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# Diameter and long paths in critical digraph

Arthur Blanc-Renaudie\*

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## Abstract

We study the random directed graph  $\vec{G}(n, p)$  in which each of the  $n(n-1)$  possible directed edges are present with probability  $p$ . We show that in the critical window the longest self avoiding oriented paths in  $\vec{G}(n, p)$  have length  $O_{\mathbb{P}}(n^{1/3})$  so  $\vec{G}(n, p)$  has diameter  $O_{\mathbb{P}}(n^{1/3})$ .

## 1 Introduction

Since its introduction more than fifty years ago [9] the interest for Erdős–Rényi random graphs has kept growing. This model and its study has also been generalized in many ways due to many critics. An important one is that many real-life networks are oriented. However, random digraphs, which are the natural oriented version of Erdős–Rényi random graphs, are much harder to study due to their lack of symmetry, and thus less understood.

For  $n \in \mathbb{N}, p \in [0, 1]$ , the random digraph  $\vec{G}(n, p)$  is a random directed graph on  $\{1, 2, \dots, n\}$  in which each of the  $n(n-1)$  possible directed edges are independently present with probability  $p$ .

In this paper, we are interested in the phase transition for random digraphs. The first result on this phase transition was proved by Karp [11] and Luczak [12]. Those results was then refined by Luczak and Seierstad [13] who showed that the size of strongly connected components in random digraphs share the same critical window as Erdős–Rényi random graphs. Later Goldshmidt and Stephenson [10] proved that renormalized by  $n^{1/3}$  the strongly connected components converge.

Here we consider the whole digraph by studying its diameter and long paths. More precisely we say that  $(v_0, v_1, \dots, v_k)$  is a saop (for Self-Avoiding Oriented Path) of length  $k$  in an oriented graph  $\vec{G}$  if the oriented edges  $(v_0, v_1), \dots, (v_{k-1}, v_k)$  are different and in  $\vec{G}$ . For every  $i, j \in \vec{G}$ , let  $d_{\rightarrow}(i, j)$  be the minimal length of a saop from  $i$  to  $j$  (or by default  $-\infty$  if there is not such paths). The diameter of  $\vec{G}$  is  $\text{diam}(\vec{G}) := \max_{i, j \in \vec{G}} d_{\rightarrow}(i, j)$ .

For a sequence  $(X_n)_{n \in \mathbb{N}}$  of random variables and  $f : n \mapsto \mathbb{R}^{+*}$ , we say that  $X_n = O_{\mathbb{P}}(f(n))$  if  $(X_n/f(n))_{n \in \mathbb{N}}$  is tight, and that  $X_n = o_{\mathbb{P}}(f(n))$  if  $(X_n/f(n))_{n \in \mathbb{N}}$  converges in probability to 0.

We will assume in all the paper that we are in the critical window, that is  $p = 1/n + O(1/n^{4/3})$ . The goal of this paper is to show the next results: (see also Theorem 4 for more precise bounds)

**Theorem 1.** *As  $n \rightarrow \infty$ ,  $\text{diam}(\vec{G}(n, p)) = O_{\mathbb{P}}(n^{1/3})$ .*

**Theorem 2.** *As  $n \rightarrow \infty$ , the longest saop in  $\vec{G}(n, p)$  have length  $O_{\mathbb{P}}(n^{1/3})$ .*

One of our main motivations for the above Theorems is to show scaling limits for  $\vec{G}(n, p)$ , as proved for Erdős–Rényi random graphs by Addario–Berry, Broutin, and Goldshmidt in [1]. However, one would first need to develop a topological theory for oriented metric spaces, and generalize the Gromov–Hausdorff–Prokhorov topology for those spaces. Solving this problem would thus require a whole new set of ideas.

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**Plan of the paper** In Section 2, we prove Theorems 1 and 2 assuming an analog result for uniform rooted tree. This result is then proved in Section 3 using the chaining method and branch tightness.

