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A.S. convergence for infinite colour Pólya urns associated with stable random walks

Arthur Blanc-Renaudie*

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Abstract

We answer Problem 11.1 of Janson [6] on Pólya urns associated with stable random walk. Our proof use neither martingales nor trees, but an approximation with a differential equation.

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1 Introduction

1.1 Model and motivations

We consider *single ball addition random walk (SBARW) Pólya urns* defined as follows. Let $d \in \mathbb{N}$. Let Δ be a Borel random variable on \mathbb{R}^d . Let $(\Delta_i)_{i \in \mathbb{N}}$ be independent copies of Δ . Let $X_1 = \Delta_1$. Then, independently for every $n \geq 2$, let U_n be uniform in $\{1, \dots, n-1\}$ and let $X_n = X_{U_n} + \Delta_n$. We are interested in the random measure as $n \rightarrow \infty$

$$\mu_n := \sum_{i=1}^n \delta_{X_i}.$$

Morally, we give colors $(X_i)_{i \in \mathbb{N}}$ to the balls of a classical Pólya Urn, which corresponds to positions of some random walks with displacement Δ . Since for $n \in \mathbb{N}$, X_n corresponds to the position of such a random walk after approximately $\log n$ steps, it is not hard to show, under natural assumptions for Δ , that X_n after proper renormalisation converge in law. Our problem is: Can we say the same thing for μ_n ? We show an almost sure (a.s.) convergence when Δ is in the stable regime, thus answering Problem 11.1 from Janson [6].

This model of Urn was first studied by Bandyopadhyay and Thacker [1, 2, 3] who first studied precisely the law of (X_n) and then showed the convergence of μ_n in distribution. At the same period, Mailler and Marckert [7], also proved this convergence for μ_n , and proved an almost sure convergence when Δ have finite exponential moment. Later, Janson [6] extended this almost sure convergence assuming only a second moment.

All those authors also studied another model of urn: the *deterministic addition random walk (DARW) Pólya urns* which is similar to the SBARW model. Although our method can be used without much change to study both models, we will focus for simplicity on the SBARW model.

Remark. I also believe the method of this paper work for several generalization of SBARW urns: non integer number of balls, adding several balls at once, non uniformly distributed displacement $(\Delta_i)_{i \in \mathbb{N}}$. . . However, I am not fully aware of the various urn motivations. So I decided instead to keep the setting as elementary as possible, to better explain the method.

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1.2 Main result and differential equation

In this paper we assume that Δ is in the stable regime. That is we assume that there exists $\alpha > 0$, and a random variable Λ on \mathbb{R}^d of law $\mathcal{L}(\Lambda)$ such that as $n \rightarrow \infty$,

$$\frac{1}{n^\alpha} \sum_{i=1}^n \Delta_i \xrightarrow{(d)} \Lambda. \quad (1)$$

Note that in this case there exists $\beta > 0$ such that as $n \rightarrow \infty$,

$$\mathbb{P}(\|\Delta\| > n^\beta) = o(1/n). \quad (2)$$

For every $a > 0$, and finite Borel measure μ on \mathbb{R}^d we define its renormalization $\Theta_a(\mu)$ as the unique Borel measure such that for every Borel set $B \subset \mathbb{R}^d$ we have $\Theta_a(\mu)(B/a) = \mu(B)$. For $n \in \mathbb{N}$, we define $\tilde{\mu}_n := \Theta_{n,0}(\mu) = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$. Note that $\Theta_a(\tilde{\mu}_n) = \frac{1}{n} \sum_{i=1}^n \delta_{X_i/a}$.

The main goal of this paper is to show the following theorem.

Theorem 1. *Under (1), almost surely, we have the following weak convergence as $n \rightarrow \infty$,*

$$\Theta_{\log(n)^\alpha}(\tilde{\mu}_n) \longrightarrow \mathcal{L}(\Lambda).$$

We use a completely a different method from the ones used in [1, 2, 3, 6, 7], which are based on martingales, Yule trees and branching random walk. Our proof use none of those tools, and is instead based on an approximation with a differential equation for $(\mu_n)_{n \in \mathbb{N}}$.

