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# A hybrid shell-beam element for straight thin-walled tubular structures

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## Abstract

A novel beam model is proposed in order to consider deformations of its cross-section in the context of straight thin-walled tubes. For this purpose, the straight beam kinematics is enriched by addition of orthogonal shell-type displacement field of the tube section. **The considered displacement field is composed of three contributions: classical linear beam and linear shell terms** in conjunction with the non-linear coupling between local deformation of the section and its global rotation. Both Euler-Bernoulli and Love-Kirchhoff hypotheses for beam and shell kinematics, respectively, are adopted as well as the thin-walled assumption. First-order shear deformation for the radial variable and Fourier expansion in terms of the circumferential variable are also considered for the mid-surface displacement field. Then, the stress tensor is obtained under plane stress conditions. The virtual power principle is finally used to obtain the equations of motion satisfied by the corresponding generalized forces. Afterwards, an explicit updated Lagrangian Finite-Element approach using a lumped mass matrix is proposed for solving the tube governing equations and the stability condition of the time integration is given. Test-cases are then chosen to assess the present tube finite-element. Both static and dynamic problems are considered. First, the proposed model is compared to analytical solutions. Finally, a tube subjected to a distributed patch loading is studied. The influence of the number of Fourier modes, of warping and coupling terms is examined. The proposed model makes it possible to retrieve classical shell solution of the cross-section deformation with significant computational savings.

**Keywords:** Beam kinematics, shell kinematics, straight thin-walled tubes, explicit time integration, Finite-Element method

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## 1. Introduction

Piping systems are widely used in many industrial energy and transportation applications. Under operational or accident conditions, pipelines can be subjected to a wide variety of static or dynamic loads. The dynamic behavior of such structures has therefore received much attention in recent decades. Furthermore, the presence of fluid inside the pipes also has a significant influence on the pipe motion. The fluid-structure interaction (FSI) effects in piping systems are summarized in following comprehensive textbooks [1, 2] and reviews [3, 4]. Two main approaches have been developed in order to represent the mechanical behavior of pipelines based on shell or beam models. A very large literature exists for describing models issued from different theories. Shell models describe the pipe mid-surface motion in order to depict the cross-sectional evolution. However, the high computational cost associated with these models prohibits their use for realistic complex piping systems. In contrast, beam models describe the pipe center line motion considering rigid cross-sections rendering these models very computationally efficient without taking account of cross-sectional evolutions. What is more, some piping elements such as elbows are known since the pioneer work of von Kàrmàn [5] to present significant cross-section deformations under bending. As a consequence, formulations able to represent the pipe cross-section deformations at an affordable computational cost are of strong interest.

For this purpose, many researchers have previously proposed formulations based on finite elements for both curved and straight pipes [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In addition, Weicker *et al.* [14] have reviewed much of the

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17 aforementioned research. For example, in order to consider the pipe radial expansion into their formulation, Ohtsubo  
18 & Watanabe [6] use the thin Love-Kirchhoff shell kinematic constraints in conjunction with Fourier series expan-  
19 sions in terms of the tangential variable for longitudinal, tangential and radial displacements of the pipe mid-surface.  
20 Furthermore, Bathe & Almeida [7] have proposed a formulation based on the combination between beam and shell  
21 kinematics where cross-sectional deformations are expressed in Fourier series. In addition, warping deformations are  
22 neglected and the inextensibility in the circumferential direction is assumed by the authors. Additional improvements  
23 have been proposed in order to consider interaction effects between straight and curved pipes [18] and pressure stiffen-  
24 ing effects [19]. Then, Militello & Huespe [9] consider warping displacements into the formulation initially proposed  
25 by Bathe & Almeida in [7]. What is more, Abo-Elkhier [10] consider a more complete displacement-strain relations  
26 into this formulation. Finally, as reviewed in [14], there is a great difference among the previously mentioned works.  
27 For example, incomplete Fourier expansions are considered as well as some displacements as warping are neglected.  
28 For instance, only the second mode of ovalization is considered in [8, 11]. For more generality, complete Fourier  
29 series for all three displacements of the mid-surface are adopted in [14, 15] within the framework of the approach pro-  
30 posed by Ohtsubo & Watanabe [6]. The obtained formulation considers warping deformations, variations of tangential  
31 and longitudinal displacements in the longitudinal and transverse directions as well as the pipe radial expansion. In  
32 addition, Attia *et al.* [16] have compared three different interpolating schemes for the displacement fields in the cir-  
33 cumferential direction showing the good behavior of the Fourier series expansion in the context of straight pipes. **More**  
34 **recent researches have been published concerning enriched beam elements. For example, a non-linear beam model**  
35 **has been proposed in order to take into account the Poisson effect [20]. Moreover, a high-order beam model has been**  
36 **recently derived to consider deforming cross-sections including both Poisson effect and warping deformations [21].**  
37 **In addition, enhanced beam kinematics is also considered in [22] to allow distortion of the cross section. In addi-**  
38 **tion, a higher-order beam model able to accurately represent the buckling behaviour of three-dimensional thin-walled**  
39 **structures has been proposed in [23]. Furthermore, a geometrically exact cross-section deformable curved thin-walled**  
40 **beam finite element is proposed in [24]. In the aforementioned research, non-linear or high-order beam models are**  
41 **used for the derivation of the elements. A review of beam models can be found in the textbook [25].**

42 A new non-linear tube model is here proposed for straight thin-walled pipes. The corresponding displacement  
43 field is taken as a combination of three contributions:

- 44 • an Euler-Bernoulli beam model (for the tube axis motion),
- 45 • a Love-Kirchhoff shell kinematics (for the mid-surface evolution) in order to consider both warping and evolu-  
46 tion of the cross-section,
- 47 • a non-linear coupling between local deformation of the cross-section and the rotation of the tube axis.

48 Thanks to the consideration of this non-linear coupling, the pipe inertia evolve with cross-section deformation, possi-  
49 ble cross-section ovalization is then ensured. Furthermore, the thin-walled cylindrical Flügge shell theory is adopted.  
50 In addition, complete Fourier series are used for the shell kinematics respectively to the tangential (or circumferential)  
51 variable. As in [12], the coefficients in Fourier expansion corresponding to rigid-body motion are excluded to avoid  
52 redundancy with the beam displacements and rotations.

53 In conjunction with this new model, an explicit solver based on an updated Lagrangian Finite-Element approach  
54 is also developed here. Focus is given to the explicit time integration for considering fast-transient events occurring in  
55 accidental conditions. Due to the use of explicit schemes, additional questions arise especially concerning the mass  
56 matrix as well as the stability condition. As for all elements involving both translational and rotational degrees of  
57 freedom (DOFs), a special attention has to be given to the lumped mass matrix. Finally, stability conditions are here  
58 simply estimated using natural frequencies of both beams and shells.

59 The paper is organized as follows. Section 2 is devoted to the derivation of the proposed tube model. The tube  
60 kinematics decomposed into both beam and shell terms is presented as well as the non-linear coupling between the  
61 tube cross-section deformations and the tube axis rotations. The assumptions adopted for the derivation of the present  
62 tube model are also detailed. Afterwards, explicit finite-element resolution of the proposed model is presented in  
63 Section 3. In particular, the lumped mass matrix is described and stability conditions are given. The tube model is  
64 then assessed on several problems involving a variety of loads in Section 4. Both static and dynamic problems are  
65 examined. A series of test-cases where analytical solutions are available is first considered. Finally, the proposed tube

66 model is assessed on a distributed patch load via the comparison with shell elements. In particular, influence of the  
 67 number of Fourier modes is analyzed as well as influence of warping and of non-linear coupling.

## 68 2. Theoretical model: kinematics, strain-displacement relations and equations of motion

69 This section is devoted to the derivation of the model proposed here for straight thin-walled tubular geometries.  
 70 For this purpose, kinematics is decomposed into beam and shell contributions. Conditions of Euler-Bernoulli for  
 71 the beam contribution and Love-Kirchhoff for the shell description are assumed. In addition, a non-linear coupling  
 72 between cross-section deformation and rotation of the tube axis is considered. Finally, the tube equations of motion  
 73 obtained using the virtual power principle are given.

### 74 2.1. Enriched beam kinematics of a straight tube

75 Let us consider a straight tube of length  $L$ , mean radius  $a$  and thickness  $e$  along the axis  $\mathbf{E}_x$  as depicted in Fig.  
 (1). The axial, circumferential and radial (with respect to the mean radius) coordinates are denoted by  $x$ ,  $\theta$  and  $z$ ,

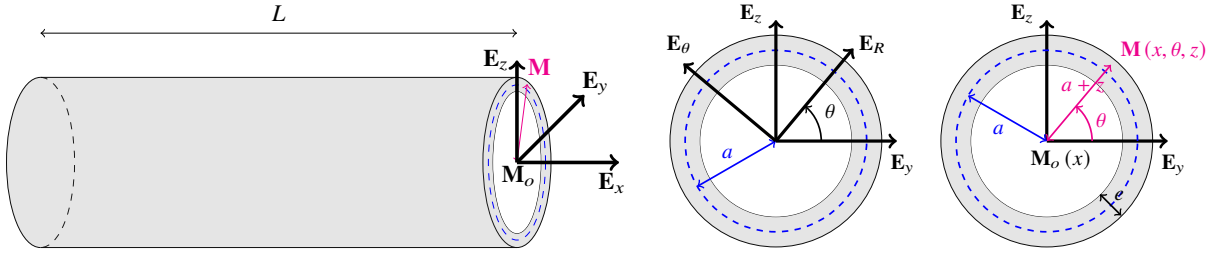


Figure 1: Sketch of a straight pipe: flat view (left), cross-sectional view - cylindrical coordinate system (center), pipe mid-surface, i.e.  $z = 0$ , and thickness (right).

76 respectively, while  $t$  denotes the time.

77  
 78 The kinematics is here decomposed into beam-type and shell-type contributions. The beam motion is given  
 79 by displacements of the tube axis, i.e.  $U_o(t, x)$ ,  $W_o(t, x)$  and  $V_o(t, x)$  expressed in the basis  $(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z)$  and by  
 80 rotations of section, i.e.  $\mathbf{R}(t, x)$ . Thereafter, the shell-type kinematics is associated with the relative displacements  
 81 with respect to the rotated section, i.e.  $u(t, x, \theta, z)$ ,  $w(t, x, \theta, z)$  and  $v(t, x, \theta, z)$  expressed in the basis  $(\mathbf{E}_x, \mathbf{E}_R, \mathbf{E}_\theta)$ ,  
 where  $-e/2 \leq z \leq e/2$  (see Figs. (2) and (3)). These displacements represent respectively the warping (associated

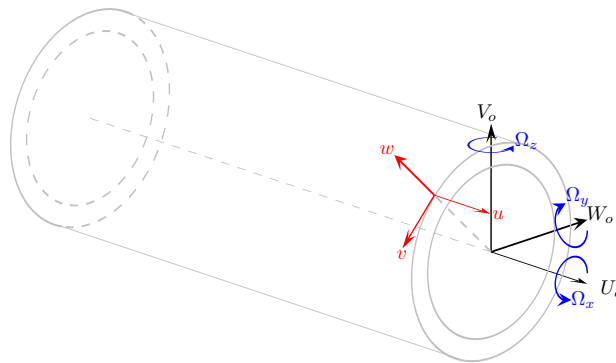


Figure 2: Beam and shell displacements:  $U_o$ ,  $V_o$ ,  $W_o$ ,  $\Omega_x$ ,  $\Omega_y$  and  $\Omega_z$  for the beam kinematics and  $u$ ,  $v$  and  $w$  for the shell kinematics.

82 with  $u$ ) and the ovalization (associated with  $v$  and  $w$ ) of the deformed section, (see Figs. (2) and (3)).  
 83

84 The coordinates of the point denoted by  $\mathbf{M}(x, \theta, z)$  in the initial state and the coordinates of the point  $\mathbf{m}(t, x, \theta, z)$   
 85 in the deformed state are respectively given by:

$$\begin{aligned} \mathbf{M} &= \mathbf{M}_o + (a + z) \mathbf{E}_R = x \mathbf{E}_x + (a + z) \mathbf{E}_R \\ \mathbf{m} &= \mathbf{M} + \mathbf{U} = (x + U_o) \mathbf{E}_x + V_o \mathbf{E}_z + W_o \mathbf{E}_y + \mathbf{R}(t, x) \cdot \left( (a + z + w) \mathbf{E}_R + u \mathbf{E}_x + v \mathbf{E}_\theta \right) \end{aligned} \quad (1)$$

86 The rotation of the section is considered to be small, therefore:

$$\mathbf{R}(t, x) = \mathbf{I} + \boldsymbol{\Omega}(t, x), \quad \boldsymbol{\Omega} \cdot \mathbf{V} = (\Omega_x(t, x) \mathbf{E}_x + \Omega_y(t, x) \mathbf{E}_y + \Omega_z(t, x) \mathbf{E}_z) \wedge \mathbf{V} \quad (2)$$

87 where  $\mathbf{V}$  is any 3-D vector and  $\mathbf{I}$  the identity matrix.

88  
 89 Even if a full 3-D motion has been developed as described in [26], only an in-plane motion in  $(\mathbf{E}_x, \mathbf{E}_z)$  is considered  
 90 in the following in order to simplify the presentation of the new model, i.e.  $W_o = 0$  and  $\Omega_x = \Omega_z = 0$  (see Fig. (3)).  
 91 Assuming small perturbations, linear strain in terms of displacement is considered. However, in order to take into  
 92 account the ovalization of the section in a straight tube, a non-linear coupling between the beam-type rotation and the  
 93 shell-type displacements is also introduced. Only the coupling between the cross-section ovalization driven by  $w$  and  
 94  $v$  and the rotation of the tube axis, i.e.  $\Omega_y$ , is taking into account, the coupling involving the warping cross-section  
 95 being here neglected. As it is explained in the following, the consideration of this coupling ensures that the inertia of  
 96 the section can evolve. In addition, the present motion considers the dynamic behavior as the proposed tube model  
 97 is a first step for the simulation of fast-transient events involving fluid-structure interaction in piping systems with  
 deformable cross-sections.

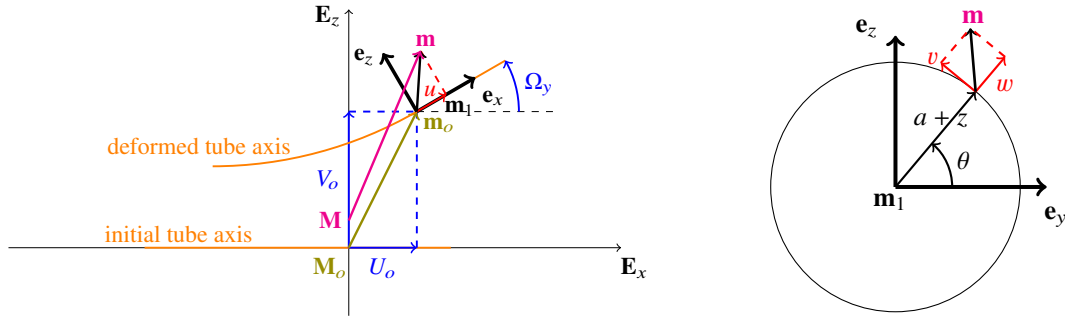


Figure 3: In-plane beam motion (left) and shell displacement in the tube cross-section (right).

98  
 99 The current position,  $\mathbf{m}(t, x, \theta, z)$ , is written in the cylindrical basis  $(\mathbf{E}_x, \mathbf{E}_R, \mathbf{E}_\theta)$  as:

$$\mathbf{m} = (x + U_o + u + [(a + z + w) \sin \theta + v \cos \theta] \Omega_y) \mathbf{E}_x + (a + z + V_o \sin \theta + w) \mathbf{E}_R + (V_o \cos \theta + v) \mathbf{E}_\theta \quad (3)$$

100 The non-linear coupling is described in the axial component by the term  $h\Omega_y$  with  $h = w \sin \theta + v \cos \theta$ . The derivatives  
 101 of the position vector  $\mathbf{m}$  with respect to the coordinates  $x$ ,  $\theta$  and  $z$  are detailed in Appendix A as well as the strain-

102 displacement relations which are given in Eq. (A.1) and summarized in the following:

$$\left\{ \begin{array}{l}
 \varepsilon_{xx} = \frac{\partial U_o}{\partial x} + (a+z) \sin \theta \frac{\partial \Omega_y}{\partial x} + \frac{\partial u}{\partial x} + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial x} + h \frac{\partial \Omega_y}{\partial x} \\
 \varepsilon_{\theta\theta} = \frac{1}{a+z} \left( \frac{\partial v}{\partial \theta} + w \right) \\
 \varepsilon_{zz} = \frac{\partial w}{\partial z} \\
 2 \varepsilon_{x\theta} = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \cos \theta + \frac{1}{a+z} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{a+z} \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial \theta} \\
 2 \varepsilon_{xz} = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \sin \theta + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial z} \\
 2 \varepsilon_{z\theta} = \frac{1}{a+z} \left( \frac{\partial w}{\partial \theta} - v \right) + \frac{\partial v}{\partial z}
 \end{array} \right. \quad (4)$$

103 The strain tensor given in Eq. (4) is decomposed into three contributions: linear beam-type kinematic terms  $\varepsilon^b$ , linear  
 104 shell-type kinematic terms  $\varepsilon^s$  and non-linear *coupling* terms  $\varepsilon^{nl}$ :

$$\begin{array}{lll}
 \varepsilon_{xx}^b = \frac{\partial U_o}{\partial x} + (a+z) \sin \theta \frac{\partial \Omega_y}{\partial x}, & \varepsilon_{xx}^s = \frac{\partial u}{\partial x}, & \varepsilon_{xx}^{nl} = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial x} + h \frac{\partial \Omega_y}{\partial x}, \\
 \varepsilon_{\theta\theta}^b = 0, & \varepsilon_{\theta\theta}^s = \frac{1}{a+z} \left( \frac{\partial v}{\partial \theta} + w \right), & \varepsilon_{\theta\theta}^{nl} = 0, \\
 \varepsilon_{zz}^b = 0, & \varepsilon_{zz}^s = \frac{\partial w}{\partial z}, & \varepsilon_{zz}^{nl} = 0, \\
 2 \varepsilon_{x\theta}^b = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \cos \theta, & 2 \varepsilon_{x\theta}^s = \frac{1}{a+z} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}, & 2 \varepsilon_{x\theta}^{nl} = \frac{1}{a+z} \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial \theta}, \\
 2 \varepsilon_{xz}^b = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \sin \theta, & 2 \varepsilon_{xz}^s = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, & 2 \varepsilon_{xz}^{nl} = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial z}, \\
 2 \varepsilon_{z\theta}^b = 0, & 2 \varepsilon_{z\theta}^s = \frac{1}{a+z} \left( \frac{\partial w}{\partial \theta} - v \right) + \frac{\partial v}{\partial z}, & 2 \varepsilon_{z\theta}^{nl} = 0.
 \end{array} \quad (5)$$

105 Considering the beam strain, the absence of shear deformation corresponding to the Euler-Bernoulli hypothesis  
 106 imposes that:

$$\varepsilon_{xz}^b = 0 \quad \text{or equivalently} \quad \Omega_y + \frac{\partial V_o}{\partial x} = 0 \quad (6)$$

107 Under this hypothesis, the beam kinematics enforces planar sections normal to the neutral axis to remain planar after  
 108 deformation. As a consequence, the deformation terms associated to the beam are only present in the  $\varepsilon_{xx}^b$  component.  
 109 Finally, this corresponds to a beam whose straight section is an undeformed circular ring. Note that one of the main  
 110 consequences of the previous hypothesis is that the non-linear coupling term is only present in the  $\varepsilon_{xx}$  component,  
 111 i.e.  $\varepsilon_{xx}^{nl} = h \partial_x \Omega_y$  while  $\varepsilon_{\theta\theta}^{nl} = \varepsilon_{zz}^{nl} = \varepsilon_{x\theta}^{nl} = \varepsilon_{xz}^{nl} = \varepsilon_{z\theta}^{nl} = 0$ .

112  
 113 Concerning the shell-type kinematics, the mid-surface of the tube corresponding to  $z = 0$  (see Fig. (4)) is taken to  
 114 be the reference surface and the local rotations of this surface are here considered and assumed to be small. Taking  
 115 the classical Love-Kirchhoff hypothesis corresponding to the absence of shear deformation for the shell kinematics,  
 116 i.e. the shell-type terms in the  $\varepsilon_{xz}^s$  and  $\varepsilon_{z\theta}^s$  components being null, leads to the two following additional relations:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \quad \text{and} \quad \frac{1}{a+z} \left( \frac{\partial w}{\partial \theta} - v \right) + \frac{\partial v}{\partial z} = 0 \quad (7)$$

117 The shell kinematics follows here the first-order shear deformation shell theory [27, 28] leading to the following  
 118 displacement field:

$$w = w_o, \quad v = v_o + z v_1, \quad u = u_o - z u_1 \quad (8)$$

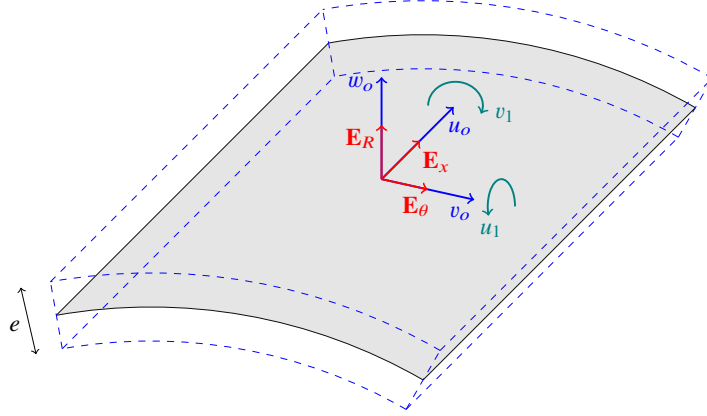


Figure 4: Shell geometry: tube mid-surface at  $z = 0$  in gray, its associated cylindrical coordinates system  $(\mathbf{E}_x, \mathbf{E}_R, \mathbf{E}_\theta)$  and the shell DOFs:  $u_o, v_o, w_o, u_1$  and  $v_1$ .

119 with  $u_1$  and  $v_1$  the rotation angles of the mid-surface of transverse normal  $\mathbf{E}_R$  in the  $(\mathbf{E}_x, \mathbf{E}_R)$  and  $(\mathbf{E}_R, \mathbf{E}_\theta)$  planes,  
 120 respectively, (see Fig. (4)). As a consequence, Eq. (7) rewrites as:

$$u_1 = \frac{\partial w_o}{\partial x} \quad \text{and} \quad v_1 = -\frac{1}{a} \left( \frac{\partial w_o}{\partial \theta} - v_o \right) \quad (9)$$

121 Under this hypothesis, the shell kinematics enforces straight fibres initially perpendicular to the middle surface to  
 122 remain straight and perpendicular to the deformed mid-surface.

123  
 124 In addition, the global strain tensor is decomposed into first-order terms related to the radial variable  $z$  and the  
 125 thin-shell hypothesis is also considered. As a consequence, terms in  $\frac{1}{a+z}$  are approximated as follows:

$$\frac{1}{a+z} = \frac{1}{a} - \frac{z}{a^2} + o\left(\frac{z}{a^2}\right) \quad (10)$$

126 neglecting the second-order and higher order terms. Consequently, the strain tensor can be decomposed as the sum of  
 127 a membrane term denoted by  $\boldsymbol{\varepsilon}^m$  and a linear term in  $z$  corresponding to curvatures denoted by  $\mathbf{k}$  as  $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^m + z \mathbf{k}$   
 128 which can be expressed in terms of both axis displacements, i.e.  $U_o$  and  $V_o$ , and mid-surface displacements, i.e.  $u_o,$   
 129  $v_o$  and  $w_o$ :

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial U_o}{\partial x} - a \sin \theta \frac{\partial^2 V_o}{\partial x^2} + \frac{\partial u_o}{\partial x} - \frac{\partial^2 V_o}{\partial x^2} (w_o \sin \theta + v_o \cos \theta) + z k_{xx} \\ \varepsilon_{\theta\theta} &= \frac{1}{a} \left( \frac{\partial v_o}{\partial \theta} + w_o \right) + z k_{\theta\theta} \\ 2 \varepsilon_{x\theta} &= \frac{1}{a} \frac{\partial u_o}{\partial \theta} + \frac{\partial v_o}{\partial x} + 2 z k_{x\theta} \end{aligned} \quad (11)$$

130 and the following curvature terms:

$$\begin{aligned} k_{xx} &= -\sin \theta \frac{\partial^2 V_o}{\partial x^2} - \frac{\partial^2 w_o}{\partial x^2} + \frac{\cos \theta}{a} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{\partial w_o}{\partial \theta} - v_o \right) \\ k_{\theta\theta} &= -\frac{1}{a^2} \left( \frac{\partial^2 w_o}{\partial \theta^2} + w_o \right) \\ 2 k_{x\theta} &= -\frac{1}{a} \left( 2 \frac{\partial^2 w_o}{\partial x \partial \theta} + \frac{1}{a} \frac{\partial u_o}{\partial \theta} - \frac{\partial v_o}{\partial x} \right) \end{aligned} \quad (12)$$

131 Notice that the shell-type membrane and curvature terms correspond exactly to those obtained with the Flügge shell  
 132 theory [29]. A brief review of thin cylindrical shell theories can be found in [14] where it is shown that the different  
 133 contributions on thin-shell theory all agree concerning the expressions of the membrane strain. However, different  
 134 expressions for curvature have been obtained relying upon the assumptions used in the derivation. For example, the  
 135 shell-type curvature terms adopted in Ohtsubo and Watanabe [6], Millard and Roche [8], Berton [11] or in Weicker *et*  
 136 *al.* [14] are different from the present ones and correspond to those developed by Timoshenko [30]. However, it has  
 137 to be noticed that with the additional assumption of the pipe radial inextensibility, the mentioned thin cylindrical shell  
 138 theories lead to the same strain-displacement relations. In the present work, the pipe is not assumed to be inextensible  
 139 in order to capture its radial expansion/contraction. Finally, in all these previous works, the coupling terms between  
 140 the section deformation and the axis rotation have not been taken into account. The coupling contributes only in the  
 141 uni-axial components  $\varepsilon_{xx}^m$  and  $k_{xx}$ .