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## 2 Reduction to a tree problem

### 2.1 The long path probability, and why it implies Theorems 1 and 2

We show here that the next technical proposition implies Theorem 1 and 2. In the rest of the section, we will explain why it comes from an analog result for uniform rooted trees. For every  $n, m \in \mathbb{N}$ , let  $\llbracket n, m \rrbracket := [n, m] \cap \mathbb{N}$ . For every  $v \in \llbracket 1, n \rrbracket$ , we write  $\mathcal{L}^\uparrow(v)$  (resp.  $\mathcal{L}^\downarrow(v)$ ) for the length of the longest saop in  $\vec{G}(n, p)$  starting from (resp. ending at)  $v$ .

**Proposition 3.** *For every  $\delta > 0$ , there exists  $C \in \mathbb{R}^+$ , such that for every  $n \in \mathbb{N}$ ,  $v \in \llbracket 1, n \rrbracket$ ,*

$$\mathbb{P}(\mathcal{L}^\uparrow(v) > \delta n^{1/3}) = \mathbb{P}(\mathcal{L}^\downarrow(v) > \delta n^{1/3}) \leq Cn^{-1/3}.$$

Given Proposition 3, the BK inequality implies Theorem 2, and so Theorem 1:

*Proof of Theorem 2.* We say that  $(u, v)$  is an inner edge if disjointly:  $(u, v) \in \vec{G}(n, p)$ , there is a saop of length at least  $n^{1/3}$  ending at  $u$ , and there is a saop of length at least  $n^{1/3}$  starting from  $v$ . By Proposition 3, and by the BK inequality, for some constant  $C$ , for every  $u, v \in \llbracket 1, n \rrbracket$ ,

$$\mathbb{P}((u, v) \text{ is an inner edge}) \leq pC^2n^{-2/3}.$$

Thus, by Markov's inequality the number of inner edges is  $O_{\mathbb{P}}(n^{1/3})$ . Theorem 2 follows as in a saop of length  $Ln^{1/3}$  there is at least  $(L - 2)n^{1/3} - 3$  inner edges.  $\square$

Refining the above proof, the BK inequality also directly provides sub-gamma upper-bound for  $\mathcal{L}(\vec{G}(n, p))$  the length of the longest saop in  $\vec{G}(n, p)$ , and so for  $\text{diam}(\vec{G}(n, p)) \leq \mathcal{L}(\vec{G}(n, p))$ .

**Theorem 4.** *There exists  $c, C > 0$  such that for every  $n \in \mathbb{N}$ , for every  $x > C$ ,*

$$\mathbb{P}(\text{diam}(\vec{G}(n, p)) > xn^{1/3}) \leq \mathbb{P}(\mathcal{L}(\vec{G}(n, p)) > xn^{1/3}) \leq e^{-cx \log x}.$$

*Remark.* I don't think the above bound is optimal, as for Erdős–Rényi random graphs a bound in  $e^{-cx^{3/2}}$  was proved by Nachmias and Peres [14, Proposition 1.4].

*Proof.* For  $k \in \mathbb{N}$ , we say that  $(u_1, v_1), (u_2, v_2), \dots, (u_k, v_k) \in \llbracket 1, n \rrbracket$  are disjointly inner edges, if disjointly for  $1 \leq i \leq k$ ,  $(u_i, v_i)$  is an inner edge. By Proposition 3, and by the BK inequality, for some  $C \in \mathbb{R}^+$ , for every  $k \in \mathbb{N}$ ,  $u_1, v_1, u_2, v_2, \dots, u_k, v_k \in \llbracket 1, n \rrbracket$ ,

$$\mathbb{P}((u_1, v_1), \dots, (u_k, v_k) \text{ are disjointly inner edges}) \leq p^k C^{2k} n^{-2k/3}. \quad (1)$$

Then given a saop of length  $L$ , we may look at the  $k$ -tuples of edges in this path that are separated by at least  $2\lceil n^{1/3} \rceil$  edges from each other and separated by at least  $\lceil n^{1/3} \rceil$  edges from the start and end of the saop. By a counting argument, we see that if  $L > 9kn^{1/3}$  then there is at least  $(L/2)^k$  such  $k$ -tuples of edges. Moreover, note that such tuples of edges are disjointly inner edges.