We can think of our urn problem as the following informal approximative differential equation:

$$\frac{d}{dt} \mu_t \approx \frac{1}{t} \mu_t * \Delta, \quad (3)$$

where $*$ denotes the convolution between two measures. To make use of this differential equation, we consider some times $0 = T_0 < T_1 < T_2 < \dots < T_n < \dots$, and use for $n \in \mathbb{N}$,

$$\mu_{T_n} = \sum_{i=1}^n (\mu_{T_i} - \mu_{T_{i-1}}).$$

This allows us to split the complex estimate of $(\mu_n)_{n \in \mathbb{N}}$ into many simpler ones. To this end, we follow (3), and show that approximatively for $i \in \mathbb{N}$,

$$\mu_{T_i} - \mu_{T_{i-1}} \approx (T_i - T_{i-1}) \frac{1}{t} \mu_t * \Delta. \quad (4)$$

We will detail how we prove this approximation in the next section.

For this approach to work, $(T_i)_{i \in \mathbb{N}}$ must be properly chosen. Indeed, the faster $(T_i)_{i \in \mathbb{N}}$ grows, the harder it is to study $\mu_n(T_i) - \mu_n(T_{i-1})$. On the other hand, the slower $(T_i)_{i \in \mathbb{N}}$ grows, the less precise we are in (4), and by sum the less precise we are on $(\mu_n)_{n \in \mathbb{N}}$.

This approach is very similar to the chaining method developed by Talagrand [9], which, with proper divisions, found many applications to prove nearly optimal bound of stochastic processes. Recently, the chaining method was also highly developed to study random geometry, and notably of trees constructed by stick-breaking (see e.g. the introduction of my thesis [4] for a detailed explanation on the method) or by aggregation (see e.g. Sénizergues thesis [8]). Since Pólya urns are closely related to recursive trees it is not surprising that the exact same methods can be used here.

1.3 Main ideas of the proof

First, let us introduce some notations and random variables. To ease the writing, the random variables defined below, unless mentioned otherwise are defined independent and independent of the previously defined random variables.

For $n \in \mathbb{N}$ let $T_n := \lfloor e^{3n^{1/3}} \rfloor$. Note that $(T_{n+1} - T_n)/T_{n+1} \sim n^{-2/3}$. Let $(R_n)_{n \in \mathbb{N}}$ be Bernoulli random variables with parameter $((T_{n+1} - T_n)/T_{n+1})_{n \in \mathbb{N}}$. Let $(\tilde{\Delta}_i)_{i \in \mathbb{N}}$ be copies of Δ . For $n \in \mathbb{N}$, let $Y_n := R_n \tilde{\Delta}_n$. For $n \in \mathbb{N}$, let \tilde{U}_n be uniform in $\{1, \dots, n\}$, and let $\tilde{X}_i := X_{U_n}$.

The following result is our main technical idea to prove Theorem 1. Indeed, it will give us that a.s. given $(X_i)_{i \in \mathbb{N}}$,

$$\tilde{X}_{T_n} \stackrel{(d)}{\approx} Y_1 + Y_2 + \dots + Y_n$$

and Theorem 1 will follow (see Section 3).

Proposition 2. *Almost surely $\mathbf{X} := (X_i)_{i \in \mathbb{N}}$ satisfy the following property:*

"For every $n \in \mathbb{N}$ large enough we can construct a coupling between $\tilde{X}_{T_n} + Y_n$ and $\tilde{X}_{T_{n+1}}$ such that those two variable are at distance for $\|\cdot\|_\infty$ at least n^{-2} with probability at most $3n^{-4/3}$."

Note that, in the above, the existence of the coupling is random as we see for $n \in \mathbb{N}$, the law of $\tilde{X}_{U_n} + Y_n$ and of $\tilde{X}_{U_{n+1}}$ as random variables on the set of probability measure on \mathbb{R}^d . And the law of those random variables are determined by $(X_i)_{i \in \mathbb{N}}$.

Morally the principle behind Proposition 2 is the following: with probability at least $1 - n^{-4/3}$ a uniform variable in $(X_i)_{1 \leq i \leq T_{n+1}}$ is either in $(X_i)_{1 \leq i \leq T_n}$, or it is in $(X_i)_{T_n < i \leq T_{n+1}}$ and $U_i \leq T_n$. So $\tilde{X}_{T_{n+1}} \stackrel{(d)}{\approx} \tilde{X}_{T_n} + Y_n$.

