Transformations on Irreducible Binary Polynomials

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Abstract. Using the natural action of $GL_2(\mathbb{F}_2) \simeq \mathfrak{S}_3$ over $\mathbb{F}_2[X]$, one can define different classes of polynomials strongly analogous to self-reciprocal irreducible polynomials. We give transformations to construct polynomials of each kind of invariance and we deal with the question of explicit infinite sequences of invariant irreducible polynomials in $\mathbb{F}_2[X]$. We generalize results obtained by Varshamov, Wiedemann, Meyn and Cohen and we give sequences of invariant irreducible polynomials. Moreover we explain what happens when the given constructions fail. We also give a result on the order of the polynomials of one of the classes: the alternate irreducible polynomials.
Keywords: irreducible polynomials, finite fields, sequences of irreducible invariant polynomials.

1 Introduction

In [6], we studied the natural action of the 6 elements group $\mathfrak{S}_3 \simeq GL_2(\mathbb{F}_2) \simeq PGL_2(\mathbb{F}_2)$ on the set

$$\mathcal{I} = \{P \in \mathbb{F}_2[X] |\ P \text{ irreducible}\} \setminus \{X, X+1\}.$$

This action of $\mathfrak{S}_3$ on $\mathcal{I}$, is defined by the two operations

$$P^+(X) = P(X+1)$$
$$P^*(X) = X^{\deg P} P(\frac{1}{X}).$$

The action of all other elements of $\mathfrak{S}_3$ are obtained by compositions of these two operations. We shall write for example P^{*+} for $(P^*)^+$.

Definition 1 *The **hexagon** of $P \in \mathcal{I}$ is the orbit of P :*

$$Hex(P) = \{P^\sigma | \sigma \in \mathfrak{S}_3\}.$$

*The **degree** of a hexagon is the degree of its polynomials. When $Card(Hex(P)) < 6$ we say that the hexagon is **degenerate**.*

Such an orbit has 1, 2, 3 or 6 elements according to the isotropy subgroup of P. In $\mathfrak{S}_3$, apart from the trivial subgroups, we have 3 subgroups of order 2, and one of order 3. Each of them will give a family of invariant irreducible polynomials. The case of 1 element hexagon is easy : $\mathcal{I}(2) = \{X^2+X+1\}$ is the only 1 element degenerate hexagon. This polynomial is the only fixed point of the action.

Let us define for any integer $n > 1$

$$\mathcal{I}(n) = \{P \in \mathcal{I} | \deg P = n\},$$

then this grading of $\mathcal{I}$ is stable under the action (but this is not true for $n = 1$).

2 Self-reciprocity and the 3 Elements Degenerate Hexagons

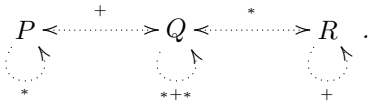
Self-reciprocal polynomials are invariant under the action of one subgroup of order 2 (generated by $*$ operation). In fact there is no reason to focus on one subgroup because they are all conjugate.

2.1 The Invariant Polynomials

Definition 2 *Let P be a polynomial, P is said to be **self-reciprocal** (resp. **periodic**, **median**) if $P^* = P$ (resp. $P^+ = P$, $P^{*+*} = P^{+*+} = P$). We easily check that these polynomials are of even degree.*

Definition 3 *Let P and Q, $P \neq Q$, be two polynomials of $\mathcal{I}$, $\{P, Q\}$ is said to be a **reciprocal pair** (resp. **periodic pair**, **median pair**) if $Q = P^*$ (resp. $Q = P^+$, $Q = P^{*+*}$). As the polynomials of a pair have the same degree, by extension, it will also be the degree of the pair.*

Definition 4 *A **3 elements degenerate hexagon** is made of a self-reciprocal irreducible polynomial P, a median irreducible polynomial Q and a periodic irreducible polynomial R such that:*



The next theorem extends Meyn's one (see [4], Theorem 1) to the periodic and median polynomials:

Theorem 1 *i) Each self-reciprocal (resp. periodic, median) irreducible polynomial of degree $2n$ ($n \geq 1$) is a factor of the polynomial*

$$H_{r,n}(X) = X^{2^n+1} + 1$$

$$\textit{(resp. } H_{p,n}(X) = X^{2^n} + X + 1, \quad H_{m,n}(X) = X^{2^n} + X^{2^n-1} + 1\textit{).}$$

ii) Each irreducible factor of degree ≥ 2 of $H_{r,n}$ (resp. $H_{p,n}$, $H_{m,n}$) is a self-reciprocal (resp. periodic, median) polynomial of degree $2d$, where d divides n such that n/d is odd.

Proof. In [4], Meyn proves the theorem for self-reciprocal polynomials. We trivially extend it to the periodic (resp. median) polynomials noting that, on the one hand, $H_{p,n} = H_{r,n}^{+*}$ (resp. $H_{m,n} = H_{r,n}^{+}$) and on the other hand, a periodic (resp. median) irreducible polynomial is obtained from a self-reciprocal one, applying the transformation $+^*$ (resp. $+$) on it.

2.2 The Quadratic Transformations

Definition 5 *We define the maps ϕ_p, ϕ_m, ϕ_r: $\mathbb{F}_2[X] \to \mathbb{F}_2[X]$ by*

$$\phi_p(P) = P(X^2 + X)$$
$$\phi_m(P) = \phi_p(P)^* = X^{2n}\, P(\frac{X+1}{X^2})$$
$$\phi_r(P) = \phi_p(P)^{*+} = (X^2+1)^n\, P(\frac{X}{X^2+1}).$$

The image by ϕ_p (resp. ϕ_m, ϕ_r) of a polynomial of $\mathcal{I}(n)$ is a periodic (resp. median, self-reciprocal) polynomial of degree $2n$ but is not always irreducible.

