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Abstract

We consider the stabilization of a class of linear evolution systems $z' = Az + Bv$ under the observation $y = Cz$ by means of a finite dimensional control v. The control is based on the design of a Luenberger observer which can be infinite or finite dimensional (of dimension large enough). In the infinite dimensional case, the operator A is supposed to generate an analytical semigroup with compact resolvent and the operators B and C are unbounded operators whereas in the finite dimensional case, A is assumed to be a self-adjoint operator with compact resolvent, B and C are supposed to be bounded operators. In both cases, we show that if (A, B) and (A, C) verify the Fattorini-Hautus Criterion, then we can construct an observer-based control v of finite dimension (greater or equal than largest geometric multiplicity of the unstable eigenvalues of A) such that the evolution problem is exponentially stable. As an application, we study the stabilization of the N dimensional convection-diffusion system with Dirichlet boundary control and an internal observation.

Keywords: Parabolic systems, Feedback control, Stabilization, Luenberger observers.
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Contents

1 Introduction and main results

Given $b \in L^2(0,1)$, consider the one-dimensional controlled heat equation

$$\begin{cases} \partial_t z(t,x) = \partial_{xx} z(t,x) + b(x)v(t), \quad t > 0, \quad x \in (0,1), \\ \partial_x z(t,0) = 0, \quad z(t,1) = 0, \\ z(0) = z^0. \end{cases} \tag{1.1}$$

Obviously, the open-loop system (i.e. for $v = 0$) is exponentially stable, with a decay rate defined by the smallest eigenvalue of the underlying operator describing the free dynamics (namely the positive definite self-adjoint operator $-\partial_{xx}$ with Neumann boundary condition at $x = 0$ and Dirichlet boundary condition at $x = 1$). Based on the observation

$$y(t) = \int_0^1 c(x) z(t,x)\, \mathrm{d}x, \tag{1.2}$$

where $c \in L^2(0,1)$, a natural question that arises is to know wether it is possible to design a finite dimensional feedback control v, such that the closed-loop system (1.1) is exponentially stable with an arbitrary prescribed decay rate $\sigma > 0$. It a recent work, a positive and constructive answer to this question has been proposed by Katz and Fridman [11], using an observer-based feedback control. More precisely, the authors proposed feasible design conditions for the construction of such controls for a more general $1D$ reaction-diffusion equation with variable coefficients (i.e. for a free dynamics described by an operator of the form $\partial_x \left(p(x)\partial_x \cdot\right) - q(x)\cdot$).

In this paper, our objective is to generalize this result to a large class of parabolic systems, possibly multi-dimensional and involving unbounded control and/or observation operators. More precisely, given three Hilbert spaces $\mathbb{H}$ (the state space), $\mathbb{U}$ (the control space) and $\mathbb{Y}$ (the observation space), consider the linear infinite dimensional system

$$\begin{cases} z'(t) = Az(t) + Bv(t), \\ z(0) = z^0, \\ y(t) = Cz(t), \end{cases} \tag{1.3}$$

where $A : \mathcal{D}(A) \longrightarrow \mathbb{H}$ is an unbounded operator, $B \in \mathcal{L}\left(\mathbb{U}, (\mathcal{D}(A^*))'\right)$ and $C \in \mathcal{L}\left(\mathcal{D}(A), \mathbb{Y}\right)$. Given $\sigma > 0$, the goal of this paper is to prove the existence of an observer-based control v such that the solution of (1.3) is exponentially stable, with a decay rate $-\sigma$:

$$\|z(t)\|_{\mathbb{H}} \leqslant M e^{-\sigma t} \|z^0\|_{\mathbb{H}}.$$

We will investigate two classes of systems, depending on whether the observer used is infinite-dimensional (IDO case) or finite-dimensional (FDO case). We will make the following assumptions on A, B and C in these two cases (below, $\rho(A)$ denote the resolvent set of A):

- **Infinite-Dimensional Observer (IDO)**

$$\text{A is an analytic operator with compact resolvent on } \mathbb{H}, \tag{H1.A}$$

$$(\mu_0 \,\mathrm{Id} - A)^{-\gamma} B \in \mathcal{L}(\mathbb{U}, \mathbb{H}) \quad \text{is a linear bounded operator for some } \gamma \in [0,1) \text{ and } \mu_0 \in \rho(A), \tag{H1.B}$$

$$C(\mu_0 \,\mathrm{Id} - A)^{-\widehat{\gamma}} \in \mathcal{L}(\mathbb{H}, \mathbb{Y}) \quad \text{is a linear bounded operator for some } \widehat{\gamma} \in [0,1) \text{ and } \mu_0 \in \rho(A), \tag{H1.C}$$

$$\begin{cases} \forall \varepsilon \in \mathcal{D}(A^*),\ \forall \lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \geqslant -\sigma, \quad A^*\varepsilon = \overline{\lambda}\varepsilon \quad \text{and} \quad B^*\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0, \\ \forall \varepsilon \in \mathcal{D}(A),\ \forall \lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \geqslant -\sigma, \quad A\varepsilon = \lambda\varepsilon \quad \text{and} \quad C\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \end{cases} \tag{H1.D}$$

- **Finite-Dimensional Observer (FDO)**

$$\text{A is a self-adjoint operator with compact resolvent,} \tag{H2.A}$$

$$B \in \mathcal{L}(\mathbb{U}, \mathbb{H}), \tag{H2.B}$$

$$C \in \mathcal{L}(\mathbb{H}, \mathbb{Y}), \tag{H2.C}$$

$$\begin{cases} \forall \varepsilon \in \mathcal{D}(A),\ \forall \lambda \in \mathbb{R},\ \lambda \geqslant -\sigma, \quad A\varepsilon = \lambda\varepsilon \quad \text{et} \quad B^*\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0, \\ \forall \varepsilon \in \mathcal{D}(A),\ \forall \lambda \in \mathbb{R},\ \lambda \geqslant -\sigma, \quad A\varepsilon = \lambda\varepsilon \quad \text{et} \quad C\varepsilon = 0 \quad \Longrightarrow \quad \varepsilon = 0. \end{cases} \tag{H2.D}$$

It is worth mentioning that assumption (H1.D) (and its counterpart (H2.D) in the self-adjoint case) is the well-known Fattorini-Hautus criterion for exponential stabilization (see [6], [8] and [2]).

For every $\nu > 0$, we set

$$\Sigma_\nu^+ := \{\lambda_j \in \sigma(A)\ ;\ \mathrm{Re}\,\lambda_j \geqslant -\nu\}, \quad \Sigma_\nu^- := \{\lambda_j \in \sigma(A)\ ;\ \mathrm{Re}\,\lambda_j < -\nu\}, \tag{1.4}$$

where $\sigma(A)$ is the spectrum of A. Condition (H1.A) in the IFD case and (H2.A) in the FDO case imply that Σ_ν^+ describes a finite set. We define the projection

$$P_\nu^+ = -\frac{1}{2\imath\pi}\int_{\Gamma_\nu^+} (\zeta\,\mathrm{Id} - A)^{-1}\,d\zeta, \qquad P_\nu^- = \mathrm{Id} - P_\nu^+, \tag{1.5}$$

where Γ_ν^+ is a curve enclosing Σ_ν^+ but no other point of the spectrum of A and oriented counterclockwise (see [10, V.5, p.272]). We set

$$z_\nu^\pm := P_\nu^\pm z, \qquad \forall z \in \mathbb{H}.$$

We also introduce the finite dimensional operators

$$A_\nu^\pm := AP_\nu^\pm, \qquad B_\nu^\pm := P_\nu^\pm B, \qquad C_\nu^\pm := C\iota_\nu^\pm,$$

where $\iota_\nu^\pm$ is the embedding operator from $\mathbb{H}_\nu^\pm := P_\nu^\pm\mathbb{H}$ to $\mathbb{H}$. Finally, we denote by Q_ν^+ the orthogonal projection from $\mathbb{Y}$ onto $\mathbb{Y}_\nu^+ := CP_\nu^+\mathbb{H}$.

We are now in position to state our main results in the (IDO) and (FDO) cases.

