that P_{α} is proper.

Since \mathfrak{q} is P_{β} -convex then we may easily show that $P_{\alpha} = \mathfrak{q} \cup P_{\beta}$. If $ab \in P_{\alpha}$ then it follows from the fact that \mathfrak{q} is prime and P_{β} is an ordering that $a \in P_{\alpha}$ or $-b \in P_{\alpha}$. The proof is done. \square

Is is not possible to differentiate the supports of precentral orderings from those of central ones:

Proposition 3.21. *Let $\text{supp} : \text{Spec}_r A \rightarrow \text{Spec} A$ be the support map. We have*

$$\text{supp}(\text{Spec}_c A) = \text{C-Spec} A.$$

Proof. From Proposition 3.14 one has $\text{supp}(\text{Spec}_c A) \subset \text{C-Spec} A$.

Assume $\mathfrak{p} \in \text{C-Spec} A$. By [6, Prop. 4.2.9] there exists an ordering $P' \in \text{Spec}_r \mathcal{K}(A)$ such that \mathfrak{p} is $(P' \cap A)$ -convex. To end the proof use Proposition 3.1 and Lemma 3.20 where β is associated to $(P' \cap A)$ and $\mathfrak{q} = \mathfrak{p}$. \square

This allows us to say that the existence of a central ordering, or in other words the fact that $\text{Spec}_c A \neq \emptyset$, is equivalent to the conditions given in Proposition 3.16.

4. CENTRAL VERSUS PRECENTRAL

This section is the heart of the paper. We aim to compare central and precentral orderings, two classes of orderings which share the same supports by Propositions 3.14 and 3.21. Since a precentral ordering of A contains a proper cone of $\mathcal{K}(A)$ and a central ordering contains an ordering of $\mathcal{K}(A)$, it follows that a central ordering is precentral as previously noted. However the converse implication does not hold, and the goal of this section is to study the difference.

We assume in the sequel that $\mathcal{K}(A)$ is a formally real field since otherwise we do not have any precentral nor central ordering.

First note that for closed points of varieties, both notions coincide.

Proposition 4.1. *Let V be an irreducible affine algebraic variety over R and let $x \in V(R)$. Then α_x is central if and only if α_x is precentral.*

Proof. Assume α_x is precentral. By Proposition 3.14 then $\mathfrak{m}_x = \text{supp}(\alpha_x) \in \text{C-Spec} R[V]$. Since $R[V]/\mathfrak{m}_x = R$ is real closed, α_x is the unique ordering of $\text{Spec}_r R[V]$ with support \mathfrak{m}_x then it follows from Proposition 3.21 that α_x is central. \square

One may readily generalize to an abstract setting:

Proposition 4.2. *Let $\alpha \in \text{Spec}_r A$ be such that the residue field $k(\text{supp}(\alpha))$ admits a unique ordering. Then, α is central if and only if α is precentral.*

Since $\text{Spec}_c A \subset \text{Spec}_{pc} A \subset \text{Spec}_r A$, we already know that A is precentral whenever A is central and the latter condition is satisfied for instance when A is the coordinate ring of an irreducible affine algebraic variety over R which is central (see Proposition 3.18).

Our aim is to give characterizations of central and precentral orderings. To start, let us note that, since $\text{Spec}_c A$ and $\text{Spec}_{pc} A$ are closed subsets, a point α belongs to $\text{Spec}_c A$ (resp. $\text{Spec}_{pc} A$) if and only if $U \cap \text{Spec}_c A \neq \emptyset$ (resp. $U \cap \text{Spec}_{pc} A \neq \emptyset$) for any open subset U containing α . One may replace in that statement U with basic open subsets like $\mathcal{S}(f_1, \dots, f_k)$ which are a basis of neighbourhoods.

Lemma 4.3. *Let $\alpha \in \text{Spec}_r A$ and f_1, \dots, f_k in A such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$. Let us consider the following properties:*

- (1) $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k] \subset P_\alpha$.
- (2) $\sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper in $\mathcal{K}(A)$.
- (3) $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper in A .
- (4) $\mathcal{S}(f_1, \dots, f_k) \cap \text{Spec}_c A \neq \emptyset$.
- (5) $\mathcal{S}(f_1, \dots, f_k) \cap \text{Spec}_{pc} A \neq \emptyset$.

One has (1) \implies (2) \iff (3) \iff (4) \implies (5).

Proof. The equivalence between (2) and (3) is clear. We prove (2) implies (4). Assume that $\mathcal{S}(f_1, \dots, f_k) \subset (\text{Spec}_r A \setminus \text{Spec}_c A)$. It follows that $\{\beta \in \text{Spec}_r \mathcal{K}(A) \mid f_1(\beta) > 0, \dots, f_k(\beta) > 0\} = \emptyset$.

Since the f_i are non zero, one may equivalently say that $\{\beta \in \text{Spec}_r \mathcal{K}(A) \mid f_1(\beta) \geq 0, \dots, f_k(\beta) \geq 0\} = \emptyset$. By the Positivstellensatz recalled in Theorem 2.11, one gets an identity $1 + p = 0$ in $\mathcal{K}(A)$ with $p \in \sum \mathcal{K}(A)^2[f_1, \dots, f_k]$. It follows that $-1 \in A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ and it proves that (2) implies (4) by contraposition.

We prove (4) implies (2). Let $\alpha \in \mathcal{S}(f_1, \dots, f_k) \cap \text{Spec}_c A$. There exists $\beta \in \text{Spec}_r A$ such that $\text{supp}(\beta) = (0)$ and $\beta \rightarrow \alpha$. We have $\beta \in \text{Spec}_r \mathcal{K}(A)$ by Proposition 3.1. For $i = 1, \dots, k$, we have $f_i \in P_\alpha \setminus \text{supp}(\alpha)$ and thus $f_i \in P_\beta \setminus \{0\}$. It follows that $\sum \mathcal{K}(A)^2[f_1, \dots, f_k] \subset P_\beta$ (viewed in $\mathcal{K}(A)$) and thus $\sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper.

We prove (1) implies (2). Assume that $\sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is not proper. It follows that $-1 \in A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ and since $-1 \notin P_\alpha$ we get that $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k] \not\subset P_\alpha$.

Since $\text{Spec}_c A \subset \text{Spec}_{pc} A$ then (4) implies (5). \square

The point (2) does not necessarily imply (1) as one can see looking for example at a point distinct from the origin in the stick of the Cartan umbrella (Example 3.6) and $f_1 = \dots = f_k = 1$. Likewise (3) implies (1) does not hold in general, nevertheless it becomes true after quantification on the family f_1, \dots, f_k and we derive the following characterizations for central orderings:

Proposition 4.4. *Let $\alpha \in \text{Spec}_r A$. The following properties are equivalent:*

- (1) $\alpha \in \text{Spec}_c A$.
- (2) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$, the cone $\sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper in $\mathcal{K}(A)$.
- (3) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$, the cone $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper in A .
- (4) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$, the intersection $\mathcal{S}(f_1, \dots, f_k) \cap \text{Spec}_c A$ is non-empty.
- (5) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$, the cone $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k] \subset P_\alpha$.

Proof. The equivalence between (2), (3) and (4) is given by Lemma 4.3.

Let us prove that (1) implies (5). Let $\alpha \in \text{Spec}_c A$. There exists $\beta \in \text{Spec}_r A$ such that $\text{supp}(\beta) = (0)$ and $\beta \rightarrow \alpha$. We have $\beta \in \text{Spec}_r \mathcal{K}(A)$ by Proposition 3.1. Let $f_1, \dots, f_k \in P_\alpha \setminus \text{supp}(\alpha)$ then $\forall i$, $f_i \in P_\beta \setminus \{0\}$ and it follows that $\sum \mathcal{K}(A)^2[f_1, \dots, f_k] \subset P_\beta$ (viewed in $\mathcal{K}(A)$) and thus

$$A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k] \subset P_\beta \subset P_\alpha.$$

By Lemma 4.3, we have that (5) implies (2).

As already said, we know that (4) implies (1) since $\text{Spec}_c A$ is closed for the topology of $\text{Spec}_r A$. \square

Here appears the notion of stability index we recalled in section 2.4. Namely, the stability index $s(U)$ of an open subset U of $\text{Spec}_r A$ is the infimum of the numbers $k \in \mathbb{N}$ such that for any basic open subset S of $\text{Spec}_r A$ satisfying $S \subset U$, there exist f_1, \dots, f_k in A with $S = \mathcal{S}(f_1, \dots, f_k)$.

It leads to new characterizations for central orderings:

Theorem 4.5. *Let $\alpha \in \text{Spec}_r A$. The following properties are equivalent:*

- (1) $\alpha \in \text{Spec}_c A$.
- (2) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$ and $k \leq s(\text{Spec}_r A \setminus \text{Spec}_c A)$, the cone $\sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper in $\mathcal{K}(A)$.
- (3) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$ and $k \leq s(\text{Spec}_r A \setminus \text{Spec}_c A)$, the cone $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ is proper in A .
- (4) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$ and $k \leq s(\text{Spec}_r A \setminus \text{Spec}_c A)$, the intersection $\mathcal{S}(f_1, \dots, f_k) \cap \text{Spec}_c A$ is non-empty.

- (5) For any $f_1, \dots, f_k \in A$ such that $\alpha \in \mathcal{S}(f_1, \dots, f_k)$ and $k \leq s(\text{Spec}_r A \setminus \text{Spec}_c A)$, the cone $A \cap \sum \mathcal{K}(A)^2[f_1, \dots, f_k] \subset P_\alpha$.

Proof. We know that (1) is equivalent to (4) of Proposition 4.4. By definition of the stability index, (4) of Proposition 4.4 is equivalent to (4): indeed, if $\alpha \in S$ with S a basic open subset of $\text{Spec}_r A$ which cannot be described by less than $s(\text{Spec}_r A \setminus \text{Spec}_c A) + 1$ inequalities, then we must have $S \cap \text{Spec}_c A \neq \emptyset$. By Proposition 4.4, (1) implies (5). By Lemma 4.3, (5) implies (2) and (2), (3), (4) are equivalent. \square

Note that if the stability index $s(\text{Spec}_r A \setminus \text{Spec}_c A)$ happens to be zero and more generally if $k = 0$ in Theorem 4.5, then $\sum \mathcal{K}(A)^2[f_1, \dots, f_k]$ reduces to $\sum \mathcal{K}(A)^2$ as the cone in $\mathcal{K}(A)$ generated by the empty family. In this case, assertion (5) of Theorem 4.5 is equivalent to say that P_α is precentral.

