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Self-Management of ROV Umbilical using sliding element: a general 3D-model

Christophe Viel*

July 10, 2023

Abstract

The umbilical of Remote Operated Vehicle (ROV) has two main problems: it is subject to entanglement with obstacles or itself, and its shape is difficult to predict for navigation. To address these issues, this article proposes a passive self-management of an ROV's umbilical by adding one or two elements, like ballasts, buoys, or an oriented thruster, to stretch it and gives it a predictable shape. These elements can be fixed or move freely on the umbilical. In opposite with [26, 27], we propose a general model which can estimate the shape of the umbilical in three-dimension regardless of the orientation of the force applied to the elements, allowing to consider the presence of currents, passive or motorized elements on the tether, and the presence or not of TMS. Several examples of umbilical configuration are proposed, each one adapted to ROV exploration missions like near the surface, seafloor, wall, dive in presence of obstacles... The model is compared with results proposed in other works like [26, 27]. The limits of the method are discussed.

Part I

Introduction, state-of-art, contribution

1 Introduction

Underwater umbilicals allow to connect an underwater Remotely Operated Vehicle (ROV) to a control unit or Human-Machine Interface typically located on a boat. The umbilical, or tether, fulfills three objectives: real-time bidirectional data transmission, *i.e.* including video feedback, control inputs, and instrument measurements... (see [6, 23]), providing

energy to the ROV, and preventing the loss of the robot during exploration [21]. However, umbilicals have drawbacks such as collision risks, impact on ROV maneuverability due to umbilical inertia and drag forces, entanglement, and cable breakage. Designing the umbilical involves balancing the constraints of the umbilical, battery power, and real-time feedback for optimal ROV performance [7].

Understanding the shape of the umbilical holds two main advantages. Firstly, designing the umbilical's parameters before diving helps avoid weight-related issues and reduces the risk of entanglement by using a cable that is appropriately sized and not excessively long. Secondly, real-time knowledge of the umbilical shape during the dive enables operators to prevent self-entanglement or collisions with known obstacles in a mapped environment or detected by sensors such as sonar or vision systems. In the literature, umbilicals have been modeled and instrumented to provide feedback on their position and shape. Two main categories of methods exist: vision-based detection of the umbilical [19, 18, 20] and/or sensors placed directly on or in the umbilical to obtain shape feedback [13, 10], or direct modeling of the umbilical using boat and ROV positions [14, 15, 12, 27, 5], sometimes accounting for sea current information. While the first category offers accurate real-time shape estimation, it often requires expensive umbilical equipment and complex sensor setups, making modeling methods more appealing for cost-effective obstacle avoidance. In opposite, this second category of methods have the advantage of being implementable for all kinds of umbilicals, but are often less accurate and cannot always provide results in real-time.

Various methods are available for modeling the cable's shape and dynamics. These range from simple geometrical models like the catenary curve [24, 8, 9] to segment-based models with geometrical constraints [14]. Geometrical models are suitable for simulating a large number of segments in real-time and are memory-efficient when precise physical modeling is not necessary. For cases requiring accurate cable

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dynamics, the Lumped-mass-spring method [4, 15, 16] and the segmental method are commonly used [11, 12, 2, 5]. The former represents the umbilical as mass points interconnected by massless elastic elements, while the latter treats the cable as a continuous system and solves the resulting partial differential equations numerically.

The umbilical can also be equipped. A Tether Management System (TMS) [1] is a subsea winch controlled by a human operator and attached to the ROV cage. The TMS helps regulate the length of the tether cable to maintain the umbilical taut. However, this system adds weight to the cable and can be operationally complex. Some research explores automation or alternative solutions, such as using an Unmanned Surface Vehicle (USV) [25, 17], a secondary ROV [20], multiple ROVs, or a motorized plug/float assembly [3]. In [9, 8], an estimation of the catenary tether shape connecting a pair of underwater robots is performed using data acquired from inertial measurement units (IMUs) attached to the cable near its ends. To provide a fast time computational model of the cable, the main author [27, 26] has proposed to equip the umbilical with ballasts or buoys moving freely on the cable, allowing it to be stretched and given a predictable shape. The umbilical can therefore be assimilated to predictable straight lines, simple to model and compute calculate in real time. Nonetheless, these systems still rely on knowledge of various ROV parameters, such as its position.

This paper proposes proposes a passive self-management of an ROV’s umbilical by adding one or two elements, like ballasts, buoys or an oriented thruster, to stretch it and gives it a predictable shape. These elements can be fixed or move freely on the umbilical, restrained to a specific part of the cable with stops to prevent the elements from staying constantly in contact. In opposite with our previous work [26, 27], a general model is proposed which can estimate the shape of the umbilical in three-dimension regardless of the orientation of the force applied to the elements, allowing to consider presence of currents, passive or motorized elements on the tether, and presence or not of TMS.

The main contributions are

- a self-management of the umbilical to stretch it and so avoid self-entanglement. This one can be passive using only sliding ballasts and buoys using only weight and Archimedes force, or active using motorized thruster or TMS;
- a method simple to add to existing ROVs with a light and simple setup;
- a general model of the umbilical in three-

dimensions which can consider presence of currents or external forces, helping the operator to prevent collisions with environmental obstacles.

The umbilical is modeled using geometrical relations and the Fundamental Principle of Static (FPS). Depending on the umbilical configurations, the resolution will be performed analytically or numerically. Numerical performance are discussed

The paper is divided in five parts. Part I introduces the problem in Section 1 and the related work in Section 2.

Part II grouped the umbilical model, the assumptions taken, definitions and properties of the umbilical and elements studied. The problematic and assumptions considered are described in Section 3. The geometrical model of the umbilical and its different configurations are exposed in Section 4. The dynamics of the system is studied in Section 5.

Part III presents theorems allowing to solve the system in function of the current configuration of the umbilical.

Part IV discutes of the numerical resolution of the problem, proposes possible umbilical configuration/equipment adapted for specific missions, discuss of the limits of the model based on simulation and experimentation.

Section 10 concludes this work.

2 Related work

2.1 Cable modeling

Various methods are available for modeling the shape and dynamics of cables. The simplest model is the catenary curve [24], which describes a non-rigid flexible cable with greater weight in water than buoyancy force. However, for longer or heavier cables, additional parameters such as bending stiffness need to be considered. In other methods, like the one presented in [14], neutrally buoyant cables are assumed, allowing the gravity and buoyancy forces to be ignored. These methods involve modeling the umbilical as a chain of segments with geometric constraints to account for its stiffness. While these geometric models may not be physically accurate or consider cable dynamics, they offer fast calculations and are memory efficient.

For a dynamic and physically accurate cable model, two main types of methods exist as discussed in [2]: the lumped-mass-spring method[4, 15, 16] and the segmental method [11, 12]. The lumped-mass-spring method represents the umbilical as mass points connected by massless elastic elements. This approach is suitable for modeling elastic cables but requires significant computational resources. On the other hand, the segmental

method treats the cable as a continuous system and numerically solves the resulting partial differential equations. These two methods primarily focus on cable dynamics in simplified environments, considering forces such as gravity, buoyancy, hydrodynamic drag, environmental inertial force, axial tension, twisting force, and bending force. In [12], a three-dimensional ROV-cable model is presented, utilizing the Euler-Bernoulli beam theory modified to allow for cable compression. The model’s accuracy is verified through experimental validation. Building on the beam equation, [5] derives a realistic 3D model of tether dynamics and approximates it via the Finite Element Method (FEM), which can be executed in real time. The method is however only tested in simulation.

In [27, 26] the umbilical is equipped with ballasts or buoys moving freely on the cable giving it a predictable shape. The umbilical can therefore be assimilated to predictable straight lines, simple to model and compute calculate in real time. The method however stills rely on knowledge of parameters like ROV’s position and forces applied on the sliding elements, and solves the problem for specific configurations. In particular, [26] considers only a 2 dimensional case without horizontal current.

2.2 Cable instrumentation

The umbilical can be equipped and instrumented to provide feedback on its position and shape. In the literature, two main categories of methods are discussed: vision-based detection of the umbilical [19, 18, 20], and the use of sensors embedded directly in the umbilical [13, 10]. These methods offer accurate real-time modeling but require specific, expensive, and complex equipment.

In [13], the "Smart Tether" method is introduced, which uses IMU sensor nodes embedded in the umbilical to provide real-time information about its shape and motion. However, this method is costly and introduces irregularities along the cable, making winding problematic. In [10], optic fibers are braided within the umbilical and use interferometry properties to monitor the cable’s 3D curve in real time. Again, this solution is expensive (around 200,000 euros for 50m length).

For shallow-water exploration, [9, 8, 20] proposes an estimation of the shape of a catenary for a negatively buoyant cable, connecting a pair of underwater robots. First estimate with camera-based detection and tracking of a constant distance between the ROVs in [20, 19, 18], the method is then in [9, 8] based on the calculation of local tangents thanks to the data acquired from inertial measurement units (IMUs), which are attached to the cable near its ends. In opposite with works in [19], the cable shape’s identification is not affected by the limits of the camera’s field of view and by the image projection. Combining it with

the study in [27], an improvement is proposed in [28] by equipping the umbilical with moving ballasts and buoys to give it a predictable shape with straight lines, in opposite with the catenary curve trickier to estimate.

Additional components such as TMS, ballasts, buoys, or intermediate cables can be used as dampers to mitigate undesired forces on the ROV caused by waves, currents, or the weight of the umbilical. For deep and ultra-deep-water operations, [22] proposes alternative configurations to minimize umbilical tension and the risk of snap, such as installing a series of floaters along the umbilical. However, floaters increase the ROV’s offset and sensitivity to currents. The TMS is a common equipment choice, acting as an underwater winch attached to the ROV’s cage to regulate the tether length and maintain tension. When placed underwater, it also serves as a ballast to reduce ROV offset due to currents and waves. However, TMS is not suitable for umbilicals equipped with buoys or ballasts.

Due to the complexity of TMS operation, efforts have been made to automate or replace it with other vehicles such as USVs or secondary ROVs [20, 3, 25]. In [3], a motorized plug/float assembly moves along the umbilical to adjust its buoyancy, even allowing the ROV to temporarily function as an AUV. The cable length is regulated by a winch on the boat. While this system offers flexibility, it is large, expensive, and not suitable for all ROV applications. In [25, 17], an integrated system consisting of a USV with an embedded winch, an umbilical, and an ROV is proposed to provide various cable management options. The distance between the USV and the ROV can be adjusted to control the tension in the umbilical and avoid collisions with underwater obstacles. A segmental method-based mechanical behavior model is proposed in the study. In [27, 26], the addition of ballast and a buoy or two buoys moving freely on the umbilical allows to avoid self-entanglement of the cable by stretching it, and gives it a predictable shape. The buoys move on their own to keep the cable taut without a motorized system, making this method easy to adapt to existing ROVs with few constraints on their navigation. [27, 26] proposes different shape of umbilical obtained with ballasts an buoy, allowing to expose close to the surface, seafloor exploration, or dive easily in a cluttered environment.

Part II

Umbilical model, assumption, properties

3 Problematic and assumption

3.1 Notation

Consider the indirect referential \mathcal{R} where the vertical axis is oriented to the ground such that $y = 0$ corresponds to the sea level and $y_1 > y_2$ means that y_1 is deeper than y_2 . The umbilical is attached between $O = (0, 0, 0)$ and $R = (x, y, z)$, the boat's and ROV's coordinates. Let define $\vec{e}_x = [1 \ 0 \ 0]^T$, $\vec{e}_y = [0 \ 1 \ 0]^T$ and $\vec{e}_z = [0 \ 0 \ 1]^T$. Let $\vec{a} \cdot \vec{b}$ be the scalar product of two vector $\vec{a} \in \mathbb{R}^3$ and $\vec{b} \in \mathbb{R}^3$. For the two arguments (x, y) , the function atan2 is defined as $\phi = \text{atan2}(y, x)$.

For a variable a , let note $a_{\mathcal{R}_b}$ the variable in a referential \mathcal{R}_b . In order to lighten notations, the notations $v_{\mathcal{R}_b} = [a_{\mathcal{R}_b} \ b_{\mathcal{R}_b} \ c_{\mathcal{R}_b}]^T$ and $v_{\mathcal{R}_b} = [a \ b \ c]^T_{\mathcal{R}_b}$ are equivalent. By default and unless otherwise specified in the current section, parameters are expressed in \mathcal{R} ($x_{\mathcal{R}}$ is noted x). For two lists $L_1 = \{a, b\}$ and $L_2 = \{c, d\}$, $L_1 \cup L_2 = \{a, b, c, d\}$ is the union of the lists and $L_3 = \{L_1, L_2\} = \{\{a, b\}, \{c, d\}\}$ is a list of lists.

3.2 Problematic

In the absence of tension, a cable can move freely and take on an irregular shape. To prevent it from becoming entangled with itself or its environment, a common technique for shallow dives is to suspend a ballast at a fixed length on the umbilical, stretching the portion between the boat and the ballast. However, the part between the ballast and the ROV remains loose, especially when the ROV is close to the boat, where it takes the shape of a bell. In order to keep the umbilical taut independently of the robot position, we propose to equip the umbilical with other elements to stretch it and give it a geometrical shape that is simple to model and predict. In opposite with [26, 27], the model proposed here consider three-dimensional case and a general orientation of the forces applied on the umbilical.

In [26, 27], additional elements attached to the cables were ballasts and buoys, *i.e.* the study considered only vertical forces. They can be defined as "fixed" or "sliding". A fixed element can usually only stretch one part of it, both in particular configuration. A sliding element however finds its position at its minimum potential energy where it stretches both parts of the cable simultaneously, corresponding to the lowest/highest point in case of ballast/buoy. In this study,

the Definition 1 defines the term "element" such that it takes into account both horizontal and vertical forces, making it possible to take into account the presence of currents in addition to the forces exerted by the ballast/buoy, or to propose a motorized element pushing in a customized direction.

Definition 1. An element i on the umbilical is defined by a vector \vec{F}_i expressed as

$$\vec{F}_i = \begin{bmatrix} \cos(\phi_i) \cos(\psi_{Fi}) \\ \sin(\phi_i) \cos(\psi_{Fi}) \\ \sin(\psi_{Fi}) \end{bmatrix} F_i \quad (1)$$

where F_i is the norm of the force applied on the cable, ψ_{Fi} is the orientation in the plan $O\vec{x}\vec{y}$ and $\phi_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the angle of elevation from the plan $O\vec{x}\vec{y}$. Let's also define the notation $\vec{F}_i = [F_{ix} \ F_{iy} \ F_{iz}]^T$.

An element is considered

- "Sliding" if its point of application changes with the cable orientation. The movements of the sliding element are limited to a defined portion of the cable of length l_i . Let l_{i1} and l_{i2} be the two parts of the cable l_i on each side of the element i such $l_i = l_{i1} + l_{i2}$ with $l_i \geq l_{ij} \geq 0$ for $j \in \{1, 2\}$.
- "Fixed" if its point of application is constant on the cable. In this case, consider the element is at the end of its portion of cable, *i.e.* ones has $l_{i1} = l_i$ and $l_{i2} = 0$.

A combination of sliding elements or fixed and sliding elements is an interesting solution to stretch the tether. Each sliding element must be restrained to a specific part of the cable with stops to prevent the elements from staying in contact at the same minimum potential energy. The umbilical is so divided into several parts with defined lengths. In the optic to make this study the most general possible, consider here the length of these parts can be modified using TMS at the end of the tether or at the stop position, but that the length of these parts is always known.

In this paper, the umbilical is equipped with one or two elements at most, with always at least one sliding element, plus a possible anchor directly after the boat. Cases with more elements are not treated here because the solution will be too complex and/or none efficient. This point will be discussed in Section 9.3.

3.3 Assumption

Consider the umbilical is equipped maximum two elements at most, plus a possible anchor directly after the boat (not necessarily on the seafloor). The following assumption are considered:

- A1)** The forces applied on the umbilical due to its mass/buoyancy are negligible compare to the action made by the elements and the anchor.
- A2)** The umbilical length is such that it is reasonable to neglect its length variation, considered as constant.
- A3)** When the umbilical is taut, its geometry can be assimilated to straight lines between defined points, here the elements, the boat and the ROV.
- A4)** Consider the ROV, boat and anchor are strong enough to compensate action of the cables and the elements, and therefore remain stationary. (x, y, z) is so fixed when ROV is not moving.
- A5)** Consider the sliding elements move freely on the umbilical without friction.
- A6)** The end of a cable connected to an anchor or to the boat is assumed to be perfectly fixed. A cable connected at both ends by the boat and the anchor does not need to meet the Assumption A1–A3.
- A7)** The lengths l_i of the different parts of the umbilical are supposed to be constant or known in real time. Possible TMS positioned on the boat, anchor or the ROV can be placed to effect these lengths.
- A8)** The influence of currents and external perturbations are considered in the elements i .

The model proposed in this paper assumes that the umbilical can be assimilated to straight lines by the action of the elements (neglecting a slight bending of the cable). Excluding the configurations where an element reaches the surface (for buoys) or touches the seabed (for ballasts), these cases do not exist if two sliding elements are used, only if fixed elements are involved.

4 Description umbilical model

4.1 Geometrical model

The description made below consider a case with two sliding elements and an anchor, from which the other configurations can be deduced. Indeed:

- a fixed element i can be assimilated to a particular case of a sliding element i where $l_{i2} = 0$ (see Definition 1);
- the absence of anchor correspond to $l_0 = 0$;
- a configuration with an unique element can be obtained taking $l_2 = 0$ and $F_2 = 0$.

The parameters are illustrated in Figure 1 in 2D, with an illustration of a 3D case in Figure 2. The umbilical of length l is divided in three parts. Let $l_0 = \|OA\| \geq 0$ be the part between the boat O and the anchor A , $l_1 = \|AS\| \geq 0$ the part between the anchor A and a stop S , and $l_2 = \|SR\| \geq 0$ the part between the stop S and the ROV R . The total length of the boat is $l = L + l_0$ with $L = l_1 + l_2$. The sliding elements B_1 and B_2 can respectively move freely on l_1 and l_2 .

Following Assumption A6, the part l_0 is supposed to stay perfectly vertical. The sliding elements B_1 and B_2 create angles in the geometry of the umbilical. Note that the stop S does not create an angle by itself, so $\|B_1B_2\|$ is a straight line.

The orientation of $\|AB_1\|$ is described by the oriented angle α_1 and ψ_1 such that

$$\begin{aligned} \overrightarrow{AB_1} = & l_{11} (\cos(\alpha_1) \cos(\psi_1) \vec{e}_x \\ & + \sin(\alpha_1) \cos(\psi_1) \vec{e}_y + \sin(\psi_1) \vec{e}_z) \end{aligned} \quad (2)$$

In the same way, one can define the orientation of $\|B_1B_2\|$ and $\|B_2R\|$ with angles (α_2, ψ_2) and (α_3, ψ_3) such that

$$\begin{aligned} \overrightarrow{B_1B_2} = & (l_{12} + l_{21}) (\cos(\alpha_2) \cos(\psi_2) \vec{e}_x \\ & + \sin(\alpha_2) \cos(\psi_2) \vec{e}_y + \sin(\psi_2) \vec{e}_z) \end{aligned} \quad (3)$$

$$\begin{aligned} \overrightarrow{B_2R} = & l_{22} (\cos(\alpha_3) \cos(\psi_3) \vec{e}_x \\ & + \sin(\alpha_3) \cos(\psi_3) \vec{e}_y + \sin(\psi_3) \vec{e}_z). \end{aligned} \quad (4)$$

Note that $\alpha_1 \in [-\pi, \pi]$, $\alpha_2 \in [-\pi, \pi]$, $\alpha_3 \in [-\pi, \pi]$ and $\psi_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\psi_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The vectors $\overrightarrow{B_1S}$ and $\overrightarrow{SB_2}$ can also be expressed

$$\begin{aligned} \overrightarrow{B_1S} = & l_{12} (\cos(\alpha_2) \cos(\psi_2) \vec{e}_x \\ & + \sin(\alpha_2) \cos(\psi_2) \vec{e}_y + \sin(\psi_2) \vec{e}_z) \end{aligned} \quad (5)$$

$$\begin{aligned} \overrightarrow{SB_2} = & l_{21} (\cos(\alpha_2) \cos(\psi_2) \vec{e}_x \\ & + \sin(\alpha_2) \cos(\psi_2) \vec{e}_y + \sin(\psi_2) \vec{e}_z). \end{aligned} \quad (6)$$

In a configuration where the umbilical is taut, the system can be expressed such

$$\begin{cases} x = l_{11} \cos(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \cos(\alpha_2) \cos(\psi_2) \\ \quad + l_{22} \cos(\alpha_3) \cos(\psi_3) \\ y = l_0 + l_{11} \sin(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \sin(\alpha_2) \cos(\psi_2) \\ \quad + l_{22} \sin(\alpha_3) \cos(\psi_3) \\ z = l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) \end{cases} \quad (7)$$

Remark 1. When $\alpha_1 = \alpha_2$ and $\psi_1 = \psi_2$, the portion of cable l_1 is perfectly straight. Same remark for l_2 when $\alpha_2 = \alpha_3$ and $\psi_2 = \psi_3$.

Let's define $\mathcal{C}_{A,r}$ the sphere of center $A = [0 \ l_0 \ 0]$ and radius $r = l_1 + l_2$. Let's also define $\bar{\mathcal{C}}_{A,r}$ the truncated sphere such as for all $(x, y, z) \in \bar{\mathcal{C}}_{A,r}$, one has $(x, y, z) \in \mathcal{C}_{A,r}$ and $y \geq 0$, i.e. $\bar{\mathcal{C}}_{A,r} = \{(x, y, z) \in \mathcal{C}_{A,r} | y \in [0, r]\}$. $\bar{\mathcal{C}}_{A,r}$ corresponds to the underwater sphere where the ROV can move.

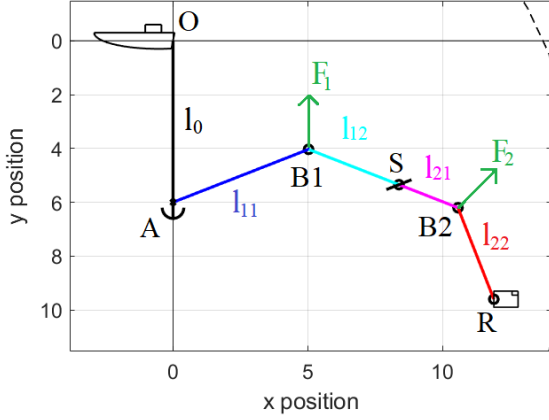


Figure 1: Parameters of the umbilical with two sliding elements B_1 and B_2 and an anchor A , in 2 dimension. \vec{F}_1 and \vec{F}_2 are the forces applied on the elements B_1 and B_2 . Here \vec{F}_1 is a pure vertical force made for example by a buoy, and \vec{F}_2 is a combination of a vertical and horizontal forces, for example a buoy plus horizontal currents.

4.2 Configuration Areas

Depending on the ROV's position, the sliding elements move and can come in contact 1) with the anchor A or the stop S for the element 1, 2) with the stop S or the ROV R for the element 2. To simplify the notations, let note S_i and S_{i+1} the stops that surround the element i such that the element i is in contact with S_i when $l_{i1} = 0$ and in contact with the S_{i+1} when $l_{i2} = 0$. Thus, one has $S_1 = A$ and $S_2 = S$ for the element 1, and $S_2 = S$ and $S_3 = R$ for the element 2.

The different cable combinations representing when elements do or do not touch a stop are called "configurations" or "areas". The system is considered to be inside an area if the ROV coordinates (x, y, z) are inside this area.

Eight areas corresponding to specific umbilical configurations can be observed, plus the forbidden area F, as illustrate in Figures 3 and 4.

The different areas are described below:

- Area A0: the two elements move freely on l_1 and l_2 , i.e. $0 < l_{11} < l_1$, $0 < l_{12} < l_1$, $0 < l_{21} < l_2$, $0 < l_{22} < l_2$.
- Area A1: the element 1 moves freely on l_1 , i.e. $0 < l_{11} < l_1$, $0 < l_{12} < l_1$. Moreover:
 - Area A11: the element 2 is in contact with R , thus $l_{21} = l_2$, $l_{22} = 0$.
 - Area A12: the element 2 is in contact with S , thus $l_{21} = 0$, $l_{22} = l_2$.
- Area A2: the element 2 moves freely on l_2 , i.e. $0 < l_{21} < l_2$, $0 < l_{22} < l_2$. Moreover:

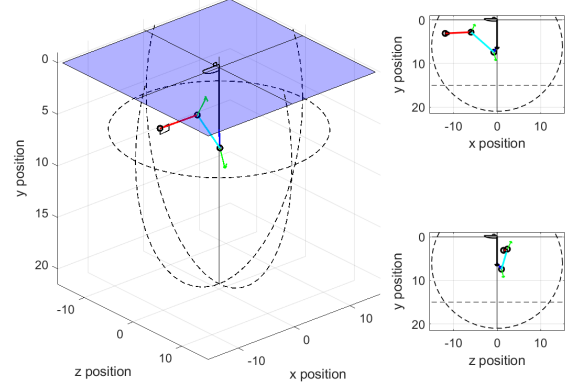


Figure 2: Illustration of the umbilical with two sliding elements and an anchor, in 3-Dimension.. $\vec{F}_1 = [0.5 \ 2 \ 0.5]$ and $\vec{F}_2 = [0.5 \ -1 \ 0.5]$.

- Area A21: the element 1 is in contact with A , thus $l_{11} = 0$, $l_{12} = l_1$.
- Area A22: the element 1 is in contact with S , thus $l_{11} = l_1$, $l_{12} = 0$.
- Area D: both elements are in contact with a stop such that
 - Area D1: the element 1 is in contact with S and element 2 is in contact with R , $l_{11} = l_1$, $l_{12} = 0$, $l_{21} = l_2$, $l_{22} = 0$.
 - Area D2: the two elements are in contact with S , i.e. $l_{11} = l_1$, $l_{12} = 0$, $l_{21} = 0$, $l_{22} = l_2$.
 - Area D3: the element 1 is in contact with A and element 2 is in contact with S , i.e. $l_{11} = 0$, $l_{12} = l_1$, $l_{21} = 0$, $l_{22} = l_2$.
- Area F. The forbidden area F includes 1) the inaccessible area for the ROV due to the length of its umbilical, 2) the areas where the umbilical cannot be taut by the elements, 3) case where an element is in contact with the seabed/surface.

In function of the number and characteristic of the elements on the umbilical, some configurations cannot exist. Area A0 for example cannot exist in presence of two forces in the same direction (example: two buoys) because one of the element will always be in contact with a stop, see Theorem 1. When the element 1 is fixed, Areas 21 and Areas 22 does not exist.

The Table 1 summarizes the existing area in function of the configurations.

Definition 2. When the cable l_1 is perfectly straight, i.e. $\alpha_1 = \alpha_2$ and $\psi_1 = \psi_2$, the sliding element 1 has no obstacle on this path. Since we consider there is no friction (Assumption A5) and to remove any ambiguity, we also assume that the sliding element can only be in contact with a stop, even in the case where \vec{F}_1 and

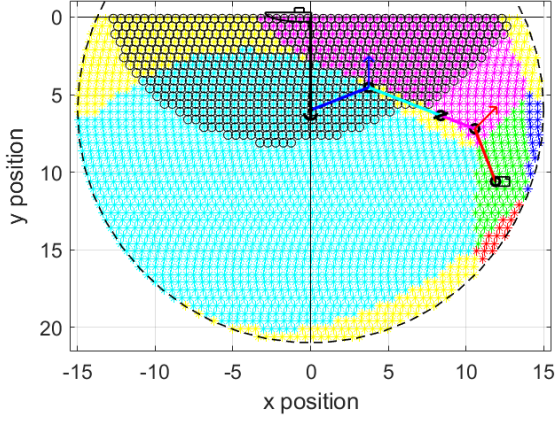


Figure 3: Areas for an umbilical with two sliding elements and an anchor. Green: A0. Blue: A11. Cyan: A12. Red: A21. Magenta: A22. Yellow: areas D. Black: area F where the umbilical cannot be stretched. Blue arrow : force \vec{F}_1 . Red arrow: force \vec{F}_2 .

\vec{AB}_1 are perpendicular. Same remark for element 2 if l_2 is straight. In this condition:

(1) The configuration $(\alpha_1 = \alpha_2) \& (\psi_1 = \psi_2)$ is exclude in areas A0, A11, A12, A22, D1, D2, but is assigned to area A21 or D3;

(2) The configuration $(\alpha_2 = \alpha_3) \& (\psi_2 = \psi_3)$ is exclude in areas A12, A0, A21, A22, D2, D3, but is assigned to area A21 or D1.

Theorem 1. The area A0 doesn't exist if

(1) $\frac{\vec{F}_1}{\|\vec{F}_1\|} = \frac{\vec{F}_2}{\|\vec{F}_2\|}$, i.e. the forces have the same orientation,

(2) $\frac{\vec{F}_1}{\|\vec{F}_1\|} = -\frac{\vec{F}_2}{\|\vec{F}_2\|}$ and $F_1 \neq F_2$, i.e. the forces' orientations are opposite and they don't have the same strength.

The proof of Theorem 1 is provided in Appendix F.3.1.

4.3 Graph of areas

When a sliding element moves on the cable, it comes in contact with a stop or becomes free again: the umbilical pass so from one configuration to another. One can define a graph of communication between the different areas, as summarizes in Figure 5. When some areas don't exist (see Table 1), the graph is cut off from the associated branches, but remains globally connected. Moreover, since an element cannot pass instantaneously from one stop to another without passing by a sliding phase, this graph is not fully-connected, i.e. the umbilical cannot always pass directly from any area to any other. Two areas directly connected are called neighbors.

This graph will be used to reduce the number of configurations to calculate to find the next shape of

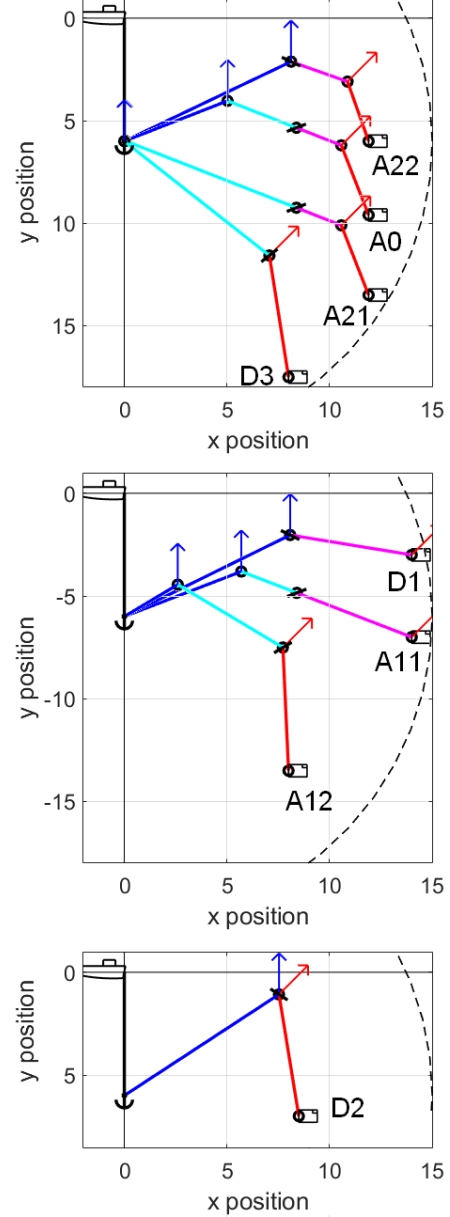


Figure 4: Illustration of the umbilical in all areas.

case	Element 1	Element 2	Existing areas	Remark
1	sliding	sliding	A0*, A11, A12, A21, A22, D1, D2, D3, F	A0*: depend of the direction of the element force
2	fixed	sliding	A22, D1, D2, F	$l_{11} = l_1$. Else, see case 5
3	sliding	fixed	A12, D2, D3, F	$l_{21} = 0$. Else, see case 5
4	fixed	fixed	D3, F	equivalent to one element in S . $l_{11} = l_1$ and $l_{21} = 0$
5	sliding	/	A11, D1, D3, F	One element configuration.
6	fixed	/	F	the cable cannot be taut excepted as a straight line

Table 1: Existing areas in function of the number and type of element. Note that a fixed element is always fixed in S . Else, it is assimilate to the anchor A or the ROV R (Assumption A4).

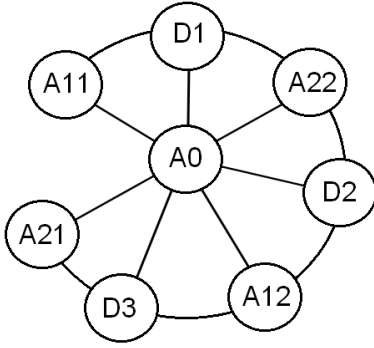


Figure 5: Graph of connection between areas.

the umbilical.