Proposition 1 *Let $\mathcal{P} \subset \mathbb{F}_2[X]$ be the subset of periodic polynomials, then $\mathcal{P}$ is a sub algebra of $\mathbb{F}_2[X]$ and $\phi_p : \mathbb{F}_2[X] \to \mathcal{P}$ is an algebra isomorphism.*

Proof. Only ϕ_p surjectivity deserves proof. Let Q be a periodic polynomial, its degree is an even integer $2n$. Then, $Q + (X^2+X)^n$ is a periodic polynomial of degree $< 2n$. Iterating this process, we obtain a polynomial P of degree n such that $\phi_p(P) = Q$.

We call **trace** of a polynomial $P \neq 0$ of degree n, and write $Tr(P)$, the X^{n-1} coefficient.

The following theorem and corollary can be seen as an extension of Meyn's Lemma (see [4], Lemma 4):

Theorem 2 *Let $P \in \mathcal{I}(n)$. If $Tr(P) = 1$, then $\phi_p(P)$ is a periodic irreducible polynomial of degree $2n$, else, $\phi_p(P)$ is the product of two irreducible polynomials of degree n, which form a periodic pair.*

Conversely, let Q be an irreducible periodic polynomial of degree $2n$ (resp. $\{R, R^+\}$ a periodic pair of degree n), then, there exists a unique $P \in \mathcal{I}(n)$ such that $\phi_p(P) = Q$ (resp. $\phi_p(P) = RR^+$).

Proof. Let $P \in \mathcal{I}(n)$ and let h be a root of $\phi_p(P)$, then $h^2 + h$ is a root of P by definition and generates the field $\mathbb{F}_{2^n}$. This implies that $\mathbb{F}_2[h] \supset \mathbb{F}_2[h^2+h] = \mathbb{F}_{2^n}$. In turn h cannot be a root of a polynomial whose degree is $< n$. We conclude that the irreducible decomposition of $\phi_p(P)$ contains only polynomials of degree $\geq n$. This leaves only two cases: $\phi_p(P)$ is irreducible of degree $2n$ or is the product of two irreducible polynomials A and B of degree n.

Suppose we are in the second case : AB being periodic, we have two possibilities: $B = A^+$ or A and B are themselves periodic. This last event cannot

occur because if $\phi_p(P) = AB$ with A and B periodic and irreducible, then by Proposition 1, there exist U and V such that $\phi_p(U) = A$ and $\phi_p(V) = B$. Then $\phi_p(P) = \phi_p(U)\phi_p(V) = \phi_p(UV)$ and $P = UV$. This would contradict the irreducibility of P.

We now prove that the trace dispatches the two cases. Suppose $\phi_p(P) = AB$, then we have $h \in \mathbb{F}_{2^n}$. We call $a = h^2+h \in \mathbb{F}_{2^n}$, then $Tr(a) = Tr(h^2)+Tr(h) = 0$ and $Tr(P) = 0$ because a is a root of P. Conversely if $Tr(P) = 0$ then $Tr(a) = 0$ for any root a of P. Then it is well known that the equation $X^2 + X = a$ has roots in $\mathbb{F}_{2^n}$ (using the "half-trace") . This means that $\phi_p(P)$ has a root in $\mathbb{F}_{2^n}$, so it is reducible.

We prove now the "conversely" part of the theorem.
Let Q be an irreducible periodic polynomial of degree $2n$ and let h be a root of Q, then, $h+1$ is also a root of Q and, from Theorem 1, we know that $h+1 = h^{2^n}$. Now, if

$$\mathcal{E} = \{h^{2^i} | 0 \leq i < n\},$$

$$Q = \prod_{h \in \mathcal{E}} (X + h)(X + h + 1) = \prod_{h \in \mathcal{E}} (X^2 + X + h(h + 1)).$$

We take

$$P = \prod_{h \in \mathcal{E}} (X + h(h + 1)),$$

it is then clear that $\phi_p(P) = Q$. Let $h(h+1)$ be a root of P, then any other root can be written $h^{2^k}(h^{2^k}+1) = [h(h+1)]^{2^k}$ for some integer $k \geq 1$. In other terms, the roots of P are conjugate by Frobenius. This implies that $P \in \mathbb{F}_2[X]$. If P is reducible, say $P = ST$ then $\phi_p(P) = \phi_p(S)\phi_p(T) = Q$. This decomposition is trivial because Q is irreducible. This forces S or T to be trivial and P to be irreducible. Now, if P is not unique, there exists another irreducible polynomial S such that $\phi_p(S) = Q$, then $h(h + 1)$ is a root of S by definition, so S and P have the same roots and consequently, $S = P$.

In the same manner, let $\{R, R^+\}$ be a periodic pair of degree n and $\mathcal{F}$ be the set of the n roots of R then taking

$$P = \prod_{a \in \mathcal{F}} (X + a(a + 1)),$$

we obtain a $P \in \mathbb{F}_2[X]$ such that $\phi_p(P) = RR^+$.

Suppose $P = ST$ with two non constant polynomials in $\mathbb{F}_2[X]$, then $\phi_p(P) = \phi_p(S)\phi_p(T) = RR^+$. Consequently, we can write $\phi_p(S) = R$ and this is a contradiction because $\phi_p(S)$ is periodic and R is not, so P is irreducible. The unicity of P is proved like previously.