We also notice that being a precentral ordering is equivalent to satisfy condition (5) of Theorem 4.5 only for $k = 0$. Then, we give similar characterizations for precentral orderings, namely, one recover conditions (2), (3) and (4) of Theorem 4.5 for $k \leq 1$.

Theorem 4.6. *Let $\alpha \in \text{Spec}_r A$. The following properties are equivalent:*

- (1) $\alpha \in \text{Spec}_{pc} A$.
- (2) For any $f \in A$ such that $\alpha \in \mathcal{S}(f)$, the cone $\sum \mathcal{K}(A)^2[f]$ is proper.
- (3) For any $f \in A$ such that $\alpha \in \mathcal{S}(f)$, the cone $A \cap \sum \mathcal{K}(A)^2[f]$ is proper.
- (4) For any $f \in A$ such that $\alpha \in \mathcal{S}(f)$, the intersection $\mathcal{S}(f) \cap \text{Spec}_c A$ is non-empty.

Proof. The equivalence between (2), (3) and (4) are clear from Lemma 4.3.

Assume $\alpha \notin \text{Spec}_{pc} A$. There exists $g \in \mathcal{C}$ such that $g \notin P_\alpha$. Remark that $g \neq 0$. We have $-g \in (P_\alpha \setminus \text{supp}(\alpha))$ and thus $\alpha \in \mathcal{S}(-g)$. Suppose there exists $\beta \in (\text{Spec}_c A \cap \mathcal{S}(-g))$. By definition of a central ordering and by Proposition 3.1, there is $\gamma \in \text{Spec}_r A$ such that $\text{supp}(\gamma) = (0)$ and $\gamma \rightarrow \beta$. Since $(P_\beta \setminus \text{supp}(\beta)) \subset (P_\gamma \setminus \{0\})$ then $-g \in (P_\gamma \setminus \{0\})$, it is impossible because $g \in \mathcal{C} \subset P_\gamma$. We get $\mathcal{S}(-g) \subset (\text{Spec}_r A \setminus \text{Spec}_c A)$ and it proves (4) implies (1).

Assume there exists $g \in A$ such that $\alpha \in \mathcal{S}(g)$ and $\mathcal{S}(g) \subset (\text{Spec}_r A \setminus \text{Spec}_c A)$. We have $-g \notin P_\alpha$. Moreover $\forall \beta \in \text{Spec}_c A$, $-g \in P_\beta$ and thus $\forall \beta \in \text{Spec}_r A$ such that $\text{supp}(\beta) = (0)$, $-g \in P_\beta$. From Proposition 3.1 and the Positivstellensatz, cf. Theorem 2.11, we get $-g \in \sum \mathcal{K}(A)^2$. It shows that $\alpha \notin \text{Spec}_{pc} A$ and it proves (1) implies (4). \square

With this characterization and that of Theorem 4.5, one may view precentral orderings as central orderings of "level 1", and going further in that direction would lead to the consideration of central orderings of "level k ". We decide not to develop such a formalism until we find some relevant applications.

The value of the stability index of the non-central locus appears to be related to the existence of a precentral ordering that is not central. Namely, one has $\text{Spec}_{pc} A = \text{Spec}_c A$ whenever $s(\text{Spec}_r A \setminus \text{Spec}_c A) \leq 1$.

In the geometric case, using Bröcker-Scheiderer Theorem and Theorem 4.5, one gets a family of geometric rings where any precentral ordering is central :

Corollary 4.7. *Let V be an irreducible affine algebraic variety over R such that $\dim V \leq 2$. Then, $\text{Spec}_{pc} R[V] = \text{Spec}_c R[V]$.*

Proof. One has $s(\text{Spec}_r R[V] \setminus \text{Spec}_c R[V]) \leq s(\widetilde{V(R)} \setminus \widetilde{V_{reg}(R)})$. And from Bröcker-Scheiderer Theorem one gets that the stability index of $V(R) \setminus V_{reg}(R)$ is at most 1. \square

Let us now give an example of a precentral ordering which is not central.

Example 4.8. Let V be the irreducible affine algebraic variety over \mathbb{R} with coordinate ring $A = \mathbb{R}[V] = \mathbb{R}[x, y, z, t_1, t_2]/(z^2 + t_1x^2 + t_2y^2)$. The real part of the singular locus of V is contained in the real plane in t_1 and t_2 and we are going to describe $S = V(\mathbb{R}) \setminus (\text{Cent } V(\mathbb{R}))$ in this plane. By

[6, Prop. 7.6.2], S is the locus of points of $V(\mathbb{R})$ where the local semi-algebraic dimension is < 4 . Seeing $V(\mathbb{R})$ as a variety with parameters t_1 and t_2 then we can show that $S = S(t_1, t_2)$, the open right upper quadrant of the plane, and the local dimension at points of S in $V(\mathbb{R})$ is equal to two. Let us consider the four elements of the real spectrum $-1_{+\uparrow}, -1_{+\downarrow}, 1_{+\uparrow}, 1_{+\downarrow}$ which, as described in [6, Ex. 10.4.3], have support the ideal (x, y, z) (which define the plane in t_1, t_2), the first two specializing to the point $(-1, 0)$ and the second two specializing to the point $(1, 0)$. These four orderings define a fan F (see [6, Defn. 10.4.2]). It is easy to check that $F \cap \tilde{S} = \{1_{+\uparrow}\}$. Since $1_{+\uparrow} \in \tilde{S}$ then it cannot be central. This can also be seen algebraically by considering the property (5) of Theorem 4.5 since $t_i > 0$ on $1_{+\uparrow}$ for $i = 1, 2$ and since $-1 = t_1(x/z)^2 + t_2(y/z)^2 \in \sum \mathcal{K}(V)^2[t_1, t_2]$.

Assume now that there exists $f \in \mathbb{R}[V]$ such that $1_{+\uparrow} \in \mathcal{S}(f)$. If we assume $\mathcal{S}(f) \subset \tilde{S}$, then $\#(F \cap \mathcal{S}(f)) = 1$ and by [2, V Cor. 1.9] we get a contradiction. It follows that $\#(F \cap \mathcal{S}(f)) > 1$. Since $F \setminus \{1_{+\uparrow}\} \subset \text{Spec}_c A$ then using (4) of Theorem 4.6 we get that $1_{+\uparrow} \in \text{Spec}_{pc} A$.

To end, let us note that it is possible to give a similar example in dimension 3 by intersecting our variety with the hypersurface with equation $z - xy = 0$.

This example shows also that the precentral spectrum is not necessarily constructible. Indeed, if $\text{Spec}_{pc} \mathbb{R}[V]$ were a constructible subset then, by the correspondence between semialgebraic subsets S and constructible subsets \tilde{S} and using Proposition 4.1, one would get $\text{Spec}_{pc} \mathbb{R}[V] = \text{Spec}_{pc} \widetilde{\mathbb{R}[V]} \cap V(\mathbb{R}) = \text{Spec}_c \widetilde{\mathbb{R}[V]} \cap V(\mathbb{R}) = \text{Spec}_c \mathbb{R}[V]$ a contradiction.

Although the central spectrum and the precentral spectrum seem to be close, it is not true that for any $\alpha \in \text{Spec}_{pc} A$, there exists $\gamma \in \text{Spec}_c A$ such that $\alpha \rightarrow \gamma$ as one can see using again the previous example 4.8. Indeed, take $\alpha \in \text{Spec}_{pc} A$ be the ordering of all polynomial functions which are nonnegative on a $+\infty$ neighbourhood of the transcendent curve of equation $t_2 = e^{-t_1}$ in the plane (t_1, t_2) . This ordering does not admit any strict specialization and it is not central since the non central locus is $\tilde{S}(t_1, t_2)$. Moreover, arguing again with a 4-elements fan (which cannot intersect any given principal open subset at a single element), one shows that α is precentral.

From (4) of Theorem 4.6, a precentral ordering of $\text{Spec}_r A$ is an ordering that cannot be separated from the central locus $\text{Spec}_c A$ by principal open subsets. In the spirit of Example 4.8, it is possible to create precentral but non-central orderings with an higher level of non-separation with the central locus. Namely, let k be an integer ≥ 2 and let V be the irreducible affine algebraic variety over \mathbb{R} with coordinate ring $\mathbb{R}[V] = \mathbb{R}[x_1, \dots, x_k, z, t_1, \dots, t_k]/(z^2 + t_1 x_1^2 + \dots + t_k x_k^2)$. There exists $\alpha \in (\text{Spec}_{pc} A \setminus \text{Spec}_c A)$ such that for any basic open subset S of $\text{Spec}_r A$ given by $t \leq k - 1$ strict inequalities then $S \cap \text{Spec}_c A \neq \emptyset$. Moreover we have $\alpha \in \mathcal{S}(t_1, \dots, t_k) \subset (\text{Spec}_r A \setminus \text{Spec}_c A)$.

In the equivalent properties of Theorem 4.6, recall that we get rid of the condition (5) of Theorem 4.5:

$$(5) \quad \forall f \in A \text{ such that } \alpha \in \mathcal{S}(f), A \cap \sum \mathcal{K}(A)^2[f] \subset P_\alpha.$$

Moreover, there is another condition which arises naturally as the following one:

$$(6) \quad \forall f \in A \text{ such that } \alpha \in \mathcal{S}(f), \exists \beta \in \text{Spec}_c A \text{ with } \text{supp}(\alpha) = \text{supp}(\beta) \text{ such that } \beta \in \mathcal{S}(f).$$

Indeed, by Theorem 4.6 we know that

$$\alpha \in \text{Spec}_{pc} A \iff \forall f \in A \text{ such that } \alpha \in \mathcal{S}(f), \exists \beta \in \text{Spec}_c A \text{ such that } \beta \in \mathcal{S}(f),$$

and by Propositions 3.14 and 3.21 we also know that

$$\alpha \in \text{Spec}_{pc} A \implies \exists \beta \in \text{Spec}_c A \text{ such that } \text{supp}(\alpha) = \text{supp}(\beta).$$

So, a natural question is study how $\alpha \in \text{Spec}_{pc} A$ is related to (6).