It follows from (1) and by Markov's inequality, that for every  $k \in \mathbb{N}$ ,  $L > 9kn^{1/3}$ ,

$$\mathbb{P}(\mathcal{L}(\vec{G}(n, p)) \geq L) \leq n^{2k} p^k C^{2k} n^{-2k/3} / (L/2)^k = (2pn^{-2/3}C/L)^k.$$

The desired result follows by taking  $L = \lceil xn^{1/3} \rceil$  and  $k = \lfloor x/9 \rfloor$ .  $\square$

## 2.2 The depth-first search (DFS) exploration

To show Proposition 3 we need a construction from [10] for  $\vec{G}(n, p)$ . Since we only look at the paths going to 1, we only need part of it. So, we aim for concision, and refer to [10] for details. We define the DFS exploration of a graph as follows: We define inductively for  $i \in \llbracket 0, n \rrbracket$  an ordered set (or stack)  $\mathcal{O}_i$  of open vertices at time  $i$ , and a set  $\mathcal{A}_i$  of vertices already explored at time  $i$ .

*Algorithm 1. DFS exploration of a graph on  $\llbracket 1, n \rrbracket$ .*

- Set  $\mathcal{O}_0 = (1)$ ,  $\mathcal{A}_0 = \emptyset$ .
- For  $0 \leq i \leq n-1$ : Let  $v_{i+1}$  be the first vertex of  $\mathcal{O}_i$ . Let  $\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{v_{i+1}\}$ . Let  $\mathfrak{N}_i$  be the set of neighbours of  $v_i$  that are not in  $\mathcal{O}_i \cup \mathcal{A}_i$ . Construct  $\mathcal{O}_{i+1}$  by removing  $v_i$  from the start of  $\mathcal{O}_i$  and by adding in increasing order  $\mathfrak{N}_i$  to the start of  $\mathcal{O}_i \setminus \{v_i\}$ . If now  $\mathcal{O}_{i+1} = \emptyset$ , add to it the lowest-labelled element of  $\llbracket 1, n \rrbracket \setminus \mathcal{A}_i$ .

Note that alongside the exploration we construct a permutation  $(v_i)_{1 \leq i \leq n}$  of  $\llbracket 1, n \rrbracket$ , which we call the DFS-order. Our main motivation for using this DFS exploration is the next construction:

*Algorithm 2. Decomposition of  $\vec{G} = \vec{G}(n, p)$  from Proposition 2.1 (iii) of [10]:*

- Let  $G(n, p)$  be a Erdős–Rényi random graph.
- Let  $(v_i)_{1 \leq i \leq n}$  be the DFS order for  $G(n, p)$ .
- For every  $i < j$  put  $(v_j, v_i)$  in  $\vec{G}(n, p)$  if and only if  $\{v_j, v_i\} \in G(n, p)$ .
- Independently for every  $i < j$ , put  $(v_i, v_j)$  in  $\vec{G}(n, p)$  with probability  $p$ .

Along this construction we split the oriented edges of  $\vec{G}(n, p)$  in three categories (see Figure 1):

- The down-edges: Let  $\downarrow(\vec{G}) := \{(v_i, v_j), i \in \llbracket 1, n \rrbracket, j \in \mathfrak{N}_i\}$  be the set of down-edges of  $\vec{G}$ . Note that  $(\llbracket 1, n \rrbracket, \downarrow(\vec{G}))$  is a forest oriented from the leaves toward the roots.
- The surplus-edges: Let  $\leftrightarrow(\vec{G}) := \{(v_j, v_i), i < j\} \setminus \downarrow(\vec{G})$  be the set of surplus edges of  $\vec{G}$ . One can recover  $G(n, p)$  by removing the orientations in  $\downarrow(\vec{G}) \cup \leftrightarrow(\vec{G})$ .
- The right-edges: Let  $\rightsquigarrow(\vec{G}) := \{(v_i, v_j), i < j\}$  be the set of right-edges.