Corollary 1 *Let $P \in \mathcal{I}(n)$, if $Tr(P) = 1$, then, $\phi_r(P)$(resp. $\phi_m(P)$) is a self-reciprocal (resp. median) irreducible polynomial of degree $2n$, else, it is the product of two irreducible polynomials of degree n, which form a reciprocal (resp. median) pair.*

Conversely, let Q be an irreducible self-reciprocal (resp. median) polynomial of degree $2n$, then, there exists $P \in \mathcal{I}(n)$ such that $\phi_r(P) = Q$ (resp. $\phi_m(P) = Q$). In the same way, let $\{R, R^*\}$ (resp. $\{R, R^{+*+}\}$) be a reciprocal (resp. median) pair of degree n, then, there exists a unique $P \in \mathcal{I}(n)$ such that $\phi_r(P) = RR^*$ (resp. $\phi_m(P) = RR^{+*+}$).

Proof. We prove the corollary only for the self-reciprocal case, the median case being similar.

If $Tr(P) = 1$, from Theorem 2, we know that $\phi_p(P)$ is periodic in $\mathcal{I}(2n)$. Using Definitions 4 and 5, we see that $\phi_r(P)$ is self-reciprocal in $\mathcal{I}(2n)$. If $Tr(P) = 0$, there exists a periodic pair $\{S, S^+\}$ of degree n such that $\phi_p(P) = SS^+$. Now, as the transformations * and $+$ are distributive with regard to the multiplication in $\mathbb{F}_2[X]$, $\phi_r(P) = (SS^+)^{*+} = S^{*+}S^{+*+} = S^{*+}(S^{*+})^*$, which is a reciprocal pair of degree n.

Using Theorem 2, Definitions 4 and 5, the proof of the second part is obvious.

We get a simple way to construct 3 and 6 elements hexagons: let P be an irreducible polynomial of degree n such that $Tr(P) = 1$, then $\{\phi_r(P), \phi_m(P), \phi_p(P)\}$ is a 3 elements degenerate hexagon of degree $2n$. This is illustrated by the left-side diagram. If $Tr(P) = 0$, we have the right-side diagram, where $\{Q_1, Q_2\}$, $\{Q_3, Q_4\}$ and $\{Q_5, Q_6\}$ are the pairs such that $\phi_r(P) = Q_1Q_2$, $\phi_m(P) = Q_3Q_4$ and $\phi_p(P) = Q_5Q_6$ respectively.

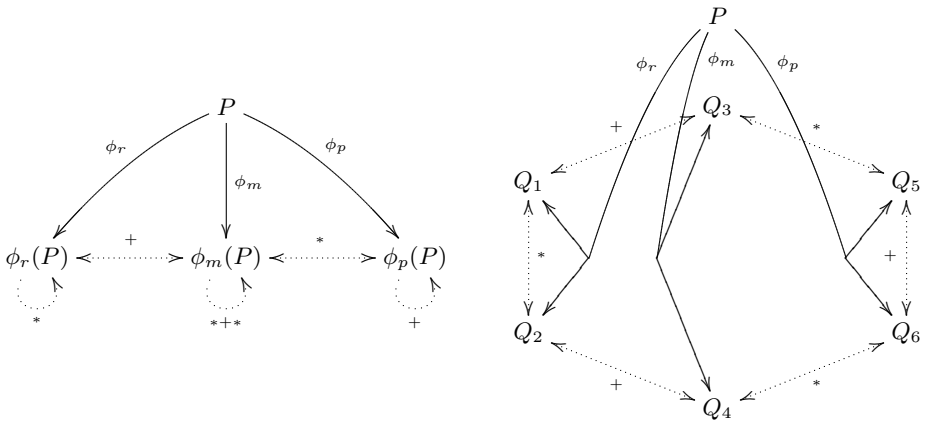


Fig. 1. Construction of hexagons by ϕ_r, ϕ_m and ϕ_p from an irreducible polynomial P such that $Tr(P) = 1$ (*left*) and $Tr(P) = 0$ (*right*).

2.3 Infinite Sequences of Self-reciprocal, Median and Periodic Irreducible Polynomials

Let $P \in \mathcal{I}(n)$, we denote $c_1(P)$ the coefficient of X in P.

Sequences of invariant irreducible polynomials appear implicitely in Varshamov [8] and more explicitely in Wiedemann [9], Meyn [4] and Cohen [1]. These sequences appear in papers devoted to a more general problem: the construction of irreducible polynomials (see Kyuregyan [2] and [3] for recent references).

Our approach shows that, actually, there are three strongly analogous families of invariant irreducible polynomials corresponding to the three conjugated 2-groups of $\mathfrak{S}_3$. The following theorem is an easy extension of the construction of sequences of self-reciprocal irreducible polynomials given by Meyn and Cohen to the median and periodic cases.

Theorem 3 *Let $P \in \mathcal{I}(n)$ be such that $c_1(P) = Tr(P) = 1$. Starting from $\phi_r(P)$ (resp. $\phi_m(P)$, $\phi_p(P)$) and iterating the transformation $P \to \phi_r(P)$ (resp. $P \to \phi_m(P^+)$, $P \to \phi_p(P^{*+})$) one generates an infinite sequence of self-reciprocal (resp. median, periodic) irreducible polynomials of degree $2^i n$, $i > 0$.*

Proof. We know by Theorem 2 that $\phi_r(P)$ is self-reciprocal and irreducible. By explicit computation, we see that $Tr(\phi_r(P)) = 1$ and because of self-reciprocity, $c_1(\phi_r(P)) = 1$. So by recurrence, we get a sequence of self-reciprocal irreducible polynomials. Then, using the Definitions 4 and 5, we obtain sequences of median and periodic polynomials.