Proposition 4.9. *Let $\alpha \in \text{Spec}_r A$. Then condition (6) is equivalent to the following*

$$\forall f \in A \text{ such that } \alpha \in \mathcal{S}(f), \text{supp}(\alpha) \text{ is } A \cap \sum \mathcal{K}(A)^2[f]\text{-convex.}$$

Moreover, one has the following implications

$$\alpha \in \text{Spec}_c A \implies (5) \implies (6) \implies \alpha \in \text{Spec}_{pc} A.$$

Proof. Let us show that condition (6) implies the one of the proposition. Let $f \in A$ and $\mathfrak{p} \in \text{Spec} A$. Assume there exists $\beta \in \text{Spec}_c A$ with $\mathfrak{p} = \text{supp}(\beta)$ such that $\beta \in \mathcal{S}(f)$. By Proposition 3.21 then $\mathfrak{p} \in \text{C-Spec} A$. There exists γ of support (0) such that $\gamma \rightarrow \beta$. We have $f(\gamma) > 0$ and thus $\sum \mathcal{K}(A)^2[f] \subset P_\gamma$ in $\mathcal{K}(A)$. Since \mathfrak{p} is P_β -convex then \mathfrak{p} is convex for P_γ and hence also for $A \cap \sum \mathcal{K}(A)^2[f]$.

Conversely, take $\alpha \in \text{Spec}_r A$ such that $\text{supp}(\alpha) = \mathfrak{p}$, $\alpha \in \mathcal{S}(f)$ and \mathfrak{p} is $A \cap \sum \mathcal{K}(A)^2[f]$ -convex. By [6, Prop. 4.2.9] there exists an ordering $\gamma \in \text{Spec}_r \mathcal{K}(A)$ such that \mathfrak{p} is $P_\gamma \cap A$ -convex and $A \cap \sum \mathcal{K}(A)^2[f] \subset P_\gamma$. By Propositions 3.1 and 3.20 there exists $\beta \in \text{Spec}_r A$ such that $P_\gamma \cap A \rightarrow P_\beta$ and $\text{supp}(\beta) = \mathfrak{p}$. Clearly $\beta \in \text{Spec}_c A$ and since $A \cap \sum \mathcal{K}(A)^2[f] \subset P_\beta$ then $f(\beta) \geq 0$. Since $\alpha \in \mathcal{S}(f)$ and $\text{supp}(\beta) = \text{supp}(\alpha) = \mathfrak{p}$ then $\beta \in \mathcal{S}(f)$. We have shown the first assertion.

Let us now prove the implications. The first one comes directly from characterizations of Proposition 4.4. The second one relies on the fact that $\text{supp}(\alpha)$ is P_α -convex and if $A \cap \sum \mathcal{K}(A)^2[f] \subset P_\alpha$, then $\text{supp}(\alpha)$ is $A \cap \sum \mathcal{K}(A)^2[f]$ -convex. And the last implication is also immediate using [6, Prop. 4.2.7] and Theorem 4.6. \square

Example 4.8 allows us to study some converse implications. Namely, the precentral ordering $\alpha = 1_{+\uparrow}$ considered there satisfies condition (6) (by Proposition 4.9 and since $-1_{+\uparrow}$, $-1_{+\downarrow}$ and $1_{+\downarrow}$ are central orderings with the same support as α) but not condition (5). Indeed, one has identity $-1 = t_1(x/z)^2 + t_2(y/z)^2 \in \sum \mathcal{K}(V)^2[t_1, t_2]$. Multiplying by t_1 , one gets $-t_1 \in \mathbb{R}[V] \cap \sum \mathcal{K}(V)^2[t_1, t_2]$ but $\alpha \in \mathcal{S}(t_1 t_2)$ and $-t_1 \notin P_\alpha$.

It happens also that the converse of the first implication doesn't hold. Indeed, consider the slightly modified example: $A = \mathbb{R}[V] = \mathbb{R}[x_1, x_2, x_3, z, t_1, t_2, t_3]/(z^2 + t_1 x_1^2 + t_2 x_2^2 + t_3 x_3^2)$. Let us take $\beta \in \text{Spec} A \setminus \text{Spec}_c A = \mathcal{S}(t_1, t_2, t_3)$ such that $\beta \in \text{Spec}_{pc} A$ and also such that any basic open defined by two or less inequalities which contains β has to intersect $\text{Spec}_c A$. Let us argue by contradiction and assume that β does not satisfy property (5). Then, there is $f \in A$ such that $\beta \in \mathcal{S}(f)$ and $A \cap \sum \mathcal{K}(A)^2[f] \not\subset P_\beta$. One has an identity $a = s + ft \notin P_\beta$ with $s, t \in \sum \mathcal{K}(A)^2$, namely $\beta \in \mathcal{S}(f, -a)$ and hence $\mathcal{S}(f, -a)$ intersects $\text{Spec}_c A$. Let $\beta' \in \mathcal{S}(f, -a) \cap \text{Spec}_c A$, there exists γ of support (0) such that $\gamma \rightarrow \beta'$. Relatively to the ordering γ , one gets $s + ft > 0$ whereas $a < 0$, a contradiction. Hence β satisfies (5) and gives a counterexample to the first implication. And concerning the converse of the last implication, we did not succeed to prove or disprove it.

To end this section whose aim was to compare central and precentral orderings, we recall that a central domain is obviously precentral and we prove that the converse is also true in the geometric case.

Proposition 4.10. *Let V be an irreducible affine algebraic variety over R . The following properties are equivalent:*

- (1) V is central.
- (2) $R[V]$ is central.
- (3) $R[V]$ is precentral.

Proof. By Proposition 3.18 we are left to prove (3) implies (1). Assume $R[V]$ is precentral. We have $\text{Spec}_r R[V] = \text{Spec}_{pc} R[V] = \widetilde{V(R)}$ and thus $\text{Spec}_{pc} R[V] \cap V(R) = V(R)$. Using Proposition 4.1 then $\text{Spec}_{pc} R[V] \cap V(R) = \text{Spec}_c R[V] \cap V(R)$ and the proof is done. \square

5. APPLICATIONS: CENTRAL AND PRECENTRAL POSITIVSTELLENSÄTZE AND \mathcal{C} -SETS

Unless otherwise stated, A is a domain whose fraction field $\mathcal{K}(A)$ formally real.

Artin answer to Hilbert 17th problem says that a nonnegative polynomial in $A = R[x_1, \dots, x_n]$ is a sum of squares in $\mathcal{K}(A) = R(x_1, \dots, x_n)$: the trace on A of the intersection of all cones of $\mathcal{K}(A)$

coincides with the intersection of all orderings in A :

$$A \cap \bigcap_{P \in \text{Cone}(\mathcal{K}(A))} P = A \cap \bigcap_{P \in \text{Spec}_r \mathcal{K}(A)} P = \bigcap_{P \in \text{Spec}_r A} P.$$

Recall that we are interested in getting Hilbert 17th property (with an equivalence rather than simply an implication) for a general domain A , namely finding what kind of positivity is equivalent to the algebraic certificate of an f in the cone $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$.

In this direction, the classical Hilbert 17th property can be reformulated with the language of central cones and orderings of A . Since $A = R[x_1, \dots, x_n]$ is a central and precentral ring, the following sequence of inclusions

$$\mathcal{C} = \bigcap_{P \in \text{Cone}_c(A)} P \subset \bigcap_{P \in \text{Spec}_{pc} A} P \subset \bigcap_{P \in \text{Spec}_c A} P,$$

are in fact equalities. We may wonder if these equalities still stand when A is a general domain.

In the sequel we give some central or precentral certificates of vanishing (Nullstellensätze) and of positivity (Positivstellensätze) on several subsets of $\text{Spec}_r A$. It would allow to characterize the \mathcal{C} -subsets of $\text{Spec}_r A$ containing $\text{Spec}_r \mathcal{K}(A)$. Let us recall from Definition 3.11 that \mathcal{C} -subsets E are those satisfying

$$\mathcal{C} = \bigcap_{P \in E} P.$$

Even if A is a field, \mathcal{C} -subsets seem far too numerous and difficult to characterize. However, in the case of geometric function fields there is a unique \mathcal{C} -subset similarly to the result of Theorem 3.12.

Proposition 5.1. *Let V be an irreducible affine algebraic variety over R . Then $\text{Spec}_r \mathcal{K}(V)$ is its unique own \mathcal{C} -subset.*

Proof. We have already noted that $\text{Spec}_r \mathcal{K}(V)$ is a \mathcal{C} -set.

The set of nonsingular points of $V(R)$ is the set of real closed points of an affine algebraic variety birational to V , so we may assume $V(R)$ nonsingular. Let E be a proper \mathcal{C} -subset of $\text{Spec}_r \mathcal{K}(V)$. Let $\alpha \in \text{Spec}_r \mathcal{K}(V) \setminus E$. Since E is closed, we can find $f_i \in R[V]$, $i = 1, \dots, r$, such that $\alpha \in \{\beta \in \text{Spec}_r \mathcal{K}(V) \mid f_1(\beta) > 0, \dots, f_r(\beta) > 0\} \subset \text{Spec}_r \mathcal{K}(V) \setminus E$. Working now in $\text{Spec}_r R[V]$, it follows from Proposition 3.1 that $\mathcal{S}(f_1, \dots, f_r) \neq \emptyset$. By Artin-Lang theorem [6, Thm. 4.1.2], then $S(f_1, \dots, f_r) = \mathcal{S}(f_1, \dots, f_r) \cap V(R) \neq \emptyset$. Let $x \in S(f_1, \dots, f_r)$. Similarly to the end of the proof of Proposition 3.12, considering a small open ball centred at x then there exists $f \in R[V]$ such that $x \in S(f)$ and $S(f) \subset S(f_1, \dots, f_r)$. It follows that $f(E) < 0$ and thus $-f \in \sum \mathcal{K}(V)^2$. By [6, Prop. 7.6.2] and since $x \in \text{Cent } V(R)$ then there exists $\beta \in \text{Spec}_r \mathcal{K}(V)$ such that $\beta \rightarrow \alpha_x$. Since $f(x) > 0$ then $f(\beta) > 0$ and it gives a contradiction. \square

This result does not remain true in the non geometric setting. Indeed, thanks to an example communicated by Claus Scheiderer, one may consider the example of the field of Laurent powers series $K = \mathbb{R}((x))((y))$ whose real spectrum is a four elements set (which is a fan). It appears that any f in K which is nonnegative for three elements of $\text{Spec}_r K$ is nonnegative for the fourth. In other words, any subset of cardinality equal to 3 of $\text{Spec}_r K$ is a \mathcal{C} -set.

This shows that the \mathcal{C} -subsets of $\text{Spec}_r A$ containing $\text{Spec}_r \mathcal{K}(A)$ are natural objects to focus on.

5.1. Central and precentral Nullstellensätze. We start by studying certificates of vanishing. Let I be an ideal of A . Denote by $\mathcal{Z}^r(I)$ the set of all $\alpha \in \text{Spec}_r A$ such that $I \subset \text{supp}(\alpha)$. Then we write $\mathcal{Z}^c(I) = \mathcal{Z}^r(I) \cap \text{Spec}_c A$ and $\mathcal{Z}^{pc}(I) = \mathcal{Z}^r(I) \cap \text{Spec}_{pc} A$. Let $W \subset \text{Spec}_r A$, we denote by $\mathcal{I}(W)$ the set of $f \in A$ such that $W \subset \mathcal{Z}^r(f)$; it is clearly an ideal of A .