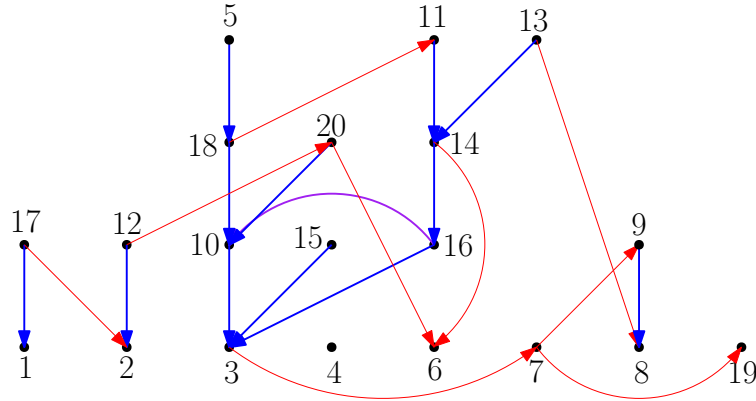


Figure 1: A DFS decomposition of a random digraph  $\vec{G}$  on  $\llbracket 1, 20 \rrbracket$ . The DFS order of  $\vec{G}$  is  $1 \prec 17 \prec 2 \prec 12 \prec 3 \prec 10 \prec 18 \prec 5 \prec 20 \prec 15 \prec 14 \prec 11 \prec 13 \prec 16 \prec 4 \prec 6 \prec 7 \prec 8 \prec 9 \prec 19$ . The down-edges  $\downarrow(\vec{G})$  are blue, the surplus-edges  $\leftrightarrow(\vec{G})$  are purple, and the right-edges  $\rightsquigarrow(\vec{G})$  are red. The longest saop has length 9 and is  $(5, 18, 11, 14, 16, 10, 3, 7, 9, 8)$ .

## 2.3 Analog problem for uniform rooted trees, and why it implies Proposition 3

Recall Algorithm 2. We write  $\mathcal{C}(1)$  for the connected component of 1 in  $G(n, p)$ . Note that all saop in  $\vec{G}$  ending at 1 must start in  $\mathcal{C}(1)$ . So we may focus on  $\mathcal{C}(1)$ . Since with small probability  $\mathcal{C}(1)$  contains a cycle, we may restrict to the case where  $\mathcal{C}(1)$  is a tree. In this case, given  $\#\mathcal{C}(1)$ , it is

well known that  $\mathcal{C}(1)$  is a uniform rooted tree. For those reasons to show Proposition 3 it suffices to prove the following analog problem (see the end of the section for more details):

For  $m \in \mathbb{N}$ ,  $p \in [0, 1]$ , we construct a random directed graph  $\vec{\mathcal{T}}_{m,p}$  as follows: Let  $\mathcal{T}_m$  be a uniform tree with vertices  $\llbracket 1, m \rrbracket$ . Root  $\mathcal{T}_m$  at 1. Let  $\vec{\mathcal{T}}_m$  be  $\mathcal{T}_m$  with the edges oriented toward 1. Let  $\rightsquigarrow (\vec{\mathcal{T}}_{m,p})$  be an independent copy of  $\vec{G}(m, p)$ . Let  $\vec{\mathcal{T}}_{m,p}$  be the union of  $\vec{\mathcal{T}}_m$  and  $\rightsquigarrow (\vec{\mathcal{T}}_{m,p})$ .

We write  $\mathcal{L}(\vec{\mathcal{T}}_{m,p})$  for the length of the longest saop in  $\vec{\mathcal{T}}_{m,p}$ . We need the following result:

**Proposition 5.** *There exists some constants  $C > 0$  such that for every  $m \in \mathbb{N}$ ,  $p > 0$ ,  $x > 0$ ,*

$$\mathbb{P}(\mathcal{L}(\vec{\mathcal{T}}_{m,p}) > x\sqrt{m}) \leq C(1/x^2 + p^{2/3}m).$$

*Remark.* The method of the proof can yield a much better bound, but I do not think it is needed.