2.4 Equivalent Transformations

In Definition 5 we defined ϕ_m and ϕ_r from ϕ_p in a natural way. Other transformations can be used instead of ϕ_p. They are obtained by choosing one "right" action by an element of the group $\mathfrak{S}_3$ before applying ϕ_p.

We construct in this way new transformation ϕ'_p and consequently new ϕ'_m and ϕ'_r. We recover in this way the transformation ϕ'_r quoted in Section 2.3 which is more frequently used in the mathematical literature for constructing self-reciprocal polynomial. The results of the preceding sections can be transposed with small changes. The following table lists these transformations:

ϕ'_p	$\phi'_m = * \circ \phi'_p$	$\phi'_r = + \circ * \circ \phi'_p$
$P(X^2+X+1) = \phi_p(P^+)$	$X^{2n}P(\frac{X^2+X+1}{X^2})$	$(X^2+1)^n P(\frac{X^2+X+1}{X^2+1})$
$(X^2+X+1)^n P(\frac{1}{X^2+X+1}) = \phi_p(P^{*+})$	$(X^2+X+1)^n P(\frac{X^2}{X^2+X+1})$	$(X^2+X+1)^n P(\frac{X^2+1}{X^2+X+1})$
$(X^2+X+1)^n P(\frac{X^2+X}{X^2+X+1}) = \phi_p(P^{+*+})$	$(X^2+X+1)^n P(\frac{X+1}{X^2+X+1})$	$(X^2+X+1)^n P(\frac{X}{X^2+X+1})$
$(X^2+X)^n P(\frac{X^2+X+1}{X^2+X}) = \phi_p(P^{+*})$	$(X+1)^n P(\frac{X^2+X+1}{X+1})$	$X^n P(\frac{X^2+X+1}{X})$
$(X^2+X)^n P(\frac{1}{X^2+X}) = \phi_p(P^*)$	$(X+1)^n P(\frac{X^2}{X+1})$	$X^n P(\frac{X^2+1}{X})$

Table 1. Equivalent quadratic transformations.

3 Alternate Polynomials and 2 Elements Degenerate Hexagons

We introduced in [6] the class of alternate polynomials over $\mathbb{F}_2$:

Definition 6 – *A polynomial P is said to be **alternate** if $P^{*+} = P^{+*} = P$.*
– *Let $P \in \mathcal{I}$, if P is not alternate, then $\{P, P^{*+}, P^{+*}\}$ is said to be an **alternate triplet**. P, P^{*+} and P^{+*} are of the same degree, so by extension, the degree of an alternate triplet is the degree of its polynomials.*
– *A **2 elements degenerate hexagon** is made of two distinct alternate irreducible polynomials P and Q such that:*

$$P \overset{+/*}{\prec\cdots\succ} Q \ .$$

We refer again to [6] for the details of the following :

Proposition 2 *The irreducible alternate polynomials are exactly the irreducible factors of the polynomials*

$$B_k(X) = X^{2^k+1} + X + 1,$$

for $k \in \mathbb{N}$.
*If P is alternate then $\deg P \equiv 0 \mod 3$ or $P = X^2 + X + 1$. If $\deg P = 3m$ then either P divides B_m and B^*_{2m}, or P divides B^*_m and B_{2m}. In the first case we say that P has **type** 1, in the second P has **type** 2.*

Corollary 2 *The order of an alternate irreducible polynomial of degree $3n$ divides $2^{2n} + 2^n + 1$.*

Proof. Let $P \in \mathcal{I}(n)$ be alternate. We suppose P is of type 1. Let a be a root of P, then, from Proposition 2, $a^{2^n+1}+a+1 = 0$. We put this equation to the 2^n, we get $a^{2^n(2^n+1)}+a^{2^n}+1 = 0$. Now, multiplying by a, we have $a^{2^n(2^n+1)+1}+a^{2^n+1}+a = 0$. Finally, we replace the term in the middle using the first equation, we obtain $a^{2^n(2^n+1)+1} + 1 = 0$.

If P is of type 2, then P^* is of type 1 and as P and P^* have same order, this concludes the proof.

3.1 The Cubic Transformation

Let $P \in \mathbb{F}_2[X]$ of degree n, we define the map $\psi : \mathbb{F}_2[X] \to \mathbb{F}_2[X]$ by

$$\psi(P)(X) = (X^2 + X)^n \, P(\frac{X^3 + X^2 + 1}{X^2 + X}).$$

We verify that $\psi(P)$ is alternate.

Let ϵ and ϵ^2 be the roots of $X^2 + X + 1$. For any irreducible $P \neq X^2 + X + 1$ in $\mathbb{F}_2[X]$ we have $P(\epsilon) \in \{1, \epsilon, \epsilon^2\}$. The main result of this section is:

Theorem 4 – *Let $P \in \mathcal{I}(n)$, $n > 2$. If $P(\epsilon) \neq 1$ then $\psi(P)$ is an irreducible alternate polynomial of degree $3n$, else, $\psi(P) = RST$, where $\{R, S, T\}$ is an alternate triplet of degree n.*

– *Conversely, let $Q \in \mathcal{I}(3n)$, $n > 1$ be an alternate polynomial (resp. $\{R, R^{+*}, R^{*+}\}$ an alternate triplet of degree n), then there exists a unique $P \in \mathcal{I}(n)$ such that $\psi(P) = Q$ (resp. $\psi(P) = RR^{+*}R^{*+}$).*

To complete the theorem, we precise that the irreducible (alternate) polynomials of degree 3 are obtained from X and $X + 1$:

$$\psi(X) = X^3 + X^2 + 1 \text{ and } \psi(X + 1) = X^3 + X + 1,$$

and for the particular case $X^2 + X + 1$ (whose value in ϵ is 0), we have:

$$\psi(X^2 + X + 1) = (X^2 + X + 1)^3.$$

Before going into the proof of Theorem 4, we need to establish some results.