To prove Proposition 2 we split \mathbb{R}^d into small box of the form, for $i_1, \dots, i_d \in \mathbb{Z}$,

$$\square_{i_1, i_2, \dots, i_d}^n := \left[\frac{i_1}{n^2}, \frac{i_1 + 1}{n^2} \right) \times \left[\frac{i_2}{n^2}, \frac{i_2 + 1}{n^2} \right) \times \dots \times \left[\frac{i_d}{n^2}, \frac{i_d + 1}{n^2} \right).$$

And we count given $(X_i)_{1 \leq i \leq T_n}$ the number of $(X_i)_{i \leq T_{n+1}}$ that are in the small box $\square_{i_1, i_2, \dots, i_d}^n$. Note that this number is exactly equal to

$$T_{n+1} \mathbb{P} \left(\tilde{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right). \quad (5)$$

Morally, to estimate this number we focus on the $T_n < i \leq T_{n+1}$ such that $U_i \leq T_n$. And for all $T_n < i, j \leq T_{n+1}$, given $U_i, U_j \leq T_n$ and $(X_i)_{1 \leq i \leq T_n}$, the conditional random variables X_i, X_j are independent and have law $\tilde{X}_{T_n} + \Delta$. As a result, the number (5) is essentially a sum of independent random variables and so is concentrated, and we have the following approximation,

$$\mathbb{P} \left(\tilde{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) \approx \mathbb{P} \left(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right).$$

Rigorously, we show the next result which directly implies the existence of the desired couplings.

Proposition 3. *Almost surely as $n \rightarrow \infty$*

$$\sum_{i_1, i_2, \dots, i_d \in \mathbb{Z}^d} \left| \mathbb{P} \left(\tilde{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) - \mathbb{P} \left(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) \right| \leq 3/n^{4/3}.$$

2 Proof of Proposition 3

2.1 The bulk : Proof of Lemma 4

Let $\gamma > 0$. The aim of this section is to prove the next result.

Lemma 4. For $n \in \mathbb{N}$, let $\diamond_n := [-n^\gamma, n^\gamma]^d \cap \mathbb{Z}^d$. Almost surely as $n \rightarrow \infty$

$$\sum_{i_1, i_2, \dots, i_d \in \diamond_n} \left| \mathbb{P} \left(\tilde{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) - \mathbb{P} \left(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) \right| \leq 2.9/n^{4/3}.$$

This implies together with the next result, which is proved in the next section, Proposition 3.

Lemma 5. If $\gamma > 0$ is large enough, almost surely as $n \rightarrow \infty$

$$\mathbb{P} \left(\|\tilde{X}_{T_{n+1}}\| > n^\gamma \mid \mathbf{X} \right) = o(n^{-4/3}) \quad \text{and} \quad \mathbb{P} \left(\|\tilde{X}_{T_n} + Y_n\| > n^\gamma \mid \mathbf{X} \right) = o(n^{-4/3}).$$

Before proving Lemma 4, let us recall a consequence of Bernstein's inequality tailored for our purpose (see e.g. [5] Section 2.8).

Lemma 6 (Bernstein's inequality). Let $n \in \mathbb{N}$. Let $(X_i)_{1 \leq i \leq n}$ be a family of independent Bernoulli random variables with parameter $(p_i)_{1 \leq i \leq n}$. Let $v \geq \sum_{i=1}^n p_i$. Let $S = \sum_{i=1}^n X_i - \mathbb{E}[X_i]$. For every $t \geq 0$,

$$\mathbb{P}(|S| \geq \sqrt{2vt} + t) \leq e^{-t}.$$

Let us now prove Lemma 4. We first consider a modification of $\tilde{X}_{T_{n+1}}$. For $n \in \mathbb{N}$, let $(\bar{U}_i^n)_{n \in \mathbb{N}}$ be independent uniform random variables in $\{1, \dots, n\}$ such that whenever $U_i \leq n$, $\bar{U}_i^n = U_i$. Then let

$$\bar{X}_i^n := X_i \quad \text{if } i \leq T_n \quad \text{and let} \quad \bar{X}_i^n := X_{\bar{U}_i^n} + \Delta_i \quad \text{if } i > T_n.$$

