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# Observer-based feedback-control for the stabilization of a class of parabolic systems

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## Abstract

We consider the stabilization of a class of linear evolution systems  $z' = Az + Bv$  under the observation  $y = Cz$  by means of a finite dimensional control  $v$ . The control is based on the design of a Luenberger observer which can be infinite or finite dimensional (of dimension large enough). In the infinite dimensional case, the operator  $A$  is supposed to generate an analytical semigroup with compact resolvent and the operators  $B$  and  $C$  are unbounded operators whereas in the finite dimensional case,  $A$  is assumed to be a self-adjoint operator with compact resolvent,  $B$  and  $C$  are supposed to be bounded operators. In both cases, we show that if  $(A, B)$  and  $(A, C)$  verify the Fattorini-Hautus Criterion, then we can construct an observer-based control  $v$  of finite dimension (greater or equal than largest geometric multiplicity of the unstable eigenvalues of  $A$ ) such that the evolution problem is exponentially stable. As an application, we study the stabilization of the  $N$  dimensional convection-diffusion system with Dirichlet boundary control and an internal observation.

**Keywords:** Parabolic systems, Feedback control, Stabilization, Luenberger observers.

**2010 Mathematics Subject Classification** 93B53, 93D15, 93C20.

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# 1 Introduction and main results

Given  $b \in L^2(0, 1)$ , consider the one-dimensional controlled heat equation

$$\begin{cases} \partial_t z(t, x) = \partial_{xx} z(t, x) + b(x)v(t), & t > 0, \quad x \in (0, 1), \\ \partial_x z(t, 0) = 0, \quad z(t, 1) = 0, \\ z(0) = z^0. \end{cases} \quad (1.1)$$

Obviously, the open-loop system (i.e. for  $v = 0$ ) is exponentially stable, with a decay rate defined by the smallest eigenvalue of the underlying operator describing the free dynamics (namely the positive definite self-adjoint operator  $-\partial_{xx}$  with Neumann boundary condition at  $x = 0$  and Dirichlet boundary condition at  $x = 1$ ).

Based on the observation

$$y(t) = \int_0^1 c(x)z(t, x) dx, \quad (1.2)$$

where  $c \in L^2(0, 1)$ , a natural question that arises is to know whether it is possible to design a finite dimensional feedback control  $v$ , such that the closed-loop system (1.1) is exponentially stable with an arbitrary prescribed decay rate  $\sigma > 0$ . It is a recent work, a positive and constructive answer to this question has been proposed by Katz and Fridman [11], using an observer-based feedback control. More precisely, the authors proposed feasible design conditions for the construction of such controls for a more general 1D reaction-diffusion equation with variable coefficients (i.e. for a free dynamics described by an operator of the form  $\partial_x(p(x)\partial_x) - q(x)$ ).

In this paper, our objective is to generalize this result to a large class of parabolic systems, possibly multi-dimensional and involving unbounded control and/or observation operators. More precisely, given three Hilbert spaces  $\mathbb{H}$  (the state space),  $\mathbb{U}$  (the control space) and  $\mathbb{Y}$  (the observation space), consider the linear infinite dimensional system

$$\begin{cases} z'(t) = Az(t) + Bv(t), \\ z(0) = z^0, \\ y(t) = Cz(t), \end{cases} \quad (1.3)$$

where  $A : \mathcal{D}(A) \rightarrow \mathbb{H}$  is an unbounded operator,  $B \in \mathcal{L}(\mathbb{U}, (\mathcal{D}(A^*))')$  and  $C \in \mathcal{L}(\mathcal{D}(A), \mathbb{Y})$ . Given  $\sigma > 0$ , the goal of this paper is to prove the existence of an observer-based control  $v$  such that the solution of (1.3) is exponentially stable, with a decay rate  $-\sigma$ :

$$\|z(t)\|_{\mathbb{H}} \leq M e^{-\sigma t} \|z^0\|_{\mathbb{H}}.$$

We will investigate two classes of systems, depending on whether the observer used is infinite-dimensional (IDO case) or finite-dimensional (FDO case). We will make the following assumptions on  $A$ ,  $B$  and  $C$  in these two cases (below,  $\rho(A)$  denote the resolvent set of  $A$ ):

## • Infinite-Dimensional Observer (IDO)

$$A \text{ is an analytic operator with compact resolvent on } \mathbb{H}, \quad (\text{H1.A})$$

$$(\mu_0 \text{Id} - A)^{-\gamma} B \in \mathcal{L}(\mathbb{U}, \mathbb{H}) \text{ is a linear bounded operator for some } \gamma \in [0, 1) \text{ and } \mu_0 \in \rho(A), \quad (\text{H1.B})$$

$$C(\mu_0 \text{Id} - A)^{-\hat{\gamma}} \in \mathcal{L}(\mathbb{H}, \mathbb{Y}) \text{ is a linear bounded operator for some } \hat{\gamma} \in [0, 1) \text{ and } \mu_0 \in \rho(A), \quad (\text{H1.C})$$

$$\begin{cases} \forall \varepsilon \in \mathcal{D}(A^*), \forall \lambda \in \mathbb{C}, \text{Re } \lambda \geq -\sigma, \quad A^* \varepsilon = \bar{\lambda} \varepsilon \quad \text{and} \quad B^* \varepsilon = 0 \implies \varepsilon = 0, \\ \forall \varepsilon \in \mathcal{D}(A), \forall \lambda \in \mathbb{C}, \text{Re } \lambda \geq -\sigma, \quad A \varepsilon = \lambda \varepsilon \quad \text{and} \quad C \varepsilon = 0 \implies \varepsilon = 0. \end{cases} \quad (\text{H1.D})$$

## • Finite-Dimensional Observer (FDO)

$$A \text{ is a self-adjoint operator with compact resolvent,} \quad (\text{H2.A})$$

$$B \in \mathcal{L}(\mathbb{U}, \mathbb{H}), \quad (\text{H2.B})$$

$$C \in \mathcal{L}(\mathbb{H}, \mathbb{Y}), \quad (\text{H2.C})$$

$$\begin{cases} \forall \varepsilon \in \mathcal{D}(A), \forall \lambda \in \mathbb{R}, \lambda \geq -\sigma, & A\varepsilon = \lambda\varepsilon \text{ et } B^*\varepsilon = 0 \implies \varepsilon = 0, \\ \forall \varepsilon \in \mathcal{D}(A), \forall \lambda \in \mathbb{R}, \lambda \geq -\sigma, & A\varepsilon = \lambda\varepsilon \text{ et } C\varepsilon = 0 \implies \varepsilon = 0. \end{cases} \quad (\text{H2.D})$$

It is worth mentioning that assumption (H1.D) (and its counterpart (H2.D) in the self-adjoint case) is the well-known Fattorini-Hautus criterion for exponential stabilization (see [6], [8] and [2]).

For every  $\nu > 0$ , we set

$$\Sigma_\nu^+ := \{\lambda_j \in \sigma(A) ; \operatorname{Re} \lambda_j \geq -\nu\}, \quad \Sigma_\nu^- := \{\lambda_j \in \sigma(A) ; \operatorname{Re} \lambda_j < -\nu\}, \quad (1.4)$$

where  $\sigma(A)$  is the spectrum of  $A$ . Condition (H1.A) in the IFD case and (H2.A) in the FDO case imply that  $\Sigma_\nu^+$  describes a finite set. We define the projection

$$P_\nu^+ = -\frac{1}{2i\pi} \int_{\Gamma_\nu^+} (\zeta \operatorname{Id} - A)^{-1} d\zeta, \quad P_\nu^- = \operatorname{Id} - P_\nu^+, \quad (1.5)$$

where  $\Gamma_\nu^+$  is a curve enclosing  $\Sigma_\nu^+$  but no other point of the spectrum of  $A$  and oriented counterclockwise (see [10, V.5, p.272]). We set

$$z_\nu^\pm := P_\nu^\pm z, \quad \forall z \in \mathbb{H}.$$

We also introduce the finite dimensional operators

$$A_\nu^\pm := AP_\nu^\pm, \quad B_\nu^\pm := P_\nu^\pm B, \quad C_\nu^\pm := C\iota_\nu^\pm,$$

where  $\iota_\nu^\pm$  is the embedding operator from  $\mathbb{H}_\nu^\pm := P_\nu^\pm \mathbb{H}$  to  $\mathbb{H}$ . Finally, we denote by  $Q_\nu^+$  the orthogonal projection from  $\mathbb{Y}$  onto  $\mathbb{Y}_\nu^+ := CP_\nu^+ \mathbb{H}$ .

We are now in position to state our main results in the (IDO) and (FDO) cases.