## 142 2.2. Fourier series expansion

143 The shell displacement field is now expanded in Fourier series in terms of the tangential variable  $\theta$ . In contrast to  
 144 previous works [31, 8, 13, 32, 7, 9, 10] where incomplete displacement Fourier series are considered, the longitudinal,  
 145 tangential and radial mid-surface displacements, i.e.  $u_o$ ,  $v_o$  and  $w_o$ , are here all expanded in Fourier series:

$$\left\{ \begin{array}{l} u_o(t, x, \theta) = u_o^c(t, x) + u_1^c(t, x) \cos \theta + u_1^s(t, x) \sin \theta + \sum_{i=2}^{N_f} (u_i^c(t, x) \cos(i\theta) + u_i^s(t, x) \sin(i\theta)) \\ v_o(t, x, \theta) = v_o^c(t, x) + v_1^c(t, x) \cos \theta + v_1^s(t, x) \sin \theta + \sum_{i=2}^{N_f} (v_i^c(t, x) \cos(i\theta) + v_i^s(t, x) \sin(i\theta)) \\ w_o(t, x, \theta) = w_o^c(t, x) + w_1^c(t, x) \cos \theta + w_1^s(t, x) \sin \theta + \sum_{i=2}^{N_f} (w_i^c(t, x) \cos(i\theta) + w_i^s(t, x) \sin(i\theta)) \end{array} \right. \quad (13)$$

146  $N_f$  represents the number of Fourier modes considered in the expansion.  $u_i^c(t, x)$ ,  $u_i^s(t, x)$ ,  $v_i^c(t, x)$ ,  $v_i^s(t, x)$ ,  $w_i^c(t, x)$  and  
 147  $w_i^s(t, x)$  are the coefficients of the Fourier series becoming the new unknowns and depending only on the longitudinal  
 148 variable  $x$  (and on the time  $t$ ). The index  $c$  represents the coefficients in cosine while the index  $s$  the coefficients in  
 149 sine in the Fourier expansion.

150 The coefficients in the expansion producing a rigid-body motion are set equal to zero in order to avoid redundancy  
 151 with the beam-type degrees of freedom. As a consequence,  $u_o^c$  corresponding to a rigid-body longitudinal displace-  
 152 ment is taken to be zero. Similarly,  $u_1^c$ ,  $u_1^s$  and  $v_o^c$  are also null as these terms correspond to a rigid-body rotation  
 153 about the  $\mathbf{E}_z$ , the  $\mathbf{E}_y$  and the  $\mathbf{E}_x$  axis, respectively. In addition, the coefficient multiplying  $\cos \theta$  in the expansion of  $v_o$   
 154 has been set to the negative of the coefficient multiplying  $\sin \theta$  in the expansion of  $w_o$ , i.e.  $v_1^c(t, x) = -w_1^s(t, x)$ . This  
 155 produces a vanishing rigid-body displacement of the cross-section in the  $\mathbf{E}_z$  direction [12] via the integration over  
 156 the tangential variable  $\theta$ . In the same way, the coefficient multiplying  $\sin \theta$  in the expansion of  $v_o$  has been set to the  
 157 coefficient multiplying  $\cos \theta$  in the expansion of  $w_o$ , i.e.  $v_1^s(t, x) = w_1^c(t, x)$  in order to produce a vanishing rigid-body  
 158 displacement of the cross-section in the  $\mathbf{E}_y$  direction.

159 In addition, for Fourier modes of higher order, i.e. greater than or equal to two, an inextensibility assumption in  
 160 the circumferential direction of the section is also taken in consideration meaning that the section perimeter remains  
 161 constant [33, 34]. In other words, the circumferential membrane deformation is null, i.e.  $(\varepsilon^m)_{\theta\theta} = 0$ , for  $i \geq 2$ . As  
 162  $(\varepsilon^m)_{\theta\theta} \equiv \frac{1}{a} (\partial_\theta v_o + w_o)$ , this allows to obtain a relation between  $v_i^c$ ,  $w_i^s$ ,  $w_i^c$  and  $v_i^s$  for  $i \geq 2$ :

$$\frac{1}{a} \sum_{i=2}^{N_f} \left( \frac{\partial}{\partial \theta} (v_i^c \cos(i\theta) + v_i^s \sin(i\theta)) + w_i^c \cos(i\theta) + w_i^s \sin(i\theta) \right) = 0 \quad (14)$$



163 Thus, the shell-type displacements of the tube mid-surface can be expressed as:

$$\begin{cases} u_o = & \sum_{i=2}^{N_f} (u_i^c \cos(i\theta) + u_i^s \sin(i\theta)) \\ v_o = & -w_1^s \cos \theta + w_1^c \sin \theta + \sum_{i=2}^{N_f} \left( \frac{1}{i} w_i^s \cos(i\theta) - \frac{1}{i} w_i^c \sin(i\theta) \right) \\ w_o = & w_o^c + w_1^c \cos \theta + w_1^s \sin \theta + \sum_{i=2}^{N_f} (w_i^c \cos(i\theta) + w_i^s \sin(i\theta)) \end{cases} \quad (15)$$

164  $w_o^c$  represents the radial extensibility of the cross-section.  $w_1^c$  and  $w_1^s$  are associated with vanishing lateral rigid-body  
165 displacements.  $w_i^c$  and  $w_i^s$  for  $i \geq 2$  represent cross-section distortions. In addition,  $u_i^c$  and  $u_i^s$  for  $i \geq 2$  represent the  
166 warping cross-section deformations.

167

168 Finally, the tube displacement field is given by:

$$\mathbf{U} = (U_o + u + [(a + z + w) \sin \theta + v \cos \theta] \Omega_y) \mathbf{E}_x + (V_o \sin \theta + w) \mathbf{E}_R + (V_o \cos \theta + v) \mathbf{E}_\theta \quad (16)$$

169 respecting for the beam-type displacement terms the Euler-Bernoulli hypothesis

$$\Omega_y = -\frac{\partial V_o}{\partial x}$$

170 and with the shell-type displacement field:

$$w = w_o, \quad v = v_o + zv_1, \quad u = u_o - zu_1 \quad (17)$$

171 where  $u_o$ ,  $v_o$  and  $w_o$  are expressed in Eq. (15) and with

$$u_1 = \frac{\partial w_o}{\partial x} \quad \text{and} \quad v_1 = -\frac{1}{a} \left( 2w_1^s \cos \theta - 2w_1^c \sin \theta + \sum_{i=2}^{N_f} \frac{i^2 - 1}{i} (w_i^s \cos(i\theta) - w_i^c \sin(i\theta)) \right) \quad (18)$$

172 respecting the Love-Kirchhoff hypothesis.

### 173 2.3. Virtual power principle

174 The equations of motion can be obtained through the application of the virtual power principle which expresses  
175 the equilibrium between the forces of inertia, external and internal forces:

$$\mathcal{P}_a = \mathcal{P}_e + \mathcal{P}_i \quad (19)$$

176 with  $\mathcal{P}_a$ ,  $\mathcal{P}_e$  and  $\mathcal{P}_i$  the power of inertia, external and internal forces, respectively. The power of inertia forces is  
177 defined in the current (deformed) configuration as:

$$\mathcal{P}_a = \int_{\Omega} \rho \frac{\partial^2 \mathbf{U}}{\partial t^2} \cdot \delta \mathbf{U} \, dV \quad (20)$$

178 in which the integration is performed over the current volume  $\Omega$  occupied by the solid body and where  $\rho$  is the solid  
179 density,  $\frac{\partial^2 \mathbf{U}}{\partial t^2}$  the acceleration vector,  $\delta \mathbf{U}$  the virtual displacement field.

180

181 The power of external forces (without considering volume forces) is defined as:

$$\mathcal{P}_e = \int_{\partial\Omega} \mathbf{T} \cdot \delta \mathbf{U} \, dS \quad (21)$$

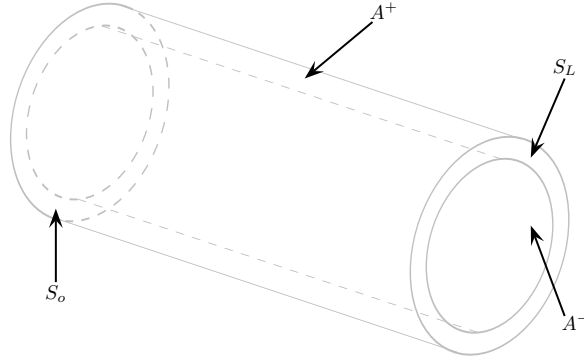


Figure 5: Tube surfaces decomposed by the two cross-sections  $S_o$  and  $S_L$  and by  $A^-$  the inner and  $A^+$  the outer lateral sections.

182  $\mathbf{T}$  is the vector of applied forces on the boundaries. The edge of the tube is decomposed into the two end surfaces  
 183 denoted by  $S_o$  and  $S_L$  where the elementary surface  $dS$  is set to be equal to  $a d\theta dz$  instead of  $(a + z) d\theta dz$  due to the  
 184 thin-walled assumption, and the inner and outer skins of the tube denoted by  $A^-$  and  $A^+$  where  $A^\pm = 2\pi L(a \pm e/2)$  as  
 185 depicted in Fig. (5).

186

187 Finally, the power of internal forces is given by:

$$\mathcal{P}_i = - \int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} dV \quad (22)$$

188 where  $\boldsymbol{\sigma}$  is the Cauchy stress tensor and  $\delta \boldsymbol{\varepsilon}$  the virtual stress tensor. Using the circular geometry of the considered  
 189 pipe leads to the following expression for the power of internal forces:

$$\mathcal{P}_i = - \int_0^L \left( \int_0^{2\pi} \int_{-e/2}^{e/2} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} a d\theta dz \right) dx \quad (23)$$

190 Due to the thin-shell assumption, i.e.  $z \ll a$ , the integration is performed through the elementary volume  $a d\theta dz dx$   
 191 instead of  $(a + z) d\theta dz dx$ .

192 The strain-stress relationship is here obtained using a local linear elastic behavior:

$$\boldsymbol{\sigma} = \frac{E}{1 - \nu^2} \left( (1 - \nu) \boldsymbol{\varepsilon} + \nu \text{trac}(\boldsymbol{\varepsilon}) \mathbf{I} \right) \quad (24)$$

193 Using plane stress conditions in the  $(\mathbf{E}_\theta, \mathbf{E}_x)$  plane, i.e.  $\sigma_{zz} = \sigma_{xz} = \sigma_{\theta z} = 0$ , this leads to:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{\theta\theta} \\ \sigma_{x\theta} \end{pmatrix} = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 - \nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{x\theta} \end{pmatrix} \quad (25)$$

194 with  $\nu$  the Poisson's ratio and  $E$  the Young's modulus of the pipe wall material.

195

196 The power of inertia forces as well as the power of internal forces are expressed in Appendix B (see Eqs. (B.5)  
 197 and (B.8) respectively) when considering all the Fourier modes and the non-linear coupling terms. In addition,  
 198 the governing equation of the tube are expressed in Eq. (B.14) and the corresponding generalized forces in Eqs.  
 199 (B.9,B.10,B.11,B.12,B.13).

200

201 In the following, details are given when considering only the mode 0 in the Fourier series and then only the Fourier  
 202 mode 2. As previously mentioned, Fourier mode 0 corresponds to the radial expansion/contraction of the tube cross-  
 203 section whereas Fourier mode 2 corresponds to its ovalization. Comments are also given when higher Fourier modes  
 204 are considered.

205 *2.4. Governing equations and generalized forces for Fourier mode 0 for linear-type motion*

206 When only the first Fourier mode ( $N_f = 0$ ) is considered, the shell kinematics is only carried by  $w_o$  (i.e.  $u_o = v_o =$   
207  $0$ ) and  $u_1$  (i.e.  $v_1 = 0$ ) (cf. Eqs. (15) and (18)). In addition, the non-linear coupling term is not taken into account at  
208 this stage. As a consequence, the shell-type displacement field is given by:

$$w_o(t, x, \theta) = w_o^c(t, x), \quad u_1 = \frac{\partial w_o^c}{\partial x} \quad (26)$$

209 leading to the tube displacement field:

$$\mathbf{U} = \left( U_o - z \frac{\partial w_o^c}{\partial x} - (a + z) \sin \theta \frac{\partial V_o}{\partial x} \right) \mathbf{E}_x + (V_o \sin \theta + w_o^c) \mathbf{E}_R + (V_o \cos \theta) \mathbf{E}_\theta \quad (27)$$

210 As a consequence, the non-zero components of the strain tensor are given by:

$$\varepsilon_{xx} = \frac{\partial U_o}{\partial x} - a \sin \theta \frac{\partial^2 V_o}{\partial x^2} + z \left( -\sin \theta \frac{\partial^2 V_o}{\partial x^2} - \frac{\partial^2 w_o^c}{\partial x^2} \right) \quad \text{and} \quad \varepsilon_{\theta\theta} = \left( \frac{1}{a} - \frac{z}{a^2} \right) w_o^c \quad (28)$$

211 The power of internal forces can thus be expressed as:

$$\mathcal{P}_i = - \int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, dV = - \int_0^L \left( N \frac{\partial \delta U_o}{\partial x} - M_y \frac{\partial^2 \delta V_o}{\partial x^2} - (m_w)_o^c \frac{\partial^2 \delta w_o^c}{\partial x^2} + (t_w)_o^c \delta w_o^c \right) dx \quad (29)$$

212 where the generalized forces are defined as:

$$\begin{aligned} N &= \int_S \sigma_{xx} \, dS \quad ; \quad M_y = \int_S \sigma_{xx} (a + z) \sin \theta \, dS \\ (t_w)_o^c &= \int_S \sigma_{\theta\theta} \left( \frac{1}{a} - \frac{z}{a^2} \right) \, dS \quad ; \quad (m_w)_o^c = \int_S \sigma_{xx} z \, dS \end{aligned} \quad (30)$$

213 with  $dS = a d\theta dz$ . The term  $z/a^2$  in  $(t_w)_o^c$  is neglected in the following due to the thin-walled assumption.

214 In addition, the power of inertia forces can be expressed as:

$$\begin{aligned} \mathcal{P}_a &= \int_{\Omega} \rho \frac{\partial^2 \mathbf{U}}{\partial t^2} \cdot \delta \mathbf{U} \, dV \\ &= \int_0^L \rho \left( S \frac{\partial^2 U_o}{\partial t^2} \delta U_o + S \left( \frac{\partial^2 V_o}{\partial t^2} - \frac{I_y^b}{S} \frac{\partial^4 V_o}{\partial t^2 \partial x^2} \right) \delta V_o + S \left( \frac{\partial^2 w_o^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_o^c}{\partial t^2 \partial x^2} \right) \delta w_o^c \right) dx \end{aligned} \quad (31)$$

215 with the pipe cross-section  $S = 2\pi a e$  and the moment of inertia  $I_y^b = \pi a e \left( a^2 + \frac{e^2}{12} \right) \approx \pi a^3 e$ .

216 Without considering external forces, i.e.  $\mathcal{P}_e = 0$ , the equations of motion write:

$$\begin{cases} \rho S \frac{\partial^2 U_o}{\partial t^2} & - \frac{\partial N}{\partial x} & = 0 \\ \rho S \frac{\partial^2 V_o}{\partial t^2} - \rho I_y^b \frac{\partial^4 V_o}{\partial t^2 \partial x^2} & - \frac{\partial^2 M_y}{\partial x^2} & = 0 \\ \rho S \left( \frac{\partial^2 w_o^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_o^c}{\partial t^2 \partial x^2} \right) & - \frac{\partial^2 (m_w)_o^c}{\partial x^2} + (t_w)_o^c & = 0 \end{cases} \quad (32)$$

217 Note that for a plate element with uniform material properties, the ratio between the moment of inertia and the area of  
218 the cross-section corresponds to  $e^2/12$  which thus takes the same role as  $I_y/S$  for the shell-type DOFs.

219 The static counterpart of Eq. (32) corresponds to:

$$\frac{\partial N}{\partial x} = 0; \quad \frac{\partial^2 M_y}{\partial x^2} = 0; \quad -\frac{\partial^2 (m_w)_o^c}{\partial x^2} + (t_w)_o^c = 0 \quad (33)$$

220 For the considered behavior given in Eq. (25), the generalized forces are given by:

$$\begin{aligned}
N &= \frac{ES}{1-\nu^2} \left( \frac{\partial U_o}{\partial x} + \nu \frac{w_o^c}{a} \right) ; \quad M_y = -\frac{EI_y^b}{1-\nu^2} \frac{\partial^2 V_o}{\partial x^2} ; \\
(t_w)_o^c &= \frac{ES}{1-\nu^2} \frac{1}{a} \left( \nu \frac{\partial U_o}{\partial x} + \frac{w_o^c}{a} \right) ; \quad (m_w)_o^c = -\frac{ES}{1-\nu^2} \frac{e^2}{12} \frac{\partial^2 w_o^c}{\partial x^2}
\end{aligned} \tag{34}$$

221 where  $N$  corresponds to the beam-type axial force with an additional term involving the pipe radial contraction/expansion  
222 corresponding to the Poisson effect,  $M_y$  to the beam in-plane bending moment whereas  $(t_w)_o^c$  corresponds to the radial  
223 force divided by the mean radius and  $(m_w)_o^c$  to the bending moment of the mid-surface in the  $(\mathbf{E}_x, \mathbf{E}_R)$  plane. Eqs.  
224 (32) and (34) correspond to Eqs. (B.15) and (B.16) described in Appendix B.

225 Note that, without considering the Poisson effect, the two first equations in Eq. (32) correspond to the classical  
226 straight Euler-Bernoulli beam governing equations which, for example, are given in [35]. In addition, it has to be  
227 noticed that Eq. (32) correspond to the axisymmetric thin-shell cylinder equations of motion as expressed in [36]  
228 where the terms  $e^2/12a$  is neglected due to the thin-walled assumption and thus the integration over  $a$   $d\theta dz$  instead of  
229  $(a+z)$   $d\theta dz$ . Finally, neglecting the  $\frac{\partial^2}{\partial x^2} (m_w)_o^c$  term in Eq. (32) makes it possible to retrieve the equations of motion  
230 proposed by Walker & Phillips [37] for an axisymmetric small-deformed elastic plane-stressed tube as also reported  
231 in the review [3].

### 232 2.5. Governing equations and generalized forces for Fourier mode 2

233 Only the second Fourier mode is here considered corresponding to the cross-section ovalization. As in [8, 11], only  
234 the shell membrane deformations are considered (i.e. for shell-type DOFs  $\mathbf{k} = \mathbf{0}$  except  $k_{\theta\theta} \neq 0$ ). The assumption  
235 of the section inextensibility is taken into account (see Eq. (14)) as well as the non-linear coupling terms. As a  
236 consequence, for an in-plane motion, the considered shell kinematics of the mid-surface writes:

$$u_o(t, x, \theta) = 0, \quad w_o(t, x, \theta) = w_2^c(t, x) \cos(2\theta), \quad v_o(t, x, \theta) = -\frac{1}{2} w_2^c(t, x) \sin(2\theta) \tag{35}$$

237 The strain tensor is thus given by:

$$\begin{cases} \varepsilon_{xx} = \frac{\partial U_o}{\partial x} - [(a+z) \sin \theta + h_o] \frac{\partial^2 V_o}{\partial x^2} \\ \varepsilon_{\theta\theta} = 3 \frac{z}{a^2} \cos(2\theta) w_2^c \\ 2 \varepsilon_{x\theta} = -\frac{1}{2} \sin(2\theta) \frac{\partial w_2^c}{\partial x} \end{cases} \quad \text{with } h_o = \left( \cos(2\theta) \sin \theta - \frac{1}{2} \sin(2\theta) \cos \theta \right) w_2^c \tag{36}$$

238 After some algebraic manipulations, the power of internal forces leads to the following expression:

$$\mathcal{P}_i = - \int_0^L \left( N \frac{\partial \delta U_o}{\partial x} - M_y \frac{\partial^2 \delta V_o}{\partial x^2} + (t_w)_2^c \delta w_2^c + (n_w)_2^c \frac{\partial \delta w_2^c}{\partial x} \right) dx \tag{37}$$

239 where the generalized forces are defined as:

$$\begin{aligned}
N &= \int_S \sigma_{xx} dS \\
M_y &= \int_S \sigma_{xx} [(a+z) \sin \theta + h_o] dS \\
(t_w)_2^c &= \int_S \left( \frac{1}{2} \sigma_{xx} (\sin(2\theta) \cos \theta - 2 \cos(2\theta) \sin \theta) \frac{\partial^2 V_o}{\partial x^2} + 3 \frac{z}{a^2} \sigma_{\theta\theta} \cos(2\theta) \right) dS \\
(n_w)_2^c &= - \int_S \frac{1}{2} \sigma_{x\theta} \sin(2\theta) dS
\end{aligned} \tag{38}$$

240 Without considering external forces, i.e.  $\mathcal{P}_e = 0$ , the equations of motion write:

$$\begin{cases} \rho S \frac{\partial^2 U_o}{\partial t^2} - \frac{\partial N}{\partial x} = 0 \\ \rho S \frac{\partial^2 V_o}{\partial t^2} - \rho I_y \frac{\partial^4 V_o}{\partial t^2 \partial x^2} - \frac{\partial^2 M_y}{\partial x^2} = 0 \\ \rho S \frac{1}{2} \left( \frac{5}{4} \frac{\partial^2 w_2^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_2^c}{\partial t^2 \partial x^2} \right) - \frac{\partial (n_w)_2^c}{\partial x} + (t_w)_2^c = 0 \end{cases} \quad (39)$$

241 with the moment of inertia given by:

$$I_y = \pi a^3 e \left( 1 + \frac{e^2}{12a^2} - \frac{3}{2} \frac{w_2^c}{a} + \frac{5}{8} \left( \frac{w_2^c}{a} \right)^2 \right) \approx \pi a^3 e \left( 1 - \frac{3}{2} \frac{w_2^c}{a} + \frac{5}{8} \left( \frac{w_2^c}{a} \right)^2 \right)$$

242 where  $I_y^b \approx \pi a^3 e$ .

243 In a similar way, the equilibrium equations in static are obtained as:

$$\frac{\partial N}{\partial x} = 0 \quad ; \quad \frac{\partial^2 M_y}{\partial x^2} = 0 \quad ; \quad \frac{\partial (n_w)_2^c}{\partial x} - (t_w)_2^c = 0 \quad (40)$$

244 With the behavior law given in Eq. (25), the generalized forces are expressed as:

$$\begin{aligned} N &= + \frac{ES}{1-\nu^2} \frac{\partial U_o}{\partial x} \quad ; \quad \xi = \frac{w_2^c}{a} \\ M_y &= - \frac{EI_y^b}{1-\nu^2} \left( 1 - \frac{3}{2} \xi + \frac{5}{8} \xi^2 \right) \frac{\partial^2 V_o}{\partial x^2} \\ (t_w)_2^c &= + \frac{ES}{1-\nu^2} \frac{1}{2} \left( \left( -\frac{3}{4} + \frac{5}{8} \xi \right) a \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 + 3 \frac{e^2}{4a^2} \frac{1}{a} \xi \right) \\ (n_w)_2^c &= + \frac{ES}{1+\nu} \frac{1}{16} a \frac{\partial \xi}{\partial x} \end{aligned} \quad (41)$$

245 As previously,  $N$  corresponds the beam-type axial force and  $M_y$  to the beam in-plane bending moment. For the shell  
246 DOFs,  $(t_w)_2^c$  corresponds to a force divided by the mean radius and  $(n_w)_2^c$  to a shear-type force. Eqs. (39) and (41)  
247 correspond to Eqs. (B.17) and (B.18) described in Appendix B.

248 Due to the non-linear coupling between the tube cross-section deformation (here linked to  $w_2^c$ ) and the rotation of  
249 the tube axis, i.e.  $-\partial_x V_o$ , the in-plane bending moment  $M_y$  depends on terms in  $w_2^c$  and, in a similar way, terms in  
250  $\partial_x V_o$  are involved in the expression of  $(t_w)_2^c$ . Let consider the curvature  $\kappa$  defined as:

$$\kappa = \frac{a^2}{e} \left( -\frac{\partial^2 V_o}{\partial x^2} \right)$$

251 In the case of Euler-Bernoulli beams and in the case of Fourier mode 0 described previously, it appears that the  
252 bending moment  $M_y$  is linear with respect to the curvature  $\kappa$  as the inertia remains constant. In contrast, in the present  
253 model involving both the Fourier mode 2 and the non-linear coupling, due to the terms in  $\xi$  in the expression of the  
254 bending moment, the inertia of the section can change with its deformation leading to a non-linear relation between  
255  $M_y$  and  $\kappa$ . Comparison between the different moment-curvature and ovalization-curvature relationships encountered  
256 in the literature is performed in [34]. The ovalization-curvature are obtained taking  $(t_w)_2^c$  equal to zero. For example,  
257 considering only the first-order terms in the expressions of  $M_y$  and  $(t_w)_2^c$ , i.e.

$$M_y = - \frac{EI_y^b}{1-\nu^2} \left( 1 - \frac{3}{2} \xi \right) \frac{\partial^2 V_o}{\partial x^2} \quad \text{and} \quad (t_w)_2^c = \frac{ES}{1-\nu^2} \frac{3}{8} \left( -a \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 + \frac{e^2}{a^3} \xi \right)$$

258 leads to the following moment-curvature and ovalization-curvature relationships:

$$M_y = K \left( 1 - \frac{3}{2}\xi \right) \kappa \quad \text{and} \quad \xi = \kappa^2$$

259 with  $K$  a constant. This exactly corresponds to the relationships initially derived by Brazier [33]. Then, considering  
260 the second-order terms in Eq. 41 leads to the following moment-curvature and ovalization-curvature relationships:

$$M_y = K \left( 1 - \frac{3}{2}\xi + \frac{5}{8}\xi^2 \right) \kappa \quad \text{and} \quad \xi = \frac{\kappa^2}{1 + \frac{5}{6}\kappa^2}$$

261 which corresponds to the relationships referred to as modified Brazier initially straight tubes in [34].