Then let $\bar{X}_{T_{n+1}}$ be a uniform random variable in $(\bar{X}_i^n)_{1 \leq i \leq T_{n+1}}$. Note that

$$\mathbb{P}(\tilde{X}_{T_{n+1}} \neq \bar{X}_{T_{n+1}}) \leq \frac{1}{T_{n+1}} \sum_{T_n < i \leq T_{n+1}} \mathbf{1}_{U_i \in (T_n, T_{n+1}]}$$

The above sum is a sum of independent Bernoulli random variables with parameter bounded by $(T_{n+1} - T_n)/T_{n+1}$. We can thus apply Lemma 6 with $v = (T_{n+1} - T_n)^2/T_{n+1}$ and $t = T_{n+1}/n$. We get that a.s. as $n \rightarrow \infty$,

$$\mathbb{P}(\tilde{X}_{T_{n+1}} \neq \bar{X}_{T_{n+1}}) \leq (1 + o(1)) \frac{(T_{n+1} - T_n)^2}{T_{n+1}^2} \sim n^{-4/3}.$$

Thus to prove Lemma 4, it suffices to prove the following result.

Lemma 7. Almost surely as $n \rightarrow \infty$

$$\sum_{i_1, i_2, \dots, i_d \in \diamond_n} \left| \mathbb{P} \left(\bar{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) - \mathbb{P} \left(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) \right| \leq 1.8/n^{4/3}.$$

Proof. Note that conditionally on $(X_i)_{1 \leq i \leq T_n}$, for every $i_1, i_2, \dots, i_d \in \mathbb{Z}^n$,

$$T_{n+1} \mathbb{P} \left(\bar{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X} \right) = \sum_{i=1}^{T_{n+1}} \mathbf{1}_{\bar{X}_i^n \in \square_{i_1, i_2, \dots, i_d}^n},$$

is a sum of independent Bernoulli random variables. Furthermore, conditionally on $(X_i)_{1 \leq i \leq T_n}$, the random variables $\bar{X}_{T_{n+1}}$ and $\tilde{X}_{T_n} + Y_n$ have exactly the same law. As a result,

$$\begin{aligned} \frac{1}{T_{n+1}} \sum_{i=1}^{T_{n+1}} \mathbb{P}(\bar{X}_i^n \in \square_{i_1, i_2, \dots, i_d}^n \mid (X_i)_{1 \leq i \leq T_n}) &= \mathbb{P}(\bar{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid (X_i)_{1 \leq i \leq T_n}) \\ &= \mathbb{P}(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X}) \end{aligned}$$

Thus, we may apply Bernstein's inequality (Lemma 6) with

$$v_{i_1, i_2, \dots, i_d}^n := T_{n+1} \mathbb{P}(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X})$$

to estimate

$$\delta_{i_1, i_2, \dots, i_d}^n := \left| \mathbb{P}(\bar{X}_{T_{n+1}} \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X}) - \mathbb{P}(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X}) \right|.$$

We choose for $n \in \mathbb{N}$, $t_n := \log(n)^2$, and we obtain for every $n \in \mathbb{N}$, $i_1, i_2, \dots, i_d \in \diamond_n$,

$$\mathbb{P}(T_{n+1} \delta_{i_1, i_2, \dots, i_d}^n > \sqrt{2v_{i_1, i_2, \dots, i_d}^n \log(n)^2} + \log(n)^2) \leq n^{-\log(n)}.$$

This is summable over all $n \in \mathbb{N}$, $i_1, i_2, \dots, i_d \in \diamond_n$, so by the Borel–Cantelli lemma almost surely for every n large enough, $i_1, i_2, \dots, i_d \in \diamond_n$,

$$T_{n+1} \delta_{i_1, i_2, \dots, i_d}^n \leq \sqrt{2v_{i_1, i_2, \dots, i_d}^n \log(n)^2} + \log(n)^2.$$

Summing over all $i_1, i_2, \dots, i_d \in \diamond_n$ we get writing $|\diamond_n|$ for the cardinal of \diamond_n , and writing $\delta_n := \sum_{i_1, i_2, \dots, i_d \in \diamond_n} \delta_{i_1, i_2, \dots, i_d}^n$ for the sum of the lemma,

$$T_{n+1} \delta_n \leq \sum_{i_1, i_2, \dots, i_d \in \diamond_n} \sqrt{2v_{i_1, i_2, \dots, i_d}^n \log(n)^2} + |\diamond_n| \log(n)^2.$$