## 1.1 Infinite-Dimensional Observer (IDO)

**Theorem 1.1.** *Let  $\sigma > 0$  be given and assume that assumptions (H1.A), (H1.B), (H1.C) and (H1.D) hold true. Then, there exist two operators  $K_\sigma^+ \in \mathcal{L}(\mathbb{H}, B^*(P_\sigma^+)^*\mathbb{H})$  and  $L_\sigma^+ \in \mathcal{L}(CP_\sigma^+\mathbb{H}, P_\sigma^+\mathbb{H})$  such that the observer-based feedback control defined by*

$$v(t) = K_\sigma^+ \hat{z}(t), \quad (1.6)$$

where the infinite dimensional observer  $\hat{z}$  solves

$$\begin{cases} \hat{z}'(t) = A\hat{z}(t) + Bv(t) + L_\sigma^+ Q_\sigma^+ (C\hat{z}(t) - y(t)), \\ \hat{z}(0) = 0, \end{cases} \quad (1.7)$$

ensures that for any  $z^0 \in \mathbb{H}$ , the solution  $z$  of the closed loop (1.3)-(1.6)-(1.7), that is

$$\begin{cases} z'(t) = Az(t) + BK_\sigma^+ \hat{z}(t), \\ z(0) = z^0, \end{cases}$$

satisfies

$$\|z(t)\|_{\mathbb{H}} \leq Me^{-\sigma t} \|z^0\|_{\mathbb{H}}, \quad \forall t > 0. \quad (1.8)$$

To prove this result, we introduce the error  $e := z - \hat{z}$  and we check that systems (1.3) and (1.7) yield

$$\begin{pmatrix} z \\ e \end{pmatrix}' = \begin{pmatrix} A + BK_\sigma^+ & -BK_\sigma^+ \\ 0 & A + L_\sigma^+ Q_\sigma^+ C \end{pmatrix} \begin{pmatrix} z \\ e \end{pmatrix}.$$

The result follows then by choosing the operators  $K_\sigma^+$  and  $L_\sigma^+$  in such a way that both  $A + BK_\sigma^+$  and  $A + L_\sigma^+ Q_\sigma^+ C$  generate analytic semigroups with decay rate less than  $-\sigma$ . This is achieved by solving two Riccati equations, using the method proposed by Badra-Takahashi [1] (for  $K_\sigma^+$ ) combined to a duality argument (for  $L_\sigma^+$ ).

## 1.2 Finite-Dimensional Observer (FDO)

In this case, the stabilizing control is based on a finite dimensional observer of size  $\sigma^* > 0$ , where  $\sigma^* > \sigma$  and need to be chosen large enough.

**Theorem 1.2.** *Let  $\sigma > 0$  be given and assume that assumptions (H2.A), (H2.B), (H2.C) and (H2.D) hold true. Let  $K_\sigma^+ \in L(\mathbb{H}, B^*P_\sigma^+\mathbb{H})$  and  $L_\sigma^+ \in \mathcal{L}(CP_\sigma^+\mathbb{H}, P_\sigma^+\mathbb{H})$  be the operators defined in Theorem 1.1. Then, there exists  $\sigma^* > \sigma$  such that the observer-based feedback control defined by*

$$v(t) = K_\sigma^+ \widehat{z}_*(t) \in B^*P_\sigma^+\mathbb{H} \subset B^*P_{\sigma^*}^+\mathbb{H}, \quad (1.9)$$

where the finite dimensional observer  $\widehat{z}_* \in P_{\sigma^*}^+\mathbb{H}$  solves

$$\begin{cases} \widehat{z}'_*(t) = A_{\sigma^*}^+ \widehat{z}_*(t) + B_{\sigma^*}^+ v(t) + L_\sigma^+ Q_\sigma^+ (C_\sigma^+ \widehat{z}_*(t) - y(t)), \\ \widehat{z}_*(0) = 0, \end{cases} \quad (1.10)$$

ensures that for any  $z^0 \in \mathbb{H}$ , the solution  $z$  of the closed loop (1.3)-(1.9)-(1.10), that is

$$\begin{cases} z'(t) = Az(t) + BK_\sigma^+ \widehat{z}_*(t), \\ z(0) = z^0, \end{cases}$$

satisfies

$$\|z(t)\|_{\mathbb{H}} \leq M e^{-\sigma t} \|z^0\|_{\mathbb{H}}, \quad \forall t > 0. \quad (1.11)$$

To prove this result, we proceed as follows. Introducing the auxiliary variables

$$e := z_{\sigma^*}^+ - \widehat{z}_* \quad \mathbb{X} = \begin{pmatrix} \widehat{z}_* \\ e \end{pmatrix},$$

we show that the equations satisfied by the state  $z$  and the observer  $\widehat{z}_*$  yield

$$\begin{cases} \mathbb{X}'(t) = \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z_{\sigma^*}^-(t)), \\ (z_{\sigma^*}^-)'(t) = A_{\sigma^*}^- z_{\sigma^*}^-(t) + B_{\sigma^*}^- \mathbb{K}_\sigma^+ \mathbb{X}(t), \end{cases} \quad (1.12)$$

where

$$\mathbb{A} = \begin{pmatrix} A_{\sigma^*}^+ + B_{\sigma^*}^+ K_\sigma^+ & -L_\sigma^+ Q_\sigma^+ C_\sigma^+ \\ 0 & A_{\sigma^*}^+ + L_\sigma^+ Q_\sigma^+ C_\sigma^+ \end{pmatrix}, \quad \mathbb{L}(z_{\sigma^*}^-) = \begin{pmatrix} -L_\sigma^+ Q_\sigma^+ C z_{\sigma^*}^- \\ L_\sigma^+ Q_\sigma^+ C z_{\sigma^*}^- \end{pmatrix}, \quad \mathbb{K}_\sigma^+ = (K_\sigma^+, 0).$$

We prove then that the matrix  $\exp(t\mathbb{A})$  is exponentially stable with a decay rate less than  $-\sigma$ . Next, thanks to the first equation in (1.12), we use Duhamel's formula to express  $\mathbb{X}$  in terms of  $z_{\sigma^*}^-$ . Plugging the obtained relation in the second equation of (1.12), we obtain an integral equation for  $z_{\sigma^*}^-$ . We use a fixed point argument to prove the well-posedness of this integral equation in the weighted space

$$L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-) := \{f \in L^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-) \text{ such that } e^{\sigma(\cdot)} f(\cdot) \in L^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)\}.$$

This provides the expected result, that is the exponential decay of the controlled system with a decay rate less than  $-\sigma$ .

## 1.3 Related works

As already mentioned, the closest reference to our work is [11], in which the authors considered the case of a one dimensional heat equation. Their strategy is based on a (modal) splitting of the system into two parts: a finite dimensional unstable one and a stable infinite dimensional one. A Luenberger observer of large enough dimension is then constructed and the stability of the closed loop system is proved using a Lyapunov function.

Contrarily to the proof proposed here, the arguments used in [11] are valid only in dimension one and heavily rely on the type of the considered equation. Let us also mention that in [12], the authors used a similar approach to prove the stabilization of a one dimension convection diffusion equation in the case of a boundary control. The use of modal splitting for the stabilization of infinite dimensional systems has also been achieved in some specific settings, like Burgers equations [20, 4, 19], Navier-Stokes system [1, 2, 7, 16, 18], semi-linear wave equation [5] and populations dynamics [14, 15].

## 1.4 Outline

In Section 2, we prove Theorem 1.1, which provides the stabilizing observer-based feedback-control through an infinite dimensional observer. In Section 3, we construct a finite dimensional observer to design a similar feedback-control. For this case, we need to assume that the operator  $A$  is self-adjoint and the control and observation operators are bounded. Finally, in Section 4, these abstract results are applied to obtain a stabilizing control for a reaction-diffusion system.