## 262 2.6. Higher Fourier modes

263 The derivation of both governing equations and generalized forces has been presented previously when only  
264 considering Fourier mode 0 or Fourier mode 2. Taking into account higher Fourier modes together makes this deriva-  
265 tion more complex, especially due to the coupling between different modes. The corresponding tube governing  
266 equations are derived in Appendix B and are expressed in Eq. (B.14). The generalized forces are given in Eqs.  
267 (B.9), (B.10), (B.11), (B.12) and (B.13). In addition, mathematical tools dedicated to formal computation such as  
268 Sympy (<https://www.sympy.org>) and wxMaxima (<https://wxmaxima-developers.github.io/wxmaxima/>)  
269 have been used for verification.

## 270 2.7. Further discussion on the kinematics of the present element and comparison with previous beam elements

271 As shown in the previous sections, the present tube element is here derived in the context of thin-walled tubu-  
272 lar geometries. Even if a full 3-D motion has been considered in the derivation of this element, only an in-plane  
273 motion is presented for simplicity. As described previously, the kinematics is obtained through the enrichment of  
274 straight beam kinematics by the addition of orthogonal shell-type displacement field of the tube cross-section. The  
275 corresponding displacement field is here composed of three contributions: linear beam and shell terms in conjunction  
276 with a non-linear coupling between cross-section deformation and rotation of the tube axis. First, concerning the  
277 beam-type DOFs of the proposed tube model, as shown in the derivation of the generalized forces (cf. Appendix B.2),  
278 the classical Euler-Bernoulli beam is recovered plus the consideration of the Poisson effect which is derived from the  
279 link between beam and shell. Then, concerning the shell-type motion, the present tube element incorporates all the  
280 elements reviewed in Weicker [14] including the elements previously proposed in [6, 7, 8, 9, 10, 11, 13] as complete  
281 Fourier series for warping, ovalization and radial deformations of the mid-surface are considered. In addition, as  
282 shown in Appendix B.4, the present tube kinematics incorporates the circular shell kinematics as the corresponding  
283 circular cylindrical shells equations of motion can be retrieved from the present formulation. What is more, via the  
284 consideration of the non-linear coupling, the inertia of the section can change with its deformation as described in  
285 Brazier [33]. As a consequence, the present tube element seems to present the simplest kinematics able to consider  
286 both deformations of its cross-section and changes of inertia as only one non-linear coupling term is considered for  
287 this purpose. However, this single non-linear term is not sufficient to describe all possible local and global buckling  
288 as it is performed in [34]. In addition, the previous elements have been derived for static analysis whereas the present  
289 one is considered for explicit dynamic analysis.

290  
291 In the following section, the numerical method used to solve the derived tube model is detailed.

## 292 3. Numerical approximations: an explicit updated Lagrangian Finite-Element approach

293 The governing equations are here discretized using an explicit updated Lagrangian Finite-Element approach which  
294 is described in the following. For simplicity, the numerical methods are detailed in the case where only the 0-mode  
295 is considered in the Fourier expansion as described in Sec. 2.4 and the extension to higher Fourier modes is briefly  
296 discussed. In addition, even if the numerical method as well as the tube element have been derived for a 3-D motion,  
297 only an in-plane motion is presented in the following for simplicity. The time integration is performed using the  
298 **explicit** central difference operator whose stability condition is given considering both beam and shell contributions.

299 *3.1. Nodal unknowns and shape functions*

300 By means of the Finite-Element method, the spatial domain of the beam is discretized into a number of sub-  
 301 domains named here tube element. Each element has two nodes and the vector of nodal generalized displacements is  
 302 defined as:

$$\mathbf{d}_e = \left( \mathbf{d}_e^1, \mathbf{d}_e^2 \right)^T \quad \text{where} \quad \mathbf{d}_e^i = \left( U_o^i, V_o^i, \Omega_y^i, (w_o^c)^i, u_1^i \right)^T \quad (42)$$

303 when only Fourier mode 0 is here considered. The subscript  $e$  associates a given quantity with a typical tube element  
 304 whereas the superscript  $i = 1, 2$  associates a given quantity with the corresponding node of the considered tube  
 305 element. The length of the element is denoted by  $\ell_e$ . Due to the assumption of small deformations, the length of the  
 306 tube element is assumed to be the same as the initial undeformed element, i.e.  $\ell_e^n = \ell_e^o$  where the superscript  $n$  denotes  
 307 the time level. Shape functions are used to approximate the displacement field inside the tube element of any arbitrary  
 308 point located at the beam axis for the beam-type DOFs and at the mid-surface of the tube for the shell-type DOFs. In  
 309 addition, the displacement field is assumed to satisfy the conditions imposed by the tube kinematics, in particular the  
 310 Euler-Bernoulli and the Love-Kirchhoff hypotheses:

$$\Omega_y + \frac{\partial V_o}{\partial x} = 0 \quad \text{and} \quad u_1 = \frac{\partial w_o^c}{\partial x} \quad (43)$$

311 To this end, for the beam-type DOFs, linear and cubic interpolating polynomials are chosen to approximate, respec-  
 312 tively, the axial and transverse neutral fiber displacements within each tube element whereas cubic shape functions  
 313 are used for the mid-surface displacements for the shell-type DOFs. As a consequence the displacement field within  
 314 the tube element is approximated using  $\mathbf{N}_e$  the matrix of shape functions as follows:

$$\bar{\mathbf{d}}_e = \mathbf{N}_e \mathbf{d}_e \quad \text{with} \quad \mathbf{N}_e = \left( \mathbf{N}_e^1 \mathbf{N}_e^2 \right) \quad \text{and} \quad \mathbf{N}_e^i = \begin{pmatrix} N_{U_o^i} & 0 & 0 & 0 & 0 \\ 0 & N_{V_o^i} & N_{\Omega_y^i} & 0 & 0 \\ 0 & -\frac{dN_{V_o^i}}{dx} & -\frac{dN_{\Omega_y^i}}{dx} & 0 & 0 \\ 0 & 0 & 0 & N_{(w_o^c)^i} & N_{u_1^i} \\ 0 & 0 & 0 & \frac{dN_{(w_o^c)^i}}{dx} & \frac{dN_{u_1^i}}{dx} \end{pmatrix} \quad (44)$$

315 with the following shape functions:

$$\begin{cases} N_{U_o^1} & = & 1 - \lambda \\ N_{V_o^1} = N_{(w_o^c)^1} & = & (\lambda - 1)^2 (2\lambda + 1) \\ N_{\Omega_y^1} = -N_{u_1^1} & = & -\ell_e \lambda (\lambda - 1)^2 \end{cases} \quad \text{and} \quad \begin{cases} N_{U_o^2} & = & \lambda \\ N_{V_o^2} = N_{(w_o^c)^2} & = & \lambda^2 (3 - 2\lambda) \\ N_{\Omega_y^2} = -N_{u_1^2} & = & -\ell_e \lambda^2 (\lambda - 1) \end{cases}$$

316 with  $\lambda = x/\ell_e$  where  $\ell_e$  corresponds to the length of the tube element.

317 We can also note that the tube displacement field given in Eq. (27) can be expressed as:

$$\mathbf{U} = \mathbf{T} \bar{\mathbf{d}}_e \quad \text{with} \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & (a+z) \sin \theta & 0 & -z \\ 0 & \sin \theta & 0 & 1 & 0 \\ 0 & \cos \theta & 0 & 0 & 0 \end{pmatrix} \quad (45)$$

318 as the non-linear coupling term is here neglected.

319 For the discretized pipe, the virtual power principle (see Eqs. (20) and (29)) writes:

$$\sum_e \left( \int_{V_e} \rho \frac{\partial^2 \mathbf{U}}{\partial t^2} \cdot \delta \mathbf{U} \, dV \right) = - \sum_e \left[ \int_0^{\ell_e} \left( N \frac{\partial \delta U_o}{\partial x} + M_y \left( -\frac{\partial^2 \delta V_o}{\partial x^2} \right) + (t_w)_o^c \delta w_o^c - (m_w)_o^c \frac{\partial^2 \delta w_o^c}{\partial x^2} \right) dx \right] \quad (46)$$

320 with  $V_e$  the volume of the tube element and considering the sum over the tube elements. This can be rewritten in the  
 321 following form:

$$\sum_e \left[ \int_{V_e} \left( \rho (\delta \mathbf{U})^T \frac{\partial^2 \mathbf{U}}{\partial t^2} \right) dV \right] = - \sum_e \left[ \int_0^{\ell_e} \left( (\delta \mathbf{E})^T \mathbf{F} \right) dx \right] \quad (47)$$

322 with the generalized force vector  $\mathbf{F}$  and the generalized strain vector  $\mathbf{E}$  defined as:

$$\mathbf{F} = (N, M_y, (t_w)_o^c, (m_w)_o^c)^T \quad \text{and} \quad \mathbf{E} = \left( \frac{\partial U_o}{\partial x}, -\frac{\partial^2 V_o}{\partial x^2}, w_o^c, -\frac{\partial^2 w_o^c}{\partial x^2} \right)^T$$

323 which are related using the constitutive matrix denoted here by  $\mathbf{D}$  as follows:

$$\mathbf{F} = \mathbf{D}\mathbf{E} \quad \text{with} \quad \mathbf{D} = \frac{ES}{1-\nu^2} \begin{pmatrix} 1 & 0 & \frac{\nu}{a} & 0 \\ 0 & \frac{I_y}{S} & 0 & 0 \\ \frac{\nu}{a} & 0 & \frac{1}{a^2} & 0 \\ 0 & 0 & 0 & \frac{e^2}{12} \end{pmatrix}$$

324 With the assumed displacement field within the tube element, the vector  $\mathbf{E}$  can be expressed using the nodal displacement vector, i.e.  $\mathbf{E} = \mathbf{B}_e \mathbf{d}_e$ , with the strain-displacement matrix  $\mathbf{B}_e$  given by:

$$\mathbf{B}_e = (\mathbf{B}_e^1 \mathbf{B}_e^2) \quad \text{with} \quad \mathbf{B}_e^i = \begin{pmatrix} \frac{dN_{U_o^i}}{dx} & 0 & 0 & 0 & 0 \\ 0 & \frac{d^2 N_{V_o^i}}{dx^2} & \frac{d^2 N_{\Omega_y^i}}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & N_{(w_o^c)^i} & N_{u_1^i} \\ 0 & 0 & 0 & \frac{d^2 N_{(w_o^c)^i}}{dx^2} & \frac{d^2 N_{u_1^i}}{dx^2} \end{pmatrix}$$

326 With the previous notations, the virtual power principle can be expressed as:

$$\sum_e [(\delta \mathbf{d}_e)^T \mathbf{M}_e \dot{\mathbf{d}}_e] = - \sum_e [(\delta \mathbf{d}_e)^T \mathbf{f}_{e,\text{int}}] \quad (48)$$

327 with the consistent mass matrix  $\mathbf{M}_e$  and the nodal internal forces vector  $\mathbf{f}_{e,\text{int}}$  defined by:

$$\mathbf{M}_e = \int_{V_e} \rho \mathbf{N}_e^T \mathbf{T}^T \mathbf{T} \mathbf{N}_e dV \quad \text{and} \quad \mathbf{f}_{e,\text{int}} = \int_0^{\ell_e} \mathbf{B}_e^T \mathbf{F} dx$$

328 The internal forces are approximated using a Gauss quadrature approach:

$$\mathbf{f}_{e,\text{int}} = \int_0^{\ell_e} \mathbf{B}_e^T \mathbf{F} dx \approx \sum_{j=1}^{N_G} \omega_j \mathbf{B}_e^T(\lambda_j) \mathbf{F}(\lambda_j) \quad (49)$$

329 with  $N_G$  the number of Gauss points used for the present tube element,  $\omega_j$  and  $\lambda_j$  the weight and location of each  
 330 Gauss point, respectively. The number of Gauss points as well as the corresponding weights and locations depends on  
 331 the order of the polynomials used in  $\mathbf{B}_e^T \mathbf{F}$ . Due to the presence of the cubic interpolating functions  $N_{(w_o^c)^i}$  for  $i = 1, 2$ ,  
 332 a polynomial of power six is involved as  $\mathbf{F} = \mathbf{D}\mathbf{B}_e \mathbf{d}_e$ . A four-point Gauss rule is thus necessary to integrate this  
 333 six-order polynomial function in an exact manner, i.e.  $N_G = 4$ , with the following weights and locations:

$$\begin{aligned} \lambda_1 &= 0.0694318442, & \omega_1 &= 0.1739274226 \ell_e \\ \lambda_2 &= 0.3300094782, & \omega_2 &= 0.3260725774 \ell_e \\ \lambda_3 &= 0.6699905218, & \omega_3 &= 0.3260725774 \ell_e \\ \lambda_4 &= 0.9305615580, & \omega_4 &= 0.1739274226 \ell_e \end{aligned}$$

334 In the case of the standard Euler-Bernoulli beam element, only two Gauss points are required as described in [35, 38,  
 335 39].

### 336 3.2. Consistent and lumped mass matrices

337 Based on the assumed displacement field within the tube element, the consistent mass matrix has the following  
 338 explicit form:

$$\mathbf{M}_e = \begin{pmatrix} \mathbf{M}_e^{11} & \mathbf{M}_e^{12} \\ \mathbf{M}_e^{21} & \mathbf{M}_e^{22} \end{pmatrix} \quad \text{with} \quad \mathbf{M}_e^{ij} = \int_{V_e} \rho (\mathbf{N}_e^i)^T \mathbf{T}^T \mathbf{T} \mathbf{N}_e^j dV$$



339 as  $\mathbf{M}_e^{ji} = (\mathbf{M}_e^{ij})^T$ , the consistent mass matrix is symmetric and its components are given by:

$$\begin{aligned}
\mathbf{M}_e^{11} &= \rho S \ell_e \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{13}{35} + \frac{6}{5} \frac{I_y}{S \ell_e} & -\frac{11 \ell_e}{210} - \frac{1}{10} \frac{I_y}{S \ell_e} & 0 & 0 \\ 0 & -\frac{11 \ell_e}{210} - \frac{1}{10} \frac{I_y}{S \ell_e} & \frac{\ell_e^2}{105} + \frac{2}{15} \frac{I_y}{S} & 0 & 0 \\ 0 & 0 & 0 & \frac{13}{35} + \frac{6}{5} \frac{e^2}{12 \ell_e^2} & -\frac{11 \ell_e}{210} - \frac{1}{10} \frac{e^2}{12 \ell_e} \\ 0 & 0 & 0 & -\frac{11 \ell_e}{210} - \frac{1}{10} \frac{e^2}{12 \ell_e} & \frac{\ell_e^2}{105} + \frac{2}{15} \frac{e^2}{12} \end{pmatrix} \\
\mathbf{M}_e^{22} &= \rho S \ell_e \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{13}{35} + \frac{6}{5} \frac{I_y}{S \ell_e} & \frac{11 \ell_e}{210} + \frac{1}{10} \frac{I_y}{S \ell_e} & 0 & 0 \\ 0 & \frac{11 \ell_e}{210} + \frac{1}{10} \frac{I_y}{S \ell_e} & \frac{\ell_e^2}{105} + \frac{2}{15} \frac{I_y}{S} & 0 & 0 \\ 0 & 0 & 0 & \frac{13}{35} + \frac{6}{5} \frac{e^2}{12 \ell_e^2} & \frac{11 \ell_e}{210} + \frac{1}{10} \frac{e^2}{12 \ell_e} \\ 0 & 0 & 0 & \frac{11 \ell_e}{210} + \frac{1}{10} \frac{e^2}{12 \ell_e} & \frac{\ell_e^2}{105} + \frac{2}{15} \frac{e^2}{12} \end{pmatrix} \\
\mathbf{M}_e^{12} &= \rho S \ell_e \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{9}{70} - \frac{6}{5} \frac{I_y}{S \ell_e} & \frac{13 \ell_e}{420} - \frac{1}{10} \frac{I_y}{S \ell_e} & 0 & 0 \\ 0 & -\frac{13 \ell_e}{420} + \frac{1}{10} \frac{I_y}{S \ell_e} & -\frac{\ell_e^2}{140} - \frac{1}{30} \frac{I_y}{S} & 0 & 0 \\ 0 & 0 & 0 & \frac{9}{70} - \frac{6}{5} \frac{e^2}{12 \ell_e^2} & \frac{13 \ell_e}{420} - \frac{1}{10} \frac{e^2}{12 \ell_e} \\ 0 & 0 & 0 & -\frac{13 \ell_e}{210} + \frac{1}{10} \frac{e^2}{12 \ell_e} & -\frac{\ell_e^2}{140} - \frac{1}{30} \frac{e^2}{12} \end{pmatrix} \quad \text{and} \quad \mathbf{M}_e^{21} = (\mathbf{M}_e^{12})^T
\end{aligned}$$

340 Note that the first  $3 \times 3$  block-matrix of each component of the mass matrix of the present tube element corresponds  
341 exactly to the component of the consistent mass matrix of the Euler-Bernoulli beam element as described in [35, 38].  
342 As an explicit time integration is retained in the following, the use of lumped or diagonal mass matrix in place of  
343 the consistent mass matrix is classically used to enhance the computational efficiency of the explicit scheme. For  
344 the Euler-Bernoulli beam element described in [35] with both translational and rotational DOFs, the lumping of the  
345 mass matrix is performed as follows. Concerning the translational terms, a ‘‘row-sum’’ approach is used. This also  
346 corresponds to the element mass preservation by dividing the total mass among the diagonal entries of the mass matrix  
347 and setting all other entries to zero. This leads to the diagonal term:  $\rho S \ell_e / 2$ . In contrast, for the rotational terms, the  
348 lumping is performed by ignoring the off-diagonal terms of the consistent mass matrix and scaling the term involving  
349 inertia in the diagonal term as performed in [38, 35]. The same procedure is here used for both beam and shell  
350 translational and rotational DOFs leading to the following lumped mass matrix:

$$\tilde{\mathbf{M}}_e = \begin{pmatrix} \tilde{\mathbf{M}}_e^{11} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{M}}_e^{22} \end{pmatrix} \quad \text{with} \quad \tilde{\mathbf{M}}_e^{11} = \tilde{\mathbf{M}}_e^{22} = \rho S \ell_e \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\ell_e^2}{105} + \frac{I_y}{S} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\ell_e^2}{105} + \frac{e^2}{12} \end{pmatrix} \quad (50)$$

351 where the same scaling is used for the terms  $I_y/S$  and  $e^2/12$  in the rotational beam and shell DOFs, respectively.

352 The local coordinate system for each tube element has been used for expressing the nodal displacement vector  $\mathbf{d}_e$ .  
353 In order to express the unknowns in the global coordinate system, the transformation matrix is used  $\mathbf{P}_e$ . Note that  
354 the present lumped mass matrix is independent of the coordinate system, i.e.  $\mathbf{P}_e \tilde{\mathbf{M}}_e \mathbf{P}_e^T = \tilde{\mathbf{M}}_e$ . The assembly is then  
355 obtained through the sum over the elements after the connectivity transformation:

$$\mathbf{P}_e \mathbf{d}_e = \mathbf{C}_e \mathbf{d} \quad (51)$$

356 with  $\mathbf{d}$  the global displacement expressed in generalized coordinates and  $\mathbf{C}_e$  the connectivity matrix associated with  
357 the finite element in question. As the two displacement vectors  $\mathbf{P}_e \mathbf{d}_e$  and  $\mathbf{d}$  are expressed in the same coordinate

358 system, the  $\mathbf{C}_e$  matrices are simply Boolean matrices. Finally, this leads to the following relation:

$$\mathbf{d}^T \mathbf{M} \ddot{\mathbf{d}} = -\mathbf{d}^T \mathbf{f}_{\text{int}} \quad \text{or equivalently} \quad \tilde{\mathbf{M}} \ddot{\mathbf{d}} = -\mathbf{f}_{\text{int}} \quad (52)$$

359 with  $\tilde{\mathbf{M}} = \sum_e \left( \mathbf{C}_e^T \tilde{\mathbf{M}}_e \mathbf{C}_e \right)$  the global lumped mass matrix and  $\mathbf{f}_{\text{int}} = \sum_e \left( \mathbf{C}_e^T \mathbf{P}_e \mathbf{f}_{e,\text{int}} \right)$  the global internal force vector.

### 360 3.3. Explicit central difference scheme

361 The explicit time integration of Eq. (52) is here performed using the central difference operator leading to:

$$\begin{cases} \dot{\mathbf{d}}^{n+1/2} &= \dot{\mathbf{d}}^n + \Delta t^n \ddot{\mathbf{d}}^n / 2 \\ \mathbf{d}^{n+1} &= \mathbf{d}^n + \Delta t^n \dot{\mathbf{d}}^{n+1/2} \\ \ddot{\mathbf{d}}^{n+1} &= \left( \tilde{\mathbf{M}}^{n+1} \right)^{-1} \left( \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1} \right) \\ \dot{\mathbf{d}}^{n+1} &= \dot{\mathbf{d}}^{n+1/2} + \Delta t^n \ddot{\mathbf{d}}^{n+1} / 2 \end{cases} \quad (53)$$

362 with  $\mathbf{d}$ ,  $\dot{\mathbf{d}}$  and  $\ddot{\mathbf{d}}$  the nodal displacements, velocities and accelerations,  $\mathbf{f}_{\text{int}}$  and  $\mathbf{f}_{\text{ext}}$  the nodal internal and external forces, respectively, the superscript  $n$  denoting the time level. As the inertia of the section can evolve, the lumped mass matrix is updated at each time step.

365 The explicit central difference scheme is one of the most popular explicit time integration operator due to its simplicity and its second-order accuracy in time. In addition, in the case of constant time step, this scheme is known to be non-dissipative [40].

### 368 3.4. Extension to higher Fourier modes

369 The previous derivation has been detailed for  $N_f = 0$  in the Fourier expansion described previously and for an in-plane motion. Considering the Fourier mode 1 leads to the integration of the two following additional DOFs  $w_1^s$  and  $\frac{\partial w_1^s}{\partial x}$  for an in-plane motion in the generalized displacement vector  $\mathbf{d}_e^i$ . Cubic interpolating shape functions are thus used for  $w_1^s$ . Then, considering the  $k$ -th Fourier mode for  $k \geq 2$  induces to take into account three additional DOFs for an in-plane motion, i.e.  $u_k^m$ ,  $w_k^m$  and  $\frac{\partial w_k^m}{\partial x}$  with  $m = s$  if  $k$  is impair and  $m = c$  if  $k$  is pair. As a consequence, linear shape functions are used for  $u_k^m$  whereas cubic shape function for  $w_k^m$ . For the same reason as previously, a 4-point Gauss quadrature approach is still used for the computation of the internal forces (cf. Eq. (49)) even if higher Fourier modes are considered. In addition, the lumped mass matrix of the tube element is modified in Eq. (50) as follows:

$$\tilde{\mathbf{M}}_e^{11} = \tilde{\mathbf{M}}_e^{22} = \rho S \ell_e \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{\ell_e^2}{105} + \frac{I_y}{S} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{\ell_e^2}{105} + \frac{e^2}{12} & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\ell_e^2}{105} + \frac{e^2}{12} & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{\ell_e^2}{105} + \frac{e^2}{12} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (54)$$

377 where the first  $3 \times 3$  block diagonal matrix corresponds to the standard Euler-Bernoulli lumped mass matrix [35, 39],  
378 the following  $2 \times 2$  block diagonal matrix corresponds to the Fourier mode 0, the next  $2 \times 2$  block diagonal to the  
379 Fourier mode 1 and the next  $3 \times 3$  to the Fourier mode  $i$ . The diagonal term for the transitional DOFs for both  
380 beam and shell is  $\rho S \ell_e / 2$ , whereas the term of the rotational DOFs for beam and shell are  $\rho S \ell_e (\ell_e^2 / 105 + I_y / S)$  and  
381  $\rho S \ell_e (\ell_e^2 / 105 + e^2 / 12)$  respectively.

382 *3.5. Stability condition and time step evaluation*

383 For the explicit time integration of an undamped system with the central difference operator, the time step  $\Delta t^n$   
 384 used in Eq. (53) has to respect the following stability condition [41, 42]:

$$\Delta t^n = C \Delta t_{\text{crit}}^n \quad \text{with} \quad \Delta t_{\text{crit}}^n = \frac{2}{\omega_{\text{max}}^n} \quad (55)$$

385 where  $0 < C \leq 1$  is the Courant number,  $\Delta t_{\text{crit}}^n$  the critical time step and  $\omega_{\text{max}}^n$  the maximum eigenvalue of the system  
 386 which is classically bounded by the largest element eigenvalue:

$$\omega_{\text{max}}^n \leq \max_e (\omega_{e,\text{max}}^n) \quad (56)$$

387 with  $\omega_{e,\text{max}}^n$  the maximum eigenvalue of the  $e$ -th element, unrestrained.