*Proof of Proposition 3 given Proposition 5.* By an union bound we get:

$$\mathbb{P}(\mathcal{L}^\uparrow(1) > \delta n^{1/3}) \leq \mathbb{P}(\mathcal{C}(1) \text{ is not a tree}) + \sum_{m=1}^{\infty} \mathbb{P}(\mathcal{L}^\uparrow(1) > \delta n^{1/3}, \mathcal{C}(1) \text{ is a tree}, \#\mathcal{C}(1) = m). \quad (2)$$

By classical results on Erdős-Rényi random graphs, in the critical window,  $\mathcal{C}(1)$  is not a tree with probability  $O(n^{-1/3})$  (see e.g. Aldous [4]). And by Proposition 5 for some universal constant  $C$ , for every  $m \geq 1$ ,

$$\begin{aligned} \mathbb{P}(\mathcal{L}^\uparrow(1) > \delta n^{1/3}, \mathcal{C}(1) \text{ is a tree}, \#\mathcal{C}(1) = m) &\leq \mathbb{P}(\mathcal{C}(1) \text{ is a tree}, \#\mathcal{C}(1) = m) \mathbb{P}(\mathcal{L}(\vec{\mathcal{T}}_{m,p}) > \delta n^{1/3}) \\ &\leq \mathbb{P}(\#\mathcal{C}(1) = m) C m (1/(\delta^2 n^{2/3}) + p^{2/3}). \end{aligned}$$

It follows by sum that, writing  $S$  for the sum in (2),

$$S \leq C(1/(\delta^2 n^{2/3}) + p^{2/3}) \mathbb{E}[\#\mathcal{C}(1)].$$

Moreover, it is well known that in the critical window  $\mathbb{E}[\#\mathcal{C}(1)] = O(n^{1/3})$  (see e.g. Aldous [4]). Thus  $S = O(n^{-1/3})$ , and Proposition 3 follows from (2).  $\square$

### 3 Proof of Proposition 5

In this section  $m, p$  are fixed, and we often omit in the notations the dependence in  $m, p$ .

To prove Proposition 5 we use the chaining method based on leaf tightness. This idea is not new and comes from the pioneer papers of Aldous [2, 3] on uniform rooted trees. It has been refined over the last recent years in e.g. [5, 8, 15, 7], and I refer to the introduction of my thesis [7] for a detailed discussion on the subject.

The main idea is that we may approximate all paths from  $\vec{\mathcal{T}}_{m,p}$  by looking only at the paths starting from a finite number of vertices. To go in the details, we need to introduce a few notations. We consider for  $i \in \llbracket 1, m \rrbracket$ ,  $\mathcal{L}(i)$  the length of the longest saop starting from  $i$  in  $\vec{\mathcal{T}}_{m,p}$ . Similarly for  $i \in \llbracket 1, m \rrbracket$ ,  $S \subset \llbracket 1, m \rrbracket$ , let  $\mathcal{L}(i, S)$  be the length of the longest saop starting from  $i$  in  $\vec{\mathcal{T}}_{m,p} \cap S$ . Let  $\mathfrak{h}(S) - 1$  be the length of the longest saop in  $\vec{\mathcal{T}}_m \cap S$ . Also let  $\downarrow(i)$  be the set of ancestors of  $i$  in  $\mathcal{T}_m$ . Our chaining argument is based on the next result:

**Lemma 6.** *For all  $S \subset \llbracket 1, m \rrbracket$ ,  $j \in \llbracket 1, m \rrbracket$ ,*

$$\mathcal{L}(j) \leq \mathcal{L}\left(j, \llbracket 1, m \rrbracket \setminus \bigcup_{i \in S} \downarrow(i)\right) + 1 + \max_{i \in S} \mathcal{L}(i).$$

*Proof.* Note that a saop starting from  $j$  must meet  $\bigcup_{i \in S} \downarrow(i)$  in at most  $\mathcal{L}(S \setminus \bigcup_{i \in S} \downarrow(i)) + 1$  steps, and after it meet say  $\downarrow(i)$ , it coincides with a part of a path starting from  $i$  and so must last at more  $\mathcal{L}(i)$  steps.  $\square$

