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# Edge coloring of graphs of signed class 1 and 2

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## Abstract

Recently, Behr [1] introduced a notion of the chromatic index of signed graphs and proved that for every signed graph  $(G, \sigma)$  it holds that

$$\Delta(G) \leq \chi'(G, \sigma) \leq \Delta(G) + 1,$$

where  $\Delta(G)$  is the maximum degree of  $G$  and  $\chi'$  denotes its chromatic index.

In general, the chromatic index of  $(G, \sigma)$  depends on both the underlying graph  $G$  and the signature  $\sigma$ . In the paper we study graphs  $G$  for which  $\chi'(G, \sigma)$  does not depend on  $\sigma$ . To this aim we introduce two new classes of graphs, namely  $1^\pm$  and  $2^\pm$ , such that graph  $G$  is of class  $1^\pm$  (respectively,  $2^\pm$ ) if and only if  $\chi'(G, \sigma) = \Delta(G)$  (respectively,  $\chi'(G, \sigma) = \Delta(G) + 1$ ) for all possible signatures  $\sigma$ .

We prove that all wheels, necklaces, complete bipartite graphs  $K_{r,t}$  with  $r \neq t$  and almost all cacti graphs are of class  $1^\pm$ . Moreover, we give sufficient and necessary conditions for a graph to be of class  $2^\pm$ , i.e. we show that these graphs must have odd maximum degree and give examples of such graphs with arbitrary odd maximum degree bigger than 1.

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## 1. Introduction

In this paper we only consider simple, finite and undirected graphs. Graph  $G$  has a set of vertices  $V(G)$  and a set of edges  $E(G)$  with  $n(G)$ ,  $m(G)$  denoting their cardinalities, respectively. By  $\deg_G(v)$  we denote degree of a vertex  $v$  in a graph  $G$  and by  $\Delta(G)$  the maximum degree of all vertices of  $G$ . *Incidence* is a pair  $(v, e)$ , where  $v$  is a vertex,  $e$  is an edge and  $v$  is one of the endpoints of  $e$ . Incidence  $(v, e)$  will be shortly denoted by  $v: e$  and the set of all graph's incidences will be denoted by  $I(G)$ . All other definitions and symbols are consistent with those defined in Diestel [2].

Signed graphs were introduced in 1950s by Harary [3] as a generalization of simple graphs. Their main purpose was to better model social relations of disliking, indifference and liking. A *signed graph* is a pair  $(G, \sigma)$  where  $G$  is a graph and  $\sigma: E(G) \rightarrow \{\pm 1\}$  is a function.  $G$  and  $\sigma$  are called the *underlying graph* of  $(G, \sigma)$  and the *signature* of  $(G, \sigma)$ , respectively. Edge  $e \in E(G)$  will be called *positive* (respectively, *negative*) if and only if  $\sigma(e) = 1$  (respectively,  $\sigma(e) = -1$ ). Cycle in  $(G, \sigma)$  is called *positive* (respectively, *negative*) if product of signs of its edges is positive (respectively, negative). A signed graph with only positive cycles is *balanced* otherwise it is *unbalanced*.

Switching of a set  $V' \subseteq V(G)$  of a signed graph  $(G, \sigma)$  is an operation resulting in a signed graph  $(G, \sigma')$  such that for every edge  $uv \in E(G)$  we have

$$\sigma'(uv) = \begin{cases} -\sigma(uv), & \text{if exactly one of } u, v \text{ belongs to } V', \\ \sigma(uv), & \text{otherwise.} \end{cases}$$

For example, switching a vertex, i.e. switching a subset of vertices of cardinality one, negates the signs of its incident edges. If a signed graph  $(G, \sigma')$  can be obtained by switching some of the vertices of  $(G, \sigma)$ , we say that  $(G, \sigma')$  and  $(G, \sigma)$  are *switching equivalent*. It is well-known that switching equivalence is an equivalence relation in the set of all signed graphs with a fixed underlying graph.

In 2020, Behr [1] introduced a concept of edge coloring of signed graphs as a generalization of ordinary graph edge coloring. Let  $n$  be a positive integer and

$$M_n = \begin{cases} \{0, \pm 1, \dots, \pm k\}, & \text{if } n = 2k + 1, \\ \{\pm 1, \dots, \pm k\}, & \text{if } n = 2k. \end{cases}$$

An *n-edge-coloring* of a signed graph  $(G, \sigma)$  is a function  $f: I(G) \rightarrow M_n$  such that  $f(u: uv) = -\sigma(uv)f(v: uv)$  for each edge  $uv \in E(G)$  and  $f(u: e_1) \neq f(u: e_2)$  for all edges  $e_1 \neq e_2$  such that  $u: e_1, u: e_2 \in I(G)$ . By  $\chi'(G, \sigma)$  we denote *the chromatic index* of a signed graph  $(G, \sigma)$ , i.e. the smallest  $n$  for which  $(G, \sigma)$  has an *n-edge-coloring*.

Behr [1] proved that every signed path can be colored using 2 colors and a signed cycle can be colored with 2 colors if and only if it's balanced. The main result of Behr's article [1] is the generalized Vizing's theorem. We will call it the Behr's theorem to emphasize the importance of Behr's achievement.

**Theorem 1** (Behr [1]).  $\Delta(G) \leq \chi'(G, \sigma) \leq \Delta(G) + 1$  for all signed graphs  $(G, \sigma)$ .  $\square$

In this paper we introduce two new classes of graphs, namely  $1^\pm$  and  $2^\pm$ , such that graph  $G$  is of class  $1^\pm$  (respectively,  $2^\pm$ ) if and only if  $\chi'(G, \sigma) = \Delta(G)$  (respectively,  $\chi'(G, \sigma) = \Delta(G) + 1$ ) for all possible signatures  $\sigma$ . In Section 2 we relate these classes to previously known notions. We also show sufficient and necessary conditions for a regular signed graph  $(G, \sigma)$  to be  $\Delta(G)$ -edge-colorable and use these results to obtain sufficient and necessary conditions for a regular signed graph to be of class  $2^\pm$ . Next, we show that graphs of class  $2^\pm$  must be of odd maximum degree and there are such graphs for any greater than 1 odd value of the degree. In Section 3 we prove that all cacti except cycles are of class  $1^\pm$ . Section 4 deals with wheels—they are all of class  $1^\pm$ . In Section 5 we show that all necklaces except cycles are of class  $1^\pm$ . In Section 6 we deal with complete bipartite graphs by showing that graphs  $K_{r,t}$  are of class  $1^\pm$  for any  $r \neq t$ . The paper ends with some open problems and conjectures.