388 Therefore, to express the stability condition, the expression of the maximum frequency of the proposed tube  
 389 element is required. Instead of solving the eigenvalue problem which can be complex in the case of the present element  
 390 especially when considering higher Fourier modes, this is here performed using the natural frequencies of beams and  
 391 cylindrical shells which are given in [43]. **Even if only an in-plane motion is presented in the previous sections for**  
 392 **simplicity, we recall that a full 3-D motion is derived. As a consequence, all the corresponding contributions have**  
 393 **been also considered for numerical stability. Finally,** the maximum element eigenvalue corresponds to the maximum  
 394 of the eigenvalues associated to all of the contributions of the present tube element:

$$\omega_{e,\text{max}}^n = \max (\omega_{e,\text{b}}^n, \omega_{e,\text{s}}^n) \quad (57)$$

395 where for the beam-type contribution, i.e. beam axial and beam flexural, as described in [35, 39]:

$$\omega_{e,\text{b}}^n = \max \left( \frac{2}{\ell_e} \sqrt{\frac{E}{\rho}}, \frac{2}{\ell_e^2} \sqrt{\frac{EI_y}{\rho S}} \pi^4 \right) \quad (58)$$

396 and the natural frequencies of cylindrical shells as expressed in [43] in terms of a dimensionless frequency parameter  
 397  $\lambda$  are given by:

$$\omega_{e,\text{s}}^n = \frac{2\lambda}{a} \sqrt{\frac{E}{\rho(1-\nu^2)}} \quad (59)$$

398 with the following expressions of the dimensionless frequency parameter  $\lambda$ :

$$\lambda = \begin{cases} \lambda^r = 1 & \text{for the radial tube mode} \\ \lambda^a = \frac{a}{\ell_e} \sqrt{(1-\nu^2)} \pi^2 & \text{for the axial tube mode} \\ \lambda^t = \frac{a}{\ell_e} \sqrt{\left(\frac{1-\nu^2}{1+\nu}\right) \frac{\pi^2}{2}} & \text{for the torsional tube mode} \\ \lambda^f = \frac{a^2}{\ell_e^2} \sqrt{(1-\nu^2) \frac{\pi^4}{2}} & \text{for the flexural tube mode} \\ \lambda_{ij}^o = \frac{\sqrt{(1-\nu^2) \left(\frac{j\pi a}{\ell_e}\right)^4 + \frac{e^2}{12a^2} \left(i^2 + \left(\frac{j\pi a}{\ell_e}\right)^2\right)^4}}{i^2 + \left(\frac{j\pi a}{\ell_e}\right)^2} & \text{for the ovalization tube mode} \end{cases} \quad (60)$$

399 with  $i = 6$  and  $j = 1$  as  $i$  and  $j$  correspond to the number of circumferential and axial modes, respectively.

400 As a consequence, the maximum element eigenvalue is defined as:

$$\omega_{e,\max}^n = \max\left(\frac{2}{\ell_e} \sqrt{\frac{E}{\rho}}, \frac{2}{\ell_e^2} \sqrt{\frac{EI_y}{\rho S}} \pi^4, \frac{2\lambda_{\max}}{a} \sqrt{\frac{E}{\rho(1-\nu^2)}}\right) \quad (61)$$

401 where  $\lambda_{\max} = \max(\lambda^r, \lambda^a, \lambda^t, \lambda^f, \lambda_{ij}^o)$  given in Eq. (60) making it possible to express the stability condition used for  
402 the time step evaluation at each time iteration using Eq. (55).

#### 403 4. Numerical results

404 The new tube finite-element proposed here and detailed in the previous sections is then assessed via comparisons  
405 with analytical or numerical reference solutions. For this purpose, several test-cases on a single straight pipe under  
406 various loading conditions are considered dealing with extension, breathing due to internal pressure, pure bending  
407 and punctual force on the tube cross-section. The pipe configuration used for all the considered tests is described in  
Table 1 and respects both the thin-shell and the Euler-Bernoulli hypotheses. In the following, in order to assess the

$L$ (m)	$a$ (m)	$e$ (m)	$E$ (GPa)	$\nu$	$\rho$ (kg/m <sup>3</sup> )
10	1	0.1	200	0.3	7800

Table 1: Geometric and material properties of the pipe configuration used for the present tests.

408 grid-independence of the numerical solution, two computations are performed: the first one with a single tube element  
409 and the second one with ten tube elements. The verification of the present beam model is assessed on both static and  
410 dynamic problems. All the computations considered here are performed using a Courant number  $C = 0.5$  in Eq. (55).  
411 For all considered test-cases, it appears that the critical time step is linked to the pipe cross-section ovalization (cf.  
412 Eq. (61)) leading to the following expression:  
413

$$\Delta t = C \frac{a}{\lambda_{ij}^o} \sqrt{\frac{\rho(1-\nu^2)}{E}} \quad (62)$$

414 with  $\lambda_{ij}^o$  the dimensionless ovalization frequency parameter expressed in Eq. (60). Finally, the following values of  
415 the time step are obtained:  $\Delta t = 9.039 \times 10^{-5}$  s for one element and  $\Delta t = 9.862 \times 10^{-6}$  s for ten elements. For  
416 the considered static problems, as an explicit time integration is used for solving the tube equations of motion, the  
417 imposed loading is not imposed suddenly but with a smooth increase in time in order to avoid spurious numerical  
418 oscillations. The associated rise time which is denoted here by  $\tau$  is set to be significantly greater than the maximum  
419 time scale of each contribution of the tube element, i.e. beam axial, beam flexural, tube radial, tube torsional, tube  
420 flexural and tube ovalization, expressed in Sec. 3.5. In other words,  $\tau$  is taken to be:

$$\tau = N \mathcal{T} \quad \text{with} \quad \mathcal{T} = \max\left(\ell_e \sqrt{\frac{\rho}{E}}, \ell_e^2 \sqrt{\frac{\rho S}{EI_y \pi^4}}, \frac{a}{\lambda_{\min}} \sqrt{\frac{\rho(1-\nu^2)}{E}}\right) \quad (63)$$

421 where  $\lambda_{\min} = \min(\lambda^r, \lambda^a, \lambda^t, \lambda^f, \lambda_{ij}^o)$  given in Eq. (60) and  $N \gg 1$ .

##### 422 4.1. Test 1: Pipe contraction due to a uniform and static extension via the Poisson effect

423 The first test-case considered here consists in a simple extension. For this purpose, a symmetric displacement is  
424 imposed at the two boundaries of the tube, i.e.  $\pm u_D/2$  at each side. Due to the Poisson effect which is taken into  
425 account in the present tube model, the pipe elongation is accompanied by a pipe cross-section reduction.

426 First, the analytical solution of the present test-case is derived from the proposed model considering only a uniform  
427 radial displacement which corresponds to the 0 mode in Fourier series as described in Sec. 2.4. In the present case of a

428 uniform extension, only the beam-type axial displacement  $U_o$  and the shell-type radial displacement  $w_o^c$  are non-zero.  
 429 Considering a uniform radial displacement, the non-zero components of the strain tensor given in Eq. (11) and the  
 430 non-zero generalized forces defined in Eq. (34) are expressed as:

$$\begin{cases} \varepsilon_{xx} = \frac{\partial U_o}{\partial x} \\ \varepsilon_{\theta\theta} = \left(\frac{1}{a} - \frac{z}{a^2}\right) w_o^c \end{cases} \quad \text{and} \quad \begin{cases} N = \frac{ES}{1-\nu^2} \left( \frac{\partial U_o}{\partial x} + \nu \frac{w_o^c}{a} \right) \\ (t_w)_o^c = \frac{ES}{1-\nu^2} \frac{1}{a} \left( \nu \frac{\partial U_o}{\partial x} + \frac{w_o^c}{a} \right) \end{cases} \quad (64)$$

431 Finally, the static equilibrium equations expressed in Eq. (33) write:

$$\frac{\partial N}{\partial x} = 0, \quad (t_w)_o^c = 0 \quad (65)$$

432 leading to the two following relations verified by the axial and the radial displacements:

$$w_o^c = -\nu a \frac{\partial U_o}{\partial x} \quad \text{and} \quad \frac{\partial^2 U_o}{\partial x^2} = 0 \quad (66)$$

433 Finally, using the value of the axial displacements at the two pipe boundaries, i.e.  $U_o(0) = -\frac{u_D}{2}$  and  $U_o(L) = \frac{u_D}{2}$ ,  
 434 leads to:

$$U_o = \frac{u_D}{2L} (2x - L) \quad \text{and} \quad w_o^c = -\nu a \frac{u_D}{L} \quad (67)$$

435 In the present test-case, the elongation is taken as:  $u_D/L = 0.01$  satisfying the small deformation assumption and  
 436 leading to the following constant value:  $w_o^c/a = -3 \times 10^{-3}$ . Then, the numerical solutions are compared to the  
 437 analytical solution previously described in order to assess the numerical method used for approximating the governing  
 438 equations of the tube model.

439 The time evolution of the imposed displacement at the boundaries depicted in Fig. (6) starts with a smooth increase  
 in order to avoid spurious numerical oscillations. The associated rise time is set to be equal to  $\tau = 100 \mathcal{T}$  in Eq. (63).

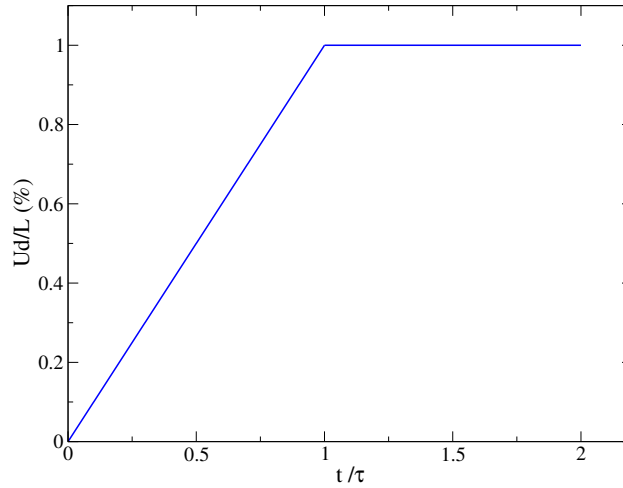


Figure 6: Test 1: Time evolution of the axial imposed normalized displacement.

440 The numerical solutions of the radial displacement obtained with one single element and ten elements are compared  
 441 with the analytical solution in Fig. (7). It is shown that the steady-state solution is in good agreement with the  
 442 analytical one as expected.  
 443

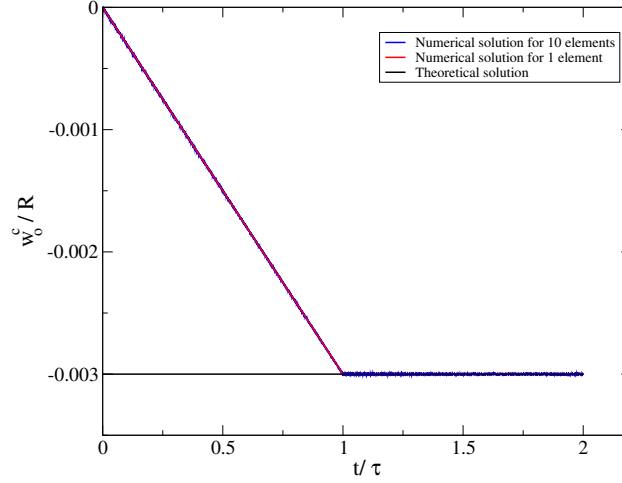


Figure 7: Test 1: Comparison between numerical and analytical solutions of the radial displacements for the elongation test.

#### 4.2. Test 2: Radial pipe expansion due to an internal pressure

The second test-case consists in a tube under an internal pressure. A uniform pressure denoted by  $P_{\text{int}}$  is applied inside the pipe and creates a uniform expansion of the tube cross-section.

As in the previous case, the analytical solution of the present test is derived from the enriched beam model developed in this paper and considering only the mode 0 in the Fourier expansion. Once again, under an internal pressure loading, only the beam-type axial displacement  $U_o$  and the shell-type radial displacement  $w_o^c$  are non-zero. In addition, let us consider that  $w_o^c$  is uniform and that  $U_o$  is null. As a consequence, the only non-zero component of the strain tensor  $\varepsilon_{\theta\theta}$  expressed in Eq. (11) and the generalized force associated with the radial pipe cross-section expansion  $(t_w)_o^c$  given in Eq. (34) can be expressed as:

$$\varepsilon_{\theta\theta} = \left( \frac{1}{a} - \frac{z}{a^2} \right) w_o^c \quad \text{and} \quad (t_w)_o^c = \frac{ES}{1-\nu^2} \frac{1}{a^2} w_o^c \quad (68)$$

The power of the internal pressure applied to the inner tube surface writes as:

$$\mathcal{P}_e = \int_{A^-} P_{\text{int}} \delta w_o^c dA^- = \int_0^L \int_0^{2\pi} P_{\text{int}} \delta w_o^c \left( a - \frac{e}{2} \right) d\theta dx = \int_0^L 2\pi P_{\text{int}} \left( a - \frac{e}{2} \right) \delta w_o^c dx \quad (69)$$

Knowing the external loading, the equilibrium equation derived from Eq. (33) becomes:

$$(t_w)_o^c = 2\pi P_{\text{int}} \left( a - \frac{e}{2} \right) \quad (70)$$

As a consequence, the radial displacement due to the internal pressure can be expressed as:

$$w_o^c = \frac{P_{\text{int}} (1-\nu^2)}{E} \frac{a}{e} \left( a - \frac{e}{2} \right) \quad (71)$$

The numerical solutions are now compared with this analytical solution. In the present test-case, a uniform internal pressure  $P_{\text{int}}$  of 100 bar is imposed following a time evolution similar to the one depicted in Fig. (6) with a smooth linear increase in order to avoid spurious numerical oscillations where  $\tau = 100 \mathcal{T}$  as defined in Eq. (63). In addition, a condition of zero axial displacement, i.e.  $U_o = 0$ , is imposed at the two boundary nodes. For a pressure of 100 bar, the analytical solution expressed in Eq. 71 leads to the following value:  $w_o^c = 4.3225 \times 10^{-4}$  m.

The comparison between the numerical solutions and the analytical radial displacement is shown in Fig. (8). The numerical results for one element is registered at the right boundary node and at the middle node for the computation with ten elements. As in the previous test-case, a good agreement between the numerical solutions and the analytical one is achieved.

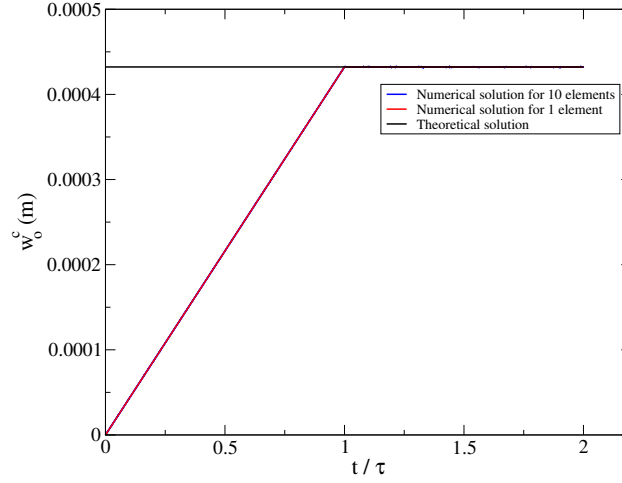


Figure 8: Test 2: Comparison between numerical and analytical solutions of the radial displacements for the breathing test - the analytical solution corresponds to the value:  $w_o^c = 4.3225 \times 10^{-4}$  m.

#### 4.3. Test 3: Dynamic radial tube motion

The present test-case is similar to the previous one except that a dynamic internal pressure loading is now considered. Following the same assumptions as previously, the equation of motion given in Eq. (32) writes as:

$$\rho S \frac{\partial^2 w_o^c}{\partial t^2} + (t_w)_o^c = 2\pi P_{\text{int}} \left( a - \frac{e}{2} \right) \quad \text{with} \quad (t_w)_o^c = \frac{ES}{1-\nu^2} \frac{1}{a^2} w_o^c \quad (72)$$

with the initial condition:  $w_o^c(t=0) = 0$  m. The analytical solution of this equation writes as:

$$w_o^c(t) = w_o^{c,\text{sta}} \left( 1 - \cos(\omega t) \right) \quad \text{with} \quad w_o^{c,\text{sta}} = \frac{P_{\text{int}} (1-\nu^2)}{E} \frac{a}{e} \left( a - \frac{e}{2} \right) \quad \text{and} \quad \omega = \frac{1}{a} \sqrt{\frac{E}{\rho (1-\nu^2)}} \quad (73)$$

$w_o^{c,\text{sta}}$  corresponds to the static solution, i.e. the analytical solution of the previous test, and  $\omega$  corresponds to the radial pulsation of a cylindrical shell as expressed in [43].

The computation of the present test-case is performed by imposing the internal pressure of 100 bar at the first time step. The corresponding numerical solutions obtained with both one element and ten elements shown in Fig. (9) correspond to a periodic function. The period obtained from the numerical solution is compared to its analytical counterpart:  $T_{\text{the}} \approx 1.1836761 \times 10^{-3}$  s versus  $T_{\text{num}} \approx 1.183688 \times 10^{-3}$  s, showing the good agreement between the two results.

In the two previous cases, only mode 0 in the Fourier expansion is considered which corresponds to the radial expansion/contraction of the tube cross-section. In the next test-case, only Fourier mode 2 is considering in order to represent the cross-section ovalization.

#### 4.4. Test 4: Ovalization due to a pipe in-plane bending

The present test-case consists in a pure in-plane bending test where a symmetric rotation is imposed at the two boundaries of the tube, i.e.  $\pm\Omega$  at each side. Due to the bending, it is expected that the tube tends to ovalize.

The analytical solution of the present test is derived from the present beam model considering only mode 2 in the Fourier series as it is detailed in Sec. 2.5. Considering in addition that  $w_2^c$  is uniform and that  $w_2^c/a \ll 1$ , the non-zero

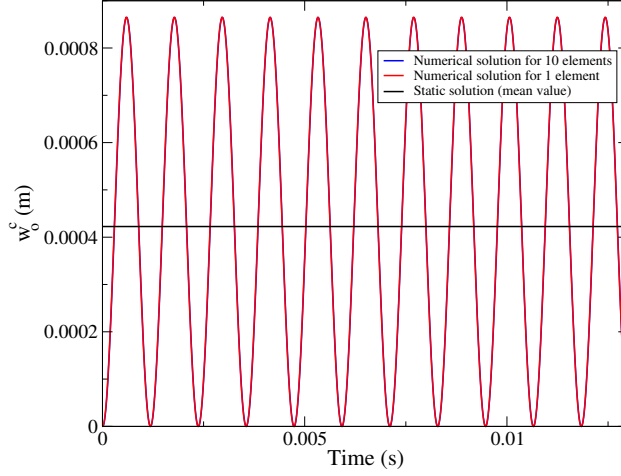


Figure 9: Test 3: Time evolution of the radial displacement in the case of an internal pressure loading.

485 forces defined in Eq. (41) can be expressed as:

$$\begin{cases} M_y &= -\frac{E}{1-\nu^2}\pi a^3 e \left(1 - \frac{3}{2}\frac{w_2^c}{a}\right) \frac{\partial^2 V_o}{\partial x^2} \\ (t_w)_2^c &= +\frac{E}{1-\nu^2}\frac{3\pi a e}{4} \left(-a \left(\frac{\partial^2 V_o}{\partial x^2}\right)^2 + \frac{e^2}{a^2}\frac{w_2^c}{a^2}\right) \end{cases} \quad (74)$$

486 and the equilibrium equations given in Eq. (40) write:

$$\frac{\partial^2 M_y}{\partial x^2} = 0, \quad (t_w)_2^c = 0 \quad (75)$$

487 with the following boundary conditions:

$$\begin{cases} V_o(0) = 0 \\ V_o(L) = 0 \end{cases} \quad \text{and} \quad \begin{cases} -\Omega_y(0) = \frac{\partial V_o}{\partial x}(0) = -\Omega \\ -\Omega_y(L) = \frac{\partial V_o}{\partial x}(L) = \Omega \end{cases} \quad (76)$$

488 As a consequence, the beam-type vertical displacement  $V_o$  and the shell-displacement  $w_2^c$  are given by:

$$V_o = \Omega \frac{x}{L} (x - L) \quad \text{and} \quad w_2^c = 4\Omega^2 a \frac{a^4}{e^2 L^2} \quad (77)$$

489 In the present test-case, the rotation at the two pipe ends is taken as  $\Omega = 0.1$  radian.

490 The numerical solutions are now compared to the analytical reference previously described. The time evolution  
491 of the imposed rotation at the two tube boundaries is once again similar as the one shown in Fig. (6). As previously, a  
492 smooth increase of the imposed rotation is considered in order to avoid spurious oscillations. In the present case, the  
493 rise time  $\tau$  is taken as  $\tau = 3000 \mathcal{T}$  as defined in Eq. (63). In addition, the beam-type vertical displacement is set to be  
494 zero at the two pipe ends, i.e.  $V_o(0) = 0$  and  $V_o(L) = 0$ .

495 The time evolution of the shell displacement corresponding to the second Fourier mode associated with the pipe  
496 cross-section ovalization is depicted in Fig. (10). The numerical results are taken at the outlet end of the tube for both  
497 one and ten elements. Once again, the analytical solution corresponds to the numerical steady-state solution. Note  
498 that, during the period  $0 \leq t \leq \tau$  where the imposed rotation  $\Omega$  evolves linearly in time, the radial displacement  $w_2^c$   
499 follows a quadratic behavior. This is explained as  $w_2^c$  evolves with the square of  $\Omega$  as shown in Eq. (77).



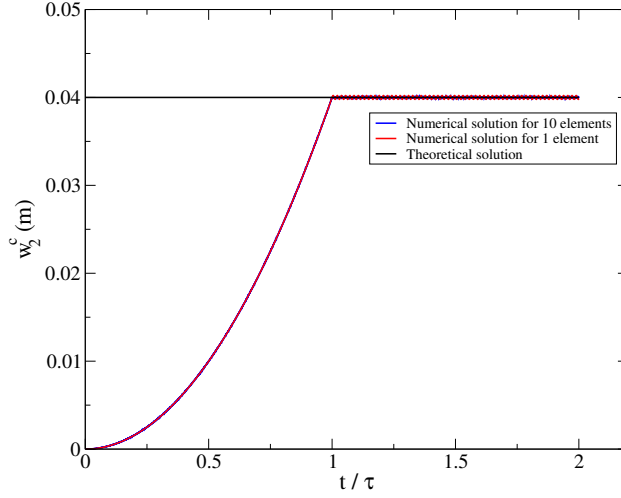


Figure 10: Test 4: Comparison between numerical and analytical solutions of the shell-type radial displacement corresponding to the second Fourier mode.

#### 4.5. Test 5: Cross-section deformation of a clamped tube subject to a distributed patch load

This test-case consists in a tube clamped at its two ends and subjected to a downward distributed patch load of amplitude  $F_z = 10^{10}$  N acting at the mid-region of its mid-surface within an area given by  $0.4L \leq x \leq 0.6L$ ,  $\pi/2 - \theta_p \leq \theta \leq \pi/2 + \theta_p$  at  $z = 0$  with  $\theta_p = \pi/100$  as depicted in Fig. (11). A reference numerical solution is first

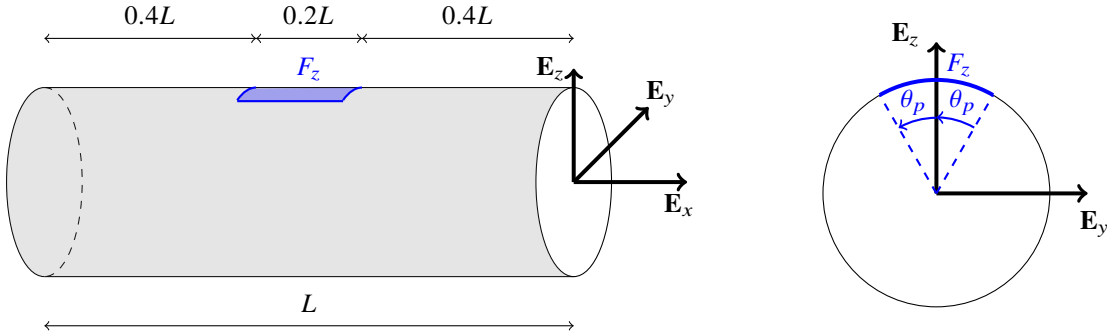


Figure 11: Test 5: Clamped tube subjected to a distributed patch load - patch loading area and direction of the vertical force.

obtained with shell elements in order to identify the tube cross-section deformation due to the present loading. For this purpose, this reference numerical solution is obtained using Code\_Aster (<https://www.code-aster.org>) with the 9-node linear and bi-quadratic quadrilateral shell elements named Coque\_3D [44]. The corresponding computation is performed with 9 082 elements, 32 068 nodes and 171 345 DOFs. Afterwards, the ability of the present tube model to retrieve the tube deformation is assessed. To this end, the distributed loading is applied with a smooth increase and  $\tau = 100\mathcal{T}$  in Eq. (63) similarly as in Fig. (6). Due to the longitudinal variation of the distributed patch loading, 10 elements are used in the present computations, i.e.  $\ell_e = L/10$ . The external force is here applied to the present tube model through the virtual power principle where the power of this external force can be expressed as:

$$\mathcal{P}_e = -F_z \int_{0.4L}^{0.6L} \int_{\pi/2-\theta_p}^{\pi/2+\theta_p} \left( \delta V_o + \sin \theta \delta w_o + \cos \theta \delta v_o \right) d\theta dx$$

In practice, the external force is applied on the two tube elements located at the mid-length of the pipe, i.e.  $0.4L \leq x \leq 0.5L$  for the first element and  $0.5L \leq x \leq 0.6L$  for the second one. The different tests performed for the analysis of the present tube model on this test-case are described in the following.