4.4 Potential energy

The movement of the element 1 are limited by the cable l_1 attached to the anchor $A = [0 \ l_0 \ 0]^T$. If the cable l_1 were subjected only to the actions of \vec{F}_1 , element 1 would slide since it came in contact with S , and then the part AS would behave like a pendulum, whose minimum potential energy $E_{p \min, 1}$ is reached at the position $p_{E_{p \min, 1}} = A + l_1 \frac{\vec{F}_1}{F_1}$, *i.e.* the point furthest from A in direction $\frac{\vec{F}_1}{F_1}$. Using A as reference, the potential energy $E_{p, 1}$ of element 1 can so be expressed as

$$E_{p, 1} = l_1 F_1 - \overrightarrow{AB_1} \cdot \vec{F}_1 \quad (8)$$

In the same way, the element 2 is limited to the cable l_2 attached in S to the cable l_1 . Following the same reasoning, the minimal potential energy $E_{p, 2}$ of element 2 can so be expressed as

$$E_{p, 2} = (l_1 + l_2) F_2 - \overrightarrow{AB_2} \cdot \vec{F}_2 \quad (9)$$

with $\overrightarrow{AB_2} = \overrightarrow{AB_1} + \overrightarrow{B_1 S} + \overrightarrow{SB_2}$. The potential energy of the system can so be expressed as

$$E_p = E_{p, 1} + E_{p, 2}. \quad (10)$$

This energy can be used to find the current configuration of the umbilical.

4.5 Deduction of the current area

The Part III will provide theorems giving a solution to the system (7) if the ROV is inside a corresponding area. If not solution can be found for the current coordinate, then the ROV is not inside this area. However, it can happen that several solutions are found in different areas, and only one corresponds to the reality. This section allows to select the current solution between the several solution found.

Consider in this section that if a solution is possible in an area for the current coordinate, then this solution is found. The current configuration can be found using the following steps:

1. **Reduction of the number of areas possible:**
 - (a) **Areas possible:** Table 1 defines the number of existing areas to be evaluated according to the elements on the umbilical.
 - (b) **Areas neighbors:** when the area of the previous position is known, if the new coordinates, forces and cable length are close to the previous configuration, only the current zone and its neighbors as defined in Section 4.3 need to be evaluated.
2. **Evaluation of the solution in the different areas:** a solution is searched in each remaining area using Theorems exposed in Sections 6 and 7.
3. **Selection of the areas:** if only one solution is found, this solution is the current configuration. Else, the true solution is the one with the smallest potential energy E_p exposed in Section 4.4.

4. **Final test:** check if the solution is a submerged configuration, *i.e.* the coordinate y is positive for the points B_1 , S and B_2 .

5. **If not solution is found:** the system is in area F.

Remark 2. not solution is found: in presence of at least one fixed element, sometime the umbilical cannot be stretched for some position of the ROV. In these conditions, the system (7) cannot find a solution and is considered inside area F. Example: element 1 is a fixed ballast pushing vertically with $l_1 > l_2$ (see [27]). The coordinate $(0, l_0, 0)$ requires the cable l_1 to be folded back to reach the position.

5 Dynamic of the system

The systems presented in the following sections are studied in equilibrium. It should be noted that each time the ROV moves, part of the umbilical temporarily becomes loosen, which can lead to entanglement and not follow the proposed model. As a loose cable is complex to model and/or computationally heavy, we propose to maintain a quasi-static equilibrium by controlling the ROV to shorten transient phases, thus minimizing the gap between proposed models and reality. In this way, the ROV is controlled to move more slowly than the elements on the cable: as long as their behavior is faster than the ROV speed, the umbilical will remain taut overall. Details of this approach is described in [27]. Same solution can be taken to manage the waves.

5.1 Fundamental Principle of Static

The dynamics of the system is studied at its equilibrium. Consider in this section than neither of the two elements touches the surface, the seafloor. or an obstacle. Since there is not friction (Assumption A5), the Fundamental Principle of Static (FPS) on B_i can be expressed as

$$\vec{F}_i + \vec{T}_i + \vec{T}_{i+1} = 0 \quad (11)$$

where \vec{T}_i and \vec{T}_{i+1} are the tension of the umbilical applied on element i . Let's note $T_i = \|\vec{T}_i\| > 0$ and $\|\vec{T}_{i+1}\| = T_{i+1} > 0$.

For an element $i \in \{1, 2\}$, if $B_1 \neq B_2$ and $B_1 \neq A$ if $i = 1$ or $B_2 \neq R$ if $i = 2$, one can perform the following FPS in B_i :

$$\begin{cases} -\cos(\psi_i) \cos(\alpha_i) T_i + \cos(\psi_{i+1}) \cos(\alpha_{i+1}) T_{i+1} \\ \quad + F_i \cos(\psi_{Fi}) \cos(\phi_i) & = 0 \\ -\cos(\psi_i) \sin(\alpha_i) T_i + \cos(\psi_{i+1}) \sin(\alpha_{i+1}) T_{i+1} \\ \quad + F_i \cos(\psi_{Fi}) \sin(\phi_i) & = 0 \\ -\sin(\psi_i) T_{1i} + \sin(\psi_{i+1}) T_{i+1} + F_i \sin(\psi_{Fi}) & = 0 \end{cases} \quad (12)$$

with $T_i > 0$, $T_{i+1} > 0$, $F_i > 0$. (12) is also valid if element i is fixed.

In particular case where both elements are in contact, *i.e.* $B_1 = B_2$, one gets a single FPS in B_1 :

$$\begin{cases} -\cos(\psi_1) \cos(\alpha_1) T_1 + \cos(\psi_3) \cos(\alpha_3) T_3 \\ \quad + F_{\Sigma x} & = 0 \\ -\cos(\psi_1) \sin(\alpha_1) T_1 + \cos(\psi_3) \sin(\alpha_3) T_3 \\ \quad + F_{\Sigma y} & = 0 \\ -\sin(\psi_1) T_1 + \sin(\psi_3) T_3 + F_{\Sigma z} & = 0 \end{cases} \quad (13)$$

where $\vec{F}_{\Sigma} = \vec{F}_1 + \vec{F}_2$ can be expressed as

$$\begin{bmatrix} F_{\Sigma x} \\ F_{\Sigma y} \\ F_{\Sigma z} \end{bmatrix} = \sum_{i=1}^2 \left(\begin{bmatrix} \cos(\phi_i) \cos(\psi_{Fi}) \\ \sin(\phi_i) \cos(\psi_{Fi}) \\ \sin(\psi_{Fi}) \end{bmatrix} F_i \right) \quad (14)$$

From Assumption A4, the ROV, boat and anchor are strong enough to compensate action of the elements. Thus, when a element is in contact with the anchor A , boat O or the ROV R , element's action can be fuse with them and so no FPS are performed.

From the areas properties exposed in Section 4.2, one has

- The PFD (12) with $i = 1$ is valid in areas A0, A11, A12, A22, D1;
- The PFD (12) with $i = 2$ is valid in areas A0, A12, A21, A22, D3;
- The PFD (13) is valid in area D2.

It can be deduced that T_1 does not exist in areas A21 and D3, T_2 does not exist in area D2 and T_3 does not exist in areas A11 and D1.

In function of if the systems (12) and (13) exist, *i.e.* in function of the areas, the expressions of the forces T_1 , T_2 and T_3 changed. Let's note $T_i^{S_j}$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$ the expression of T_1 , T_2 and T_3 in function of the different cases exposed below. In all case, one has

$$F_{1xy} = \cos(\psi_{F1}) F_1 \quad (15)$$

$$F_{2xy} = \cos(\psi_{F2}) F_2. \quad (16)$$

$$T_i^{S_j} = \frac{T_i^{S_j}}{\cos(\psi_i)} \quad (17)$$

- **If not area D2:** the systems (12) is valid. Following steps described in Appendix A.4, if $\cos(\alpha_1) \neq 0$ and/or $\cos(\alpha_3) \neq 0$, $\sin(\alpha_2 - \alpha_1) \neq 0$ and/or $\sin(\alpha_3 - \alpha_2)$, one gets $T_{1xy} = T_{1xy}^{S1}, T_{2xy} = T_{2xy}^{S1}$ and $T_{3xy} = T_{3xy}^{S1}$ where

$$T_{1xy}^{S1} = \begin{cases} 0 & \text{if A21 or D3,} \\ \frac{1}{\cos(\alpha_1)} (\cos(\alpha_2) T_{2xy}^{S1} + F_{1xy} \cos(\phi_1)) & \text{else,} \end{cases} \quad (18)$$

$$T_{3xy}^{S1} = \begin{cases} 0 & \text{if A11 or D1,} \\ \frac{1}{\cos(\alpha_3)} (\cos(\alpha_2) T_{2xy}^{S1} - F_{2xy} \cos(\phi_2)) & \text{else,} \end{cases} \quad (19)$$

with

$$T_{2xy}^{S1} = \begin{cases} -\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy} & \text{if A0, A11, A12 or D1,} \\ -\frac{\sin(\phi_2 - \alpha_3)}{\sin(\alpha_3 - \alpha_2)} F_{2xy} & \text{if A21, A22 or D3.} \end{cases} \quad (20)$$

- **If area D2:** the systems (13) is valid. Following the steps described in Appendix A.4.4, if $\cos(\alpha_1) \neq 0$ and $\sin(\alpha_3 - \alpha_1) \neq 0$, one has $T_{1xy} = T_{1xy}^{S2}$ and $T_{3xy} = T_{3xy}^{S2}$ where

$$T_{1xy}^{S2} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_3) T_{3xy}^{S2} + F_{\Sigma x}) \quad (21)$$

$$T_{3xy}^{S2} = \frac{F_{\Sigma x} \sin(\alpha_1) - F_{\Sigma y} \cos(\alpha_1)}{\sin(\alpha_3 - \alpha_1)} \quad (22)$$

and $T_{2xy}^{S2} = 0$ because doesn't exist.

In the case where $\alpha_3 = \alpha_1 + k_2\pi$ with $k_2 \in \mathbb{Z}$, one has $T_{3xy} = T_{1xy} = \frac{\sqrt{F_{\Sigma x}^2 + F_{\Sigma y}^2}}{2}$. If $\alpha_1 = s\frac{\pi}{2}$, one has $T_{3xy} = F_{\Sigma x} \frac{1}{\cos(\alpha_3)}$ and $T_{1xy} = s [\sin(\alpha_3) T_{3xy} + F_{\Sigma y}]$.

- **Particular cases:** if not area D2, in the case where $\alpha_1 = \alpha_2 + k_2\pi$ or $\alpha_3 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$, one has respectively $T_{1xy} = T_{2xy} = \frac{F_{1xy}}{2}$ or $T_{3xy} = T_{2xy} = \frac{F_{3xy}}{2}$. Moreover, if $\alpha_1 = s\frac{\pi}{2}$ or $\alpha_3 = s\frac{\pi}{2}$, one has respectively $T_{2xy} = F_{1xy} \frac{\cos(\phi_1)}{\cos(\alpha_2)}$ and $T_{1xy} = sF_{1xy} [\tan(\alpha_2) \cos(\phi_1) + \sin(\phi_1)]$ or $T_{2xy} = F_{2xy} \frac{\cos(\phi_2)}{\cos(\alpha_2)}$ and $T_{3xy} = sF_{2xy} [\tan(\alpha_2) \cos(\phi_2) + \sin(\phi_2)]$.

These systems will be used in next sections to solve the model of the umbilical.

5.2 Sliding elements properties

Since the sliding element move freely on the tether, the tensions of the cable try to balance on both sides of the element. Thus, some properties can be defined, as described in Definition 3, Theorem 2 and illustrated in Figure 6.

Definition 3. For the sliding element i , the following properties can be expressed:

(1) If $(\vec{F}_i \cdot \vec{T}_i < 0) \& (\vec{F}_i \cdot \vec{T}_{i+1} < 0)$ and $(l_i > l_{i1} > 0) \& (l_i > l_{i2} > 0)$, the sliding element i can move freely and is not in contact with a stop. Since the element slides without friction (Assumption A5), one has the property

$$T_i = T_{i+1}. \quad (23)$$

(2) If $l_{i1} = 0$ and $(\vec{F}_i \cdot \vec{T}_{i+1} < 0)$, the element i is in stable contact with S_i . In this configuration, it can be assimilate to a fixed element. Reciprocally for S_{i+1} if $l_{i2} = 0$ and $(\vec{F}_i \cdot \vec{T}_i < 0)$.

(3) If $\vec{F}_i \cdot \vec{T}_i > 0$, then the element i slides in the direction $\vec{B}_i \vec{S}_i$ and so cannot stay in contact with S_{i+1} . Reciprocally in direction $\vec{B}_i \vec{S}_{i+1}$ if $\vec{F}_i \cdot \vec{T}_{i+1} > 0$.

(4) If $(\vec{F}_i \cdot \vec{T}_i > 0) \& (\vec{F}_i \cdot \vec{T}_{i+1} > 0)$, the cable is loosen.

Theorem 2. For $i \in \{1, 2\}$, if the sliding element i is not in contact with a stop, then

$$\cos(\psi_i) \sin(\phi_i - \alpha_i) = \cos(\psi_{i+1}) \sin(\phi_i - \alpha_{i+1}). \quad (24)$$

The proof of Theorem 2 can be found in Appendix A.

Remark 3. The condition $(\vec{F}_i \cdot \vec{T}_i < 0)$ in Definition 3 can be replaced by the condition $(\vec{F}_i \cdot \vec{B}_i \vec{S}_i < 0)$. Same for $(\vec{F}_i \cdot \vec{T}_{i+1} < 0)$ with $(\vec{F}_i \cdot \vec{B}_i \vec{S}_{i+1} < 0)$.

These properties will be used the next sections to solve the system (7) and estimate if a configuration is possible (example: in area A12, the solution must respected the condition (1) for the element 1. Else, the solution is invalid).

Part III

Resolution model in the areas

6 Resolution model in areas A0, A1 and A2

Due to the strong non-linearity of system (7), the model is too complex to be solved analytically in areas A0, A1 and A2, named here areas A. Thus, theorems exposed in this section proposes a theoretical solution and a numerical resolution of system (7) and a value

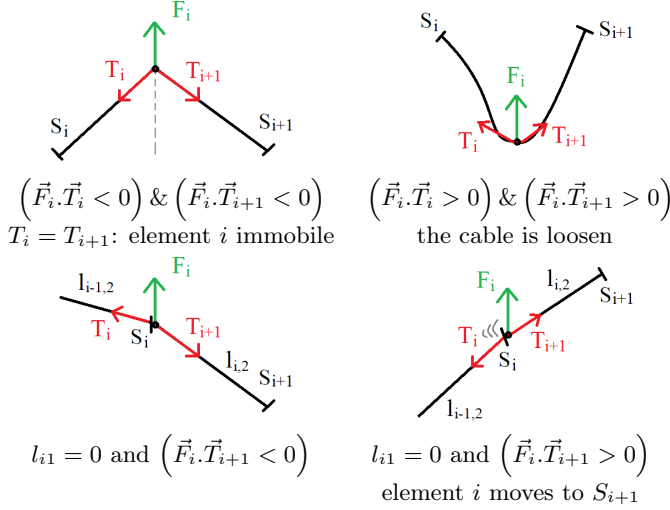


Figure 6: Illustration of the forces applied on the umbilical with a sliding element. The different cases correspond to the cases exposed in Definition 3.

of all its parameters in the different areas. To simplify the resolution, the solution is evaluated in several references obtained using a rotation of \mathcal{R} .

The transformation of the referential is described in the subsection 6.1. The analytical and numerical resolutions are described in Section 6.3, based on set of parameters for each areas described in theorems proposed in Sections 6.5.1, 6.5.2, 6.6.1, 6.6.2 and 6.7. The calculation of the lengths are summarized in subsection 6.4.

Remark the proposed theorems are valid only if the ROV is in their respective areas. The deduction of the area where the ROV is currently located is exposed in Section 4.5. Comments on the numerically resolution of the system will be made in Section 8.

6.1 Change of referential

To find a solution in the different areas, the system exposed in previous sections is easier to solve in others referential where some of these components become equal to zero.

In areas A0, A1 and A2, a simple around the vertical axis $O\vec{y}$ is used to obtain a referential \mathcal{R}_j where the component $\psi_{F1|\mathcal{R}_j}$ or $\psi_{F2|\mathcal{R}_j}$ is equal to zero. The Definition 4 defines the referential \mathcal{R}_0 , \mathcal{R}_1 and \mathcal{R}_2 respectively associated to the resolution of the system in the areas A0, A1 and A2.

Definition 4. The referential \mathcal{R}_j for $j \in \{0, 1, 2\}$ is defined as a rotation of the referential \mathcal{R} around the axis $O\vec{y}$ of an angle $\mu_j = \text{atan2}(F_{jz}|\mathcal{R}, F_{jx}|\mathcal{R})$ if $j \in \{1, 2\}$, $\mu_0 = \text{atan2}(F_{1z}|\mathcal{R}, F_{1x}|\mathcal{R})$ if $j = 0$. The

rotation matrix M_j is expressed as

$$M_j = \begin{bmatrix} \cos(\mu_j) & 0 & \sin(\mu_j) \\ 0 & 1 & 0 \\ -\sin(\mu_j) & 0 & \cos(\mu_j) \end{bmatrix}. \quad (25)$$

Let's define $L_{\alpha, \psi|\mathcal{R}_j} = \{\alpha_{1|\mathcal{R}_j}, \alpha_{2|\mathcal{R}_j}, \alpha_{3|\mathcal{R}_j}, \psi_{1|\mathcal{R}_j}, \psi_{2|\mathcal{R}_j}, \psi_{3|\mathcal{R}_j}\}$ the list of the angles in \mathcal{R}_j . Moreover,

(1) The lengths $l_{11}, l_{12}, l_{21}, l_{22}$ are unaffected by the change of referential;

(2) The coordinate of the ROV can be evaluated such that

$$\begin{bmatrix} x & y & z \end{bmatrix}_{\mathcal{R}_j}^T = M_j \begin{bmatrix} x & y & z \end{bmatrix}_{\mathcal{R}}^T \quad (26)$$

and since the rotation is only around axis $O\vec{y}$, the length l_0 and its orientation are unaffected, so the system (7) can be rewritten in \mathcal{R}_j using $L_{\alpha, \psi|\mathcal{R}_j}$;

(3) The force $\vec{F}_i|\mathcal{R}_j$ with $i \in \{1, 2\}$ and its parameters $\phi_i|\mathcal{R}_j$ and $\psi_{Fi|\mathcal{R}_j}$ defined in the referential \mathcal{R}_j with $j \in \{0, 1, 2\}$ can be expressed as

$$\vec{F}_i|\mathcal{R}_j = M_j \vec{F}_i|\mathcal{R} \quad (27)$$

$$\phi_i|\mathcal{R}_j = \text{atan2}(F_{iy}|\mathcal{R}_j, F_{ix}|\mathcal{R}_j) \quad (28)$$

$$\psi_{Fi|\mathcal{R}_j} = \frac{\pi}{2} - \text{acos}\left(\frac{F_{iz}|\mathcal{R}_j}{F_i}\right). \quad (29)$$

and the systems of FPS expressed in (12) can be expressed using $\phi_i|\mathcal{R}_j$, $\psi_{Fi|\mathcal{R}_j}$ and $L_{\alpha, \psi|\mathcal{R}_j}$. Thus, the Definition 3 and Theorem 2 are still valid replacing $\phi_i|\mathcal{R}$, $\psi_{Fi|\mathcal{R}}$ and $L_{\alpha, \psi|\mathcal{R}}$ by $\phi_i|\mathcal{R}_j$, $\psi_{Fi|\mathcal{R}_j}$ and $L_{\alpha, \psi|\mathcal{R}_j}$.

The return in the referential \mathcal{R} can be performed using for example Definition 5.

Definition 5. For $i \in \{1, 2, 3\}$, let's define the following vector in referential in a referential \mathcal{R}_j with a rotation matrix M_j :

$$\vec{v}_i|\mathcal{R}_j = \begin{bmatrix} \cos(\alpha_i) \cos(\psi_i) \\ \sin(\alpha_i) \cos(\psi_i) \\ \sin(\psi_i) \end{bmatrix}_{\mathcal{R}_j} \quad (30)$$

In referential \mathcal{R} , this vector becomes

$$\vec{v}_i|\mathcal{R} = \text{pinv}(M_j) \vec{v}_i|\mathcal{R}_j \quad (31)$$

an the angles in \mathcal{R} can be expressed as

$$\alpha_i|\mathcal{R} = \text{atan2}(v_{iy}|\mathcal{R}, v_{ix}|\mathcal{R}) \quad (32)$$

$$\psi_i|\mathcal{R} = \text{asin}(v_{iz}|\mathcal{R}) \quad (33)$$

where $\vec{v}_i = [v_{ix} \ v_{iy} \ v_{iz}]$.

6.2 Additional notations

For a referential \mathcal{R}_i , let's define the energy of the system (7)

$$\mathbf{E}_{\mathcal{R}_i} = \left\| \overrightarrow{AB_1}|\mathcal{R}_i + \overrightarrow{B_1B_2}|\mathcal{R}_i + \overrightarrow{B_2R}|\mathcal{R}_i - \begin{bmatrix} x & (y - l_0) & z \end{bmatrix}_{\mathcal{R}_i}^T \right\|^2 \quad (34)$$

This energy will be used to solve the umbilical model.

Since a solution will be required to be evaluated in each area, the different angles and lengths are grouped in a set of parameters expressed in Definition 6.

Definition 6. Let's define the area $A_m \in \{A0, A11, A12, A21, A22\}$ and note \mathcal{R}_m the referential associated to area A_m as defined in Definition 4. For $(x, y, z) \in \bar{\mathcal{C}}_{A,r}$, let $\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_k})$ be the set of parameters

$$\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m}) = \{ \alpha_{1|\mathcal{R}_k}, \alpha_{2|\mathcal{R}_m}, \alpha_{3|\mathcal{R}_k}, \\ \psi_{1|\mathcal{R}_m}, \psi_{2|\mathcal{R}_k}, \psi_{3|\mathcal{R}_m}, \\ l_{11}, l_{12}, l_{21}, l_{22} \} \quad (35)$$

evaluated for the angle $\alpha_{1|\mathcal{R}_k}$ and $(x, y, z)_{\mathcal{R}_m}$ using respectively

- (1) Theorem 4 for area A11,
- (2) Theorem 5 for area A12,
- (3) Theorem 6 for area A21,
- (4) Theorem 7 for area A22,
- (5) For area A0, Theorem 8 if $\psi_{F2|\mathcal{R}_0} = 0$, Theorem 9 else.

Let also note $\mathbf{E}_{\mathcal{R}_k}(\mathcal{L}_{\alpha,\psi,l}^{A_m})$ the evaluation of (34) using parameters inside $\mathcal{L}_{\alpha,\psi,l}^{A_m}$.

In order to lighten notations, $\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m})$ will be written $\mathcal{L}_{\alpha,\psi,l}^{A_m}$ in the next sections of the paper. Theorems exposed in Definition 6 provide sets of parameters $\mathcal{L}_{\alpha,\psi,l}^{A_m}$ which will be used in Theorem 3 to solve the system in areas A. These theorems will be described in the next sections.

6.3 Resolution of the system in areas A

Due to the strong non-linearity of system (7) and in opposite to areas D described in Section G, the model is too complex to be solved analytically in areas A. The theoretical solution can however be expressed, and a numerical resolution is proposed to find a solution to the system. These two aspects are described in Theorem 3.

Theorem 3. Consider an area $A_m \in \{A0, A11, A12, A21, A22\}$, its associated referential \mathcal{R}_m and the set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m})$ evaluated for an angle $\alpha_{1|\mathcal{R}_m}$ as defined in Definition 6. For the coordinate $(x, y, z) \in \bar{\mathcal{C}}_{A,r}$, a solution of system (7) exists in the area A_m if there exists an angle $\alpha_{1|\mathcal{R}_m} \in [-\pi, \pi]$ such as the associated $\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m})$ guarantee

$$E_{\min}^{A_m} \leq \eta \quad (36)$$

where

$$E_{\min}^{A_m} = \min_{\substack{\alpha_{1|\mathcal{R}_m} \in [-\pi, \pi] \\ \mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m}) \neq \emptyset}} \mathbf{E}_{\mathcal{R}_k} \left(\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m}) \right). \quad (37)$$

and with $\eta = 0$ in theoretical case, $\eta \geq 0$ a chosen parameter for numerical resolution.

If a solution exist, this solution $\mathcal{S}_{\alpha,\psi,l}^{A_m}$ can be expressed as

$$\mathcal{S}_{\alpha,\psi,l}^{A_m} = \left\{ \mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m}) \mid \mathbf{E}_{\mathcal{R}_m} \left(\mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m}) \right) = E_{\min}^{A_m}, \right. \\ \left. \mathcal{L}_{\alpha,\psi,l}^{A_m}(\alpha_{1|\mathcal{R}_m}) \neq \emptyset, \forall \alpha_{1|\mathcal{R}_m} \in [-\pi, \pi] \right\}. \quad (38)$$

In theory, the unique solution of the system (7) is $\alpha_{1|\mathcal{R}_m}$ solution of (34), which is equivalent to solve (36) with $\eta = 0$ in Theorem 3.

The proof of Theorem 3 is connected to the proofs of Theorems 4, 5, 6, 7, 8 and 9 which will be exposed in the next sections, each one for a specific areas.

6.4 Length calculation

The calculation of the length of cables l_{11} , l_{12} , l_{21} and l_{22} can be performed if the angles α_i and ψ_i for $i \in \{1, 2, 3\}$ are known. In function of the umbilical configuration, some lengths are already known, see Section 4.2. This section details the calculation of the remaining lengths in function of the areas. Note that the umbilical lengths are identical in all references, so they can be calculated with the angles of any referential. Thus, in order to lighten notations, the referential is not be specify in the notation of α_1 , α_2 , α_3 , ψ_1 , ψ_2 , ψ_3 in this section.

Let's define the following terms:

$$\begin{aligned} a_{11} &= \cos(\psi_1) \sin(\alpha_1) & b_{11} &= \cos(\psi_1) \cos(\alpha_1) \\ a_{12} &= \cos(\psi_2) \sin(\alpha_2) & b_{12} &= \cos(\psi_2) \cos(\alpha_2) \\ a_{22} &= \cos(\psi_3) \sin(\alpha_3) & b_{22} &= \cos(\psi_3) \cos(\alpha_3). \end{aligned}$$

6.4.1 lengths in Area A0

Following step described in Appendix B.3, one gets

$$l_{11} = \frac{x - (b_{12}(l_1 + l_2) + (b_{22} - b_{12})B_{11})}{(b_{11} - b_{12}) - (b_{22} - b_{12})A_{11}} \quad (39)$$

$$l_{22} = B_{11} - l_{11}A_{11} \quad (40)$$

with $A_{11} = \left(\frac{a_{11} - a_{12}}{a_{22} - a_{12}} \right)$ and $B_{11} = \frac{y - (l_0 + a_{12}(l_1 + l_2))}{a_{22} - a_{12}}$. Moreover, one has $l_{12} = l_1 - l_{11}$ and $l_{21} = l_2 - l_{22}$.

6.4.2 lengths in Area A1

In area A1, the length l_{21} and l_{22} are known: one has $l_{22} = 0$ if area A11 and $l_{22} = l_2$ if area A12, and $l_{21} =$

$l_2 - l_{22}$. Following step described in Appendix B.2, one gets

$$l_{11} = \frac{y - (l_0 + a_{22}l_{22} + a_{12}(l_{21} + l_1))}{a_{11} - a_{12}} \quad \text{if } a_{11} - a_{12} \neq 0 \quad (41)$$

$$l_{11} = \frac{x - (b_{22}l_{22} + b_{12}(l_{21} + l_1))}{b_{11} - b_{12}} \quad \text{if } a_{11} - a_{12} = 0. \quad (42)$$

and $l_{12} = l_1 - l_{11}$. Note that the Definition 2 excludes the case $a_{11} = a_{12}$ and $b_{11} = b_{12}$ simultaneously in area A2.

Remark 4. For the numerical computation, (42) can be used instead of (41) when $|a_{22} - a_{12}|$ is very close to zero.

6.4.3 lengths in Area A2

In area A2, the length l_{11} and l_{12} are known: one has $l_{11} = 0$ if area A21 and $l_{11} = l_1$ if area A22, and $l_{12} = l_1 - l_{11}$. Following step described in Appendix B.1, one gets

$$l_{22} = \frac{y - (l_0 + a_{11}l_{11} + a_{12}(l_{12} + l_2))}{a_{22} - a_{12}} \quad \text{if } a_{22} - a_{12} \neq 0 \quad (43)$$

$$l_{22} = \frac{x - (b_{11}l_{11} + b_{12}(l_{12} + l_2))}{b_{22} - b_{12}} \quad \text{if } a_{22} - a_{12} = 0. \quad (44)$$

and $l_{21} = l_2 - l_{22}$. Note that the Definition 2 excludes the case $a_{22} = a_{12}$ and $b_{22} = b_{12}$ simultaneously in area A2.

Remark 5. For the numerical computation, (44) can be used instead of (43) when $|a_{22} - a_{12}|$ is very close to zero.

6.5 Solution in Area A1

The Theorems 4 and 5 proposes a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A11}(\alpha_1|\mathcal{R}_1)$ and $\mathcal{L}_{\alpha,\psi,l}^{A12}(\alpha_1|\mathcal{R}_1)$, possibly solution in area A11. The solution is evaluated in referential \mathcal{R}_1 to simplify the resolution. The solution in \mathcal{R} can be found using a rotation of referential as described in Definition 4.

Based on properties made in Section 4.2 and Definition 3 for the sliding element 1, and the element 2 in contact with S or R , the Definition 7 provides conditions which must be satisfied to guarantee the physical and geometrical feasibility of solutions in areas A11 and A12.

Definition 7. From Definition 3, a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A11}$ or $\mathcal{L}_{\alpha,\psi,l}^{A12}$ is a possible solution of system (7) in respectively area A11 or area A12 iff the following condition C^{A1} is respected

$$C^{A1} = (C_1^{A1}) \& (C_2^{A1}) \& (C_3^{A1}) \& (C_4^{A1}) \quad (45)$$

with

$$C_1^{A1} = (0 \leq l_{11} \leq l_1) \& (0 \leq l_{12} \leq l_1) \quad (46)$$

$$C_2^{A1} = (T_{1|\mathcal{R}_1}^{S1} > 0) \& (T_{2|\mathcal{R}_1}^{S1} > 0) \& (T_{3|\mathcal{R}_1}^{S1} \geq 0) \quad (47)$$

$$C_3^{A1} = (\vec{F}_1 \cdot \overrightarrow{AB_1} > 0) \& (\vec{F}_1 \cdot \overrightarrow{B_1S} < 0) \quad (48)$$

$$C_4^{A1} = \begin{cases} \text{True} & \text{if element 2 is fixed or } l_2 = 0, \\ (\vec{F}_2 \cdot \overrightarrow{SB_2} > 0) & \text{else if A11} \\ (\vec{F}_2 \cdot \overrightarrow{B_2R} < 0) & \text{else if A12} \end{cases} \quad (49)$$

where $T_{1|\mathcal{R}_1}^{S1}$, $T_{2|\mathcal{R}_1}^{S1}$ and $T_{3|\mathcal{R}_1}^{S1}$ are evaluated using (17).

6.5.1 Solution in Area A11

The Theorem 4 proposes a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A11}(\alpha_1|\mathcal{R}_1)$, possibly solution in area A11.

Theorem 4. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area A11. Considering the assumptions A1 to A9. In referential \mathcal{R}_1 , for a defined angle $\alpha_1|\mathcal{R}_1 \in [-\pi, \pi]$, a possible solution $\mathcal{L}_{\alpha,\psi,l}^{A11}$ of the system (7) exists if $\mathcal{L}_{\alpha,\psi,l}^{A11} = L_{\alpha,\psi} \cup L_l$ respects the conditions C_1^{A1} defined in Definition 7 with $L_{\alpha,\psi} = \{\alpha_1|\mathcal{R}_1, \alpha_2|\mathcal{R}_1, \alpha_3|\mathcal{R}_1, \psi_1|\mathcal{R}_1, \psi_2|\mathcal{R}_1, \psi_3|\mathcal{R}_1\}$ such that

$$\alpha_2|\mathcal{R}_1 = -\alpha_1|\mathcal{R}_1 + 2\phi_1|\mathcal{R}_1 + \pi \quad (50)$$

$$\alpha_3|\mathcal{R}_1 = \alpha_2|\mathcal{R}_1 \quad (51)$$

$$\psi_1|\mathcal{R}_1 = \psi_2|\mathcal{R}_1 \quad (52)$$

$$\psi_2|\mathcal{R}_1 = a \sin\left(\frac{z_{\mathcal{R}_1}}{l_1 + l_2}\right) \quad (53)$$

$$\psi_3|\mathcal{R}_1 = \psi_2|\mathcal{R}_1. \quad (54)$$

and $L_l = \{l_{11}, l_{12}, l_{21}, l_{22}\}$ where $l_{21} = l_2$, $l_{22} = 0$ and (l_{11}, l_{12}) can be evaluated in \mathcal{R}_1 using parameters in $L_{\alpha,\psi}$ and Section 6.4.2.