Let us take $P \in \mathcal{I}(n)$ with $n > 2$, $K = \mathbb{F}_{2^n}$ the splitting field of P, and h a root of $\psi(P)$, then $h \neq 0, 1$ and

$$a = \frac{h^3 + h^2 + 1}{h^2 + h} = h + \frac{1}{h} + \frac{1}{h + 1} \tag{1}$$

is a root of P. This implies $K \subset K(h)$.

As $h \neq 0, 1$, h is a root of the polynomial

$$T_a(X) = X^3 + (1 + a)X^2 + aX + 1. \tag{2}$$

3.2 Proof of the First Part of Theorem 4

Proposition 3 *Let w be a cubic root of $(\epsilon + a)(\epsilon + a^2)$, then the roots of (2) are*

$$h_i = 1 + a + \epsilon^i w + \frac{b}{\epsilon^i w},$$

with $i = 0$, 1 or 2, and $b = a^2 + a + 1$. Moreover, they verify the relations $h_1 = 1/(h_0 + 1)$ and $h_2 = 1 + 1/h_0$.

Proof. The formulas for the h_i are obtained by the classical Cardan's method:

The first step is to cancel the X^2 coefficient in

$$X^3 + (a + 1)X^2 + aX + 1 = 0.$$

We take $X' = X + 1 + a$ and $b = 1 + a + a^2$:

$$X'^3 + bX' + b = 0.$$

The second step is to use two variables u, v and take $X' = u + v$:

$$u^3 + v^3 + (u + v)(uv + b) + b = 0.$$

Choosing $uv = b$, if we can solve

$$\begin{cases} uv &= b \\ u^3 + v^3 &= b \end{cases}$$

in $\overline{K}$, we will have the roots of (2), they would be the

$$1 + a + u + v.$$

We can write:

$$\begin{cases} u^3 v^3 &= b^3 \\ u^3 + v^3 &= b, \end{cases}$$

which is equivalent to a second degree problem. If $Y = u^3$ then:

$$Y^2 + bY + b^3 = 0,$$

and dividing by b^2 we get:

$$\frac{Y^2}{b^2} + \frac{Y}{b} + b = 0.$$

We take $Z = Y/b$ (because $b \neq 0$):

$$Z^2 + Z + b = 0$$

$$Z^2 + Z + 1 + a + a^2 = 0$$

$$(Z + a)^2 + (Z + a) + 1 = 0.$$

So the solutions are $Z = a + \epsilon$ and $Z = a + \epsilon^2$. And we find:

$$u^3 = (1 + a + a^2)(a + \epsilon^2),$$

which can be written equivalently:

$$\begin{aligned} u^3 &= (a + \epsilon)(a + \epsilon^2)(a + \epsilon^2) \\ &= (a + \epsilon)(a^2 + \epsilon). \end{aligned}$$

So we can choose u as one of the three cubic roots of $(a+\epsilon)(a^2+\epsilon)$. The quantity $v = b/u$ is then determined uniquely. The roles of v and u can be exchanged.

We now establish the relations between the roots:

$$\begin{aligned} 1 + \frac{1}{h_0} &= h_0^2 + (1 + a)h_0 + a + 1 \quad \text{(from (2))} \\ &= w^2 + \frac{b^2}{w^2} + (1 + a)(w + \frac{b}{w}) + a + 1 \\ &= \frac{w^3}{w} + \frac{b^2 w}{w^3} + (1 + a)(w + \frac{b}{w}) + a + 1 \\ &= [\frac{b^2}{w^3} + 1 + a]w + [\frac{w^3}{b} + 1 + a]\frac{b}{w} + a + 1. \end{aligned}$$

Then using $b = (\epsilon + a)(\epsilon^2 + a)$ and $w^3 = (\epsilon + a)(\epsilon + a^2)$ we find:

$$1 + \frac{1}{h_0} = \epsilon^2 w + \frac{b}{\epsilon^2 w} + a + 1 = h_2.$$

The first relation can be obtained by the same trick.

Lemma 1 *If $P(\epsilon) = 1$ and n is even (resp. odd), then $(\epsilon + a)(\epsilon + a^2)$ has cubic roots in $K = \mathbb{F}_{2^n}$ (resp. $K(\epsilon)$).*

Proof. Case n even: If $n = 2m$

$$\begin{aligned} P(\epsilon) &= (\epsilon + a)(\epsilon + a^2)(\epsilon + a^4)(\epsilon + a^8)\dots(\epsilon + a^{2^{2m-2}})(\epsilon + a^{2^{2m-1}}) \\ &= (\epsilon + a)(\epsilon + a^2)((\epsilon + a)(\epsilon + a^2))^4 \dots ((\epsilon + a)(\epsilon + a^2))^{2^{2m-2}}. \end{aligned}$$

So $P(\epsilon) = [(\epsilon + a)(\epsilon + a^2)]^k$ with

$$k = 1 + 4 + \dots + 2^{2(m-1)} = \frac{2^n - 1}{3}.$$

Let w be a cubic root of $(\epsilon + a)(\epsilon + a^2)$ in some extension of $\mathbb{F}_2$ (it always exists). Then, from what we have just said:

$$w^{2^n - 1} = w^{3 \cdot \frac{2^n - 1}{3}} = P(\epsilon) = 1,$$

so $w \in K$.