1.1 Infinite-Dimensional Observer (IDO)

Theorem 1.1. *Let $\sigma > 0$ be given and assume that assumptions* (H1.A), (H1.B), (H1.C) *and* (H1.D) *hold true. Then, there exist two operators $K_\sigma^+ \in \mathcal{L}(\mathbb{H}, B^*(P_\sigma^+)^*\mathbb{H})$ and $L_\sigma^+ \in \mathcal{L}(CP_\sigma^+\mathbb{H}, P_\sigma^+\mathbb{H})$ such that the observer-based feedback control defined by*

$$v(t) = K_\sigma^+\widehat{z}(t), \tag{1.6}$$

where the infinite dimensional observer $\widehat{z}$ solves

$$\begin{cases} \widehat{z}'(t) = A\widehat{z}(t) + Bv(t) + L_\sigma^+Q_\sigma^+(C\widehat{z}(t) - y(t)), \\ \widehat{z}(0) = 0, \end{cases} \tag{1.7}$$

ensures that for any $z^0 \in \mathbb{H}$, the solution z of the closed loop (1.3)-(1.6)-(1.7)*, that is*

$$\begin{cases} z'(t) = Az(t) + BK_\sigma^+\widehat{z}(t), \\ z(0) = z^0, \end{cases}$$

satisfies

$$\|z(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\|z^0\|_{\mathbb{H}}, \qquad \forall t > 0. \tag{1.8}$$

To prove this result, we introduce the error $e := z - \widehat{z}$ and we check that systems (1.3) and (1.7) yield

$$\begin{pmatrix} z \\ e \end{pmatrix}' = \begin{pmatrix} A + BK_\sigma^+ & -BK_\sigma^+ \\ 0 & A + L_\sigma^+Q_\sigma^+C \end{pmatrix}\begin{pmatrix} z \\ e \end{pmatrix}.$$

The result follows then by choosing the operators K_σ^+ and L_σ^+ in such a way that both $A+BK_\sigma^+$ and $A+L_\sigma^+Q_\sigma^+C$ generate analytic semigroups with decay rate less than $-\sigma$. This is achieved by solving two Riccati equations, using the method proposed by Badra-Takahashi [1] (for K_σ^+) combined to a duality argument (for L_σ^+).

1.2 Finite-Dimensional Observer (FDO)

In this case, the stabilizing control is based on a finite dimensional observer of size $\sigma^\star > 0$, where $\sigma^\star > \sigma$ and need to be chosen large enough.

Theorem 1.2. *Let $\sigma > 0$ be given and assume that assumptions* (H2.A), (H2.B), (H2.C) *and* (H2.D) *hold true. Let $K_\sigma^+ \in L(\mathbb{H}, B^*P_\sigma^+\mathbb{H})$ and $L_\sigma^+ \in \mathcal{L}(CP_\sigma^+\mathbb{H}, P_\sigma^+\mathbb{H})$ be the operators defined in Theorem 1.1. Then, there exists $\sigma^\star > \sigma$ such that the observer-based feedback control defined by*

$$v(t) = K_\sigma^+ \widehat{z}_\star(t) \in B^*P_\sigma^+\mathbb{H} \subset B^*P_{\sigma^\star}^+\mathbb{H}, \tag{1.9}$$

where the finite dimensional observer $\widehat{z}_\star \in P_{\sigma^\star}^+\mathbb{H}$ solves

$$\begin{cases} \widehat{z}_\star'(t) = A_{\sigma^\star}^+ \widehat{z}_\star(t) + B_{\sigma^\star}^+ v(t) + L_\sigma^+ Q_\sigma^+ (C_{\sigma^\star}^+ \widehat{z}_\star(t) - y(t)), \\ \widehat{z}_\star(0) = 0, \end{cases} \tag{1.10}$$

ensures that for any $z^0 \in \mathbb{H}$, the solution z of the closed loop (1.3)-(1.9)-(1.10)*, that is*

$$\begin{cases} z'(t) = Az(t) + BK_\sigma^+ \widehat{z}_\star(t), \\ z(0) = z^0, \end{cases}$$

satisfies

$$\|z(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\|z^0\|_{\mathbb{H}}, \qquad \forall t > 0. \tag{1.11}$$

To prove this result, we proceed as follows. Introducing the auxiliary variables

$$e := z_{\sigma^\star}^+ - \widehat{z}_\star \qquad\qquad \mathbb{X} = \begin{pmatrix} \widehat{z}_\star \\ e \end{pmatrix},$$

we show that the equations satisfied by the state z and the observer $\widehat{z}_\star$ yield

$$\begin{cases} \mathbb{X}'(t) = \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z_{\sigma^\star}^-(t)), \\ (z_{\sigma^\star}^-)'(t) = A_{\sigma^\star}^- z_{\sigma^\star}^-(t) + B_{\sigma^\star}^- \mathbb{K}_\sigma^+ \mathbb{X}(t), \end{cases} \tag{1.12}$$

where

$$\mathbb{A} = \begin{pmatrix} A_{\sigma^\star}^+ + B_{\sigma^\star}^+ K_\sigma^+ & -L_\sigma^+ Q_\sigma^+ C_{\sigma^\star}^+ \\ 0 & A_{\sigma^\star}^+ + L_\sigma^+ Q_\sigma^+ C_{\sigma^\star}^+ \end{pmatrix}, \quad \mathbb{L}(z_{\sigma^\star}^-) = \begin{pmatrix} -L_\sigma^+ Q_\sigma^+ C z_{\sigma^\star}^- \\ L_\sigma^+ Q_\sigma^+ C z_{\sigma^\star}^- \end{pmatrix}, \quad \mathbb{K}_\sigma^+ = (K_\sigma^+, 0).$$

We prove then that the matrix $\exp(t\mathbb{A})$ is exponentially stable with a decay rate less than $-\sigma$. Next, thanks to the first equation in (1.12), we use Duhamel's formula to express $\mathbb{X}$ in terms of $z_{\sigma^\star}^-$. Plugging the obtained relation in the second equation of (1.12), we obtain an integral equation for $z_{\sigma^\star}^-$. We use a fixed point argument to prove the well-posedness of this integral equation in the weighted space

$$L_\sigma^\infty(0,\infty;\mathbb{H}_{\sigma^\star}^-) := \{f \in L^\infty(0,\infty;\mathbb{H}_{\sigma^\star}^-) \text{ such that } e^{\sigma(\cdot)}f(\cdot) \in L^\infty(0,\infty;\mathbb{H}_{\sigma^\star}^-)\}.$$

This provides the expected result, that is the exponential decay of the controlled system with a decay rate less than $-\sigma$.

1.3 Related works

As already mentioned, the closest reference to our work is [11], in which the authors considered the case of a one dimensional heat equation. Their strategy is based on a (modal) splitting of the system into two parts: a finite dimensional unstable one and a stable infinite dimensional one. A Luenberger observer of large enough dimension is then constructed and the stability of the closed loop system is proved using a Lyapunov function.

Contrarily to the proof proposed here, the arguments used in [11] are valid only in dimension one and heavily rely on the type of the considered equation. Let us also mention that in [12], the authors used a similar approach to prove the stabilization of a one dimension convection diffusion equation in the case of a boundary control. The use of modal splitting for the stabilization of infinite dimensional systems has also been achieved in some specific settings, like Burgers equations [20, 4, 19], Navier-Stokes system [1, 2, 7, 16, 18], semi-linear wave equation [5] and populations dynamics [14, 15].

1.4 Outline

In Section 2, we prove Theorem 1.1, which provides the stabilizing observer-based feedback-control through an infinite dimensional observer. In Section 3, we construct a finite dimensional observer to design a similar feedback-control. For this case, we need to assume that the operator A is self-adjoint and the control and observation operators are bounded. Finally, in Section 4, these abstract results are applied to obtain a stabilizing control for a reaction-diffusion system.