# 2 Infinite dimensional observer

## 2.1 Spectral decomposition of the system

In this section, we suppose that assumptions (H1.A), (H1.B), (H1.C) and (H1.D) hold true. We consider below a classical modal decomposition (it has been used, for instance, in [1, 2, 7, 18]) that we recall it for the sake of completeness. Let  $\sigma > 0$ . We first separate the spectrum of  $A$  into “unstable” and “stable” modes using the projection  $P_\sigma^+$  defined in (1.5). We set

$$\mathbb{H}_\sigma^+ = P_\sigma^+ \mathbb{H}, \quad \mathbb{H}_\sigma^- = (\text{Id} - P_\sigma^+) \mathbb{H}, \quad \mathbb{H} = \mathbb{H}_\sigma^+ \oplus \mathbb{H}_\sigma^-.$$

According to this projection, we set

$$A_\sigma^+ := A|_{\mathbb{H}_\sigma^+} : \mathbb{H}_\sigma^+ \rightarrow \mathbb{H}_\sigma^+, \quad A_\sigma^- := A|_{\mathbb{H}_\sigma^-} : \mathcal{D}(A) \cap \mathbb{H}_\sigma^- \rightarrow \mathbb{H}_\sigma^-.$$

Then the spectrum of  $A_\sigma^+$  is exactly  $\Sigma_\sigma^+$  and the spectrum of  $A_\sigma^-$  is exactly  $\Sigma_\sigma^-$  where  $\Sigma_\sigma^+$  and  $\Sigma_\sigma^-$  are defined in (1.4). We denote by  $A^*$  the adjoint operator of  $A$  and we define similarly the projection  $(P_\sigma^+)^*$  such that

$$(P_\sigma^+)^* = -\frac{1}{2i\pi} \int_{\Gamma_\sigma^+} (\zeta \text{Id} - A^*)^{-1} d\zeta. \quad (2.1)$$

The projection (2.1) provides also the following spaces

$$(\mathbb{H}_\sigma^+)^* = (P_\sigma^+)^* \mathbb{H}, \quad (\mathbb{H}_\sigma^-)^* = (\text{Id} - (P_\sigma^+)^*) \mathbb{H}, \quad \mathbb{H} = (\mathbb{H}_\sigma^+)^* \oplus (\mathbb{H}_\sigma^-)^*, \quad (2.2)$$

with

$$(A_\sigma^+)^* := A^*|_{(\mathbb{H}_\sigma^+)^*} : (\mathbb{H}_\sigma^+)^* \rightarrow (\mathbb{H}_\sigma^+)^*, \quad (A_\sigma^-)^* := A^*|_{(\mathbb{H}_\sigma^-)^*} : \mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^* \rightarrow (\mathbb{H}_\sigma^-)^*.$$

**Lemma 2.1.** *There exist  $\varepsilon > 0$  and  $M > 0$  such that for any  $\delta \geq 0$ ,  $t > 0$*

$$\begin{aligned} \left\| e^{A_\sigma^- t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} &\leq M e^{-(\sigma+\varepsilon)t}, & \left\| (\mu_0 \text{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} &\leq \frac{M}{t^\delta} e^{-(\sigma+\varepsilon)t}, \\ \left\| e^{(A_\sigma^-)^* t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} &\leq M e^{-(\sigma+\varepsilon)t}, & \left\| (\mu_0 \text{Id} - (A_\sigma^-)^*)^\delta e^{(A_\sigma^-)^* t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} &\leq \frac{M}{t^\delta} e^{-(\sigma+\varepsilon)t}. \end{aligned} \quad (2.3)$$

*Proof.* We detail the proof only for the operator  $A_\sigma^-$ , as the arguments for its adjoint are similar. The first inequality is obvious. Concerning the second one, we first note that

$$(\mu_0 \text{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t} = (\mu_0 \text{Id} - A_\sigma^-)^\delta (A_\sigma^-)^{-\delta} (A_\sigma^-)^\delta e^{A_\sigma^- t}.$$

Applying [13, Corollary 6.11] with  $B = (\mu_0 \text{Id} - A_\sigma^-)^\delta$ ,  $A = A_\sigma^-$  and  $x = (A_\sigma^-)^{-\delta} y$ , for  $y \in \mathbb{H}_\sigma^-$ , we obtain that for some constant positive  $C$ ,

$$\left\| (\mu_0 \text{Id} - A_\sigma^-)^\delta (A_\sigma^-)^{-\delta} y \right\|_{\mathbb{H}} \leq C \|y\|_{\mathbb{H}}, \quad \forall y \in \mathbb{H}_\sigma^-,$$

and thus

$$\left\| (\mu_0 \text{Id} - A_\sigma^-)^\delta (A_\sigma^-)^{-\delta} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leq C.$$

Consequently,

$$\left\| (\mu_0 \text{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leq \left\| (\mu_0 \text{Id} - A_\sigma^-)^\delta (A_\sigma^-)^{-\delta} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \left\| (A_\sigma^-)^\delta e^{A_\sigma^- t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)},$$

1 and the desired estimate follows then immediately from [13, Theorem 6.13].  $\square$

We also define

$$\mathbb{U}_\sigma^+ := B^*(\mathbb{H}_\sigma^+)^*, \quad \mathbb{U}_\sigma^- := B^*(\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*),$$

and

$$p_\sigma^+ : \mathbb{U} \rightarrow \mathbb{U}_\sigma^+, \quad p_\sigma^- : \mathbb{U} \rightarrow \mathbb{U}_\sigma^-, \quad i_\sigma^+ : \mathbb{U}_\sigma^+ \rightarrow \mathbb{U}, \quad i_\sigma^- : \mathbb{U}_\sigma^- \rightarrow \mathbb{U},$$

2 the orthogonal projections and the inclusion maps. Note that we have the following relations for the above  
3 maps:

$$i_\sigma^+ = (p_\sigma^+)^*, \quad i_\sigma^- = (p_\sigma^-)^*. \quad (2.4)$$

From [18], we can extend  $P_\sigma^+$  and  $(\text{Id} - P_\sigma^+)$  as bounded operators

$$P_\sigma^+ \in \mathcal{L}(\mathcal{D}(A^*)', \mathbb{H}_\sigma^+), \quad (\text{Id} - P_\sigma^+) \in \mathcal{L}(\mathcal{D}(A^*)', [\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*]').$$

We can thus define

$$B_\sigma^+ := P_\sigma^+ B i_\sigma^+ \in \mathcal{L}(\mathbb{U}_\sigma^+, \mathbb{H}_\sigma^+), \quad B_\sigma^- := (\text{Id} - P_\sigma^+) B i_\sigma^- \in \mathcal{L}(\mathbb{U}_\sigma^-, [\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*]').$$

We show as in [1, 2, 18] that

$$P_\sigma^+ B = B_\sigma^+ p_\sigma^+, \quad (\text{Id} - P_\sigma^+) B = B_\sigma^- p_\sigma^-.$$

4 Using the projections  $P_\sigma^+$  and  $\text{Id} - P_\sigma^+$ , system (1.3) can be split into the two sub-systems (see [1, 2, 18]).

$$(z_\sigma^+)'(t) = A_\sigma^+ z_\sigma^+(t) + B_\sigma^+ p_\sigma^+ v(t), \quad z_\sigma^+(0) = P_\sigma^+ z^0, \quad (2.5)$$

$$5 (z_\sigma^-)'(t) = A_\sigma^- z_\sigma^-(t) + B_\sigma^- p_\sigma^- v(t), \quad z_\sigma^-(0) = (\text{Id} - P_\sigma^+) z^0. \quad (2.6)$$

We also introduce the orthogonal projections  $Q_\sigma^+$  from  $\mathbb{Y}$  into  $\mathbb{Y}_\sigma^+ = C\mathbb{H}_\sigma^+$  and we define

$$C_\sigma^+ = C \iota_\sigma^+,$$

6 where  $\iota_\sigma^+$  designates the injection operator from  $\mathbb{H}_\sigma^+$  to  $\mathbb{H}$ . We are now in position to prove Theorem 1.1.

## 7 2.2 Construction of an infinite dimensional observer based control

8 Let us consider first the system

$$\begin{cases} \tilde{z}'(t) = A\tilde{z}(t) + Bu(t), \\ \tilde{z}(0) = \tilde{z}^0. \end{cases} \quad (2.7)$$

We want to construct a finite dimensional vector  $u$  such that the system (2.7) is exponentially stable. Let  $N_\sigma \in \mathbb{N}^*$  and  $(w_j)_{1 \leq j \leq N_\sigma} \subset \mathbb{U}$ , and let us suppose that the control  $u(t)$  is of the form

$$u(t) = \sum_{j=1}^{N_\sigma} u_j(t) w_j,$$

where  $u_j(t) \in \mathbb{C}$ , for  $1 \leq j \leq N_\sigma$  and  $t \geq 0$ . It is natural to introduce the mapping

$$\mathfrak{B} : \mathbb{C}^{N_\sigma} \longrightarrow \mathcal{D}(A^*)', \quad \Theta = (\theta_1, \dots, \theta_{N_\sigma}) \longmapsto \mathfrak{B}\Theta = \sum_{j=1}^{N_\sigma} \theta_j B w_j,$$

in such a way that setting

$$\mathbf{u}(t) := (u_1(t), \dots, u_{N_\sigma}(t)),$$

1 system (2.7) is equivalent to

$$\begin{cases} \tilde{z}'(t) = A\tilde{z}(t) + \mathfrak{B}\mathbf{u}(t), \\ \tilde{z}(0) = \tilde{z}^0. \end{cases} \quad (2.8)$$