Case n odd: let $n = 2m + 1$, we write $P(\epsilon)$ in two different ways, using Fermat's little theorem:

$$\begin{aligned} P(\epsilon) &= (\epsilon + a)(\epsilon + a^2)(\epsilon + a^4)(\epsilon + a^8)\dots(\epsilon + a^{2^{2m-2}})(\epsilon + a^{2^{2m-1}})(\epsilon + a^{2^{2m}}) \\ P(\epsilon) &= (\epsilon + a^{2^{2m+1}})(\epsilon + a^{2^{2m+2}})(\epsilon + a^{2^{2m+3}})(\epsilon + a^{2^{2m+4}})\dots(\epsilon + a^{2^{4m}})(\epsilon + a^{2^{4m+1}}). \end{aligned}$$

Multiplying these equalities, we get

$$\begin{aligned} P(\epsilon)^2 &= [(\epsilon + a)(\epsilon + a^2)][(\epsilon + a)(\epsilon + a^2)]^4 \dots [(\epsilon + a)(\epsilon + a^2)]^{2^{4m}} \\ &= [(\epsilon + a)(\epsilon + a^2)]^k, \end{aligned}$$

with

$$k = 1 + 4 + \dots + 2^{4m} = \frac{4^{2m+1} - 1}{3} = \frac{2^{2n} - 1}{3}.$$

Then

$$w^{2^{2n} - 1} = w^{3 \cdot \frac{2^{2n} - 1}{3}} = P(\epsilon)^2 = 1,$$

and $w \in K(\epsilon)$ because $[K(\epsilon) : \mathbb{F}_2] = 2n$.

Proposition 4 *Let $P \in \mathcal{I}(n)$ be such that $P(\epsilon) = 1$, then, $\psi(P)$ is reducible.*

Proof. If n is even this is a direct consequence of the preceding Lemma and Proposition 3.

If n is odd, $w \in K(\epsilon)$ by the preceding lemma, so by Proposition 3 we have $K(h) \subset K(\epsilon)$. If $\psi(P)$ is irreducible then for any of its root h we have $[K(h) : \mathbb{F}_2] = 3n$ and $[K(h) : K] = 3$. This is a contradiction so $\psi(P)$ is reducible.

Proposition 5 *Let $P \in \mathcal{I}(n)$, $n > 2$, the polynomial $\psi(P)$ has $3n$ distinct roots and*

$$\psi(P) = \prod_{k=0}^{n-1} T_{a^{2^k}}(X),$$

where $a \in K$ is any root of P.

Proof. If a is a root of P and if h is a root of T_a, then h is a root of $\psi(P)$. Let a' be another root of P, distinct from a, then, the set of roots of T_a is disjoint from the set of roots of $T_{a'}$ because of (1). As P is irreducible, all the roots of P are conjugate and distinct, which concludes the proof of the proposition.

Proposition 6 *Let $P \in \mathcal{I}(n)$, $n > 2$, if $\psi(P)$ is reducible in $\mathbb{F}_2[X]$ then $\psi(P) = RST$, where $\{R, S, T\}$ is an alternate triplet of degree n, moreover $P(\epsilon) = 1$.*

Proof. Let h be a root of $\psi(P)$, we have seen that $K \subset K(h)$. Thus, the degree of the minimal polynomial of h is divisible by n and is $\leq 3n$. This implies that the irreducible factors of $\psi(P)$ in $\mathbb{F}_2[X]$ are at least of degree n, $< 3n$ and multiples of n. Consequently, one of the irreducible factor of $\psi(P)$ is of degree n. Let R be such a factor.

We consider R^{*+} and R^{+*}. They are polynomials of $\mathbb{F}_2[X]$ of degree n and their roots are roots of $\psi(P)$, thus, they divide $\psi(P)$. If R is not alternate, then we obtain an alternate triplet $\{R, R^{*+}, R^{+*}\}$ of degree n, as expected.

If R is alternate let h be a root of R and a such that

$$a = \frac{h^3 + h^2 + 1}{h^2 + h},$$

then, like previously, a is a root of P and $T_a(X)|R$ because h, $\frac{1}{h+1}$ and $\frac{h+1}{h}$ are distinct roots of R. As $R \in \mathbb{F}_2[X]$ is irreducible of degree n, its roots are the h^{2^k}, with $0 \leq k \leq n-1$. So, as Frobenius commutes with our group transformations, $T_{a^{2^k}}|R$ for every k. The $T_{a^{2^k}}$ are distinct because of the preceding proposition. It follows that R has $3n$ distinct roots which is a contradiction, so R cannot be alternate.

We can explicit the decomposition:

$$\psi(P) = R(X)(X+1)^n R\Big(\frac{1}{X+1}\Big) X^n R\Big(\frac{X+1}{X}\Big),$$

with $R \in \mathbb{F}_2[X]$ irreducible of degree $n > 2$, so $\psi(P)(\epsilon) = \epsilon^{3n} R(\epsilon)^3 = R(\epsilon)^3 = 1$.

Then, taking $X = \epsilon$ in the definition of ψ, we get $\psi(P)(\epsilon) = P(\epsilon^2)$. Consequently, $P(\epsilon^2) = P(\epsilon)^2 = 1$ and $P(\epsilon) = 1$.

The Propositions 4 and 6 give a demonstration of the first part of Theorem 4.