The proof of Theorem 4 is described in Appendices C.2 and C.3.

6.5.2 Solution in Area A12

Consider also the following additional assumption:

A9) In area A1 and A0, consider that two elements 1 and 2, the elements are chosen such that in \mathcal{R}_1 , the following condition is always respected

$$\frac{2 \cos(\psi_{F2|\mathcal{R}_1}) F_{2|\mathcal{R}_1}}{\cos(\psi_{F1|\mathcal{R}_1}) F_{1|\mathcal{R}_1}} \cos(\phi_{2|\mathcal{R}_1} - \phi_{1|\mathcal{R}_1}) + 1 \neq 0. \quad (55)$$

Note that (55) is always respected if $F_2 < \frac{F_1}{2}$, see Appendix C.5.

The Theorem 5 proposes a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A12}(\alpha_{1|\mathcal{R}_1})$, possibly solution in area A12.

Theorem 5. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area A12. Considering the assumptions A1 to A9. In referential \mathcal{R}_1 , for a defined angle $\alpha_{1|\mathcal{R}_1} \in [-\pi, \pi]$, a possible solution $\mathcal{L}_{\alpha,\psi,l}^{A12}$ of the system (7) exists if $\mathcal{L}_{\alpha,\psi,l}^{A12} \neq \emptyset$ where $\mathcal{L}_{\alpha,\psi,l}^{A12}$ can be expressed as

$$\mathcal{L}_{\alpha,\psi,l}^{A12} = \left\{ S \mid \mathbf{E}_{\mathcal{R}_1}(S) = \min_{S \in S_{\alpha,\psi,l}} (\mathbf{E}_{\mathcal{R}_1}(S)), S \in S_{\alpha,\psi,l} \right\} \quad (56)$$

where $S_{\alpha,\psi,l}$ is the set of all possible solutions (possibly empty) defined as

$$S_{\alpha,\psi,l} = \left\{ \left\{ L_{\alpha,\psi,l}^{i,j} \mid C^{A1}(L_{\alpha,\psi,l}^{i,j}) = \text{True}, \right. \right. \\ \left. \left. j \in \{1, \dots, N_i\} / \forall i \in \{1, 2\} \right\} \right\} \quad (57)$$

with C^{A1} is defined in Definition 7, $L_{\alpha,\psi,l}^{i,j} = L_{\alpha,\psi}^{i,j} \cup L_l^{i,j}$ with

$$L_{\alpha,\psi}^{i,j} = L_{\alpha}^i \cup \left\{ \psi_{1|\mathcal{R}_1}^{i,j}, \psi_{2|\mathcal{R}_1}^{i,j}, \psi_{3|\mathcal{R}_1}^{i,j} \right\} \quad (58)$$

$$L_{\alpha}^i = \left\{ \alpha_{1|\mathcal{R}_1}, \alpha_{2|\mathcal{R}_1}, \alpha_{3|\mathcal{R}_1}^i \right\} \quad (59)$$

$$L_l^{i,j} = \left\{ l_{11}^{i,j}, l_{12}^{i,j}, l_{21}, l_{22} \right\} \quad (60)$$

and $0 \leq N_i \leq 4$ is the number of elements of $S_{\psi_2}(\alpha_{3|\mathcal{R}_1}^i)$, $S_{\psi_2}(\alpha_{3|\mathcal{R}_1}^i) = \left\{ s_{\psi_2}^j \mid j \in \{1, \dots, N_i\} \right\}$ can be evaluated following the Theorem 10 with the angles

$$\alpha_{2|\mathcal{R}_1} = 2\phi_{1|\mathcal{R}_1} - \alpha_{1|\mathcal{R}_1} + \pi \quad (61)$$

$$\alpha_{3|\mathcal{R}_1}^i = \alpha_{30|\mathcal{R}_1} + \phi_{1|\mathcal{R}_1} + (i-1)\pi \quad (62)$$

$$\psi_{1|\mathcal{R}_1}^{i,j} = \psi_{2|\mathcal{R}_1}^{i,j} \quad (63)$$

$$\psi_{2|\mathcal{R}_1}^{i,j} = s_{\psi_2}^j \quad (64)$$

$$\psi_{3|\mathcal{R}_1}^{i,j} = \text{atan} \left(\tan \left(\psi_{2|\mathcal{R}_1}^{i,j} \right) a_1 + b_1 \right) \quad (65)$$

with $a_1 = \frac{T_{2xy|\mathcal{R}_1}^{S1}}{T_{3xy|\mathcal{R}_1}^{S1}}$ and $b_1 = -\frac{F_2}{T_{3xy|\mathcal{R}_1}^{S1}} \sin(\psi_{F2|\mathcal{R}_1})$, $T_{2xy|\mathcal{R}_1}^{S1}$ and $T_{3xy|\mathcal{R}_1}^{S1}$ evaluated using (20) and (19),

$$\alpha_{30|\mathcal{R}_1} = \text{atan} \left(\frac{2 \frac{F_{2xy|\mathcal{R}_1}}{F_{1xy|\mathcal{R}_1}} \sin(\Delta\phi_{\mathcal{R}_1}) + \tan(\Gamma_1)}{\left(2 \frac{F_{2xy|\mathcal{R}_1}}{F_{1xy|\mathcal{R}_1}} \cos(\Delta\phi_{\mathcal{R}_1}) + 1 \right)} \right) \quad (66)$$

where $\Delta\phi_{\mathcal{R}_1} = \phi_{2|\mathcal{R}_1} - \phi_{1|\mathcal{R}_1}$ and $\Gamma_1 = \phi_{1|\mathcal{R}_1} - \alpha_{1|\mathcal{R}_1}$.

Then, the lengths can be expressed as $l_{21} = 0$, $l_{22} = l_2$ and $(l_{11}^{i,j}, l_{12}^{i,j})$ can be evaluated in \mathcal{R}_1 using parameters in $L_{\alpha,\psi}^{i,j}$ and Section 6.4.2.

The proof of Theorem 5 is described in Appendices C.2 and C.3. Remark also the angles $\psi_{i|\mathcal{R}_1}$ require the value of angles $\alpha_{i|\mathcal{R}_1}$, which must be evaluated first.

6.6 Solution in area A2

The Theorems 6 and 7 propose a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A21}(\alpha_{1|\mathcal{R}_2})$ and $\mathcal{L}_{\alpha,\psi,l}^{A22}(\alpha_{1|\mathcal{R}_2})$, possibly solution in areas A21 and A22. The solution is evaluated in referential \mathcal{R}_2 to simplify the resolution. The solution in \mathcal{R} can be found using a rotation of referential as described in Definition 4.

Based on properties made in Section 4.2 and Definition 3 for the sliding element 2, and the element 1 in contact with A or S, the Definition 8 provides conditions which must be satisfied to guarantee the physical and geometrical feasibility of solutions in areas A21 and A22.

Definition 8. From Definition 3, a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A21}$ or $\mathcal{L}_{\alpha,\psi,l}^{A22}$ is a possible solution of system (7) in respectively area A21 or area A22 iff the following condition C^{A2} is respected

$$C^{A2} = (C_1^{A2}) \& (C_2^{A2}) \& (C_3^{A2}) \& (C_4^{A2}) \quad (67)$$

with

$$C_1^{A2} = (0 \leq l_{21} \leq l_2) \& (0 \leq l_{22} \leq l_2) \quad (68)$$

$$C_2^{A2} = (T_{1|\mathcal{R}_1}^{S1} \geq 0) \& (T_{2|\mathcal{R}_1}^{S1} > 0) \& (T_{3|\mathcal{R}_1}^{S1} > 0) \quad (69)$$

$$C_3^{A2} = (\vec{F}_2 \cdot \overrightarrow{SB_2} > 0) \& (\vec{F}_2 \cdot \overrightarrow{B_2R} < 0) \quad (70)$$

$$C_4^{A2} = \begin{cases} \text{True} & \text{if element 1 is fixed,} \\ (\vec{F}_1 \cdot \overrightarrow{B_1S} < 0) & \text{else if A21} \\ (\vec{F}_1 \cdot \overrightarrow{AB_1} > 0) & \text{else if A22} \end{cases} \quad (71)$$

where $T_{1|\mathcal{R}_1}^{S1}$, $T_{2|\mathcal{R}_1}^{S1}$ and $T_{3|\mathcal{R}_1}^{S1}$ are evaluated using (17).

6.6.1 Solution in Area A21

The Theorem 6 proposes a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A21}(\alpha_{1|\mathcal{R}_2})$, possibly solution in area A21.

Theorem 6. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area A21. Considering the assumptions A1 to A8. In referential \mathcal{R}_2 , for a defined angle $\alpha_{1|\mathcal{R}_2} \in [-\pi, \pi]$, a possible solution $\mathcal{L}_{\alpha,\psi,l}^{A21}$ of the system (7) exists if $\mathcal{L}_{\alpha,\psi,l}^{A21} = L_{\alpha,\psi} \cup L_l$ respects the conditions C^{A2} defined in Definition 8 where $L_{\alpha,\psi} = \{\alpha_{1|\mathcal{R}_2}, \alpha_{2|\mathcal{R}_2}, \alpha_{3|\mathcal{R}_2}, \psi_{1|\mathcal{R}_2}, \psi_{2|\mathcal{R}_2}, \psi_{3|\mathcal{R}_2}\}$ with

$$\alpha_{2|\mathcal{R}_2} = \alpha_{1|\mathcal{R}_2} \quad (72)$$

$$\alpha_{3|\mathcal{R}_2} = -\alpha_{2|\mathcal{R}_2} + 2\phi_{1|\mathcal{R}_2} + \pi \quad (73)$$

$$\psi_{1|\mathcal{R}_2} = \psi_{2|\mathcal{R}_2} \quad (74)$$

$$\psi_{2|\mathcal{R}_2} = \text{asin} \left(\frac{z_{\mathcal{R}_2}}{l_1 + l_2} \right) \quad (75)$$

$$\psi_{3|\mathcal{R}_2} = \psi_{2|\mathcal{R}_2}. \quad (76)$$

and $L_l = \{l_{11}, l_{12}, l_{21}, l_{22}\}$ where $l_{11} = 0$ and $l_{12} = l_1$, l_{21} and l_{22} are evaluated in \mathcal{R}_2 using $L_{\alpha,\psi}$ and Section 6.4.3.

The proof of Theorem 6 is described in Appendices D.2 and D.3.

6.6.2 Solution in Area A22

The Theorem 7 proposes a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A22}(\alpha_1|\mathcal{R}_2)$, possibly solution in area A22.

Theorem 7. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area A22. Considering the assumptions A1 to A8. In referential \mathcal{R}_2 , for a defined angle $\alpha_1|\mathcal{R}_2 \in [-\pi, \pi]$, a possible solution $\mathcal{L}_{\alpha,\psi,l}^{A22}$ of the system (7) exists if $\mathcal{L}_{\alpha,\psi,l}^{A22} \neq \emptyset$ where $\mathcal{L}_{\alpha,\psi,l}^{A22}$ can be expressed as

$$\mathcal{L}_{\alpha,\psi,l}^{A22} = \left\{ S \mid \mathbf{E}_{\mathcal{R}_2}(S) = \min_{S \in S_{\alpha,\psi,l}} (\mathbf{E}_{\mathcal{R}_2}(S)), S \in S_{\alpha,\psi,l} \right\} \quad (77)$$

where $S_{\alpha,\psi,l}$ is the set of all possible solutions (possibly empty) defined as

$$S_{\alpha,\psi,l} = \left\{ \left\{ L_{\alpha,\psi,l}^{i,j} \mid C^{A22} \left(L_{\alpha,\psi,l}^{i,j} \right) = \text{True}, \right. \right. \\ \left. \left. j \in \{1, \dots, N_i\} \right\} / \forall i \in \{1, 2\} \right\} \quad (78)$$

with C^{A2} is defined in Definition 8, $L_{\alpha,\psi,l}^{i,j} = L_{\alpha,\psi}^{i,j} \cup L_l^{i,j}$

$$L_{\alpha,\psi}^{i,j} = L_{\alpha}^i \cup \left\{ \psi_{1|\mathcal{R}_2}^{i,j}, \psi_{2|\mathcal{R}_2}^{i,j}, \psi_{3|\mathcal{R}_2}^{i,j} \right\} \quad (79)$$

$$L_{\alpha}^i = \left\{ \alpha_{1|\mathcal{R}_2}^i, \alpha_{2|\mathcal{R}_2}^i, \alpha_{3|\mathcal{R}_2}^i \right\} \quad (80)$$

$$L_l^{i,j} = \left\{ l_{11}^{i,j}, l_{12}^{i,j}, l_{21}, l_{22} \right\} \quad (81)$$

and $0 \leq N_i \leq 4$ is the number of elements of $S_{\psi_2}(\alpha_{3|\mathcal{R}_2}^i)$, the set $S_{\psi_2}(\alpha_{3|\mathcal{R}_2}^i) = \left\{ s_{\psi_2}^j \mid j \in \{1, \dots, N_i\} \right\}$ can be evaluated following the Theorem 10 with the angles

$$\alpha_{2|\mathcal{R}_2}^i = 2\phi_{1|\mathcal{R}_2} - \alpha_{3|\mathcal{R}_2}^i + \pi. \quad (82)$$

$$\alpha_{3|\mathcal{R}_2}^i = \alpha_{30|\mathcal{R}_2} + \phi_{2|\mathcal{R}_2} + (i-1)\pi \quad (83)$$

$$\psi_{1|\mathcal{R}_2}^{i,j} = \text{atan} \left(\tan \left(\psi_{2|\mathcal{R}_2}^{i,j} \right) a_2 + b_2 \right) \quad (84)$$

$$\psi_{2|\mathcal{R}_2}^{i,j} = s_{\psi_2}^j \quad (85)$$

$$\psi_{3|\mathcal{R}_2}^{i,j} = \psi_{2|\mathcal{R}_2}^{i,j} \quad (86)$$

with $a_2 = \frac{T_{1xy|\mathcal{R}_2}^{S1}}{T_{2xy|\mathcal{R}_2}^{S1}}$ and $b_2 = -\frac{F_1}{T_{1xy|\mathcal{R}_2}^{S4}} \sin(\psi_{F1|\mathcal{R}_2})$, $T_{1xy|\mathcal{R}_2}^{S1}$ and $T_{2xy|\mathcal{R}_2}^{S1}$ evaluated using (18) and (20),

$$\alpha_{30|\mathcal{R}_2} = \text{atan} \left(2 \frac{F_{1xy|\mathcal{R}_2}}{F_{2xy|\mathcal{R}_2}} [\tan(\Gamma_2) \cos(\Delta\phi_{\mathcal{R}_2}) - \sin(\Delta\phi_{\mathcal{R}_2})] \right. \\ \left. + \tan(\Gamma_2) \right) \quad (87)$$

with $\Delta\phi_{\mathcal{R}_2} = \phi_{2|\mathcal{R}_2} - \phi_{1|\mathcal{R}_2}$ and $\Gamma_2 = \phi_{2|\mathcal{R}_2} - \alpha_{1|\mathcal{R}_2}$. Then, the lengths can be expressed as $l_{11} = l_1$, $l_{12} = 0$ and $(l_{21}^{i,j}, l_{22}^{i,j})$ can be evaluated in \mathcal{R}_2 using parameters in $L_{\alpha,\psi}^{i,j}$ and Section 6.4.2.

The proof of Theorem 7 is described in Appendices D.2 and D.4. Remark also the angles $\psi_{i|\mathcal{R}_2}$ require the value of angles $\alpha_{i|\mathcal{R}_2}$, which must be evaluated first.

6.7 Solution in Area A0

Based on properties made in Section 4.2 and Definition 3 for sliding elements 1 and 2, the Definition 9 provides conditions which must be satisfied to guarantee the physical and geometrical feasibility of solutions in area A0.

Definition 9. From Definition 3, a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A0}$ is a possible solution of system (7) in area A0 iff the following condition C^{A0} is respected

$$C^{A0} = (C_1^{A1}) \& (C_1^{A2}) \& (C_2^{A2}) \\ \& (C_1^{A0}) \& (C_2^{A0}) \quad (88)$$

with

$$C_1^{A0} = \left(\vec{F}_1 \cdot \overrightarrow{AB_1} > 0 \right) \& \left(\vec{F}_1 \cdot \overrightarrow{B_1S} < 0 \right) \quad (89)$$

$$C_2^{A0} = \left(\vec{F}_2 \cdot \overrightarrow{SB_2} > 0 \right) \& \left(\vec{F}_2 \cdot \overrightarrow{B_2R} < 0 \right) \quad (90)$$

where $T_{1|\mathcal{R}_0}^{S1}$, $T_{2|\mathcal{R}_0}^{S1}$ and $T_{3|\mathcal{R}_0}^{S1}$ are evaluated using (17).

The Theorems 8 and 9 propose a set of parameters $\mathcal{L}_{\alpha,\psi,l}^{A0}(\alpha_1|\mathcal{R}_0)$, possibly solution in area A0. Theorem 8 considers the particular case where $\psi_{F2|\mathcal{R}_0} = 0$, and Theorem 9 considers the general case. The solution is evaluated in referential \mathcal{R}_0 to simplify the resolution.

Theorem 8. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area A0 and not respecting the conditions of Theorem 1. Considering the assumptions A1 to A9. In referential \mathcal{R}_0 , in the particular case where the condition $\psi_{F2|\mathcal{R}_0} = 0$ is respected, one has for $i \in \{1, 2\}$

$$\alpha_{2|\mathcal{R}_0}^i = -\alpha_{1|\mathcal{R}_0}^i + 2\phi_{1|\mathcal{R}_0} + \pi \quad (91)$$

$$\alpha_{3|\mathcal{R}_0}^i = -\alpha_{2|\mathcal{R}_0}^i + 2\phi_{2|\mathcal{R}_0} + \pi \quad (92)$$

$$\psi_{1|\mathcal{R}_0}^i = a \sin \left(\frac{z_{\mathcal{R}_0}}{l_1 + l_2} \right) \quad (93)$$

$$\psi_{2|\mathcal{R}_0}^i = \psi_{1|\mathcal{R}_0}^i \quad (94)$$

$$\psi_{3|\mathcal{R}_0}^i = \psi_{1|\mathcal{R}_0}^i \quad (95)$$

so $L_{\alpha}^i = \left\{ \alpha_{1|\mathcal{R}_2}^i, \alpha_{2|\mathcal{R}_2}^i, \alpha_{3|\mathcal{R}_2}^i \right\}$ and $L_{\alpha,\psi}^i = L_{\alpha}^i \cup \left\{ \psi_{1|\mathcal{R}_0}^i, \psi_{2|\mathcal{R}_0}^i, \psi_{3|\mathcal{R}_0}^i \right\}$, the lengths $L_l^i = \left\{ l_{11}^i, l_{12}^i, l_{21}^i, l_{22}^i \right\}$ can be evaluated in \mathcal{R}_0 using parameters $L_{\alpha,\psi}^i$ and Section 6.4.2, and

(1) if $\sin(\Delta\phi_{\mathcal{R}_0}) = 0$, one has

$$\alpha_{1|\mathcal{R}_0}^i = (3 - 2i) a \cos \left(\frac{x_{\mathcal{R}_0}}{(l_1 + l_2) \cos(\psi_{1|\mathcal{R}_0}^i)} \right). \quad (96)$$

The solution $\mathcal{L}_{\alpha,\psi,l}^{A0}$ can be expressed with $L_{\alpha,\psi,l}^i = \{L_{\alpha,\psi}^i, L_l^i\}$ such as

$$\mathcal{L}_{\alpha,\psi,l}^{A0} = \begin{cases} L_{\alpha,\psi,l}^1 & \text{if } C_y^{A0} \left(L_{\alpha,\psi,l}^1 \right) = \text{True} \\ L_{\alpha,\psi,l}^2 & \text{if } C_y^{A0} \left(L_{\alpha,\psi,l}^2 \right) = \text{True} \\ \emptyset & \text{else} \end{cases} \quad (97)$$

where

$$C_y^{A0} (L_{\alpha,\psi,l}) = l_0 - y + l_{11} \sin(\alpha_{1|\mathcal{R}_0}) \cos(\psi_{1|\mathcal{R}_0}) + (l_{12} + l_{21}) \sin(\alpha_{2|\mathcal{R}_0}) \cos(\psi_{2|\mathcal{R}_0}) \quad (98)$$

(2) if $\sin(\Delta\phi_{\mathcal{R}_0}) \neq 0$, one has

$$\alpha_{1|\mathcal{R}_0}^i = \phi_{1|\mathcal{R}_0} + (1-i)\pi + \text{atan} \left(\frac{\frac{F_{2xy|\mathcal{R}_0}}{F_{1xy|\mathcal{R}_0}} + \cos(\Delta\phi_{\mathcal{R}_0})}{\sin(\Delta\phi_{\mathcal{R}_0})} \right) \quad (99)$$

with $\Delta\phi_{\mathcal{R}_0} = \phi_{2|\mathcal{R}_0} - \phi_{1|\mathcal{R}_0}$, $F_{1xy|\mathcal{R}_0}$ and $F_{2xy|\mathcal{R}_0}$ are evaluated using (15)-(16).

The solution $\mathcal{L}_{\alpha,\psi,l}^{A0}$ can be expressed with $L_{\alpha,\psi,l}^i = L_{\alpha,\psi}^i \cup L_l^i$ such as

$$\mathcal{L}_{\alpha,\psi,l}^{A0} = \begin{cases} L_{\alpha,\psi,l}^1 & \text{if } C^{A0} \left(L_{\alpha,\psi,l}^1 \right) = \text{True} \\ L_{\alpha,\psi,l}^2 & \text{if } C^{A0} \left(L_{\alpha,\psi,l}^2 \right) = \text{True} \\ \emptyset & \text{else} \end{cases} \quad (100)$$

where C^{A0} is expressed in Definition 9.

The proof of Theorem 8 is provided in Appendices F.2 and F.3.

Theorem 9. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area A0. Considering the assumptions A1 to A9. In referential \mathcal{R}_0 , for a defined angle $\alpha_{1|\mathcal{R}_0} \in [-\pi, \pi]$, a possible solution $\mathcal{L}_{\alpha,\psi,l}^{A0}$ of the system (7) exists if $\mathcal{L}_{\alpha,\psi,l}^{A0} \neq \emptyset$ where $\mathcal{L}_{\alpha,\psi,l}^{A0}$ can be expressed as

$$\mathcal{L}_{\alpha,\psi,l}^{A0} = \left\{ S \mid \mathbf{E}_{\mathcal{R}_0}(S) = \min_{S \in S_{\alpha,\psi,l}} (\mathbf{E}_{\mathcal{R}_0}(S)), S \in S_{\alpha,\psi,l} \right\} \quad (101)$$

where $S_{\alpha,\psi,l}$ is the set of all possible solutions (possibly empty) defined as

$$S_{\alpha,\psi,l} = \left\{ \left\{ L_{\alpha,\psi,l}^{i,j} \mid L_{\alpha,\psi,l}^{i,j} \in \mathbb{L}_{\alpha,\psi,l}^{i,j}, C^{A0} \left(L_{\alpha,\psi,l}^{i,j} \right) = \text{True}, C_6^{A0} \left(L_{\alpha,\psi,l}^{i,j} \right) = \text{True}, \forall j \in \{1, 2\} \right\} / \forall i \in \{1, 2\} \right\} \quad (102)$$

with C^{A0} is defined in Definition 9,

$$\mathbb{L}_{\alpha,\psi,l}^{i,j} = \begin{cases} \left\{ L_{\alpha,\psi,l}^{i,j,k} / \forall k \in \{1, 2\} \right\} & \text{if } \left(\left| \Psi_3^{i,j} \right| \leq 1 \right) \& (C_6^{A0} = 0) \\ \emptyset & \text{else} \end{cases} \quad (103)$$

where $L_{\alpha,\psi,l}^{i,j,k} = L_{\alpha,\psi}^{i,j,k} \cup L_l^{i,j,k}$ with

$$L_{\alpha,\psi}^{i,j,k} = L_{\alpha}^i \cup \left\{ \psi_{1|\mathcal{R}_0}^{i,j}, \psi_{2|\mathcal{R}_0}^{i,j}, \psi_{3|\mathcal{R}_0}^{i,j,k} \right\} \quad (104)$$

$$L_{\alpha}^i = \left\{ \alpha_{1|\mathcal{R}_0}, \alpha_{2|\mathcal{R}_0}, \alpha_{3|\mathcal{R}_0}^i \right\} \quad (105)$$

$$L_l^{i,j,k} = \left\{ l_{11}^{i,j,k}, l_{12}^{i,j,k}, l_{21}^{i,j,k}, l_{22}^{i,j,k} \right\} \quad (106)$$

$$\Psi_3^{i,j} = \cos \left(\psi_{2|\mathcal{R}_0}^j \right) \frac{\sin \left(\phi_{2|\mathcal{R}_0} - \alpha_{2|\mathcal{R}_0} \right)}{\sin \left(\phi_{2|\mathcal{R}_0} - \alpha_{3|\mathcal{R}_0}^i \right)} \quad (107)$$

$$C_6^{A0} = -\sin \left(\psi_{2|\mathcal{R}_0}^{i,j} \right) T_{2|\mathcal{R}_0}^{S1} + \sin \left(\psi_{3|\mathcal{R}_0}^{i,j,k} \right) T_{3|\mathcal{R}_0}^{S1} + F_{2xy|\mathcal{R}_0} \sin \left(\psi_{F2|\mathcal{R}_0} \right) \quad (108)$$

where $F_{2xy|\mathcal{R}_0}$, $T_{2|\mathcal{R}_0}^{S1}$ and $T_{3|\mathcal{R}_0}^{S1}$ are evaluated using (16), (17), and the angles with

$$\alpha_{2|\mathcal{R}_0} = 2\phi_{1|\mathcal{R}_0} - \alpha_{1|\mathcal{R}_0} + \pi \quad (109)$$

$$\alpha_{3|\mathcal{R}_0}^i = \alpha_{30|\mathcal{R}_0} + (1-i)\pi \quad (110)$$

$$\psi_{1|\mathcal{R}_0}^{i,j} = \psi_{2|\mathcal{R}_0}^{i,j} \quad (111)$$

$$\psi_{2|\mathcal{R}_0}^{i,j} = \psi_{20|\mathcal{R}_0} + (1-j)\pi \quad (112)$$

$$\psi_{3|\mathcal{R}_0}^{i,j,k} = (3-2k) \text{acos} \left(\Psi_3^{i,j} \right) \quad \text{if } \left| \Psi_3^{i,j} \right| \leq 1 \quad (113)$$

where

$$\alpha_{30|\mathcal{R}_0} = \phi_{2|\mathcal{R}_0} + \text{atan} \left(\frac{2 \frac{F_{2xy|\mathcal{R}_0}}{F_{1xy|\mathcal{R}_0}} \sin(\Delta\phi_{\mathcal{R}_0}) + \tan(\Gamma_0)}{2 \frac{F_{2xy|\mathcal{R}_0}}{F_{1xy|\mathcal{R}_0}} \cos(\Delta\phi_{\mathcal{R}_0}) + 1} \right) \quad (114)$$

$$\psi_{20|\mathcal{R}_0} = \text{atan} \left(\frac{(1-\Psi_2) \bar{T}_{2xy|\mathcal{R}_0}}{-2F_2 \sin(\psi_{F2|\mathcal{R}_0})} \right) \quad (115)$$

$$\Psi_2 = \frac{\sin(\phi_{2|\mathcal{R}_0} - \alpha_{2|\mathcal{R}_0})^2}{\sin(\phi_{2|\mathcal{R}_0} - \alpha_{30|\mathcal{R}_0})^2} + \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2|\mathcal{R}_0}) \right)^2 \quad (116)$$

with $\Delta\phi_{\mathcal{R}_0} = \phi_{2|\mathcal{R}_0} - \phi_{1|\mathcal{R}_0}$, $\Gamma_0 = \phi_{1|\mathcal{R}_0} - \alpha_{1|\mathcal{R}_0}$ and

$$\bar{T}_{2xy|\mathcal{R}_0} = \frac{F_{1xy|\mathcal{R}_0}}{2 \cos(\phi_{1|\mathcal{R}_0} - \alpha_{1|\mathcal{R}_0})}. \quad (117)$$

with $F_{1xy|\mathcal{R}_0}$ evaluated using (15). Then, the lengths $L_l^{i,j,k}$ can be evaluated in \mathcal{R}_0 using parameters in $L_{\alpha,\psi}^{i,j,k}$ and Section 6.4.1.

The proof of Theorem 9 is provided in Appendices F.2 and F.4. Same remarks that for the areas A1 and areas A2 can be made.

Note that the Theorem 8 can be solve analytically. In opposite, the solution of Theorem 9 will be found numerically.

6.8 Evaluation of ψ_2 in area A12 and A22

This section summarize the calculation of the angle ψ_2 in areas A12 and A22. The Definition 10 defines the parameters in function of the areas, and Theorem 10 describes the calculation.

Definition 10. Let define the expression of parameters A_i , B_i , C_i and D_i for $i \in \{1, 2\}$ in function of the areas and referential.

(1) In area A12, A_1 , B_1 , C_1 and D_1 can be expressed in \mathcal{R}_1 such that

$$\text{if } \sin(\alpha_{1|\mathcal{R}_1}) \neq \sin(\alpha_{2|\mathcal{R}_1}),$$

$$A_1 = \frac{\cos(\alpha_{1|\mathcal{R}_1}) - \cos(\alpha_{2|\mathcal{R}_1})}{\sin(\alpha_{1|\mathcal{R}_1}) - \sin(\alpha_{2|\mathcal{R}_1})} \quad (118)$$

$$B_1 = l_2 \cos(\alpha_{3|\mathcal{R}_1}) - \sin(\alpha_{3|\mathcal{R}_1}) l_2 A_1 \quad (119)$$

$$C_1 = l_1 \cos(\alpha_{2|\mathcal{R}_1}) - \sin(\alpha_{2|\mathcal{R}_1}) l_1 A_1 \quad (120)$$

$$D_1 = (y_{\mathcal{R}_1} - l_0) A_1 - x_{\mathcal{R}_1} \quad (121)$$

else,

$$A_1 = \frac{\sin(\alpha_{1|\mathcal{R}_1}) - \sin(\alpha_{2|\mathcal{R}_1})}{\cos(\alpha_{1|\mathcal{R}_1}) - \cos(\alpha_{2|\mathcal{R}_1})} \quad (122)$$

$$B_1 = l_2 \sin(\alpha_{3|\mathcal{R}_1}) - \cos(\alpha_{3|\mathcal{R}_1}) l_2 A_1 \quad (123)$$

$$C_1 = l_1 \sin(\alpha_{2|\mathcal{R}_1}) - \cos(\alpha_{2|\mathcal{R}_1}) l_1 A_1 \quad (124)$$

$$D_1 = l_0 + x_{\mathcal{R}_1} A_1 - y_{\mathcal{R}_1}. \quad (125)$$

(2) In area A22, A_2 , B_2 , C_2 and D_2 can be expressed in \mathcal{R}_2 such that

$$\text{if } \sin(\alpha_{2|\mathcal{R}_2}) \neq \sin(\alpha_{3|\mathcal{R}_2}),$$

$$A_2 = \frac{\cos(\alpha_{3|\mathcal{R}_2}) - \cos(\alpha_{2|\mathcal{R}_2})}{\sin(\alpha_{3|\mathcal{R}_2}) - \sin(\alpha_{2|\mathcal{R}_2})} \quad (126)$$

$$B_2 = l_1 \cos(\alpha_{1|\mathcal{R}_2}) - \sin(\alpha_{1|\mathcal{R}_2}) l_1 A_2 \quad (127)$$

$$C_2 = \cos(\alpha_{2|\mathcal{R}_2}) l_2 - \sin(\alpha_{2|\mathcal{R}_2}) l_2 A_2 \quad (128)$$

$$D_2 = (y_{\mathcal{R}_2} - l_0) A_2 - x_{\mathcal{R}_2} \quad (129)$$

else,

$$A_2 = \frac{\sin(\alpha_{3|\mathcal{R}_2}) - \sin(\alpha_{2|\mathcal{R}_2})}{\cos(\alpha_{3|\mathcal{R}_2}) - \cos(\alpha_{2|\mathcal{R}_2})} \quad (130)$$

$$B_2 = l_1 \sin(\alpha_{1|\mathcal{R}_2}) - \cos(\alpha_{1|\mathcal{R}_2}) l_1 A_2 \quad (131)$$

$$C_2 = l_2 \sin(\alpha_{2|\mathcal{R}_2}) - \cos(\alpha_{2|\mathcal{R}_2}) l_2 A_2 \quad (132)$$

$$D_2 = l_0 + x_{\mathcal{R}_2} A_2 - y_{\mathcal{R}_2}. \quad (133)$$

Theorem 10. Let's define the areas A_{i2} with $i \in \{1, 2\}$ where $A_{i2} \in \{A12, A22\}$. For areas A_{i2} , the parameter $\psi_{2|\mathcal{R}_i}$ can be expressed in the referential \mathcal{R}_i

such that

$$\psi_{2|\mathcal{R}_i} \in S_{\psi_2}$$

$$S_{\psi_2} = \{ \text{asin}(X_k) \mid X_k \in S_X, X_k \notin \mathbb{C}, |X_k| \leq 1 \} \quad (134)$$

where S_X is the set of solution of the polynomial

$$L_i + K_i X + J_i X^2 + I_i X^3 + H_i X^4 = 0 \quad (135)$$

where for A_i , B_i , C_i and D_i defined in Definition 10, one has

(1) If $B_i \neq 0$ and $C_i \neq 0$,

$$H_i = G_i^2 \quad (136)$$

$$I_i = 2F_i G_i \quad (137)$$

$$J_i = F_i^2 + (C_i D_i)^2 + 2E_i G_i \quad (138)$$

$$K_i = 2E_i F_i \quad (139)$$

$$L_i = E_i^2 - (C_i D_i)^2 \quad (140)$$

(2) If $B_i = 0$ and $C_i \neq 0$,

$$H_i = I_i = K_i = 0 \quad (141)$$

$$J_i = -C_i^2 \quad (142)$$

$$L_i = C_i^2 - D_i^2 \quad (143)$$

(3) If $B_i \neq 0$ and $C_i = 0$,

$$H_i = I_i = 0 \quad (144)$$

$$J_i = l_1^2 \quad (145)$$

$$K_i = -2l_1 z \quad (146)$$

$$L_i = z^2 - l_2^2 \left(1 - \frac{D_i^2}{B_i^2} \right). \quad (147)$$

The proves of Definition 10 and Theorem 10 are provided in Appendix E.