## 2. $1^\pm$ and $2^\pm$

Behr [4] defined *class ratio*  $\mathcal{C}(G)$  of graph  $G$  to be the number of possible signatures  $\sigma: E(G) \rightarrow \{\pm 1\}$  such that the signed graph  $(G, \sigma)$  is  $\Delta(G)$ -colorable, divided by the number of all possible signatures defined on  $E(G)$ , i.e.  $2^{m(G)}$ . The class ratio is a rational number satisfying  $0 \leq \mathcal{C}(G) \leq 1$  and, by previous definitions,  $G$  is of class  $1^\pm$  (respectively,  $2^\pm$ ) if and only if  $\mathcal{C}(G) = 1$  (respectively,  $\mathcal{C}(G) = 0$ ). Clearly, there are graphs that are neither of class  $1^\pm$  nor  $2^\pm$ . The obvious example is a cycle—all signed cycles are 2-colorable if are balanced and require 3 colors otherwise. It is also easy to see that if  $G$  is of class  $1^\pm$  ( $2^\pm$ ), then it is of class 1 (2), where class 1 (2) is defined as all graphs  $G$  whose ordinary chromatic index equals  $\Delta(G)$  ( $\Delta(G) + 1$ ). This relation provides justification for our notation: class  $X^\pm$  is both a signed counterpart of class  $X$  and a subclass of  $X$ .

Recall that graph  $G$  is *r-regular* if and only if  $\deg_G(v) = r$  for all  $v \in V(G)$ . Recall also that  $M \subseteq E(G)$  is a (*perfect*) *matching* in  $G$  if every vertex of  $G$  is incident to at most (exactly) one edge of  $M$ .

**Lemma 2.** *Let  $G$  be a  $k$ -regular graph and  $\sigma: E(G) \rightarrow \{\pm 1\}$  be a signature.  $\chi'(G, \sigma) = \Delta(G)$  if and only if one of the following conditions holds:*

1.  $k = 2r$  where  $r$  is a positive integer and  $(G, \sigma)$  admits a decomposition into exactly  $r$  spanning edge disjoint 2-regular balanced subgraphs.
2.  $k = 2r + 1$  where  $r$  is a positive integer and  $(G, \sigma)$  admits a decomposition into a perfect matching and exactly  $r$  spanning edge disjoint 2-regular balanced subgraphs.

*Proof.* We prove the equivalence by proving both implications.

( $\Leftarrow$ ) In the  $k = 2r$  case, each of the  $r$  spanning edge disjoint 2-regular balanced subgraphs can be colored using exactly 2 colors, so in total  $2r$  colors are sufficient to color the whole graph. In the  $k = 2r + 1$  case, the matching can be colored using color 0, each of the  $r$  spanning edge disjoint 2-regular balanced subgraphs can be colored using 2 colors, so in total  $2r + 1$  colors are also sufficient. Since  $\Delta(G) = 2r$  in the first case and  $\Delta(G) = 2r + 1$  in the second, we get  $\chi'(G, \sigma) = \Delta(G)$ .

( $\Rightarrow$ ) We consider the two cases separately:

- (1)  $G$  is a  $2r$ -regular graph, so  $m(G) = r \cdot n(G)$ . Let  $c$  be an arbitrary  $\Delta(G)$ -edge-coloring of  $(G, \sigma)$ . For every pair of opposite colors used by  $c$ , edges colored with them form a subgraph with maximum degree not exceeding 2. Clearly, all such subgraphs are edge disjoint. Each of the subgraphs has at most  $n(G)$  edges. There are exactly  $r$  such subgraphs in  $G$  and  $G$  has  $r \cdot n(G)$  edges, so every subgraph contains exactly  $n(G)$  edges. If a subgraph with maximum degree not exceeding 2 contains  $n(G)$  edges, it clearly is spanning and 2-regular. If it can be colored with two colors, it has to be balanced, too.
- (2)  $G$  is a  $(2r + 1)$ -regular graph, so  $m(G) = r \cdot n(G) + \frac{1}{2}n(G)$ . Let  $c$  be an arbitrary  $\Delta(G)$ -edge-coloring of  $(G, \sigma)$ . The set of edges colored in  $c$  with color 0 must form a matching in  $G$ . Matching in  $G$  can have maximum size of  $\frac{1}{2}n(G)$ , so there have to be at least  $r \cdot n(G)$  edges colored by  $c$  with colors different than 0. Such edges form  $r$  subgraphs with maximum degree not exceeding 2, each of them is colored with  $\{-k, k\}$  for some  $k \neq 0$ . Clearly, they are edge disjoint subgraphs with degree not exceeding 2, so they have in total at most  $r \cdot n(G)$  edges. Combining these observations, we conclude that these subgraphs have exactly  $r \cdot n(G)$  edges. This implies that they are 2-regular, spanning and balanced – since otherwise they could not be colored using only colors  $\{-k, k\}$ . This also implies that our matching has  $\frac{1}{2}n(G)$  edges, which means it is perfect.

□

Let  $\mathbf{1}_X$  denote the function  $X \ni x \mapsto 1 \in \{\pm 1\}$ .