2 Infinite dimensional observer

2.1 Spectral decomposition of the system

In this section, we suppose that assumptions (H1.A), (H1.B), (H1.C) and (H1.D) hold true. We consider below a classical modal decomposition (it has been used, for instance, in [1, 2, 7, 18]) that we recall it for the sake of completeness. Let $\sigma > 0$. We first separate the spectrum of A into "unstable" and "stable" modes using the projection P_σ^+ defined in (1.5). We set

$$\mathbb{H}_\sigma^+ = P_\sigma^+ \mathbb{H}, \quad \mathbb{H}_\sigma^- = (\mathrm{Id} - P_\sigma^+)\mathbb{H}, \quad \mathbb{H} = \mathbb{H}_\sigma^+ \oplus \mathbb{H}_\sigma^-.$$

According to this projection, we set

$$A_\sigma^+ := A_{|\mathbb{H}_\sigma^+} : \mathbb{H}_\sigma^+ \to \mathbb{H}_\sigma^+, \quad A_\sigma^- := A_{|\mathbb{H}_\sigma^-} : \mathcal{D}(A) \cap \mathbb{H}_\sigma^- \to \mathbb{H}_\sigma^-.$$

Then the spectrum of A_σ^+ is exactly Σ_σ^+ and the spectrum of A_σ^- is exactly Σ_σ^- where Σ_σ^+ and Σ_σ^- are defined in (1.4). We denote by A^* the adjoint operator of A and we define similarly the projection $(P_\sigma^+)^*$ such that

$$(P_\sigma^+)^* = -\frac{1}{2\imath\pi} \int_{\overline{\Gamma_\sigma^+}} (\zeta \,\mathrm{Id} - A^*)^{-1} \, d\zeta. \tag{2.1}$$

The projection (2.1) provides also the following spaces

$$(\mathbb{H}_\sigma^+)^* = (P_\sigma^+)^* \mathbb{H}, \quad (\mathbb{H}_\sigma^-)^* = (\mathrm{Id} - (P_\sigma^+)^*)\mathbb{H}, \quad \mathbb{H} = (\mathbb{H}_\sigma^+)^* \oplus (\mathbb{H}_\sigma^-)^*, \tag{2.2}$$

with

$$(A_\sigma^+)^* := A^*_{|(\mathbb{H}_\sigma^+)^*} : (\mathbb{H}_\sigma^+)^* \to (\mathbb{H}_\sigma^+)^*, \quad (A_\sigma^-)^* := A_{|(\mathbb{H}_\sigma^-)^*} : \mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^* \to (\mathbb{H}_\sigma^-)^*.$$

Lemma 2.1. *There exist $\varepsilon > 0$ and $M > 0$ such that for any $\delta \geqslant 0$, $t > 0$*

$$\begin{aligned} &\left\| e^{A_\sigma^- t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant M e^{-(\sigma+\varepsilon)t}, && \left\| (\mu_0 \,\mathrm{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant \frac{M}{t^\delta} e^{-(\sigma+\varepsilon)t}, \\ &\left\| e^{(A_\sigma^-)^* t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant M e^{-(\sigma+\varepsilon)t}, && \left\| (\mu_0 \,\mathrm{Id} - (A_\sigma^-)^*)^\delta e^{(A_\sigma^-)^* t} \right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant \frac{M}{t^\delta} e^{-(\sigma+\varepsilon)t}. \end{aligned} \tag{2.3}$$

Proof. We detail the proof only for the operator A_σ^-, as the arguments for its adjoint are similar. The first inequality is obvious. Concerning the second one, we first note that

$$(\mu_0 \,\mathrm{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t} = (\mu_0 \,\mathrm{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta} \left(A_\sigma^-\right)^\delta e^{A_\sigma^- t}.$$

Applying [13, Corollary 6.11] with $B = (\mu_0 \operatorname{Id} - A_\sigma^-)^\delta$, $A = A_\sigma^-$ and $x = \left(A_\sigma^-\right)^{-\delta} y$, for $y \in \mathbb{H}_\sigma^-$, we obtain that for some constant positive C,

$$\left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta} y\right\|_{\mathbb{H}} \leqslant C\|y\|_{\mathbb{H}}, \qquad \forall y \in \mathbb{H}_\sigma^-,$$

and thus

$$\left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant C.$$

Consequently,

$$\left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta e^{A_\sigma^- t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant \left\|(\mu_0 \operatorname{Id} - A_\sigma^-)^\delta \left(A_\sigma^-\right)^{-\delta}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \left\|\left(A_\sigma^-\right)^\delta e^{A_\sigma^- t}\right\|_{\mathcal{L}(\mathbb{H}_\sigma^-)},$$

and the desired estimate follows then immediately from [13, Theorem 6.13]. □

We also define

$$\mathbb{U}_\sigma^+ := B^*(\mathbb{H}_\sigma^+)^*, \quad \mathbb{U}_\sigma^- := B^*\left(\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*\right),$$

and

$$p_\sigma^+ : \mathbb{U} \to \mathbb{U}_\sigma^+, \quad p_\sigma^- : \mathbb{U} \to \mathbb{U}_\sigma^-, \quad i_\sigma^+ : \mathbb{U}_\sigma^+ \to \mathbb{U}, \quad i_\sigma^- : \mathbb{U}_\sigma^- \to \mathbb{U},$$

the orthogonal projections and the inclusion maps. Note that we have the following relations for the above maps:

$$i_\sigma^+ = (p_\sigma^+)^*, \quad i_\sigma^- = (p_\sigma^-)^*. \tag{2.4}$$

From [18], we can extend P_σ^+ and $(\operatorname{Id} - P_\sigma^+)$ as bounded operators

$$P_\sigma^+ \in \mathcal{L}(\mathcal{D}(A^*)', \mathbb{H}_\sigma^+), \quad (\operatorname{Id} - P_\sigma^+) \in \mathcal{L}(\mathcal{D}(A^*)', \left[\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*\right]').$$

We can thus define

$$B_\sigma^+ := P_\sigma^+ B i_\sigma^+ \in \mathcal{L}(\mathbb{U}_\sigma^+, \mathbb{H}_\sigma^+), \quad B_\sigma^- := (\operatorname{Id} - P_\sigma^+) B i_\sigma^- \in \mathcal{L}(\mathbb{U}_\sigma^-, \left[\mathcal{D}(A^*) \cap (\mathbb{H}_\sigma^-)^*\right]').$$

We show as in [1, 2, 18] that

$$P_\sigma^+ B = B_\sigma^+ p_\sigma^+, \quad (\operatorname{Id} - P_\sigma^+) B = B_\sigma^- p_\sigma^-.$$

Using the projections P_σ^+ and $\operatorname{Id} - P_\sigma^+$, system (1.3) can be split into the two sub-systems (see [1, 2, 18]).

$$(z_\sigma^+)'(t) = A_\sigma^+ z_\sigma^+(t) + B_\sigma^+ p_\sigma^+ v(t), \quad z_\sigma^+(0) = P_\sigma^+ z^0, \tag{2.5}$$

$$(z_\sigma^-)'(t) = A_\sigma^- z_\sigma^-(t) + B_\sigma^- p_\sigma^- v(t), \quad z_\sigma^-(0) = (\operatorname{Id} - P_\sigma^+) z^0. \tag{2.6}$$

We also introduce the orthogonal projections Q_σ^+ from $\mathbb{Y}$ into $\mathbb{Y}_\sigma^+ = C\mathbb{H}_\sigma^+$ and we define

$$C_\sigma^+ = C\iota_\sigma^+,$$

where ι_σ^+ designates the injection operator from $\mathbb{H}_\sigma^+$ to $\mathbb{H}$. We are now in position to prove Theorem 1.1.

2.2 Construction of an infinite dimensional observer based control

Let us consider first the system

$$\begin{cases} \widetilde{z}'(t) = A\widetilde{z}(t) + Bu(t), \\ \widetilde{z}(0) = \widetilde{z}^0. \end{cases} \tag{2.7}$$

We want to construct a finite dimensional vector u such that the system (2.7) is exponentially stable. Let $N_\sigma \in \mathbb{N}^*$ and $(w_j)_{1 \leqslant j \leqslant N_\sigma} \subset \mathbb{U}$, and let us suppose that the control $u(t)$ is of the form

$$u(t) = \sum_{j=1}^{N_\sigma} u_j(t) w_j,$$

where $u_j(t) \in \mathbb{C}$, for $1 \leqslant j \leqslant N_\sigma$ and $t \geqslant 0$. It is natural to introduce the mapping

$$\mathfrak{B} : \mathbb{C}^{N_\sigma} \longrightarrow \mathcal{D}(A^*)', \quad \mathbf{\Theta} = (\theta_1, \cdots, \theta_{N_\sigma}) \longmapsto \mathfrak{B}\mathbf{\Theta} = \sum_{j=1}^{N_\sigma} \theta_j B w_j,$$

in such a way that setting

$$\mathfrak{u}(t) := (u_1(t), \cdots, u_{N_\sigma}(t)),$$

system (2.7) is equivalent to

$$\begin{cases} \widetilde{z}'(t) = A\widetilde{z}(t) + \mathfrak{B}\mathfrak{u}(t), \\ \widetilde{z}(0) = \widetilde{z}^0. \end{cases} \tag{2.8}$$

It is worth noticing that the adjoint $\mathfrak{B}^* \in \mathcal{L}(\mathcal{D}(A^*), \mathbb{C}^{N_\sigma})$ is given by

$$\mathfrak{B}^*\psi = \Big(\langle w_1, B^*\psi\rangle_{\mathbb{U}}, \cdots, \langle w_{N_\sigma}, B^*\psi\rangle_{\mathbb{U}}\Big).$$