7 Resolution model in areas D1, D2 and D3

Using a transformation of the referential, the solution of system (7) in areas D (D1, D2 and D3) can be found analytically. For that, the referential is transformed using a translation and three rotations of \mathcal{R} , described in the subsection 7.2.

7.1 Force applied at stop S

Let's define $\vec{F}_{S|\mathcal{R}} = [F_{Sx|\mathcal{R}} \ F_{Sy|\mathcal{R}} \ F_{Sz|\mathcal{R}}]^T$ the force at the stop S expressed as

$$\vec{F}_{S|\mathcal{R}} = \vec{F}_{1|\mathcal{R}} \quad \text{in area D1} \quad (148)$$

$$\vec{F}_{S|\mathcal{R}} = \vec{F}_{\Sigma|\mathcal{R}} = \vec{F}_{1|\mathcal{R}} + \vec{F}_{2|\mathcal{R}} \quad \text{in area D2} \quad (149)$$

$$\vec{F}_{S|\mathcal{R}} = \vec{F}_{2|\mathcal{R}} \quad \text{in area D3.} \quad (150)$$

This force will be used in the resolution of the model.

7.2 Change of referential \mathcal{R}_D

In the areas D, all the elements are in contact with a stop. Thus, the umbilical between the anchor and the ROV becomes the triangle ASR with $AS = l_1$ and $SR = l_2$. Moreover, there is a referential where the triangle ASR and all forces applied on A , S and R can be expressed in a 2D plan ($O\bar{x}\bar{y}$). This referential, named \mathcal{R}_D , can be obtained by performing a translation and three rotation of the referential \mathcal{R} , as illustrated in Figure 7. The Definition 11 defines the referential \mathcal{R}_D associated to the resolution of the system in the areas D1, D2 and D3.

Definition 11. *The referential \mathcal{R}_D is defined as a translation and three rotations of the referential \mathcal{R} . The vector of coordinate $X_{\mathcal{R}} = [x \ y \ z]^T_{\mathcal{R}}$ and forces $\vec{F}_i|_{\mathcal{R}_D} \forall i \in \{1, 2, S\}$ can be expressed in \mathcal{R}_D such as*

$$X_{\mathcal{R}_D} = M_D (X_{\mathcal{R}} - T_{\mathcal{R}}) \quad (151)$$

$$\vec{F}_i|_{\mathcal{R}_D} = M_D \vec{F}_i|_{\mathcal{R}} \quad (152)$$

$$M_D = M_{D(3)} M_{D(2)} M_{D(1)} \quad (153)$$

where $T_{\mathcal{R}} = [0 \ l_0 \ 0]^T_{\mathcal{R}}$ and

$$M_{D(1)} = \begin{bmatrix} \cos(\mu_{D(1)}) & -\sin(\mu_{D(1)}) & 0 \\ \sin(\mu_{D(1)}) & \cos(\mu_{D(1)}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (154)$$

with $\mu_{D(1)} = -\text{atan2}(y_{\mathcal{R}_{D(1)}}, x_{\mathcal{R}_{D(1)}})$ and $X_{\mathcal{R}_{D(1)}} = X_{\mathcal{R}} + T_{\mathcal{R}}$,

$$M_{D(2)} = \begin{bmatrix} \cos(\mu_{D(2)}) & 0 & \sin(\mu_{D(2)}) \\ 0 & 1 & 0 \\ -\sin(\mu_{D(2)}) & 0 & \cos(\mu_{D(2)}) \end{bmatrix} \quad (155)$$

with $\mu_{D(2)} = \text{atan2}(z_{\mathcal{R}_{D(2)}}, x_{\mathcal{R}_{D(2)}})$ and $X_{\mathcal{R}_{D(2)}} = M_{D(1)} X_{\mathcal{R}_{D(1)}}$,

$$M_{D(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\mu_{D(3)}) & -\sin(\mu_{D(3)}) \\ 0 & \sin(\mu_{D(3)}) & \cos(\mu_{D(3)}) \end{bmatrix} \quad (156)$$

with $\mu_{D(3)} = -\text{atan2}(F_{S_z|_{\mathcal{R}_{D(3)}}}, F_{S_y|_{\mathcal{R}_{D(3)}}})$ and $\vec{F}_i|_{\mathcal{R}_{D(3)}} = M_{D(2)} M_{D(1)} \vec{F}_i|_{\mathcal{R}}$.

The parameters $\phi_i|_{\mathcal{R}_D}$ and $\psi_{F_i|_{\mathcal{R}_D}}$ defined in the referential \mathcal{R}_D can be expressed as

$$\phi_i|_{\mathcal{R}_D} = \text{atan2}(F_{iy|_{\mathcal{R}_D}}, F_{ix|_{\mathcal{R}_D}}) \quad (157)$$

$$\psi_{F_i|_{\mathcal{R}_D}} = \frac{\pi}{2} - \text{acos}\left(\frac{F_{iz|_{\mathcal{R}_D}}}{F_i}\right). \quad (158)$$

The return in the referential \mathcal{R} can be performed using for example Definition 12. Remind that the referential \mathcal{R} is not direct, and so its transformation. This explains the presence of the sign “-” in the angle of rotations.

Definition 12. *The return from referential \mathcal{R}_D to the referential \mathcal{R} can be obtained such as*

$$X_{\mathcal{R}} = \text{pinv}(M_D) X_{\mathcal{R}_D} + T_{\mathcal{R}} \quad (159)$$

$$\vec{v}_i|_{\mathcal{R}} = \text{pinv}(M_D) \vec{v}_i|_{\mathcal{R}_D} \quad (160)$$

and the angles in \mathcal{R} can be expressed as

$$\alpha_i|_{\mathcal{R}} = \text{atan2}(v_{iy|_{\mathcal{R}}}, v_{ix|_{\mathcal{R}}}) \quad (161)$$

$$\psi_i|_{\mathcal{R}} = \text{asin}(v_{iz|_{\mathcal{R}}}) \quad (162)$$

where the expression of \vec{v}_i has been described in Definition 5.

7.3 Solution in areas D

Due to the change of referential \mathcal{R}_D , the system (7) can be expressed in two dimensions. A first set of solutions can so be defined using only the geometrical properties of the system, described in Theorem 11. Since the solution found is not unique, the dynamical properties are added to find the current solution, as exposed in Theorem 11. Both theorems provides an analytical solution.

Theorem 11. *In the referential \mathcal{R}_D , let's define the following system*

$$x_{\mathcal{R}_D} = l_1 \cos(\gamma) + l_2 \cos(\beta) \quad (163)$$

$$y_{\mathcal{R}_D} = l_1 \sin(\gamma) + l_2 \sin(\beta). \quad (164)$$

where (γ, β) are unknown parameters. The solutions of (γ, β) associated to the system (163)-(164) are

(1) If $x_{\mathcal{R}_D} = 0$ and $y_{\mathcal{R}_D} = 0$, a solution is possible only if $l_1 = l_2$. Then, one has $L_{\gamma, \beta} = \{\bar{\gamma}, \bar{\beta}\}$ with $\bar{\gamma} = \text{atan2}(F_{S_y}, F_{S_x})$ and $\bar{\beta} = \pi + \bar{\gamma}$.

(2) Else if $x_{\mathcal{R}_D} \neq 0$, the solutions of the system are contained in the set $\mathcal{L}_{\gamma, \beta}$ expressed as

$$\mathcal{L}_{\gamma, \beta} = \{L_{\gamma, \beta}(\gamma) \mid \gamma \in L_{\gamma}\} \quad (165)$$

where

$$L_{\gamma, \beta}(\gamma) = \{\{\gamma, \beta\} \mid \beta \in L_{\beta}(\gamma), C_1^D(\gamma, \beta) = 0\} \quad (166)$$

with $C_1^D(\gamma, \beta)$ is the following condition

$$C_1^D(\gamma, \beta) = |y_{\mathcal{R}_D} - (l_1 \sin(\gamma) + l_2 \sin(\beta))| + |x_{\mathcal{R}_D} - (l_1 \cos(\gamma) + l_2 \cos(\beta))| \quad (167)$$

and L_{γ} and L_{β} are the following sets

$$L_{\gamma} = \left\{ \bar{\gamma} \mid \bar{\gamma} \in L_{S1}, \left| \frac{y_{\mathcal{R}_D}}{l_2} - \frac{l_1}{l_2} \sin(\bar{\gamma}) \right| \leq 1 \right\} \quad (168)$$

$$L_{S1} = \{\text{acos}(sX) \mid s \in \{-1, 1\}, X \in L_X\} \quad (169)$$

$$L_{\beta}(\bar{\gamma}) = \left\{ \bar{\beta}, \pi - \bar{\beta} \mid \bar{\beta} = \text{asin}\left(\frac{y_{\mathcal{R}_D}}{l_2} - \frac{l_1}{l_2} \sin(\bar{\gamma})\right) \right\} \quad (170)$$

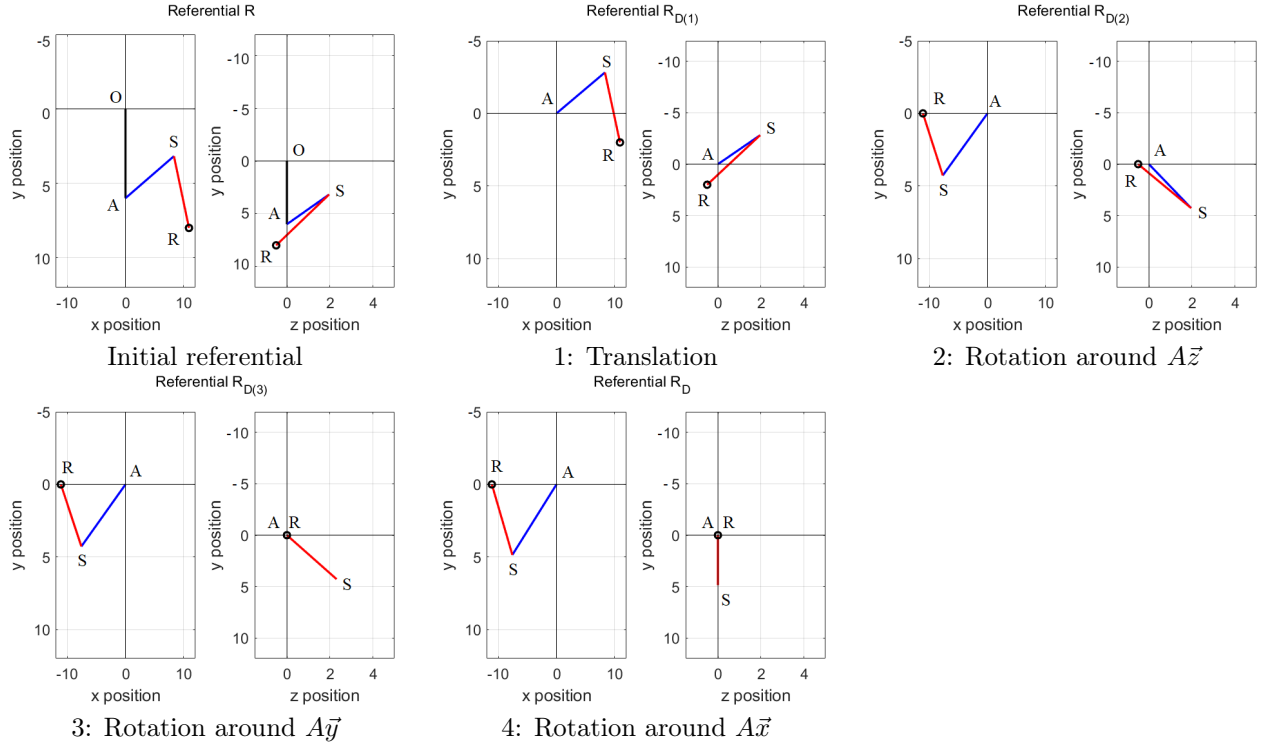


Figure 7: Transformations to pass from the referential \mathcal{R} to the referential \mathcal{R}_D . After four transformations, one observes the part ASR of the umbilical is contained inside the plan $(O\vec{x}\vec{y})_{\mathcal{R}_D}$.

with

$$\text{if } (y \neq 0) \& (\Delta \geq 0)$$

$$L_X = \begin{cases} \{X_{1D}, X_{2D}\} & \text{if } (|X_{1D}| \leq 1) \& (|X_{2D}| \leq 1) \\ \{X_{1D}\} & \text{if } (|X_{1D}| \leq 1) \& (|X_{2D}| > 1) \\ \{X_{2D}\} & \text{if } (|X_{1D}| > 1) \& (|X_{2D}| \leq 1) \\ \emptyset & \text{else} \end{cases} \quad (171)$$

else if $(y = 0)$

$$L_X = \begin{cases} \{X_{3D}\} & \text{if } (|X_{3D}| \leq 1) \\ \emptyset & \text{else} \end{cases} \quad (172)$$

where $\Delta = a_D^2 b_D^2 - (1 + b_D^2)(a_D^2 - 1)$, $a_D = \frac{x_{\mathcal{R}_D}^2 + y_{\mathcal{R}_D}^2 + l_1^2 - l_2^2}{2l_1 y_{\mathcal{R}_D}}$, $b_D = \frac{x_{\mathcal{R}_D}}{y_{\mathcal{R}_D}}$, $X_{3D} = \frac{x_{\mathcal{R}_D}^2 + l_1^2 - l_2^2}{2l_1 x_{\mathcal{R}_D}}$, and for $k \in \{1, 2\}$

$$X_{kD} = \frac{a_D b_D + s_k \sqrt{a_D^2 b_D^2 - (1 + b_D^2)(a_D^2 - 1)}}{(1 + b_D^2)} \quad (173)$$

with $\{s_1, s_2\} = \{-1, 1\}$.

The proof of Theorem 11 is provided in Appendix G.2. Note that $L_{\gamma, \beta}$ correspond to the solutions geometrically possible. If not solution in L_γ and L_β respected condition $C^D(\gamma, \beta)$, its means the configuration is not possible.

From the solutions provide by Theorem 11, Theorem 12 adds the constraints link to the PFS.

Theorem 12. Consider the system (7) with $(x, y, z) \in \bar{C}_{A,r}$ and is inside the area $D1, D2$ or $D3$. Considering the assumptions $A1$ to $A8$. In referential \mathcal{R}_D , let's define $L_{\gamma, \beta}$ the set of solution defined in Theorem 11. Then

(1) In area $D1$, $\alpha_{1|\mathcal{R}_D} = \gamma$, $\alpha_{2|\mathcal{R}_D} = \beta$, $\alpha_{3|\mathcal{R}_D} = 0$, $l_{11} = l_1$, $l_{12} = 0$, $l_{21} = l_2$, $l_{22} = 0$, where $(\gamma, \beta) \in L_{\gamma, \beta}$ and respect the conditions $\mathbf{E}_{\mathcal{R}} = 0$ and

$$\left(T_{1|\mathcal{R}_D}^{S1} > 0\right) \& \left(T_{2|\mathcal{R}_D}^{S1} > 0\right) \quad (174)$$

$$\left(\vec{F}_1 \cdot \vec{AS} > 0\right) \quad \text{if element 1 is not fixed} \quad (175)$$

$$\left(\vec{F}_2 \cdot \vec{SR} > 0\right) \quad \text{if element 2 is not fixed} \quad (176)$$

(2) In area $D2$, $\alpha_{1|\mathcal{R}_D} = \gamma$, $\alpha_{2|\mathcal{R}_D} = 0$, $\alpha_{3|\mathcal{R}_D} = \beta$, $l_{11} = l_1$, $l_{12} = 0$, $l_{21} = 0$, $l_{22} = l_2$, where $(\gamma, \beta) \in L_{\gamma, \beta}$ and respect the conditions $\mathbf{E}_{\mathcal{R}} = 0$ and

$$\left(T_{1|\mathcal{R}_D}^{S1} > 0\right) \& \left(T_{3|\mathcal{R}_D}^{S1} > 0\right) \quad (177)$$

$$\left(\vec{F}_1 \cdot \vec{AS} > 0\right) \quad \text{if element 1 is not fixed} \quad (178)$$

$$\left(\vec{F}_2 \cdot \vec{SR} < 0\right) \quad \text{if element 2 is not fixed} \quad (179)$$

(3) In area $D3$, $\alpha_{1|\mathcal{R}_D} = 0$, $\alpha_{2|\mathcal{R}_D} = \gamma$, $\alpha_{3|\mathcal{R}_D} = \beta$, $l_{11} = 0$, $l_{12} = l_1$, $l_{21} = 0$, $l_{22} = l_2$, where $(\gamma, \beta) \in L_{\gamma, \beta}$

and respect the conditions $E_{\mathcal{R}} = 0$ and

$$\left(T_{2|\mathcal{R}_D}^{S_2} > 0\right) \& \left(T_{3|\mathcal{R}_D}^{S_2} > 0\right) \quad (180)$$

$$\left(\vec{F}_1 \cdot \vec{A}\vec{S} < 0\right) \quad \text{if element 1 is not fixed} \quad (181)$$

$$\left(\vec{F}_2 \cdot \vec{S}\vec{R} < 0\right) \quad \text{if element 2 is not fixed} \quad (182)$$

with $\vec{A}\vec{S} = \vec{A}\vec{B}_1 + \vec{B}_1\vec{S}$ and $\vec{S}\vec{R} = \vec{S}\vec{B}_2 + \vec{B}_2\vec{R}$.

The proof of Theorem 12 is provided in Appendix G.3.

8 Numerical resolution

To find a solution to Theorem 3 of Section 6, we propose to solve it numerically using a global optimization. Note that a global optimization is required because the system has a lot of local minimum due to the presence of cos, sin and tan functions. In this section, some comments are made to reduce the computation time and outline some numerical problems which can occur.

- 1) In a numerical resolution, all conditions of form $X = 0$ must be replaced by $|X| < \epsilon$ where $1 \gg \epsilon > 0$. In our simulations, on takes $\epsilon = 0.01$.
- 2) The Theorem 3 induces to test all $\alpha_1 \in \Lambda$ with $\Lambda = [-\pi, \pi]$, which can induce a long computation time. However, it can be observed that
 - A global research on $\Lambda = [-\pi, \pi]$ is required only for the first estimation of the system and a first value of α_1 . Then, if the new coordinates, forces and cable length are close to the parameters of the previous, Λ can be reduced to an interval $\Lambda_{k+1} = \alpha_1(k) + \omega$ where $\omega = [-\frac{\pi}{12}, \frac{\pi}{12}]$ or smaller, close to the value $\alpha_1(k)$ found at the previous iteration k . Note that in the case where the solution was found in areas A21 or D3 where $l_{11} = 0$ and so α_1 doesn't exist, the optimization must be done for parameter α_2 , because the value of α_1 can change abruptly when the area changes.
 - Since the calculation for each α_1 are independent, these ones can be made in parallel.
- 3) In the numerical resolution, a solution of Theorem 3 is valid if its energy $E_{\min}^{A_m}$ is bellow a defined threshold $\eta \geq 0$. The choice of η can be tricky because a too high threshold can admit a solution in a zone where there should be none, and a too small threshold can exclude solution where numerical rounding would be too penalizing. η must be found empirically. In our simulation, we choose $\eta = 0.1$.
- 4) The method is very sensible to small the variation of α_1 . A final accuracy of $\delta\alpha_1 = 0.0001^\circ$ is recommend for simulation. In our simulations, three optimizations' step were performed: a first loop with an accuracy of $\delta\alpha_1 = 0.01$ is performed where all local minimal are kept regardless of η , then a second research around these local minimal with a step of $\delta\alpha_1 = 0.001$, then a last step where the minimal energy $E_{\min}^{A_m}$ and the threshold η are considered.

Part IV

Simulation, results, conclusion

9 Simulations and results

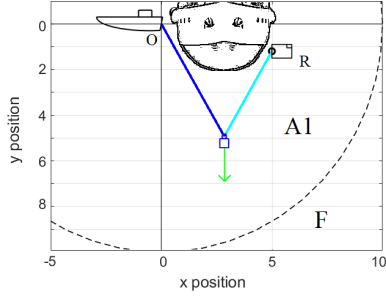
9.1 Examples of system

Simulations have been performed with Matlab R2019b, using the tool "parfor" with 4 workers (maximum of the computer used) to obtain a low parallelization of the operations.

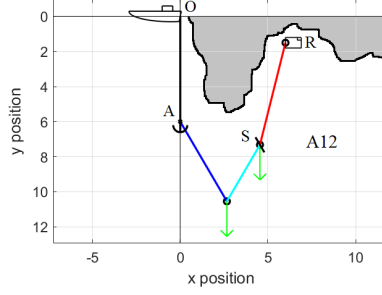
In addition to the example in Figures 3 and 4, nine examples of umbilical shape are illustrated in Figure 8 to show the possibilities and performances of the model.

Sub-Figures 8(1) and (2) propose an exploration close to the surface to avoid hull or underwater cavern ceiling. Sub-Figures 8(4) and (5) propose an exploration close to the seabed to avoid and dive between obstacles such as rocks. Sub-Figure 8(6) adds presence of current, with a more realistic representation of sub-Figure 8(4) where the anchor can move (here the "anchor" is a heavy ballast in the model). This sub-figure illustrates also the main advantage of using sliding element instead of fixed: the area marked C* is inaccessible with a fixed ballast, but not with a sliding ballast. Sub-Figures 8(3) and (9) use a motorized element to create a vertical force, allowing to explore a cavern in the wall. Finally, sub-Figures 8(7) and (8) show illustration of area A0 with theoretical forces, not necessarily useful or realistic in practice, but showing the performance of the proposed model.

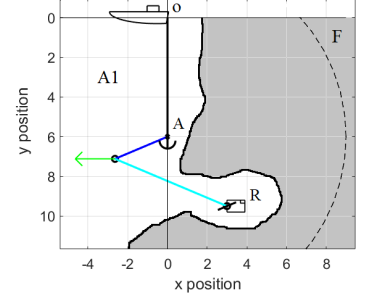
The configurations expressed in sub-Figures 8(1), (2), (4), (5) and (6) have been tested experimentally in pool, see Figure 9. Sliding elements were performed using pulley with a neutral buoyancy. Results are described in Section 9.2.1.



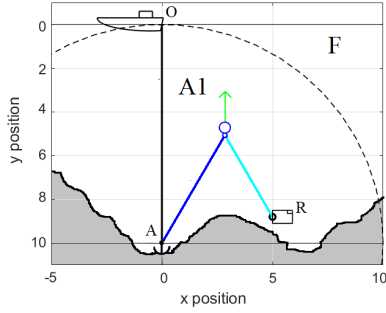
(1) 1 sliding ballast
 $\vec{F}_1 = [0, 1, 0]^T$
 $l_0 = 0, l_1 = 10$



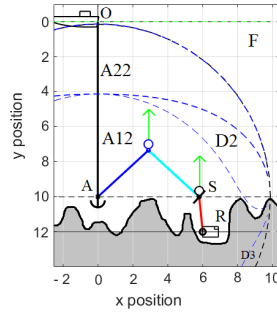
(2) 2 sliding ballasts
 $\vec{F}_1 = [0, 1, 0]^T, \vec{F}_2 = \frac{1}{2}\vec{F}_1,$
 $l_0 = 6, l_1 = l_2 = 10$



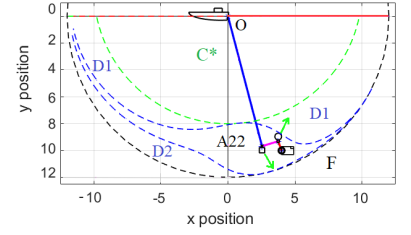
(3) 1 sliding vertical thruster
 $\vec{F}_1 = [-1, 0, 0]^T$
 $l_0 = 6, l_1 = 9$



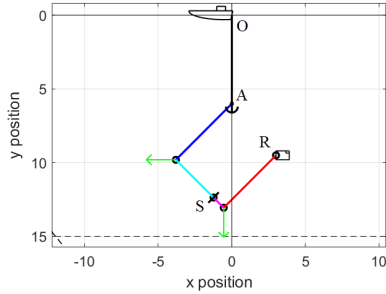
(4) 1 sliding buoy
 $\vec{F}_1 = [0, -1, 0]^T$
 $l_0 = l_1 = 10$



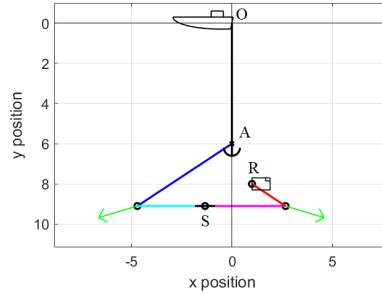
(5) 2 sliding buoys
 $\vec{F}_1 = [0, -1, 0]^T, \vec{F}_2 = 2\vec{F}_1$
 $l_0 = 10, l_1 = 6, l_2 = 4$



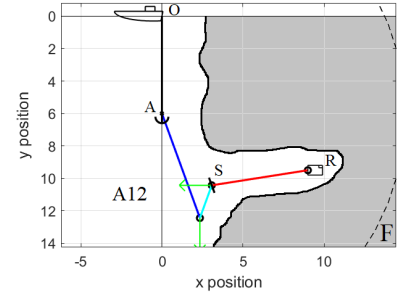
(6) 1 fixed or sliding ballast, 1 sliding buoy, both with horizontal current
 $\vec{F}_1 = [0.5, 2, 0]^T, \vec{F}_2 = [0.5, -1, 0]^T$
 $l_0 = 0, l_1 = 10, l_2 = 2$
 $C^* = F$ if fixed ballast,
 $C^* = A11$ if sliding ballast



(7) Theoretical configuration to illustrate A0 with opposing forces
 $\vec{F}_1 = [-1, 0, 0]^T, \vec{F}_2 = [0, 1, 0]^T$
 $l_0 = 6, l_1 = 9, l_2 = 6$

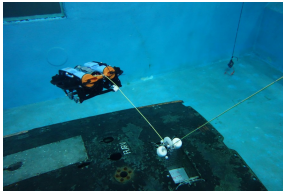


(8) Theoretical configuration to illustrate A0 with opposing forces
 $\vec{F}_1 = [-1, 0.3, 0]^T, \vec{F}_2 = [1, 0.3, 0]^T$
 $l_0 = 6, l_1 = 9, l_2 = 6$

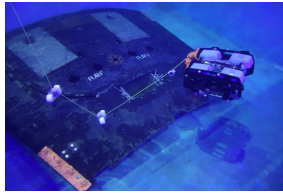


(9) 1 sliding ballast, 1 fixed vertical thruster
 $\vec{F}_1 = [0, 1, 0]^T, \vec{F}_2 = [-3, 0, 0]^T$
 $l_0 = 6, l_1 = 9, l_2 = 6$

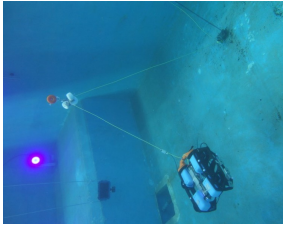
Figure 8: Example of configurations possible obtained with the proposed model. Areas possible/visible are noted in the illustrations. Grey areas: theoretical obstacles to illustrate the main interest of the illustrated configuration. In subfigure (1)-(3), only area A1 exist (with A11 = A12 because only one element). Note that it's the ratio between two lengths or distances that matters, not the unit.



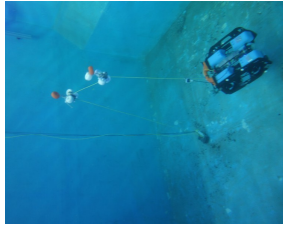
Experiment of (1)
1 sliding ballast



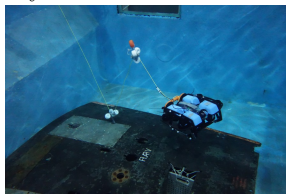
Experiment of (2)
2 sliding ballasts



Experiment of (4)
1 sliding buoy



Experiment of (5)
2 sliding buoys



Experiment of (6) : 1 sliding ballast and 1 sliding buoy

Figure 9: Configurations exposed in Figure 8 tested in pools. A robot BlueROV2 is used for the experiments.

9.2 Results and comments

9.2.1 Experiments results and comparison with previous models

Experimental results of configurations illustrated in Figure 8(1), (4) and (6) are described in [27, 28]. Experimental results of configurations Figure 8(2) and (5) are described in [26]. In these previous works, each configuration had its own model, and their accuracy with reality has already been demonstrated. Tests have been performed in a pool of size $3\text{ m} \times 4\text{ m}$, with a depth of 3 m (not the pool where the photos were taken).

Our new general model allows to find exactly the same geometrical results as the previous work. The main difference is that the computation time is lower for the previous models, as the assumptions made in these works allow simplifications that enable analytical resolution in all cases. It should be noted, however, that most of these models cannot consider the presence of horizontal currents or three-dimensional cases, and testing a new configuration would mean redoing a complete study of the system. Conversely, the new general model enables us to calculate the cable shape for all configurations using one or two elements.

9.2.2 A third element

A third element between R and the ROV can be considered if, like the anchor, its position relative to the ROV remains fixed: its coordinates become those of the ROV in the calculations. This can be used to offset the cable attachment point, for example with a buoy or ballast, for better diving between rocks.

9.2.3 Observations made

A hierarchy has been observed between the different areas. First, when area A0 exists, this configuration always has priority over all other areas. Then, the areas A (A11, A12, A21 and A22) have priority over the areas D (D1, D2 and D3). We have tested this hierarchy by prioritizing one solution over another, and using the potential energy in case of conflict between two solutions inside areas A or inside areas D. Same results have been observed that using only the potential energy. Couple with the graph theory, this hierarchy could be an effective method to solve computation time by testing first area A0, then areas A if no solution is found and finally areas D. Unfortunately, no theoretical study has been found to prove this hierarchy, so there's no guarantee of the reliability of this method.

9.2.4 Computation time

The computation time is proportional to the number of areas to evaluate, thus we consider here configurations with two sliding elements, *i.e.* with the more number of areas. Between each iteration, a displacement of 0.5 m for a cable of $L = 15\text{ m}$ is performed. The system is solved using resolution described in Part III.

Past the first estimation of the system (see the numerical implementation described in Section 8), the computation time at each iteration is between 0.8 s and 1.2 s with Matlab R2019b and its tool “parfor” with 4 workers. This computation time is too long for a real-time implementation. A possible way to further speed up the real-time performance of our implementation would be transpiling the Matlab code to C++ or Python. An implementation on GPU to obtain a better calculation parallelization would strongly improve computation time. Other methods of resolution using for example finite element (with only two elements) could also be proposed.

9.3 Limits of the method and overview

The model provides a good vision of the system in offline, allowing to build a umbilical equipment and observe its behavior. The model has however some limitations.