3.3 Proof of the Second Part of Theorem 4

Proposition 7 *let $Q \in \mathcal{I}(3n)$, $n > 1$ be an alternate polynomial (resp. $\{R, R^{+*}, R^{*+}\}$ an alternate triplet of degree n), then, there exists a unique $P \in \mathcal{I}(n)$ such that $\psi(P) = Q$ (resp. $\psi(P) = RR^{+*}R^{*+}$).*

Proof. Let a be a root of Q, suppose $type(Q) = 1$ (type 2 is similar), then, $1/(1+a)$ and $1+1/a$ are also roots of Q, moreover, from Proposition 2, we know that $1 + 1/a = a^{2^n}$ and that $1/(1 + a) = a^{2^{2n}}$. Now, we take

$$\mathcal{E} = \{a^{2^i} | 0 \leq i < n\},$$

from what we have just seen, we can write

$$\begin{aligned} Q &= \prod_{a \in \mathcal{E}} (X + a)(X + 1 + \frac{1}{a})(X + \frac{1}{1 + a}) \\ &= (X^2 + X)^n \prod_{a \in \mathcal{E}} \Big(\frac{X^3 + X^2 + 1}{X^2 + X} + \frac{a^3 + a^2 + 1}{a^2 + a} \Big). \end{aligned}$$

We take

$$P = \prod_{a \in \mathcal{E}} \Big(X + \frac{a^3 + a^2 + 1}{a^2 + a} \Big),$$

it is then clear that $\psi(P) = Q$. Let $\frac{a^3+a^2+1}{a^2+a}$ be a root of P, then any other root can be written $a^{2^k}(1 + 1/a^{2^k})(1/(1 + a^{2^k})) = [a(1 + 1/a)(1/(1 + a))]^{2^k}$ for some integer $k \geq 1$. In other terms, the roots of P are conjugate by Frobenius. This implies that $P \in \mathbb{F}_2[X]$.

To show that P is irreducible, we suppose that $P = ST$, with two non constant polynomials in $\mathbb{F}_2[X]$, then, as $\deg ST = degS + \deg T$:

$$\psi(P) = \psi(S)\psi(T) = Q,$$

and this decomposition is non trivial. This is impossible because Q is irreducible, so P is irreducible.

And to show that P is unique, we suppose that there exists another irreducible polynomial P_1 such that $\psi(P_1) = Q$, then, $\frac{a^3+a^2+1}{a^2+a}$ is a root of P_1 by definition, so P and P_1 have the same roots and $P = P_1$.

In the same way, let $\{R, R^{+*}, R^{*+}\}$ be an alternate triplet of degree n and $\mathcal{F}$ the set of the n roots of R, then, taking

$$P = \prod_{a \in \mathcal{F}} \Big(X + \frac{a^3 + a^2 + 1}{a^2 + a} \Big),$$

we obtain a P such that $\psi(P) = RR^{+*}R^{*+}$ and $P \in \mathbb{F}_2[X]$.

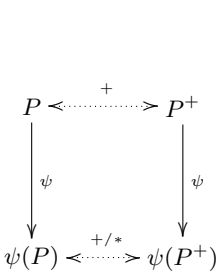
If P is reducible, we can write $P = ST$, with two non constant polynomials in $\mathbb{F}_2[X]$, then

$$\psi(P) = \psi(S)\psi(T) = RR^{+*}R^{*+}.$$

Consequently, we can write $\psi(S) = R$ and this is a contradiction because $\psi(S)$ is alternate and R is not, so P is irreducible. Finally, we can show that P is unique as we did in the first part.

This completes the proof of Theorem 4.

As we did in the previous section, we propose a simple way to construct 2 elements hexagons: let $P \in \mathcal{I}(n)$ be such that $P(\epsilon) \neq 1$, then, using the fact that $\psi \circ + = + \circ \psi$ (the demonstration is obvious), we deduce that $\{\psi(P), \psi(P^+)\}$ is a 2 elements degenerate hexagon of degree $3n$. This is illustrated by the left-side diagram. On the other hand, if $P(\epsilon) = 1$, we have the right-side diagram, where $\{Q_1, Q_3, Q_5\}$ and $\{Q_2, Q_4, Q_6\}$ are the alternate triplets such that $\psi(P) = Q_1Q_3Q_5$ and $\psi(P^+) = Q_2Q_4Q_6$ respectively:



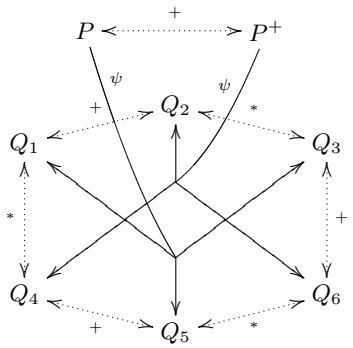


Fig. 2. Construction of hexagons by ψ from an irreducible polynomial P such that $P(\epsilon) \neq 1$ (*left*) and $P(\epsilon) = 1$ (*right*).

3.4 Infinite Sequences of Irreducible Alternate Polynomials

Proposition 8 *If P is an irreducible alternate polynomial of degree > 2, then $\psi(P)$ is an irreducible alternate polynomial.*

Proof. Let P be an irreducible alternate polynomial, by Theorem 4, $P = \psi(Q)$ for some irreducible Q, then $P(\epsilon) = \epsilon$ or ϵ^2. By the same theorem $\psi(P)$ is irreducible and alternate.

Theorem 5 *Let $P \in \mathcal{I}(n)$ be such that $P(\epsilon) = \epsilon$ or ϵ^2, then the iteration ψ on P generates a sequence of alternate irreducible polynomials of degree $3^i n$, $i > 0$.*

Proof. Theorem 4 and Proposition 8 prove this theorem.