515 *4.5.1. Influence of the number of Fourier modes*

516 First, the influence of the number of modes considered in the Fourier expansion in terms of the circumferential  
 517 variable expressed in Eq. (15) is examined. For this purpose, the numerical solutions of the tube model using  $N_f = 2$ ,  
 518  $N_f = 4$  and  $N_f = 6$  are compared to the reference shell-type solution. Fig. (12) shows the tube cross-section  
 deformation at  $x = 0.5L$  obtained with the different computations as well as the initial undeformed circular cross-

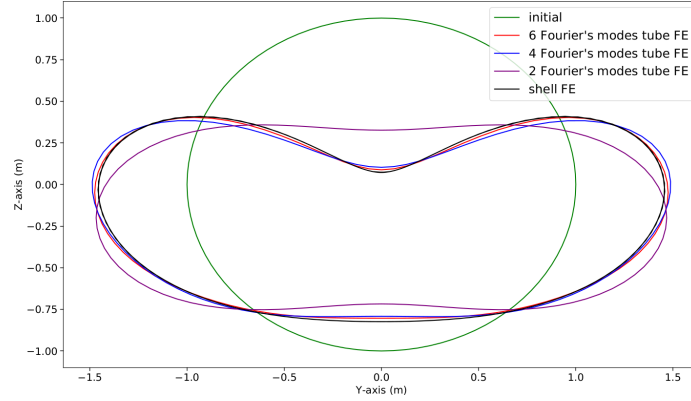


Figure 12: Test 5: Influence of the number of Fourier modes on the tube cross-section deformation at  $x = 0.5L$ .

519 section. The tube cross-section obtained with  $N_f = 2$  has an elliptical shape with a slight asymmetry which is due  
 520 to the inextensibility hypothesis as discussed in [34]. In contrast, the tube cross-section obtained with  $N_f = 4$  and  
 521  $N_f = 6$  is in good agreement with the reference shell solution with an improved concordance with  $N_f = 6$ . Note that  
 522 this result is in accordance with the following references [6, 15, 45] where  $N_f = 6$  is also retained by the authors. In  
 523 the following,  $N_f = 6$  is thus considered.  
 524

525 *4.5.2. Influence of the warping terms*

526 The influence of the consideration of the warping terms in the tube displacement field is now studied. The tube  
 cross-sections at  $x = 0.5L$  obtained with and without the warping terms are compared in Fig. (13). It is clearly shown

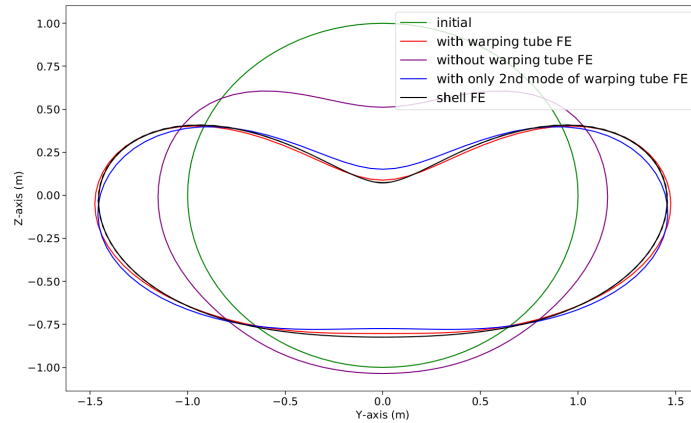


Figure 13: Test 5: Influence of the warping terms on the tube cross-section deformation at  $x = 0.5L$ .

527 that considering the warping terms is necessary to retrieve the reference deformation. This shows the interaction  
 528 between warping and ovalization terms through the distortion  $\varepsilon_{x\theta}$ . Note that this coupling is also observed in the  
 529 generalized forces as terms in  $u_i^{c/s}$  are involved in the forces  $(t_w)_i^{c/s}$  (see Eq. (B.13)). In addition, it has to be noticed  
 530

531 that the second Fourier mode for the warping terms  $u_o$  seems to be sufficient to be in agreement with the shell solution.  
 532 However, in the following the warping terms are considered in conjunction with  $N_f = 6$  for  $u_o$ ,  $v_o$  and  $w_o$ .

### 533 4.5.3. Influence of the non-linear coupling terms

534 The influence of the consideration of the non-linear coupling terms in the tube kinematics is now analyzed. Fig. (14) shows the cross-section deformations at  $x = 0.5L$  obtained with or without the non-linear terms. This demon-

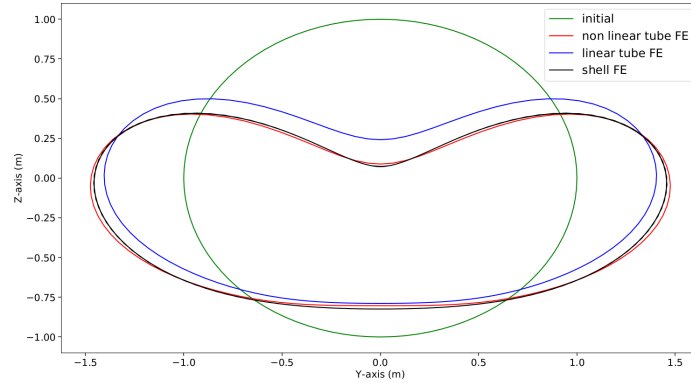


Figure 14: Test 6: Influence of the non-linear coupling terms on the tube cross-section deformation at  $x = 0.5L$ .

535 strates the requirement of considering the non-linear coupling to retrieve the reference shell deformation. To complete  
 536 the analysis, the same comparison is now performed at different locations along the pipe, i.e.  $x = 0.6L$ ,  $x = 0.7L$ ,  
 537  $x = 0.8L$  and  $x = 0.9L$ . In addition, the warping of the tube cross-section is also examined. Fig. (15) shows the  
 538 corresponding results. At  $x = 0.6L$ , the highly deformed cross-section presents also a significant warping as well  
 539 as a notable rotation which is not observed at  $x = 0.5L$  due to the symmetry of the present problem. Once again,  
 540 considering the non-linear coupling makes it possible to improve the agreement between the tube and the shell solu-  
 541 tions. The more the sections are far from the pipe center the less is the corresponding deformation (ovalization and  
 542 warping). In the same way, the influence of the non-linear coupling is reduced as the section moves away. In all the  
 543 observed locations, the solutions considering the non-linear coupling is in good agreement with the reference solution.  
 544

545  
 546 The previous analysis is performed on the tube cross-section deformation including both ovalization and warping.  
 547 Focus is now given to the local generalized stress field for both linear and non-linear tube models. This is performed  
 548 through the comparison with the numerical solutions obtained with shell finite-elements.

### 549 4.5.4. Analysis of the longitudinal stress component and comparison with shell elements

550 Comparison with shell-type solution is now performed on the longitudinal stress component  $\sigma_{xx}$  given by:

$$\sigma_{xx} = \sigma_{xx}^m + z\sigma_{xx}^k \quad \text{with} \quad \sigma_{xx}^m = \frac{E}{1-\nu^2} (\varepsilon_{xx}^m - \nu \varepsilon_{\theta\theta}^m) \quad \text{and} \quad \sigma_{xx}^k = \frac{E}{1-\nu^2} (k_{xx} - \nu k_{\theta\theta})$$

551 The profile of this contribution along the circumferential direction at  $x = 0.5L$  is depicted in Fig. (16) for the shell-  
 552 type modeling, the non-linear and the linear tube model in conjunction with six Fourier modes at  $z = 0$ , i.e. at the  
 553 mid-surface, for the membrane term and at  $z = e/2$  which corresponds to the outer surface. Concerning the membrane  
 554 term  $\sigma_{xx}^m$ , it has to be noticed that the three models lead to a similar oscillatory response with a first peak associated  
 555 with a compressive stress at  $\theta = \pi/2$  which corresponds to the location of the external load center and a second  
 556 compressive stress peak at  $\theta = 3\pi/2$ . In addition, two peaks associated with a tensile stress are also obtained but  
 557 their locations differ from one model to the other. With the shell model, these two peaks are located at  $\theta = \pi$  and  
 558  $\theta = 2\pi$ . However, with the present tube model (for both linear and non-linear formulations), the analysis of the strain  
 559 tensor components shows that these tensile stress peaks are located at the locations of the two local extrema of the  $\varepsilon_{xx}^m$   
 560 component. Due to the inextensibility assumption for Fourier modes of order greater than or equal to two considered

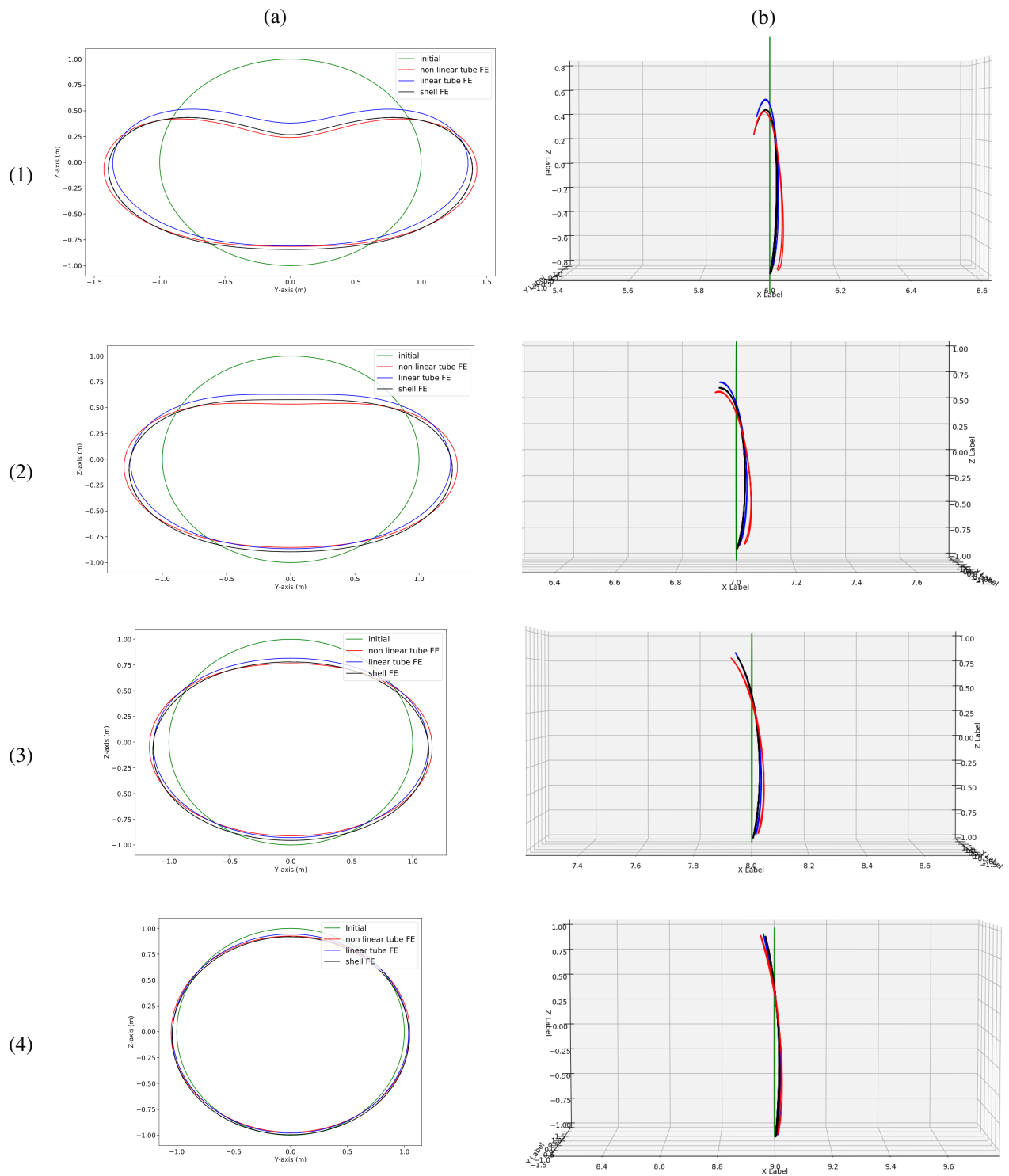


Figure 15: Test 5: Influence of the non-linear coupling on the tube cross-section (a) ovalization and (b) warping at different locations along the pipe: (1) at  $x = 0.6L$ , (2) at  $x = 0.7L$ , (3) at  $x = 0.8L$  and (4) at  $x = 0.9L$ .

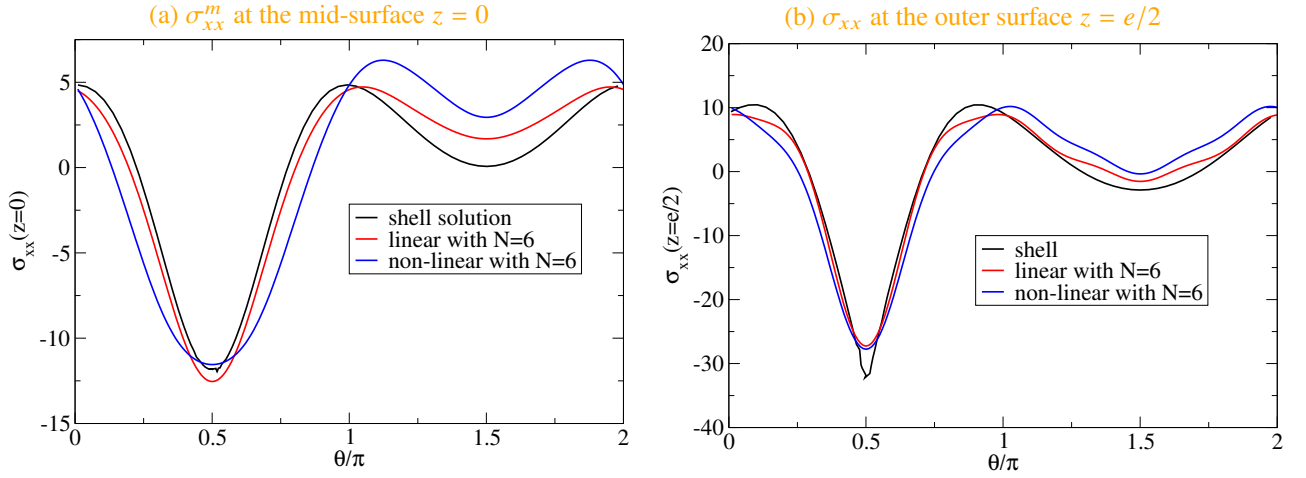


Figure 16: Test 5: Longitudinal stress component  $\sigma_{xx}$  at  $x = 0.5L$  (a) for  $z = 0$  and (b) for  $z = e/2$ , comparison between shell elements, non-linear and linear tube model with 6 Fourier modes, the stress values are divided by  $10^9$  Pa.

561 in the present formulation, only the first Fourier mode is involved in  $\varepsilon_{\theta\theta}^m$  which, finally, has no influence on these two  
562 *tensile stress* peaks. Nevertheless, a quite good agreement is obtained with the three models. This agreement seems to  
563 be furthermore improved for the  $\sigma_{xx}$  component at the outer surface. Once again, two *compressive stress* peaks and  
564 two *tensile stress* peaks are observed in the shell solution which are retrieved with the present tube model with both  
565 linear and non-linear formulations.

566  
567 The previous analysis demonstrates the accuracy of the present tube model in comparison with a shell solution.  
568 The two types of numerical solution are now compared in terms of computational efficiency.

#### 569 4.5.5. Further discussion on the computational efficiency of the present tube model

570 In order to estimate the computational savings providing by the use of the present tube model with respect to  
571 the use of shell elements, focus is here firstly given to the number of DOFs involved in the two types of model  
572 considered here. Classically when using quadrangle shell elements, it corresponds to 4-node elements with 6 DOFs  
573 per node (3 translations and 3 rotations). Let us denote by  $\mathcal{N}$  the number of shell elements used to discretize the pipe  
574 circumference. A lower estimation of  $\mathcal{N}$  is 16 where only 4 nodes are used for the discretization of a quarter of the pipe  
575 circumference which corresponds to a quite coarse description of the pipe perimeter. Using only one shell in the pipe  
576 longitudinal direction leads to 192 DOFs. In contrast, when using one single tube element, only 2 nodes are considered  
577 with 42 DOFs per nodes (or 22 DOFs for an in-plane motion), i.e. 84 DOFs (or 44 DOFs for an in-plane motion).  
578 As a consequence, with similar length and stability condition, the use of the present tube model makes it possible to  
579 reduce the computational effort by a factor of 2.28 (or 4.56 for an in-plane motion). Obviously, the computational  
580 savings are increased when the pipe circumference is more accurately discretized. For example, in the present test-  
581 case,  $\mathcal{N} = 64$  nodes are used for the pipe perimeter leading to a ratio of 9.14 (or 17.45 for an in-plane motion) between  
582 the DOFs with shell elements and the DOFs with one single tube element showing the significant reduction of the  
583 number of DOFs. Then, another aspect which has to be considered for the evaluation of computational efficiency of  
584 the present tube model in comparison with standard shell elements is the time for constructing generalized forces. For  
585 this purpose, the size and sparseness of the stiffness matrices when using the tube element and the standard 4-node  
586 shell finite element are here studied and detailed in Appendix C. It is shown that the stiffness matrix of the present  
587 tube element has 1948 non-zero components using the non-linear formulation and 476 non-zero components when it  
588 is derived following the linear formulation for one single element. In comparison, the shell stiffness matrix has 208  
589 non-zero components for one single element showing a ratio of 9.36 for the non-linear tube and a ratio of 2.29 for the  
590 linear tube. As it is discussed previously  $\mathcal{N} = 16$  shell elements represent a coarse description of the pipe perimeter.  
591 However, it leads to a greater time for constructing the stiffness matrix when using shell elements in comparison

with only one single tube element for the same section. Finally, focus is now given to the comparison between the critical time step obtained with the present tube model and the one obtained with shell elements as it directly drives the computational cost for dynamic analysis. In the present case, as discussed previously, the critical time step is given by the cross-section ovalization (cf. Eq. (62)) leading to the following value:  $\Delta t_{\text{tube}} \approx 9.862 \times 10^{-6}$  s for  $\ell_e = L/10$  (corresponding to 10 elements in the pipe length). For shell elements, the critical time step should be rigorously given by the most restrictive mode among traction, bending and torsion which is in general due to the rotational DOFs. In explicit dynamics, the rotary mass can be scaled to permit larger time steps without a loss of stability [46, 41]. As a consequence, the critical time step is given by the traction mode with the following expression:

$$\Delta t_{\text{shell}} = C \ell_e \sqrt{\frac{\rho(1-\nu^2)}{E}} \quad (78)$$

leading to the value:  $\Delta t_{\text{tube}} \approx 9.25 \times 10^{-6}$  s for  $\ell_e = 2\pi a/64$  (corresponding to 64 elements in the pipe perimeter) with a Courant number  $C = 0.5$ . This leads to a similar value for the critical time step between the proposed tube element and shell elements. This analysis clearly demonstrates the interest of the present tube model in terms of computational efficiency. In addition, it has to be noticed that for FSI problems, the present tube model has to be coupled with a 1-D fluid model for the internal fluid. In that case, only one 1-D fluid cell is required in conjunction with the present tube model as described in [39]. In contrast, it is necessary to mesh the 3-D internal fluid domain inside the tube when it is discretized with shell elements. Once again it strongly increases the computational effort associated with the use of shell elements in contrast to the one of the present tube model. What is more, it also makes it possible to avoid the numerical problems due to the rezoning procedure with large pipe cross-section deformations where robustness problems can appear due to the apparition of high distortions of meshes in the fluid domain. This is an additional major interest of using the present tube model.

## 5. Conclusion and perspectives

A novel straight thin-walled tube element has been here proposed in order to take deformations of its cross-section and inertia evolution into account. The present kinematics is decomposed into Euler-Bernoulli beam, Love-Kirchhoff shell contributions and a non-linear coupling between cross-section deformation and neutral axis rotation. Concerning the shell kinematics, the Flügge thin cylindrical shell theory is used. Finally, Fourier expansion in terms of the circumferential variable is applied for the mid-surface DOFs in which coefficients producing a rigid-body motion are excluded to avoid redundancy with beam DOFs. In addition, for Fourier modes of order greater than or equal to two, an inextensibility of the section is assumed. Following this derivation, the proposed tube element DOFs depend only on the longitudinal variable while also considering radial expansion, warping as well as ovalization of its cross-section. In addition, the consideration of the non-linear coupling terms ensures that the inertia of the tube section can evolve with its deformation. A comparison between the proposed formulation and the previously published ones is discussed showing that the present formulation consider complete Fourier expansion for all the shell DOFs which thus can be seen as a generalization of the previous approaches. In addition, the present formulation seems to be the simplest one able to consider the cross-section deformations as well as the changes of inertia as only one non-linear coupling term is taken into account for this purpose. Note that this non-linear term is not sufficient to describe all possible local and global buckling. Then, the tube equations of motion are obtained and expressed through the virtual power principle. Afterwards, an explicit updated Lagrangian Finite-Element solver is proposed for the present tube model. In particular, the lumped mass matrix is described as well as the stability condition of the explicit scheme taking into account both beam and shell contributions. Then, the proposed tube model is assessed on several test-cases. A first series of simple cases where analytical solutions of the tube model can be obtained is considered in order to evaluate the numerical method. Finally, a more complex test-case involving a distributed patch loading is studied. The numerical solution of the present tube model is here compared with a reference solution obtained with shell elements. The influence of the Fourier modes number as well as the warping terms and the non-linear coupling is examined. The shell-type cross-section deformation is retrieved with the present tube model. Finally, a discussion concerning the computational efficiency of the proposed element is also provided in order to illustrate the corresponding savings in comparison with the use of shell elements.

638 The present tube element has been proposed and derived in the context of straight thin-walled tubular geometries.  
639 However, realistic piping systems involve elbows and bends. For this reason, the present model has first to be extended  
640 to curved thin-walled tubes via the introduction of the elbow's radius of curvature and the associated local frame  
641 of reference where the tube displacement field has to be expressed. In addition, only a linear elastic behavior is  
642 considered in the present study. Plasticity has thus to be taken into account in the future in order to perform inelastic  
643 analysis of piping systems where additional questions arise. In particular, a yield criterion taking into account all  
644 of the generalized forces of the tube element has to be defined. In addition, focus has also to be given to the local  
645 detection of the tube region where plasticity occurs. Finally, for considering FSI problems, the present tube model has  
646 also to be coupled with a 1-D fluid model in a similar way as it is proposed in [39] where a standard Euler-Bernoulli  
647 beam element is used. In this kind of coupling, the cross-section deformation has a direct impact on the wave speed  
648 of the internal fluid pressure waves whereas the internal fluid pressure variation has an influence on the tube section  
649 evolution. These different issues are under consideration for current research in order to use the present tube element  
650 for the simulation of fast-transient events occurring in industrial piping systems such as pipe whipping for example.

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- 656 [1] M. P. Païdoussis, Fluid-Structure Interactions, Slender structures and axial flow, Volume 2, Elsevier Academic Press, 2003.
- 657 [2] M. P. Païdoussis, Fluid-Structure Interactions, Slender structures and axial flow, Volume 1, 2nd edition, Elsevier Academic Press, 2014.
- 658 [3] A. S. Tijsseling, Fluid-structure interaction in liquid-filled pipe systems: A review, *J. Fluids Struct.* 10 (1996) 395–420.
- 659 [4] S. Li, B. W. Karney, G. Liu, FSI research in pipeline systems A review of the literature, *J. Fluids Struct.* 57 (2015) 277–297.
- 660 [5] T. von Kármán, Über die Formänderung dünnwandiger Rohre, insbesondere federnder Ausgleichsrohre, *Z. Ver. deut. Ing.* 55 (1911) 1889–  
661 1895, (in German).
- 662 [6] H. Ohtsubo, O. Watanabe, Stress analysis of pipe bends by ring elements, *ASME J. Pressure Vessel Technol.* 100 (1) (1978) 122–122.
- 663 [7] K.-J. Bathe, C. A. Almeida, A simple and effective pipe elbow element, linear analysis, *ASME J. Appl. Mech.* 47 (1980) 93–100.
- 664 [8] A. Millard, R. Roche, Elementary solutions for the propagation of ovalization along straight pipes and elbows, *Int. J. Pres. Ves. & Piping*  
665 16 (1) (1984) 101–129.
- 666 [9] C. Militello, A. E. Huespe, A displacement-based pipe elbow element, *Comput. Struct.* 29 (2) (1988) 339–343.
- 667 [10] M. Abo-Elkhier, Analysis of pipe bends using pipe elbow element, *Comput. Struct.* 37 (1) (1990) 9–15.
- 668 [11] M.-N. Berton, A simplified method for elastic calculation for pipes based on the beam theory and allowing for section ovalization, *Int. J. Pres.*  
669 *Ves. & Piping* 51 (1) (1992) 53–83.
- 670 [12] S. A. Karamanos, J. L. Tassoulas, Tubular members I: stability analysis and preliminary results, *ASCE J. Eng. Mech.* 122 (1) (1996) 64–71.
- 671 [13] E. M. M. Fonseca, F. J. M. Q. de Melo, C. A. M. Oliveira, Determination of flexibility factors in curved pipes with end restraints using a  
672 semi-analytical formulation, *Int. J. Pres. Ves. & Piping* 79 (12) (2002) 829–840.
- 673 [14] K. Weicker, R. Salahifar, M. Mohareb, Shell analysis of thin walled pipes. Part I - Field equations and solution, *Int. J. Pres. Ves. & Piping* 87  
674 (2010) 402–413.
- 675 [15] K. Weicker, R. Salahifar, M. Mohareb, Shell analysis of thin walled pipes. Part II - Finite element formulation, *Int. J. Pres. Ves. & Piping* 87  
676 (2010) 414–423.
- 677 [16] S. Attia, M. Mohareb, M. Martens, N. Y. Ghodsi, Y. Li, S. Adeeb, Shell finite element formulation for geometrically nonlinear analysis of  
678 straight thin-walled pipes, *Int. J. Nonlinear Mech.* 137 (2021) 103829.
- 679 [17] S. Attia, M. Mohareb, M. Martens, N. Y. Ghodsi, Y. Li, S. Adeeb, Shell finite element formulation for geometrically nonlinear analysis of  
680 curved thin-walled pipes, *Thin-Walled Struct.* 173 (2022) 108971.
- 681 [18] K.-J. Bathe, C. A. Almeida, A simple and effective pipe elbow element, interaction effects, *ASME J. Appl. Mech.* 49 (1982) 165–171.
- 682 [19] K.-J. Bathe, C. A. Almeida, A simple and effective pipe elbow element, pressure stiffening effects, *ASME J. Appl. Mech.* 49 (1982) 914–915.
- 683 [20] E. Ruocco, J. N. Reddy, A new nonlinear 5-parameter beam model accounting for the Poisson effect, *Int. J. Nonlinear Mech.* 142 (2022)  
684 103996.
- 685 [21] M. Chadha, M. D. Todd, The mathematical theory of a higher-order geometrically-exact beam with a deforming cross-section, *Int. J. Solids*  
686 *Struct.* 202 (2020) 854–880.
- 687 [22] I. Sokolov, S. Krylov, I. Harari, Extension of non-linear beam models with deformable cross sections, *Comput. Mech.* 56 (2015) 999–1021.
- 688 [23] R. F. Vieira, F. B. E. Virtuoso, E. B. R. Pereira, Buckling of thin-walled structures through a higher order beam model, *Comput. Struct.* 180  
689 (2017) 104–116.
- 690 [24] L. Duan, J. Zhao, A geometrically exact cross-section deformable thin-walled beam finite element based on generalized beam theory, *Comput.*  
691 *Struct.* 218 (2019) 32–59.
- 692 [25] J. N. Reddy, *Theories and analyses of beams and axisymmetric circular plates*, CRC Press, 2022.
- 693 [26] Y. Pascal Abdellaoui, Modèles mécaniques de poutre enrichis pour la simulation de tubes minces sous pression, Ph.D. thesis, Université  
694 Paris-Saclay (2022).