First, as already discuss in Section 9.2.4, the computation time is actually the major weakness of our

method. Suggestions for improvement have been made in Section 9.2.4.

The method is also limited to two elements: except case exposed in Section 9.2.2, each element added would multiple the number of possible areas by three, making the solution too complex to found and/or none efficient in calculation time.

More elements could be added with an more efficient computation time by

- Use of a less general model, for example in 2D and/or with elements having a specific orientation, or using more fixed elements, like in [27, 26].
- Changing the method of resolution: solving the problem with theorems exposed in Part III may be heavy for some areas. By retaining assumption and properties proven in Part II, another type of resolution such as finite-element might be more efficient.

It can however be remark that in practice, install more than two elements can be useless or more unstable.

Another practical problem is knowing the forces applied to the elements. While the forces of weight, buoyancy or motorization are simple to know, it is more complex to know the currents of other external forces in practice. However, this problem is common to all umbilical models.

It can also be noted that the system is solved with quasi-static equilibrium hypothesis. Solutions to respect the hypotheses of quasi-static have been proposed in Section 5 and a resolution of the system in dynamic will be the subject of futures studied.

10 Conclusion

This paper presents a novel approach to passively manage the umbilical of an ROV by adding additional elements such as ballasts, buoys, or an oriented thruster. The purpose is to stretch the umbilical in a controlled manner, ensuring a predictable shape and preventing entanglement with obstacles or itself. These elements can either be fixed or move freely along the umbilical, with stops in place to restrict constant contact. Unlike previous studies, this proposal introduces a general model capable of estimating the three-dimensional shape of the umbilical regardless of the applied force's orientation, allowing to consider the presence of currents, passive or motorized elements on the tether, and the inclusion or exclusion of a TMS. The geometry and forces applied upon the umbilical are calculated using a quasi-static assumption.

The model provides a good vision of the system, allowing to build a umbilical equipment and observe its behavior. Example of different set up, adapted to specific missions, are proposed and tested in pool. The

limits of the method have been discussed, specifically the computation time which can be improved. Suggestions for improvement have been made.

These improvements will be the subject of future works. Future works will also study the dynamics instead of a quasi-static equilibrium, with also variations of the sea current and uncertainties on parameters. Finally, measurements at sea during a true mission will be performed. the curvature of the cable will also be taken into account.

Acknowledgment

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A Study of PFS

Consider a referential where $|\psi_1|$, $|\psi_2|$, $|\psi_3|$, $|\psi_{F1}|$ and $|\psi_{F2}|$ are different of $\frac{\pi}{2}$, so one has $T_{1xy} > 0$, $T_{2xy} > 0$, $T_{3xy} > 0$, $F_{1xy} > 0$, $F_{2xy} > 0$ with $T_{ixy} = \cos(\psi_i) T_i$ for $i \in \{1, 2, 3\}$ and $F_{jxy} = \cos(\psi_{Fj}) F_j$ for $j \in \{1, 2\}$.

Remark. A rotation of the referential \mathcal{R} can be required to fulfill these conditions.

In the plan Oxy, (12) can be rewritten for $i \in \{1, 2\}$ as

$$\begin{cases} -\cos(\alpha_1) T_{1xy} + \cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1) = 0 \\ -\sin(\alpha_1) T_{1xy} + \sin(\alpha_2) T_{2xy} + F_{1xy} \sin(\phi_1) = 0 \end{cases} \quad (183)$$

$$\begin{cases} -\cos(\alpha_2) T_{2xy} + \cos(\alpha_3) T_{3xy} + F_{2xy} \cos(\phi_2) = 0 \\ -\sin(\alpha_2) T_{2xy} + \sin(\alpha_3) T_{3xy} + F_{2xy} \sin(\phi_2) = 0 \end{cases} \quad (184)$$

Let's (183) and (184) in the following subsections.

A.1 Study of (183)

Consider first (183). Suppose in a first time that $\alpha_1 \notin \{\frac{\pi}{2}k | k \in \mathbb{Z}\}$, thus $\cos(\alpha_1) \neq 0$. The case $\cos(\alpha_1) = 0$ is treated in Appendix A.1.1. From (183), one gets

$$\begin{cases} T_{1xy} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1)) \\ 0 = [-\tan(\alpha_1) \cos(\alpha_2) + \sin(\alpha_2)] T_{2xy} \\ \quad + [-\tan(\alpha_1) \cos(\phi_1) + \sin(\phi_1)] F_{1xy} \end{cases} \quad (185)$$

If $\alpha_1 = \phi_1 + k\pi$ with $k \in \mathbb{Z}$, one has $[-\tan(\alpha_1) \cos(\phi_1) + \sin(\phi_1)] = 0$. Since $T_{2xy} > 0$ and $F_{1xy} > 0$, one must have $\alpha_1 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$ to equilibrate the second line of (185). These specific cases correspond to case where the cable is perfectly straight ($\alpha_1 = \alpha_2$) or perfectly folded ($\alpha_1 = \alpha_2 + \pi$). Case $\alpha_1 = \alpha_2$ is excluded in most configuration by Definition 2. Case $\alpha_1 = \alpha_2 + \pi$ induce $\cos(\alpha_1) = -\cos(\alpha_2)$ and $\sin(\alpha_1) = -\sin(\alpha_2)$. From (183), it leads to $\phi_1 = \alpha_1$. Then, it can be deduced that $T_{1xy} = T_{2xy} = \frac{F_{1xy}}{2}$.

Consider now we are not in these particulars case. Then, one has

$$\begin{cases} T_{1xy} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1)) \\ T_{2xy} = -\frac{[-\tan(\alpha_1) \cos(\phi_1) + \sin(\phi_1)]}{[-\tan(\alpha_1) \cos(\alpha_2) + \sin(\alpha_2)]} F_{1xy} \end{cases} \quad (186)$$

One gets

$$T_{1xy} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1)) \quad (187)$$

and T_{2xy} can be rewritten such as

$$\begin{aligned} T_{2xy} &= -\frac{\left[-\frac{\sin(\alpha_1)}{\cos(\alpha_1)} \cos(\phi_1) + \sin(\phi_1)\right]}{\left[-\frac{\sin(\alpha_1)}{\cos(\alpha_1)} \cos(\alpha_2) + \sin(\alpha_2)\right]} F_{1xy} \\ &= -\frac{[-\sin(\alpha_1) \cos(\phi_1) + \cos(\alpha_1) \sin(\phi_1)]}{[-\sin(\alpha_1) \cos(\alpha_2) + \cos(\alpha_1) \sin(\alpha_2)]} F_{1xy} \\ &= -\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy}. \end{aligned} \quad (188)$$

From (186) and (188), one gets an expression of T_{1xy} and T_{2xy} .

Since $T_{1xy} > 0$ and $T_{2xy} > 0$, let's now study the ratio $\frac{T_{1xy}}{T_{2xy}}$:

$$\begin{aligned} \frac{T_{1xy}}{T_{2xy}} &= \frac{\frac{1}{\cos(\alpha_1)} \left(-\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy} \cos(\alpha_2) + F_{1xy} \cos(\phi_1)\right)}{-\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy}} \\ &= \frac{-1}{\cos(\alpha_1)} \left(-\cos(\alpha_2) + \frac{\sin(\alpha_2 - \alpha_1)}{\sin(\phi_1 - \alpha_1)} \cos(\phi_1)\right) \\ &= -\left(\frac{-\cos(\alpha_2) \sin(\phi_1 - \alpha_1) + \sin(\alpha_2 - \alpha_1) \cos(\phi_1)}{\cos(\alpha_1) \sin(\phi_1 - \alpha_1)}\right). \end{aligned} \quad (189)$$

Put $A = -\cos(\alpha_2) \sin(\phi_1 - \alpha_1) + \sin(\alpha_2 - \alpha_1) \cos(\phi_1)$. It can be shown that

$$\begin{aligned} A &= -\cos(\alpha_2) [\sin(\phi_1) \cos(\alpha_1) - \sin(\alpha_1) \cos(\phi_1)] \\ &\quad + [\sin(\alpha_2) \cos(\alpha_1) - \sin(\alpha_1) \cos(\alpha_2)] \cos(\phi_1) \\ &= -\cos(\alpha_2) \sin(\phi_1) \cos(\alpha_1) \\ &\quad + \sin(\alpha_1) \cos(\alpha_2) \cos(\phi_1) \\ &\quad + \sin(\alpha_2) \cos(\phi_1) \cos(\alpha_1) \\ &\quad - \sin(\alpha_1) \cos(\alpha_2) \cos(\phi_1) \\ &= [\sin(\alpha_2) \cos(\phi_1) - \cos(\alpha_2) \sin(\phi_1)] \cos(\alpha_1) \\ &= \cos(\alpha_1) \sin(\alpha_2 - \phi_1). \end{aligned} \quad (190)$$

Injecting (190) inside (189), one gets

$$\begin{aligned} \frac{T_{1xy}}{T_{2xy}} &= -\left(\frac{\cos(\alpha_1) \sin(\alpha_2 - \phi_1)}{\cos(\alpha_1) \sin(\phi_1 - \alpha_1)}\right) \\ &= \left(\frac{\sin(\phi_1 - \alpha_2)}{\sin(\phi_1 - \alpha_1)}\right) \end{aligned} \quad (191)$$

and since $T_{1xy} = \cos(\psi_1) T_1$ and $T_{2xy} = \cos(\psi_2) T_2$, one has

$$\frac{T_1}{T_2} = \frac{\cos(\psi_2)}{\cos(\psi_1)} \left(\frac{\sin(\phi_1 - \alpha_2)}{\sin(\phi_1 - \alpha_1)}\right) \quad (192)$$

Consider now the case where the element 1 is sliding and not in contact with a stop. Thus, from Definition 3, $T_1 = T_2$ and so (192) becomes

$$\sin(\alpha_2 - \phi_1) \cos(\psi_2) = \sin(\alpha_1 - \phi_1) \cos(\psi_1) \quad (193)$$

A.1.1 Particular case $\cos(\alpha_1) \neq 0$

Consider now the particular case where $\alpha_1 = s\frac{\pi}{2}$ with $s \in \{-1, 1\}$. From (183), one has

$$\begin{cases} 0 = \cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1) \\ 0 = -sT_{1xy} + \sin(\alpha_2) T_{2xy} + F_{1xy} \sin(\phi_1) \end{cases} \quad (194)$$

and so $T_{2xy} = F_{1xy} \frac{\cos(\phi_1)}{\cos(\alpha_2)}$. Then, one gets from the second line of (194)

$$-sT_{1xy} + \sin(\alpha_2) F_{1xy} \frac{\cos(\phi_1)}{\cos(\alpha_2)} + F_{1xy} \sin(\phi_1) = 0$$

$$F_{1xy} [\tan(\alpha_2) \cos(\phi_1) + \sin(\phi_1)] = sT_{1xy}$$

Let's now study the ratio $\frac{T_{1xy}}{T_{2xy}}$:

$$\frac{T_{1xy}}{T_{2xy}} = s \frac{F_{1xy} [\tan(\alpha_2) \cos(\phi_1) + \sin(\phi_1)]}{F_{1xy} \frac{\cos(\phi_1)}{\cos(\alpha_2)}}$$

$$\frac{T_{1xy}}{T_{2xy}} = s \frac{\sin(\alpha_2) \cos(\phi_1) + \cos(\alpha_2) \sin(\phi_1)}{\cos(\phi_1)}$$

$$\frac{T_{1xy}}{T_{2xy}} = s \frac{\sin(\alpha_2 - \phi_1)}{\cos(\phi_1)}$$

$$\frac{T_{1xy}}{T_{2xy}} = s \frac{\sin(\alpha_2 - \phi_1)}{\cos(\phi_1 + s\frac{\pi}{2} - s\frac{\pi}{2})} \quad (195)$$

and since $\alpha_1 = s\frac{\pi}{2}$, one gets

$$\frac{T_{1xy}}{T_{2xy}} = -s \frac{\sin(\alpha_2 - \phi_1)}{\sin(\phi_1 - \alpha_1 + s\frac{\pi}{2})}$$

$$\frac{T_{1xy}}{T_{2xy}} = -s^2 \frac{\sin(\alpha_2 - \phi_1)}{\sin(\phi_1 - \alpha_1)}$$

$$\frac{T_{1xy}}{T_{2xy}} = \frac{\sin(\alpha_2 - \phi_1)}{\sin(\alpha_1 - \phi_1)} \quad (196)$$

which lead to the same result than (191).

A.2 Study of (184)

Consider now (184). Suppose in a first time that $\alpha_3 \notin \{\frac{\pi}{2}k | k \in \mathbb{Z}\}$, thus $\cos(\alpha_3) \neq 0$. The case $\cos(\alpha_3) = 0$ can be solved following the same steps than in Appendix A.1.1. From (184), one gets

$$\begin{cases} T_{3xy} = \frac{1}{\cos(\alpha_3)} (\cos(\alpha_2) T_{2xy} - F_{2xy} \cos(\phi_2)) \\ 0 = [-\sin(\alpha_2) + \tan(\alpha_3) \cos(\alpha_2)] T_{2xy} \\ \quad + [-\tan(\alpha_3) \cos(\phi_2) + \sin(\phi_2)] F_{2xy} \end{cases}$$

Again, from the second line of (184), one deduce than if $\alpha_2 = \phi_2 + k\pi$ with $k \in \mathbb{Z}$, since $T_{2xy} > 0$ and $F_{2xy} > 0$, one must have $\alpha_3 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$. These specific cases correspond to case where the cable is perfectly straight ($\alpha_3 = \alpha_2$) or perfectly folded ($\alpha_3 = \alpha_2 + \pi$). Case $\alpha_2 = \alpha_3$ is excluded in most configuration by Definition 2. Case $\alpha_3 = \alpha_2 + \pi$ induces $\cos(\alpha_3) = -\cos(\alpha_2)$ and $\sin(\alpha_3) = -\sin(\alpha_2)$. From (184), it leads to $\phi_2 = \alpha_2$. Then, it can be deduced that $T_{2xy} = T_{3xy} = \frac{F_{3xy}}{2}$.

Consider first we are not in these particulars case. Then, one has

$$\begin{cases} T_{3xy} = \frac{1}{\cos(\alpha_3)} (\cos(\alpha_2) T_{2xy} - F_{2xy} \cos(\phi_2)) \\ T_{2xy} = -F_{2xy} \frac{[-\tan(\alpha_3) \cos(\phi_2) + \sin(\phi_2)]}{[\tan(\alpha_3) \cos(\alpha_2) - \sin(\alpha_2)]} \end{cases} \quad (197)$$

One gets

$$T_{3xy} = \frac{1}{\cos(\alpha_3)} (\cos(\alpha_2) T_{2xy} - F_{2xy} \cos(\phi_2)) \quad (198)$$

and T_{2xy} can be expressed such as

$$T_{2xy} = -F_{2xy} \frac{[-\tan(\alpha_3) \cos(\phi_2) + \sin(\phi_2)]}{[\tan(\alpha_3) \cos(\alpha_2) - \sin(\alpha_2)]}$$

$$= -F_{2xy} \frac{\left[-\frac{\sin(\alpha_3)}{\cos(\alpha_3)} \cos(\phi_2) + \sin(\phi_2)\right]}{\left[\frac{\sin(\alpha_3)}{\cos(\alpha_3)} \cos(\alpha_2) - \sin(\alpha_2)\right]}$$

$$= -F_{2xy} \frac{[-\sin(\alpha_3) \cos(\phi_2) + \cos(\alpha_3) \sin(\phi_2)]}{[\sin(\alpha_3) \cos(\alpha_2) - \cos(\alpha_3) \sin(\alpha_2)]}$$

$$= -F_{2xy} \frac{\sin(\phi_2 - \alpha_3)}{\sin(\alpha_3 - \alpha_2)}. \quad (199)$$

From (197) and (199), one gets an expression of T_{2xy} and T_{3xy} .

Following the same steps than (189)-(192) in Section A.1, one gets

$$\frac{T_3}{T_2} = \frac{\cos(\psi_2)}{\cos(\psi_3)} \left(\frac{\sin(\phi_2 - \alpha_2)}{\sin(\phi_2 - \alpha_3)} \right) \quad (200)$$

Consider now the case where the element 2 is sliding and not in contact with a stop. Thus, from Definition 3, one has $T_2 = T_3$ and (200) becomes

$$\cos(\psi_3) \sin(\phi_2 - \alpha_3) = \cos(\psi_2) \sin(\phi_2 - \alpha_2) \quad (201)$$

A.3 Study of (183) and (184)

Suppose we are in configuration where (183) and (184) are both valid, for example in the areas A0, A12 and A22. Using the expression of T_{2xy} in (188) and (199), one get

$$\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_1 - \alpha_2)} F_{1xy} = F_{2xy} \frac{\sin(\phi_2 - \alpha_3)}{\sin(\alpha_2 - \alpha_3)}$$

$$\frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin(\alpha_2 - \alpha_3) = \sin(\phi_2 - \alpha_3) \sin(\alpha_1 - \alpha_2). \quad (202)$$

A.4 Calculation of T_1 , T_2 and T_3

In function of if systems (183) and (184) are valid or not, the calculation of T_1 , T_2 and T_3 changes. This section exposes the different case possible. In all case, one has $\forall i \in \{1, 2, 3\}$

$$T_i = \frac{T_{ixy}}{\cos(\psi_i)}, \quad (203)$$

$$F_{1xy} = \cos(\psi_{F1}) F_1 \text{ and } F_{2xy} = \cos(\psi_{F2}) F_2.$$

Let's find the expression of T_{1xy} , T_{2xy} and T_{3xy} . Note that all the particulars cases excluded in following sections are grouped in Appendix A.4.5.

A.4.1 If system (183) is valid and (183) is not valid (areas A11, D1)

Suppose here the case where the two elements are not in contact, *i.e.* $B_1 \neq B_2$. We exclude the case $\alpha_1 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$ because it corresponds to case where the cable is perfectly straight ($\alpha_1 = \alpha_2$) or perfectly folded ($\alpha_1 = \alpha_2 + \pi$). Since $\sin(\alpha_2 - \alpha_1) \neq 0$ and based on the steps described in Appendix A.1, one has if $\alpha_1 \neq s\frac{\pi}{2}$ with $s \in \{-1, 1\}$

$$T_{1xy} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1)) \quad (204)$$

$$T_{2xy} = -\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy} \quad (205)$$

and T_3 does not exist.

A.4.2 If system (183) is not valid and (183) is not valid (areas A21, D3)

Suppose here the case where the two elements are not in contact, *i.e.* $B_1 \neq B_2$. We exclude the case $\alpha_3 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$ because it corresponds to case where the cable is perfectly straight ($\alpha_3 = \alpha_2$) or perfectly folded ($\alpha_3 = \alpha_2 + \pi$). Since $\sin(\alpha_2 - \alpha_3) \neq 0$ and based on the steps described in Appendix A.2, one has one has if $\alpha_3 \neq s\frac{\pi}{2}$ with $s \in \{-1, 1\}$

$$T_{2xy} = -\frac{\sin(\phi_2 - \alpha_3)}{\sin(\alpha_3 - \alpha_2)} F_{2xy} \quad (206)$$

$$T_{3xy} = \frac{1}{\cos(\alpha_3)} (\cos(\alpha_2) T_{2xy} - F_{2xy} \cos(\phi_2)) \quad (207)$$

and T_1 does not exist.

A.4.3 Systems (183) and (183) are valid (areas A0, A12, A22)

Suppose here the case where the two elements are not in contact, *i.e.* $B_1 \neq B_2$. We exclude the case $\alpha_1 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$ because it corresponds to case where the cable is perfectly straight ($\alpha_1 = \alpha_2$) or perfectly folded ($\alpha_1 = \alpha_2 + \pi$). Since $|\sin(\alpha_2 - \alpha_1)| \neq 0$ and based on the steps described in Appendices A.1 and A.2, one has if $\alpha_1 \neq s\frac{\pi}{2}$ with $s \in \{-1, 1\}$

$$T_{1xy} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_2) T_{2xy} + F_{1xy} \cos(\phi_1)) \quad (208)$$

$$T_{3xy} = \frac{1}{\cos(\alpha_3)} (\cos(\alpha_2) T_{2xy} - F_{2xy} \cos(\phi_2)). \quad (209)$$

with

$$T_{2xy} = \begin{cases} -\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy} & \text{if areas A12 or A0} \\ -\frac{\sin(\phi_2 - \alpha_3)}{\sin(\alpha_3 - \alpha_2)} F_{2xy} & \text{if areas A22} \end{cases} \quad (210)$$

A.4.4 Two elements in contact (area D2)

Consider now the case where the two elements are in contact, *i.e.* $B_1 = B_2$. We exclude the case $\alpha_1 = \alpha_3 + k_2\pi$ with $k_2 \in \mathbb{Z}$ because it corresponds to case where the cable is perfectly straight ($\alpha_1 = \alpha_3$) or perfectly folded ($\alpha_1 = \alpha_3 + \pi$).

The system (13) can be rewrite such as

$$\begin{cases} -\cos(\alpha_1) T_{1xy} + \cos(\alpha_3) T_{3xy} + F_{\Sigma x} = 0 \\ -\sin(\alpha_1) T_{1xy} + \sin(\alpha_3) T_{3xy} + F_{\Sigma y} = 0 \end{cases} \quad (211)$$

Following the same step that (186)-(188) in Appendix A.1, one gets from (211)

$$T_{1xy} = \frac{1}{\cos(\alpha_1)} (\cos(\alpha_3) T_{3xy} + F_{\Sigma x}) \quad (212)$$

and

$$-\frac{\sin(\alpha_1)}{\cos(\alpha_1)} (\cos(\alpha_3) T_{3xy} + F_{\Sigma x}) + \sin(\alpha_3) T_{3xy} + F_{\Sigma y} = 0$$

$$[\cos(\alpha_1) \sin(\alpha_3) - \sin(\alpha_1) \cos(\alpha_3)] T_{3xy} + [F_{\Sigma y} \cos(\alpha_1) - F_{\Sigma x} \sin(\alpha_1)] = 0$$

$$\begin{aligned} T_{3xy} &= -\frac{F_{\Sigma y} \cos(\alpha_1) - F_{\Sigma x} \sin(\alpha_1)}{[\cos(\alpha_1) \sin(\alpha_3) - \sin(\alpha_1) \cos(\alpha_3)]} \\ T_{3xy} &= \frac{F_{\Sigma x} \sin(\alpha_1) - F_{\Sigma y} \cos(\alpha_1)}{\sin(\alpha_3 - \alpha_1)}. \end{aligned} \quad (213)$$

and T_2 does not exist.

A.4.5 Particular cases

In all previous configurations:

If not area D2:

In the case where $\alpha_1 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$, one has $T_{1xy} = T_{2xy} = \frac{F_{1xy}}{2}$.

In the case where $\alpha_3 = \alpha_2 + k_2\pi$ with $k_2 \in \mathbb{Z}$, one has $T_{3xy} = T_{2xy} = \frac{F_{3xy}}{2}$.

If $\alpha_1 = s\frac{\pi}{2}$, one has $T_{2xy} = F_{1xy} \frac{\cos(\phi_1)}{\cos(\alpha_2)}$ and $T_{1xy} = sF_{1xy} [\tan(\alpha_2) \cos(\phi_1) + \sin(\phi_1)]$.

If $\alpha_3 = s\frac{\pi}{2}$, one has $T_{2xy} = F_{2xy} \frac{\cos(\phi_2)}{\cos(\alpha_2)}$ and $T_{3xy} = sF_{2xy} [\tan(\alpha_2) \cos(\phi_2) + \sin(\phi_2)]$.

If area D2:

In the case where $\alpha_3 = \alpha_1 + k_2\pi$ with $k_2 \in \mathbb{Z}$, one has $T_{3xy} = T_{1xy} = \frac{\sqrt{F_{\Sigma x}^2 + F_{\Sigma y}^2}}{2}$.

If $\alpha_1 = s\frac{\pi}{2}$, one has $T_{3xy} = F_{\Sigma x} \frac{1}{\cos(\alpha_3)}$ and $T_{1xy} = s[\sin(\alpha_3) T_{3xy} + F_{\Sigma y}]$.

B Proof of length calculation

B.1 Area A2

In area A2, the length l_{11} and l_{12} are known. We desire to find l_{21} and l_{22} is function of the angles. Put $a_{11} = \cos(\psi_1) \sin(\alpha_1)$, $a_{12} = \cos(\psi_2) \sin(\alpha_2)$ and $a_{22} = \cos(\psi_3) \sin(\alpha_3)$.

Suppose first $a_{11} = a_{12}$. By injecting a_{11} , a_{12} and a_{22} in the second line of (7), one gets

$$\begin{aligned} y - l_0 &= a_{11}l_{11} + a_{12}(l_{12} + l_{21}) + a_{22}l_{22} \\ y - (l_0 + a_{11}l_{11} + a_{12}l_{12}) &= a_{12}(l_2 - l_{22}) + a_{22}l_{22} \\ y - (l_0 + a_{11}l_{11} + a_{12}(l_{12} + l_2)) &= (a_{22} - a_{12})l_{22} \\ l_{22} &= \frac{y - (l_0 + a_{11}l_{11} + a_{12}(l_{12} + l_2))}{a_{22} - a_{12}} \end{aligned} \quad (214)$$

with $l_{12} = l_1 - l_{12}$.

Consider now first $a_{11} = a_{12}$. By injecting a_{11} , a_{12} and a_{22} in the first line of (7) and following the same step than (214), one gets

$$l_{22} = \frac{x - (b_{11}l_{11} + b_{12}(l_{12} + l_2))}{b_{22} - b_{12}} \quad (215)$$

with $b_{11} = \cos(\psi_1) \cos(\alpha_1)$, $b_{12} = \cos(\psi_2) \cos(\alpha_2)$ and $b_{22} = \cos(\psi_3) \cos(\alpha_3)$.

In both case, one has $l_{21} = l_2 - l_{21}$. Note that the Definition 2 excludes the case $a_{22} = a_{12}$ and $b_{22} = b_{12}$ simultaneously in area A2.

B.2 Area A1

In area A1, the length l_{21} and l_{22} are known. We desire to find l_{11} and l_{12} is function of the angles.

Suppose first $a_{11} = a_{12}$. By injecting a_{11} , a_{12} and a_{22} in the second line of (7), one gets

$$\begin{aligned} y - l_0 &= a_{11}l_{11} + a_{12}(l_{12} + l_{21}) + a_{22}l_{22} \\ y - (l_0 + a_{22}l_{22} + a_{12}l_{21}) &= a_{11}l_{11} + a_{12}(l_1 - l_{11}) \\ y - (l_0 + a_{22}l_{22} + a_{12}(l_{21} + l_1)) &= (a_{11} - a_{12})l_{11} \\ l_{11} &= \frac{y - (l_0 + a_{22}l_{22} + a_{12}(l_{21} + l_1))}{a_{11} - a_{12}} \end{aligned} \quad (216)$$

Consider now first $a_{11} = a_{12}$. By injecting a_{11} , a_{12} and a_{22} in the first line of (7) and following the same step than (214), one get

$$l_{11} = \frac{x - (b_{22}l_{22} + b_{12}(l_{21} + l_1))}{b_{11} - b_{12}}. \quad (217)$$

In both case, one has $l_{12} = l_1 - l_{12}$. Note that the Definition 2 excludes the case $a_{11} = a_{12}$ and $b_{11} = b_{12}$ simultaneously in area A1, so there is always a solution.

B.3 Area A0

In opposite with areas A1 and A2, we have no specific information on l_{11} and l_{22} in the area A0. Let's start using the result (214) of Appendix (B.1). Injecting $l_{12} = l_1 - l_{12}$ in (214), one gets

$$\begin{aligned} l_{22} &= \frac{y - (l_0 + a_{11}l_{11} + a_{12}(l_{12} + l_2))}{a_{22} - a_{12}} \\ &= \frac{y - (l_0 + a_{11}l_{11} + a_{12}(l_1 - l_{11} + l_2))}{a_{22} - a_{12}} \\ &= \frac{y - (l_0 + (a_{11} - a_{12})l_{11} + a_{12}(l_1 + l_2))}{a_{22} - a_{12}} \\ &= \frac{y - (l_0 + a_{12}(l_1 + l_2))}{a_{22} - a_{12}} - \left(\frac{a_{11} - a_{12}}{a_{22} - a_{12}} \right) l_{11} \\ &= B_{11} - l_{11}A_{11} \end{aligned} \quad (218)$$

with $A_{11} = \left(\frac{a_{11} - a_{12}}{a_{22} - a_{12}} \right)$ and $B_{11} = \frac{y - (l_0 + a_{12}(l_1 + l_2))}{a_{22} - a_{12}}$.

Consider now the second line of (7) and put $b_{11} = \cos(\psi_1) \cos(\alpha_1)$, $b_{12} = \cos(\psi_2) \cos(\alpha_2)$, $b_{22} = \cos(\psi_3) \cos(\alpha_3)$. Then,

$$\begin{aligned} x &= b_{11}l_{11} + b_{12}(l_{12} + l_{21}) + b_{22}l_{22} \\ x &= b_{11}l_{11} + b_{12}(l_1 - l_{11} + l_2 - l_{22}) + b_{22}l_{22} \end{aligned}$$

$$x - b_{12}(l_1 + l_2) = (b_{11} - b_{12})l_{11} + (b_{22} - b_{12})l_{22}$$

Using (218), one gets

$$\begin{aligned} x - b_{12}(l_1 + l_2) &= (b_{11} - b_{12})l_{11} \\ &\quad + (b_{22} - b_{12})(B_{11} - l_{11}A_{11}) \end{aligned}$$

$$\begin{aligned} x - [b_{12}(l_1 + l_2) + (b_{22} - b_{12})B_{11}] \\ &= [(b_{11} - b_{12}) - (b_{22} - b_{12})A_{11}]l_{11} \end{aligned}$$

$$l_{11} = \frac{x - (b_{12}(l_1 + l_2) + (b_{22} - b_{12})B_{11})}{(b_{11} - b_{12}) - (b_{22} - b_{12})A_{11}} \quad (219)$$

One can found l_{22} using (218) and (219), then l_{12} and l_{21} with $l_{12} = l_1 - l_{12}$ and $l_{21} = l_2 - l_{22}$

C Calculation in area A1

C.1 Referential \mathcal{R}_1

To find a solution in the area A1, we have observe that the system is much more easier to solve if $\psi_{F1} = 0$: it will allow to simplify the system (12) to find a solution for (201) independent of the elevation angles ψ_1 and ψ_2 . Thus, in this section, we decide to perform a rotation of the referential \mathcal{R} around the vertical axis $O\vec{y}$ of an angle $\mu_1 = \text{atan2}(F_{1z}, F_{1x})$ with $F_{1|\mathcal{R}} = [F_{1x} \ F_{1y} \ F_{1z}]^T$ to obtain the referential

\mathcal{R}_1 , whose the characteristics are described in the Definition 4. In \mathcal{R}_1 , one has $\psi_{F1|\mathcal{R}_1} = 0$. Once a solution will be found in \mathcal{R}_1 , the angles and lengths will be re-evaluated in \mathcal{R} using for example Definition 5.

In this section, since all the calculations are performed in \mathcal{R}_1 , the notations are lightened such that a parameter $\beta_{i|\mathcal{R}_1}$ is simply written β_i .

C.2 Common angles in A1

In area A1, the element 1 is sliding such that it correspond to the Definition 3 (1). Thus, one has $T_1 = T_2$ and from Theorem 2

$$\cos(\psi_1) \sin(\phi_1 - \alpha_1) = \cos(\psi_2) \sin(\phi_1 - \alpha_2) \quad (220)$$

Moreover, since $\psi_{F1} = 0$, the third line of system (12) becomes

$$\begin{aligned} -\sin(\psi_1) T_1 + \sin(\psi_2) T_2 &= 0 \\ \sin(\psi_2) &= \sin(\psi_1) \end{aligned} \quad (221)$$

Since $\psi_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it can be deduced from (221) that in area A1:

$$\psi_1 = \psi_2. \quad (222)$$

Injecting (222) inside(220), one has $\sin(\phi_1 - \alpha_1) = \sin(\phi_1 - \alpha_2)$, leading to two possible solutions: $\alpha_1 = \alpha_2$ or

$$\begin{aligned} \alpha_1 - \phi_1 &= \pi - (\alpha_2 - \phi_1) \\ \alpha_2 &= -\alpha_1 + 2\phi_1 + \pi \end{aligned} \quad (223)$$

However, since $\psi_{F1} = 0$, *i.e.* there is no force perpendicular to the plan for element 1, the only solution respecting the condition of Definition 3 (1) is (223), $\alpha_1 = \alpha_2$ corresponding to the case where the cables l_{12} and l_{21} form a straight line where the sliding element cannot be immobilized, excluded by Definition 2 since $\psi_1 = \psi_2$.