**Corollary 3.** *If  $G$  is a graph with even  $\Delta(G)$ , then  $G$  is not of class  $2^\pm$ .*

*Proof.* Every graph has a regular supergraph with the same maximum degree. Let  $H$  be a  $\Delta(G)$ -regular supergraph of  $G$ . It follows from Petersen's 2-factor theorem [5] that  $H$  admits a decomposition into  $r$  edge disjoint 2-regular, spanning subgraphs. Since all edges in a signed graph  $(H, \mathbf{1}_{E(H)})$  are positive, these subgraphs are balanced. Lemma 2 implies that  $\chi'(H, \mathbf{1}_{E(H)}) = \Delta(H)$ . Hence  $\chi'(G, \mathbf{1}_{E(G)}) \leq \chi'(H, \mathbf{1}_{E(H)}) = \Delta(H) = \Delta(G)$ , which shows that  $G$  is not of class  $2^\pm$ . □

**Theorem 4.** *Let  $G$  be a graph.  $G$  is of class  $2^\pm$  if and only if  $\Delta(G)$  is odd and there is no matching  $M$  in  $G$  such that  $\Delta(G \setminus M) < \Delta(G)$ .*

*Proof.* It follows from Corollary 3 that if graph is of class  $2^\pm$  then its maximum degree must be odd.

( $\Rightarrow$ ) Let  $G$  be a graph with odd maximum degree. Suppose that there is a matching  $M \subseteq E(G)$  such that  $\Delta(G \setminus M) < \Delta(G)$ . We will show that  $G$  is not of class  $2^\pm$ . Clearly,  $\Delta(G \setminus M)$  is even since  $\Delta(G \setminus M) = \Delta(G) - 1$ . The proof of Corollary 3 shows that a signed graph  $(G \setminus M, \mathbf{1}_{E(G) \setminus M})$  is  $\Delta(G \setminus M)$ -colorable. Since  $\Delta(G \setminus M)$  is even,  $\Delta(G \setminus M)$ -colorings of  $G \setminus M$  does not use color 0. We can extend any of these colorings to a coloring of  $G$  by using color 0 on  $M$ , so  $(G, \mathbf{1}_{E(G)})$  is  $\Delta(G)$ -edge-colorable. This completes the proof.

( $\Leftarrow$ ) Let us assume  $\Delta(G)$  is odd, there is no matching  $M \subseteq E(G)$  such that  $\Delta(G \setminus M) < \Delta(G)$  and  $G$  is not of class  $2^\pm$ . We show that this leads to a contradiction. There must exist a signature  $\sigma$  such that  $(G, \sigma)$  is  $\Delta(G)$ -edge-colorable. Let  $c$  be an arbitrary  $\Delta(G)$ -edge-coloring of  $(G, \sigma)$ . Edges colored in  $c$  with color 0 must form a matching  $M$ . Clearly,  $\Delta(G \setminus M) = \Delta(G)$ , so graph  $(G \setminus M, \sigma|_{E(G) \setminus M})$  requires at least  $\Delta(G)$  colors different than 0. It follows that  $(G, \sigma)$  cannot be colored using  $\Delta(G)$  colors, so  $G$  is of class  $2^\pm$ .  $\square$

It's easy to observe that signed graphs with  $\Delta = 1$  can be colored using color 0 regardless of their signature, so graphs with  $\Delta = 1$  are of class  $1^\pm$ .

**Theorem 5.** *For every  $k \geq 1$  there is a graph  $G$  of class  $2^\pm$  such that  $\Delta(G) = 2k + 1$ .*

*Proof.* We will describe the procedure of construction of a graph  $G$  such that  $\Delta(G) = 2k + 1$  and  $G$  is of class  $2^\pm$ . We start by constructing a complete bipartite graph  $H$  with parts  $\{u_1, \dots, u_{2k}\}$  and  $\{v_1, \dots, v_{2k}\}$ . Next we create a graph  $H'$  by adding edges  $u_{2i-1}u_{2i}$  for  $1 \leq i \leq k$  to  $H$ , a new vertex  $v$  and edges  $v_iv$  for  $1 \leq i \leq 2k$ . It is easy to see that

$$\deg_{H'}(w) = \begin{cases} 2k, & \text{if } w = v, \\ 2k + 1, & \text{otherwise.} \end{cases}$$

Now we construct the graph  $G$  by creating two disjoint copies of  $H'$  and connecting their lowest degree vertices with path  $P_3$  in such a way that these vertices are the endpoints of the path (see Figure 1 for an example). By  $v'_1, v_c, v'_2$  we denote vertices of that path and by  $v'_1v_c, v_cv'_2$ —its edges. Let's observe that  $\deg_G(v_c) = 2$  and other vertices of  $G$  have degree  $2k + 1$ .

To complete the proof it suffices to show that  $G$  is of class  $2^\pm$ . Clearly,  $\Delta(G)$  is odd, so Theorem 4 tells us that we need to show that there is no matching  $M \subseteq E(G)$  such that  $\Delta(G \setminus M) < \Delta(G)$ . Let's assume there is such a matching  $M$  in  $G$ . We show that it leads to a contradiction. Clearly,  $M$  must cover all the vertices of  $G$  other than  $v_c$ —it must cover all the vertices of both copies of  $H'$ . Let's observe that  $H'$  has an odd number of vertices, so there is no perfect matching in it. It means that in order to cover vertices of both copies of  $H'$ ,  $M$  must contain edges  $v'_1v_c$  and  $v_cv'_2$ . Then  $M$  contains two edges incident with  $v_c$ , so  $M$  is not a matching, a contradiction that completes the proof.  $\square$

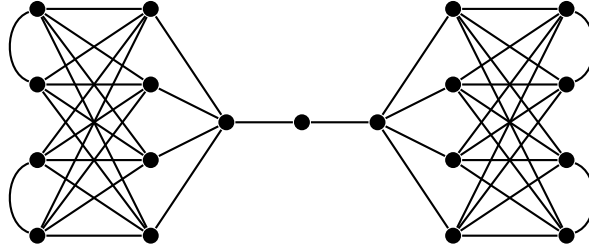


Figure 1: Example graph of class  $2^\pm$  with  $\Delta = 5$ .