It is worth noticing that the adjoint  $\mathfrak{B}^* \in \mathcal{L}(\mathcal{D}(A^*), \mathbb{C}^{N_\sigma})$  is given by

$$\mathfrak{B}^*\psi = \left( \langle w_1, B^*\psi \rangle_{\mathbb{U}}, \dots, \langle w_{N_\sigma}, B^*\psi \rangle_{\mathbb{U}} \right).$$

2 Using the projection  $P_\sigma^+$ , we get that (2.8) is equivalent to

$$\begin{cases} (\tilde{z}_\sigma^+)'(t) = A_\sigma^+ \tilde{z}_\sigma^+(t) + \mathfrak{B}_\sigma^+ \mathbf{u}(t), & \tilde{z}_\sigma^+(0) = P_\sigma^+ \tilde{z}^0, & \mathfrak{B}_\sigma^+ = P_\sigma^+ \mathfrak{B}, \\ (\tilde{z}_\sigma^-)'(t) = A_\sigma^- \tilde{z}_\sigma^-(t) + \mathfrak{B}_\sigma^- \mathbf{u}(t), & \tilde{z}_\sigma^-(0) = P_\sigma^- \tilde{z}^0, & \mathfrak{B}_\sigma^- = P_\sigma^- \mathfrak{B}, \end{cases} \quad (2.9)$$

3 where  $\tilde{z}_\sigma^\pm = P_\sigma^\pm \tilde{z}$ . We need to show that the finite dimensional part (2.9)<sub>1</sub> is exactly controllable. Let

$$N_\sigma \geq \max_{\operatorname{Re} \lambda_j \geq -\sigma} \ell_j, \quad (2.10)$$

4 where  $\ell_j$  is the geometric multiplicity of the eigenvalue  $\lambda_j$  of the operator  $A$ . From [1, Theorem 5] and the  
5 first condition in (H1.D), there exists a family  $(w_j)_{1 \leq j \leq N_\sigma} \subset \mathbb{U}_\sigma^+ \subset \mathbb{U}$  such that (2.9)<sub>1</sub> is exactly controllable.  
6 Moreover, it is proved that  $u$  is expressed by means of a linear feedback operator

$$u = K_\sigma^+ \tilde{z}, \quad K_\sigma^+(\cdot) = - \sum_{j=1}^{N_\sigma} \langle w_j, B^* \Pi P_\sigma^+(\cdot) \rangle_{\mathbb{U}} w_j = - \sum_{j=1}^{N_\sigma} (\mathfrak{B}^*(\Pi P_\sigma^+(\cdot)))_j w_j, \quad (2.11)$$

7 where  $\Pi \in \mathcal{L}(\mathbb{H}_\sigma^+, (\mathbb{H}_\sigma^+)^*)$  is the unique solution of the algebraic Riccati equation: for all  $\xi, \zeta \in \mathbb{H}_\sigma^+$

$$\begin{cases} \langle \xi, \zeta \rangle_{\mathbb{H}} + \langle (A_\sigma^+ + \sigma \operatorname{Id})\xi, \Pi \zeta \rangle_{\mathbb{H}} + \langle \Pi \xi, (A_\sigma^+ + \sigma \operatorname{Id})\zeta \rangle_{\mathbb{H}} - \sum_{j=1}^{N_\sigma} \langle B^* \Pi \xi, w_j \rangle_{\mathbb{U}} \langle B^* \Pi \zeta, w_j \rangle_{\mathbb{U}} = 0, \\ \langle \Pi \xi, \zeta \rangle_{\mathbb{H}} = \langle \xi, \Pi \zeta \rangle_{\mathbb{H}}, \quad \text{and} \quad \forall \xi \neq 0, \langle \Pi \xi, \xi \rangle_{\mathbb{H}} > 0. \end{cases} \quad (2.12)$$

This choice ensures that the solution of the finite dimensional system (2.9)<sub>1</sub>

$$(\tilde{z}_\sigma^+)'(t) = A_\sigma^+ \tilde{z}_\sigma^+(t) - \mathfrak{B}_\sigma^+ (\mathfrak{B}^* (\Pi P_\sigma^+ \tilde{z}(t)))$$

is exponentially stable i.e.

$$\|\tilde{z}_\sigma^+(t)\|_{\mathbb{H}} \leq M e^{-(\sigma+\varepsilon)t} \|\tilde{z}^0\|_{\mathbb{H}}, \quad t > 0.$$

8 It follows from Duhamel's formula that the whole system (2.9) is exponentially stable (see [1]). We can construct  
9  $L_\sigma^+$  similarly considering the system

$$\begin{cases} \tilde{z}_\star'(t) = A^* \tilde{z}_\star(t) + C^* u_\star(t), \\ \tilde{z}_\star(0) = \tilde{z}_\star^0. \end{cases} \quad (2.13)$$

Using similar arguments and the second condition in (H1.D), we show that there exists a family  $(w_j^*)_{1 \leq j \leq N_\sigma} \subset \mathbb{Y}_\sigma^+$  such that

$$u_\star = L_\star \tilde{z}_\star, \quad L_\star(\cdot) = - \sum_{j=1}^{N_\sigma} \langle w_j^*, C \Pi_\star (P_\sigma^+)^*(\cdot) \rangle_{\mathbb{Y}} w_j^* = - \sum_{j=1}^{N_\sigma} (\mathfrak{C}_\star^*(\Pi_\star (P_\sigma^+)^*(\cdot)))_j w_j^*,$$



where

$$\mathfrak{C}_\star : \mathbb{C}^{N_\sigma} \longrightarrow \mathcal{D}(A^\star)', \quad \Theta = (\theta_1, \dots, \theta_{N_\sigma}) \longrightarrow \mathfrak{C}_\star \Theta = \sum_{j=1}^{N_\sigma} \theta_j C^\star w_j^\star,$$

1 where  $\Pi_\star \in \mathcal{L}((\mathbb{H}_\sigma^+)^*, \mathbb{H}_\sigma^+)$  is the unique solution of the algebraic Riccati equation: for all  $\xi, \zeta \in (\mathbb{H}_\sigma^+)^*$

$$\begin{cases} \langle \xi, \zeta \rangle_{\mathbb{H}} + \langle ((A_\sigma^+)^* + \sigma \text{Id})\xi, \Pi_\star \zeta \rangle_{\mathbb{H}} + \langle \Pi_\star \xi, ((A_\sigma^+)^* + \sigma \text{Id})\zeta \rangle_{\mathbb{H}} - \sum_{j=1}^{N_\sigma} \langle C \Pi_\star \xi, w_j \rangle_{\mathbb{Y}} \langle C \Pi_\star \zeta, w_j \rangle_{\mathbb{Y}} = 0, \\ \langle \Pi_\star \xi, \zeta \rangle_{\mathbb{H}} = \langle \xi, \Pi_\star \zeta \rangle_{\mathbb{H}}, \quad \text{and} \quad \forall \xi \neq 0, \langle \Pi_\star \xi, \xi \rangle_{\mathbb{H}} > 0. \end{cases} \quad (2.14)$$

2 Hence, we define

$$L_\sigma^+(\cdot) = L_\star^*(\cdot) = - \sum_{j=1}^{N_\sigma} \langle w_j^\star, \cdot \rangle_{\mathbb{Y}} \chi_j, \quad (2.15)$$

3 with

$$\chi_j = P_\sigma^+ \Pi_\star C^\star w_j^\star \in \mathbb{H}_\sigma^+. \quad (2.16)$$

4 With this choice, we get that  $(A + L_\sigma^+ C)^\star$  and hence  $A + L_\sigma^+ Q_\sigma^+ C$  is exponentially stable with decay rate less  
5 than  $-\sigma$ . Finally, using  $K_\sigma^+$  and  $L_\sigma^+$  we construct the observer  $\hat{z}$  satisfying (1.6)-(1.7), that is

$$\begin{cases} \hat{z}'(t) = A \hat{z}(t) + B K_\sigma^+ \hat{z}(t) + L_\sigma^+ Q_\sigma^+ (C \hat{z}(t) - y(t)), \\ \hat{z}(0) = 0. \end{cases} \quad (2.17)$$

### 6 2.3 Stability of the closed-loop system

7 We define the error  $e = z - \hat{z}$ . Then, we obtain

$$\begin{cases} e'(t) = (A + L_\sigma^+ Q_\sigma^+ C)e(t), & e(0) = z^0, \\ z'(t) = (A + B K_\sigma^+)z(t) - B K_\sigma^+ e(t), & z(0) = z^0. \end{cases} \quad (2.18)$$

8 We prove that  $e$  is exponentially stable with decay rate  $-\sigma$ .

9 **Proposition 2.2.** *Systems (2.17) and (2.18) are exponentially stable with decay rate  $-\sigma$ .*