C.3 Calculation in area A11

In area A11, one has $l_{22} = 0$ and $l_{21} = l_2$. Since $l_{22} = 0$, the angle α_3 and ψ_3 are respectively merged with angle α_2 and ψ_2 so

$$\alpha_2 = \alpha_3 \quad (224)$$

$$\psi_2 = \psi_3. \quad (225)$$

Thus, from the third line of (7), since $l_{22} = 0$, $l_{21} = l_2$ and $\psi_1 = \psi_2$ from (222), one gets

$$\begin{aligned} l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) &= z \\ (l_{11} + l_{12} + l_{21} + l_{22}) \sin(\psi_2) &= z \\ (l_1 + l_2) \sin(\psi_2) &= z \\ \sin(\psi_2) &= \frac{z}{(l_1 + l_2)} \end{aligned} \quad (226)$$

and since $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, one has

$$\psi_2 = \text{asin}\left(\frac{z}{l_1 + l_2}\right). \quad (227)$$

The calculation of α_1 can be performed as it will be described in Appendix C.6.

C.4 Calculation in area A12

In area A12, one has $l_{21} = 0$ and $l_{22} = l_2$.

C.4.1 Relation between α_1 and α_3

Let's find first a relation between α_1 and α_3 . Using the PFD (12) for $i = \{1, 2\}$ and following the steps described in Appendix (A), one gets the result (202) in Appendix (A.3):

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin(\alpha_2 - \alpha_3) \\ = \sin(\phi_2 - \alpha_3) \sin(\alpha_1 - \alpha_2). \end{aligned} \quad (228)$$

Injecting (223) inside (228), one gets

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin(-\alpha_1 + 2\phi_1 - \alpha_3) \\ = \sin(\phi_2 - \phi_1 + \phi_1 - \alpha_3) \sin(\alpha_1 - (-\alpha_1 + 2\phi_1)) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin((\phi_1 - \alpha_1) + (\phi_1 - \alpha_3)) \\ = \sin((\phi_2 - \phi_1) + (\phi_1 - \alpha_3)) \sin(2(\alpha_1 - \phi_1)) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(\Gamma_1 + \Omega_1) \\ = -\sin(\Delta\phi + \Omega_1) \sin(2\Gamma_1) \end{aligned} \quad (229)$$

where $\Delta\phi = \phi_2 - \phi_1$, $\Gamma_1 = \phi_1 - \alpha_1$ and $\Omega_1 = \phi_1 - \alpha_3$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) [\sin(\Gamma_1) \cos(\Omega_1) + \sin(\Omega_1) \cos(\Gamma_1)] \\ = -[\sin(\Delta\phi) \cos(\Omega_1) + \sin(\Omega_1) \cos(\Delta\phi)] \\ \times 2 \sin(\Gamma_1) \cos(\Gamma_1) \end{aligned}$$

$$\begin{aligned} \tan(\Gamma_1) + \tan(\Omega_1) \\ = -2 \frac{F_{2xy}}{F_{1xy}} [\sin(\Delta\phi) + \tan(\Omega_1) \cos(\Delta\phi)] \end{aligned}$$

$$\begin{aligned} 2 \frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1) \\ = -\left(2 \frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right) \tan(\Omega_1) \end{aligned}$$

Then, if $\xi = \frac{2F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1 \neq 0$ (as suppose by Assumption A9), one gets

$$\begin{aligned}\tan(\Omega_1) &= -\frac{2\frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2\frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)} \\ \tan(\phi_1 - \alpha_3) &= -\frac{2\frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2\frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)} \\ \tan(\alpha_3 - \phi_1) &= \frac{2\frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2\frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)}.\end{aligned}\quad (230)$$

From (230), one deduce α_3 can take the two following values:

$$\alpha_3 \in \{\alpha_{31}, \alpha_{32}\} \quad (231)$$

with

$$\begin{aligned}\alpha_{31} &= \phi_1 + \text{atan}\left(\frac{2\frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2\frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)}\right) \\ \alpha_{32} &= \phi_1 + \pi + \text{atan}\left(\frac{2\frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2\frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)}\right).\end{aligned}\quad (232)$$

C.4.2 Study of ψ_2 and ψ_3

Let's study now the values of ψ_2 and ψ_3 .

Calculation of ψ_3 Considering the third line of system (12) for $i = 2$ and remind that for $j \in \{1, 2, 3\}$, $T_{jxy} = \cos(\psi_j) T_j$:

$$\begin{aligned}-\sin(\psi_2) T_2 + \sin(\psi_3) T_3 + F_2 \sin(\psi_{F2}) &= 0 \\ -\sin(\psi_2) \frac{T_{2xy}}{\cos(\psi_2)} + \sin(\psi_3) \frac{T_{3xy}}{\cos(\psi_3)} + F_2 \sin(\psi_{F2}) &= 0 \\ -\tan(\psi_2) T_{2xy} + \tan(\psi_3) T_{3xy} &= -F_2 \sin(\psi_{F2}) \\ \tan(\psi_3) T_{3xy} &= \tan(\psi_2) T_{2xy} - F_2 \sin(\psi_{F2})\end{aligned}$$

$$\begin{aligned}\tan(\psi_3) &= \tan(\psi_2) \frac{T_{2xy}}{T_{3xy}} - \frac{F_2 \sin(\psi_{F2})}{T_{3z}} \\ \tan(\psi_3) &= \tan(\psi_2) a_1 + b_1\end{aligned}\quad (234)$$

with $a_1 = \frac{T_{2xy}}{T_{3xy}}$ and $b_1 = -\frac{F_2}{T_{3xy}} \sin(\psi_{F2})$, where the expression of T_{2xy} and T_{3xy} can be found in (210) and (209) in Appendix A.1.

Since $\psi_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, one gets an evaluation of the value of ψ_3 :

$$\psi_3 = \text{atan}(\tan(\psi_1) a_1 + b_1). \quad (235)$$

Alternative calculation of $\cos(\psi_3)$

From the third line of (7), since $l_{21} = 0$, $l_{22} = l_2$ and $\psi_1 = \psi_2$ from (222), one gets

$$\begin{aligned}l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) &= z \\ (l_{11} + l_{12} + l_{21}) \sin(\psi_2) + l_2 \sin(\psi_3) &= z \\ l_1 \sin(\psi_2) + l_2 \sin(\psi_3) &= z \\ \sin(\psi_3) &= \frac{z - l_1 \sin(\psi_2)}{l_2}\end{aligned}\quad (236)$$

Since $\psi_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, (236) leads to

$$\cos(\psi_3) = \sqrt{1 - \left(\frac{z - l_1 \sin(\psi_2)}{l_2}\right)^2}. \quad (237)$$

This form will be used in the calculation of ψ_2 in next part below.

Calculation of ψ_2

We desire now to find an expression of ψ_2 in function only of α_1 , α_2 and α_3 . a_{11} , a_{12} , a_{21} and a_{22} are defined in Section 6.4.

Suppose first $a_{11} - a_{12} \neq 0$, so $\sin(\alpha_1) - \sin(\alpha_2) \neq 0$ since $\psi_1 = \psi_2$. From (41), one may write

$$\begin{aligned}l_{11} &= \\ \frac{y - (l_0 + \cos(\psi_3) \sin(\alpha_3) l_{22} + \cos(\psi_2) \sin(\alpha_2) (l_{21} + l_1))}{\cos(\psi_1) \sin(\alpha_1) - \cos(\psi_2) \sin(\alpha_2)}\end{aligned}\quad (238)$$

In area A12, one has $l_{21} = 0$, $l_{22} = l_2$ and $\psi_1 = \psi_2$. Using it, one gets

$$l_{11} = \frac{y - (l_0 + \cos(\psi_3) \sin(\alpha_3) l_{22} + \cos(\psi_2) \sin(\alpha_2) l_1)}{\cos(\psi_2) (\sin(\alpha_1) - \sin(\alpha_2))} \quad (239)$$

Using it and $l_{12} = l_1 - l_{11}$, the first line of system (7) can be rewritten as

$$\begin{aligned}(l_{11} (\cos(\alpha_1) - \cos(\alpha_2)) + l_1 \cos(\alpha_2)) \cos(\psi_2) \\ + l_2 \cos(\psi_3) \cos(\alpha_3) = x\end{aligned}\quad (240)$$

Injecting (239) inside (240), one gets

$$\begin{aligned}\frac{y - (l_0 + \cos(\psi_3) \sin(\alpha_3) l_{22} + \cos(\psi_2) \sin(\alpha_2) l_1)}{\cos(\psi_2) (\sin(\alpha_1) - \sin(\alpha_2))} \\ \times (\cos(\alpha_1) - \cos(\alpha_2)) \cos(\psi_2) \\ + l_1 \cos(\alpha_2) \cos(\psi_2) + l_2 \cos(\psi_3) \cos(\alpha_3) = x \\ (y - l_0 - \cos(\psi_3) \sin(\alpha_3) l_2 - \cos(\psi_2) \sin(\alpha_2) l_1) \\ \times \left(\frac{\cos(\alpha_1) - \cos(\alpha_2)}{\sin(\alpha_1) - \sin(\alpha_2)}\right) \\ + l_1 \cos(\alpha_2) \cos(\psi_2) + l_2 \cos(\psi_3) \cos(\alpha_3) = x\end{aligned}$$

$$\begin{aligned}
& (y - l_0) \hat{A}_1 - \cos(\psi_3) \sin(\alpha_3) l_2 \hat{A}_1 \\
& - \cos(\psi_2) \sin(\alpha_2) l_1 \hat{A}_1 + l_1 \cos(\alpha_2) \cos(\psi_2) \\
& \quad + l_2 \cos(\psi_3) \cos(\alpha_3) = x
\end{aligned} \tag{241}$$

with $\hat{A}_1 = \left(\frac{\cos(\alpha_1) - \cos(\alpha_2)}{\sin(\alpha_1) - \sin(\alpha_2)} \right)$. Then,

$$\begin{aligned}
& \left[(y - l_0) \hat{A}_1 - x \right] \\
& + \left[l_1 \cos(\alpha_2) - \sin(\alpha_2) l_1 \hat{A}_1 \right] \cos(\psi_2) \\
& + \left[l_2 \cos(\alpha_3) - \sin(\alpha_3) l_2 \hat{A}_1 \right] \cos(\psi_3) = 0 \\
& \hat{D}_1 + \hat{C}_1 \cos(\psi_2) + \hat{B}_1 \cos(\psi_3) = 0
\end{aligned} \tag{242}$$

with

$$\hat{B}_1 = l_2 \cos(\alpha_3) - \sin(\alpha_3) l_2 \hat{A}_1 \tag{243}$$

$$\hat{C}_1 = l_1 \cos(\alpha_2) - \sin(\alpha_2) l_1 \hat{A}_1 \tag{244}$$

$$\hat{D}_1 = (y - l_0) \hat{A}_1 - x \tag{245}$$

In the case where $\sin(\alpha_1) = \sin(\alpha_2)$ (or numerically close), the same steps can be followed as described in Appendix C.4.3 to obtain

$$\bar{D}_1 + \bar{C}_1 \cos(\psi_2) + \bar{B}_1 \cos(\psi_3) = 0 \tag{246}$$

with

$$\bar{A}_1 = \frac{\sin(\alpha_1) - \sin(\alpha_2)}{\cos(\alpha_1) - \cos(\alpha_2)} \tag{247}$$

$$\bar{B}_1 = l_2 \sin(\alpha_3) - \cos(\alpha_3) l_2 \bar{A}_1 \tag{248}$$

$$\bar{C}_1 = l_1 \sin(\alpha_2) - \cos(\alpha_2) l_1 \bar{A}_1 \tag{249}$$

$$\bar{D}_1 = l_0 + x \bar{A}_1 - y. \tag{250}$$

Let introduce A_1, B_1, C_1 and D_1 where

$$\text{if } \sin(\alpha_2) \neq \sin(\alpha_1), A_1 = \hat{A}_1$$

$$B_1 = \hat{B}_1$$

$$C_1 = \hat{C}_1$$

$$D_1 = \hat{D}_1$$

$$\text{else, } A_1 = \bar{A}_1$$

$$B_1 = \bar{B}_1$$

$$C_1 = \bar{C}_1$$

$$D_1 = \bar{D}_1.$$

and so the general form

$$D_1 + \cos(\psi_2) C_1 + \cos(\psi_3) B_1 = 0. \tag{251}$$

Using (237) inside (251), one gets

$$C_1 \cos(\psi_2) + B_1 \sqrt{1 - \left(\frac{z - l_1 \sin(\psi_2)}{l_2} \right)^2} = -D_1 \tag{252}$$

Following steps described in Appendix E, an expression of ψ_2 in function of α_1 can be found using (252) and A_1, B_1, C_1 and D_1 .

The calculation of α_1 can be performed as it will be described in Appendix C.6.

C.4.3 Sub-case $\sin(\alpha_1) = \sin(\alpha_2)$

Suppose now $a_{11} - a_{12} = 0$ with $\alpha_2 \neq \alpha_1$, so $\sin(\alpha_1) = \sin(\alpha_2)$ since $\psi_1 = \psi_2$. From (42), one may write

$$\begin{aligned}
l_{11} = & \\
& \frac{x - (\cos(\psi_3) \cos(\alpha_3) l_{22} + \cos(\psi_2) \cos(\alpha_2) (l_{21} + l_1))}{\cos(\psi_1) \cos(\alpha_1) - \cos(\psi_2) \cos(\alpha_2)}
\end{aligned} \tag{253}$$

In area A12, one has $l_{21} = 0, l_{22} = l_2$. and $\psi_1 = \psi_2$. Using it, one gets

$$l_{11} = \frac{x - (\cos(\psi_3) \cos(\alpha_3) l_2 + \cos(\psi_2) \cos(\alpha_2) l_1)}{\cos(\psi_2) (\cos(\alpha_1) - \cos(\alpha_2))}. \tag{254}$$

The second line of system (7) can be rewritten as

$$\begin{aligned}
& l_{11} \cos(\psi_2) \sin(\alpha_1) + l_{12} \cos(\psi_2) \sin(\alpha_2) \\
& + l_2 \cos(\psi_3) \sin(\alpha_3) = y - l_0.
\end{aligned} \tag{255}$$

Using $l_{12} = l_1 - l_{11}$ and (254), (255) becomes

$$\begin{aligned}
& \left(\frac{x - (\cos(\psi_3) \cos(\alpha_3) l_2 + \cos(\psi_2) \cos(\alpha_2) l_1)}{\cos(\psi_2) (\cos(\alpha_1) - \cos(\alpha_2))} \right) \\
& \times (\cos(\psi_2) \sin(\alpha_1) - \cos(\psi_2) \sin(\alpha_2)) \\
& + l_1 \cos(\psi_2) \sin(\alpha_2) + l_2 \cos(\psi_3) \sin(\alpha_3) = y - l_0
\end{aligned}$$

$$\begin{aligned}
& (x - (\cos(\psi_3) \cos(\alpha_3) l_2 + \cos(\psi_2) \cos(\alpha_2) l_1)) \\
& \times \left(\frac{\sin(\alpha_1) - \sin(\alpha_2)}{\cos(\alpha_1) - \cos(\alpha_2)} \right) \\
& + l_1 \cos(\psi_2) \sin(\alpha_2) + l_2 \cos(\psi_3) \sin(\alpha_3) = y - l_0
\end{aligned}$$

$$\begin{aligned}
& x \bar{A}_1 - \cos(\psi_3) \cos(\alpha_3) l_2 \bar{A}_1 - \cos(\psi_2) \cos(\alpha_2) l_1 \bar{A}_1 \\
& + l_1 \cos(\psi_2) \sin(\alpha_2) + l_2 \cos(\psi_3) \sin(\alpha_3) = y - l_0
\end{aligned}$$

with $\bar{A}_1 = \frac{\sin(\alpha_1) - \sin(\alpha_2)}{\cos(\alpha_1) - \cos(\alpha_2)}$. Then,

$$\begin{aligned}
& [l_0 + x \bar{A} - y] + [l_1 \sin(\alpha_2) - \cos(\alpha_2) l_1 \bar{A}_1] \cos(\psi_2) \\
& + [l_2 \sin(\alpha_3) - \cos(\alpha_3) l_2 \bar{A}_1] \cos(\psi_3) = 0
\end{aligned}$$

$$\bar{D}_1 + \bar{C}_1 \cos(\psi_2) + \bar{B}_1 \cos(\psi_3) = 0$$

where

$$\bar{A}_1 = \left(\frac{\sin(\alpha_1) - \sin(\alpha_2)}{\cos(\alpha_1) - \cos(\alpha_2)} \right) \tag{256}$$

$$\bar{B}_1 = l_2 \sin(\alpha_3) - \cos(\alpha_3) l_2 \bar{A}_1 \tag{257}$$

$$\bar{C}_1 = l_1 \sin(\alpha_2) - \cos(\alpha_2) l_1 \bar{A}_1 \tag{258}$$

$$\bar{D}_1 = l_0 + x \bar{A}_1 - y. \tag{259}$$

From this point, using $\bar{B}_1, \bar{C}_1, \bar{D}_1$ instead of $\hat{B}_1, \hat{C}_1, \hat{D}_1$, one can follow the steps described in Appendix C.4.1.

C.5 Particular case $2 \frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1 = 0$

To avoid the singularity $2 \frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1 = 0$, the elements must respect the condition

$$\cos(\Delta\phi) \neq -\frac{\cos(\psi_{F1}) F_1}{2 \cos(\psi_{F2}) F_2}. \quad (260)$$

Note that $\psi_{F1|\mathcal{R}_1} = 0$. Thus (260) is always satisfied in \mathcal{R}_1 if

$$1 < \left| \frac{F_1}{2 \cos(\psi_{F2}) F_2} \right|, \quad (261)$$

and (261) is always true if

$$1 < \frac{F_1}{2F_2}. \quad (262)$$

C.6 Calculation of α_1

Based on the system (7), let's define the three equations

$$E_X = (l_{11} \cos(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \cos(\alpha_2) \cos(\psi_2) + l_{22} \cos(\alpha_3) \cos(\psi_3) - x)^2 \quad (263)$$

$$E_Y = (l_{11} \sin(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \sin(\alpha_2) \cos(\psi_2) + l_{22} \sin(\alpha_3) \cos(\psi_3) - y + l_0)^2 \quad (264)$$

$$E_Z = (l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) - z)^2 \quad (265)$$

Following steps described in Appendix B.2, the first or the second line of (7) is used to find an expression of l_{11} and l_{12} in function of α_1 . Thus, the expression of l_{11} and l_{12} make automatically E_X or E_Y equal to zero. Then, the third line of (7) is used to find an expression of ψ_2 and ψ_3 in both areas A11 and A12, see Appendices C.3 and C.4.2, thus value of ψ_2 and ψ_3 make also automatically E_Z equal to zero.

Then, using the first or second line of (7) and injecting inside

- (222)-(223)-(224)-(225)-(227) for area A11,
- (222)-(223)-(231)-(252)-(235) for area A12,

a solution of α_1 can be found. Since E_Z and E_X or E_Y are equal to zero, the same solution can be found by solving the following equation (266):

$$0 = E_X^2 + E_Y^2 + E_Z^2. \quad (266)$$

In practice, this solution will be found numerically using an optimization: using (266) allows to test the proposed solution of α_1 independently of the expression of l_{11}, l_{12}, ψ_2 or ψ_3 .

Moreover, the solution must respect the following conditions:

1. The evaluation of the lengths must be bounded such that $(l_{11}, l_{12}) \in [0, l_1]^2$ and $(l_{21}, l_{22}) \in [0, l_2]^2$;
2. The evaluation of the forces must be strictly positive:
 - if area A11, $T_1^{S1} > 0$ and $T_2^{S1} > 0$,
 - if area A12, $T_1^{S1} > 0$, $T_2^{S1} > 0$ and $T_3^{S1} > 0$.
3. The Definition 3 is respected for the two elements, *i.e.*
 - the element 1 is sliding so $\vec{F}_1 \cdot \overrightarrow{B_1 \hat{A}} < 0$ and $\vec{F}_1 \cdot \overrightarrow{B_1 \hat{S}} < 0$, where $\vec{F}_1 \cdot \overrightarrow{B_1 \hat{A}} < 0$ is equivalent to $\vec{F}_1 \cdot \overrightarrow{AB_1} > 0$;
 - if the element 2 is not fixed:
 - in area A11, the element 2 is in contact with R , so $\vec{F}_2 \cdot \overrightarrow{B_2 \hat{S}} < 0$, which is equivalent to $\vec{F}_2 \cdot \overrightarrow{SB_2} > 0$;
 - in area A12, the element 2 is in contact with S , so $\vec{F}_2 \cdot \overrightarrow{B_2 \hat{R}} < 0$,

where $\overrightarrow{AB_1}, \overrightarrow{B_1 \hat{S}}, \overrightarrow{SB_2}$ and $\overrightarrow{B_2 \hat{R}}$ have been expressed in Section 4.1.

D Calculation in area A2

D.1 Referential \mathcal{R}_2

To find a solution in the area A2, we have observed that the system is much more easier to solve if $\psi_{F2} = 0$: it will allow to simplify the system (12) to find a solution for (201) independent of the elevation angles ψ_2 and ψ_3 . Thus, in this section, we decide to perform a rotation of the referential \mathcal{R} around the vertical axis $O\vec{y}$ of an angle $\mu_2 = \text{atan2}(F_{2z}, F_{2x})$ with $F_{2|\mathcal{R}} = [F_{2x} \ F_{2y} \ F_{2z}]^T$ to obtain the referential \mathcal{R}_2 , whose characteristics are described in the Definition 4. In \mathcal{R}_2 , one has $\psi_{F2|\mathcal{R}_2} = 0$. Once a solution will be found in \mathcal{R}_2 , the angles and lengths will be re-evaluated in \mathcal{R} using for example Definition 5.

In this section, since all the calculations are performed in \mathcal{R}_2 , the notations are lightened such that a parameter $\beta_{i|\mathcal{R}_2}$ is simply written β_i .

D.2 Common angles in A2

In area A2, the element 2 is sliding such that it corresponds to the Definition 3 (1). Thus, one has $T_2 = T_3$ and from Theorem 2

$$\cos(\psi_2) \sin(\phi_2 - \alpha_2) = \cos(\psi_3) \sin(\phi_2 - \alpha_3). \quad (267)$$

Moreover, since $\psi_{F_2} = 0$, the third line of system (12) for $i = 2$ becomes

$$\begin{aligned} -\sin(\psi_2)T_2 + \sin(\psi_3)T_3 &= 0 \\ \sin(\psi_2) &= \sin(\psi_3) \end{aligned} \quad (268)$$

Since $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it can be deduced from (268) that in area A2:

$$\psi_3 = \psi_2. \quad (269)$$

Injecting (269) inside(267), one has $\sin(\phi_2 - \alpha_2) = \sin(\phi_2 - \alpha_3)$, leading to two possible solutions: $\alpha_2 = \alpha_3$ or

$$\begin{aligned} \alpha_2 - \phi_2 &= \pi - (\alpha_3 - \phi_2) \\ \alpha_3 &= -\alpha_2 + 2\phi_2 + \pi \end{aligned} \quad (270)$$

However, since $\psi_{F_2} = 0$, *i.e.* there is no force perpendicular to the plan for element 2, the only solution respecting the condition of Definition 3 (1) is (270), $\alpha_2 = \alpha_3$ corresponding to the case where the cables l_{12} and l_{21} form a straight line where the sliding element cannot be immobilized, excluded by Definition 2 since $\psi_3 = \psi_2$.

D.3 Calculation in area A21

In area A21, one has $l_{11} = 0$ and $l_{12} = l_1$. Since $l_{11} = 0$, the angle α_1 and ψ_1 are respectively merged with angle α_2 and ψ_2 so

$$\alpha_2 = \alpha_1 \quad (271)$$

$$\psi_1 = \psi_2 \quad (272)$$

Thus, from the third line of (7), since $l_{11} = 0$, $l_{12} = l_1$ and $\psi_2 = \psi_3$ from (269), one gets

$$\begin{aligned} l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) &= z \\ (l_{11} + l_{12} + l_{21} + l_{22}) \sin(\psi_2) &= z \\ (l_1 + l_2) \sin(\psi_2) &= z \\ \sin(\psi_2) &= \frac{z}{(l_1 + l_2)} \end{aligned} \quad (273)$$

and since $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, one has

$$\psi_2 = \text{asin}\left(\frac{z}{l_1 + l_2}\right). \quad (274)$$

The calculation of α_1 can be performed as it will be described in Appendix D.5.

D.4 Calculation in area A22

In area A22, one has $l_{11} = l_1$ and $l_{12} = 0$.

D.4.1 Relation between α_1 and α_3

Let's find first a relation between α_1 and α_3 . Using the PFD (12) for $i \in \{1, 2\}$ and following the steps described in Appendix (A), one gets the result (202) in Appendix (A.3):

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin(\alpha_2 - \alpha_3) \\ = \sin(\phi_2 - \alpha_3) \sin(\alpha_1 - \alpha_2). \end{aligned} \quad (275)$$

Injecting (270) inside (275), one gets

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \phi_2 + \phi_2 - \alpha_1) \sin(-\alpha_3 + 2\phi_2 + \pi - \alpha_3) \\ = \sin(\phi_2 - \alpha_3) \sin(\alpha_1 - (-\alpha_3 + 2\phi_2 + \pi)) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin((\phi_2 - \alpha_1) - (\phi_2 - \phi_1)) (-\sin(2(\phi_2 - \alpha_3))) \\ = \sin(\phi_2 - \alpha_3) (-\sin(\pi - [(\alpha_1 - \phi_2) - (\phi_2 - \alpha_3)])) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin((\phi_2 - \alpha_1) - (\phi_2 - \phi_1)) \sin(2(\phi_2 - \alpha_3)) \\ = \sin(\phi_2 - \alpha_3) \sin((\alpha_1 - \phi_2) - (\phi_2 - \alpha_3)) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_2 - \Delta\phi) \sin(2\Omega_2) = \sin(\Omega_2) \sin(-\Gamma_2 - \Omega_2) \end{aligned} \quad (276)$$

where $\Delta\phi = \phi_2 - \phi_1$, $\Gamma_2 = \phi_2 - \alpha_1$, $\Omega_2 = \phi_2 - \alpha_3$. Then

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} [\sin(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi) \cos(\Gamma_2)] \\ \times 2 \sin(\Omega_2) \cos(\Omega_2) \\ = -\sin(\Omega_2) [\sin(\Gamma_2) \cos(\Omega_2) + \sin(\Omega_2) \cos(\Gamma_2)] \end{aligned}$$

$$\begin{aligned} 2 \frac{F_{1xy}}{F_{2xy}} [\sin(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi) \cos(\Gamma_2)] \\ = -[\sin(\Gamma_2) + \tan(\Omega_2) \cos(\Gamma_2)] \end{aligned}$$

$$\begin{aligned} 2 \frac{F_{1xy}}{F_{2xy}} [\tan(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi)] \\ = -[\tan(\Gamma_2) + \tan(\Omega_2)] \end{aligned}$$

$$\begin{aligned} \tan(\Omega_2) = \\ -\left(2 \frac{F_{1xy}}{F_{2xy}} [\tan(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi)] + \tan(\Gamma_2)\right) \end{aligned}$$

$$\begin{aligned} \tan(\phi_2 - \alpha_3) = \\ -\left(2 \frac{F_{1xy}}{F_{2xy}} [\tan(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi)] + \tan(\Gamma_2)\right) \end{aligned}$$

$$\tan(\alpha_3 - \phi_2) = 2 \frac{F_{1xy}}{F_{2xy}} [\tan(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi)] + \tan(\Gamma_2)$$

$$\tan(\alpha_3 - \phi_2) = 2 \frac{F_{1xy}}{F_{2xy}} [\tan(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi)] + \tan(\Gamma_2). \quad (277)$$

From (277), one deduce α_3 can take the two following values α_{3k} for $k \in \{1, 2\}$ with

$$\begin{aligned} \alpha_{3k} &= \phi_2 + \pi(k-1) \\ &+ \operatorname{atan}\left(2 \frac{F_{1xy}}{F_{2xy}} [\tan(\Gamma_2) \cos(\Delta\phi) - \sin(\Delta\phi)] + \tan(\Gamma_2)\right) \end{aligned} \quad (278)$$

Note that from (270), α_2 can take two values based on α_3 .

D.4.2 Study of ψ_1 and ψ_2

Let's study now the values of ψ_1 and ψ_2 .

Calculation of ψ_1

Considering the third line of system (12) for $i = 1$ and remind that for $j \in \{1, 2, 3\}$, $T_{jxy} = \cos(\psi_j) T_j$:

$$\begin{aligned} -\sin(\psi_1) T_1 + \sin(\psi_2) T_2 + F_1 \sin(\psi_{F1}) &= 0 \\ -\sin(\psi_1) \frac{T_{1xy}}{\cos(\psi_1)} + \sin(\psi_2) \frac{T_{2xy}}{\cos(\psi_2)} + F_1 \sin(\psi_{F1}) &= 0 \\ -\tan(\psi_1) T_{1xy} + \tan(\psi_2) T_{2xy} + F_1 \sin(\psi_{F1}) &= 0 \end{aligned}$$

$$\begin{aligned} \tan(\psi_1) &= \tan(\psi_2) \frac{T_{2xy}}{T_{1xy}} + \frac{F_1}{T_{1xy}} \sin(\psi_{F1}) \\ \tan(\psi_1) &= \tan(\psi_2) a_2 + b_2 \end{aligned} \quad (279)$$

where $a_2 = \frac{T_{2xy}}{T_{1xy}}$ and $b_2 = -\frac{F_1}{T_{1xy}} \sin(\psi_{F1})$, where the expression of T_{1xy} and T_{2xy} can be found in (204) and (210) in Appendix A.1.

Since $\psi_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, one gets an evaluation of the value of ψ_1 :

$$\psi_1 = \operatorname{atan}(\tan(\psi_2) a_2 + b_2). \quad (280)$$

Alternative calculation of $\cos(\psi_1)$

From the third line of (7), since $l_{11} = l_1$, $l_{12} = 0$ and $\psi_2 = \psi_3$ from (269), one gets

$$l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) = z$$

$$l_1 \sin(\psi_1) + (l_{21} + l_{22}) \sin(\psi_2) = z$$

$$\sin(\psi_1) = \frac{z - l_2 \sin(\psi_2)}{l_1}. \quad (281)$$

Since $\psi_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, (281) leads to

$$\cos(\psi_1) = \sqrt{1 - \left(\frac{z - l_2 \sin(\psi_2)}{l_1}\right)^2}. \quad (282)$$

This form will be used in the calculation of ψ_1 in next part below.

Calculation of ψ_2

We desire now to find an expression of ψ_2 in function only of α_1 , α_2 and α_3 . a_{11} , a_{12} , a_{21} and a_{22} are defined in Section 6.4.