Using the projection P_σ^+, we get that (2.8) is equivalent to

$$\begin{cases} (\widetilde{z}_\sigma^+)'(t) = A_\sigma^+ \widetilde{z}_\sigma^+(t) + \mathfrak{B}_\sigma^+ \mathfrak{u}(t), \quad \widetilde{z}_\sigma^+(0) = P_\sigma^+ \widetilde{z}^0, \quad \mathfrak{B}_\sigma^+ = P_\sigma^+ \mathfrak{B}, \\ (\widetilde{z}_\sigma^-)'(t) = A_\sigma^- \widetilde{z}_\sigma^-(t) + \mathfrak{B}_\sigma^- \mathfrak{u}(t), \quad \widetilde{z}_\sigma^-(0) = P_\sigma^- \widetilde{z}^0, \quad \mathfrak{B}_\sigma^- = P_\sigma^- \mathfrak{B}, \end{cases} \tag{2.9}$$

where $\widetilde{z}_\sigma^\pm = P_\sigma^\pm \widetilde{z}$. We need to show that the finite dimensional part $(2.9)_1$ is exactly controllable. Let

$$N_\sigma \geqslant \max_{\mathrm{Re}\,\lambda_j \geqslant -\sigma} \ell_j, \tag{2.10}$$

where ℓ_j is the geometric multiplicity of the eigenvalue λ_j of the operator A. From [1, Theorem 5] and the first condition in (H1.D), there exists a family $(w_j)_{1\leqslant j\leqslant N_\sigma} \subset \mathbb{U}_\sigma^+ \subset \mathbb{U}$ such that $(2.9)_1$ is exactly controllable. Moreover, it is proved that u is expressed by means of a linear feedback operator

$$u = K_\sigma^+ \widetilde{z}, \quad K_\sigma^+(\cdot) = -\sum_{j=1}^{N_\sigma} \langle w_j, B^*\Pi P_\sigma^+(\cdot)\rangle_{\mathbb{U}} w_j = -\sum_{j=1}^{N_\sigma} (\mathfrak{B}^*(\Pi P_\sigma^+(\cdot)))_j w_j, \tag{2.11}$$

where $\Pi \in \mathcal{L}(\mathbb{H}_\sigma^+, (\mathbb{H}_\sigma^+)^*)$ is the unique solution of the algebraic Riccati equation: for all $\xi, \zeta \in \mathbb{H}_\sigma^+$

$$\begin{cases} \langle \xi, \zeta\rangle_{\mathbb{H}} + \langle (A_\sigma^+ + \sigma\,\mathrm{Id})\xi, \Pi\zeta\rangle_{\mathbb{H}} + \langle \Pi\xi, (A_\sigma^+ + \sigma\,\mathrm{Id})\zeta\rangle_{\mathbb{H}} - \displaystyle\sum_{j=1}^{N_\sigma} \langle B^*\Pi\xi, w_j\rangle_{\mathbb{U}} \langle B^*\Pi\zeta, w_j\rangle_{\mathbb{U}} = 0, \\ \langle \Pi\xi, \zeta\rangle_{\mathbb{H}} = \langle \xi, \Pi\zeta\rangle_{\mathbb{H}}, \quad \text{and} \quad \forall \xi \neq 0,\ \langle \Pi\xi, \xi\rangle_{\mathbb{H}} > 0. \end{cases} \tag{2.12}$$

This choice ensures that the solution of the finite dimensional system $(2.9)_1$

$$(\widetilde{z}_\sigma^+)'(t) = A_\sigma^+ \widetilde{z}_\sigma^+(t) - \mathfrak{B}_\sigma^+(\mathfrak{B}^*(\Pi P_\sigma^+ \widetilde{z}(t)))$$

is exponentially stable i.e.

$$\|\widetilde{z}_\sigma^+(t)\|_{\mathbb{H}} \leqslant M e^{-(\sigma+\varepsilon)t} \|\widetilde{z}^0\|_{\mathbb{H}}, \quad t > 0.$$

It follows from Duhamel's formula that the whole system (2.9) is exponentially stable (see [1]). We can construct L_σ^+ similarly considering the system

$$\begin{cases} \widetilde{z}_\star'(t) = A^*\widetilde{z}_\star(t) + C^* u_\star(t), \\ \widetilde{z}_\star(0) = \widetilde{z}_\star^0. \end{cases} \tag{2.13}$$

Using similar arguments and the second condition in (H1.D), we show that there exists a family $(w_j^\star)_{1\leqslant j\leqslant N_\sigma} \subset \mathbb{Y}_\sigma^+$ such that

$$u_\star = L_\star \widetilde{z}_\star, \quad L_\star(\cdot) = -\sum_{j=1}^{N_\sigma} \langle w_j^\star, C\Pi_\star(P_\sigma^+)^*(\cdot)\rangle_{\mathbb{Y}} w_j^\star = -\sum_{j=1}^{N_\sigma} (\mathfrak{C}_\star^*(\Pi_\star(P_\sigma^+)^*(\cdot)))_j w_j^\star,$$

where

$$\mathfrak{C}_\star : \mathbb{C}^{N_\sigma} \longrightarrow \mathcal{D}(A^*)', \quad \mathbf{\Theta} = (\theta_1, \cdots, \theta_{N_\sigma}) \longrightarrow \mathfrak{C}_\star \mathbf{\Theta} = \sum_{j=1}^{N_\sigma} \theta_j C^* w_j^\star,$$

where $\Pi_\star \in \mathcal{L}((\mathbb{H}_\sigma^+)^*, \mathbb{H}_\sigma^+)$ is the unique solution of the algebraic Riccati equation: for all $\xi, \zeta \in (\mathbb{H}_\sigma^+)^*$

$$\begin{cases} \langle \xi, \zeta \rangle_{\mathbb{H}} + \langle ((A_\sigma^+)^* + \sigma \,\mathrm{Id})\xi, \Pi_\star \zeta \rangle_{\mathbb{H}} + \langle \Pi_\star \xi, ((A_\sigma^+)^* + \sigma \,\mathrm{Id})\zeta \rangle_{\mathbb{H}} - \displaystyle\sum_{j=1}^{N_\sigma} \langle C\Pi_\star \xi, w_j \rangle_{\mathbb{Y}} \langle C\Pi_\star \zeta, w_j \rangle_{\mathbb{Y}} = 0, \\ \langle \Pi_\star \xi, \zeta \rangle_{\mathbb{H}} = \langle \xi, \Pi_\star \zeta \rangle_{\mathbb{H}}, \quad \text{and} \quad \forall \xi \neq 0,\ \langle \Pi_\star \xi, \xi \rangle_{\mathbb{H}} > 0. \end{cases} \tag{2.14}$$

Hence, we define

$$L_\sigma^+(\cdot) = L_\star^*(\cdot) = -\sum_{j=1}^{N_\sigma} \langle w_j^\star, \cdot \rangle_{\mathbb{Y}} \chi_j, \tag{2.15}$$

with

$$\chi_j = P_\sigma^+ \Pi_\star C^* w_j^\star \in \mathbb{H}_\sigma^+. \tag{2.16}$$

With this choice, we get that $(A + L_\sigma^+ C)^*$ and hence $A + L_\sigma^+ Q_\sigma^+ C$ is exponentially stable with decay rate less than $-\sigma$. Finally, using K_σ^+ and L_σ^+ we construct the observer $\widehat{z}$ satisfying (1.6)-(1.7), that is

$$\begin{cases} \widehat{z}'(t) = A\widehat{z}(t) + BK_\sigma^+ \widehat{z}(t) + L_\sigma^+ Q_\sigma^+ (C\widehat{z}(t) - y(t)), \\ \widehat{z}(0) = 0. \end{cases} \tag{2.17}$$

2.3 Stability of the closed-loop system

We define the error $e = z - \widehat{z}$. Then, we obtain

$$\begin{cases} e'(t) = (A + L_\sigma^+ Q_\sigma^+ C)e(t), & e(0) = z^0, \\ z'(t) = (A + BK_\sigma^+)z(t) - BK_\sigma^+ e(t), & z(0) = z^0. \end{cases} \tag{2.18}$$

We prove that e is exponentially stable with decay rate $-\sigma$.

Proposition 2.2. *Systems* (2.17) *and* (2.18) *are exponentially stable with decay rate* $-\sigma$.