Note that the inclusion $\mathcal{Z}^c(I) \subset \mathcal{Z}^{pc}(I)$ can be strict, take $I = (x, y, z)$ in Example 4.8. One may nevertheless go further to get an abstract central and precentral Nullstellensätze:

Proposition 5.2. (*Central and precentral Nullstellensätze*) *Let I be an ideal of A . One has*

$$\mathcal{I}(\mathcal{Z}^c(I)) = \mathcal{I}(\mathcal{Z}^{pc}(I)) = \sqrt[\mathcal{C}]{I}.$$

Proof. Since $\text{Spec}_c A \subset \text{Spec}_{pc} A$ then $\mathcal{Z}^c(I) \subset \mathcal{Z}^{pc}(I)$ and thus $\mathcal{I}(\mathcal{Z}^c(I)) \supset \mathcal{I}(\mathcal{Z}^{pc}(I))$.

Clearly $\sqrt[\mathcal{C}]{I} \subset \mathcal{I}(\mathcal{Z}^{pc}(I))$. Indeed, let $a \in \sqrt[\mathcal{C}]{I}$. We have $a^{2m} + b \in I$ with $b \in \mathcal{C}$. Let $\alpha \in \mathcal{Z}^{pc}(I)$: one has $\text{supp}(\alpha) \supset I$, and hence $a^{2m} + b \in \text{supp}(\alpha)$. By Proposition 3.14 we have $\text{supp}(\alpha) \in \text{C-Spec } A$. A central ideal is \mathcal{C} -convex and we get $a^{2m} \in \text{supp}(\alpha)$. By radicality of $\text{supp}(\alpha)$ then $a \in \text{supp}(\alpha)$.

Let $f \in \mathcal{I}(\mathcal{Z}^c(I))$. Let us show that, if \mathfrak{p} is a central prime ideal containing I , then $f \in \mathfrak{p}$. By [17, Prop. 3.14] we will then get $f \in \sqrt[\mathcal{C}]{I}$. Using Proposition 3.21 there exists $\alpha \in \text{Spec}_c A$ such that $\text{supp}(\alpha) = \mathfrak{p}$. Clearly $\alpha \in \mathcal{Z}^c(I)$ and thus $f(\alpha) = 0$ i.e $f \in \text{supp}(\alpha) = \mathfrak{p}$. We get $\mathcal{I}(\mathcal{Z}^c(I)) \subset \sqrt[\mathcal{C}]{I}$ and it ends the proof. \square

5.2. Precentral Positivstellensätze. One may carry on further to get central and precentral Positivstellensätze having in mind that the algebraic nature of precentrality seems much more convenient than the geometric nature of centrality.

To get geometric central Positivstellensätze, our strategy is first to establish abstract precentral Positivstellensätze, then derive some abstract central ones and finally get geometric central ones.

A set of the form $\overline{\mathcal{S}}(f_1, \dots, f_k) = \{\alpha \in \text{Spec}_r A \mid f_1(\alpha) \geq 0, \dots, f_k(\alpha) \geq 0\}$ for some elements f_1, \dots, f_k of A , is called a basic closed subset of $\text{Spec}_r A$. We denote by $\mathcal{S}^c(f_1, \dots, f_k)$, $\overline{\mathcal{S}}^c(f_1, \dots, f_k)$, $\mathcal{S}^{pc}(f_1, \dots, f_k)$ and $\overline{\mathcal{S}}^{pc}(f_1, \dots, f_k)$, the sets $\mathcal{S}(f_1, \dots, f_k)$, $\overline{\mathcal{S}}(f_1, \dots, f_k)$ intersected respectively with $\text{Spec}_c A$ and $\text{Spec}_{pc} A$.

With these notations, one gets:

Theorem 5.3. (*Precentral Positivstellensätze*) *Let f_1, \dots, f_r in A and $f \in A$. One has:*

- (1) $f \geq 0$ on $\overline{\mathcal{S}}^{pc}(f_1, \dots, f_r)$ if and only if

$$fq = p + f^{2m}$$

where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.

- (2) $f > 0$ on $\overline{\mathcal{S}}^{pc}(f_1, \dots, f_r)$ if and only if

$$fq = 1 + p$$

where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.

- (3) $f = 0$ on $\overline{\mathcal{S}}^{pc}(f_1, \dots, f_r)$ if and only if

$$f^{2m} + p = 0$$

where p is in $\mathcal{C}[f_1, \dots, f_r]$.

Proof. (1) Let H be the set $\mathcal{C} \cup \{f_1, \dots, f_r, -f\}$ and M the monoid generated by f . Then, there is no $\alpha \in \text{Spec}_{pc} A$ such that $-f(\alpha) \geq 0$, $f_1(\alpha) \geq 0, \dots, f_r(\alpha) \geq 0$ and $f(\alpha) \neq 0$. if and only if there is no $\alpha \in \text{Spec}_r A$ such that $H \subset \alpha$ and $-f(\alpha) \neq 0$.

Then, from the formal Positivstellensatz recalled in Theorem 2.11, it is equivalent to have an identity of the form $p - fq + f^{2m} = 0$ where $p, q \in \mathcal{C}[f_1, \dots, f_r]$.

- (2) Let H be the set $\mathcal{C} \cup \{f_1, \dots, f_r, -f\}$ and M the monoid generated by 1. There is no $\alpha \in \text{Spec}_{pc} A$ such that $f_1(\alpha) \geq 0, \dots, f_r(\alpha) \geq 0$, $-f(\alpha) \geq 0$ and $1(\alpha) \neq 0$ if and only if there is no $\alpha \in \text{Spec}_r A$ such that $H \subset \alpha$ and $1(\alpha) \neq 0$ and conclude using the formal Positivstellensatz, cf. Theorem 2.11.

- (3) Let H be the set $\mathcal{C} \cup \{f_1, \dots, f_r\}$ and M the monoid generated by f . Since there is no $\alpha \in \text{Spec}_{pc} A$ such that $f_1(\alpha) \geq 0, \dots, f_r(\alpha) \geq 0$ and $f(\alpha) \neq 0$ if and only if there is no $\alpha \in \text{Spec}_r A$ such that $H \subset \alpha$ and $f(\alpha) \neq 0$, we get the proof using again the formal Positivstellensatz, cf. Theorem 2.11. \square

As a particular case, one gets that $\text{Spec}_{pc} A$ is an \mathcal{C} -subset of $\text{Spec}_r A$:

Proposition 5.4. *(Precentral Hilbert 17th property) Let $f \in A$. The following properties are equivalent:*

- (1) $f \geq 0$ on $\text{Spec}_{pc} A$.
- (2) There exist p, q in \mathcal{C} such that $fq = p + f^{2m}$.
- (3) There exist $p, q \in \mathcal{C}$ such that $q^2 f = p$ and $\mathcal{Z}^{pc}(q) \subset \mathcal{Z}^{pc}(f)$.
- (4) $f \in \mathcal{C}$.

Proof. Let us show (1) \Rightarrow (2). Assume $f \geq 0$ on $\text{Spec}_{pc} A$. Since $f \geq 0$ on $\overline{\mathcal{S}}^{pc}(1)$ then by Theorem 5.3 one gets (2).

Let us show (2) implies (3). Assume $fq = p + f^{2m}$ with $p, q \in \mathcal{C}$. One may assume that $q \neq 0$ or equivalently that $p + f^{2m} \neq 0$ since otherwise $f = 0$ (by hypothesis $\mathcal{K}(A)$ is formally real and hence the null ideal is real in A). One gets $f(p + f^{2m}) = f^2 q$ which gives $f = \frac{(f^2 q)(p + f^{2m})}{(p + f^{2m})^2} \in \mathcal{C}$. Set $s = (f^2 q)(p + f^{2m})$ and $t = p + f^{2m}$. We have $t^2 f = s$ and $s, t \in \mathcal{C}$. We have $\mathcal{Z}^{pc}(t) \subset \mathcal{Z}^{pc}(f)$: if $\alpha \in \mathcal{Z}^{pc}(p + f^{2m})$ then $p + f^{2m} \in \text{supp}(\alpha)$ which is a central ideal and by definition we get $f \in \text{supp}(\alpha)$. This shows the desired implication.

Trivially (3) implies (4).

To end, let us show that (4) implies (1). Assume $f \in \mathcal{C}$ then it is clear from the definition of a central cone that $f \geq 0$ on $\text{Spec}_{pc} A$. \square

Hence, in any domain A with formally real fraction field, the intersection of all central cones coincides with the intersection of all precentral orderings.

5.3. Central Positivstellensätze. From the precentral Positivstellensätze, one may deduce some central ones. Let us remind first that Proposition 5.2 gives that $f = 0$ on $\mathcal{Z}^{pc}(I)$ if and only if $f = 0$ on $\mathcal{Z}^c(I)$. One also have :

- Lemma 5.5.** (1) $f > 0$ on $\text{Spec}_{pc} A$ if and only if $f > 0$ on $\text{Spec}_c A$.
(2) $f \geq 0$ on $\text{Spec}_{pc} A$ if and only if $f \geq 0$ on $\text{Spec}_c A$.

Proof. The direct implications are clear since $\text{Spec}_c A \subset \text{Spec}_{pc} A$.

Let $f > 0$ on $\text{Spec}_c A$. Then, $f > 0$ on $\text{Spec}_r \mathcal{K}(A)$ and hence $f \in \sum \mathcal{K}(A)^2$. Hence, $f \geq 0$ on $\text{Spec}_{pc} A$. Assume there exists $\alpha \in \text{Spec}_{pc} A$ such that $f(\alpha) = 0$, we have $f \in \text{supp}(\alpha)$. By Proposition 3.14 then $\text{supp}(\alpha)$ is a central ideal. By Proposition 3.21, there is $\beta \in \text{Spec}_c A$ such that $\text{supp}(\alpha) = \text{supp}(\beta)$ and thus $f(\beta) = 0$, a contradiction.