Suppose first $a_{22} - a_{12} \neq 0$, so $\sin(\alpha_3) - \sin(\alpha_2) \neq 0$ since $\psi_3 = \psi_2$. From (43), one may write

$$l_{22} = \frac{y - (l_0 + \cos(\psi_1) \sin(\alpha_1) l_{11} + \cos(\psi_2) \sin(\alpha_2) (l_{12} + l_2))}{\cos(\psi_3) \sin(\alpha_3) - \cos(\psi_2) \sin(\alpha_2)}. \quad (283)$$

In area A22, one has $l_{11} = l_1$, $l_{12} = 0$ and $\psi_3 = \psi_2$. Using it, one gets

$$l_{22} = \frac{y - (l_0 + \cos(\psi_1) \sin(\alpha_1) l_1 + \cos(\psi_2) \sin(\alpha_2) l_2)}{\cos(\psi_2) (\sin(\alpha_3) - \sin(\alpha_2))}. \quad (284)$$

and the first line of system (7) can be rewritten as

$$l_1 \cos(\psi_1) \cos(\alpha_1) + (l_2 - l_{22}) \cos(\psi_2) \cos(\alpha_2) + l_{22} \cos(\psi_3) \cos(\alpha_3) = x$$

$$l_1 \cos(\psi_1) \cos(\alpha_1) + l_2 \cos(\psi_2) \cos(\alpha_2) + l_{22} \cos(\psi_2) (\cos(\alpha_3) - \cos(\alpha_2)) = x \quad (285)$$

Injecting (284) inside (285), one gets

$$l_1 \cos(\psi_1) \cos(\alpha_1) + l_2 \cos(\psi_2) \cos(\alpha_2) + \left(\frac{y - (l_0 + \cos(\psi_1) \sin(\alpha_1) l_1 + \cos(\psi_2) \sin(\alpha_2) l_2)}{\cos(\psi_2) (\sin(\alpha_3) - \sin(\alpha_2))} \right) \times \cos(\psi_2) (\cos(\alpha_3) - \cos(\alpha_2)) = x$$

$$l_1 \cos(\psi_1) \cos(\alpha_1) + l_2 \cos(\psi_2) \cos(\alpha_2) + [y - (l_0 + \cos(\psi_1) \sin(\alpha_1) l_1 + \cos(\psi_2) \sin(\alpha_2) l_2)] \times \left(\frac{\cos(\alpha_3) - \cos(\alpha_2)}{\sin(\alpha_3) - \sin(\alpha_2)} \right) = x$$

$$l_1 \cos(\psi_1) \cos(\alpha_1) + l_2 \cos(\psi_2) \cos(\alpha_2) + (y - l_0) \hat{A}_2 - \cos(\psi_1) \sin(\alpha_1) l_1 \hat{A}_2 - \cos(\psi_2) \sin(\alpha_2) l_2 \hat{A}_2 = x \quad (286)$$

with $\hat{A}_2 = \left(\frac{\cos(\alpha_3) - \cos(\alpha_2)}{\sin(\alpha_3) - \sin(\alpha_2)} \right)$. Then

$$\begin{aligned} & (y - l_0) \hat{A}_2 - x \\ & + \cos(\psi_1) \left[l_1 \cos(\alpha_1) - \sin(\alpha_1) l_1 \hat{A}_2 \right] \\ & + \cos(\psi_2) \left[\cos(\alpha_2) l_2 - \sin(\alpha_2) l_2 \hat{A}_2 \right] = 0 \\ & \hat{D}_2 + \cos(\psi_1) \hat{B}_2 + \cos(\psi_2) \hat{C}_2 = 0 \end{aligned} \quad (287)$$

with

$$\hat{B}_2 = l_1 \cos(\alpha_1) - \sin(\alpha_1) l_1 \hat{A}_2 \quad (288)$$

$$\hat{C}_2 = \cos(\alpha_2) l_2 - \sin(\alpha_2) l_2 \hat{A}_2 \quad (289)$$

$$\hat{D}_2 = (y - l_0) \hat{A}_2 - x \quad (290)$$

In the case where $\sin(\alpha_3) = \sin(\alpha_2)$ (or numerically close), the same steps can be followed as described in Appendix D.4.3 to obtain

$$\bar{D}_2 + \cos(\psi_1) \bar{B}_2 + \cos(\psi_2) \bar{C}_2 = 0 \quad (291)$$

with

$$\bar{A}_2 = \left(\frac{\sin(\alpha_3) - \sin(\alpha_2)}{\cos(\alpha_3) - \cos(\alpha_2)} \right) \quad (292)$$

$$\bar{B}_2 = l_1 \sin(\alpha_1) - \cos(\alpha_1) l_1 \bar{A}_2 \quad (293)$$

$$\bar{C}_2 = l_2 \sin(\alpha_2) - \cos(\alpha_2) l_2 \bar{A}_2 \quad (294)$$

$$\bar{D}_2 = l_0 + x \bar{A}_2 - y. \quad (295)$$

Finally, the case $\alpha_3 = \alpha_2$ is excluded by Definition 2 because $\psi_2 = \psi_3$.

Let introduce A_2, B_2, C_2 and D_2 where

$$\begin{aligned} \text{if } \sin(\alpha_3) \neq \sin(\alpha_2), & A_2 = \hat{A}_2 \\ & B_2 = \hat{B}_2 \\ & C_2 = \hat{C}_2 \\ & D_2 = \hat{D}_2 \end{aligned}$$

$$\begin{aligned} \text{else, } & A_2 = \bar{A}_2 \\ & B_2 = \bar{B}_2 \\ & C_2 = \bar{C}_2 \\ & D_2 = \bar{D}_2. \end{aligned}$$

and so the general form

$$D_2 + \cos(\psi_1) B_2 + \cos(\psi_2) C_2 = 0. \quad (296)$$

Using (282) inside (296), one gets

$$C_2 \cos(\psi_2) + B_2 \sqrt{1 - \left(\frac{z - l_2 \sin(\psi_2)}{l_1} \right)^2} = -D_2 \quad (297)$$

Following steps described in Appendix E, an expression of ψ_2 in function of α_1 can be found using (297) and A_2, B_2, C_2 and D_2 .

The calculation of α_1 can be performed as it will be described in Appendix D.5.

D.4.3 Sub-case $\sin(\alpha_3) = \sin(\alpha_2)$

Suppose now $a_{22} - a_{12} = 0$ with $\alpha_2 \neq \alpha_3$, so $\sin(\alpha_3) = \sin(\alpha_2)$ since $\psi_3 = \psi_2$. From (44), one may write

$$l_{22} = \frac{x - (\cos(\psi_1) \cos(\alpha_1) l_{11} + \cos(\psi_2) \cos(\alpha_2) (l_{12} + l_2))}{\cos(\psi_3) \cos(\alpha_3) - \cos(\psi_2) \cos(\alpha_2)}. \quad (298)$$

In area A22, one has $l_{11} = l_1, l_{12} = 0$ and $\psi_3 = \psi_2$. Using it, one gets

$$l_{22} = \frac{x - (\cos(\psi_1) \cos(\alpha_1) l_1 + \cos(\psi_2) \cos(\alpha_2) l_2)}{\cos(\psi_2) (\cos(\alpha_3) - \cos(\alpha_2))}. \quad (299)$$

and the second line of system (7) can be rewritten as

$$\begin{aligned} & l_1 \cos(\psi_1) \sin(\alpha_1) + (l_2 - l_{22}) \cos(\psi_2) \sin(\alpha_2) \\ & + l_{22} \cos(\psi_2) \sin(\alpha_3) = y - l_0 \end{aligned}$$

$$\begin{aligned} & l_1 \cos(\psi_1) \sin(\alpha_1) + l_2 \cos(\psi_2) \sin(\alpha_2) \\ & + l_{22} \cos(\psi_2) (\sin(\alpha_3) - \sin(\alpha_2)) = y - l_0 \end{aligned} \quad (300)$$

Injecting (299) inside (300), one gets

$$\begin{aligned} & \frac{x - (\cos(\psi_1) \cos(\alpha_1) l_1 + \cos(\psi_2) \cos(\alpha_2) l_2)}{\cos(\psi_2) (\cos(\alpha_3) - \cos(\alpha_2))} \\ & \times \cos(\psi_2) (\sin(\alpha_3) - \sin(\alpha_2)) \\ & + l_1 \cos(\psi_1) \sin(\alpha_1) + l_2 \cos(\psi_2) \sin(\alpha_2) = y - l_0 \\ & (x - (\cos(\psi_1) \cos(\alpha_1) l_1 + \cos(\psi_2) \cos(\alpha_2) l_2)) \\ & \times \left(\frac{\sin(\alpha_3) - \sin(\alpha_2)}{\cos(\alpha_3) - \cos(\alpha_2)} \right) \\ & + l_1 \cos(\psi_1) \sin(\alpha_1) + l_2 \cos(\psi_2) \sin(\alpha_2) = y - l_0 \\ & (-\cos(\psi_1) \cos(\alpha_1) l_1 - \cos(\psi_2) \cos(\alpha_2) l_2) \bar{A}_2 \\ & + l_1 \cos(\psi_1) \sin(\alpha_1) + l_2 \cos(\psi_2) \sin(\alpha_2) \\ & = y - l_0 - x \bar{A}_2 \end{aligned} \quad (301)$$

with $\bar{A}_2 = \left(\frac{\sin(\alpha_3) - \sin(\alpha_2)}{\cos(\alpha_3) - \cos(\alpha_2)} \right)$. Then

$$\begin{aligned} & [l_0 + x \bar{A}_2 - y] + \cos(\psi_1) [l_1 \sin(\alpha_1) - \cos(\alpha_1) l_1 \bar{A}_2] \\ & + \cos(\psi_2) [l_2 \sin(\alpha_2) - \cos(\alpha_2) l_2 \bar{A}_2] = 0 \\ & \bar{D}_1 + \cos(\psi_1) \bar{B}_2 + \cos(\psi_2) \bar{C}_2 = 0 \end{aligned} \quad (302)$$

where

$$\bar{B}_2 = l_1 \sin(\alpha_1) - \cos(\alpha_1) l_1 \bar{A}_2 \quad (303)$$

$$\bar{C}_2 = l_2 \sin(\alpha_2) - \cos(\alpha_2) l_2 \bar{A}_2 \quad (304)$$

$$\bar{D}_2 = l_0 + x \bar{A}_2 - y. \quad (305)$$

From this point, using $\bar{B}_2, \bar{C}_2, \bar{D}_2$ instead of $\hat{B}_2, \hat{C}_2, \hat{D}_2$, one can follow the steps described in Appendix D.4.1.

D.5 Calculation of α_1

Based on the system (7), let's define the three equations

$$E_X = (l_{11} \cos(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \cos(\alpha_2) \cos(\psi_2) + l_{22} \cos(\alpha_3) \cos(\psi_3) - x)^2 \quad (306)$$

$$E_Y = (l_{11} \sin(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \sin(\alpha_2) \cos(\psi_2) + l_{22} \sin(\alpha_3) \cos(\psi_3) - y + l_0)^2 \quad (307)$$

$$E_Z = (l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) - z)^2 \quad (308)$$

Following steps described in Appendix B.1, the first or the second line of (7) is used to find an expression of l_{11} and l_{12} in function of α_1 . Thus, the expression of l_{11} and l_{12} make automatically E_X or E_Y equal to zero. Then, the third line of (7) is used to find an expression of ψ_2 and ψ_1 in both areas A21 and A22, see Appendices D.3 and D.4.2, thus value of ψ_2 and ψ_1 make also automatically E_Z equal to zero.

Then, using the first or second line of (7) and injecting inside

- (269)-(270)-(271)-(272)-(274) for area A21,
- (269)-(270)-(278)-(280)-(297) for area A22,

a solution of α_1 can be found. Since E_Z and E_X or E_Y are equal to zero, the same solution can be found by solving the following equation (309):

$$0 = E_X^2 + E_Y^2 + E_Z^2. \quad (309)$$

In practice, this solution will be found numerically using an optimization: using (309) allows to tests the proposed solution of α_1 independently of the expression of l_{11} , l_{12} , ψ_1 or ψ_2 .

Moreover, the solution must respect the following conditions:

1. The evaluation of the lengths must be bounded such that $(l_{11}, l_{12}) \in [0, l_1]^2$ and $(l_{21}, l_{22}) \in [0, l_2]^2$;
2. The evaluation of the forces must be strictly positive:
 - if area A21, $T_2^{S1} > 0$ and $T_3^{S1} > 0$,
 - if area A22, $T_1^{S1} > 0$, $T_2^{S1} > 0$ and $T_3^{S1} > 0$.
3. The Definition 3 is respected for the two elements, *i.e.*
 - the element 2 is sliding so $\vec{F}_2 \cdot \vec{B}_2 \vec{S} < 0$ and $\vec{F}_2 \cdot \vec{B}_2 \vec{R} < 0$, where $\vec{F}_2 \cdot \vec{B}_2 \vec{S} < 0$ is equivalent to $\vec{F}_2 \cdot \vec{S} \vec{B}_2 > 0$;
 - if the element 1 is not fixed:

- in area A21, the element 1 is in contact with A , so $\vec{F}_1 \cdot \vec{B}_1 \vec{S} < 0$;
- in area A22, the element 1 is in contact with S , so $\vec{F}_1 \cdot \vec{B}_1 \vec{A} < 0$, which is equivalent to $\vec{F}_1 \cdot \vec{A} \vec{B}_1 > 0$,

where $\vec{A} \vec{B}_1$, $\vec{B}_1 \vec{S}$, $\vec{S} \vec{B}_2$ and $\vec{B}_2 \vec{R}$ have been expressed in Section 4.1.

E Study of ψ_2 in area A12 and A22

Consider the parameter A_i , B_i , C_i and D_i for $i \in \{1, 2\}$ where $i = 1$ in area A12 and $i = 2$ in area A22. Let's find a solution of ψ_2 for areas A12 and A22.

E.1 General case: $B_i \neq 0$ and $C_i \neq 0$

Consider first the general case where $B_i \neq 0$ and $C_i \neq 0$. From (252) in area A12 or (297) in area A22, one has

$$C_i \cos(\psi_i) + B_i \sqrt{1 - \left(\frac{z - l_1 \sin(\psi_i)}{l_2}\right)^2} = -D_i \quad (310)$$

$$\sqrt{1 - \left(\frac{z - l_1 \sin(\psi_i)}{l_2}\right)^2} = -\frac{(C_i \cos(\psi_i) + D_i)}{B_i} \quad (311)$$

which lead to

$$1 - \left(\frac{z - l_1 \sin(\psi_i)}{l_2}\right)^2 = \frac{(C_i \cos(\psi_i) + D_i)^2}{B_i^2}$$

$$\begin{aligned} l_2^2 - \left(z^2 - 2l_1 z \sin(\psi_i) + l_1^2 \sin^2(\psi_i)\right) \\ = \frac{l_2^2}{B_i^2} \left(C_i^2 \cos^2(\psi_i) + 2C_i D_i \cos(\psi_i) + D_i^2\right) \end{aligned}$$

$$\begin{aligned} l_2^2 - z^2 + 2l_1 z \sin(\psi_i) - l_1^2 \sin^2(\psi_i) \\ = \frac{l_2^2}{B_i^2} C_i^2 \cos^2(\psi_i) + 2C_i D_i \frac{l_2^2}{B_i^2} \cos(\psi_i) + \frac{l_2^2}{B_i^2} D_i^2. \end{aligned} \quad (312)$$

Let's put $X = \sin(\psi_i)$ and remark that $\cos(\psi_i) = \sqrt{1 - X^2}$ since $\psi_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, (312) becomes

$$\begin{aligned} l_2^2 - z^2 + 2l_1 z X - l_1^2 X^2 \\ = \frac{l_2^2}{B_i^2} C_i^2 (1 - X^2) + 2C_i D_i \frac{l_2^2}{B_i^2} \sqrt{1 - X^2} \\ + \frac{l_2^2}{B_i^2} D_i^2 \end{aligned}$$

$$\begin{aligned} & \left[l_2^2 - z^2 - \frac{l_2^2}{B_i^2} (C_i^2 + D_i^2) \right] + (2l_1 z) X \\ & + \left(\frac{l_2^2}{B_i^2} C_i^2 - l_1^2 \right) X^2 \\ & = 2C_i D_i \frac{l_2^2}{B_i^2} \sqrt{1 - X^2} \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \left[(l_2^2 - z^2) \frac{B_i^2}{l_2^2} - (C_i^2 + D_i^2) \right] + \left(l_1 \frac{B_i^2}{l_2^2} z \right) X \\ & + \frac{1}{2} \left(C_i^2 - \frac{B_i^2}{l_2^2} l_1^2 \right) X^2 \\ & = C_i D_i \sqrt{1 - X^2} \end{aligned}$$

$$E_i + F_i X + G_i X^2 = C_i D_i \sqrt{1 - X^2} \quad (313)$$

where

$$E_i = \frac{1}{2} \left[(l_2^2 - z^2) \frac{B_i^2}{l_2^2} - (C_i^2 + D_i^2) \right] \quad (314)$$

$$F_i = l_1 \frac{B_i^2}{l_2^2} z \quad (315)$$

$$G_i = \frac{1}{2} \left(C_i^2 - \frac{B_i^2}{l_2^2} l_1^2 \right) \quad (316)$$

The equality (313) can be squared up to obtain

$$\begin{aligned} & E_i + F_i X + G_i X^2 = C_i D_i \sqrt{1 - X^2} \\ & (E_i + F_i X + G_i X^2)^2 = (C_i D_i)^2 - (C_i D_i)^2 X^2 \end{aligned}$$

$$\begin{aligned} & E_i^2 + F_i^2 X^2 + G_i^2 X^4 + 2E_i F_i X \\ & + 2E_i G_i X^2 + 2F_i G_i X^3 \\ & = (C_i D_i)^2 - (C_i D_i)^2 X^2 \end{aligned}$$

$$\begin{aligned} & \left(E_i^2 - (C_i D_i)^2 \right) + (2E_i F_i) X \\ & + \left(F_i^2 + (C_i D_i)^2 + 2E_i G_i \right) X^2 \\ & + (2F_i G_i) X^3 + (G_i^2) X^4 = 0 \end{aligned}$$

$$L_i + K_i X + J_i X^2 + I_i X^3 + H_i X^4 = 0 \quad (317)$$

where

$$H_i = G_i^2 \quad (318)$$

$$I_i = 2F_i G_i \quad (319)$$

$$J_i = F_i^2 + (C_i D_i)^2 + 2E_i G_i \quad (320)$$

$$K_i = 2E_i F_i \quad (321)$$

$$L_i = E_i^2 - (C_i D_i)^2 \quad (322)$$

(317) is a polynomial of degree 4, which can be solved to find the solution of $X = \sin(\psi_i)$. For S_X

the set of real solutions of (317), the solution of ψ_i is inside the set S_{ψ_i} such that

$$\begin{aligned} & \psi_i \in S_{\psi_i} \\ & S_{\psi_i} = \{ \text{asin}(X_k) \mid X_k \in S_X, X_k \notin \mathbb{C}, |X_k| \leq 1 \}. \end{aligned} \quad (323)$$

If S_{ψ_i} is not a singleton, the true solution of ψ_i will be found with a test in the same time that the value of α_1 .

E.2 Case $B_i = 0$ and $C_i \neq 0$

If $B_i = 0$, (310) becomes

$$C_i \cos(\psi_i) = -D_i. \quad (324)$$

To obtain a common form between all the sub-case of this study, we put (324) on the following form:

$$\begin{aligned} & C_i \cos(\psi_i) = -D_i \\ & C_i^2 \cos^2(\psi_i) = D_i^2 \\ & C_i^2 (1 - \sin^2(\psi_i)) = D_i^2 \\ & -C_i^2 X^2 + (C_i^2 - D_i^2) = 0 \end{aligned}$$

with $X = \sin(\psi_i)$ and one can define a 4-degree polynomial $L_i + K_i X + J_i X^2 + I_i X^3 + H_i X^4 = 0$ where

$$H_i = I_i = K_i = 0 \quad (325)$$

$$J_i = -C_i^2 \quad (326)$$

$$L_i = C_i^2 - D_i^2 \quad (327)$$

and the solution of ψ_i can be found as exposed in Appendix E.1.

E.3 Case: $B_i \neq 0$ and $C_i = 0$

If $C_i = 0$, (310) becomes

$$\begin{aligned} & C_i \cos(\psi_i) + B_i \sqrt{1 - \left(\frac{z - l_1 \sin(\psi_i)}{l_2} \right)^2} = -D_i \\ & \sqrt{1 - \left(\frac{z - l_1 \sin(\psi_i)}{l_2} \right)^2} = -\frac{D_i}{B_i}. \end{aligned} \quad (328)$$

The equality (328) can be squared up to obtain

$$\begin{aligned} & 1 - \left(\frac{z - l_1 \sin(\psi_i)}{l_2} \right)^2 = \frac{D_i^2}{B_i^2} \\ & z^2 - 2l_1 z \sin(\psi_i) + l_1^2 \sin^2(\psi_i) = l_2^2 \left(1 - \frac{D_i^2}{B_i^2} \right) \\ & \left[z^2 - l_2^2 \left(1 - \frac{D_i^2}{B_i^2} \right) \right] - 2l_1 z X + l_1^2 X^2 = 0 \end{aligned} \quad (329)$$

with $X = \sin(\psi_i)$. To take the same form that the solution found in Appendix E.1, one can define a 4-degree polynomial

$$L_i + K_i X + J_i X^2 + I_i X^3 + H_i X^4$$

where

$$H_i = 0 \quad (330)$$

$$I_i = 0 \quad (331)$$

$$J_i = l_1^2 \quad (332)$$

$$K_i = -2l_1 z \quad (333)$$

$$L_i = z^2 - l_2^2 \left(1 - \frac{D_i^2}{B_i^2}\right) \quad (334)$$

and the solution of ψ_i can be found as exposed in Appendix E.1.

F Calculation in area A0

F.1 Referential \mathcal{R}_0

The two elements are sliding in the area A0. Thus, similarly to area A1 and A2, we have observe that the system is much more easier to solve if $\psi_{F1} = 0$ or $\psi_{F2} = 0$ to simplify the system (12) and find a solution for (201) independent of the elevation angles ψ_1 and ψ_2 . Since choosing between the referential \mathcal{R}_1 and \mathcal{R}_2 exposed in Appendix F and D doesn't matter much, we decide to perform a rotation of the referential \mathcal{R} around the vertical axis $O\vec{y}$ of an angle $\mu_1 = \text{atan2}(F_{1z}, F_{1x})$ with $F_{1|\mathcal{R}} = [F_{1x} \ F_{1y} \ F_{1z}]^T$ to obtain the referential \mathcal{R}_0 , similar to the referential \mathcal{R}_1 . The characteristics of the referential \mathcal{R}_0 are described in the Definition 4. In \mathcal{R}_0 , one has $\psi_{F1|\mathcal{R}_0} = 0$. Once a solution will be found in \mathcal{R}_0 , the angles and lengths will be re-evaluated in \mathcal{R} using for example Definition 5.

In this section, since all the calculations are performed in \mathcal{R}_0 , the notations are lightened such that a parameter $\beta_{i|\mathcal{R}_0}$ is simply written β_i .

F.2 Relation between $\alpha_1, \alpha_2, \psi_1$ and ψ_2

In area A0, the element 1 is sliding such that it correspond to the Definition 3 (1). Thus, one has $T_1 = T_2$ and from Theorem 2

$$\cos(\psi_1) \sin(\phi_1 - \alpha_1) = \cos(\psi_2) \sin(\phi_1 - \alpha_2) \quad (335)$$

Moreover, since $\psi_{F1} = 0$, the third line of system (12) becomes

$$\begin{aligned} -\sin(\psi_1) T_1 + \sin(\psi_2) T_2 &= 0 \\ \sin(\psi_2) &= \sin(\psi_1) \end{aligned} \quad (336)$$

Since $\psi_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it can be deduced from (336) that in area A0:

$$\psi_1 = \psi_2. \quad (337)$$

Injecting (337) inside(335), one has $\sin(\phi_1 - \alpha_1) = \sin(\phi_1 - \alpha_2)$, leading to two possible solutions: $\alpha_1 = \alpha_2$ or

$$\begin{aligned} \alpha_1 - \phi_1 &= \pi - (\alpha_2 - \phi_1) \\ \alpha_2 &= -\alpha_1 + 2\phi_1 + \pi \end{aligned} \quad (338)$$

However, since $\psi_{F1} = 0$, *i.e.* there is no force perpendicular to the plan for element 1, the only solution respecting the condition of Definition 3 (1) is (338), $\alpha_1 = \alpha_2$ corresponding to the case where the cables l_{12} and l_{21} form a straight line where the sliding element cannot be immobilized, excluded by Definition 2 since $\psi_1 = \psi_2$.

Two cases must now be considered: $\psi_{F2} = 0$ and $\psi_{F2} \neq 0$. The two cases will be studied in the next sections.

F.3 Case 1: $\psi_{F2} = 0$

Consider first the specific case where $\psi_{F2} = 0$. In area A0, the element 2 is sliding such that it correspond to the Definition 3 (1). Thus, one has $T_2 = T_3$ and from Theorem 2

$$\cos(\psi_2) \sin(\phi_2 - \alpha_2) = \cos(\psi_3) \sin(\phi_2 - \alpha_3). \quad (339)$$

Moreover, if $\psi_{F2} = 0$, the third line of system (12) for $i = 2$ becomes

$$\begin{aligned} -\sin(\psi_2) T_2 + \sin(\psi_3) T_3 &= 0 \\ \sin(\psi_2) &= \sin(\psi_3) \end{aligned} \quad (340)$$

Since $\psi_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\psi_3 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it can be deduced from (340) that in area A0:

$$\psi_3 = \psi_2. \quad (341)$$

Injecting (341) inside(339), one has $\sin(\phi_2 - \alpha_2) = \sin(\phi_2 - \alpha_3)$, leading to two possible solutions: $\alpha_2 = \alpha_3$ (exclude by Definition 2 since $\psi_3 = \psi_2$) or

$$\begin{aligned} \alpha_2 - \phi_2 &= \pi - (\alpha_3 - \phi_2) \\ \alpha_3 &= -\alpha_2 + 2\phi_2 + \pi \end{aligned} \quad (342)$$

From (336) and (341), one deduces that

$$\psi_1 = \psi_2 = \psi_3. \quad (343)$$

Using it inside the third line of (7), one gets

$$\begin{aligned} l_{11} \sin(\psi_1) + (l_{12} + l_{21}) \sin(\psi_2) + l_{22} \sin(\psi_3) &= z \\ (l_{11} + l_{12} + l_{21} + l_{22}) \sin(\psi_1) &= z \\ \sin(\psi_1) &= \frac{z}{(l_1 + l_2)} \end{aligned} \quad (344)$$

Let's find first a solution to α_1 . Using the PFD (12) for $i \in \{1, 2\}$ and following the steps described in

Appendix (A), one gets the result (202) in Appendix (A.3):

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin(\alpha_2 - \alpha_3) \\ = \sin(\phi_2 - \alpha_3) \sin(\alpha_1 - \alpha_2). \end{aligned} \quad (345)$$

In a first time, we inject (338) in (345). Following steps described in Appendix C.4.1 for area A12, one gets

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(\Gamma_1 + (\phi_1 - \alpha_3)) \\ = -\sin(\phi_2 - \alpha_3) \sin(2\Gamma_1) \end{aligned} \quad (346)$$

with $\Delta\phi = \phi_2 - \phi_1$ and $\Gamma_1 = \phi_1 - \alpha_1$.

Combining now (338) and (342), one gets

$$\begin{aligned} \alpha_3 &= -\alpha_2 + 2\phi_2 + \pi \\ &= -(-\alpha_1 + 2\phi_1 + \pi) + 2\phi_2 + \pi \\ &= \alpha_1 + 2(\phi_2 - \phi_1) \\ &= \alpha_1 + 2\Delta\phi. \end{aligned} \quad (347)$$

Injecting (347) inside (346), one has

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(\Gamma_1 + (\phi_1 - \alpha_3)) \\ = \sin(\phi_2 - \alpha_3) \sin(-2\Gamma_1) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(\phi_1 - \alpha_1 + \phi_1 - (\alpha_1 + 2\Delta\phi)) \\ = \sin(\phi_2 - (\alpha_1 + 2\Delta\phi)) \sin(-2\Gamma_1) \end{aligned} \quad (348)$$

Let's study first the sub-case where $\sin(\Delta\phi) = 0$ in Appendix F.3.1 before study the general case in Appendix F.3.2.

F.3.1 Case $\sin(\Delta\phi) = 0$

Consider first the case where $\Delta\phi = \pi k$ with $k \in \mathbb{Z}$, *i.e.* $\sin(\Delta\phi) = 0$. One has from (347) that $\alpha_1 = \alpha_3$. (342) becomes

$$\frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(2(\phi_1 - \alpha_1)) = \sin(\phi_2 - \alpha_1) \sin(-2\Gamma_1)$$

$$\frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(2(\phi_1 - \alpha_1)) = \sin(\phi_2 - \alpha_1) \sin(-2\Gamma_1)$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(2\Gamma_1) &= \sin(\phi_2 - \phi_1 + \phi_1 - \alpha_1) \\ &\quad \times \sin(-2\Gamma_1) \end{aligned}$$

$$\frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) = -\sin(\Delta\phi + \Gamma_1).$$

Let's define $s \in \{-1, 1\}$ with $s = -1$ if $\Delta\phi = 0$, $s = 1$ else. one has

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) &= s \sin(\Gamma_1) \\ F_{1xy} &= s F_{2xy} \\ \cos(\psi_{F_1}) F_1 &= s \cos(\psi_{F_2}) F_2 \end{aligned}$$

and since we are in $\mathcal{R}_0 = \mathcal{R}_1$ (so $\psi_{F_1} = 0$) and in the specific case where $\psi_{F_2} = 0$, one has

$$F_1 = s F_2. \quad (349)$$

Since $F_1 > 0$ and $F_2 > 0$, case $s = -1$ is impossible, (349) proves that the area A0 doesn't exist when $\Delta\phi = 0$, corresponding to $\phi_2 = \phi_1$. Since $\psi_{F_1} = \psi_{F_2} = 0$, one deduce area A0 is not possible when $\frac{\vec{F}_1}{\|\vec{F}_1\|} = \frac{\vec{F}_2}{\|\vec{F}_2\|}$ or simply $\vec{F}_1 = \vec{F}_2$. Moreover, (349) show that the area A0 doesn't exist when $\sin(\Delta\phi) = 0$ and $F_1 \neq F_2$. Since case $\Delta\phi = 0$ is already treated, it corresponding here to $\Delta\phi = \pi \Leftrightarrow \phi_2 = \pi - \phi_1$ *i.e.* when $\frac{\vec{F}_1}{\|\vec{F}_1\|} = -\frac{\vec{F}_2}{\|\vec{F}_2\|}$ and $F_1 \neq F_2$.

Consider case where $F_1 = F_2$ and $\Delta\phi = \pi k \forall k \in \mathbb{Z}^*$: one has $\alpha_1 = \alpha_3 = -\alpha_2$ from (338) and (347), and $\psi_1 = \psi_2 = \psi_3$ from (343). Thus, one gets from the first line of system (7):

$$\begin{aligned} x &= l_{11} \cos(\alpha_1) \cos(\psi_1) + (l_{12} + l_{21}) \cos(\alpha_2) \cos(\psi_2) \\ x &= (l_{11} + l_{12} + l_{21} + l_{22}) \cos(\alpha_1) \cos(\psi_1) \end{aligned}$$

$$\cos(\alpha_1) = \frac{x}{(l_1 + l_2) \cos(\psi_1)} \quad (350)$$

where the value of ψ_1 is known using (344). The sign of α_1 can be deduced from the second line of (7).

F.3.2 Case $\sin(\Delta\phi) \neq 0$

Suppose now $\Delta\phi \neq \pi k$ with $k \in \mathbb{Z}$, so $\sin(\Delta\phi) \neq 0$. (348) becomes

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(2\phi_1 - 2\alpha_1 - 2\phi_2 + 2\phi_1) \\ = \sin(\phi_2 - \alpha_1 - 2\phi_2 + 2\phi_1) \sin(-2\Gamma_1) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(-2\alpha_1 - 2\phi_2 + 4\phi_1) \\ = \sin(-\alpha_1 - \phi_2 + 2\phi_1) \sin(-2\Gamma_1) \end{aligned}$$

$$\begin{aligned} \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(2(-\alpha_1 - \phi_2 + 2\phi_1)) \\ = \sin(-\alpha_1 - \phi_2 + 2\phi_1) \sin(-2\Gamma_1) \end{aligned}$$

$$\begin{aligned}
& 2 \frac{F_{1xy}}{F_{2xy}} \sin(\Gamma_1) \sin(-\alpha_1 - \phi_2 + 2\phi_1) \\
& \times \cos(-\alpha_1 - \phi_2 + 2\phi_1) \\
& = -2 \sin(-\alpha_1 - \phi_2 + 2\phi_1) \sin(\Gamma_1) \cos(\Gamma_1)
\end{aligned}$$

$$\frac{F_{1xy}}{F_{2xy}} \cos((\phi_1 - \alpha_1) - (\phi_2 - \phi_1)) = -\cos(\Gamma_1) \quad (351)$$

and by using $\Delta\phi = \phi_2 - \phi_1$ and $\Gamma_1 = \phi_1 - \alpha_1$, one gets

$$\begin{aligned}
& \frac{F_{1xy}}{F_{2xy}} \cos(\Gamma_1 - \Delta\phi) = -\cos(\Gamma_1) \\
& \frac{F_{1xy}}{F_{2xy}} [\cos(\Gamma_1) \cos(\Delta\phi) \\
& + \sin(\Gamma_1) \sin(\Delta\phi)] = -\cos(\Gamma_1) \\
& \cos(\Delta\phi) + \tan(\Gamma_1) \sin(\Delta\phi) = -\frac{F_{2xy}}{F_{1xy}} \\
& \tan(\Gamma_1) = -\frac{\frac{F_{2xy}}{F_{1xy}} + \cos(\Delta\phi)}{\sin(\Delta\phi)} \\
& \tan(\phi_1 - \alpha_1) = -\frac{\frac{F_{2xy}}{F_{1xy}} + \cos(\Delta\phi)}{\sin(\Delta\phi)} \\
& \tan(\alpha_1 - \phi_1) = \frac{\frac{F_{2xy}}{F_{1xy}} + \cos(\Delta\phi)}{\sin(\Delta\phi)}. \quad (352)
\end{aligned}$$

From (352), one deduces α_1 can take only two values α_{1k} for $k \in \{1, 2\}$ where

$$\alpha_{1k} = \phi_1 + \pi(k-1) + \text{atan} \left(\frac{\frac{F_{2xy}}{F_{1xy}} + \cos(\Delta\phi)}{\sin(\Delta\phi)} \right) \quad (353)$$

As shown in this Appendix, others angles can be deduced with α_1 and the lengths l_{ij} , $(i, j) \in \{1, 2\}^2$ can be found following steps described in Appendix B.3. The valid value of α_1 between the two solutions proposed by (353) can be found by evaluating the other parameters and keeping the only solution respecting the conditions $(l_{11}, l_{12}) \in [0, l_1]^2$, $(l_{21}, l_{22}) \in [0, l_2]^2$, $T_1 > 0$, $T_2 > 0$, $T_3 > 0$ and the conditions of Definition 3:

- the element 1 is sliding, so $\vec{F}_1 \cdot \overrightarrow{B_1 A} < 0$ and $\vec{F}_1 \cdot \overrightarrow{B_1 S} < 0$,
- the element 2 is sliding, so $\vec{F}_2 \cdot \overrightarrow{B_2 S} < 0$ and $\vec{F}_2 \cdot \overrightarrow{B_2 R} < 0$,

where $\overrightarrow{AB_1}$, $\overrightarrow{B_1 S}$, $\overrightarrow{SB_2}$ and $\overrightarrow{B_2 S}$ have been expressed in Section 4.1.