- 695 [27] M. Amabili, *Nonlinear Vibrations and Stability of Shells and Plates*, Cambridge University Press, New York, USA, 2008.
- 696 [28] A. W. Leissa, *Vibration of shells*, Tech. Rep. NASA SP-288, NASA (1973).
- 697 [29] W. Flügge, *Stresses in Shells*, Springer Berlin Heidelberg, Germany, 1973.
- 698 [30] S. Timoshenko, *Theory Of Plates And Shells*, McGraw Hill, New-York, USA, 1959.
- 699 [31] J. D. Wood, The flexure of a uniformly pressurized, circular, cylindrical shell, *ASME J. Appl. Mech.* 25 (4) (1958) 453–458.
- 700 [32] L. Madureira, F. Q. Melo, Stress analysis of curved pipes with a hybrid formulation, *Int. J. Pres. Ves. & Piping* 81 (3) (2004) 243–249.
- 701 [33] L. G. Brazier, On the flexure of thin cylindrical shells and other “thin” sections, *Proc. Royal Society London, Series 116 (773)* (1927) 104–114.
- 702
- 703 [34] S. A. Karamanos, Bending instabilities of elastic tubes, *Int. J. Solids Struct.* 39 (2002) 2059–2085.
- 704 [35] F. Daude, P. Galon, A. Shams, *Numerical investigations of a two-way coupled fluid-structure interaction approach for fast transients in fluid-filled flexible piping systems*, In preparation.
- 705
- 706 [36] H. Cao, M. Mohareb, I. Nistor, Finite element for the dynamic analysis of pipes subjected to water hammer, *J. Fluids Struct.* 93 (2020) 102845.
- 707
- 708 [37] J. S. Walker, J. W. Phillips, Pulse propagation in fluid-filled tubes, *ASME J. Appl. Mech.* 44 (1977) 31–35.
- 709 [38] P. Galon, M. Lepareux, Documentation théorique : Modélisation des tuyauteries dans *Europlexus*, Tech. Rep. DEN/DM2S/SEMT/DYN/RT/01-021/A, CEA (2001).
- 710
- 711 [39] F. Daude, P. Galon, A Finite-Volume approach for compressible single- and two-phase flows in flexible pipelines with fluid-structure interaction, *J. Comput. Phys.* 362 (C) (2018) 375–408.
- 712
- 713 [40] S. W. Key, Transient response by time integrations: review of implicit and explicit operators, in: J. Donea (Ed.), *Advanced Structural Dynamics*, Applied Science Publishers, UK, 1980, pp. 71–95.
- 714
- 715 [41] T. J. R. Hughes, *The Finite Element Method - Linear Static and Dynamic Finite Element Analysis*, Dover Publications, Inc., 2000.
- 716 [42] T. Belytschko, W. K. Liu, B. Moran, K. I. Elkhodary, *Nonlinear Finite Elements for Continua and Structures*, 2d edition, Wiley, 2014.
- 717 [43] R. D. Blevins, *Formulas For Dynamics, Acoustics And Vibration*, Wiley, 2016.
- 718 [44] T. De Soza, *Modélisation Coque\_3D*, documentation Code\_Aster (2015).
- 719 [45] H. D. Hibbit, E. K. Leung, An approach to detailed inelastic analysis of thin walled pipelines, in: T. J. R. Hughes, A. Pifko, A. Jay (Eds.), *Nonlinear Finite Element Analysis of Plates and Shells*, ASME, AMD 48, 1981, pp. 83–118.
- 720
- 721 [46] T. J. R. Hughes, M. Cohen, M. Haroun, *Reduced and selective integration techniques in the Finite Element analysis of plates*, *Nuclear Eng. Design* 46 (1978) 203–222.
- 722
- 723 [47] J.-S. Chang, W.-J. Chiou, Natural frequencies and critical velocities of fixed-fixed laminated circular cylindrical shells conveying fluids, *Comput. Struct.* 57 (5) (1995) 929–939.
- 724
- 725 [48] M. Ji, K. Inaba, F. Triawan, Vibration characteristics of cylindrical shells filled with fluid based on first-order shell theory, *J. Fluids Struct.* 85 (2019) 275–291.
- 726

## 727 Appendix A. Derivatives with respect to coordinates $x$ , $\theta$ and $z$

728 As given in Eq. (1), the coordinates of the point  $\mathbf{m}$  in the deformed configuration following an in-plane motion in the  $(\mathbf{E}_x, \mathbf{E}_z)$  plane, i.e.  $W_o = 0$  and  $\Omega_x = \Omega_z = 0$ , are expressed as:

$$\mathbf{m} = (x + U_o + u + [(a + z + w) \sin \theta + v \cos \theta] \Omega_y) \mathbf{E}_x + (V_o \sin \theta + a + z + w - u \Omega_y \sin \theta) \mathbf{E}_R + (V_o \cos \theta + v - u \Omega_y \cos \theta) \mathbf{E}_\theta$$

730 The coupling terms involving the tube axis rotation and the warping degree-of-freedom, i.e.  $u \Omega_y$ , are here neglected leading to:

$$\mathbf{m} = (x + U_o + (a + z) \Omega_y \sin \theta + u + \Omega_y h) \mathbf{E}_x + (V_o \sin \theta + a + z + w) \mathbf{E}_R + (V_o \cos \theta + v) \mathbf{E}_\theta$$

732 with  $h = (w \mathbf{E}_R + v \mathbf{E}_\theta) \cdot \mathbf{E}_z = w \sin \theta + v \cos \theta$ . The derivatives of the coordinates of the point  $\mathbf{m}$  with respect to the variables  $x$ ,  $\theta$  and  $z$  are given by:

$$\begin{cases} \partial_x \mathbf{m} = \left( 1 + \frac{\partial U_o}{\partial x} + \frac{\partial u}{\partial x} + (a + z) \frac{\partial \Omega_y}{\partial x} \sin \theta + \frac{\partial}{\partial x} (\Omega_y h) \right) \mathbf{E}_x + \left( \frac{\partial V_o}{\partial x} \sin \theta + \frac{\partial w}{\partial x} \right) \mathbf{E}_R + \left( \frac{\partial V_o}{\partial x} \cos \theta + \frac{\partial v}{\partial x} \right) \mathbf{E}_\theta \\ \partial_z \mathbf{m} = \left( \frac{\partial u}{\partial z} + \Omega_y \sin \theta + \Omega_y \frac{\partial h}{\partial z} \right) \mathbf{E}_x + \left( 1 + \frac{\partial w}{\partial z} \right) \mathbf{E}_R + \frac{\partial v}{\partial z} \mathbf{E}_\theta \\ \partial_\theta \mathbf{m} = \left( \frac{\partial u}{\partial \theta} + (a + z) \Omega_y \cos \theta + \Omega_y \frac{\partial h}{\partial \theta} \right) \mathbf{E}_x + \left( \frac{\partial w}{\partial \theta} - v \right) \mathbf{E}_R + \left( \frac{\partial v}{\partial \theta} + a + z + w \right) \mathbf{E}_\theta \end{cases}$$

734 The metrics associated with this gradient are defined as:

$$g_{ij} \equiv \partial_i \mathbf{m} \cdot \partial_j \mathbf{m}$$



735 with  $i = x, \theta, z$  and  $j = x, \theta, z$ . Then, neglecting both the second-order beam-type displacement terms and the second-  
736 order shell-type displacement terms leads to the following metrics expressions:

$$\left\{ \begin{array}{l} g_{xx} = 1 + 2 \left( \frac{\partial U_o}{\partial x} + \frac{\partial u}{\partial x} + (a+z) \frac{\partial \Omega_y}{\partial x} \sin \theta \right) + 2 \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial x} + 2h \frac{\partial \Omega_y}{\partial x} \\ g_{\theta\theta} = (a+z)^2 + 2(a+z) \left( \frac{\partial v}{\partial \theta} + w \right) \\ g_{zz} = 1 + 2 \frac{\partial w}{\partial z} \\ g_{x\theta} = \frac{\partial u}{\partial \theta} + (a+z) \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \cos \theta + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial \theta} + (a+z) \frac{\partial v}{\partial x} \\ g_{xz} = \frac{\partial u}{\partial z} + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \sin \theta + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial z} + \frac{\partial w}{\partial x} \\ g_{z\theta} = \frac{\partial w}{\partial \theta} - v + (a+z) \frac{\partial v}{\partial z} \end{array} \right.$$

737 The strain tensor is directly obtained with the metrics using the relations:

$$\left\{ \begin{array}{l} 2 \varepsilon_{xx} = g_{xx} - 1 \\ 2 \varepsilon_{\theta\theta} = \frac{1}{(a+z)^2} g_{\theta\theta} - 1 \\ 2 \varepsilon_{zz} = g_{zz} - 1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} 2 \varepsilon_{x\theta} = \frac{1}{a+z} g_{x\theta} \\ 2 \varepsilon_{xz} = g_{xz} \\ 2 \varepsilon_{z\theta} = \frac{1}{a+z} g_{z\theta} \end{array} \right.$$

738 As a consequence, the strain-displacement relations are given by:

$$\left\{ \begin{array}{l} \varepsilon_{xx} = \frac{\partial U_o}{\partial x} + (a+z) \sin \theta \frac{\partial \Omega_y}{\partial x} + \frac{\partial u}{\partial x} + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial x} + h \frac{\partial \Omega_y}{\partial x} \\ \varepsilon_{\theta\theta} = \frac{1}{a+z} \left( \frac{\partial v}{\partial \theta} + w \right) \\ \varepsilon_{zz} = \frac{\partial w}{\partial z} \\ 2 \varepsilon_{x\theta} = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \cos \theta + \frac{1}{a+z} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{1}{a+z} \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial \theta} \\ 2 \varepsilon_{xz} = \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \sin \theta + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \left( \Omega_y + \frac{\partial V_o}{\partial x} \right) \frac{\partial h}{\partial z} \\ 2 \varepsilon_{z\theta} = \frac{1}{a+z} \left( \frac{\partial w}{\partial \theta} - v \right) + \frac{\partial v}{\partial z} \end{array} \right. \quad (\text{A.1})$$

## 739 Appendix B. Virtual power principle

740 The virtual power principle writes:

$$\mathcal{P}_a = \mathcal{P}_i + \mathcal{P}_e \quad \text{with} \quad \mathcal{P}_a = \int_{\Omega} \rho \frac{\partial^2 \mathbf{U}}{\partial t^2} \cdot \delta \mathbf{U} \, dV \quad \text{and} \quad \mathcal{P}_i = - \int_{\Omega} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, dV \quad (\text{B.1})$$

741 with the elementary volume  $dV = a \, d\theta dz dx$  and the elementary surface  $dS = a \, d\theta dz$ .

742 *Appendix B.1. Power of inertia*

743 Based on the displacement field expressed in Eq. (16), the acceleration and virtual displacement vectors are given  
744 by:

$$\begin{cases} \frac{\partial^2 \mathbf{U}}{\partial t^2} = \left( \frac{\partial^2 U_o}{\partial t^2} + \frac{\partial^2 u}{\partial t^2} + [(a+z) \sin \theta + h] \frac{\partial^2 \Omega_y}{\partial t^2} + 2 \frac{\partial h}{\partial t} \frac{\partial \Omega_y}{\partial t} + \Omega_y \frac{\partial^2 h}{\partial t^2} \right) \mathbf{E}_x + \left( \frac{\partial^2 V_o}{\partial t^2} \sin \theta + \frac{\partial^2 w}{\partial t^2} \right) \mathbf{E}_R + \left( \frac{\partial^2 V_o}{\partial t^2} \cos \theta + \frac{\partial^2 v}{\partial t^2} \right) \mathbf{E}_\theta \\ \delta \mathbf{U} = (\delta U_o + \delta u + [(a+z) \sin \theta + h] \delta \Omega_y + \Omega_y \delta h) \mathbf{E}_x + (\delta V_o \sin \theta + \delta w) \mathbf{E}_R + (\delta V_o \cos \theta + \delta v) \mathbf{E}_\theta \end{cases}$$

745 leading to the following scalar product:

$$\frac{\partial^2 \mathbf{U}}{\partial t^2} \cdot \delta \mathbf{U} = \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \frac{\partial^2 u}{\partial t^2} \delta u + [(a+z) \sin \theta + h]^2 \frac{\partial^2 \Omega_y}{\partial t^2} \delta \Omega_y + \frac{\partial^2 V_o}{\partial t^2} \delta V_o + \frac{\partial^2 V_o}{\partial t^2} \delta h + \frac{\partial^2 h}{\partial t^2} \delta V_o + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w$$

746 Using the Euler-Bernoulli condition (cf. Eq. (6)) and the relation  $\delta h = \delta w \sin \theta + \delta v \cos \theta$  leads to the following  
747 expression:

$$\mathcal{P}_a = \int_{\Omega} \rho \left[ \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \left( \frac{\partial^2 V_o}{\partial t^2} - [(a+z) \sin \theta + h]^2 \frac{\partial^4 V_o}{\partial t^2 \partial x^2} + \frac{\partial^2 h}{\partial t^2} \right) \delta V_o + \frac{\partial^2 u}{\partial t^2} \delta u + \left( \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \cos \theta \right) \delta v + \left( \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \sin \theta \right) \delta w \right] dV$$

748 Then, using the first-order decomposition of the shell-type displacements (cf. Eq. (8)) makes it possible to rewrite the  
749 power of inertia as:

$$\mathcal{P}_a = \int_{\Omega} \rho \left[ \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \left( \frac{\partial^2 V_o}{\partial t^2} - [(a+z) \sin \theta + h]^2 \frac{\partial^4 V_o}{\partial t^2 \partial x^2} + \frac{\partial^2 h}{\partial t^2} \right) \delta V_o + \left( \frac{\partial^2 u_o}{\partial t^2} - z \frac{\partial^2 u_1}{\partial t^2} \right) \delta u_o - z \left( \frac{\partial^2 u_o}{\partial t^2} - z \frac{\partial^2 u_1}{\partial t^2} \right) \delta u_1 + \left( \frac{\partial^2 v_o}{\partial t^2} + z \frac{\partial^2 v_1}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \cos \theta \right) \delta v_o + z \left( \frac{\partial^2 v_o}{\partial t^2} + z \frac{\partial^2 v_1}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \cos \theta \right) \delta v_1 + \left( \frac{\partial^2 w_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \sin \theta \right) \delta w_o \right] dV$$

750 As  $\int_{-e/2}^{e/2} z dz = 0$ , this relation can be simplified to:

$$\mathcal{P}_a = \int_{\Omega} \rho \left[ \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \left( \frac{\partial^2 V_o}{\partial t^2} - [(a+z) \sin \theta + h]^2 \frac{\partial^4 V_o}{\partial t^2 \partial x^2} + \frac{\partial^2 h_o}{\partial t^2} \right) \delta V_o + \frac{\partial^2 u_o}{\partial t^2} \delta u_o + z^2 \frac{\partial^2 u_1}{\partial t^2} \delta u_1 + \left( \frac{\partial^2 v_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \cos \theta \right) \delta v_o + z^2 \frac{\partial^2 v_1}{\partial t^2} \delta v_1 + \left( \frac{\partial^2 w_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \sin \theta \right) \delta w_o \right] dV \quad (\text{B.2})$$

751 with  $h_o = w_o \sin \theta + v_o \cos \theta$ . Then, using the Love-Kirchhoff conditions (cf. Eq. (9)) leads to:

$$\mathcal{P}_a = \int_{\Omega} \rho \left[ \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \left( \frac{\partial^2 V_o}{\partial t^2} - [(a+z) \sin \theta + h]^2 \frac{\partial^4 V_o}{\partial t^2 \partial x^2} + \frac{\partial^2 h_o}{\partial t^2} \right) \delta V_o + \frac{\partial^2 u_o}{\partial t^2} \delta u_o + z^2 \frac{\partial^3 w_o}{\partial t^2 \partial x} \frac{\partial \delta w_o}{\partial x} + \left( \frac{\partial^2 v_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \cos \theta \right) \delta v_o + \frac{z^2}{a^2} \left( \frac{\partial^3 w_o}{\partial t^2 \partial \theta} - \frac{\partial^2 v_o}{\partial t^2} \right) \left( \frac{\partial \delta w_o}{\partial \theta} - \delta v_o \right) + \left( \frac{\partial^2 w_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \sin \theta \right) \delta w_o \right] dV$$

752 After integration by parts this rewrites as:

$$\begin{aligned} \mathcal{P}_a = & \int_{\Omega} \rho \left[ \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \left( \frac{\partial^2 V_o}{\partial t^2} - [(a+z) \sin \theta + h]^2 \frac{\partial^4 V_o}{\partial t^2 \partial x^2} + \frac{\partial^2 h_o}{\partial t^2} \right) \delta V_o + \right. \\ & \frac{\partial^2 u_o}{\partial t^2} \delta u_o + \left( \frac{\partial^2 v_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \cos \theta - \frac{z^2}{a^2} \left( \frac{\partial^3 w_o}{\partial t^2 \partial \theta} - \frac{\partial^2 v_o}{\partial t^2} \right) \right) \delta v_o + \\ & \left. \left( \frac{\partial^2 w_o}{\partial t^2} + \frac{\partial^2 V_o}{\partial t^2} \sin \theta - z^2 \frac{\partial^4 w_o}{\partial t^2 \partial x^2} - \frac{z^2}{a^2} \left( \frac{\partial^4 w_o}{\partial t^2 \partial \theta^2} - \frac{\partial^3 v_o}{\partial t^2 \partial \theta} \right) \right) \delta w_o \right] dV \end{aligned}$$

753 The mid-surface displacement field are expressed as Fourier expansions (cf. Eq. (15)). In addition, note that the  
754 integral of the product of two Fourier expansions given by:

$$\begin{aligned} \alpha &= \alpha_o^c + \sum_{i=1}^{N_f} (\alpha_i^c \cos(i\theta) + \alpha_i^s \sin(i\theta)) \\ \beta &= \beta_o^c + \sum_{i=1}^{N_f} (\beta_i^c \cos(i\theta) + \beta_i^s \sin(i\theta)) \end{aligned}$$

755 is as follows:

$$\int_0^{2\pi} \alpha \beta d\theta = 2\pi \alpha_o^c \beta_o^c + \pi \sum_{i=1}^{N_f} (\alpha_i^c \beta_i^c + \alpha_i^s \beta_i^s)$$

756 As a consequence, the power of inertia is expressed as:

$$\begin{aligned} \mathcal{P}_a = & \int \rho \left[ 2\pi a e \frac{\partial^2 U_o}{\partial t^2} \delta U_o + \left( 2\pi a e \frac{\partial^2 V_o}{\partial t^2} - I_y \frac{\partial^4 V_o}{\partial t^2 \partial x^2} \right) \delta V_o + \right. \\ & \pi a e \sum_{i=2}^{N_f} \left( \frac{\partial^2 u_i^c}{\partial t^2} \delta u_i^c + \frac{\partial^2 u_i^s}{\partial t^2} \delta u_i^s \right) + \pi a e \left( 1 + \frac{e^2}{12a^2} \right) \sum_{i=1}^{N_f} \frac{1}{i^2} \left( \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) - \pi a e \frac{\partial^2 V_o}{\partial t^2} \delta w_1^s \\ & + \pi a e \frac{e^2}{12a^2} \left( \frac{\partial^2 w_1^c}{\partial t^2} \delta w_1^c + \frac{\partial^2 w_1^s}{\partial t^2} \delta w_1^s - \sum_{i=2}^{N_f} \left( \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) \right) + \\ & 2\pi a e \frac{\partial^2 w_o^c}{\partial t^2} \delta w_o^c + \pi a e \sum_{i=1}^{N_f} \left( \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) + \pi a e \frac{\partial^2 V_o}{\partial t^2} \delta w_1^s \\ & - 2\pi a e \frac{e^2}{12} \frac{\partial^4 w_o^c}{\partial t^2 \partial x^2} \delta w_o^c - \pi a e \frac{e^2}{12} \sum_{i=1}^{N_f} \left( \frac{\partial^4 w_i^c}{\partial t^2 \partial x^2} \delta w_i^c + \frac{\partial^4 w_i^s}{\partial t^2 \partial x^2} \delta w_i^s \right) + \pi a e \frac{e^2}{12a^2} \sum_{i=1}^{N_f} \left( i^2 \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + i^2 \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) \\ & \left. + \pi a e \frac{e^2}{12a^2} \left( \frac{\partial^2 w_1^c}{\partial t^2} \delta w_1^c + \frac{\partial^2 w_1^s}{\partial t^2} \delta w_1^s - \sum_{i=2}^{N_f} \left( \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) \right) \right] dx \end{aligned}$$

757 as  $\int_0^{2\pi} h_o d\theta = 0$  and with the area moment of inertia:

$$I_y = \int_S \left( (a+z) \sin \theta + h \right)^2 dS$$

758 which can be expressed as:

$$\begin{aligned} I_y = & \pi a^3 e \left( 1 + 2 \frac{w_o^c}{a} - \frac{3 w_2^c}{2a} + \left( \frac{3 w_1^s}{4a} \right)^2 + \left( \frac{w_o^c}{a} - \frac{3 w_2^c}{4a} \right)^2 + \right. \\ & \frac{1}{4} \sum_{i \geq 2} \left( \left( \frac{i+2}{i+1} \frac{w_{i+1}^s}{a} - \frac{i-2+2\delta_{i2}}{i-1} \frac{w_{i-1}^s}{a} \right)^2 + \left( \frac{i+2}{i+1} \frac{w_{i+1}^c}{a} - \frac{i-2+2\delta_{i2}}{i-1} \frac{w_{i-1}^c}{a} \right)^2 \right) \\ & \left. + \frac{e^2}{12a^2} \left( 1 + 4 \frac{w_1^c}{a} \right) \right) \end{aligned} \quad (\text{B.3})$$

759 Note that the moment of inertia for a beam is given by:

$$I_y^b = \pi a^3 e \left( 1 + \frac{e^2}{12a^2} \right) \quad (\text{B.4})$$

760 After some algebraic manipulations, this can be rewritten under the following form:

$$\begin{aligned} \mathcal{P}_a = & \int \rho \left[ S \frac{\partial^2 U_o}{\partial t^2} \delta U_o + S \left( \frac{\partial^2 V_o}{\partial t^2} - \frac{I_y}{S} \frac{\partial^4 V_o}{\partial t^2 \partial x^2} \right) \delta V_o + \right. \\ & \pi a e \sum_{i=2}^{N_f} \left( \frac{\partial^2 u_i^c}{\partial t^2} \delta u_i^c + \frac{\partial^2 u_i^s}{\partial t^2} \delta u_i^s \right) + 2\pi a e \left( \frac{\partial^2 w_o^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_o^c}{\partial t^2 \partial x^2} \right) \delta w_o^c + \\ & 2\pi a e \left( 1 + 2 \frac{e^2}{12a^2} \right) \left( \frac{\partial^2 w_1^c}{\partial t^2} \delta w_1^c + \frac{\partial^2 w_1^s}{\partial t^2} \delta w_1^s \right) - \pi a \frac{e^3}{12} \left( \frac{\partial^4 w_1^c}{\partial t^2 \partial x^2} \delta w_1^c + \frac{\partial^4 w_1^s}{\partial t^2 \partial x^2} \delta w_1^s \right) \\ & + \pi a e \sum_{i=2}^{N_f} \frac{i^2 + 1}{i^2} \left( \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) + \pi a \frac{e^3}{12a^2} \sum_{i=2}^{N_f} \frac{(i^2 - 1)^2}{i^2} \left( \frac{\partial^2 w_i^c}{\partial t^2} \delta w_i^c + \frac{\partial^2 w_i^s}{\partial t^2} \delta w_i^s \right) \\ & \left. - \pi a \frac{e^3}{12} \sum_{i=2}^{N_f} \left( \frac{\partial^4 w_i^c}{\partial t^2 \partial x^2} \delta w_i^c + \frac{\partial^4 w_i^s}{\partial t^2 \partial x^2} \delta w_i^s \right) \right] dx \end{aligned} \quad (\text{B.5})$$

761 with  $S = 2\pi a e$ .