Morally the previous lemma tells us that for  $S', S \subset \llbracket 1, m \rrbracket$  to estimate  $\max_{i \in S'} \mathcal{L}(i)$  given  $\max_{i \in S} \mathcal{L}(i)$  it suffices to estimate the length of the paths in a much smaller subgraph of  $\vec{\mathcal{T}}_{m,p}$ . Those paths then tends to be much smaller. This allows us to inductively estimate for well chosen sets  $S_1 \subset S_2 \subset \dots \subset S_K = \llbracket 1, m \rrbracket$ , the  $\max_{i \in S} \mathcal{L}(i)$ . To do so, we need an upper-bound on  $\mathcal{L}(i, S)$  for  $i \in \llbracket 1, m \rrbracket$ ,  $S \subset \llbracket 1, m \rrbracket$ . This is provided by the next lemma:

**Lemma 7.** *For every  $\mathcal{T}_m$ -measurable random variables  $S \subset \llbracket 1, m \rrbracket$ ,  $i \in \llbracket 1, m \rrbracket$ ,  $k \in \mathbb{N}$ ,*

$$\mathbb{P}(\mathcal{L}(i, S) > (k+1)\mathfrak{h}(S) | \mathcal{T}_m) \leq (p\mathfrak{h}(S)m)^k.$$

*Proof.* Recall that a saop in  $S$  which only use oriented edges from  $\vec{\mathcal{T}}_m$  has length at most  $\mathfrak{h}(S) - 1$ . Thus it is enough to upper-bound the probability that there exists a path in  $S$  starting from  $i$  which uses at least  $k$  edges from  $\rightsquigarrow (\vec{\mathcal{T}}_{m,p})$ . By an union bound this probability is bounded by

$$\sum \mathbb{P}(\forall i \in \llbracket 0, k-1 \rrbracket, (j_a, i_{a+1}) \in \rightsquigarrow (\vec{\mathcal{T}}_{m,p}) | \mathcal{T}_m),$$

where the sum above is over all tuples  $i_0, j_0, i_1, \dots, i_k$  such that  $(j_0, i_0) \neq (j_1, i_1) \neq \dots (j_k, i_k)$ , and such that for every  $a \in \llbracket 0, k-1 \rrbracket$ , there is a path from  $i_a$  to  $j_a$  in  $\vec{\mathcal{T}}_m \cap S$ . Note that there is at most  $\mathfrak{h}(S)$  choices for  $j_0$ , then at most  $m$  choices for  $i_1$ , then at most  $\mathfrak{h}(S)$  choices for  $j_1$ , and so on. . . And by definition of  $\rightsquigarrow (\vec{\mathcal{T}}_{m,p})$ , the above probability is  $p^k$ . The desired result follows.  $\square$

To finish the proof of Proposition 5, we need the next estimates. For  $\delta > 0$ , let  $N(\mathcal{T}_m, \delta)$  be the size of the minimal set  $S \subset \llbracket 1, m \rrbracket$  such that  $\mathfrak{h}(\llbracket 1, m \rrbracket \setminus \bigcup_{i \in S} \downarrow(i)) \leq x$ . This covering number morally represents how many branches one need to delete in  $\mathcal{T}_m$  to have a forest of height at most  $x$ .

**Lemma 8.** *There exists a constant  $C$  such that for every  $m \in \mathbb{N}$ ,  $\delta > 0$ ,*

$$\mathbb{P}(N(\mathcal{T}_m, \delta\sqrt{m}) \geq 1/\delta^{100}) \leq C\delta^4,$$

*and such that for every  $m \in \mathbb{N}$ ,  $x > 0$ ,*

$$\mathbb{P}(\mathfrak{h}(\llbracket 1, m \rrbracket) \geq x\sqrt{m}) \leq C/x^4.$$

*Proof.* The second part is well known. The first may be proved following readily Aldous [2]. (Aldous only prove that some events hold with high probability but the same method still applies.) See also [6] where similar bounds are proved in a wider context. We omit the details.  $\square$