10 *Proof.* Since  $(A + L_\sigma^+ Q_\sigma^+ C)$  is of negative type strictly less than  $-\sigma$ , then there exists  $0 < \varepsilon'' < \varepsilon$  such that

$$\|e(t)\|_{\mathbb{H}} \leq M e^{-t(\sigma + \varepsilon'')} \|z^0\|_{\mathbb{H}}. \quad (2.19)$$

Going back to system (2.5) with the control given by (1.6), we have, since  $K_\sigma^+ z_\sigma^- = 0$ ,

$$(z_\sigma^+)'(t) = (A_\sigma^+ + B_\sigma^+ K_\sigma^+) z_\sigma^+(t) - B_\sigma^+ K_\sigma^+ e(t), \quad z_\sigma^+(0) = P_\sigma^+ z^0.$$

Moreover, there exists  $\varepsilon' > 0$  with  $\varepsilon'' < \varepsilon' < \varepsilon$  such that  $(A_\sigma^+ + B_\sigma^+ K_\sigma^+)$  is exponentially stable with rate  $-\sigma - \varepsilon'$ .  
We have

$$z_\sigma^+(t) = e^{t(A_\sigma^+ + B_\sigma^+ K_\sigma^+)} P_\sigma^+ z^0 - \int_0^t e^{(t-s)(A_\sigma^+ + B_\sigma^+ K_\sigma^+)} B_\sigma^+ K_\sigma^+ e(s) ds.$$

11 From (2.19), we see that

$$\|z_\sigma^+(t)\|_{\mathbb{H}} \leq M e^{-t(\sigma + \varepsilon'')} \|z^0\|_{\mathbb{H}}. \quad (2.20)$$

We deal now with the infinite dimensional part  $z_\sigma^-$  of the state. From (2.6) with the control given by (1.6), we have

$$(z_\sigma^-)'(t) = A_\sigma^- z_\sigma^-(t) + B_\sigma^- p_\sigma^- K_\sigma^+ z_\sigma^+(t) - B_\sigma^- p_\sigma^- K_\sigma^+ e(t), \quad z_\sigma^-(0) = (\text{Id} - P_\sigma^+) z^0.$$

Using Duhamel's formula, we obtain that

$$z_\sigma^-(t) = e^{tA_\sigma^-} (\text{Id} - P_\sigma^+) z^0 + \int_0^t e^{(t-s)A_\sigma^-} B_\sigma^- p_\sigma^- K_\sigma^+ (z_\sigma^+(s) - e(s)) ds.$$

- 1 We note that since the resolvent commutes with the projection  $P_\sigma^+$  and  $e^{(t-s)A_\sigma^-}$ , we obtain for  $\mu_0 \in \rho(A)$  and  
 2  $\gamma \in [0, 1)$ ,

$$\begin{aligned} e^{(t-s)A_\sigma^-} B_\sigma^- &= e^{(t-s)A_\sigma^-} (\mu_0 \text{Id} - A)^\gamma (\text{Id} - P_\sigma^+) (\mu_0 \text{Id} - A)^{-\gamma} B i_\sigma^- \\ &= e^{(t-s)A_\sigma^-} (\mu_0 \text{Id} - A_\sigma^-)^\gamma (\text{Id} - P_\sigma^+) (\mu_0 \text{Id} - A)^{-\gamma} B i_\sigma^- \\ &= (\mu_0 \text{Id} - A_\sigma^-)^\gamma e^{(t-s)A_\sigma^-} (\text{Id} - P_\sigma^+) (\mu_0 \text{Id} - A)^{-\gamma} B i_\sigma^-. \end{aligned} \quad (2.21)$$

- 3 Using (H1.B), (2.3), (2.19), (2.20) and (2.21), we get

$$\begin{aligned} \|z_\sigma^-(t)\|_{\mathbb{H}} &\leq M \|z^0\|_{\mathbb{H}} \left( e^{-t(\sigma+\varepsilon)} + \int_0^t \frac{1}{(t-s)^\gamma} e^{-(t-s)(\sigma+\varepsilon)} e^{-s(\sigma+\varepsilon'')} ds \right) \\ &\leq M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}} \left( 1 + \int_0^t \frac{1}{(t-s)^\gamma} e^{(t-s)(\varepsilon''-\varepsilon)} ds \right). \end{aligned} \quad (2.22)$$

- 4 Then, since  $\varepsilon'' < \varepsilon$ , we obtain

$$\|z_\sigma^-(t)\|_{\mathbb{H}} \leq M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}}. \quad (2.23)$$

- 5 Then from (2.19), (2.20) and (2.23), we obtain that  $z$ ,  $\widehat{z}$  and the error  $e$  are exponentially stable.  $\square$

- 6 That concludes the proof of Theorem 1.1.

**Remark 2.3.** According to (1.6) and (2.11), the control reads

$$v(t) = \sum_{i=1}^{N_\sigma} K_i (\widehat{z}_\sigma^+) w_i,$$

with  $K_i \in \mathcal{L}(\mathbb{H}_\sigma^+, \mathbb{C})$  and  $w_i \in \mathbb{U}_\sigma^+$ ,  $i = 1, \dots, N_\sigma$ . From decomposition (2.2) and the fact that  $(\mathbb{H}_\sigma^+)^\perp = (\mathbb{H}_\sigma^-)^*$ , we have that if  $\zeta \in (\mathbb{H}_\sigma^+)^*$ , then

$$\forall \phi \in \mathbb{H}_\sigma^+, \quad \langle \phi, \zeta \rangle_{\mathbb{H}} = 0 \implies \zeta = 0.$$

Since  $\dim((\mathbb{H}_\sigma^+)^*) = \dim \mathcal{L}(\mathbb{H}_\sigma^+, \mathbb{C})$ , we infer that there exists a unique  $\zeta_i \in (\mathbb{H}_\sigma^+)^* \subset \mathcal{D}(A^*)$  such that

$$K_i (\widehat{z}_\sigma^+) = \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}}.$$

In other words, the control can also be written in the form

$$v(t) = \sum_{i=1}^{N_\sigma} \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}} w_i.$$

In the special case where there is only one unstable simple eigenvalue (with an eigenspace spanned by an eigenfunction  $\varepsilon_1 \in \mathbb{H}$ ), the above relations take simpler forms. Indeed, we have then

$$w_1 = B^* \varepsilon_1$$

and for all  $\varphi \in \mathbb{H}_\sigma^+ = \text{Span}\{\varepsilon_1\}$ :

$$K_1(\varphi) = -\langle B^* \varepsilon_1, B^* \Pi P_\sigma^+ \varphi \rangle_{\mathbb{U}} = \langle \zeta_1, \varphi \rangle_{(\mathbb{H}_\sigma^+)^*, \mathbb{H}_\sigma^+}, \quad \zeta_1 = -(P_\sigma^+)^* \Pi B B^* \varepsilon_1.$$

### 3 Finite dimensional observer

#### 3.1 Spectral decomposition of the system

In this section, we assume hypotheses (H2.A), (H2.B), (H2.C) and (H2.D) to hold true.

Consider  $\nu > 0$  and let us introduce the projection operators  $P_\nu^+$  as in (1.5) where in this case  $\Gamma_\nu^+$  is a circle enclosing  $\Sigma_\nu^+$  but no other point of the spectrum of  $A$  and oriented counterclockwise (see [10, V.5, p.272]). Since  $A$  is a self-adjoint operator, then  $P_\nu^+$  is well defined. Moreover from the expression of the projections, it follows that

$$(P_\nu^+)^* = P_\nu^+.$$

Thus,  $P_\nu^+$  is orthogonal projection of norm equal to 1. We set

$$\mathbb{H}_\nu^+ = P_\nu^+ \mathbb{H}, \quad \mathbb{H}_\nu^- = (\text{Id} - P_\nu^+) \mathbb{H}, \quad \mathbb{H} = \mathbb{H}_\nu^+ \oplus \mathbb{H}_\nu^-,$$

and

$$A_\nu^+ := A|_{\mathbb{H}_\nu^+} : \mathbb{H}_\nu^+ \rightarrow \mathbb{H}_\nu^+, \quad A_\nu^- := A|_{\mathbb{H}_\nu^-} : \mathcal{D}(A) \cap \mathbb{H}_\nu^- \rightarrow \mathbb{H}_\nu^-.$$

We also define as before

$$\mathbb{U}_\nu^+ := B^* \mathbb{H}_\nu^+, \quad \mathbb{U}_\nu^- := B^* (\mathcal{D}(A) \cap \mathbb{H}_\nu^-),$$

and

$$p_\nu^+ : \mathbb{U} \rightarrow \mathbb{U}_\nu^+, \quad p_\nu^- : \mathbb{U} \rightarrow \mathbb{U}_\nu^-, \quad i_\nu^+ : \mathbb{U}_\nu^+ \rightarrow \mathbb{U}, \quad i_\nu^- : \mathbb{U}_\nu^- \rightarrow \mathbb{U},$$

the orthogonal projections and the inclusion maps. Note that we have the following relations for the above maps:

$$i_\nu^+ = (p_\nu^+)^*, \quad i_\nu^- = (p_\nu^-)^*. \quad (3.1)$$

We can thus define

$$B_\nu^+ := P_\nu^+ B i_\nu^+ \in \mathcal{L}(\mathbb{U}_\nu^+, \mathbb{H}_\nu^+), \quad B_\nu^- := (\text{Id} - P_\nu^+) B i_\nu^- \in \mathcal{L}(\mathbb{U}_\nu^-, \mathbb{H}_\nu^-).$$

It is proved in [1] (see also [2] and [18]) that

$$P_\nu^+ B = B_\nu^+ p_\nu^+, \quad (\text{Id} - P_\nu^+) B = B_\nu^- p_\nu^-.$$

We introduce also the orthogonal projection  $Q_\nu^+$  from  $\mathbb{Y}$  into  $\mathbb{Y}_\nu^+ = C\mathbb{H}_\nu^+$  and define

$$C_\nu^+ = C \iota_\nu^+,$$

where  $\iota_\nu^+$  designates the injection operator from  $\mathbb{H}_\nu^+$  to  $\mathbb{H}$ .