### 3. Cacti

A connected graph  $G$  is called *cactus* if and only if every edge of  $G$  belongs to at most one cycle. Clearly every tree is a cactus as it doesn't have cycles at all. It is a well known fact [6] that graph  $G$  is a cactus if and only if there exists a sequence of graphs  $G_1, \dots, G_k$  called a *decomposition of  $G$* , such that:

1.  $G_i$  is a cycle or a path  $P_2$  for  $1 \leq i \leq k$ ;
2.  $G_i$  has exactly one vertex in common with  $G_1 \cup \dots \cup G_{i-1}$  for  $2 \leq i \leq k$ ;
3.  $G = G_1 \cup \dots \cup G_k$ .

We prove that almost all signed cacti are of class  $1^\pm$ .

**Theorem 6.** *Let  $G$  be a cactus. If  $G$  is not a cycle, then  $G$  is of class  $1^\pm$ .*

*Proof.* Let  $\sigma: E(G) \rightarrow \{\pm 1\}$  be a function. If  $\Delta(G) \leq 2$ , then  $G$  is a path, so our claim is obvious. Therefore we assume that  $\Delta(G) \geq 3$  in the remainder of the proof.

Let  $G_1, \dots, G_k$  be an arbitrary decomposition of  $G$  meeting the three above conditions. By  $G'_l$  we denote  $G_1 \cup \dots \cup G_l$ . Clearly,  $G'_k = G$ . We will use induction on  $l$  to prove that  $(G'_l, \sigma|_{E(G'_l)})$  is  $\Delta(G)$ -edge-colorable. Let us observe that  $G'_1$  is a cycle or path, so  $(G'_1, \sigma|_{E(G'_1)})$  can be colored using  $\Delta(G) \geq 3$  colors. Let's assume that  $(G'_l, \sigma|_{E(G'_l)})$  is  $\Delta(G)$ -edge-colorable for any  $l < x$ . We will show that  $(G'_x, \sigma|_{E(G'_x)})$  can be colored using at most  $\Delta(G)$  colors. Let us note that  $G'_x = G'_{x-1} \cup G_x$  and  $(G'_{x-1}, \sigma|_{E(G'_{x-1})})$  is  $\Delta(G)$ -edge-colorable. By  $c$  we denote an arbitrary  $\Delta(G)$ -edge-coloring of  $(G'_{x-1}, \sigma|_{E(G'_{x-1})})$ . It follows from the definition of the decomposition of  $G$  that  $G_x$  is either a path  $P_2$  or a cycle and  $G_x$  has exactly one vertex in common with  $G'_{x-1}$ . Let us consider the two cases separately.

1.  $G_x$  is a path  $P_2$ . By  $u, v$  we denote vertices of  $G_x$  and without loss of generality we assume  $u \in V(G'_{x-1})$ . Clearly, there are at most  $\Delta(G) - 1$  edges incident to  $u$  in  $G'_{x-1}$ , so there must be a color  $\alpha \in M_{\Delta(G)}$  not used on any of these edges by  $c$ . Let's color  $u: uv$  with  $\alpha$  and  $v: uv$  with  $-\sigma(uv)\alpha$ . That way  $(G'_x, \sigma|_{E(G'_x)})$  can be colored using  $\Delta(G)$  colors.
2.  $G_x$  is a cycle. By  $u \in V(G_x)$  we denote a vertex such that  $u \in V(G'_{x-1})$ . By  $uv_1, uv_2$  we denote two different edges of  $G_x$ . Clearly, there are at

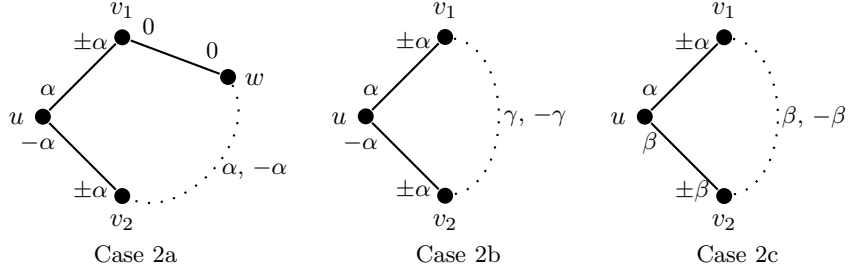


Figure 2: Three possible edge colorings of  $(G_x, \sigma|_{E(G'_x)})$  considered in the case 2 of the proof of Theorem 6.

most  $\Delta(G) - 2$  edges incident to  $u$  in  $G'_{x-1}$ , so there must be two different colors  $\alpha, \beta \in M_{\Delta(G)}$  not used on any of these edges by  $c$ . Let us consider three possible subcases.

- (a)  $\alpha = -\beta$  and  $\Delta(G) = 3$ . Since  $\Delta(G) = 3$ ,  $0 \in M_{\Delta(G)}$  and  $\alpha \neq 0$ ,  $\beta \neq 0$ . By  $w$  we denote vertex in  $G_x$  such that  $w$  is adjacent to  $v_1$  and  $w \neq u$ . Incidences  $v_1: v_1w$  and  $w: v_1w$  of  $G'_x$  can be colored with color 0. Let us observe that other edges of  $G_x$  form a path, so their incidences can be colored with colors  $\pm\alpha$  in the coloring of  $(G'_x, \sigma|_{E(G'_x)})$ .
- (b)  $\alpha = -\beta$  and  $\Delta(G) > 3$ . Since  $\Delta(G) > 3$ , there must be colors  $\gamma, -\gamma \in M_{\Delta(G)}$  such that  $\gamma \neq 0$ ,  $\gamma \neq \alpha$ ,  $\gamma \neq \beta$ . Let us observe that edges  $v_1u, uv_2$  form a path in  $G_x$ , so their incidences can be colored with colors  $\pm\alpha$  in the coloring of  $(G'_x, \sigma|_{E(G'_x)})$ . Other edges of  $G_x$  also form a path—can be colored with colors  $\pm\gamma$ .
- (c)  $\alpha \neq -\beta$ . Without loss of generality let us assume  $\beta \neq 0$ . Incidence  $u: uv_1$  can be colored with  $\alpha$  in the coloring of  $(G'_x, \sigma|_{E(G'_x)})$  and  $v_1: uv_1$ —with  $-\sigma(uv_1)\alpha$ . Edges of  $G_x$  other than  $uv_1$  form a path and can be colored with  $\pm\beta$  in such a way that  $u: uv_2$  gets color  $\beta$ .