Let $f \geq 0$ on $\text{Spec}_c A$. Assume that $f(\beta) < 0$ with β a precentral ordering. By characterization of Theorem 4.6, one gets the existence of γ central such that $f(\gamma) < 0$, contradiction. \square

On the other hand, beware that in general $\mathcal{S}^{pc}(f) \neq \mathcal{S}^c(f)$ as one can see using Example 4.8 with $f = t_1$. Likewise, beware that in general $\overline{\mathcal{S}}^{pc}(f) \neq \overline{\mathcal{S}}^c(f)$. Indeed, if we work now in the domain $B = A[t]/(tt_1 - 1)$ where A is the domain of Example 4.8, one has also an element $g = -t_2$ such that $g \geq 0$ on $\overline{\mathcal{S}}^c(f)$ but not on $\overline{\mathcal{S}}^{pc}(f)$. These observations shows that it is possible to derive a Hilbert 17th property (with an equivalence) from the precentral one, although it is not possible to derive central Positivstellensätze from Theorem 5.3.

Proposition 5.6. *(Central Hilbert 17th Property) Let $f \in A$. The following properties are equivalent:*

- (1) $f \geq 0$ on $\text{Spec}_c A$.
- (2) There exist p, q in \mathcal{C} such that $fq = p + f^{2m}$.
- (3) There exist $p, q \in \mathcal{C}$ such that $q^2 f = p$ and $\mathcal{Z}^c(q) \subset \mathcal{Z}^c(f)$.
- (4) $f \in \mathcal{C}$.

Proof. Using Lemma 5.5 and Proposition 5.4, we see that (1), (2) and (4) are equivalent. Clearly (3) implies (4).

Let us show that (2) implies (3). Assume $fq = p + f^{2m}$ with $p, q \in \mathcal{C}$. Repeating the arguments used in the proof of (2) implies (3) of Proposition 5.4 we get an identity $t^2f = s$ with $s, t \in \mathcal{C}$ and $\mathcal{Z}^{pc}(t) \subset \mathcal{Z}^{pc}(f)$ and it is easy to see that we also have $\mathcal{Z}^c(t) \subset \mathcal{Z}^c(f)$. \square

One may also give an interpretation of the previous result with cones, namely : in any domain with formally real fraction field the intersection of all central cones coincides with the intersection of all central orderings.

Note furthermore that this implies that $\text{Spec}_c A$ is an \mathcal{C} -subset of $\text{Spec}_r A$. We complete now this observation by showing that $\text{Spec}_c A$ and $\text{Spec}_{pc} A$ are respectively the smallest and the largest \mathcal{C} -subset of $\text{Spec}_r A$ that contains $\text{Spec}_r \mathcal{K}(A)$. This gives Theorem C of the introduction:

Theorem 5.7. *Let E be a closed subset of $\text{Spec}_r A$ containing $\text{Spec}_r \mathcal{K}(A)$. The following properties are equivalent :*

- (1) E is a \mathcal{C} -subset of $\text{Spec}_r A$.
- (2) $\text{Spec}_c A \subset E \subset \text{Spec}_{pc} A$.

Proof. From Propositions 5.4 and 5.6 then it follows that (2) implies (1).

Assume (1) is satisfied. Since $\text{Spec}_r \mathcal{K}(A) \subset E$ and E is closed then $\text{Spec}_c A \subset E$. By hypothesis, we have $E \subset \overline{\mathcal{S}}(f)$ for any $f \in \mathcal{C}$. It follows that $E \subset \bigcap_{f \in \mathcal{C}} \overline{\mathcal{S}}(f) = \text{Spec}_{pc}(A)$. \square

From the proof of the previous theorem, we deduce:

Corollary 5.8. *The set of \mathcal{C} -subsets of $\text{Spec}_r A$ has a biggest element given by $\text{Spec}_{pc} A$.*

In the geometric case, thanks to Proposition 5.1 we may drop the assumption that $\text{Spec}_r \mathcal{K}(A) \subset E$ in the statement of Theorem 5.7.

Corollary 5.9. *Let V be an irreducible affine algebraic variety over R . Let E be a closed subset of $\text{Spec}_r R[V]$. The following properties are equivalent :*

- (1) E is a \mathcal{C} -subset of $\text{Spec}_r R[V]$.
- (2) $\text{Spec}_c R[V] \subset E \subset \text{Spec}_{pc} R[V]$.

Proof. Assume E is a \mathcal{C} -subset of $\text{Spec}_r R[V]$ that do not contain $\text{Spec}_r \mathcal{K}(V)$. Then, $E \cap \text{Spec}_r \mathcal{K}(V)$ is a proper \mathcal{C} -subset of $\text{Spec}_r \mathcal{K}(V)$ and it contradicts Proposition 5.1. We conclude using Theorem 5.7. \square

Although we do not obtain central Positivstellensätze in the general case, under a condition on the stability index, the central and precentral spectra coincide and one gets from Theorem 5.3:

Proposition 5.10. *(Central Positivstellensätze in low dimension) Assume that $s(\text{Spec}_r A \setminus \text{Spec}_c A) \leq$*

1. *Let f_1, \dots, f_r in A and $f \in A$. One has:*

- (1) $f \geq 0$ on $\overline{\mathcal{S}}^c(f_1, \dots, f_r)$ if and only if

$$fq = p + f^{2m}$$

where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.

- (2) $f > 0$ on $\overline{\mathcal{S}}^c(f_1, \dots, f_r)$ if and only if

$$fq = 1 + p$$

where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.

- (3) $f = 0$ on $\overline{\mathcal{S}}^c(f_1, \dots, f_r)$ if and only if

$$f^{2m} + p = 0$$

where p is in $\mathcal{C}[f_1, \dots, f_r]$.

Remark

- (i) Assertion (1) is false in the case $s(\text{Spec}_r A \setminus \text{Spec}_c A) = 2$. Consider the domain $B = A[t]/(tt_2 - 1)$ where A is the ring in Example 4.8. As previously noticed, working in $\text{Spec}_r B$, $-t_2 \geq 0$ on $\overline{\mathcal{S}}^c(t_1)$ but not on $\overline{\mathcal{S}}^{pc}(t_1)$. By Theorem 5.3, we cannot have an identity of the form $-t_2q = p + (t_2)^{2m}$ where p, q are in $\mathcal{C}[t_1]$.
- (ii) Likewise, using Example 4.8, assertion (3) is false in the case $s(\text{Spec}_r A \setminus \text{Spec}_c A) = 2$. Indeed, $t_1t_2 = 0$ on $\overline{\mathcal{S}}^c(t_1, t_2)$ but not on $\overline{\mathcal{S}}^{pc}(t_1, t_2)$.
- (iii) One may give a counter example to (2) in the case $s(\text{Spec}_r A \setminus \text{Spec}_c A) = 2$. Indeed, let us consider the domain $B = A[t, s]/(tt_1 - 1, st_2 - 1)$ where A is the ring in Example 4.8. It can be shown that $-t_2 > 0$ on $\overline{\mathcal{S}}^c(t_1)$ but not on $\overline{\mathcal{S}}^{pc}(t_1)$.
- (iv) In the full case i.e if $\overline{\mathcal{S}}^c(f_1, \dots, f_r) = \text{Spec}_c A$ then all the assertions of the proposition are valid without assumption on the stability index by Propositions 5.2 and 5.6.

Besides, without assumption on the stability index, assertion (3) is always valid whenever we consider a basic closed subset with a single inequality.

Proposition 5.11. *Let f, g in A . Then,*

$$f = 0 \text{ on } \overline{\mathcal{S}}^c(g) \text{ if and only if } f^{2m} + p = 0 \text{ where } p \text{ is in } \mathcal{C}[g].$$

Proof. One implication is clear because $\text{Spec}_c A \subset \text{Spec}_{pc} A$.

Assume that $f = 0$ on $\overline{\mathcal{S}}^c(g)$.

Let $\alpha \in \overline{\mathcal{S}}^{pc}(g)$. If $g(\alpha) = 0$, then g belongs to the support of α which is known to be also the support of a central ordering β . By assumption $f(\beta) = 0$ and hence $f(\alpha) = 0$.

If $g(\alpha) > 0$, then there is a central ordering β' such that $g(\beta') > 0$ by (4) of Theorem 4.6. Hence, there is an ordering γ of support (0) such that $\gamma \rightarrow \beta'$ and thus $g(\gamma) > 0$. By assumption $f(\gamma) = 0$ and hence $f = 0$, in particular $f(\alpha) = 0$.

It follows that $f = 0$ on $\overline{\mathcal{S}}^{pc}(g)$ and we get the proof by (3) of Theorem 5.3. \square

5.4. Geometric central Positivstellensätze. Let us now write down the cases of geometric rings.

These are obtained from the central abstract results of the previous subsection together with the so-called Artin-Lang property (cf [6, Thm. 4.1.2]).

We first give a slightly more detailed version of [6, Thm. 6.1.9]. Recall that Theorem 3.12 says that $\text{Cent } V(R)$ is the unique \mathcal{C} -subset of $V(R)$.

Proposition 5.12. *(Geometric Hilbert 17th Property) Let V be an irreducible affine algebraic variety over R with $V_{\text{reg}}(R) \neq \emptyset$. Let $f \in R[V]$ and $\mathcal{C} = R[V] \cap \sum \mathcal{K}(V)^2$. The following properties are equivalent:*

- (1) $f \geq 0$ on $\text{Cent } V(R)$.
- (2) There exist p, q in \mathcal{C} such that $fq = p + f^{2m}$.
- (3) There exist $p, q \in \mathcal{C}$ such that $q^2f = p$ and $\mathcal{Z}(q) \cap \text{Cent } V(R) \subset \mathcal{Z}(f) \cap \text{Cent } V(R)$.
- (4) $f \in \mathcal{C}$.

Proof. By Artin-Lang property (cf [6, Thm. 4.1.2]), if $f \geq 0$ on $\text{Cent } V(R)$, then $\widehat{f} \geq 0$ on $\text{Spec}_c R[V]$ since $\text{Cent } V(R) = \text{Spec}_c R[V]$. Remark also that for $g \in R[V]$ we have $\mathcal{Z}(g) \cap \text{Cent } V(R) = \mathcal{Z}^c(g)$. One use Proposition 5.6 to conclude. \square

One also has:

Proposition 5.13. *(Geometric central Positivstellensätze for surfaces) Let V be an irreducible affine algebraic variety over R with $V_{\text{reg}}(R) \neq \emptyset$ and such that $\dim(V(R)) \leq 2$. Let f, f_1, \dots, f_r in $R[V]$ and $\mathcal{C} = R[V] \cap \sum \mathcal{K}(V)^2$. One has:*

(1) $f \geq 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent } V(R)$ if and only if

$$fq = p + f^{2m}$$

where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.

(2) $f > 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent } V(R)$ if and only if

$$fq = 1 + p$$

where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.