Proof. Since $(A + L_\sigma^+ Q_\sigma^+ C)$ is of negative type strictly less than $-\sigma$, then there exists $0 < \varepsilon'' < \varepsilon$ such that

$$\|e(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}}. \tag{2.19}$$

Going back to system (2.5) with the control given by (1.6), we have, since $K_\sigma^+ z_\sigma^- = 0$,

$$(z_\sigma^+)'(t) = (A_\sigma^+ + B_\sigma^+ K_\sigma^+) z_\sigma^+(t) - B_\sigma^+ K_\sigma^+ e(t), \quad z_\sigma^+(0) = P_\sigma^+ z^0.$$

Moreover, there exists $\varepsilon' > 0$ with $\varepsilon'' < \varepsilon' < \varepsilon$ such that $(A_\sigma^+ + B_\sigma^+ K_\sigma^+)$ is exponentially stable with rate $-\sigma - \varepsilon'$. We have

$$z_\sigma^+(t) = e^{t(A_\sigma^+ + B_\sigma^+ K_\sigma^+)} P_\sigma^+ z^0 - \int_0^t e^{(t-s)(A_\sigma^+ + B_\sigma^+ K_\sigma^+)} B_\sigma^+ K_\sigma^+ e(s) ds.$$

From (2.19), we see that

$$\|z_\sigma^+(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon'')} \|z^0\|_{\mathbb{H}}. \tag{2.20}$$

We deal now with the infinite dimensional part z_σ^- of the state. From (2.6) with the control given by (1.6), we have

$$(z_\sigma^-)'(t) = A_\sigma^- z_\sigma^-(t) + B_\sigma^- p_\sigma^- K_\sigma^+ z_\sigma^+(t) - B_\sigma^- p_\sigma^- K_\sigma^+ e(t), \quad z_\sigma^-(0) = (\mathrm{Id} - P_\sigma^+) z^0.$$

Using Duhamel's formula, we obtain that

$$z_\sigma^-(t) = e^{tA_\sigma^-}(\mathrm{Id} - P_\sigma^+)z^0 + \int_0^t e^{(t-s)A_\sigma^-} B_\sigma^- p_\sigma^- K_\sigma^+ \left(z_\sigma^+(s) - e(s)\right)\, ds.$$

We note that since the resolvent commutes with the projection P_σ^+ and $e^{(t-s)A_\sigma^-}$, we obtain for $\mu_0 \in \rho(A)$ and $\gamma \in [0,1)$,

$$\begin{aligned} e^{(t-s)A_\sigma^-} B_\sigma^- &= e^{(t-s)A_\sigma^-}(\mu_0 \mathrm{Id} - A)^\gamma (\mathrm{Id} - P_\sigma^+)(\mu_0 \mathrm{Id} - A)^{-\gamma} B i_\sigma^- \\ &= e^{(t-s)A_\sigma^-}(\mu_0 \mathrm{Id} - A_\sigma^-)^\gamma (\mathrm{Id} - P_\sigma^+)(\mu_0 \mathrm{Id} - A)^{-\gamma} B i_\sigma^- \\ &= (\mu_0 \mathrm{Id} - A_\sigma^-)^\gamma e^{(t-s)A_\sigma^-}(\mathrm{Id} - P_\sigma^+)(\mu_0 \mathrm{Id} - A)^{-\gamma} B i_\sigma^-. \end{aligned} \tag{2.21}$$

Using (H1.B), (2.3), (2.19), (2.20) and (2.21), we get

$$\begin{aligned} \|z_\sigma^-(t)\|_{\mathbb{H}} &\leqslant M\|z^0\|_{\mathbb{H}} \left(e^{-t(\sigma+\varepsilon)} + \int_0^t \frac{1}{(t-s)^\gamma} e^{-(t-s)(\sigma+\varepsilon)} e^{-s(\sigma+\varepsilon'')}\, ds \right) \\ &\leqslant M e^{-t(\sigma+\varepsilon'')}\|z^0\|_{\mathbb{H}} \left(1 + \int_0^t \frac{1}{(t-s)^\gamma} e^{(t-s)(\varepsilon''-\varepsilon)}\, ds \right). \end{aligned} \tag{2.22}$$

Then, since $\varepsilon'' < \varepsilon$, we obtain

$$\|z_\sigma^-(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon'')}\|z^0\|_{\mathbb{H}}. \tag{2.23}$$

Then from (2.19), (2.20) and (2.23), we obtain that z, $\widehat{z}$ and the error e are exponentially stable. □

That concludes the proof of Theorem 1.1.

Remark 2.3. *According to* (1.6) *and* (2.11)*, the control reads*

$$v(t) = \sum_{i=1}^{N_\sigma} K_i\left(\widehat{z}_\sigma^+\right) w_i,$$

with $K_i \in \mathcal{L}(\mathbb{H}_\sigma^+, \mathbb{C})$ *and* $w_i \in \mathbb{U}_\sigma^+$, $i = 1, \ldots, N_\sigma$. *From decomposition* (2.2) *and the fact that* $(\mathbb{H}_\sigma^+)^\perp = (\mathbb{H}_\sigma^-)^*$, *we have that if* $\zeta \in (\mathbb{H}_\sigma^+)^*$, *then*

$$\forall \phi \in \mathbb{H}_\sigma^+, \quad \langle \phi, \zeta \rangle_{\mathbb{H}} = 0 \implies \zeta = 0.$$

Since $\dim\left((\mathbb{H}_\sigma^+)^*\right) = \dim \mathcal{L}(\mathbb{H}_\sigma^+, \mathbb{C})$, *we infer that there exists a unique* $\zeta_i \in (\mathbb{H}_\sigma^+)^* \subset \mathcal{D}(A^*)$ *such that*

$$K_i\left(\widehat{z}_\sigma^+\right) = \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}}.$$

In other words, the control can also be written in the form

$$v(t) = \sum_{i=1}^{N_\sigma} \langle \widehat{z}, \zeta_i \rangle_{\mathbb{H}} w_i.$$

In the special case where there is only one unstable simple eigenvalue (with an eigenspace spanned by an eigenfunction $\varepsilon_1 \in \mathbb{H}$*), the above relations take simpler forms. Indeed, we have then*

$$w_1 = B^* \varepsilon_1$$

and for all $\varphi \in \mathbb{H}_\sigma^+ = \mathrm{Span}\{\varepsilon_1\}$*:*

$$K_1(\varphi) = -\langle B^*\varepsilon_1, B^*\Pi P_\sigma^+ \varphi \rangle_{\mathbb{U}} = \langle \zeta_1, \varphi \rangle_{(\mathbb{H}_\sigma^+)^*, \mathbb{H}_\sigma^+}, \qquad \zeta_1 = -(P_\sigma^+)^* \Pi B B^* \varepsilon_1.$$

3 Finite dimensional observer

3.1 Spectral decomposition of the system

In this section, we assume hypotheses (H2.A), (H2.B), (H2.C) and (H2.D) to hold true.
Consider $\nu > 0$ and let us introduce the projection operators P_ν^+ as in (1.5) where in this case Γ_ν^+ is a circle enclosing Σ_ν^+ but no other point of the spectrum of A and oriented counterclockwise (see [10, V.5, p.272]). Since A is a self-adjoint operator, then P_ν^+ is well defined. Moreover from the expression of the projections, it follows that

$$(P_\nu^+)^* = P_\nu^+.$$

Thus, P_ν^+ is orthogonal projection of norm equal to 1. We set

$$\mathbb{H}_\nu^+ = P_\nu^+ \mathbb{H}, \qquad \mathbb{H}_\nu^- = (\mathrm{Id} - P_\nu^+)\mathbb{H}, \qquad \mathbb{H} = \mathbb{H}_\nu^+ \oplus \mathbb{H}_\nu^-,$$

and

$$A_\nu^+ := A_{|\mathbb{H}_\nu^+} : \mathbb{H}_\nu^+ \to \mathbb{H}_\nu^+, \qquad A_\nu^- := A_{|\mathbb{H}_\nu^-} : \mathcal{D}(A) \cap \mathbb{H}_\nu^- \to \mathbb{H}_\nu^-.$$

We also define as before

$$\mathbb{U}_\nu^+ := B^* \mathbb{H}_\nu^+, \qquad \mathbb{U}_\nu^- := B^* \left(\mathcal{D}(A) \cap \mathbb{H}_\nu^- \right),$$

and

$$p_\nu^+ : \mathbb{U} \to \mathbb{U}_\nu^+, \quad p_\nu^- : \mathbb{U} \to \mathbb{U}_\nu^-, \quad i_\nu^+ : \mathbb{U}_\nu^+ \to \mathbb{U}, \quad i_\nu^- : \mathbb{U}_\nu^- \to \mathbb{U},$$

the orthogonal projections and the inclusion maps. Note that we have the following relations for the above maps:

$$i_\nu^+ = (p_\nu^+)^*, \qquad i_\nu^- = (p_\nu^-)^*. \tag{3.1}$$

We can thus define

$$B_\nu^+ := P_\nu^+ B i_\nu^+ \in \mathcal{L}(\mathbb{U}_\nu^+, \mathbb{H}_\nu^+), \quad B_\nu^- := (\mathrm{Id} - P_\nu^+) B i_\nu^- \in \mathcal{L}(\mathbb{U}_\nu^-, \mathbb{H}_\nu^-).$$

It is proved in [1] (see also [2] and [18]) that

$$P_\nu^+ B = B_\nu^+ p_\nu^+, \qquad (\mathrm{Id} - P_\nu^+) B = B_\nu^- p_\nu^-.$$

We introduce also the orthogonal projection Q_ν^+ from $\mathbb{Y}$ into $\mathbb{Y}_\nu^+ = C\mathbb{H}_\nu^+$ and define

$$C_\nu^+ = C \iota_\nu^+,$$

where ι_ν^+ designates the injection operator from $\mathbb{H}_\nu^+$ to $\mathbb{H}$.