□

The decomposition of an  $n$ -vertex cactus can be done in  $O(n)$  time. This implies that the coloring procedure being a part of the above proof is also linear.

#### 4. Wheels

Wheel  $W_n$  is a graph on  $n$  vertices consisting of a cycle  $C_{n-1}$ , a vertex  $v \notin V(C_{n-1})$  and edges between  $v$  and all the vertices of a cycle  $C_{n-1}$ .

**Theorem 7.** *All wheels are of class  $1^\pm$ .*

*Proof.* We consider three cases separately:

1.  $n = 4$ . Wheel  $W_4$  is a complete graph  $K_4$ . It's easy to observe that  $W_4$  has a decomposition into three perfect matchings and edges of any two of

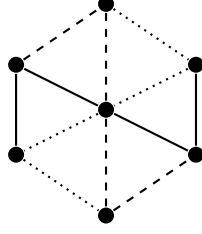


Figure 3: The decomposition of  $W_7$  into three paths. Distinct paths are marked with solid, dashed and dotted lines.

them create a spanning cycle. Let us consider an arbitrary signed graph  $(W_4, \sigma)$  and its decomposition  $D$  into three perfect matchings. If there is an odd number of negative edges in  $(W_4, \sigma)$ , there must be a matching in  $D$  with an odd number of negative edges. Remaining edges create a balanced spanning cycle. In the opposite case—when there is an even number of negative edges in  $(W_4, \sigma)$ , there must be a matching in  $D$  with an even number of negative edges. Remaining edges create a balanced spanning cycle. In both cases,  $(W_4, \sigma)$  has a decomposition into a perfect matching and a balanced spanning cycle. Obviously,  $W_4$  is 3-regular, so it follows from Lemma 2 that  $(W_4, \sigma)$  is  $\Delta$ -edge-colorable.

2.  $n = 2k + 1, k \geq 2$ . Let  $V(W_n) = \{u, v_0, \dots, v_{n-2}\}$  and  $E(G) = \{uv_0, \dots, uv_{n-2}\} \cup \{v_0v_1, v_1v_2, \dots, v_{n-2}v_0\}$ . For  $0 \leq i \leq k - 2$ , let  $G_i$  be a subgraph of  $W_n$ , such that  $V(G_i) = \{v_i, v_{i+1}, u, v_{i+k}, v_{i+k+1}\}$  and  $E(G_i) = \{v_{i+1}v_i, v_iv_i, uv_{i+k}, v_{i+k}v_{i+k+1}\}$ . Clearly,  $G_i$  is a path and for any different  $i_1, i_2$  paths  $G_{i_1}, G_{i_2}$  are edge disjoint.

Let  $P$  be a subgraph of  $W_n$ , such that  $V(P) = \{v_{k-1}, v_k, u, v_{n-2}, v_0\}$  and  $E(P) = \{v_kv_{k-1}, v_{k-1}u, uv_{n-2}, v_{n-2}v_0\}$ . Clearly,  $P$  is a path. We will show that paths  $P, G_i$  are edge disjoint for any  $0 \leq i \leq k - 2$ . It's sufficient to show that  $\{v_iv_i, uv_{i+k}\} \cap \{v_{k-1}u, uv_{n-2}\} = \emptyset$  and  $\{v_{i+1}v_i, v_{i+k}v_{i+k+1}\} \cap \{v_kv_{k-1}, v_{n-2}v_0\} = \emptyset$ . The first equality follows from the fact that  $i < k - 1 < i + k < n - 2$ . Let's assume that the second equality is false, so one of the following cases must hold:

- (a)  $v_{i+1}v_i = v_kv_{k-1}$ . Impossible because  $i < k - 1 < k$ .
- (b)  $v_{i+1}v_i = v_{n-2}v_0$ . Impossible because  $i < i + 1 < n - 2$ .
- (c)  $v_{i+k}v_{i+k+1} = v_kv_{k-1}$ . Impossible because  $k - 1 < i + k < i + k + 1$ .
- (d)  $v_{i+k}v_{i+k+1} = v_{n-2}v_0$ . Impossible because  $i + k \neq 0$  and  $i + k + 1 \neq 0$ .

Let's observe that  $m(\bigcup_{i=0}^{k-2} G_i) + m(P) = 4(k - 1) + 4 = 4k = m(W_n)$ , so

$\bigcup_{i=0}^{k-2} G_i \cup P = W_n$  and  $W_n$  has a decomposition into exactly  $k = \Delta(W_n)/2$  paths. It follows that any signed graph with  $W_n$  as an underlying graph can be colored using  $\Delta(W_n)$  colors because edges of each path can be colored with exactly two colors.