(3) $f = 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent } V(R)$ if and only if

$$f^{2m} + p = 0$$

where p is in $\mathcal{C}[f_1, \dots, f_r]$.

Proof. Since $\dim(V(R)) \leq 2$, one has $s(\text{Spec}_r R[V] \setminus \text{Spec}_c R[V]) \leq 1$.

Let us show just (1) since one proceeds likewise for the other properties. We show the non obvious implication. Let us assume that $f \geq 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent } V(R)$. By Artin-Lang property [6, Thm. 4.1.2], one deduces that $f \geq 0$ on $\overline{\mathcal{S}^c}(f_1, \dots, f_r)$. One concludes then by application of (1) Proposition 5.10. \square

5.4.1. *In the Nash setting.* Recall that a Nash function on R^n is a semialgebraic function of class C^∞ (typically $\sqrt{1+x^2}$ is a Nash function on R). Let us denote by $\mathcal{N}(R^n)$ the ring of all Nash functions on R^n . Let us consider an irreducible Nash set $V = \mathcal{Z}(I)$ given by a prime ideal $I \subset \mathcal{N}(R^n)$. Let us denote by A or $\mathcal{N}(V)$ the quotient ring $\mathcal{N}(R^n)/I$ which can be seen as the ring of Nash functions over V . This ring is an excellent ring as one can see using the same argument as in the proof of [2, VIII Prop. 8.4], namely the criterion stated in [2, VII Prop. 2.4].

As for the polynomials, one may define the central locus $\text{Cent}(V)$ of V as the Euclidean closure of the set $\text{Reg}(V)$ of Nash regular points of V (and it coincides with the algebraic central locus when V is algebraic). Namely, a point $x \in V$ associated to the maximal ideal m_x is said to be regular if the local ring A_{m_x} is regular.

Note that since a Nash function is semialgebraic, it gives sense to $\widetilde{\text{Cent}(V)} \subset \widetilde{R^n}$. Moreover, recall from [6, Prop. 8.8.1] that the canonical morphism $\text{Spec}_r \mathcal{N}(R^n) \rightarrow \text{Spec}_r R[x_1, \dots, x_n] = \widetilde{R^n}$ is an homeomorphism and induces another homeomorphism

$$\text{Spec}_r \mathcal{N}(V) \simeq \widetilde{V}.$$

A key tool in the polynomial case to relate geometry to algebra is the tilde operator. For instance, we have already seen that it commutes with the topological closure. Namely, for any semialgebraic subset S of R^n , by [6, Thm. 7.2.3] one has $\widetilde{S^E} = \widetilde{S}$. Roughly speaking, this commutation is still valid in the Nash case:

Lemma 5.14. *We have $\widetilde{\text{Cent}(V)} = \text{Spec}_c \mathcal{N}(V)$.*

Proof. As recalled, we see both quantities $\widetilde{\text{Cent}(V)}$ and $\text{Spec}_c \mathcal{N}(V)$ in $\widetilde{R^n} = \text{Spec}_r R[x_1, \dots, x_n]$.

At the Zariski spectrum level, one has $\text{Sing}(\mathcal{N}(V)) = \mathcal{V}(J)$, whereas at the geometrical level one has $\text{Sing}(V) = \mathcal{Z}(J)$, where J is an ideal of $\mathcal{N}(R^n)$ containing I . Hence, at the real spectrum level, one gets $\widetilde{\text{Sing}(\mathcal{N}(V))} = \widetilde{\text{Sing}(V)}$ where the first tilde sends a Zariski constructible subset of $\text{Spec} \mathcal{N}(V)$ to a constructible subset of $\widetilde{R^n}$ (see just before Proposition 3.19) and the second tilde is the usual one from R^n to $\widetilde{R^n}$.

Then, one derives $\widetilde{\text{Reg}(\mathcal{N}(V))} = \widetilde{\text{Reg}(V)}$ and $\widetilde{\text{Reg}(\mathcal{N}(V))} = \widetilde{\text{Reg}(V)} = \widetilde{\text{Reg}(V)^E}$. Using Proposition 3.19 we get $\text{Spec}_c \mathcal{N}(V) = \text{Cent}(V)$. \square

Then, one gets a similar statement than in the algebraic case.

Proposition 5.15. (*Central Nash Hilbert 17th Property*) *Let V be an irreducible Nash set. Let $f \in A = \mathcal{N}(V)$ and $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$. The following properties are equivalent:*

- (1) $f \geq 0$ on $\text{Cent } V$.
- (2) *There exist p, q in \mathcal{C} such that $fq = p + f^{2m}$.*
- (3) *There exist $p, q \in \mathcal{C}$ such that $q^2f = p$ and $(\mathcal{Z}(q) \cap \text{Cent } V) \subset (\mathcal{Z}(f) \cap \text{Cent } V)$.*
- (4) $f \in \mathcal{C}$.

Proof. Saying $f \geq 0$ on $\text{Cent } V$ is equivalent to say that the semialgebraic subset $S = \{f < 0\} \cap \text{Cent } V$ of R^n is empty. By the Artin-Lang property (cf [6, Thm. 4.1.2]) and Lemma 5.14, it is equivalent to the emptiness of $\tilde{S} = \mathcal{S}(-f) \cap \text{Spec}_c A = \mathcal{S}^c(-f) \subset \text{Spec}_r A$ i.e $f \geq 0$ on $\text{Spec}_c A$. One may conclude using Proposition 5.6. \square

Similarly to Definition 3.11, we define \mathcal{C} -subsets of V and $\text{Spec}_r \mathcal{N}(V)$. Using Lemma 5.14 and Artin-Lang property then the same proofs than those of Theorem 3.12, Proposition 5.1 and Corollary 5.9 gives:

Proposition 5.16. *Let V be an irreducible Nash set.*

- (1) *The central locus $\text{Cent } V$ is the unique \mathcal{C} -subset of V .*
- (2) *E is \mathcal{C} -subset of $\text{Spec}_r \mathcal{N}(V)$ if and only if $\text{Spec}_c \mathcal{K}(\mathcal{N}(V)) \subset E \subset \text{Spec}_{pc} \mathcal{N}(V)$*

Proposition 5.17. (*Nash Central Positivstellensätze for surfaces*) *Let f, f_1, \dots, f_r in $A = \mathcal{N}(V)$ the ring of Nash functions on an irreducible Nash set V of dimension ≤ 2 . Let $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$. Then,*

- (1) $f \geq 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent}(V)$ if and only if $fq = p + f^{2m}$ where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.
- (2) $f > 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent}(V)$ if and only if $fq = 1 + p$ where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.
- (3) $f = 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent}(V)$ if and only if $f^{2m} + p = 0$ where p is in $\mathcal{C}[f_1, \dots, f_r]$.

Proof. To use the general framework we have introduced, we have first to show that the stability index corresponds to the dimension also in the Nash setting. We use the Artin-Mazur description of Nash functions (cf [6, Thm. 8.4.4]) which states that for any Nash functions $f_1, \dots, f_r : R^n \rightarrow R$ there is a nonsingular algebraic set $X \subset R^q$ of dimension n , an open semialgebraic subset W of X , a Nash diffeomorphism $\sigma : R^n \rightarrow W$ and some polynomial function g_1, \dots, g_r on W such that $f_i = g_i \circ \sigma$. Hence, any description of a basic open subset $\{f_1 > 0, \dots, f_r > 0\}$ in $V \subset R^n$ where the f_i 's are Nash functions can be translated via σ into a basic open subset $\{g_1 > 0, \dots, g_r > 0\}$ in W where the g_i 's are polynomial functions. Then, one may apply the Bröcker-Scheiderer Theorem for the stability index of algebraic varieties.

Let us show now the first assertion, one proceeds likewise for the others.

Saying that $f \geq 0$ on $\overline{\mathcal{S}}(f_1, \dots, f_r) \cap \text{Cent}(V)$ means that the semialgebraic subset $S = \{f < 0, f_1 \geq 0, \dots, f_r \geq 0\} \cap \text{Cent}(V)$ of V is empty. By the Artin-Lang property (cf [6, Thm. 4.1.2]) and Lemma 5.14, it is equivalent to the emptiness of $\tilde{S} = \mathcal{S}^c(-f) \cap \overline{\mathcal{S}}^c(f_1, \dots, f_r) \subset \text{Spec}_r A$. It means $f \geq 0$ on $\overline{\mathcal{S}}^c(f_1, \dots, f_r)$. We may then use Proposition 5.10 to conclude. \square

5.4.2. *In the analytic setting.* Less classical than in the real algebraic setting, one may define the central locus of an irreducible real analytic set as the Euclidean closure of the set of all regular points.

Our main reference in this section will be [2], where the main results we need are stated, nevertheless we want to emphasize that [18] used the notion of central points in the analytic settings and also that in [1] the Łojasiewicz radical, an alternative to the usual central radical, is introduced.

Let Ω be a real analytic variety and C be a compact global semianalytic subset of Ω . Let $\mathcal{O}(\Omega_C)$ be the ring of all germs of all real analytic functions at C ; it is a Noetherian ring (see [2, VIII 8.]). For X_C a semianalytic germ of Ω_C , we set $A = \mathcal{O}(X_C) = \mathcal{O}(\Omega_C)/\mathcal{I}(X_C)$ to be the ring of analytic function germs of X_C where $\mathcal{I}(X_C)$ is the ideal of functions germs vanishing at X_C . In all the following, we consider X_C irreducible, meaning that A is a domain and $\mathcal{K}(A)$ is the field of germs of meromorphic functions.

We are mainly interested in germs of analytic functions at a point and also by the analytic functions on a compact subset both cases covered by our framework. Note that without the compactness assumption, the ring A would not be so nice, for instance not Noetherian.

From now on, we consider that X_C is a subanalytic set germ of Ω_C . For $x \in X_C$, we say that x is regular if the ring $\mathcal{O}(X_C)_{m_x}$ is regular where m_x is the ideal associated to x . One may define the analytic central locus germ $\text{Cent}(X_C)$ of X_C to be the Euclidean closure of the set of regular points in X_C .

Since C is compact, by [2, VIII Thm. 7.2], $\text{Cent}(X_C)$ is a closed semianalytic subset germ of X_C and it can be defined by a *finite* number of a conjunction of inequalities.

Again since C is compact, one may use the key tool taken from [2, VIII Prop. 8.2] that we recall for the convenience of the reader:

Proposition. *The tilde correspondence, which assigns to $\cup_i \{f_{i1} > 0, \dots, f_{ir_i} > 0, g_{i1} = 0, \dots, g_{is_i} = 0\}$ the set*

$$\cup_i \{\alpha \in \widetilde{X_C} \mid f_{i1}(\alpha) > 0, \dots, f_{ir_i}(\alpha) > 0, g_{i1}(\alpha) = 0, \dots, g_{is_i}(\alpha) = 0\},$$

induces an isomorphism between the boolean algebra of semianalytic subset germs of X_C onto that of constructible subsets of $\widetilde{X_C} = \text{Spec}_r \mathcal{O}(X_C)$.