Consider now $\sigma^\star > \sigma > 0$. We take $\nu = \sigma$ or $\nu = \sigma^\star$ in the maps and spaces defined previously. Since A is self-adjoint with compact resolvent, we deduce the existence of $\varepsilon > 0$ such that for all $t \geqslant 0$

$$\| e^{A_\sigma^- t} \|_{\mathcal{L}(\mathbb{H}_\sigma^-)} \leqslant e^{-(\sigma+\varepsilon)t}, \tag{3.2}$$

and

$$\| e^{A_{\sigma^\star}^- t} \|_{\mathcal{L}(\mathbb{H}_{\sigma^\star}^-)} \leqslant e^{-\sigma^\star t}. \tag{3.3}$$

The system (1.3) splits into

$$(z_{\sigma^\star}^+)'(t) = A_{\sigma^\star}^+ z_{\sigma^\star}^+(t) + B_{\sigma^\star}^+ p_{\sigma^\star}^+ v(t), \quad z_{\sigma^\star}^+(0) = P_{\sigma^\star}^+ z^0, \tag{3.4}$$

$$(z_{\sigma^\star}^-)'(t) = A_{\sigma^\star}^- z_{\sigma^\star}^-(t) + B_{\sigma^\star}^- p_{\sigma^\star}^- v(t), \quad z_{\sigma^\star}^-(0) = (\mathrm{Id} - P_{\sigma^\star}^+) z^0. \tag{3.5}$$

3.2 Finite dimensional observer based control and stability of the closed-loop system

We are now in position to prove Theorem 1.2. The matrices K_σ^+ and L_σ^+ being respectively given by (2.11) and (2.15), we define the finite dimensional observer-based feedback control by

$$v(t) = K_\sigma^+ \widehat{z}_\star(t) \in B^* P_\sigma^+ \mathbb{H} \subset B^* P_{\sigma^\star}^+ \mathbb{H}, \tag{3.6}$$

where the finite dimensional observer $\widehat{z}_\star \in P_{\sigma^\star}^+ \mathbb{H}$ solves

$$\begin{cases} \widehat{z}_\star'(t) = A_{\sigma^\star}^+ \widehat{z}_\star(t) + B_{\sigma^\star}^+ v(t) + L_\sigma^+ Q_\sigma^+ (C_{\sigma^\star}^+ \widehat{z}_\star(t) - y(t)), \\ \widehat{z}_\star(0) = 0. \end{cases} \tag{3.7}$$

Our goal is to prove that the coupled system (1.3), (3.6) and (3.7) is exponentially stable. We define the error $e = z_{\sigma^\star}^+ - \widehat{z}_\star$ which satisfies the following system

$$\begin{cases} e'(t) = A_{\sigma^\star}^+ e(t) + L_\sigma^+ Q_\sigma^+ (C_{\sigma^\star}^+ e(t) + C z_{\sigma^\star}^-(t)), \\ e(0) = z_{\sigma^\star}^+(0). \end{cases} \tag{3.8}$$

We prove that $\widehat{z}$ and e are exponentially stable with decay rate $-\sigma$. Let us set

$$\mathbb{X} = \begin{pmatrix} \widehat{z}_\star \\ e \end{pmatrix}, \qquad \mathbb{X}^0 = \begin{pmatrix} 0 \\ z_{\sigma^\star}^+(0) \end{pmatrix}.$$

Using (3.7) and (3.8), $\mathbb{X}$ satisfies the system

$$\begin{cases} \mathbb{X}'(t) = \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z_{\sigma^\star}^-(t)), \\ \mathbb{X}(0) = \mathbb{X}^0, \end{cases} \tag{3.9}$$

where

$$\mathbb{A} = \begin{pmatrix} A_{\sigma^\star}^+ + B_{\sigma^\star}^+ K_\sigma^+ & -L_\sigma^+ Q_\sigma^+ C_{\sigma^\star}^+ \\ 0 & A_{\sigma^\star}^+ + L_\sigma^+ Q_\sigma^+ C_{\sigma^\star}^+ \end{pmatrix}, \quad \mathbb{L}(z_{\sigma^\star}^-) = \begin{pmatrix} -L_\sigma^+ Q_\sigma^+ C z_{\sigma^\star}^- \\ L_\sigma^+ Q_\sigma^+ C z_{\sigma^\star}^- \end{pmatrix}. \tag{3.10}$$

Let us first prove that $\mathbb{A}$ is stable matrix. In the sequel, the constant M is a generic constant that can change from a line to another but need to be independent of $\sigma^\star$.

Lemma 3.1. *The matrices $A_{\sigma^\star}^+ + B_{\sigma^\star}^+ K_\sigma^+$ and $A_{\sigma^\star}^+ + L_\sigma^+ Q_\sigma^+ C_{\sigma^\star}^+$ are exponentially stable with a decay rate less than $-\sigma$.*

Proof. Let $\xi^0 \in \mathbb{H}_{\sigma^\star}^+$ be given. To prove that $A_{\sigma^\star}^+ + B_{\sigma^\star}^+ K_\sigma^+$ is exponentially stable, we only need to show that the solution $\xi(t)$ of the finite dimensional system

$$\begin{cases} \xi'(t) = (A_{\sigma^\star}^+ + B_{\sigma^\star}^+ K_\sigma^+)\xi(t), \\ \xi(0) = \xi^0, \end{cases} \tag{3.11}$$

is exponentially decaying. Consider then the infinite dimensional system

$$\begin{cases} x'(t) = (A + B K_\sigma^+)x(t), \\ x(0) = \xi^0. \end{cases} \tag{3.12}$$

From Section 2.2, we see that the system (3.12) is exponentially stable of decay rate $-\sigma - \varepsilon'$. It implies that

$$\|x(t)\|_{\mathbb{H}} \leqslant M e^{-t(\sigma+\varepsilon')} \|\xi^0\|_{\mathbb{H}}. \tag{3.13}$$

On the other hand, applying $P^+_{\sigma^\star}$ to (3.12) and recalling that $K^+_\sigma x(t)$ acts only on the projected part of $x(t)$ on $\mathbb{H}^+_\sigma$, we obtain that

$$\left\{\begin{array}{l}(x^+_{\sigma^\star})'(t) = A^+_{\sigma^\star}x^+_{\sigma^\star}(t) + B^+_{\sigma^\star}K^+_\sigma x(t) = (A^+_{\sigma^\star} + B^+_{\sigma^\star}K^+_\sigma)x^+_{\sigma^\star}(t),\\ x^+_{\sigma^\star}(0) = \xi^0.\end{array}\right. \tag{3.14}$$

This shows that $x^+_{\sigma^\star}(t)$ is the unique solution $\xi(t)$ of (3.11), and we get from (3.13), that

$$\|\xi(t)\|_{\mathbb{H}} = \|x^+_{\sigma^\star}(t)\|_{\mathbb{H}} = \|P^+_{\sigma^\star}x(t)\|_{\mathbb{H}} \leqslant Me^{-t(\sigma+\varepsilon')}\|\xi^0\|_{\mathbb{H}},$$

for all $\xi^0 \in \mathbb{H}^+_{\sigma^\star}$. Hence

$$\left\|e^{t(A^+_{\sigma^\star}+B^+_{\sigma^\star}K^+_\sigma)}\xi^0\right\|_{\mathbb{H}} \leqslant Me^{-t(\sigma+\varepsilon')}\|\xi^0\|_{\mathbb{H}},$$

and the matrix $A^+_{\sigma^\star} + B^+_{\sigma^\star}K^+_\sigma$ is exponentially stable with a decay rate less than $-\sigma$. We use the same argument for $A^+_{\sigma^\star} + L^+_\sigma Q^+_\sigma C^+_{\sigma^\star}$ by considering its adjoint $(A^+_{\sigma^\star})^* + (C^+_{\sigma^\star})^*(L^+_\sigma)^*$ that has exactly the same form as the one previously studied. □

Since $\mathbb{A}$ is a triangular matrix, using Lemma 3.1 and Duhamel's formula, we obtain that $\mathbb{A}$ is stable with exponential rate strictly less than $-\sigma$.