F.4 Case 2: $\psi_{F2} \neq 0$

If $\psi_{F2} \neq 0$, from the third line of system (12) for $i = 2$ and in opposite with (340) of the case 1 developed in Appendix F.3, one has $\sin(\psi_3) \neq \sin(\psi_2)$. Let's so find an expression for the parameters based on this condition.

F.4.1 Relation between α_1 and α_3

Let's find first a relation between α_1 and α_3 . Using the PFD (12) for $i = \{1, 2\}$ and following the steps described in Appendix (A), one gets the result (202) in Appendix (A.3):

$$\begin{aligned}
& \frac{F_{1xy}}{F_{2xy}} \sin(\phi_1 - \alpha_1) \sin(\alpha_2 - \alpha_3) \\
& = \sin(\phi_2 - \alpha_3) \sin(\alpha_1 - \alpha_2). \quad (354)
\end{aligned}$$

Then, since the referential \mathcal{R}_0 respect the same condition than the referential \mathcal{R}_1 , we can follow the same step than in Appendix C.4.1 for area A12 and find the relation

$$\tan(\Omega_1) = -\frac{2 \frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2 \frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)}. \quad (355)$$

if $\xi = \frac{2F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1 \neq 0$, so Assumption A9 is respected.

From (355), one deduce α_3 can take the two following values α_{3k} for $k \in \{1, 2\}$ with

$$\alpha_{3k} = \phi_2 + \pi(k-1) + \alpha_{30}$$

with

$$\alpha_{30} = \text{atan} \left(\frac{2 \frac{F_{2xy}}{F_{1xy}} \sin(\Delta\phi) + \tan(\Gamma_1)}{\left(2 \frac{F_{2xy}}{F_{1xy}} \cos(\Delta\phi) + 1\right)} \right) \quad (356)$$

F.4.2 Study of ψ_2 and ψ_3

In area A0, the element 2 is sliding such that it correspond to the Definition 3 (1). Thus, one has $T_2 = T_3$ and from Theorem 2

$$\cos(\psi_3) = \cos(\psi_2) \frac{\sin(\phi_2 - \alpha_2)}{\sin(\phi_2 - \alpha_3)}. \quad (357)$$

Moreover, if $\psi_{F2} \neq 0$, the third line of system (12) with $i = 2$ becomes

$$\begin{aligned}
& -\sin(\psi_2) T_2 + \sin(\psi_3) T_3 + F_2 \sin(\psi_{F2}) = 0 \\
& (-\sin(\psi_2) + \sin(\psi_3)) T_2 = -F_2 \sin(\psi_{F2}) \quad (358)
\end{aligned}$$

Remind that for $i \in \{1, 2, 3\}$, $T_{ixy} = \cos(\psi_i) T_i$. Thus, 358 becomes

$$\sin(\psi_3) = -\frac{F_2}{T_{2xy}} \sin(\psi_{F2}) \cos(\psi_2) + \sin(\psi_2). \quad (359)$$

Using (188) in Appendix A and 338, T_{2xy} can be expressed in \mathcal{R}_0 as

$$\begin{aligned}
\bar{T}_{2xy|\mathcal{R}_0} &= -\frac{\sin(\phi_1 - \alpha_1)}{\sin(\alpha_2 - \alpha_1)} F_{1xy} \\
&= -\frac{\sin(\phi_1 - \alpha_1)}{\sin(-\alpha_1 + 2\phi_1 + \pi - \alpha_1)} F_{1xy} \\
&= -\frac{\sin(\phi_1 - \alpha_1)}{\sin(2(\phi_1 - \alpha_1) + \pi)} F_{1xy} \\
&= \frac{\sin(\phi_1 - \alpha_1)}{\sin(2(\phi_1 - \alpha_1))} F_{1xy} \\
&= \frac{\sin(\phi_1 - \alpha_1)}{2\sin(\phi_1 - \alpha_1)\cos(\phi_1 - \alpha_1)} F_{1xy} \\
&= \frac{F_{1xy}}{2\cos(\phi_1 - \alpha_1)} \quad (360)
\end{aligned}$$

so depend only of the angle α_1 . With (360) to evaluate $\bar{T}_{2xy|\mathcal{R}_0}$, let's back to 359.

By squaring the two sides of 359 and since $\sin(\psi_3)^2 = 1 - \cos(\psi_3)^2$, one gets

$$\begin{aligned}
1 - \cos(\psi_3)^2 &= \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2 \cos(\psi_2)^2 + \sin(\psi_2)^2 \\
&\quad - 2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2}) \cos(\psi_2) \sin(\psi_2). \quad (361)
\end{aligned}$$

Using now (357), (361) can be expressed only with ψ_2 :

$$\begin{aligned}
1 - \cos(\psi_2)^2 \frac{\sin(\phi_2 - \alpha_2)^2}{\sin(\phi_2 - \alpha_3)^2} &= \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2 \cos(\psi_2)^2 + \left(1 - \cos(\psi_2)^2\right) \\
&\quad - 2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2}) \cos(\psi_2) \sin(\psi_2) \\
\left[1 - \frac{\sin(\phi_2 - \alpha_2)^2}{\sin(\phi_2 - \alpha_3)^2} - \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2\right] \cos(\psi_2)^2 &= -2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2}) \cos(\psi_2) \sin(\psi_2) \\
\left[1 - \frac{\sin(\phi_2 - \alpha_2)^2}{\sin(\phi_2 - \alpha_3)^2} - \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2\right] \cos(\psi_2) &= -2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2}) \sin(\psi_2) \quad (362)
\end{aligned}$$

Since $\sin(\psi_{F2}) \neq 0$, (362) becomes

$$\tan(\psi_2) = \frac{\left[1 - \frac{\sin(\phi_2 - \alpha_2)^2}{\sin(\phi_2 - \alpha_3)^2} - \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2\right]}{-2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})}. \quad (363)$$

Remind that $\alpha_3 \in \{\alpha_{31}, \alpha_{32}\}$ with $\alpha_{31} = \alpha_{30}$ and $\alpha_{32} = \alpha_{30} + \pi$. Since $\sin(\phi_2 - \alpha_{30} + \pi)^2 = \sin(\phi_2 - \alpha_{30})^2$, one may write

$$\tan(\psi_2) = \frac{\left[1 - \frac{\sin(\phi_2 - \alpha_2)^2}{\sin(\phi_2 - \alpha_{30})^2} - \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2\right]}{-2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})}. \quad (364)$$

and one has $\psi_2 \in \{\psi_{21}, \psi_{22}\}$ where ψ_{2k} can be expressed as

$$\begin{aligned}
\psi_{2k} &= (k-1)\pi \\
&\quad + \operatorname{atan}\left(\frac{\left[1 - \frac{\sin(\phi_2 - \alpha_2)^2}{\sin(\phi_2 - \alpha_{30})^2} - \left(\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})\right)^2\right]}{-2\frac{F_2}{\bar{T}_{2xy|\mathcal{R}_0}} \sin(\psi_{F2})}\right). \quad (365)
\end{aligned}$$

Then, ψ_3 can be obtained using (357) such as

$$\psi_3 \in \left\{ \left\{ S_{\psi_3}^{i,j} / j \in \{1, 2\} \right\} / i \in \{1, 2\} \right\} \quad (366)$$

$$\begin{aligned}
S_{\psi_3}^{i,j} &= \{ \psi_{30}(\psi_{2j}, \alpha_{3i}, k) \mid k \in \{1, 2\} \}, \\
\left| \cos(\psi_{2j}) \frac{\sin(\phi_2 - \alpha_2)}{\sin(\phi_2 - \alpha_{3i})} \right| &\leq 1 \quad (367)
\end{aligned}$$

where

$$\psi_{30}(\psi_2, \alpha_3, k) = (3 - 2k) \operatorname{acos}\left(\cos(\psi_2) \frac{\sin(\phi_2 - \alpha_2)}{\sin(\phi_2 - \alpha_3)}\right) \quad (368)$$

Note there are two solutions for ψ_2 , so four possible solutions for ψ_3 . Since α_3 has also two solutions, one may have four possible solutions for ψ_2 and eight for ψ_3 .

F.4.3 Calculation of α_1

Following steps described in Appendix B.3, the first and the second lines of (7) are used to find an expression of l_{11} and l_{12} in function of α_1 . Then, using the last line of (7) and injecting inside (337)-(338)-(356)-(365)-(366) and the expression $\bar{T}_{2xy|\mathcal{R}_0} = \frac{F_{1xy}}{2\cos(\phi_1 - \alpha_1)}$ for T_{2xy} , several solutions of α_1 can be found (since there are several solutions for some parameters like ψ_2 or ψ_3). The current solution of α_1 can be found by evaluating the other parameters and keeping the only solution respecting the following conditions:

1. The evaluation of the lengths must be bounded such that $(l_{11}, l_{12}) \in [0, l_1]^2$ and $(l_{21}, l_{22}) \in [0, l_2]^2$.
2. The evaluation of the forces must be strictly positive: $T_1^{S1} > 0$, $T_2^{S1} > 0$ and $T_3^{S1} > 0$.
3. $-\sin(\psi_2) T_2^{S1} + \sin(\psi_3) T_3^{S1} + F_{2xy} \sin(\psi_{F2}) = 0$ where T_{2xy}^{S1} is evaluated using (20) and not (360).

4. The Definition 3 is respected for the two elements, *i.e.*

- The element 1 is sliding, so $\vec{F}_1 \cdot \overrightarrow{B_1 A} < 0$ and $\vec{F}_1 \cdot \overrightarrow{B_1 S} < 0$,
- The element 2 is sliding, so $\vec{F}_2 \cdot \overrightarrow{B_2 S} < 0$ and $\vec{F}_2 \cdot \overrightarrow{B_2 R} < 0$.

Remind that $\overrightarrow{AB_1}$, $\overrightarrow{B_1 S}$, $\overrightarrow{SB_2}$ and $\overrightarrow{B_2 R}$ have been expressed in Section 4.1. In practice, the solution of α_1 will be found numerically.

G Calculation in area D

G.1 Referential \mathcal{R}_D

In the areas D (D1, D2 and D3), all the elements are in contact with a stop. Thus, the umbilical between the anchor and the ROV becomes the triangle ASR with $AS = l_1$ and $SR = l_2$. Let's define the force $\vec{F}_{S|\mathcal{R}}$ applied at the point S such as

$$\vec{F}_{S|\mathcal{R}} = \vec{F}_{1|\mathcal{R}} \quad \text{in area D1} \quad (369)$$

$$\vec{F}_{S|\mathcal{R}} = \vec{F}_{\Sigma|\mathcal{R}} = \vec{F}_{1|\mathcal{R}} + \vec{F}_{2|\mathcal{R}} \quad \text{in area D2} \quad (370)$$

$$\vec{F}_{S|\mathcal{R}} = \vec{F}_{2|\mathcal{R}} \quad \text{in area D3} \quad (371)$$

The triangle ASR can be expressed in a 2D plan ($O\vec{x}\vec{y}$) by performing the following four transformations of the referential \mathcal{R} , illustrated in Figure 7:

1. $\mathcal{R} \rightarrow \mathcal{R}_{D(1)}$: a translation of the referential is performed to center the anchor A at the origin of the new referential $\mathcal{R}_{D(1)}$, *i.e.* to obtain $A_{\mathcal{R}_{D(1)}} = (0, 0, 0)_{\mathcal{R}_{D(1)}}$:

$$X_{\mathcal{R}_{D(1)}} = X_{\mathcal{R}} - T_{\mathcal{R}} \quad (372)$$

$$\vec{F}_{i|\mathcal{R}_{D(1)}} = \vec{F}_{i|\mathcal{R}} \quad (373)$$

with $i \in \{1, 2, S\}$, $X_{\mathcal{R}} = [x \ y \ z]_{\mathcal{R}}^T$ and $T_{\mathcal{R}} = [0 \ l_0 \ 0]_{\mathcal{R}}^T$. Note the forces stay unchanged.

2. $\mathcal{R}_{D(1)} \rightarrow \mathcal{R}_{D(2)}$: a rotation around the lateral axis $(A\vec{z})_{\mathcal{R}_{D(1)}}$ of an angle $\mu_{D(1)} = -\text{atan2}(y_{\mathcal{R}_{D(1)}}, x_{\mathcal{R}_{D(1)}})$ is performed to align $\overrightarrow{AR}_{D(2)}$ with the horizontal plan, *i.e.* to obtain $\frac{\overrightarrow{AR}_{D(2)}}{\|\overrightarrow{AR}_{D(2)}\|} \cdot (A\vec{z})_{\mathcal{R}_{D(2)}} = 0$:

$$X_{\mathcal{R}_{D(2)}} = M_{D(1)} X_{\mathcal{R}_{D(1)}} \quad (374)$$

$$\vec{F}_{i|\mathcal{R}_{D(2)}} = M_{D(1)} \vec{F}_{i|\mathcal{R}_{D(1)}} \quad (375)$$

$$M_{D(1)} = \begin{bmatrix} \cos(\mu_{D(1)}) & -\sin(\mu_{D(1)}) & 0 \\ \sin(\mu_{D(1)}) & \cos(\mu_{D(1)}) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (376)$$

Note that the sign “-” in the angle of rotation $\mu_{D(1)}$ is due to the fact the referential \mathcal{R} is not direct, and so $\mathcal{R}_{D(1)}$. One has also $A_{\mathcal{R}_{D(1)}} = A_{\mathcal{R}_{D(2)}}$ and $(A\vec{z})_{\mathcal{R}_{D(1)}} = (A\vec{z})_{\mathcal{R}_{D(2)}}$.

3. $\mathcal{R}_{D(2)} \rightarrow \mathcal{R}_{D(3)}$: a rotation around the vertical axis $(A\vec{y})_{\mathcal{R}_{D(2)}}$ of an angle $\mu_{D(2)} = \text{atan2}(z_{\mathcal{R}_{D(2)}}, x_{\mathcal{R}_{D(2)}})$ is performed to put $R_{D(3)}$ on the horizontal axis $(A\vec{x})_{\mathcal{R}_{D(3)}}$, *i.e.* to obtain $\frac{\overrightarrow{AR}_{D(3)}}{\|\overrightarrow{AR}_{D(3)}\|} = (1, 0, 0)_{\mathcal{R}_{D(3)}}$:

$$X_{\mathcal{R}_{D(3)}} = M_{D(2)} X_{\mathcal{R}_{D(2)}} \quad (377)$$

$$\vec{F}_{i|\mathcal{R}_{D(3)}} = M_{D(2)} \vec{F}_{i|\mathcal{R}_{D(2)}} \quad (378)$$

$$M_{D(2)} = \begin{bmatrix} \cos(\mu_{D(2)}) & 0 & \sin(\mu_{D(2)}) \\ 0 & 1 & 0 \\ -\sin(\mu_{D(2)}) & 0 & \cos(\mu_{D(2)}) \end{bmatrix}. \quad (379)$$

Note that $A_{\mathcal{R}_{D(2)}} = A_{\mathcal{R}_{D(3)}}$ and $(A\vec{y})_{\mathcal{R}_{D(2)}} = (A\vec{y})_{\mathcal{R}_{D(3)}}$.

4. $\mathcal{R}_{D(3)} \rightarrow \mathcal{R}_D$. From the previous steps, the edge S is the only one which is not necessary inside the plan $(O\vec{x}\vec{y})_{\mathcal{R}_{D(2)}}$. Its position depends of the orientation of the force $\vec{F}_{S|\mathcal{R}_{D(3)}}$ applied at the point S . In the case where the orientation of this force is contained in a plan, S will be in the same plan. Thus, a rotation around the vertical axis $A_{\mathcal{R}_{D(3)}} \vec{x}$ of an angle $\mu_{D(3)} = -\text{atan2}(F_{S_z|\mathcal{R}_{D(3)}}, F_{S_y|\mathcal{R}_{D(3)}})$ is performed to put S_D in the plan $(O\vec{x}\vec{y})_{\mathcal{R}_D}$, *i.e.* $\frac{\overrightarrow{AS}_D}{\|\overrightarrow{AS}_D\|} \cdot (A\vec{z})_{\mathcal{R}_D} = 0$:

$$X_{\mathcal{R}_D} = M_{D(3)} X_{\mathcal{R}_{D(3)}} \quad (380)$$

$$\vec{F}_{i|\mathcal{R}_D} = M_{D(3)} \vec{F}_{i|\mathcal{R}_{D(3)}} \quad (381)$$

$$M_{D(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\mu_{D(3)}) & -\sin(\mu_{D(3)}) \\ 0 & \sin(\mu_{D(3)}) & \cos(\mu_{D(3)}) \end{bmatrix}. \quad (382)$$

Note that $A_{\mathcal{R}_{D(3)}} = A_{\mathcal{R}_D}$ and $(A\vec{x})_{\mathcal{R}_{D(3)}} = (A\vec{x})_{\mathcal{R}_D}$. Again, the sign “-” in $\mu_{D(3)}$ is due to the fact the referential \mathcal{R} is not direct, and so $\mathcal{R}_{D(1)}$, $\mathcal{R}_{D(2)}$ and $\mathcal{R}_{D(3)}$.

All the transformations performed below can be summarize as following:

$$X_{\mathcal{R}_D} = M_D (X_{\mathcal{R}} + T_{\mathcal{R}}) \quad (383)$$

$$\vec{F}_{i|\mathcal{R}_D} = M_D \vec{F}_{i|\mathcal{R}} \quad (384)$$

$$M_D = M_{D(3)} M_{D(2)} M_{D(1)} \quad (385)$$

where the calculation of M_D required the calculation of $X_{D(1)}$, $X_{D(2)}$ and $\vec{F}_{S|\mathcal{R}_{D(3)}}$.

Once a solution will be found in \mathcal{R}_D , the angles and length will be re-evaluated in \mathcal{R} using for example Definition 12.

In this section, since all the calculations are performed in \mathcal{R}_D , the notations are lightened such that a parameter $\beta_{i|\mathcal{R}_D}$ is simply written β_i .

G.2 Geometrical model in areas D

Due to the change of referential, the system (7) can be expressed in two dimensions with $l_0 = 0$, so $z = 0$, $\psi_1 = \psi_2 = \psi_3 = 0$. Moreover, since $AS = l_1$ and $SR = l_2$, one may rewrite (7) as

$$x = l_1 \cos(\gamma) + l_2 \cos(\beta) \quad (386)$$

$$y = l_1 \sin(\gamma) + l_2 \sin(\beta) \quad (387)$$

where

$$\begin{aligned} \gamma = \alpha_1 \quad \beta = \alpha_2 & \text{ in area D1,} \\ \gamma = \alpha_1 \quad \beta = \alpha_3 & \text{ in area D2,} \\ \gamma = \alpha_2 \quad \beta = \alpha_3 & \text{ in area D3.} \end{aligned}$$

Note that l_{11} , l_{12} , l_{21} and l_{22} are already known in function of the area, see Section 4.2. Let's find the value of γ and β . Moreover:

- In area D1, since $l_{22} = 0$, one can take $\alpha_3 = 0$.
- In area D2, since $l_{12} = l_{21} = 0$, one can take $\alpha_2 = 0$.
- In area D3, since $l_{11} = 0$, one can take $\alpha_1 = 0$. Note that since its value is not important, one can also take $\alpha_1 = -\alpha_2 + 2\phi_1 + \pi$ as in area A1 or A0, because if the element 1 moves to make $l_{11} \neq 0$, the configuration will be in area A1 or A0.

G.2.1 Case $y \neq 0$

Consider the case where $y \neq 0$, one has

$$x = l_1 \cos(\gamma) + l_2 \cos(\beta) \quad (388)$$

$$y = l_1 \sin(\gamma) + l_2 \sin(\beta) \quad (389)$$

From (389), one gets

$$\sin(\beta) = \frac{y}{l_2} - \frac{l_1}{l_2} \sin(\gamma). \quad (390)$$

Consider now (388):

$$x - l_1 \cos(\gamma) = l_2 \cos(\beta)$$

$$(x - l_1 \cos(\gamma))^2 = l_2^2 \cos^2(\beta)$$

$$x^2 - 2xl_1 \cos(\gamma) + l_1^2 \cos^2(\gamma) = l_2^2 \left(1 - \sin^2(\beta)\right) \quad (391)$$

Injecting (390) inside (391), one obtains

$$\begin{aligned} & x^2 - 2xl_1 \cos(\gamma) + l_1^2 \cos^2(\gamma) \\ &= l_2^2 \left(1 - \left(\frac{y}{l_2} - \frac{l_1}{l_2} \sin(\gamma)\right)^2\right) \end{aligned}$$

$$\begin{aligned} & x^2 - 2xl_1 \cos(\gamma) + l_1^2 \cos^2(\gamma)^2 \\ &= l_2^2 - \left(y^2 - 2yl_1 \sin(\gamma) + l_1^2 \sin^2(\gamma)\right) \end{aligned}$$

$$\begin{aligned} & x^2 + y^2 - 2xl_1 \cos(\gamma) + l_1^2 - l_2^2 \\ &= 2yl_1 \sin(\gamma). \end{aligned} \quad (392)$$

Since $y \neq 0$, (392) becomes

$$\sin(\gamma) = \frac{x^2 + y^2 - 2xl_1 \cos(\gamma) + l_1^2 - l_2^2}{2yl_1}$$

$$\sin(\gamma) = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2yl_1} - \frac{x}{y} \cos(\gamma)$$

$$\sin(\gamma) = a_D - b_D \cos(\gamma) \quad (393)$$

with $a_D = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2yl_1}$ and $b_D = \frac{x}{y}$. Put $X = \cos(\gamma)$. Taking the square of (393), one can deduce that

$$\begin{aligned} \sin^2(\gamma) &= (a_D - b_D \cos(\gamma))^2 \\ 1 - X^2 &= (a_D - b_D X)^2 \\ 1 - X^2 &= a_D^2 - 2a_D b_D X + b_D^2 X^2 \\ 0 &= a_D^2 - 1 - 2a_D b_D X + (1 + b_D^2) X^2 \\ 0 &= C_D - B_D X + A_D X^2 \end{aligned} \quad (394)$$

with $C_D = a_D^2 - 1$, $B_D = 2a_D b_D$ and $A_D = 1 + b_D^2$. Let define the determinant Δ such as

$$\begin{aligned} \Delta &= B_D^2 - 4A_D C_D \\ &= 4 \left((a_D b_D)^2 - (1 + b_D^2) (a_D^2 - 1) \right). \end{aligned}$$

If $\Delta < 0$, there is not solution and so the configuration is not possible. Else, two solutions of (394) are possible

$$\begin{aligned} X_{1D} &= \frac{B_D - \sqrt{B_D^2 - 4A_D C_D}}{2A_D} \\ &= \frac{a_D b_D - \sqrt{a_D^2 b_D^2 - (1 + b_D^2) (a_D^2 - 1)}}{(1 + b_D^2)} \end{aligned} \quad (395)$$

and

$$X_{2D} = \frac{a_D b_D + \sqrt{a_D^2 b_D^2 - (1 + b_D^2) (a_D^2 - 1)}}{(1 + b_D^2)} \quad (396)$$

Remind $X = \cos(\gamma)$, so X_{1D} is valid only if $|X_{1D}| \leq 1$, same for X_{2D} if $|X_{2D}| \leq 1$. Let's note L_X the list of valid solution of (394) such that

$$L_X = \begin{cases} \{X_{1D}, X_{2D}\} & \text{if } (|X_{1D}| \leq 1) \& (|X_{2D}| \leq 1) \\ \{X_{1D}\} & \text{if } (|X_{1D}| \leq 1) \& (|X_{2D}| > 1) \\ \{X_{2D}\} & \text{if } (|X_{1D}| > 1) \& (|X_{2D}| \leq 1) \\ \emptyset & \text{else} \end{cases} \quad (397)$$

From (390) and (393), note that the solution must also respected the condition $C_{a1} \leq 1$ where

$$C_{a1} = \left| \frac{y}{l_2} - \frac{l_1}{l_2} \sin(\gamma) \right| \quad (398)$$

Thus, we define the list of valid solution L_γ such that

$$\gamma \in L_\gamma \quad (399)$$

where

$$L_\gamma = \{\bar{\gamma} | \bar{\gamma} \in L_{S1}, C_{a1}(\bar{\gamma}) \leq 1\}, \quad (400)$$

$$L_{S1} = \{\text{acos}(sX) \mid s \in \{-1, 1\}, X \in L_X\}. \quad (401)$$

Moreover, (390) allows to obtain L_β the set of solution of β such that

$$\beta \in L_\beta \quad (402)$$

where for each γ , there is two solutions for β :

$$L_\beta = \{\bar{\beta}, \pi - \bar{\beta} \mid \bar{\beta} \in L_{S2}\} \quad (403)$$

$$L_{S2} = \left\{ \text{asin} \left(\frac{y}{l_2} - \frac{l_1}{l_2} \sin(\bar{\gamma}) \right) \mid \bar{\gamma} \in L_\gamma \right\}. \quad (404)$$

The choice of γ and β will be discussed in Appendix G.3 in the case where L_γ and L_β are not singleton.

G.2.2 Case $x \neq 0$ and $y = 0$

In the particular case where $y = 0$, one has

$$x = l_1 \cos(\gamma) + l_2 \cos(\beta) \quad (405)$$

$$0 = l_1 \sin(\gamma) + l_2 \sin(\beta) \quad (406)$$

From (406), one gets

$$\sin(\beta) = -\frac{l_1}{l_2} \sin(\gamma) \quad (407)$$

Consider now (405):

$$x - l_1 \cos(\gamma) = l_2 \cos(\beta). \quad (408)$$

It can be deduce that

$$\begin{aligned} (x - l_1 \cos(\gamma))^2 &= l_2^2 \cos(\beta)^2 \\ x^2 - 2xl_1 \cos(\gamma) + l_1^2 \cos(\gamma)^2 &= l_2^2 (1 - \sin(\beta)^2) \end{aligned} \quad (409)$$

Injecting (407) inside (409), one obtains

$$\begin{aligned} x^2 - 2xl_1 \cos(\gamma) + l_1^2 \cos(\gamma)^2 &= l_2^2 \left(1 - \left(-\frac{l_1}{l_2} \sin(\gamma) \right)^2 \right) \\ x^2 - 2xl_1 \cos(\gamma) + l_1^2 \cos(\gamma)^2 &= l_2^2 - l_1^2 \sin(\gamma)^2 \\ x^2 + l_1^2 - l_2^2 &= 2xl_1 \cos(\gamma) \end{aligned} \quad (410)$$

Since $x \neq 0$, (410) becomes

$$\cos(\gamma) = \frac{x^2 + l_1^2 - l_2^2}{2xl_1}. \quad (411)$$

Put $X_{3D} = \frac{x^2 + l_1^2 - l_2^2}{2xl_1}$. Remark (411) provides a valid solution only if $|X_{3D}| \leq 1$.

$$L_X = \begin{cases} \{X_{3D}\} & \text{if } (|X_{3D}| \leq 1) \\ \emptyset & \text{else} \end{cases}$$

Note that from (408) the solution must respected $C_{a1} \leq 1$ (defined in (398) with $y = 0$). Thus, we define the list of valid solution $L_{\gamma 2}$ such that

$$\gamma \in L_{\gamma 2} \quad (412)$$

where

$$L_{\gamma 2} = \{\bar{\gamma} | \bar{\gamma} \in L_{S1}, C_{a1}(\bar{\gamma}) \leq 1\}, \quad (413)$$

$$L_{S1} = \{\text{acos}(sX) \mid s \in \{-1, 1\}, X \in L_X\}. \quad (414)$$

Moreover, (390) allows to obtain L_β the set of solution of β such that

$$\beta \in L_\beta \quad (415)$$

where

$$L_\beta = \{\bar{\beta}, \pi - \bar{\beta} \mid \bar{\beta} \in L_{S2}\} \quad (416)$$

$$L_{S2} = \left\{ \text{asin} \left(\frac{y}{l_2} - \frac{l_1}{l_2} \sin(\bar{\gamma}) \right) \mid \bar{\gamma} \in L_{\gamma 2} \right\}. \quad (417)$$

Remark the results has the same form that in Appendix G.2.1. The choice of γ and β will be discussed in Appendix G.3 in the case where L_γ and L_β are not singleton.

G.2.3 Case $x = 0$ and $y = 0$

If $x = 0$ and $y = 0$, one has

$$0 = l_1 \cos(\gamma) + l_2 \cos(\beta) \quad (418)$$

$$0 = l_1 \sin(\gamma) + l_2 \sin(\beta) \quad (419)$$

and so one gets

$$\cos(\beta) = \frac{l_1}{l_2} \cos(\gamma) \quad (420)$$

$$l_1 \sin(\gamma) = -l_2 \sin(\beta). \quad (421)$$

From (421), one can express

$$\begin{aligned}
l_1^2 \sin(\gamma)^2 &= l_2^2 \sin(\beta)^2 \\
l_1^2 \sin(\gamma)^2 &= l_2^2 (1 - \cos(\beta)^2) \\
l_1^2 \sin(\gamma)^2 &= l_2^2 - l_1^2 \cos(\gamma)^2 \\
l_1^2 - l_2^2 &= 0
\end{aligned} \tag{422}$$

Thus, a solution exist only if $l_1 = l_2$, else the configuration is impossible. If $l_1 = l_2$, one can deduce from (420)-(421) that $\gamma = \pi + \beta$. Moreover, from the PFS, the orientation is the same of F_s , so one has $\gamma = \text{atan2}(F_{Sy}, F_{Sx})$.

G.3 Solution of γ and β

From Appendix G.2, two sets L_γ and L_β of possible solutions of γ and β has been defined. If in the case where these sets are not singleton, the current solution can be find by choosing the set (γ, β) respecting the following geometrical condition:

$$\begin{aligned}
0 &= |x - (l_1 \cos(\gamma) + l_2 \cos(\beta))| \\
&+ |y - (l_1 \sin(\gamma) + l_2 \sin(\beta))|.
\end{aligned} \tag{423}$$

Moreover, since several areas D can have a valid geometrical solution for the same coordinate (x, y, z) (several triangles are possible between A and R), the solution must also respect the following conditions based on the forces applied on the system and the properties of the sliding element (Definition 3):

1. The evaluation of the forces must be strictly positive:

- if area D1, $T_1^{S1} > 0$ and $T_2^{S1} > 0$;
- if area D2, $T_1^{S1} > 0$ and $T_3^{S1} > 0$;
- if area D3, $T_2^{S2} > 0$ and $T_3^{S2} > 0$;

where T_1^{S1} , T_2^{S1} and T_3^{S1} are evaluated using (17)

2. The Definition 3 is respected for the two elements, *i.e.*

- if area D1:
 - if element 1 is not fixed, $\vec{F}_1 \cdot \vec{AS} > 0$,
 - if element 2 is not fixed, $\vec{F}_2 \cdot \vec{SR} > 0$.
- if area D2:
 - if element 1 is not fixed, $\vec{F}_1 \cdot \vec{AS} > 0$,
 - if element 2 is not fixed, $\vec{F}_2 \cdot \vec{SR} < 0$.
- if area D3:
 - if element 1 is not fixed, $\vec{F}_1 \cdot \vec{AS} < 0$,
 - if element 2 is not fixed, $\vec{F}_2 \cdot \vec{SR} < 0$.

with $\vec{AS} = \vec{AB}_1 + \vec{B}_1\vec{S}$ and $\vec{SR} = \vec{SB}_2 + \vec{B}_2\vec{R}$ where \vec{AB}_1 , $\vec{B}_1\vec{S}$, \vec{SB}_2 and $\vec{B}_2\vec{R}$ have been expressed in Section 4.1.

If not solution of L_γ and L_β respected the previous conditions, its means the configuration is not possible. Else, a solution is found analytically.