Beware that without the compactness hypothesis, the tilde operation is no more well defined and we only have a weak Artin-Lang property.

From this, one may derive the counterpart of the classical properties on the tilde operator between semialgebraic subsets of \mathbb{R}^n and constructible subsets of $\widetilde{\mathbb{R}^n}$. One also gets an Artin-Lang property, namely $S = \emptyset$ if and only if $\widetilde{S} = \emptyset$ for sets S as described in the proposition. Let us write down the compatibility of the tilde operator with closure, an important fact in our context.

Lemma 5.18. *Let S be a semianalytic subset germ of X_C . Then, $\overline{\widetilde{S}} = \widetilde{S^E}$.*

Proof. One has obviously $\widetilde{S} \subset \widetilde{S^E}$. By compactness of C , we get from [2, VIII Thm. 7.2] that $\overline{S^E}$ is a compact subanalytic subset germ and it can be written as a *finite* union of sets of the form $\{f_1 \geq 0, \dots, f_r \geq 0\}$, hence $\widetilde{S^E}$ is closed and $\widetilde{S} \subset \widetilde{S^E}$.

Let us show the converse inclusion. Let us consider $\alpha \in \widetilde{S^E}$. Let V be an open subset containing α , one may assume that $V = \widetilde{U}$. Hence, $\alpha \in \widetilde{S^E} \cap \widetilde{U} = \widetilde{S^E \cap U}$. Since $\overline{S^E} \cap U \neq \emptyset$, one has also $\overline{S^E} \cap U \neq \emptyset$. Since U is open, one gets $S \cap U \neq \emptyset$ and hence $\widetilde{S} \cap \widetilde{U} \neq \emptyset$. We have shown that $\widetilde{S^E} \subset \widetilde{S}$. \square

We recall from [2, VIII Thm. 8.4] that $A = \mathcal{O}(X_C)$ is an excellent ring. From Proposition 3.19 and Lemma 5.18, one gets that

$$\widetilde{\text{Cent}(X_C)} = \text{Spec}_c \mathcal{O}(X_C).$$

Then, one gets a similar statements than in the algebraic and Nash cases.

The same proof as in Proposition 5.15 gives:

Proposition 5.19. *(Central Analytic Hilbert 17th Property) Let $f \in A = \mathcal{O}(X_C)$ and $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$. The following properties are equivalent:*

- (1) $f \geq 0$ on $\text{Cent } X_C$.
- (2) There exist p, q in \mathcal{C} such that $fq = p + f^{2m}$.
- (3) There exist $p, q \in \mathcal{C}$ such that $q^2 f = p$ and $(\mathcal{Z}(q) \cap \text{Cent } X_C) \subset (\mathcal{Z}(f) \cap \text{Cent } X_C)$.
- (4) $f \in \mathcal{C}$.

We define an \mathcal{C} -subset of X_C or $\text{Spec}_r \mathcal{O}(X_C)$ similarly to Definition 3.11. The same arguments used to get Proposition 5.16 gives:

Proposition 5.20. (1) *The central locus $\text{Cent } X_C$ is the unique \mathcal{C} -subset of X_C .*
 (2) *E is \mathcal{C} -subset of $\text{Spec}_r \mathcal{O}(X_C)$ if and only if $\text{Spec}_c \mathcal{K}(\mathcal{O}(X_C)) \subset E \subset \text{Spec}_{pc} \mathcal{O}(X_C)$*

To conclude, we proceed as in the proof of Proposition 5.17. Note again that the stability index coincides with the dimension by [2, VIII Thm. 6.3] in the analytic setting, and we get:

Proposition 5.21. *(Analytic Central Positivstellensätze for surfaces): Let f, f_1, \dots, f_r in $A = \mathcal{O}(X_C)$ and assume $\dim X_C \leq 2$. Let $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$. Then,*

- (1) *$f \geq 0$ on $\{f_1 \geq 0, \dots, f_r \geq 0\} \cap \text{Cent}(X_C)$ if and only if $f q = p + f^{2m}$ where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.*
- (2) *$f > 0$ on $\{f_1 \geq 0, \dots, f_r \geq 0\} \cap \text{Cent}(X_C)$ if and only if $f q = 1 + p$ where p, q are in $\mathcal{C}[f_1, \dots, f_r]$.*
- (3) *$f = 0$ on $\{f_1 \geq 0, \dots, f_r \geq 0\} \cap \text{Cent}(X_C)$ if and only if $f^{2m} + p = 0$ where p is in $\mathcal{C}[f_1, \dots, f_r]$.*

6. SUMS OF SQUARES OF RATIONAL CONTINUOUS FUNCTIONS ON THE CENTRAL SPECTRUM

From [13], it is possible to find a sum of squares solution of Hilbert 17th problem such that the rational functions appearing can be extended continuously to R^n for the Euclidean topology i.e are rational continuous functions on R^n . We denote by $\mathcal{K}^0(R^n)$ the ring of rational continuous functions on R^n . Rational continuous functions are called regulous when they are still rational continuous by restriction to any subvariety. Rational continuous and regulous functions are introduced and studied in [12], [9], [16] and [5]. Above result involving sums of squares of rational continuous functions can be generalized as it is done in [10, Thm. 6.1], namely $f \in \mathcal{K}^0(R^n)$ is nonnegative on R^n if and only if $f \in \sum \mathcal{K}^0(R^n)^2$. The proof of this result in [10] is over \mathbb{R} but it is also valid over any real closed field in place of \mathbb{R} .

We may wonder if it is possible to get a continuity property in Theorem 5.4 and Proposition 5.6. As continuity is a topological notion, we choose here to only look at a continuous version of Proposition 5.6, the one associated with central orderings of a topological nature. To do that, we recall some material about abstract continuity on the real spectrum which is mainly taken from [2].

Any $f \in A$ may be associated to a function defined on $\text{Spec}_r A$, assigning $\alpha \mapsto f(\alpha) \in R_\alpha$ with $\alpha : A \rightarrow R_\alpha$. It does not give functions in the usual sense since the target space R_α 's vary. One may then define abstract semialgebraic functions on $\text{Spec}_r A$ given by a first order formula with parameters in A (see [2, II 5]). For instance, for $p \in A$ and $q \in A \setminus \{0\}$, one may define the abstract semialgebraic function f by setting $f(\alpha) = \frac{p(\alpha)}{q(\alpha)}$ whenever $q(\alpha) \neq 0$ and $f(\alpha) = 0$ otherwise. One may also define functions on a proconstructible subset Y of $\text{Spec}_r A$. Finally, one says that an abstract semialgebraic function f is continuous on Y if, for any specialization $\beta \rightarrow \alpha$ in Y , one has $f(\beta) \in W_{\beta\alpha}$ and $\lambda_{\beta\alpha}(f(\beta)) = f(\alpha)$ where $W_{\beta\alpha}$ and $\lambda_{\beta\alpha}$ are respectively the valuation ring and the place associated to the specialization $\beta \rightarrow \alpha$ (see [2, II 3.10]). Let f be an abstract semialgebraic function on $\text{Spec}_c A$, we say that f is rational continuous on $\text{Spec}_c A$ if f is continuous on $\text{Spec}_c A$ and if there exist $p \in A$ and $q \in A \setminus \{0\}$, such that $f(\alpha) = \frac{p(\alpha)}{q(\alpha)}$ whenever $\alpha \in \text{Spec}_c A \setminus \mathcal{Z}^c(q)$. We denote by $\mathcal{K}^0(\text{Spec}_c A)$ the ring of rational continuous functions on $\text{Spec}_c A$.

Proposition 6.1. *The ring $\mathcal{K}^0(\text{Spec}_c A)$ is a domain whose fraction field is $\mathcal{K}(A)$.*

Proof. Consider the map $\mathcal{K}^0(\text{Spec}_c A) \rightarrow \mathcal{K}(A)$ which send $f \in \mathcal{K}^0(\text{Spec}_c A)$ to the class of p/q in $\mathcal{K}(A)$ with $p \in A$, $0 \neq q \in A$, and $f(\alpha) = \frac{p(\alpha)}{q(\alpha)}$ whenever $\alpha \in \text{Spec}_c A \setminus \mathcal{Z}^c(q)$. We have to prove this map is injective and thus we assume the class of p/q in $\mathcal{K}(A)$ is 0. It follows that pq vanishes on $\text{Spec}_c A$ i.e $pq \in \mathcal{I}(\mathcal{Z}^c((0)))$. Since $\mathcal{K}(A)$ is formally real then (0) is a central ideal of A (Proposition 3.4) and thus, using the central Nullstellensatz (Proposition 5.2) we get $pq = 0$ in A . Since A is a domain and $q \neq 0$ then it follows that $p = 0$. Let $\alpha \in \text{Spec}_c A$. If $\alpha \notin \mathcal{Z}^c(q)$ then $f(\alpha) = \frac{p(\alpha)}{q(\alpha)} = 0$. Assume now $\alpha \in \mathcal{Z}^c(q)$. By definition of a central ordering and by Proposition 3.1, there is $\beta \in \text{Spec}_c A$

such that $\text{supp}(\beta) = (0)$ and $\beta \rightarrow \alpha$. Clearly $\beta \notin \mathcal{Z}^c(q)$ and thus $f(\beta) = \frac{p(\beta)}{q(\beta)} = 0 \in W_{\beta\alpha}$ and $\lambda_{\beta\alpha}(f(\beta)) = f(\alpha) = 0$. It follows that $f = 0$ in $\mathcal{K}^0(\text{Spec}_c A)$ as required.

Since $A \subset \mathcal{K}^0(\text{Spec}_c A) \subset \mathcal{K}(A)$ then it follows that $\mathcal{K}(\mathcal{K}^0(\text{Spec}_c A)) = \mathcal{K}(A)$. \square

If V is an irreducible affine algebraic variety over R , we denote by $\mathcal{K}^0(\text{Cent } V(R))$ the ring of rational continuous functions on $\text{Cent } V(R)$. These functions are defined in the same way as for R^n (see [11]). It is clear that the restriction to $\text{Cent } V(R)$ of an element of $\mathcal{K}^0(\text{Spec}_c R[V])$ belongs to $\mathcal{K}^0(\text{Cent } V(R))$.