3.5 Computation of the Type of the Alternate Irreducible Polynomials

Another consequence of Theorem 4 is that it gives a simple way to compute the type:

Theorem 6 *Let Q be an alternate irreducible polynomial of degree > 2 then its type is 1 (resp. 2) if and only if $Q(\epsilon) = \epsilon$ (resp. $Q(\epsilon) = \epsilon^2$).*

Proof. If $\deg Q = 3$, we verify the theorem by calculus. We now suppose that $\deg Q = 3n$, $n > 1$. We know from the preceding that $Q = \psi(P) \in \mathcal{I}(3n)$ for some irreducible polynomial P and $Q(\epsilon) = P(\epsilon)^2$.

If n is even, let h_0 be a root of Q, then with the notations used in Proposition 3 we have $h_0 = 1 + a + w + \frac{b}{w}$. Consequently:

$$h_0^{2^n} = 1 + a + w^{2^n} + \frac{b}{w^{2^n}}.$$

From the demonstration of Lemma 1 we have $w^{2^n} = P(\epsilon)w$, and

$$h_0^{2^n} = 1 + a + P(\epsilon)w + \frac{b}{P(\epsilon)w}.$$

If $Q(\epsilon) = \epsilon$ (resp. ϵ^2), then $P(\epsilon) = \epsilon^2$ (resp. ϵ) and using again Theorem 4:

$$h_0^{2^n} = 1 + \frac{1}{h_0} \quad (\text{ resp. } h_0^{2^n} = \frac{1}{h_0 + 1}\ ).$$

This is equivalent to say that Q is of type 1 (resp. type 2).

If n is odd the demonstration follows the same line except that we must use $w^{2^{2n}} = P(\epsilon)^2 w = Q(\epsilon)w$. We compute:

$$h_0^{2^{2n}} = 1 + a + Q(\epsilon)w + \frac{b}{Q(\epsilon)w}.$$

If $Q(\epsilon) = \epsilon$ (resp. ϵ^2), then $P(\epsilon) = \epsilon^2$ (resp. ϵ). We obtain:

$$h_0^{2^{2n}} = \frac{1}{h_0 + 1} \quad (\text{ resp. } h_0^{2^{2n}} = 1 + \frac{1}{h_0}\ ).$$

This implies (by iteration for example) that:

$$h_0^{2^n} = 1 + \frac{1}{h_0} \quad (\text{ resp. } h_0^{2^n} = \frac{1}{h_0 + 1}\ ),$$

and Q has type 1 (resp. 2).

3.6 Equivalent Transformations

Other cubic transformations can be used instead of ψ. As previously, they are obtained by a right action of the group elements on ψ.

ψ'
$(X^2+X)^n P(\frac{X^3+X+1}{X^2+X}) = \psi(P^+)$
$(X^3+X+1)^n P(\frac{X^2+X}{X^3+X+1}) = \psi(P^{*+})$
$(X^3+X+1)^n P(\frac{X^3+X^2+1}{X^3+X+1}) = \psi(P^{+*+})$
$(X^3+X^2+1)^n P(\frac{X^3+X+1}{X^3+X^2+1}) = \psi(P^{+*})$
$(X^3+X^2+1)^n P(\frac{X^2+X}{X^3+X^2+1}) = \psi(P^*)$

Table 2. Equivalent cubic transformations.

4 Conclusion

To conclude, in this paper, we defined the different classes of irreducible polynomials obtained by the action of $GL_2(\mathbb{F}_2)$ on $\mathbb{F}_2[X]$. We gave transformations to get invariant polynomials of each class, ways to generate infinite sequences of them and we showed the perfect analogy between self-reciprocal irreducible polynomials and alternate ones.

Now, the perspectives would be to study the action of $GL_2(\mathbb{F}_q)$ on $\mathbb{F}_q[X]$. For that, the first step would be the action of $GL_2(\mathbb{F}_{2^n})$ on $\mathbb{F}_{2^n}[X]$.

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References

1. Cohen, S.D.: The Explicit Construction of Irreducible Polynomials over Finite Fields. Designs, Codes and Cryptography 2, pp. 169-174 (1992)
2. Kyuregyan, M.: Recurrent Methods for Constructing Irreducible Polynomials over GF(2s). Finite Fields and Their Applications 8:3, pp. 52-68 (2002)
3. Kyuregyan, M.: Iterated Constructions of Irreducible Polynomials over Finite Fields with Linearly Independant Roots. Finite Fields and Their Applications 10:1, pp. 323-341 (2004)
4. Meyn, H.: On the Construction of Irreducible Self-reciprocal Polynomials over Finite Fields. Appl. Algebra Eng. Comm. Comp. 1, pp. 43-53 (1990)
5. Meyn, H., Götz, W.: Self-reciprocal Polynomials over Finite Fields. Publ. I.R.M.A. Strasbourg 413/S-21, pp. 82-90 (1990)
6. Michon, J.-F., Ravache, P.: On Different Families of Invariant Irreducible Polynomials over GF(2). Finite Fields and Their Applications 16:3, pp. 163-174 (2010)
7. SAGE, http://www.sagemath.org
8. Varshamov, R.R.: A General Method of Synthesis for Irreducible Polynomials over Galois Fields. Soviet Math. Dokl. 29, pp. 334-336 (1984)
9. Wiedemann, D.: An Iterated Quadratic Extension of GF(2). Fibonacci Quart. 26, pp. 290-295 (1988)