3.  $n = 2k, k \geq 3$ . Let us consider wheel  $W_{n-1}$ . By  $u$  we denote the center vertex of  $W_{n-1}$ —the one with degree  $\Delta(W_{n-1})$  and by  $v_1, v_2$  we denote

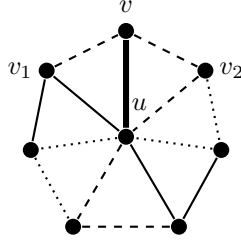


Figure 4: The decomposition of  $W_8$  into three paths and a single edge. Edge is marked with a bold line and distinct paths are marked with solid, dashed and dotted lines.

arbitrary vertices adjacent to  $u$  such that  $v_1v_2 \in E(W_{n-1})$ . It follows from the previous case that  $W_{n-1}$  admits a decomposition  $D$  into  $k - 1$  paths. We construct graph  $W'_{n-1}$  from  $W_{n-1}$  by adding new vertex  $v$  and replacing edge  $v_1v_2$  by path  $v_1, v, v_2$ . It's easy to observe that  $W'_{n-1}$  also admits some decomposition  $D'$  into  $k - 1$  paths. It can be constructed directly from  $D$ —path containing edge  $v_1v_2$  in  $D$  contains edges  $v_1v, vv_2$  in  $D'$ . Clearly, graph  $W_n$  can be obtained from  $W'_{n-1}$  by adding edge  $vu$ . It follows that  $W_n$  admits a decomposition into  $k - 1$  paths and a single edge, so every signed graph with  $W_n$  as an underlying graph can be colored using  $2k - 1 = \Delta(W_n)$  colors.

□

## 5. Necklaces

Necklace is a connected graph having a decomposition into  $k \geq 2$  paths, in which all vertices except selected two— $u, v$ —are different. Vertices  $u, v$  are starting and ending vertices of all paths. If  $k = 2$ , a necklace is a cycle, so it's possible that a signed graph on such necklace is negative and requires  $\Delta + 1$  colors to be properly colored. We prove necklaces are  $\Delta$ -colorable in all other cases.

**Theorem 8.** *Let  $G$  be a necklace. If  $G$  is not a cycle, then  $G$  is of class  $1^\pm$ .*

*Proof.*  $G$  is not a cycle, so  $\Delta(G) > 3$ . By  $D = G_1, \dots, G_{\Delta(G)}$  we denote the decomposition of  $G$  into paths. By  $u, v$  we denote the starting and ending vertices of all the paths from  $D$ . Without loss of generality, we assume that  $m(G_i) \leq m(G_{i+1})$  for  $1 \leq i < \Delta(G)$ . It follows that if there is a path of length 1 in  $D$ , it must be  $G_1$ . By  $u_i^j$  we denote a vertex belonging to path  $G_i$  such that the distance between  $u_i^j$  and  $u$  in  $G_i$  is  $j$ . The definition of  $v_i^j$  is analogous. By  $S = (G, \sigma)$  we denote an arbitrary signed graph with  $G$  being its underlying graph. We consider two cases separately:

1.  $\Delta(G) = 2k + 1, k \geq 1$ . We will construct necklace  $S$  starting with necklace  $S_0$  that contains three paths from  $D$ — $G_1, G_2, G_3$  and extending it with

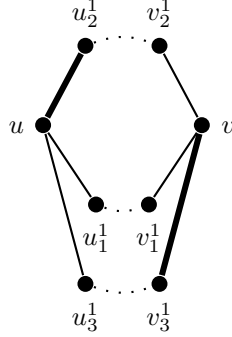


Figure 5: Necklace  $S_0$  considered in case 1 of the proof of Theorem 8. Edges colored with color 0 are marked with bold lines. Remaining edges span a path.

consecutive pairs of paths. By  $S_p$  we denote the necklace constructed from  $S_{p-1}$  by adding paths  $G_{2p+2}, G_{2p+3}$ . Clearly,  $\Delta(S_0) = 3$ . We color edges  $uu_2^1, vv_3^1$  with color 0. Let us observe that the remaining edges of  $S_0$  span a path that contains following vertices:  $u_2^1, \dots, v_2^1, v, v_1^1, \dots, u_1^1, u, u_3^1, \dots, v_3^1$ . It can be colored with colors  $\pm 1$ .

Necklace  $S_p$  is constructed from  $S_{p-1}$  by adding paths  $G_{2p+2}, G_{2p+3}$ . Clearly,  $\Delta(S_p) = \Delta(S_{p-1}) + 2$ , so there are two new colors  $\pm\alpha$  available for coloring of  $S_p$ . Let us assume there is an edge  $uu_i^1$  in  $S_{p-1}$  colored with color 0. We can color edge  $uu_{2p+2}^1$  with color 0 and path  $u_i^1, u, u_{2p+3}^1, \dots, v_{2p+3}^1, v, v_{2p+2}^1, \dots, u_{2p+2}^1$  with colors  $\pm\alpha$ . That way edge  $uu_i^1$ , previously colored with 0, was recolored. Let us observe that this gives us a  $\Delta(S_p)$ -edge-coloring of  $S_p$ . Let us also observe that there is still an edge incident to  $u$  such that it's colored with color 0 in the coloring of  $S_p$ , so such an edge must exist in the colorings of all  $S_0, \dots, S_p$ .

2.  $\Delta(G) = 2k, k \geq 2$ . We will construct necklace  $S$  starting with necklace  $S_0$  that contains four paths from  $D-G_1, \dots, G_4$  and extending it with consecutive pairs of paths. By  $S_p$  we denote the necklace constructed from  $S_{p-1}$  by adding paths  $G_{2p+3}, G_{2p+4}$ . Clearly,  $\Delta(S_0) = 4$ . Let us observe that vertices  $u_3^1, u, u_1^1, \dots, v_1^1, v, v_2^1$  span a path in  $S_0$  and its edges can be colored with colors  $\pm 1$ . We can assume that incidence  $u: uu_3^1$  gets color 1. Let us observe that the remaining edges also span a path, and its edges can be colored with colors  $\pm 2$ .