Let us define the following subcone of $\mathcal{C} = A \cap \sum \mathcal{K}(A)^2$ which is convenient to deal with continuity:

$$\mathcal{C}^0 = A \cap \sum \mathcal{K}^0(\text{Spec}_c A)^2.$$

Adding a continuity property in Theorem 5.4 and Proposition 5.6 would suggest, for a given $f \in A$, that $f(\text{Spec}_c A) \geq 0$ if and only if $f \in \mathcal{C}^0$. It would say that $\text{Spec}_c A$ is a kind of \mathcal{C}^0 -set subset of $\text{Spec}_r A$. As already mentioned, we know that it is true when $A = R[x_1, \dots, x_n]$. Surprisingly, this equivalence is false i.e, in general, an element of A nonnegative on $\text{Spec}_c A$ is not necessarily a sum of squares in $\mathcal{K}^0(\text{Spec}_c A)$ as it is shown by the following:

Example 6.2. Let V be the Whitney umbrella with coordinate ring $A = \mathbb{R}[V] = \mathbb{R}[x, y, z]/(y^2 - zx^2)$. Its normalization V' is smooth and we have $B = \mathbb{R}[V'] = \mathbb{R}[x, Y, z]/(Y^2 - z)$. The ring morphism $A \rightarrow B$ associated to the normalization map $\pi' : V' \rightarrow V$ is given by $x \mapsto x, y \mapsto Yx$ and $z \mapsto z$. Let $f = z \in A$. Since $f = (x/y)^2 \in \mathcal{K}(A)^2$ then it follows from Proposition 5.6 that $f \geq 0$ on $\text{Spec}_c A$.

We prove now that $f \notin \mathcal{C}^0$. Assume $f \in \mathcal{C}^0$ then we get, by restriction to $\text{Cent } V(\mathbb{R})$,

$$f = \sum_{i=1}^n f_i^2$$

with $f_i \in \mathcal{K}^0(\text{Cent } V(\mathbb{R}))$. By composition with $\pi'_{|\mathbb{R}} : V'(\mathbb{R}) \rightarrow V(\mathbb{R})$ then we get

$$g = f \circ \pi' = \sum_{i=1}^n g_i^2$$

with $g_i = f_i \circ \pi'_{|\mathbb{R}} \in \mathcal{K}^0(V'(\mathbb{R}))^2$ (note that $\pi'_{|\mathbb{R}}$ is surjective on $\text{Cent } V(\mathbb{R})$). Since V' is smooth then the g_i are regulus functions and thus are still rational continuous by restriction to a subvariety [12]. So we restrict our identity $g = \sum_{i=1}^n g_i^2$ to the curve $C \subset V'(\mathbb{R})$ with equations $x = 0$ and $Y^2 = z$ and since the curve is smooth then the restriction of the g_i to C are regular functions [16]. On C we get

$$g = z = \sum_{i=1}^n \left(\frac{a_{i,1}(z) + a_{i,2}(z)Y}{b_{i,1}(z) + b_{i,2}(z)Y} \right)^2$$

with the $a_{i,j}$ and $b_{i,j}$ polynomials in z . But the fractions $\frac{a_{i,1}(z) + a_{i,2}(z)Y}{b_{i,1}(z) + b_{i,2}(z)Y}$ are composition by $\pi'_{|\mathbb{R}}$ of continuous functions on the superior half of the z -axis in $V(\mathbb{R})$ and thus it follows from an easy computation that $a_{i,2} = b_{i,2} = 0$ for $i = 1, \dots, n$. The previous identity becomes impossible and the proof is done.

Let us see now that we get this continuity property if and only if some p and q appearing in the identities of the second and the third statements of Proposition 5.6 belong to \mathcal{C}^0 .

Proposition 6.3. *Let $f \in A$. The following properties are equivalent:*

- (1) *There exist p, q in \mathcal{C}^0 such that $fq = p + f^{2m}$.*
- (2) *There exist $p, q \in \mathcal{C}^0$ such that $q^2 f = p$ and $\mathcal{Z}^c(q) \subset \mathcal{Z}^c(f)$.*
- (3) *$f \in \mathcal{C}^0$.*

Proof. Let us show (1) implies (2). Suppose there exist p, q in \mathcal{C}^0 such that $fq = p + f^{2m}$. One may assume that $f \neq 0$ and then $p + f^{2m} \neq 0$ since $\mathcal{K}(A)$ is formally real. We set $P = (f^2q)(p + f^{2m})$ and $Q = p + f^{2m}$. Clearly, $P, Q \in \mathcal{C}^0$. Following the proof of Theorem 5.4 and Proposition 5.6 we get $Q^2f = P$ and $\mathcal{Z}^c(Q) \subset \mathcal{Z}^c(f)$.

We prove (2) implies (3). Assume there exist $P, Q \in \mathcal{C}^0$ such that $Q^2f = P$ and $\mathcal{Z}^c(Q) \subset \mathcal{Z}^c(f)$. Hence, one may write $f = \sum f_i^2$ where $f_i = \frac{g_i}{Q}$ with $g_i \in A$. Clearly $f_i \in \mathcal{K}(A)$. Considering that $f_i(\alpha) = 0$ whenever $Q(\alpha) = 0$, and $f_i(\alpha) = \frac{g_i(\alpha)}{Q(\alpha)}$ whenever $Q(\alpha) \neq 0$, then f_i is now defined on $\text{Spec}_c A$.

We show now that $f_i \in \mathcal{K}^0(\text{Spec}_c A)$ and we are left to prove it is continuous on $\text{Spec}_c A$. Let $\beta \rightarrow \alpha$ be a specialization in $\text{Spec}_c(A)$.

The case $Q(\beta) = 0$ is trivial. Indeed, $Q(\alpha) = 0$ since β specializes to α and in that case we have set $f_i(\beta) = 0$ and $f_i(\alpha) = 0$. Then, obviously $f_i(\beta) \in W_{\beta\alpha}$ and $\lambda_{\beta\alpha}(f_i(\beta)) = f_i(\alpha)$.

Let us assume in the following that $Q(\beta) \neq 0$. In that case, we have set $f_i(\beta) = \frac{g_i(\beta)}{Q(\beta)}$. Since $W_{\beta\alpha}$ is β -convex and $f(\beta) = \sum f_i^2(\beta)$ is in $W_{\beta\alpha}$, one gets that $f_i^2(\beta) \in W_{\beta\alpha}$. Using that a valuation ring is integrally closed, one has $f_i(\beta) \in W_{\beta\alpha}$, the first desired condition. Let us show now the second condition : $\lambda_{\beta\alpha}(f_i(\beta)) = f_i(\alpha)$.

First case: we assume $Q(\alpha) \neq 0$. Since $Qf_i \in A$, one has $Q(\beta)f_i(\beta) \in W_{\beta\alpha}$, and $\lambda_{\beta\alpha}(Q(\beta)f_i(\beta)) = Q(\alpha)f_i(\alpha)$ and hence

$$\lambda_{\beta\alpha}(Q(\beta))\lambda_{\beta\alpha}(f_i(\beta)) = Q(\alpha)f_i(\alpha)$$

Since, $\lambda_{\beta\alpha}(Q(\beta)) = Q(\alpha) \neq 0$, one gets the desired condition $\lambda_{\beta\alpha}(f_i(\beta)) = f_i(\alpha)$.

Second case: assume that $Q(\alpha) = 0$. Since $\mathcal{Z}^c(Q) \subset \mathcal{Z}^c(f)$, hence $f(\alpha) = 0$. Since $f = \sum f_i^2$, one gets

$$0 = f(\alpha) = \lambda_{\beta\alpha}(f(\beta)) = \sum \lambda_{\beta\alpha}(f_i(\beta))^2$$

This shows that, for any i , $\lambda_{\beta\alpha}(f_i(\beta)) = 0$, which gives $\lambda_{\beta\alpha}(f_i(\beta)) = f_i(\alpha)$ and we have proved that (2) implies (3).

(3) implies (1). Assume $f \in \mathcal{C}^0$. We set $q = f \in \mathcal{C}^0$ and $p = 0$ and we get the identity of (1) namely $fq = p + f^{2m}$ for $m = 1$. \square

One sufficient condition to fit in the hypothesis of this proposition is to assume nonnegativity of our element f on the whole real spectrum (not only on the central spectrum). This way, one gets a continuous solution of the Hilbert 17th problem for abstract rings :

Theorem 6.4. *Let $f \in A$. If $f \geq 0$ on $\text{Spec}_r A$, then $f \in \mathcal{C}^0$.*

Proof. Assume $f \geq 0$ on $\text{Spec}_r A$. By the formal Positivstellensatz we get an identity $fq = p + f^{2m}$ with $p, q \in \sum A^2 \subset \mathcal{C}^0$. We conclude using Proposition 6.3. \square

Using the Artin-Lang property, one may derive a geometric version of Theorem 6.4 (left to the reader) and of Proposition 6.3, namely:

Proposition 6.5. *Let V be an irreducible affine algebraic variety over R with $V_{\text{reg}}(R) \neq \emptyset$. Let $f \in R[V]$. The following properties are equivalent:*

- (1) *There exist p, q in $R[V] \cap \sum \mathcal{K}^0(\text{Cent } V(R))^2$ such that $fq = p + f^{2m}$.*
- (2) *There exist $P, Q \in R[V] \cap \sum \mathcal{K}^0(\text{Cent } V(R))^2$ such that $Q^2f = P$ and $\mathcal{Z}(Q) \cap \text{Cent } V(R) \subset \mathcal{Z}(f) \cap \text{Cent } V(R)$.*
- (3) *$f \in R[V] \cap \sum \mathcal{K}^0(\text{Cent } V(R))^2$.*

Let V be the Cartan umbrella of coordinate ring $\mathbb{R}[V] = \mathbb{R}[x, y, z]/(x^3 - z(x^2 + y^2))$ and $f = x^2 + y^2 - z^2$. As already discussed in Example 3.6, there is $Q = x^2 + y^2 \in \mathcal{C}^0$ and $P = 3x^4y^2 + 3x^2y^4 + y^6 \in \mathcal{C}^0$ such that $Q^2f = P$. Since, $\mathcal{Z}(Q) \cap \text{Cent } V(\mathbb{R}) \subset \mathcal{Z}(f) \cap \text{Cent } V(\mathbb{R})$ one gets that $f \in \sum \mathcal{K}^0(\text{Cent } V(\mathbb{R}))^2$.

Note that f is not nonnegative on all $V(\mathbb{R})$ which says that the converse implication in Theorem 6.4 is not true i.e $\text{Spec}_r A$ is not a \mathcal{C}^0 -set.

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