We can now prove the exponential stability of the full closed-loop system (3.7) and (1.3):

$$\left\{\begin{array}{rcl}\mathbb{X}'(t) &=& \mathbb{A}\mathbb{X}(t) + \mathbb{L}(z^-_{\sigma^\star}(t)),\\ (z^-_{\sigma^\star})'(t) &=& A^-_{\sigma^\star}z^-_{\sigma^\star}(t) + B^-_{\sigma^\star}p^-_{\sigma^\star}\mathbb{K}^+_\sigma\mathbb{X}(t),\end{array}\right. \tag{3.15}$$

where $\mathbb{K}^+_\sigma = (K^+_\sigma, 0)$ and with the initial conditions

$$\left\{\begin{array}{rcl}\mathbb{X}(0) &=& \mathbb{X}^0,\\ z^-_{\sigma^\star}(0) &=& (\mathrm{Id} - P^+_{\sigma^\star})z^0.\end{array}\right.$$

From Duhamel's formula, the two first equations in (3.15) also read

$$\left\{\begin{array}{rcl}\mathbb{X}(s) &=& e^{s\mathbb{A}}\mathbb{X}(0) + \displaystyle\int_0^s e^{(s-\tau)\mathbb{A}}\mathbb{L}(z^-_{\sigma^\star}(\tau))d\tau,\\ z^-_{\sigma^\star}(t) &=& e^{tA^-_{\sigma^\star}}z^-_{\sigma^\star}(0) + \displaystyle\int_0^t e^{(t-s)A^-_{\sigma^\star}}\left(B^-_{\sigma^\star}p^-_{\sigma^\star}\mathbb{K}^+_\sigma\mathbb{X}(s)\right)\,ds.\end{array}\right.$$

Substituting the first equation above into the second one yields

$$z^-_{\sigma^\star}(t) = e^{tA^-_{\sigma^\star}}z^-_{\sigma^\star}(0) + \int_0^t e^{(t-s)A^-_{\sigma^\star}}B^-_{\sigma^\star}p^-_{\sigma^\star}\mathbb{K}^+_\sigma\left(e^{s\mathbb{A}}\mathbb{X}(0) + \int_0^s e^{(s-\tau)\mathbb{A}}\mathbb{L}(z^-_{\sigma^\star}(\tau))d\tau\right)\,ds. \tag{3.16}$$

Setting

$$Z^0 := e^{tA^-_{\sigma^\star}}z^-_{\sigma^\star}(0) + \int_0^t e^{(t-s)A^-_{\sigma^\star}}\left(B^-_{\sigma^\star}p^-_{\sigma^\star}\mathbb{K}^+_\sigma\left(e^{s\mathbb{A}}\mathbb{X}(0)\right)\right)\,ds,$$

relation (3.16) can be written

$$z^-_{\sigma^\star}(t) = \int_0^t\int_0^s e^{(t-s)A^-_{\sigma^\star}}B^-_{\sigma^\star}p^-_{\sigma^\star}\mathbb{K}^+_\sigma\left(e^{(s-\tau)\mathbb{A}}\mathbb{L}(z^-_{\sigma^\star}(\tau))d\tau ds\right) + Z^0. \tag{3.17}$$

To prove the stability of $z^-_{\sigma^\star}$, we prove the existence of a unique solution to (3.17) in a weighted space by using a fixed point argument. More precisely, let us define the following map

$$\begin{array}{rcl}\Phi : L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star}) &\longrightarrow& L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})\\ g &\longmapsto& \Phi(g) := \displaystyle\int_0^t\int_0^s e^{(t-s)A^-_{\sigma^\star}}B^-_{\sigma^\star}p^-_{\sigma^\star}\mathbb{K}^+_\sigma e^{(s-\tau)\mathbb{A}}\mathbb{L}(g(\tau))\,d\tau ds + Z^0.\end{array}$$

Then, equation (3.16) simply reads

$$\Phi(z^-_{\sigma^\star}) = z^-_{\sigma^\star}.$$

First, we prove that the function Φ is well defined. Given $g \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$. Since C is bounded, $\mathbb{L}$ is also bounded. Hence using the last relation, (3.3) and (H2.B), we obtain that there exists $\varepsilon'' > 0$ such that

$$\begin{aligned}\|\Phi(g)(t)\|_{\mathbb{H}} \leqslant M \int_0^t \int_0^s e^{-(t-s)\sigma^\star} e^{-(s-\tau)(\sigma+\varepsilon'')} e^{-\tau\sigma} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}}\, d\tau\, ds \\ + M \left\|z^0\right\|_{\mathbb{H}} \int_0^t e^{-(t-s)\sigma^\star} e^{-s\sigma}\, ds + e^{-t\sigma^\star} \|z^0\|_{\mathbb{H}}.\end{aligned}$$

Consequently

$$\|\Phi(g)(t)\|_{\mathbb{H}} \leqslant I_1 + I_2 + e^{-t\sigma^\star} \|z^0\|_{\mathbb{H}}, \tag{3.18}$$

where we have set

$$I_1 := M \int_0^t e^{-(t-s)\sigma^\star} e^{-s\sigma} \left(\int_0^s e^{-(s-\tau)\varepsilon''} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}}\, d\tau \right) ds,$$

$$I_2 := M \left\|z^0\right\|_{\mathbb{H}} \int_0^t e^{-(t-s)\sigma^\star} e^{-s\sigma}\, ds.$$

Noticing that

$$\int_0^s e^{-(s-\tau)\varepsilon''}\, d\tau = \frac{1}{\varepsilon''} \left[1 - e^{-s\varepsilon''}\right] \leqslant \frac{1}{\varepsilon''},$$

we get that

$$\int_0^s e^{-(s-\tau)\varepsilon''} \|e^{\tau\sigma} g(\tau)\|_{\mathbb{H}}\, d\tau \leqslant \frac{1}{\varepsilon''} \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^*})},$$

and hence

$$I_1 \leqslant M \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})} \frac{e^{-t\sigma}}{\varepsilon''} \int_0^t e^{(t-s)(\sigma-\sigma^\star)}\, ds.$$

Since

$$\int_0^t e^{(t-s)(\sigma-\sigma^\star)}\, ds = \frac{1 - e^{-t(\sigma^*-\sigma)}}{(\sigma^\star - \sigma)} \leqslant \frac{1}{(\sigma^\star - \sigma)},$$

we have

$$I_1 \leqslant \frac{M e^{-\sigma t}}{\varepsilon''(\sigma^\star - \sigma)} \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})}$$

and similarly

$$I_2 \leqslant M e^{-\sigma t} \left(\int_0^t e^{(t-s)(\sigma-\sigma^\star)}\, ds \right) \left\|z^0\right\|_{\mathbb{H}} \leqslant \frac{M e^{-\sigma t}}{(\sigma^\star - \sigma)} \left\|z^0\right\|_{\mathbb{H}}.$$

Using the above estimates in (3.18), we get that

$$\|\Phi(g)(t)\|_{\mathbb{H}} \leqslant M e^{-\sigma t} \left(\frac{1}{\varepsilon''(\sigma^\star - \sigma)} \|g\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})} + \left\|z^0\right\|_{\mathbb{H}} + \frac{1}{\sigma^* - \sigma} \left\|z^0\right\|_{\mathbb{H}} \right),$$

and hence $\Phi(g) \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$.

It remains to show that Φ is a contraction mapping. Given $g_1, g_2 \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$, the same calculations as above show that

$$\|\Phi(g_1) - \Phi(g_2)\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})} \leqslant \frac{M}{\varepsilon''(\sigma^\star - \sigma)} \|g_1 - g_2\|_{L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})}.$$

The application Φ is thus a contraction provided that $\sigma^\star$ is chosen large enough to ensure that

$$\frac{M}{\varepsilon''(\sigma^\star - \sigma)} < 1.$$

Then, applying the fixed point theorem we get that there exists a unique $z^-_{\sigma^\star} \in L^\infty_\sigma(0,\infty;\mathbb{H}^-_{\sigma^\star})$ such that $\Phi(z^-_{\sigma^\star}) = z^-_{\sigma^\star}$ and

$$\|z^-_{\sigma^\star}(t)\|_{\mathbb{H}} \leqslant Me^{-\sigma t}\left\|z^0\right\|_{\mathbb{H}}.$$

Moreover, going back to the first equation in (3.15), and using Duhamel's formula again, we easily obtain that

$$\|\mathbb{X}(t)\|_{\mathbb{H}\times\mathbb{H}} \leqslant Me^{-\sigma t}\left\|z^0\right\|_{\mathbb{H}}.$$

This completes the proof of Theorem 1.2.