Consider now  $\sigma^* > \sigma > 0$ . We take  $\nu = \sigma$  or  $\nu = \sigma^*$  in the maps and spaces defined previously. Since  $A$  is self-adjoint with compact resolvent, we deduce the existence of  $\varepsilon > 0$  such that for all  $t \geq 0$

$$\|e^{A_\sigma^- t}\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leq e^{-(\sigma+\varepsilon)t}, \quad (3.2)$$

and

$$\|e^{A_{\sigma^*}^- t}\|_{\mathcal{L}(\mathbb{H}_{\sigma^*}^-)} \leq e^{-\sigma^* t}. \quad (3.3)$$

The system (1.3) splits into

$$(z_{\sigma^*}^+)'(t) = A_{\sigma^*}^+ z_{\sigma^*}^+(t) + B_{\sigma^*}^+ p_{\sigma^*}^+ v(t), \quad z_{\sigma^*}^+(0) = P_{\sigma^*}^+ z^0, \quad (3.4)$$

$$(z_{\sigma^*}^-)'(t) = A_{\sigma^*}^- z_{\sigma^*}^-(t) + B_{\sigma^*}^- p_{\sigma^*}^- v(t), \quad z_{\sigma^*}^-(0) = (\text{Id} - P_{\sigma^*}^+) z^0. \quad (3.5)$$

## 3.2 Finite dimensional observer based control and stability of the closed-loop system

We are now in position to prove Theorem 1.2. The matrices  $K_\sigma^+$  and  $L_\sigma^+$  being respectively given by (2.11) and (2.15), we define the finite dimensional observer-based feedback control by

$$v(t) = K_\sigma^+ \widehat{z}_*(t) \in B^* P_\sigma^+ \mathbb{H} \subset B^* P_\sigma^+ \mathbb{H}, \quad (3.6)$$

where the finite dimensional observer  $\widehat{z}_* \in P_\sigma^+ \mathbb{H}$  solves

$$\begin{cases} \widehat{z}'_*(t) = A_{\sigma^*}^+ \widehat{z}_*(t) + B_{\sigma^*}^+ v(t) + L_\sigma^+ Q_\sigma^+ (C_{\sigma^*}^+ \widehat{z}_*(t) - y(t)), \\ \widehat{z}_*(0) = 0. \end{cases} \quad (3.7)$$

Our goal is to prove that the coupled system (1.3), (3.6) and (3.7) is exponentially stable. We define the error  $e = z_{\sigma^*}^+ - \widehat{z}_*$  which satisfies the following system

$$\begin{cases} e'(t) = A_{\sigma^*}^+ e(t) + L_\sigma^+ Q_\sigma^+ (C_{\sigma^*}^+ e(t) + C z_{\sigma^*}^-(t)), \\ e(0) = z_{\sigma^*}^+(0). \end{cases} \quad (3.8)$$

We prove that  $\widehat{z}_*$  and  $e$  are exponentially stable with decay rate  $-\sigma$ . Let us set

$$\mathbb{X} = \begin{pmatrix} \widehat{z}_* \\ e \end{pmatrix}, \quad \mathbb{X}^0 = \begin{pmatrix} 0 \\ z_{\sigma^*}^+(0) \end{pmatrix}.$$

Using (3.7) and (3.8),  $\mathbb{X}$  satisfies the system

$$\begin{cases} \mathbb{X}'(t) = \mathbb{A} \mathbb{X}(t) + \mathbb{L}(z_{\sigma^*}^-(t)), \\ \mathbb{X}(0) = \mathbb{X}^0, \end{cases} \quad (3.9)$$

where

$$\mathbb{A} = \begin{pmatrix} A_{\sigma^*}^+ + B_{\sigma^*}^+ K_\sigma^+ & -L_\sigma^+ Q_\sigma^+ C_{\sigma^*}^+ \\ 0 & A_{\sigma^*}^+ + L_\sigma^+ Q_\sigma^+ C_{\sigma^*}^+ \end{pmatrix}, \quad \mathbb{L}(z_{\sigma^*}^-) = \begin{pmatrix} -L_\sigma^+ Q_\sigma^+ C z_{\sigma^*}^- \\ L_\sigma^+ Q_\sigma^+ C z_{\sigma^*}^- \end{pmatrix}. \quad (3.10)$$

Let us first prove that  $\mathbb{A}$  is stable matrix. In the sequel, the constant  $M$  is a generic constant that can change from a line to another but need to be independent of  $\sigma^*$ .

**Lemma 3.1.** *The matrices  $A_{\sigma^*}^+ + B_{\sigma^*}^+ K_\sigma^+$  and  $A_{\sigma^*}^+ + L_\sigma^+ Q_\sigma^+ C_{\sigma^*}^+$  are exponentially stable with a decay rate less than  $-\sigma$ .*

*Proof.* Let  $\xi^0 \in \mathbb{H}_{\sigma^*}^+$  be given. To prove that  $A_{\sigma^*}^+ + B_{\sigma^*}^+ K_\sigma^+$  is exponentially stable, we only need to show that the solution  $\xi(t)$  of the finite dimensional system

$$\begin{cases} \xi'(t) = (A_{\sigma^*}^+ + B_{\sigma^*}^+ K_\sigma^+) \xi(t), \\ \xi(0) = \xi^0, \end{cases} \quad (3.11)$$

is exponentially decaying. Consider then the infinite dimensional system

$$\begin{cases} x'(t) = (A + BK_\sigma^+) x(t), \\ x(0) = \xi^0. \end{cases} \quad (3.12)$$

From Section 2.2, we see that the system (3.12) is exponentially stable of decay rate  $-\sigma - \varepsilon'$ . It implies that

$$\|x(t)\|_{\mathbb{H}} \leq M e^{-t(\sigma + \varepsilon')} \|\xi^0\|_{\mathbb{H}}. \quad (3.13)$$

1 On the other hand, applying  $P_{\sigma^*}^+$  to (3.12) and recalling that  $K_{\sigma^*}^+ x(t)$  acts only on the projected part of  $x(t)$  on  
2  $\mathbb{H}_{\sigma^*}^+$ , we obtain that

$$\begin{cases} (x_{\sigma^*}^+)'(t) = A_{\sigma^*}^+ x_{\sigma^*}^+(t) + B_{\sigma^*}^+ K_{\sigma^*}^+ x(t) = (A_{\sigma^*}^+ + B_{\sigma^*}^+ K_{\sigma^*}^+) x_{\sigma^*}^+(t), \\ x_{\sigma^*}^+(0) = \xi^0. \end{cases} \quad (3.14)$$

This shows that  $x_{\sigma^*}^+(t)$  is the unique solution  $\xi(t)$  of (3.11), and we get from (3.13), that

$$\|\xi(t)\|_{\mathbb{H}} = \|x_{\sigma^*}^+(t)\|_{\mathbb{H}} = \|P_{\sigma^*}^+ x(t)\|_{\mathbb{H}} \leq M e^{-t(\sigma+\varepsilon')} \|\xi^0\|_{\mathbb{H}},$$

for all  $\xi^0 \in \mathbb{H}_{\sigma^*}^+$ . Hence

$$\left\| e^{t(A_{\sigma^*}^+ + B_{\sigma^*}^+ K_{\sigma^*}^+)} \xi^0 \right\|_{\mathbb{H}} \leq M e^{-t(\sigma+\varepsilon')} \|\xi^0\|_{\mathbb{H}},$$

3 and the matrix  $A_{\sigma^*}^+ + B_{\sigma^*}^+ K_{\sigma^*}^+$  is exponentially stable with a decay rate less than  $-\sigma$ . We use the same argument  
4 for  $A_{\sigma^*}^+ + L_{\sigma^*}^+ Q_{\sigma^*}^+ C_{\sigma^*}^+$  by considering its adjoint  $(A_{\sigma^*}^+)^* + (C_{\sigma^*}^+)^* (L_{\sigma^*}^+)^*$  that has exactly the same form as the one  
5 previously studied.  $\square$

6 Since  $\mathbb{A}$  is a triangular matrix, using Lemma 3.1 and Duhamel's formula, we obtain that  $\mathbb{A}$  is stable with  
7 exponential rate strictly less than  $-\sigma$ .