Necklace  $S_p$  is constructed from  $S_{p-1}$  by adding paths  $G_{2p+3}, G_{2p+4}$ . Clearly,  $\Delta(S_p) = \Delta(S_{p-1}) + 2$ , so there are two new colors  $\pm\alpha$  available for coloring of  $S_p$ . Let us assume there is an incidence  $u: uu_i^1$  in  $S_{p-1}$  colored with color 1. We can color incidence  $u: uu_{2p+3}^1$  with color 1 and  $u_{2p+3}^1: uu_{2p+3}^1$ —with color  $-\sigma(uu_{2p+3}^1)$ . Path  $u_i^1, u, u_{2p+4}^1, \dots, v_{2p+4}^1, v, v_{2p+3}^1, \dots, u_{2p+3}^1$  can be colored with colors  $\pm\alpha$ . That way incidence  $u: uu_i^1$ , previously colored with 1, was recolored. Let us observe that this gives us a  $\Delta(S_p)$ -edge-coloring of  $S_p$ . Let us also observe that there is

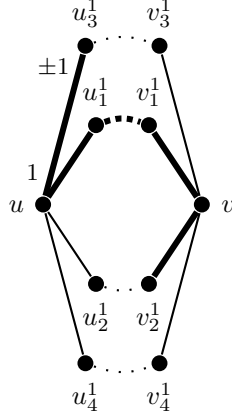


Figure 6: Necklace  $S_0$  considered in case 2 of the proof of Theorem 8. Edges colored with colors  $\pm 1$  are marked with bold lines. Remaining edges are colored with colors  $\pm 2$ .

still an incidence incident to  $u$  such that it's colored with color 1 in the coloring of  $S_p$ , so such an incidence must exist in the colorings of all  $S_0, \dots, S_p$ .

□

## 6. Complete bipartite graphs

A graph is bipartite if and only if its vertices can be divided into two sets  $V_1, V_2$  such that each edge has endpoints in both sets. These sets are usually called *parts*. Graph is a complete bipartite graph if and only if it's bipartite and for every  $u \in V_1, v \in V_2$  there is an edge  $uv$ .

**Theorem 9.** *Let  $r, t$  be positive integers and  $G$  be a complete bipartite graph  $K_{r,t}$ . If  $r \neq t$ , then  $G$  is of class  $1^\pm$ .*

*Proof.* Without loss of generality, we assume that  $r < t$ . Let  $t = 2s + k$ ,  $s \in \mathbf{N}$  and  $k \in \{0, 1\}$ . Let  $V(G) = \{u_0, \dots, u_{r-1}\} \cup \{v_0, \dots, v_{t-1}\}$  and  $E(G) = \{u_i v_j : 0 \leq i \leq r-1, 0 \leq j \leq t-1\}$ .

Let  $0 \leq j \leq s-1$ . Let  $G_j$  be a subgraph of  $G$  such that  $V(G_j) = \{u_0, \dots, u_{r-1}\} \cup \{v_{2j}, \dots, v_{(2j+r) \bmod t}\}$  and  $E(G_j) = \{u_i v_{(i+2j) \bmod t} : 0 \leq i \leq r-1\} \cup \{u_i v_{(i+2j+1) \bmod t} : 0 \leq i \leq r-1\}$ . Let us observe that:

- $\deg_{G_j}(u_i) = 2$  for  $i = 0, \dots, r-1$ ;
- $\deg_{G_j}(v_{2j}) = \deg_{G_j}(v_{(2j+r) \bmod t}) = 1$ ;
- $\deg_{G_j}(v_{(2j+i) \bmod t}) = 2$  for  $i = 1, \dots, r-1$ .

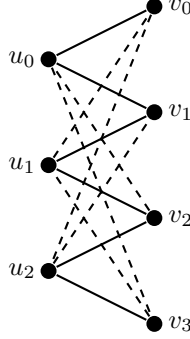


Figure 7: Paths  $G_0$  (solid lines) and  $G_1$  (dashed lines) created in the procedure of coloring of complete bipartite graph  $K_{3,4}$  described in the proof of Theorem 9.

It follows from the definition of  $G_j$  and above observations that  $G_j$  is a path. We will show that any two distinct paths  $G_{j_1}, G_{j_2}$  are edge disjoint. Let us assume it's not true. Then one of the following cases must hold for some  $i_1, i_2 \in \{0, \dots, r-1\}$ .

1.  $u_{i_1}v_{(i_1+2j_1) \bmod t} = u_{i_2}v_{(i_2+2j_2) \bmod t}$ . Clearly,  $u_{i_1} = u_{i_2}$ , so  $i_1 = i_2$ . Moreover,  $v_{(i_1+2j_1) \bmod t} = v_{(i_2+2j_2) \bmod t}$ , so  $2j_1 \equiv 2j_2 \pmod{t}$ . It follows that  $0 \equiv 2(j_1 - j_2) \pmod{t}$ . Let us consider two cases separately:
  - (a)  $t = 2s + 1$ . Then  $t \equiv 2s + 1 \pmod{t}$ , so  $0 \equiv 2s + 1 \pmod{t}$  and  $2s \equiv -1 \pmod{t}$ . We observe that if  $0 \equiv 2(j_1 - j_2) \pmod{t}$ , then  $0 \equiv 2s(j_1 - j_2) \pmod{t}$ , so  $0 \equiv j_2 - j_1 \pmod{t}$  and  $j_1 \equiv j_2 \pmod{t}$ . It follows that  $j_1 = j_2$ , a contradiction.
  - (b)  $t = 2s$ . If  $0 \equiv 2(j_1 - j_2) \pmod{t}$ , then  $0 \equiv 2(j_1 - j_2) \pmod{2s}$  and  $0 \equiv j_1 - j_2 \pmod{s}$ , so  $j_1 \equiv j_2 \pmod{s}$ . It follows that  $j_1 = j_2$ , a contradiction.
2.  $u_{i_1}v_{(i_1+2j_1+1) \bmod t} = u_{i_2}v_{(i_2+2j_2+1) \bmod t}$ . We observe that  $i_1 = i_2$ , so  $2j_1 + 1 \equiv 2j_2 + 1 \pmod{t}$  and then  $2(j_1 - j_2) \equiv 0 \pmod{t}$ . The remaining part of that case is identical to the previous case and leads to a contradiction.
3. Either  $u_{i_1}v_{(i_1+2j_1) \bmod t} = u_{i_2}v_{(i_2+2j_2+1) \bmod t}$  or  $u_{i_1}v_{(i_1+2j_1+1) \bmod t} = u_{i_2}v_{(i_2+2j_2) \bmod t}$ . Without loss of generality, we assume that the first of two holds. Clearly,  $i_1 = i_2$ , so  $2j_1 \equiv 2j_2 + 1 \pmod{t}$  and then  $2(j_1 - j_2) \equiv 1 \pmod{t}$ . We consider two cases separately:
  - (a)  $t = 2s + 1$ . Then  $2s \equiv -1 \pmod{t}$ . Let us observe that  $2s(j_1 - j_2) \equiv s \pmod{t}$ , so  $j_2 - j_1 \equiv s \pmod{t}$ . It's a contradiction because  $j_1, j_2 \in \{0, \dots, s-1\}$ .
  - (b)  $t = 2s$ . We observe that  $2(j_1 - j_2) - 1 \equiv 0 \pmod{2s}$ , so the odd number  $2(j_1 - j_2) - 1$  is divisible by an even number  $2s$ , a contradiction.