## 762 Appendix B.2. Power of internal forces

763 The power of internal forces is now expressed using the stress-strain relation. Following the decomposition  $\boldsymbol{\varepsilon} =$   
764  $\boldsymbol{\varepsilon}^m + z\mathbf{k}$ , the stress tensor can also be decomposed as:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^m + z\boldsymbol{\sigma}^k \quad \text{with} \quad \boldsymbol{\sigma}^m = \frac{E}{1-\nu^2} \left( (1-\nu) \boldsymbol{\varepsilon}^m + \nu \text{trac}(\boldsymbol{\varepsilon}^m) \mathbf{I} \right) \quad \text{and} \quad \boldsymbol{\sigma}^k = \frac{E}{1-\nu^2} \left( (1-\nu) \mathbf{k} + \nu \text{trac}(\mathbf{k}) \mathbf{I} \right)$$

765 The power of internal forces can thus be expressed as:

$$\mathcal{P}_i = - \int_{\Omega} \left( \boldsymbol{\sigma}^m : \delta \boldsymbol{\varepsilon}^m + z^2 \boldsymbol{\sigma}^k : \delta \mathbf{k} \right) dV \quad (\text{B.6})$$

766 as  $\int_{-e/2}^{e/2} z dz = 0$ . Based on the strain-displacement relationship expressed in Eq. (11), we can write:

$$\begin{aligned} \mathcal{P}_i = & - \int_{\Omega} \left[ \sigma_{xx}^m \left( \frac{\partial \delta U_o}{\partial x} - a \sin \theta \frac{\partial^2 \delta V_o}{\partial x^2} + \frac{\partial \delta u_o}{\partial x} - \frac{\partial^2 \delta V_o}{\partial x^2} h_o - \frac{\partial^2 V_o}{\partial x^2} (\delta w_o \sin \theta + \delta v_o \cos \theta) \right) + \right. \\ & z^2 \sigma_{xx}^k \left( -\sin \theta \frac{\partial^2 \delta V_o}{\partial x^2} - \frac{\partial^2 \delta w_o}{\partial x^2} + \frac{\cos \theta}{a} \frac{\partial^2 \delta V_o}{\partial x^2} \left( \frac{\partial w_o}{\partial \theta} - v_o \right) + \frac{\cos \theta}{a} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{\partial \delta w_o}{\partial \theta} - \delta v_o \right) \right) + \\ & \sigma_{\theta\theta}^m \frac{1}{a} \left( \frac{\partial \delta v_o}{\partial \theta} + \delta w_o \right) - \sigma_{\theta\theta}^k \frac{z^2}{a^2} \left( \frac{\partial^2 \delta w_o}{\partial \theta^2} + \delta w_o \right) + \\ & \left. \sigma_{x\theta}^m \left( \frac{1}{a} \frac{\partial \delta u_o}{\partial \theta} + \frac{\partial \delta v_o}{\partial x} \right) - \sigma_{x\theta}^k \frac{z^2}{a} \left( 2 \frac{\partial^2 \delta w_o}{\partial x \partial \theta} + \frac{1}{a} \frac{\partial \delta u_o}{\partial \theta} - \frac{\partial \delta v_o}{\partial x} \right) \right] dV \end{aligned} \quad (\text{B.7})$$

767 which can be rearranged as:

$$\begin{aligned} \mathcal{P}_i = & - \int_{\Omega} \left[ \sigma_{xx}^m \frac{\partial \delta U_o}{\partial x} - \sigma_{xx} \left( (a+z) \sin \theta + h_o - \frac{z}{a} \cos \theta \left( \frac{\partial w_o}{\partial \theta} - v_o \right) \right) \frac{\partial^2 \delta V_o}{\partial x^2} + \right. \\ & \sigma_{xx}^m \frac{\partial \delta u_o}{\partial x} + \frac{1}{a} \left( \sigma_{x\theta}^m - \frac{z^2}{a} \sigma_{x\theta}^k \right) \frac{\partial \delta u_o}{\partial \theta} + \\ & \left( -\sigma_{xx}^m \frac{\partial^2 V_o}{\partial x^2} \sin \theta + \frac{1}{a} \left( \sigma_{\theta\theta}^m - \frac{z^2}{a} \sigma_{\theta\theta}^k \right) \right) \delta w_o + \frac{z^2}{a} \sigma_{xx}^k \frac{\partial^2 V_o}{\partial x^2} \cos \theta \frac{\partial \delta w_o}{\partial \theta} - \frac{z^2}{a^2} \sigma_{\theta\theta}^k \frac{\partial^2 \delta w_o}{\partial \theta^2} \\ & - 2 \frac{z^2}{a} \sigma_{x\theta}^k \frac{\partial^2 \delta w_o}{\partial x \partial \theta} - z^2 \sigma_{xx}^k \frac{\partial^2 \delta w_o}{\partial x^2} \\ & \left. - \left( \sigma_{xx}^m + \frac{z^2}{a} \sigma_{xx}^k \right) \frac{\partial^2 V_o}{\partial x^2} \cos \theta \delta v_o + \left( \sigma_{x\theta}^m + \frac{z^2}{a} \sigma_{x\theta}^k \right) \frac{\partial \delta v_o}{\partial x} + \sigma_{\theta\theta}^m \frac{1}{a} \frac{\partial \delta v_o}{\partial \theta} \right] dV \end{aligned}$$

768 The mid-surface displacement virtual field is now expanded in Fourier series as in Eq. (15):

$$\begin{aligned}
\mathcal{P}_i = & - \int_{\Omega} \left[ \sigma_{xx}^m \frac{\partial \delta U_o}{\partial x} - \sigma_{xx} ((a+z) \sin \theta + h) \frac{\partial^2 \delta V_o}{\partial x^2} + \right. \\
& \sigma_{xx}^m \sum_{i=2}^{N_f} \left( \frac{\partial \delta u_i^c}{\partial x} \cos(i\theta) + \frac{\partial \delta u_i^s}{\partial x} \sin(i\theta) \right) + \frac{1}{a} \left( \sigma_{x\theta}^m - \frac{z^2}{a} \sigma_{x\theta}^k \right) \sum_{i=2}^{N_f} i \left( -\delta u_i^c \sin(i\theta) + \delta u_i^s \cos(i\theta) \right) + \\
& \left( -\sigma_{xx}^m \frac{\partial^2 V_o}{\partial x^2} \sin \theta + \frac{1}{a} \left( \sigma_{\theta\theta}^m - \frac{z^2}{a} \sigma_{\theta\theta}^k \right) \right) \left( \delta w_o^c + \delta w_1^c \cos \theta + \delta w_1^s \sin \theta + \sum_{i=2}^{N_f} (\delta w_i^c \cos(i\theta) + \delta w_i^s \sin(i\theta)) \right) \\
& + \sigma_{xx}^k \frac{z^2}{a} \frac{\partial^2 V_o}{\partial x^2} \cos \theta \left( -\delta w_1^c \sin \theta + \delta w_1^s \cos \theta + \sum_{i=2}^{N_f} i \left( -\delta w_i^c \sin(i\theta) + \delta w_i^s \cos(i\theta) \right) \right) \\
& + \sigma_{\theta\theta}^k \frac{z^2}{a^2} \left( \delta w_1^c \cos \theta + \delta w_1^s \sin \theta + \sum_{i=2}^{N_f} i^2 (\delta w_i^c \cos(i\theta) + \delta w_i^s \sin(i\theta)) \right) \\
& - 2 \frac{z^2}{a} \sigma_{x\theta}^k \left( -\frac{\partial \delta w_1^c}{\partial x} \sin \theta + \frac{\partial \delta w_1^s}{\partial x} \cos \theta + \sum_{i=2}^{N_f} i \left( -\frac{\partial \delta w_i^c}{\partial x} \sin(i\theta) + \frac{\partial \delta w_i^s}{\partial x} \cos(i\theta) \right) \right) \\
& - z^2 \sigma_{xx}^k \left( \frac{\partial^2 \delta w_o^c}{\partial x^2} + \frac{\partial^2 \delta w_1^c}{\partial x^2} \cos \theta + \frac{\partial^2 \delta w_1^s}{\partial x^2} \sin \theta + \sum_{i=2}^{N_f} \left( \frac{\partial^2 \delta w_i^c}{\partial x^2} \cos(i\theta) + \frac{\partial^2 \delta w_i^s}{\partial x^2} \sin(i\theta) \right) \right) \\
& - \left( \sigma_{xx}^m + \frac{z^2}{a} \sigma_{xx}^k \right) \frac{\partial^2 V_o}{\partial x^2} \cos \theta \left( -\delta w_1^s \cos \theta + \delta w_1^c \sin \theta + \sum_{i=2}^{N_f} \frac{1}{i} (\delta w_i^s \cos(i\theta) - \delta w_i^c \sin(i\theta)) \right) \\
& + \sigma_{\theta\theta}^m \frac{1}{a} \left( \delta w_1^c \cos \theta + \delta w_1^s \sin \theta - \sum_{i=2}^{N_f} (\delta w_i^s \sin(i\theta) + \delta w_i^c \cos(i\theta)) \right) \\
& \left. + \left( \sigma_{x\theta}^m + \frac{z^2}{a} \sigma_{x\theta}^k \right) \left( -\frac{\partial \delta w_1^s}{\partial x} \cos \theta + \frac{\partial \delta w_1^c}{\partial x} \sin \theta + \sum_{i=2}^{N_f} \frac{1}{i} \left( \frac{\partial \delta w_i^s}{\partial x} \cos(i\theta) - \frac{\partial \delta w_i^c}{\partial x} \sin(i\theta) \right) \right) \right] dV
\end{aligned}$$

769 Introducing the *generalized* forces and moments associated with the beam-type and shell-type motions, this can be  
770 rewritten as:

$$\begin{aligned}
\mathcal{P}_i = & - \int_0^L \left[ N \frac{\partial \delta U_o}{\partial x} - M_y \frac{\partial^2 \delta V_o}{\partial x^2} + \sum_{i=2}^{N_f} ((t_u)_i^c \delta u_i^c + (t_u)_i^s \delta u_i^s) + \sum_{i=2}^{N_f} \left( (n_u)_i^c \frac{\partial \delta u_i^c}{\partial x} + (n_u)_i^s \frac{\partial \delta u_i^s}{\partial x} \right) + \right. \\
& (t_w)_o^c \delta w_o^c - (m_w)_o^c \frac{\partial^2 \delta w_o^c}{\partial x^2} + \\
& (t_w)_1^c \delta w_1^c + (t_w)_1^s \delta w_1^s + (n_w)_1^c \frac{\partial \delta w_1^c}{\partial x} + (n_w)_1^s \frac{\partial \delta w_1^s}{\partial x} - (m_w)_1^c \frac{\partial^2 \delta w_1^c}{\partial x^2} - (m_w)_1^s \frac{\partial^2 \delta w_1^s}{\partial x^2} \\
& \left. \sum_{i=2}^{N_f} \left( (t_w)_i^c \delta w_i^c + (t_w)_i^s \delta w_i^s + (n_w)_i^c \frac{\partial \delta w_i^c}{\partial x} + (n_w)_i^s \frac{\partial \delta w_i^s}{\partial x} - (m_w)_i^c \frac{\partial^2 \delta w_i^c}{\partial x^2} - (m_w)_i^s \frac{\partial^2 \delta w_i^s}{\partial x^2} \right) \right] dx
\end{aligned} \tag{B.8}$$

771 with the following definition of the generalized forces and moments:

$$\left\{ \begin{array}{l}
 N = \int_S \sigma_{xx}^m dS \\
 M_y = \int_S \sigma_{xx} ((a+z) \sin \theta + h) dS \\
 (t_u)_i^c = -i \int_S \frac{1}{a} \left( \sigma_{x\theta}^m - \frac{z^2}{a} \sigma_{x\theta}^k \right) \sin(i\theta) dS \\
 (t_u)_i^s = +i \int_S \frac{1}{a} \left( \sigma_{x\theta}^m - \frac{z^2}{a} \sigma_{x\theta}^k \right) \cos(i\theta) dS \\
 (n_u)_i^c = \int_S \sigma_{xx}^m \cos(i\theta) dS \\
 (n_u)_i^s = \int_S \sigma_{xx}^m \sin(i\theta) dS \\
 (t_w)_o^c = \int_S \left( -\sigma_{xx}^m \frac{\partial^2 V_o}{\partial x^2} \sin \theta + \frac{1}{a} \left( \sigma_{\theta\theta}^m - \frac{z^2}{a} \sigma_{\theta\theta}^k \right) \right) dS \\
 (m_w)_o^c = \int_S z^2 \sigma_{xx}^k dS \\
 (t_w)_1^c = \int_S 2 \left( - \left( \sigma_{xx}^m + \frac{z^2}{a} \sigma_{xx}^k \right) \frac{\partial^2 V_o}{\partial x^2} \cos \theta \sin \theta + \frac{1}{a} \sigma_{\theta\theta}^m \cos \theta \right) dS \\
 (t_w)_1^s = \int_S \left( \left( \sigma_{xx}^m (\cos^2 \theta - \sin^2 \theta) + 2 \frac{z^2}{a} \sigma_{xx}^k \cos^2 \theta \right) \frac{\partial^2 V_o}{\partial x^2} + 2 \frac{1}{a} \sigma_{\theta\theta}^m \sin \theta \right) dS \\
 (n_w)_1^c = + \int_S \left( \sigma_{x\theta}^m + 3 \frac{z^2}{a} \sigma_{x\theta}^k \right) \sin \theta dS \\
 (n_w)_1^s = - \int_S \left( \sigma_{x\theta}^m + 3 \frac{z^2}{a} \sigma_{x\theta}^k \right) \cos \theta dS \\
 (m_w)_1^c = \int_S z^2 \sigma_{xx}^k \cos \theta dS \\
 (m_w)_1^s = \int_S z^2 \sigma_{xx}^k \sin \theta dS \\
 (t_w)_i^c = \int_S \left( + \frac{1}{i} \sigma_{xx}^m \frac{\partial^2 V_o}{\partial x^2} (\cos \theta \sin(i\theta) - i \sin \theta \cos(i\theta)) + (i^2 - 1) \frac{z^2}{a^2} \sigma_{\theta\theta}^k \cos(i\theta) - \frac{i^2 - 1}{i} \sigma_{xx}^k \frac{z^2}{a} \frac{\partial^2 V_o}{\partial x^2} \cos \theta \sin(i\theta) \right) dS \\
 (t_w)_i^s = \int_S \left( - \frac{1}{i} \sigma_{xx}^m \frac{\partial^2 V_o}{\partial x^2} (\cos \theta \cos(i\theta) + i \sin \theta \sin(i\theta)) + (i^2 - 1) \frac{z^2}{a^2} \sigma_{\theta\theta}^k \sin(i\theta) + \frac{i^2 - 1}{i} \sigma_{xx}^k \frac{z^2}{a} \frac{\partial^2 V_o}{\partial x^2} \cos \theta \cos(i\theta) \right) dS \\
 (n_w)_i^c = - \int_S \frac{1}{i} \left( \sigma_{x\theta}^m + (1 - 2i^2) \frac{z^2}{a} \sigma_{x\theta}^k \right) \sin(i\theta) dS \\
 (n_w)_i^s = + \int_S \frac{1}{i} \left( \sigma_{x\theta}^m + (1 - 2i^2) \frac{z^2}{a} \sigma_{x\theta}^k \right) \cos(i\theta) dS \\
 (m_w)_i^c = \int_S z^2 \sigma_{xx}^k \cos(i\theta) dS \\
 (m_w)_i^s = \int_S z^2 \sigma_{xx}^k \sin(i\theta) dS
 \end{array} \right.$$

772 for  $i \geq 2$  with the following components of the stress-tensor:

$$\left\{ \begin{array}{l} \sigma_{xx}^m = \frac{E}{1-\nu^2} \left( \frac{\partial U_o}{\partial x} - a \sin \theta \frac{\partial^2 V_o}{\partial x^2} + \frac{\partial u_o}{\partial x} - \frac{\partial^2 V_o}{\partial x^2} (w_o \sin \theta + v_o \cos \theta) + \nu \frac{1}{a} \left( \frac{\partial v_o}{\partial \theta} + w_o \right) \right) \\ \sigma_{\theta\theta}^m = \frac{E}{1-\nu^2} \left( \frac{1}{a} \left( \frac{\partial v_o}{\partial \theta} + w_o \right) + \nu \left( \frac{\partial U_o}{\partial x} - a \sin \theta \frac{\partial^2 V_o}{\partial x^2} + \frac{\partial u_o}{\partial x} - \frac{\partial^2 V_o}{\partial x^2} (w_o \sin \theta + v_o \cos \theta) \right) \right) \\ \sigma_{x\theta}^m = \frac{E}{1+\nu} \frac{1}{2} \left( \frac{1}{a} \frac{\partial u_o}{\partial \theta} + \frac{\partial v_o}{\partial x} \right) \\ \sigma_{xx}^k = \frac{E}{1-\nu^2} \left( -\sin \theta \frac{\partial^2 V_o}{\partial x^2} - \frac{\partial^2 w_o}{\partial x^2} + \frac{\cos \theta}{a} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{\partial w_o}{\partial \theta} - v_o \right) - \nu \frac{1}{a^2} \left( \frac{\partial^2 w_o}{\partial \theta^2} + w_o \right) \right) \\ \sigma_{\theta\theta}^k = \frac{E}{1-\nu^2} \left( -\frac{1}{a^2} \left( \frac{\partial^2 w_o}{\partial \theta^2} + w_o \right) + \nu \left( -\sin \theta \frac{\partial^2 V_o}{\partial x^2} - \frac{\partial^2 w_o}{\partial x^2} + \frac{\cos \theta}{a} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{\partial w_o}{\partial \theta} - v_o \right) \right) \right) \\ \sigma_{x\theta}^k = -\frac{E}{1+\nu} \frac{1}{2a} \left( 2 \frac{\partial^2 w_o}{\partial x \partial \theta} + \frac{1}{a} \frac{\partial u_o}{\partial \theta} - \frac{\partial v_o}{\partial x} \right) \end{array} \right.$$

773  $N$  and  $M_y$  are the axial force and the in-plane bending moment, respectively, associated with the beam-type DOFs.  
 774  $(t_u)_i^{c/s}$  and  $(n_u)_i^{c/s}$  are the forces associated with the cross-section warping.  $(t_w)_o^c$  and  $(m_w)_o^c$  are associated to the  
 775 radial expansion/contraction of the tube cross-section while  $(t_w)_1^{c/s}$ ,  $(n_w)_1^{c/s}$  and  $(m_w)_1^{c/s}$  are linked to the vanishing  
 776 rigid-body of the tube cross-section. Finally,  $(t_w)_i^{c/s}$ ,  $(n_w)_i^{c/s}$  and  $(m_w)_i^{c/s}$  for  $i \geq 2$  correspond to the forces associated  
 777 to the tube cross-section distortion.

778 The axial force and the in-plane bending moment corresponding to the beam motion are expressed as:

$$\left\{ \begin{array}{l} N = +\frac{ES}{1-\nu^2} \left( \frac{\partial U_o}{\partial x} + \nu \frac{w_o^c}{a} \right) \\ M_y = -\frac{EI_y}{1-\nu^2} \frac{\partial^2 V_o}{\partial x^2} \\ \quad + \frac{ES}{1-\nu^2} \frac{1}{2} \left( \sum_{i \geq 2} \frac{1}{2} \left( \frac{\partial u_i^c}{\partial x} \left( \frac{i+2}{i+1} w_{i+1}^s - \frac{i-2+2\delta_{i2}}{i-1} w_{i-1}^s \right) - \frac{\partial u_i^s}{\partial x} \left( \frac{i+2}{i+1} w_{i+1}^c - \frac{i-2+2\delta_{i2}}{i-1} w_{i-1}^c \right) \right) \right. \\ \quad \left. - \frac{e^2}{12} \frac{\partial^2 w_1^s}{\partial x^2} + 2\nu w_1^s \right) \end{array} \right. \quad (\text{B.9})$$

779 It has to be noticed that  $N$  corresponds to the classical axial force of a beam with an additional term related to the  
 780 cross-section expansion/contraction corresponding to the Poisson effect.  $M_y$  involve two type of terms: the first one is  
 781 similar to the beam-type bending moment with the *modified* moment of inertia given in Eq. (B.3) whereas the second  
 782 one are due to the non-linear coupling.

783 Then, the forces corresponding to the cross-section warping are expressed for  $i \geq 2$  as:

$$\left\{ \begin{array}{l} (t_u)_i^c = +\frac{ES}{1+\nu} \frac{1}{4a} \left( i^2 \left( 1 + \frac{e^2}{12a^2} \right) \frac{u_i^c}{a} + \left( 1 + \frac{e^2}{12a^2} (2i^2 - 1) \right) \frac{\partial w_i^c}{\partial x} \right) \\ (t_u)_i^s = +\frac{ES}{1+\nu} \frac{1}{4a} \left( i^2 \left( 1 + \frac{e^2}{12a^2} \right) \frac{u_i^s}{a} + \left( 1 + \frac{e^2}{12a^2} (2i^2 - 1) \right) \frac{\partial w_i^s}{\partial x} \right) \\ (n_u)_i^c = +\frac{ES}{1-\nu^2} \frac{1}{2} \left( \frac{\partial u_i^c}{\partial x} + \frac{1}{2} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{i-2+2\delta_{i2}}{i-1} w_{i-1}^s - \frac{i+2}{i+1} w_{i+1}^s \right) \right) \\ (n_u)_i^s = +\frac{ES}{1-\nu^2} \frac{1}{2} \left( \frac{\partial u_i^s}{\partial x} - \frac{1}{2} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{i-2+2\delta_{i2}}{i-1} w_{i-1}^c - \frac{i+2}{i+1} w_{i+1}^c \right) \right) \end{array} \right. \quad (\text{B.10})$$

784 The moments and forces associated to the mode 0 in the Fourier expansion of the cross-section deformation are given  
 785 by:

$$\begin{cases} (t_w)_o^c &= +\frac{ES}{1-\nu^2} \left( \frac{1}{a} \left( \frac{w_o^c}{a} + \nu \frac{\partial U_o}{\partial x} \right) + \frac{1}{2} \left( a + \frac{1}{2} \left( 2w_o^c - \frac{3}{2}w_2^c \right) \right) \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 - \nu \frac{w_1^s}{a} \frac{\partial^2 V_o}{\partial x^2} + \frac{e^2}{12a^2} \left( \frac{w_o^c}{a^2} + \nu \left( \frac{\partial^2 w_o^c}{\partial x^2} - \frac{w_1^s}{a} \frac{\partial^2 V_o}{\partial x^2} \right) \right) \right) \\ (m_w)_o^c &= -\frac{ES}{1-\nu^2} \frac{e^2}{12} \left( \frac{\partial^2 w_o^c}{\partial x^2} - \frac{\partial^2 V_o}{\partial x^2} \frac{w_1^s}{a} + \nu \frac{w_o^c}{a^2} \right) \end{cases} \quad (\text{B.11})$$

786 In addition, the moments and forces associated to the mode 1 in the Fourier expansion of the cross-section deformation  
 787 are given by:

$$\begin{cases} (t_w)_1^c &= +\frac{ES}{1-\nu^2} \left( 2 \frac{w_1^c}{a^2} + \frac{1}{2} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{\partial u_2^s}{\partial x} - \frac{e^2}{12a} \frac{\partial^2 w_2^s}{\partial x^2} \right) - \nu \frac{\partial^2 V_o}{\partial x^2} \frac{3}{4} \left( 1 + 2 \frac{e^2}{12a^2} \right) \frac{w_2^s}{a} \right. \\ &\quad \left. + \frac{1}{4} \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 \left( 2 \left( 1 + \frac{e^2}{12a^2} \right) w_1^c - \frac{4}{3} \left( 1 - 2 \frac{e^2}{12a^2} \right) w_3^s \right) \right) \\ (t_w)_1^s &= +\frac{ES}{1-\nu^2} \left( 2 \frac{w_1^s}{a^2} + \frac{1}{2} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{\partial u_2^c}{\partial x} - \frac{e^2}{12a} \left( \frac{\partial^2 w_2^c}{\partial x^2} + 2 \frac{\partial^2 w_o^c}{\partial x^2} \right) \right) - \nu \frac{\partial^2 V_o}{\partial x^2} \left( 1 + \left( 1 + \frac{e^2}{12a^2} \right) \frac{w_o^c}{a} - \frac{3}{4} \left( 1 + 2 \frac{e^2}{12a^2} \right) \frac{w_2^c}{a} \right) \right. \\ &\quad \left. + \frac{1}{4} \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 \left( 2 \left( 1 + \frac{e^2}{12a^2} \right) w_1^s - \frac{4}{3} \left( 1 - 2 \frac{e^2}{12a^2} \right) w_3^s \right) \right) \\ (n_w)_1^c &= +\frac{ES}{1+\nu} \frac{1}{4} \left( 1 + 9 \frac{e^2}{12a^2} \right) \frac{\partial w_1^c}{\partial x} \\ (n_w)_1^s &= +\frac{ES}{1+\nu} \frac{1}{4} \left( 1 + 9 \frac{e^2}{12a^2} \right) \frac{\partial w_1^s}{\partial x} \\ (m_w)_1^c &= -\frac{ES}{1-\nu^2} \frac{e^2}{24} \left( \frac{\partial^2 w_1^c}{\partial x^2} + \frac{\partial^2 V_o}{\partial x^2} \frac{3w_2^s}{4a} \right) \\ (m_w)_1^s &= -\frac{ES}{1-\nu^2} \frac{e^2}{24} \left( \frac{\partial^2 w_1^s}{\partial x^2} + \frac{\partial^2 V_o}{\partial x^2} \left( 1 + \frac{3w_2^c}{4a} \right) \right) \end{cases} \quad (\text{B.12})$$