*Proof of Proposition 5.* Fix  $m, p, x > 0$ . Without loss of generality up to chose a larger  $C$  in Proposition 5, we may assume  $x, m$  large enough and  $p^{2/3}m \geq 1/x^2 \geq 1$ . Let  $l = (p^{2/3}m)^{-1/4}$ . Note that  $l \geq 1$ . We first consider the following event:

$$\mathfrak{h}(\llbracket 1, m \rrbracket) \leq l\sqrt{m} \quad ; \quad \forall i \geq 1, N(\mathcal{T}_m, l\sqrt{m}/(2^i)) \leq (2^i l)^{100}. \quad (\text{E})$$

By Lemma 8, we have for some universal constant  $C$ ,

$$\mathbb{P}(\text{E}) \geq 1 - Cl^{-4} - C \sum_{i \geq 1} 2^{-4i} l^{-4} \geq 1 - 2Cl^{-4} = 1 - 2Cp^{2/3}m. \quad (3)$$

Now we work given  $\mathcal{T}_m$  under (E). For  $S \subset \llbracket 1, m \rrbracket$ , let  $\downarrow(S) := \bigcup_{i \in S} \downarrow(i)$ . By (E), and since  $l \geq 1$ , we may consider some increasing sets  $\emptyset = S_0 \subset S_1 \subset S_2 \subset \dots$  such that for every  $i \geq 1$ ,

$$\mathfrak{h}(\llbracket 1, m \rrbracket \setminus \downarrow(S)) \leq l\sqrt{m}/2^i \quad \text{and} \quad \#(S_{i+1} \setminus S_i) \leq (2^i l)^{100}.$$

Also since  $l \geq 1$ , we may assume without loss of generality that for  $k \geq \log(m)$ ,  $S_k = \llbracket 1, m \rrbracket$ . By Lemma 6, for every  $k \geq 0$ ,

$$\max_{i \in S_{k+1}} \mathcal{L}(i) \leq 1 + \max_{i \in S_{k+1} \setminus S_k} \mathcal{L}(i, \llbracket 1, m \rrbracket \setminus \downarrow(S_k)) + \max_{i \in S_k} \mathcal{L}(i).$$

Summing the last inequality over all  $1 \leq k \leq \log(m)$  we get

$$\mathcal{L}(\vec{\mathcal{T}}_{m,p}) \leq \log m + \sum_{k=0}^{\log m} \max_{i \in S_{k+1} \setminus S_k} \mathcal{L}(i, \llbracket 1, m \rrbracket \setminus \downarrow(S_k)). \quad (4)$$

Moreover, by an union bound in Lemma 7, for every  $k \geq 0$ , writing  $S'_k := \llbracket 1, m \rrbracket \setminus \downarrow(S_k)$ ,

$$\begin{aligned} \mathbb{P}(\exists i \in S_{k+1} \setminus S_k, \mathcal{L}(i, S'_k) > 102h(S'_k)|\mathcal{T}_m) &\leq \#(S_{k+1} \setminus S_k)(ph(S'_k)m)^{101} \\ &\leq (2^k l)^{100} (pm^{3/2}l/2^k)^{101} \\ &\leq 2^{-k} p^{2/3} m, \end{aligned}$$

using  $l = (p^{2/3}m)^{-1/4}$  and  $p^{2/3}m \leq 1$  for the last inequality. By an union bound over all  $k \geq 1$  we get that, given  $\mathcal{T}_m$ , with probability at least  $1 - 2p^{2/3}m$ , for every  $k \geq 0, i \in S_{k+1} \setminus S_k$ , we have  $\mathcal{L}(i, S'_k) \leq 3h(S'_k)$ . Hence, by (4), given  $\mathcal{T}_m$  under (E), with probability at least  $1 - 2p^{2/3}m$ ,

$$\mathcal{L}(\vec{\mathcal{T}}_{m,p}) \leq \log m + \sum_{k=0}^{\log m} \max_{i \in S_{k+1} \setminus S_k} \mathcal{L}(i, S'_k) \leq \log m + \sum_{k=0}^{\infty} \frac{102\sqrt{m}l}{2^k} \leq \log m + 102\sqrt{m}l \leq x\sqrt{m},$$

where the last inequality holds when  $p^{2/3}m \geq 1/x^2$  and  $x, m$  are large enough. The desired result follows from (3) and the above inequality.  $\square$

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