Then using the concavity of $x \mapsto \sqrt{x}$,

$$T_{n+1} \delta_n \leq \sqrt{|\diamond_n|} \left(2 \sum_{i_1, i_2, \dots, i_d \in \diamond_n} v_{i_1, i_2, \dots, i_d}^n \log(n)^2 \right)^{1/2} + |\diamond_n| \log(n)^2.$$

Moreover, we have

$$\sum_{i_1, i_2, \dots, i_d \in \diamond_n} v_{i_1, i_2, \dots, i_d}^n = T_{n+1} \sum_{i_1, i_2, \dots, i_d \in \diamond_n} \mathbb{P}(\tilde{X}_{T_n} + Y_n \in \square_{i_1, i_2, \dots, i_d}^n \mid \mathbf{X}) \leq T_{n+1}.$$

Therefore,

$$T_{n+1} \delta_n \leq \sqrt{|\diamond_n|} \sqrt{2T_{n+1} \log(n)^2} + |\diamond_n| \log(n)^2.$$

The desired result follows as $|\diamond_n| = O(n^{d\gamma})$ and $T_{n+1} = e^{3n^{1/3}(1+o(1))}$. \square

2.2 The tail : Proof of Lemma 5

We now prove Lemma 5. By (2), note that if $\gamma > 2\beta$ then $\mathbb{P}(\|Y_n\| > n^\gamma) = O(n^{-2})$ so it suffices to prove the first part of the lemma. Also writing for $n \in \mathbb{N}$,

$$E_n := \mathbb{E} \left[\mathbb{P} \left(\|\tilde{X}_{T_{n+1}}\| > n^\gamma \mid \mathbf{X} \right) \right],$$

we have by Markov inequality for every $n \in \mathbb{N}$,

$$\mathbb{P} \left(\mathbb{P} \left(\|\tilde{X}_{T_{n+1}}\| > n^\gamma \mid \mathbf{X} \right) > n^2 E_n \right) \leq 1/n^2.$$

So almost surely for every $n \in \mathbb{N}$ large enough, $\mathbb{P} \left(\|\tilde{X}_{T_{n+1}}\| > n^\gamma \mid \mathbf{X} \right) \leq n^2 E_n$.

Hence it is enough to upper-bound E_n . We have,

$$E_n = \mathbb{E} \left[\mathbb{P} \left(\|\tilde{X}_{T_{n+1}}\| > n^\gamma \mid \mathbf{X} \right) \right] = \mathbb{E} \left[\frac{1}{T_{n+1}} \sum_{i=1}^{T_{n+1}} \mathbf{1}_{\|X_i\| > n^\gamma} \right] = \frac{1}{T_{n+1}} \sum_{i=1}^{T_{n+1}} \mathbb{P}(\|X_i\| > n^\gamma).$$

Thus since $\log(T_n) \sim 3n^{1/3}$, to prove Lemma 5 it suffices to prove the following result.

Lemma 8. *If γ is large enough, as $n \rightarrow \infty$,*

$$\mathbb{P}(\|X_n\| > \log(n)^\gamma) = o(\log(n)^{-15}).$$

Proof. it is well known that for every $n \in \mathbb{N}$, X_n has the same law as $\sum_{i=1}^n B_i \Delta_i$, where $(B_i)_{i \in \mathbb{N}}$ is a family of Bernoulli random variables with parameter $(1/i)_{i \in \mathbb{N}}$ independent of $(\Delta_i)_{i \in \mathbb{N}}$. With this representation, for X_n to be larger than $\log(n)^\gamma$, there must either be at least $\log(n)^2$ term or else one of the term is larger than $\log(n)^{\gamma-2}$. So,

$$\mathbb{P}(\|X_n\| > \log(n)^\gamma) \leq \mathbb{P} \left(\sum_{i=1}^n B_i > \log(n)^2 \right) + \log(n)^2 \mathbb{P}(\Delta > \log(n)^{\gamma-2}).$$

By classical concentration inequalities on sum of independent random variables the first right-hand side term is $o(\log(n)^{-15})$ as $n \rightarrow \infty$ (one may for instance use Bernstein's inequality). And by (2), for γ large enough, $\mathbb{P}(\Delta > \log(n)^{\gamma-2}) = o(\log(n)^{-17})$. \square

3 Proof of Theorem 1

Before proving Theorem 1, let us check two easy results.