4 Stabilization of the reaction-diffusion equation

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) be a bounded domain of class $C^{1,1}$. In this section, we apply Theorem 1.1 and Theorem 1.2 for the stabilization of the heat equation. Let us consider Γ a non-empty open subset of $\partial\Omega$ and the control problem:

$$\begin{cases} \partial_t z(t,x) - \nabla\cdot(b\nabla z(t,x)) - cz(t,x) = 0 & \text{in } (0,\infty)\times\Omega, \\ z(t,x) = v(t,x) & \text{on } (0,\infty)\times\Gamma, \\ z(t,x) = 0 & \text{on } (0,\infty)\times(\partial\Omega\setminus\Gamma), \\ z(0,\cdot) = z^0 & \text{in } \Omega, \\ y(t,x) = \mathbb{1}_{\mathcal{O}} z(t,x) & \text{in } (0,\infty)\times\Omega, \end{cases} \tag{4.1}$$

where $b, c \in L^\infty(\Omega)$ and $\mathcal{O}$ an open subset of $\mathbb{R}^N$ with $\overline{\mathcal{O}} \subset \Omega$. In order to write (4.1) under the form (1.3), we introduce the following functional setting:

$$\mathbb{H} = L^2(\Omega), \quad \mathbb{U} = L^2(\Gamma),$$

$$Az = \nabla\cdot(b\nabla z) + cz, \quad \mathcal{D}(A) = H^2(\Omega)\cap H^1_0(\Omega),$$

The operator $(A,\mathcal{D}(A))$ generates an analytical semigroup, this is a direct consequence of [3, Theorem 2.12, p.115]. Thus (H1.A) is satisfied. Moreover, the operator A is self-adjoint, then in particular, we see that (H2.A) holds true. To define the control operator B, we use a standard method (see, for instance [21, pp.341-343] or [17]): we first consider the lifting operator $D_0 \in \mathcal{L}(L^2(\partial\Omega);L^2(\Omega))$ such that for any $v \in L^2(\partial\Omega)$, $w = D_0 v$ is the unique solution of the following system

$$\begin{cases} \mu_0 w - \nabla\cdot(b\nabla w) - cw = 0 & \text{in } \Omega, \\ w = v & \text{on } \partial\Omega, \end{cases}$$

where $\mu_0 \in \rho(A)$. Then, we set

$$B = (\mu_0\,\mathrm{Id} - A)D_0 : \mathbb{U} \longrightarrow (\mathcal{D}(A^*))',$$

where we have extended the operator A as an operator from $L^2(\Omega)$ into $(\mathcal{D}(A^*))'$ and where we see $\mathbb{U}$ as a closed subspace of $L^2(\partial\Omega)$ (by extending by zero in $\partial\Omega\setminus\Gamma$ any $v \in \mathbb{U}$). Using standard results on elliptic equations, we have that B satisfies (H1.B) for any $\gamma > 3/4$ (see for instance [16, Theorem 2.6]). We set also

$$\mathbb{Y} = L^2(\mathcal{O}), \quad C = \mathbb{1}_{\mathcal{O}},$$

we see that $C \in \mathcal{L}(\mathbb{H},\mathbb{Y})$. Let us recall how we can reduce (4.1) to an evolution problem (1.3). We set $\widetilde{z} = z - w$, with $w = D_0 v$. Then $\widetilde{z}$ satisfies the system

$$\begin{cases} \partial_t \widetilde{z} - \nabla\cdot(b\nabla\widetilde{z}) - c\widetilde{z} = -\partial_t w + \mu_0 w & \text{in } (0,\infty)\times\Omega, \\ \widetilde{z} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ \widetilde{z}(0,\cdot) = \widetilde{z}^0 := z^0 - w(0,\cdot) & \text{in } \Omega. \end{cases}$$

Using the Duhamel formula, we have

$$\widetilde{z}(t) = e^{tA}\widetilde{z}^0 + \int_0^t e^{(t-s)A}(-\partial_t w(s) + \mu_0 w(s))\ ds.$$

By integrating by parts, we obtain

$$z(t) = e^{tA}z^0 + \int_0^t e^{(t-s)A}(\mu_0\,\mathrm{Id} - A)w(s)\ ds,$$

that is

$$\begin{cases} z'(t) = Az(t) + Bv(t), \quad z(0) = z^0, \\ y(t) = Cz(t). \end{cases}$$

To apply Theorem 1.1, we only need to check (H2.D). We recall that

$$\mathcal{D}(A^*) = H^2(\Omega) \cap H_0^1(\Omega), \quad A^* = A.$$

Moreover, by classical results (see [21, Proposition 10.6.7]), we see that

$$D_0^* := -\frac{\partial}{\partial \nu}(\mu_0\,\mathrm{Id} - A^*)^{-1} = -\frac{\partial}{\partial \nu}(\mu_0\,\mathrm{Id} - A)^{-1},$$

and thus

$$B^*\varepsilon := -\frac{\partial \varepsilon}{\partial \nu}_{|\Gamma}.$$

Thus if ε satisfies $A^*\varepsilon = \lambda\varepsilon$ and $B^*\varepsilon = 0$, then

$$\begin{cases} \lambda\varepsilon - \nabla\cdot(b\nabla\varepsilon) - c\varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial \varepsilon}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

From standard results on the unique continuation of the Laplace operator (see for instance [9, Theorem 5.3.1, p.125]), we deduce that $\varepsilon = 0$. In the other hand, if ε satisfies $A\varepsilon = \lambda\varepsilon$ and $C\varepsilon = 0$, then

$$\begin{cases} \lambda\varepsilon - \nabla\cdot(b\nabla\varepsilon) - c\varepsilon = 0 & \text{in } \Omega, \\ \varepsilon = 0 & \text{on } \partial\Omega, \\ \varepsilon = 0 & \text{in } \mathcal{O}. \end{cases}$$

We obtain again from standard results on the unique continuation of the Laplace operator (see for instance [9, Theorem 5.3.1, p.125]), we get also that $\varepsilon = 0$. Thus (H2.D) holds for any σ.

Now, we define the observer $\widehat{z}$ using Remark 2.3. Let us define N_σ by (2.10) and $\widehat{z}$ the solution of the closed loop system

$$\begin{cases} \partial_t\widehat{z} - \nabla\cdot(b\nabla\widehat{z}) - c\widehat{z} &=& \displaystyle\sum_{i=1}^{N_\sigma} \langle \mathbb{1}_{\mathcal{O}}(\widehat{z} - z), w_i^\star\rangle_{\mathbb{Y}}\ \chi_i & \text{in } (0,\infty)\times\Omega, \\ \widehat{z} &=& \displaystyle\sum_{i=1}^{N_\sigma} \langle \widehat{z}, \zeta_i\rangle_{\mathbb{H}}\ w_i & \text{on } (0,\infty)\times\Gamma, \\ \widehat{z} &=& 0 & \text{on } (0,\infty)\times(\partial\Omega\setminus\Gamma), \\ \widehat{z}(0,\cdot) &=& 0 & \text{in } \Omega, \end{cases} \tag{4.2}$$

where $(w_i^\star)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{Y}$, $(\chi_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{H}$, $(\zeta_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathcal{D}(A^*)$ and $(w_i)_{1\leqslant i\leqslant N_\sigma} \subset \mathbb{U}$ (see equations (2.15)-(2.16) and Remark 2.3). We deduce the following result by applying Theorem 1.1:

Theorem 4.1. *Assume $\sigma > 0$. There exists a control*

$$v(t) = \sum_{i=1}^{N_\sigma} \left(\int_\Omega \widehat{z}(t)\zeta_i \, dx \right) v_i, \tag{4.3}$$

with $\zeta_k \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_k \in H^{1/2}(\Gamma)$, $k = 1, \dots, N_\sigma$ such that the coupled system (4.1) *and* (4.2) *is exponentially stable that satisfies for $z^0 \in L^2(\Omega)$ the estimate*

$$\|z(t)\|_{L^2(\Omega)} \leqslant Ce^{-\sigma t}\|z^0\|_{L^2(\Omega)}. \tag{4.4}$$

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