8 We can now prove the exponential stability of the full closed-loop system (3.7) and (1.3):

$$\begin{cases} \mathbb{X}'(t) &= \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z_{\sigma^*}^-(t)), \\ (z_{\sigma^*}^-)'(t) &= A_{\sigma^*}^- z_{\sigma^*}^-(t) + B_{\sigma^*}^- p_{\sigma^*}^- \mathbb{K}_{\sigma^*}^+ \mathbb{X}(t), \end{cases} \quad (3.15)$$

where  $\mathbb{K}_{\sigma^*}^+ = (K_{\sigma^*}^+, 0)$  and with the initial conditions

$$\begin{cases} \mathbb{X}(0) &= \mathbb{X}^0, \\ z_{\sigma^*}^-(0) &= (\text{Id} - P_{\sigma^*}^+) z^0. \end{cases}$$

From Duhamel's formula, the two first equations in (3.15) also read

$$\begin{cases} \mathbb{X}(s) &= e^{s\mathbb{A}} \mathbb{X}(0) + \int_0^s e^{(s-\tau)\mathbb{A}} \mathbb{L}(z_{\sigma^*}^-(\tau)) d\tau, \\ z_{\sigma^*}^-(t) &= e^{tA_{\sigma^*}^-} z_{\sigma^*}^-(0) + \int_0^t e^{(t-s)A_{\sigma^*}^-} (B_{\sigma^*}^- p_{\sigma^*}^- \mathbb{K}_{\sigma^*}^+ \mathbb{X}(s)) ds. \end{cases}$$

9 Substituting the first equation above into the second one yields

$$z_{\sigma^*}^-(t) = e^{tA_{\sigma^*}^-} z_{\sigma^*}^-(0) + \int_0^t e^{(t-s)A_{\sigma^*}^-} B_{\sigma^*}^- p_{\sigma^*}^- \mathbb{K}_{\sigma^*}^+ \left( e^{s\mathbb{A}} \mathbb{X}(0) + \int_0^s e^{(s-\tau)\mathbb{A}} \mathbb{L}(z_{\sigma^*}^-(\tau)) d\tau \right) ds. \quad (3.16)$$

Setting

$$Z^0 := e^{tA_{\sigma^*}^-} z_{\sigma^*}^-(0) + \int_0^t e^{(t-s)A_{\sigma^*}^-} (B_{\sigma^*}^- p_{\sigma^*}^- \mathbb{K}_{\sigma^*}^+ (e^{s\mathbb{A}} \mathbb{X}(0))) ds,$$

10 relation (3.16) can be written

$$z_{\sigma^*}^-(t) = \int_0^t \int_0^s e^{(t-s)A_{\sigma^*}^-} B_{\sigma^*}^- p_{\sigma^*}^- \mathbb{K}_{\sigma^*}^+ \left( e^{(s-\tau)\mathbb{A}} \mathbb{L}(z_{\sigma^*}^-(\tau)) d\tau ds \right) + Z^0. \quad (3.17)$$

To prove the stability of  $z_{\sigma^*}^-$ , we prove the existence of a unique solution to (3.17) in a weighted space by using  
a fixed point argument. More precisely, let us define the following map

$$\begin{aligned} \Phi : L_{\sigma}^{\infty}(0, \infty; \mathbb{H}_{\sigma^*}^-) &\longrightarrow L_{\sigma}^{\infty}(0, \infty; \mathbb{H}_{\sigma^*}^-) \\ g &\longmapsto \Phi(g) := \int_0^t \int_0^s e^{(t-s)A_{\sigma^*}^-} B_{\sigma^*}^- p_{\sigma^*}^- \mathbb{K}_{\sigma^*}^+ e^{(s-\tau)\mathbb{A}} \mathbb{L}(g(\tau)) d\tau ds + Z^0. \end{aligned}$$

Then, equation (3.16) simply reads

$$\Phi(z_{\sigma^*}^-) = z_{\sigma^*}^-.$$

First, we prove that the function  $\Phi$  is well defined. Given  $g \in L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)$ . Since  $C$  is bounded,  $\mathbb{L}$  is also bounded. Hence using the last relation, (3.3) and (H2.B), we obtain that there exists  $\varepsilon'' > 0$  such that

$$\begin{aligned} \|\Phi(g)(t)\|_{\mathbb{H}} &\leq M \int_0^t \int_0^s e^{-(t-s)\sigma^*} e^{-(s-\tau)(\sigma+\varepsilon'')} e^{-\tau\sigma} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}} d\tau ds \\ &\quad + M \|z^0\|_{\mathbb{H}} \int_0^t e^{-(t-s)\sigma^*} e^{-s\sigma} ds + e^{-t\sigma^*} \|z^0\|_{\mathbb{H}}. \end{aligned}$$

1 Consequently

$$\|\Phi(g)(t)\|_{\mathbb{H}} \leq I_1 + I_2 + e^{-t\sigma^*} \|z^0\|_{\mathbb{H}}, \quad (3.18)$$

where we have set

$$\begin{aligned} I_1 &:= M \int_0^t e^{-(t-s)\sigma^*} e^{-s\sigma} \left( \int_0^s e^{-(s-\tau)\varepsilon''} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}} d\tau \right) ds, \\ I_2 &:= M \|z^0\|_{\mathbb{H}} \int_0^t e^{-(t-s)\sigma^*} e^{-s\sigma} ds. \end{aligned}$$

Noticing that

$$\int_0^s e^{-(s-\tau)\varepsilon''} d\tau = \frac{1}{\varepsilon''} [1 - e^{-s\varepsilon''}] \leq \frac{1}{\varepsilon''},$$

we get that

$$\int_0^s e^{-(s-\tau)\varepsilon''} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}} d\tau \leq \frac{1}{\varepsilon''} \|g\|_{L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)},$$

and hence

$$I_1 \leq M \|g\|_{L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)} \frac{e^{-t\sigma}}{\varepsilon''} \int_0^t e^{(t-s)(\sigma-\sigma^*)} ds.$$

Since

$$\int_0^t e^{(t-s)(\sigma-\sigma^*)} ds = \frac{1 - e^{-t(\sigma^*-\sigma)}}{(\sigma^*-\sigma)} \leq \frac{1}{(\sigma^*-\sigma)},$$

we have

$$I_1 \leq \frac{M e^{-\sigma t}}{\varepsilon''(\sigma^*-\sigma)} \|g\|_{L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)}$$

and similarly

$$I_2 \leq M e^{-\sigma t} \left( \int_0^t e^{(t-s)(\sigma-\sigma^*)} ds \right) \|z^0\|_{\mathbb{H}} \leq \frac{M e^{-\sigma t}}{(\sigma^*-\sigma)} \|z^0\|_{\mathbb{H}}.$$

Using the above estimates in (3.18), we get that

$$\|\Phi(g)(t)\|_{\mathbb{H}} \leq M e^{-\sigma t} \left( \frac{1}{\varepsilon''(\sigma^*-\sigma)} \|g\|_{L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)} + \|z^0\|_{\mathbb{H}} + \frac{1}{\sigma^*-\sigma} \|z^0\|_{\mathbb{H}} \right),$$

2 and hence  $\Phi(g) \in L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)$ .