Lemma 9. *As $n, m \rightarrow \infty$ with $n \sim m$, writing d_{TV} for total variation distance, $d_{TV}(\tilde{\mu}_n, \tilde{\mu}_m) \rightarrow 0$.*

Proof of Lemma 9. Assume $n \leq m$. Since for every $1 \leq i \leq n$, $\tilde{X}_n = X_i$ with probability $1/n$ and $\tilde{X}_m = X_i$ with probability $1/m$ we get, $d_{TV}(\tilde{\mu}_n, \tilde{\mu}_m) \leq 1 - n/m \rightarrow 0$. \square

Lemma 10. *As $n \rightarrow \infty$, writing $S_n := Y_1 + Y_2 + \dots + Y_{n-1}$, $S_n / \log(T_n)^\alpha \xrightarrow{(d)} \Lambda$.*

Proof. Recall that for $n \in \mathbb{N}$, $S_n \stackrel{(d)}{=} \sum_{i=1}^n B_i \Delta_i$, where $(B_i)_{i \in \mathbb{N}}$ are independent Bernoulli random variables of parameter $((T_{n+1} - T_n)/T_{n+1})_{n \in \mathbb{N}}$. So given (1), it suffices to show that $n \rightarrow \infty$, $\sum_{i=1}^{n-1} B_i / \log(T_n)$ converge in probability toward 1. To this end, by the law of large number it suffices to estimate $\mathbb{E}[\sum_{i=1}^{n-1} B_i]$. We have with elementary calculus

$$\mathbb{E} \left[\sum_{i=1}^{n-1} B_i \right] = \sum_{i=1}^{n-1} \frac{T_{n+1} - T_n}{T_{n+1}} \underset{n \rightarrow \infty}{\sim} \sum_{i=1}^{n-1} \int_{T_i}^{T_{i+1}} \frac{dx}{x} \underset{n \rightarrow \infty}{\sim} \log(T_n).$$

This concludes the proof. \square

We first formulate a practical technical consequence of Proposition 2. Let \mathcal{F} be the set of all functions $f : \mathbb{R}^d \mapsto \mathbb{R}$, 1-Lipschitz for the uniform distance, bounded by 1, and of bounded support. By Proposition 2, a.s. as $n \rightarrow \infty$,

$$\max_{f \in \mathcal{F}} \left| \mathbb{E} \left[f(\tilde{X}_{T_{n+1}}) \middle| \mathbf{X} \right] - \mathbb{E} \left[f(\tilde{X}_{T_n} + Y_n) \middle| \mathbf{X} \right] \right| = O(1/n^{4/3}).$$

It follows by sum that a.s. as $n \rightarrow \infty$, $m \in \mathbb{N}$

$$\max_{f \in \mathcal{F}} \left| \mathbb{E} \left[f(\tilde{X}_{T_{n+m}}) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\tilde{X}_{T_n} + \sum_{i=0}^{m-1} Y_{n+i} \right) \middle| \mathbf{X} \right] \right| = O \left(\sum_{i=0}^{m-1} (n+i)^{-4/3} \right) = O(n^{-1/3}),$$

which we may rewrite into: a.s. as $n \leq m \rightarrow \infty$,

$$\max_{f \in \mathcal{F}} \left| \mathbb{E} \left[f(\tilde{X}_{T_m}) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\tilde{X}_{T_m} + S_m - S_n \right) \middle| \mathbf{X} \right] \right| = O(n^{-1/3}).$$

Since for every m large enough, $f \in \mathcal{F}$, the function $x \mapsto f(x/\log(T_m)^\alpha)$ is also in \mathcal{F} , this implies that a.s. as $n \leq m \rightarrow \infty$,

$$\max_{f \in \mathcal{F}} \left| \mathbb{E} \left[f \left(\frac{\tilde{X}_{T_m}}{\log(T_m)^\alpha} \right) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\frac{\tilde{X}_{T_m} + S_m - S_n}{\log(T_m)^\alpha} \right) \middle| \mathbf{X} \right] \right| = O(n^{-1/3}).$$