All of the cases lead to a contradiction, so paths  $G_{j_1}, G_{j_2}$  are edge disjoint. Clearly,  $\bigcup_{j=0}^{s-1} G_j \subseteq G$ . We will continue the proof separately for two cases:

1.  $t = 2s$ . Since  $G$  is a complete bipartite graph,  $m(G) = rt = 2rs$ . We observe that  $m(G_j) = 2r$  for  $0 \leq j \leq s-1$ . It follows that  $m(\bigcup_{j=0}^{s-1} G_j) = \sum_{j=0}^{s-1} m(G_j) = 2rs$ , so  $m(G) = m(\bigcup_{j=0}^{s-1} G_j)$ . Since paths  $G_j$  are edge disjoint,  $G = \bigcup_{j=0}^{s-1} G_j$ , so  $G$  admits a decomposition into exactly  $s$  paths. Let us observe that an arbitrary signed graph with  $G$  being its underlying graph can be colored with  $2s = \Delta(G)$  colors, since each path can be colored with just two colors. It completes a proof for this case.
2.  $t = 2s + 1$ . It's clear that  $m(G) = rt = r(2s + 1)$  and  $m(\bigcup_{j=0}^{s-1} G_j) = 2rs$ . Let  $M$  be a subgraph of  $G$  such that  $V(M) = \{u_0, \dots, u_{r-1}\} \cup \{v_{(i-1) \bmod t} : 0 \leq i \leq r-1\}$  and  $E(M) = \{u_i v_{(i-1) \bmod t} : 0 \leq i \leq r-1\}$ . It's easy to observe that all the vertices of  $M$  have degree equal to 1, so  $M$  is a matching.

We will show that  $E(M) \cap E(\bigcup_{j=0}^{s-1} G_j) = \emptyset$ . We assume that it's not true.

Then for some  $j$  and  $i_1, i_2 \in \{0, \dots, r-1\}$  one of the following cases must hold:

- (a)  $u_{i_1} v_{(i_1-1) \bmod t} = u_{i_2} v_{(i_2+2j) \bmod t}$ . Clearly,  $i_1 = i_2$ . We observe that  $i_1 - 1 \equiv i_2 + 2j \pmod{t}$ , so  $2j + 1 \equiv 0 \pmod{t}$  and then  $2js + s \equiv 0 \pmod{t}$ . Since  $t = 2s + 1$ ,  $2s \equiv -1 \pmod{t}$ , so  $j \equiv s \pmod{t}$ . It's a contradiction because  $j \in \{0, \dots, s-1\}$ .
- (b)  $u_{i_1} v_{(i_1-1) \bmod t} = u_{i_2} v_{(i_2+2j+1) \bmod t}$ . It holds that  $i_1 - 1 \equiv i_2 + 2j + 1 \pmod{t}$ , so  $2j \equiv -2 \pmod{t}$  and then  $2js \equiv -2s \pmod{t}$ . Since  $t = 2s + 1$ ,  $j + 1 \equiv 0 \pmod{t}$ . It's a contradiction because  $t = 2s + 1$  and  $j \in \{0, \dots, s-1\}$ .

All of the cases lead to a contradiction, so  $E(M) \cap E(\bigcup_{j=0}^{s-1} G_j) = \emptyset$ .

Since  $m(M) = r$ ,  $G = E(M) \cup E(\bigcup_{j=0}^{s-1} G_j)$ ,  $G$  admits a decomposition into exactly  $s$  paths and one matching. Let us observe that an arbitrary signed graph with  $G$  being its underlying graph can be colored with  $2s + 1 = \Delta(G)$  colors, since each path can be colored with two non-zero colors and a matching can be colored with color 0.

□

We briefly consider complete bipartite graphs with equal parts. Complete bipartite graph  $K_{r,r}$  is an  $r$ -regular graph, so it follows from Lemma 2 that if  $r$  is

even and  $K_{r,r}$  has an odd number of negative edges, it cannot be colored using  $\Delta$  colors. We hypothesize that for all other cases, complete bipartite graphs with equal parts are  $\Delta$ -colorable. If the conjecture is true, graphs with odd  $r$  must have a perfect matching  $M$  such that  $K_{r,r} \setminus M$  have an even number of negative edges and have a decomposition into  $(r-1)/2$  spanning 2-regular subgraphs with positive cycles being their components.

**Conjecture 10.** *Let  $r \in \mathbf{N}$  and  $S$  be a signed complete bipartite graph  $K_{r,r}$ .  $\chi'(S) = \Delta(S)$  if one of the following conditions holds:*

1.  $r$  is odd;
2.  $r$  is even and there is an even number of negative edges in  $S$ .

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