It remains to show that  $\Phi$  is a contraction mapping. Given  $g_1, g_2 \in L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)$ , the same calculations as above show that

$$\|\Phi(g_1) - \Phi(g_2)\|_{L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)} \leq \frac{M}{\varepsilon''(\sigma^*-\sigma)} \|g_1 - g_2\|_{L_\sigma^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)}.$$

The application  $\Phi$  is thus a contraction provided that  $\sigma^*$  is chosen large enough to ensure that

$$\frac{M}{\varepsilon''(\sigma^* - \sigma)} < 1.$$

Then, applying the fixed point theorem we get that there exists a unique  $z_{\sigma^*}^- \in L^\infty(0, \infty; \mathbb{H}_{\sigma^*}^-)$  such that  $\Phi(z_{\sigma^*}^-) = z_{\sigma^*}^-$  and

$$\|z_{\sigma^*}^-(t)\|_{\mathbb{H}} \leq M e^{-\sigma t} \|z^0\|_{\mathbb{H}}.$$

Moreover, going back to the first equation in (3.15), and using Duhamel's formula again, we easily obtain that

$$\|\mathbb{X}(t)\|_{\mathbb{H} \times \mathbb{H}} \leq M e^{-\sigma t} \|z^0\|_{\mathbb{H}}.$$

1 This completes the proof of Theorem 1.2.

## 2 4 Stabilization of the reaction-diffusion equation

3 Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be a bounded domain of class  $C^{1,1}$ . In this section, we apply Theorem 1.1 and Theorem 1.2  
4 for the stabilization of the heat equation. Let us consider  $\Gamma$  a non-empty open subset of  $\partial\Omega$  and the control  
5 problem:

$$\left\{ \begin{array}{l} \partial_t z(t, x) - \nabla \cdot (b \nabla z(t, x)) - cz(t, x) = 0 \quad \text{in } (0, \infty) \times \Omega, \\ z(t, x) = v(t, x) \quad \text{on } (0, \infty) \times \Gamma, \\ z(t, x) = 0 \quad \text{on } (0, \infty) \times (\partial\Omega \setminus \Gamma), \\ z(0, \cdot) = z^0 \quad \text{in } \Omega, \\ y(t, x) = \mathbb{1}_{\mathcal{O}} z(t, x) \quad \text{in } (0, \infty) \times \Omega, \end{array} \right. \quad (4.1)$$

where  $b, c \in L^\infty(\Omega)$  and  $\mathcal{O}$  an open subset of  $\mathbb{R}^N$  with  $\overline{\mathcal{O}} \subset \Omega$ . In order to write (4.1) under the form (1.3), we introduce the following functional setting:

$$\mathbb{H} = L^2(\Omega), \quad \mathbb{U} = L^2(\Gamma),$$

$$Az = \nabla \cdot (b \nabla z) + cz, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

The operator  $(A, \mathcal{D}(A))$  generates an analytical semigroup, this is a direct consequence of [3, Theorem 2.12, p.115]. Thus (H1.A) is satisfied. Moreover, the operator  $A$  is self-adjoint, then in particular, we see that (H2.A) holds true. To define the control operator  $B$ , we use a standard method (see, for instance [21, pp.341-343] or [17]): we first consider the lifting operator  $D_0 \in \mathcal{L}(L^2(\partial\Omega); L^2(\Omega))$  such that for any  $v \in L^2(\partial\Omega)$ ,  $w = D_0 v$  is the unique solution of the following system

$$\left\{ \begin{array}{l} \mu_0 w - \nabla \cdot (b \nabla w) - cw = 0 \quad \text{in } \Omega, \\ w = v \quad \text{on } \partial\Omega, \end{array} \right.$$

where  $\mu_0 \in \rho(A)$ . Then, we set

$$B = (\mu_0 \text{Id} - A)D_0 : \mathbb{U} \longrightarrow (\mathcal{D}(A^*))',$$

where we have extended the operator  $A$  as an operator from  $L^2(\Omega)$  into  $(\mathcal{D}(A^*))'$  and where we see  $\mathbb{U}$  as a closed subspace of  $L^2(\partial\Omega)$  (by extending by zero in  $\partial\Omega \setminus \Gamma$  any  $v \in \mathbb{U}$ ). Using standard results on elliptic equations, we have that  $B$  satisfies (H1.B) for any  $\gamma > 3/4$  (see for instance [16, Theorem 2.6]). We set also

$$\mathbb{Y} = L^2(\mathcal{O}), \quad C = \mathbb{1}_{\mathcal{O}},$$

we see that  $C \in \mathcal{L}(\mathbb{H}, \mathbb{Y})$ . Let us recall how we can reduce (4.1) to an evolution problem (1.3). We set  $\tilde{z} = z - w$ , with  $w = D_0 v$ . Then  $\tilde{z}$  satisfies the system

$$\left\{ \begin{array}{l} \partial_t \tilde{z} - \nabla \cdot (b \nabla \tilde{z}) - c \tilde{z} = -\partial_t w + \mu_0 w \quad \text{in } (0, \infty) \times \Omega, \\ \tilde{z} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \\ \tilde{z}(0, \cdot) = \tilde{z}^0 := z^0 - w(0, \cdot) \quad \text{in } \Omega. \end{array} \right.$$

Using the Duhamel formula, we have

$$\tilde{z}(t) = e^{tA}z^0 + \int_0^t e^{(t-s)A}(-\partial_t w(s) + \mu_0 w(s)) ds.$$

By integrating by parts, we obtain

$$z(t) = e^{tA}z^0 + \int_0^t e^{(t-s)A}(\mu_0 \text{Id} - A)w(s) ds,$$

that is

$$\begin{cases} z'(t) = Az(t) + Bv(t), & z(0) = z^0, \\ y(t) = Cz(t). \end{cases}$$

To apply Theorem 1.1, we only need to check (H2.D). We recall that

$$\mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega), \quad A^* = A.$$

Moreover, by classical results (see [21, Proposition 10.6.7]), we see that

$$D_0^* := -\frac{\partial}{\partial \nu}(\mu_0 \text{Id} - A^*)^{-1} = -\frac{\partial}{\partial \nu}(\mu_0 \text{Id} - A)^{-1},$$

and thus

$$B^* \varepsilon := -\frac{\partial \varepsilon}{\partial \nu}|_{\Gamma}.$$

Thus if  $\varepsilon$  satisfies  $A^* \varepsilon = \lambda \varepsilon$  and  $B^* \varepsilon = 0$ , then

$$\begin{cases} \lambda \varepsilon - \nabla \cdot (b \nabla \varepsilon) - c \varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial \Omega, \\ \frac{\partial \varepsilon}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

From standard results on the unique continuation of the Laplace operator (see for instance [9, Theorem 5.3.1, p.125]), we deduce that  $\varepsilon = 0$ . In the other hand, if  $\varepsilon$  satisfies  $A \varepsilon = \lambda \varepsilon$  and  $C \varepsilon = 0$ , then

$$\begin{cases} \lambda \varepsilon - \nabla \cdot (b \nabla \varepsilon) - c \varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial \Omega, \\ \varepsilon = 0 & \text{in } \mathcal{O}. \end{cases}$$

<sup>1</sup> We obtain again from standard results on the unique continuation of the Laplace operator (see for instance [9, Theorem 5.3.1, p.125]), we get also that  $\varepsilon = 0$ . Thus (H2.D) holds for any  $\sigma$ .

<sup>2</sup> Now, we define the observer  $\hat{z}$  using Remark 2.3. Let us define  $N_\sigma$  by (2.10) and  $\hat{z}$  the solution of the closed loop system

$$\begin{cases} \partial_t \hat{z} - \nabla \cdot (b \nabla \hat{z}) - c \hat{z} = \sum_{i=1}^{N_\sigma} \langle \mathbb{1}_{\mathcal{O}}(\hat{z} - z), w_i^* \rangle_{\mathbb{Y}} \chi_i & \text{in } (0, \infty) \times \Omega, \\ \hat{z} = \sum_{i=1}^{N_\sigma} \langle \hat{z}, \zeta_i \rangle_{\mathbb{H}} w_i & \text{on } (0, \infty) \times \Gamma, \\ \hat{z} = 0 & \text{on } (0, \infty) \times (\partial \Omega \setminus \Gamma), \\ \hat{z}(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (4.2)$$

<sup>3</sup> where  $(w_i^*)_{1 \leq i \leq N_\sigma} \subset \mathbb{Y}$ ,  $(\chi_i)_{1 \leq i \leq N_\sigma} \subset \mathbb{H}$ ,  $(\zeta_i)_{1 \leq i \leq N_\sigma} \subset \mathcal{D}(A^*)$  and  $(w_i)_{1 \leq i \leq N_\sigma} \subset \mathbb{U}$  (see equations (2.15)-(2.16) and Remark 2.3). We deduce the following result by applying Theorem 1.1:



1 **Theorem 4.1.** *Assume  $\sigma > 0$ . There exists a control*

$$v(t) = \sum_{i=1}^{N_\sigma} \left( \int_{\Omega} \widehat{z}(t) \zeta_i \, dx \right) v_i, \quad (4.3)$$

2 *with  $\zeta_k \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $v_k \in H^{1/2}(\Gamma)$ ,  $k = 1, \dots, N_\sigma$  such that the coupled system (4.1) and (4.2) is*  
3 *exponentially stable that satisfies for  $z^0 \in L^2(\Omega)$  the estimate*

$$\|z(t)\|_{L^2(\Omega)} \leq C e^{-\sigma t} \|z^0\|_{L^2(\Omega)}. \quad (4.4)$$

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