788 Finally, the moments and forces associated to the mode  $i$  (for  $i \geq 2$ ) in the Fourier expansion of the cross-section  
 789 deformation are expressed as:

$$\begin{cases}
 (t_w)_i^c &= \frac{ES}{1-\nu^2} \left( \frac{i^2-1}{2} \frac{e^2}{12a^2} \left( (i^2-1) \frac{w_i^c}{a^2} - \nu \frac{\partial^2 w_i^c}{\partial x^2} \right) - \frac{1}{4i} \frac{\partial^2 V_o}{\partial x^2} \left( (i-1) \frac{\partial u_{i+1}^s}{\partial x} - (i+1) \frac{\partial u_{i-1}^s}{\partial x} + (i^2-1) \frac{e^2}{12a} \left( -\frac{\partial^2 w_{i+1}^s}{\partial x^2} - \frac{\partial^2 w_{i-1}^s}{\partial x^2} \right) \right) \right. \\
 &\quad + \nu \frac{\partial^2 V_o}{\partial x^2} \left( \frac{i+1}{2i} \delta_{i2} \frac{w_{i-1}^s}{a} + \frac{i^2-1}{4} \frac{e^2}{12a^2} \left( \frac{i-2+2\delta_{i2}}{i-1} \frac{w_{i-1}^s}{a} - \frac{i+2}{i+1} \frac{w_{i+1}^s}{a} \right) \right) \\
 &\quad + \frac{1}{8i} \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 \left( -2(i+1)a\delta_{i2} - \frac{(i+1)(i-2-(1-\delta_{i2})(1-2\delta_{i3}))}{i-2} w_{i-2}^c + \frac{(i-1)^2+(i+1)^2}{i} w_i^c - \frac{(i-1)(i+3)}{i+2} w_{i+2}^c \right. \\
 &\quad \left. + (i^2-1) \frac{e^2}{12a^2} \left( 2a\delta_{i2} + \frac{(i-2)^2-1+2\delta_{i3}}{i-2} (1-\delta_{i2}) w_{i-2}^c + 2 \frac{i^2-1}{i} w_i^c + \frac{(i+2)^2-1}{i+2} w_{i+2}^c \right) \right) \\
 (t_w)_i^s &= \frac{ES}{1-\nu^2} \left( \frac{i^2-1}{2} \frac{e^2}{12a^2} \left( (i^2-1) \frac{w_i^s}{a^2} - \nu \frac{\partial^2 w_i^s}{\partial x^2} \right) + \frac{1}{4i} \frac{\partial^2 V_o}{\partial x^2} \left( (i-1) \frac{\partial u_{i+1}^c}{\partial x} - (i+1) \frac{\partial u_{i-1}^c}{\partial x} + (i^2-1) \frac{e^2}{12a} \left( -\frac{\partial^2 w_{i+1}^c}{\partial x^2} - \frac{\partial^2 w_{i-1}^c}{\partial x^2} \right) \right) \right. \\
 &\quad - \nu \frac{\partial^2 V_o}{\partial x^2} \left( \frac{i+1}{2i} \delta_{i2} \frac{w_{i-1}^c}{a} + \frac{i^2-1}{4} \frac{e^2}{12a^2} \left( \frac{i-2+2\delta_{i2}}{i-1} \frac{w_{i-1}^c}{a} - \frac{i+2}{i+1} \frac{w_{i+1}^c}{a} \right) \right) \\
 &\quad + \frac{1}{8i} \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 \left( -\frac{(i+1)(i-3+2\delta_{i3})}{i-2} (1-\delta_{i2}) w_{i-2}^s + \frac{(i-1)^2+(i+1)^2}{i} w_i^s - \frac{(i-1)(i+3)}{i+2} w_{i+2}^s \right. \\
 &\quad \left. + (i^2-1) \frac{e^2}{12a^2} \left( \frac{(i-2)^2-1+2\delta_{i3}}{i-2} (1-\delta_{i2}) w_{i-2}^s + 2 \frac{i^2-1}{i} w_i^s + \frac{(i+2)^2-1}{i+2} w_{i+2}^s \right) \right) \\
 (n_w)_i^c &= + \frac{ES}{1+\nu} \frac{1}{4} \left( \left( 1 + (2i^2-1) \frac{e^2}{12a^2} \right) \frac{u_i^c}{a} + \frac{1}{i^2} \left( 1 + (2i^2-1) \frac{e^2}{12a^2} \right) \frac{\partial w_i^c}{\partial x} \right) \\
 (n_w)_i^s &= + \frac{ES}{1+\nu} \frac{1}{4} \left( \left( 1 + (2i^2-1) \frac{e^2}{12a^2} \right) \frac{u_i^s}{a} + \frac{1}{i^2} \left( 1 + (2i^2-1) \frac{e^2}{12a^2} \right) \frac{\partial w_i^s}{\partial x} \right) \\
 (m_w)_i^c &= - \frac{ES}{1-\nu^2} \frac{e^2}{24} \left( \frac{\partial^2 w_i^c}{\partial x^2} + \frac{1}{2} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{(i+1)^2-1}{i+1} \frac{w_{i+1}^s}{a} + \frac{(i-1)^2-1+2\delta_{i2}}{i-1} \frac{w_{i-1}^s}{a} \right) - \nu (i^2-1) \frac{w_i^c}{a^2} \right) \\
 (m_w)_i^s &= - \frac{ES}{1-\nu^2} \frac{e^2}{24} \left( \frac{\partial^2 w_i^s}{\partial x^2} + \frac{1}{2} \frac{\partial^2 V_o}{\partial x^2} \left( \frac{(i+1)^2-1}{i+1} \frac{w_{i+1}^c}{a} + \frac{(i-1)^2-1+2\delta_{i2}}{i-1} \frac{w_{i-1}^c}{a} \right) - \nu (i^2-1) \frac{w_i^s}{a^2} \right)
 \end{cases} \tag{B.13}$$

790 Terms in magenta in the previous equations are due to the non-linear coupling between the axis rotation and the cross-  
 791 section deformation. First, it has to be noticed that without this coupling, the forces corresponding to mode  $i$  depends  
 792 only on DOFs of mode  $i$ . In contrary, terms with  $\partial_{x^2}^2 V_o$  involves DOFs of mode  $i-1$ ,  $i$  and  $i+1$  whereas terms with  
 793  $(\partial_{x^2}^2 V_o)^2$  involves DOFs of mode  $i-2$ ,  $i$  and  $i+2$ . Finally, the cross-section warping and the cross-section ovalization  
 794 are connected through the forces  $(t_u)_i^{c/s}$ ,  $(n_u)_i^{c/s}$ ,  $(t_w)_i^{c/s}$  and  $(n_w)_i^{c/s}$ . Terms in blue in the previous equations involved  
 795 the ratio  $e^2/12a^2$  which can be neglected in the case of very thin-walled tubes.

797 Finally, the equations of motion of the tube are:

$$\left\{ \begin{array}{l}
\rho S \frac{\partial^2 U_o}{\partial t^2} - \frac{\partial N}{\partial x} = 0 \\
\rho S \left( \frac{\partial^2 V_o}{\partial t^2} - \frac{I_y}{S} \frac{\partial^4 V_o}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 M_y}{\partial x^2} = 0 \\
\rho S \left( \frac{\partial^2 w_o^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_o^c}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_o^c}{\partial x^2} + (t_w)_o^c = 0 \\
\rho S \left( \left( 1 + 2 \frac{e^2}{12a^2} \right) \frac{\partial^2 w_1^c}{\partial t^2} - \frac{1}{2} \frac{e^2}{12} \frac{\partial^4 w_1^c}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_1^c}{\partial x^2} - \frac{\partial (n_w)_1^c}{\partial x} + (t_w)_1^c = 0 \\
\rho S \left( \left( 1 + 2 \frac{e^2}{12a^2} \right) \frac{\partial^2 w_1^s}{\partial t^2} - \frac{1}{2} \frac{e^2}{12} \frac{\partial^4 w_1^s}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_1^s}{\partial x^2} - \frac{\partial (n_w)_1^s}{\partial x} + (t_w)_1^s = 0 \\
\rho S \frac{1}{2} \left( \left( \frac{i^2 + 1}{i^2} + \frac{(i^2 - 1)^2}{i^2} \frac{e^2}{12a^2} \right) \frac{\partial^2 w_i^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_i^c}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_i^c}{\partial x^2} - \frac{\partial (n_w)_i^c}{\partial x} + (t_w)_i^c = 0 \\
\rho S \frac{1}{2} \left( \left( \frac{i^2 + 1}{i^2} + \frac{(i^2 - 1)^2}{i^2} \frac{e^2}{12a^2} \right) \frac{\partial^2 w_i^s}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_i^s}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_i^s}{\partial x^2} - \frac{\partial (n_w)_i^s}{\partial x} + (t_w)_i^s = 0 \\
\rho S \frac{1}{2} \frac{\partial^2 u_i^c}{\partial t^2} - \frac{\partial (n_u)_i^c}{\partial x} + (t_u)_i^c = 0 \\
\rho S \frac{1}{2} \frac{\partial^2 u_i^s}{\partial t^2} - \frac{\partial (n_u)_i^s}{\partial x} + (t_u)_i^s = 0
\end{array} \right. \quad (\text{B.14})$$

798 For the specific case where only mode 0 is considered in the Fourier expansion, the equations of motion write:

$$\left\{ \begin{array}{l}
\rho S \frac{\partial^2 U_o}{\partial t^2} - \frac{\partial N}{\partial x} = 0 \\
\rho S \left( \frac{\partial^2 V_o}{\partial t^2} - \frac{I_y}{S} \frac{\partial^4 V_o}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 M_y}{\partial x^2} = 0 \\
\rho S \left( \frac{\partial^2 w_o^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_o^c}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_o^c}{\partial x^2} + (t_w)_o^c = 0
\end{array} \right. \quad (\text{B.15})$$

799 with the generalized forces:

$$\left\{ \begin{array}{l}
N = + \frac{ES}{1 - \nu^2} \left( \frac{\partial U_o}{\partial x} + \nu \frac{w_o^c}{a} \right) \\
M_y = - \frac{EI_y}{1 - \nu^2} \frac{\partial^2 V_o}{\partial x^2} \\
(t_w)_o^c = + \frac{ES}{1 - \nu^2} \left( \frac{1}{a} \left( \frac{w_o^c}{a} + \nu \frac{\partial U_o}{\partial x} \right) + \frac{1}{2} (a + w_o^c) \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 + \frac{e^2}{12a^2} \left( \frac{w_o^c}{a^2} + \nu \frac{\partial^2 w_o^c}{\partial x^2} \right) \right) \\
(m_w)_o^c = - \frac{ES}{1 - \nu^2} \frac{e^2}{12} \left( \frac{\partial^2 w_o^c}{\partial x^2} + \nu \frac{w_o^c}{a^2} \right)
\end{array} \right.$$

800 with the moment of inertia given by:

$$I_y = \pi a^3 e \left( 1 + 2 \frac{w_o^c}{a} + \left( \frac{w_o^c}{a} \right)^2 + \frac{e^2}{12a^2} \right)$$

801 When neglecting the terms associated with the non-linear coupling and the terms in  $e^2/12a^2$  due to the thin-walled  
 802 assumption, this leads to:

$$\begin{cases} N &= +\frac{ES}{1-\nu^2} \left( \frac{\partial U_o}{\partial x} + \nu \frac{w_o^c}{a} \right) \\ M_y &= -\frac{EI_y}{1-\nu^2} \frac{\partial^2 V_o}{\partial x^2} \\ (t_w)_o^c &= +\frac{ES}{1-\nu^2} \frac{1}{a} \left( \frac{w_o^c}{a} + \nu \frac{\partial U_o}{\partial x} \right) \\ (m_w)_o^c &= -\frac{ES}{1-\nu^2} \frac{e^2}{12} \frac{\partial^2 w_o^c}{\partial x^2} \end{cases} \quad \text{with } I_y = \pi a^3 e = I_y^b \quad (\text{B.16})$$

803 For the specific case where only mode 2 is considered in the Fourier expansion, the equations of motion write:

$$\begin{cases} \rho S \frac{\partial^2 U_o}{\partial t^2} - \frac{\partial N}{\partial x} &= 0 \\ \rho S \left( \frac{\partial^2 V_o}{\partial t^2} - \frac{I_y}{S} \frac{\partial^4 V_o}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 M_y}{\partial x^2} &= 0 \\ \rho S \frac{1}{2} \left( \left( \frac{5}{4} + \frac{9}{4} \frac{e^2}{12a^2} \right) \frac{\partial^2 w_2^c}{\partial t^2} - \frac{e^2}{12} \frac{\partial^4 w_2^c}{\partial t^2 \partial x^2} \right) - \frac{\partial^2 (m_w)_2^c}{\partial x^2} - \frac{\partial (n_w)_2^c}{\partial x} + (t_w)_2^c &= 0 \end{cases} \quad (\text{B.17})$$

804 with the following generalized forces:

$$\begin{cases} N &= +\frac{ES}{1-\nu^2} \frac{\partial U_o}{\partial x} \\ M_y &= -\frac{EI_y}{1-\nu^2} \frac{\partial^2 V_o}{\partial x^2} \\ (t_w)_2^c &= \frac{ES}{1-\nu^2} \left( \frac{3}{2} \frac{e^2}{12a^2} \left( 3 \frac{w_2^c}{a^2} - \nu \frac{\partial^2 w_2^c}{\partial x^2} \right) + \frac{1}{16} \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 \left( -6a + 5w_2^c + 3 \frac{e^2}{12a} \left( 2 + 3 \frac{w_2^c}{a} \right) \right) \right) \\ (n_w)_2^c &= +\frac{ES}{1+\nu} \frac{1}{16} \left( 1 + 49 \frac{e^2}{12a^2} \right) \frac{\partial w_2^c}{\partial x} \\ (m_w)_2^c &= -\frac{ES}{1-\nu^2} \frac{e^2}{24} \left( \frac{\partial^2 w_2^c}{\partial x^2} - 3\nu \frac{w_2^c}{a^2} \right) \end{cases}$$

805 with the moment of inertia given by:

$$I_y = \pi a^3 e \left( 1 - \frac{3}{2} \frac{w_2^c}{a} + \frac{5}{8} \left( \frac{w_2^c}{a} \right)^2 + \frac{e^2}{12a^2} \right)$$

806 Assuming that  $\frac{\partial^2 w_2^c}{\partial x^2} = 0$  and neglecting the terms in  $e^2/12a^2$  in  $(n_w)_2^c$ ,  $(m_w)_2^c$ , in the coupling term in  $(t_w)_2^c$  and in  $I_y$ ,  
 807 leads to:

$$\begin{cases} N &= +\frac{ES}{1-\nu^2} \frac{\partial U_o}{\partial x} \\ M_y &= -\frac{EI_y}{1-\nu^2} \frac{\partial^2 V_o}{\partial x^2} \\ (t_w)_2^c &= \frac{ES}{1-\nu^2} \left( \frac{9}{2} \frac{e^2}{12a^2} \frac{w_2^c}{a^2} + \frac{1}{16} \left( \frac{\partial^2 V_o}{\partial x^2} \right)^2 (-6a + 5w_2^c) \right) \\ (n_w)_2^c &= \frac{ES}{1+\nu} \frac{1}{16} \frac{\partial w_2^c}{\partial x} \\ (m_w)_2^c &= 0 \end{cases} \quad (\text{B.18})$$

808 with  $I_y = \pi a^3 e \left( 1 - \frac{3}{2} \frac{w_2^c}{a} + \frac{5}{8} \left( \frac{w_2^c}{a} \right)^2 \right)$ .

809 *Appendix B.4. Cylindrical shell equations of motion*

810 It has also to be noticed that the present tube kinematics incorporates the circular cylindrical shell kinematics.  
811 Considering the shell terms in Eqs. (B.2) and (B.7) leads to the two following relationships:

$$\mathcal{P}_a = \int_{\Omega} \rho \left[ \frac{\partial^2 u_o}{\partial t^2} \delta u_o + z^2 \frac{\partial^2 u_1}{\partial t^2} \delta u_1 + \frac{\partial^2 v_o}{\partial t^2} \delta v_o + z^2 \frac{\partial^2 v_1}{\partial t^2} \delta v_1 + \frac{\partial^2 w_o}{\partial t^2} \delta w_o \right] dV$$

812 and

$$\begin{aligned} \mathcal{P}_i = & - \int_{\Omega} \left[ \sigma_{xx}^m \frac{\partial \delta u_o}{\partial x} + \sigma_{x\theta}^m \frac{1}{a} \frac{\partial \delta u_o}{\partial \theta} - z^2 \sigma_{xx}^k \frac{\partial \delta u_1}{\partial x} - \sigma_{x\theta}^k \frac{z^2}{a} \frac{\partial \delta u_1}{\partial \theta} + \sigma_{x\theta}^m \frac{\partial \delta v_o}{\partial x} + \left( \sigma_{\theta\theta}^m - \sigma_{\theta\theta}^k \frac{z^2}{a} \right) \frac{1}{a} \frac{\partial \delta v_o}{\partial \theta} \right. \\ & \left. + \sigma_{x\theta}^k z^2 \frac{\partial \delta v_1}{\partial x} + \sigma_{\theta\theta}^k \frac{z^2}{a} \frac{\partial \delta v_1}{\partial \theta} + \left( \sigma_{\theta\theta}^m - \sigma_{\theta\theta}^k \frac{z^2}{a} \right) \frac{1}{a} \delta w_o \right] dV \end{aligned}$$

813 As a consequence, the corresponding equations of motion write:

$$\left\{ \begin{array}{l} \rho a e \frac{\partial^2 u_o}{\partial t^2} = a \frac{\partial N_x}{\partial x} + \frac{\partial Q_{x\theta}}{\partial \theta} \\ \rho a e \frac{e^2}{12} \frac{\partial^2 u_1}{\partial t^2} = -a \frac{\partial M_x}{\partial x} - \frac{\partial N_{x\theta}}{\partial \theta} \\ \rho a e \frac{\partial^2 v_o}{\partial t^2} = a \frac{\partial Q_{x\theta}}{\partial x} + \frac{\partial Q_{\theta}}{\partial \theta} \\ \rho a e \frac{e^2}{12} \frac{\partial^2 v_1}{\partial t^2} = a \frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_{\theta}}{\partial \theta} \\ \rho a e \frac{\partial^2 w_o}{\partial t^2} = -Q_{\theta} \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{l} N_x = \int \sigma_{xx}^m dz \\ Q_{x\theta} = \int \sigma_{x\theta}^m dz \\ M_x = \int \sigma_{xx}^k z^2 dz \\ N_{x\theta} = \int \sigma_{x\theta}^k z^2 dz \\ Q_{\theta} = \int \left( \sigma_{\theta\theta}^m - \sigma_{\theta\theta}^k \frac{z^2}{a} \right) dz \\ N_{\theta} = \int \sigma_{\theta\theta}^k z^2 dz \end{array} \right. \quad (\text{B.19})$$

814 where the variables  $Q$ ,  $N$  and  $M$  correspond to the transverse shearing force, in-plane force and moments, respectively.  
815 These equations are in agreement with those expressed in [47, 48] for circular cylindrical shells when considering  
816 integration over the elementary volume  $a \, d\theta dz dx$  instead of  $(a+z) \, d\theta dz dx$ .

817 **Appendix C. Size and sparseness of the stiffness matrices**

818 Using the same notations that the ones used in Sec. 3, the (symmetric) stiffness matrix of the tube element can be  
819 written as:

$$\mathbf{K}_e = \begin{pmatrix} \mathbf{K}_e^{11} & \mathbf{K}_e^{12} \\ \mathbf{K}_e^{21} & \mathbf{K}_e^{22} \end{pmatrix} \quad \text{with} \quad \mathbf{K}_e^{ij} = \int_0^{\ell_e} \left( \mathbf{B}_e^i \right)^T \mathbf{D} \mathbf{B}_e^j dx \quad (\text{C.1})$$

820 with  $\mathbf{D}$  the (symmetric) constitutive matrix. With the expressions of the generalized forces and moments expressed in  
821 Eqs. (B.9), (B.10), (B.11), (B.12) and (B.13), we can conclude that the non-zero components of the matrix  $\mathbf{K}_e^{ij}$  are:

$$\mathbf{K}_e^{ij} = \begin{pmatrix} \mathbf{K}_{ij}^{b,b} & \mathbf{K}_{ij}^{b,s_o} & \mathbf{K}_{ij}^{b,s_1} & \mathbf{K}_{ij}^{b,s_2} & \mathbf{K}_{ij}^{b,s_3} & \mathbf{K}_{ij}^{b,s_4} & \mathbf{K}_{ij}^{b,s_5} & \mathbf{K}_{ij}^{b,s_6} \\ \mathbf{K}_{ij}^{s_o,b} & \mathbf{K}_{ij}^{s_o,s_o} & \mathbf{K}_{ij}^{s_o,s_1} & \mathbf{K}_{ij}^{s_o,s_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{ij}^{s_1,b} & \mathbf{K}_{ij}^{s_1,s_o} & \mathbf{K}_{ij}^{s_1,s_1} & \mathbf{K}_{ij}^{s_1,s_2} & \mathbf{K}_{ij}^{s_1,s_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{ij}^{s_2,b} & \mathbf{K}_{ij}^{s_2,s_o} & \mathbf{K}_{ij}^{s_2,s_1} & \mathbf{K}_{ij}^{s_2,s_2} & \mathbf{K}_{ij}^{s_2,s_3} & \mathbf{K}_{ij}^{s_2,s_4} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_{ij}^{s_3,b} & \mathbf{0} & \mathbf{K}_{ij}^{s_3,s_1} & \mathbf{K}_{ij}^{s_3,s_2} & \mathbf{K}_{ij}^{s_3,s_3} & \mathbf{K}_{ij}^{s_3,s_4} & \mathbf{K}_{ij}^{s_3,s_5} & \mathbf{0} \\ \mathbf{K}_{ij}^{s_4,b} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{ij}^{s_4,s_2} & \mathbf{K}_{ij}^{s_4,s_3} & \mathbf{K}_{ij}^{s_4,s_4} & \mathbf{K}_{ij}^{s_4,s_5} & \mathbf{K}_{ij}^{s_4,s_6} \\ \mathbf{K}_{ij}^{s_5,b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{ij}^{s_5,s_3} & \mathbf{K}_{ij}^{s_5,s_4} & \mathbf{K}_{ij}^{s_5,s_5} & \mathbf{K}_{ij}^{s_5,s_6} \\ \mathbf{K}_{ij}^{s_6,b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{ij}^{s_6,s_4} & \mathbf{K}_{ij}^{s_6,s_5} & \mathbf{K}_{ij}^{s_6,s_6} \end{pmatrix}$$

822 where the matrix  $\mathbf{K}_e^{ij}$  has a symmetric/anti-symmetric form with for  $k \geq 2$ :

$$\mathbf{K}_{ij}^b = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{b,s_o} = \begin{pmatrix} \times & \times \\ \times & \times \\ \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{b,s_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{b,s_k} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix},$$

$$\mathbf{K}_{ij}^{s_o} = \begin{pmatrix} \times & \times \\ \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{s_o,s_1} = \begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{s_o,s_2} = \begin{pmatrix} 0 & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 & 0 \end{pmatrix},$$

823

$$\mathbf{K}_{ij}^{s_1} = \begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{s_1,s_2} = \begin{pmatrix} 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{K}_{ij}^{s_1,s_3} = \begin{pmatrix} 0 & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{pmatrix},$$

824 and for  $k \geq 2$

$$\mathbf{K}_{ij}^{s_k} = \begin{pmatrix} \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \end{pmatrix}, \quad \mathbf{K}_{ij}^{s_k,s_{k+1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times & \times \\ 0 & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{K}_{ij}^{s_k,s_{k+2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times & \times \end{pmatrix},$$

825 where the non-zero components due to the non-linear coupling are colored in magenta and where the superscript  $b$  and  
826  $s_k$  correspond to the beam and the shell DOFs associated with the Fourier mode  $k$ , respectively. As a consequence, the  
827 number of the non-zero components of the stiffness matrix of the tube element is 1948 for the non-linear formulation  
828 and 476 for the linear formulation. In comparison, the stiffness of a rectangular plate with 4 nodes and 6 DOFs per  
829 node can be written as:

$$\mathbf{k}_e^s = \begin{pmatrix} \mathbf{k}_e^{ij} \end{pmatrix} \quad \text{where} \quad \mathbf{k}_e^{ij} = \begin{pmatrix} \mathbf{k}_{ij}^{tr} & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_{ij}^{rot} \end{pmatrix} \quad \text{for } 1 \leq i, j \leq 4$$

830 where the superscript  $tr$  and  $rot$  correspond to the translational and rotational DOFs, respectively, with:

$$\mathbf{k}_{ij}^{tr} = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{k}_{ij}^{rot} = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}$$

831 As a consequence, the number of the non-zero components of the shell stiffness matrix is 208.