Hence, since \tilde{X}_{T_n}, S_n are almost surely bounded, by using the Lipschitz property, a.s. as $n \rightarrow \infty$,

$$\limsup_{m \rightarrow \infty} \max_{f \in \mathcal{F}} \left| \mathbb{E} \left[f \left(\frac{\tilde{X}_{T_m}}{\log(T_m)^\alpha} \right) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\frac{S_m}{\log(T_m)^\alpha} \right) \middle| \mathbf{X} \right] \right| = O(n^{-1/3}).$$

Since the left-hand side term above does not depend on n , and since S_m and \mathbf{X} are independent, we get that a.s. for every $f \in \mathcal{F}$ as $m \rightarrow \infty$,

$$\mathbb{E} \left[f \left(\frac{\tilde{X}_{T_m}}{\log(T_m)^\alpha} \right) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\frac{S_m}{\log(T_m)^\alpha} \right) \right] \rightarrow 0.$$

Moreover by the Portmanteau Theorem and by Lemma 10 the above second expectation converges toward $\mathbb{E}[f(\Lambda)]$. So must the first: A.s. for every $f \in \mathcal{F}$ as $m \rightarrow \infty$,

$$\mathbb{E} \left[f \left(\frac{\tilde{X}_{T_m}}{\log(T_m)^\alpha} \right) \middle| \mathbf{X} \right] \rightarrow \mathbb{E}[f(\Lambda)]. \quad (6)$$

Also as $n \rightarrow \infty$, writing m_n for the larger integer with $T_{m_n} \leq n$, we have $T_{m_n} \sim n$ so by Lemma 9, for every $f \in \mathcal{F}$ as $n \rightarrow \infty$,

$$\mathbb{E} \left[f \left(\frac{\tilde{X}_{T_{m_n}}}{\log(T_{m_n})^\alpha} \right) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\frac{\tilde{X}_n}{\log(T_{m_n})^\alpha} \right) \middle| \mathbf{X} \right] \rightarrow 0.$$

Then using that all $f \in \mathcal{F}$ are Lipschitz and of bounded support, since as $n \rightarrow \infty$, $\log(T_{m_n}) \rightarrow \infty$, and $\log(T_{m_n}) \sim \log(n)$, a.s. for every $f \in \mathcal{F}$ as $n \rightarrow \infty$,

$$\mathbb{E} \left[f \left(\frac{\tilde{X}_{T_{m_n}}}{\log(T_{m_n})^\alpha} \right) \middle| \mathbf{X} \right] - \mathbb{E} \left[f \left(\frac{\tilde{X}_n}{\log(n)^\alpha} \right) \middle| \mathbf{X} \right] \rightarrow 0.$$

To conclude we finally get that by (6) a.s. for every $f \in \mathcal{F}$ as $n \rightarrow \infty$,

$$\mathbb{E} \left[f \left(\frac{\tilde{X}_n}{\log(n)^\alpha} \right) \middle| \mathbf{X} \right] \rightarrow \mathbb{E}[f(\Lambda)].$$

Theorem 1 follows by the Portmanteau theorem.

References

- [1] A. Bandyopadhyay and D. Thacker. Rate of convergence and large deviation for the infinite color Pólya urn schemes. *Statistics & Probability Letters*, 92:232–240, 2014.
- [2] A. Bandyopadhyay and D. Thacker. A new approach to Pólya urn schemes and its infinite color generalization. arXiv:1606.05317, 2016.
- [3] A. Bandyopadhyay and D. Thacker. Pólya urn schemes with infinitely many colors. *Bernoulli*, pages 3243–3267, 2017.
- [4] A. Blanc-Renaudie. *Scaling limits of inhomogeneous continuum random trees and graphs*. PhD thesis, Sorbonne Université, 2022.
- [5] S. Boucheron, G. Lugosi, and P. Massart. *Concentration Inequalities. A Nonasymptotic Theory of Independence*. Oxford university press, 2013.
- [6] S. Janson. A.s. convergence for infinite colour Pólya urns associated with random walks. *Arkiv för Matematik*, 59.1:87–123, 2021.
- [7] J-F. Marckert. and C. Mailler. Measure-valued Pólya urn processes. *Electronic Journal of Probability*, 22, 2017.
- [8] D. Sénizergues. *Structures arborescentes aléatoires : recollements d’espaces métriques et graphes stables*. PhD thesis, Université Sorbonne Paris cité., 2019.
- [9] Michel Talagrand. *The generic chaining*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. Upper and lower bounds of stochastic processes.