



**HAL**  
open science

# Convergence rate of entropy-regularized multi-marginal optimal transport costs

Luca Nenna, Paul Pegon

► **To cite this version:**

Luca Nenna, Paul Pegon. Convergence rate of entropy-regularized multi-marginal optimal transport costs. 2023. hal-04154453v1

**HAL Id: hal-04154453**

**<https://hal.science/hal-04154453v1>**

Preprint submitted on 6 Jul 2023 (v1), last revised 9 Apr 2024 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Convergence rate of entropy-regularized multi-marginal optimal transport costs

Luca Nenna\* and Paul Pegon†

July 6, 2023

We investigate the convergence rate of multi-marginal optimal transport costs that are regularized with the Boltzmann-Shannon entropy, as the noise parameter  $\varepsilon$  tends to 0. We establish lower and upper bounds on the difference with the unregularized cost of the form  $C\varepsilon \log(1/\varepsilon) + O(\varepsilon)$  for some explicit dimensional constants  $C$  depending on the marginals and on the ground cost, but not on the optimal transport plans themselves. Upper bounds are obtained for Lipschitz costs or locally semi-concave costs for a finer estimate, and lower bounds for  $\mathcal{C}^2$  costs satisfying some signature condition on the mixed second derivatives that may include degenerate costs, thus generalizing results previously in the two marginals case and for non-degenerate costs. We obtain in particular matching bounds in some typical situations where the optimal plan is deterministic.

*Keywords.* optimal transport, multi-marginal optimal transport, entropic regularization, Schrödinger problem, convex analysis, Rényi dimension.

*2020 Mathematics Subject Classification.* Primary: 49Q22 ; Secondary: 49N15, 94A17, 49K40.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>4</b>

---

\*Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France. email: luca.nenna@universite-paris-saclay.fr

†CEREMADE, Université Paris-Dauphine, Université PSL, CNRS, Mokaplan, Inria Paris, 75016 Paris, France. email: pegon@ceremade.dauphine.fr

<b>3</b>	<b>Upper bounds</b>	<b>7</b>
3.1	Upper bound for locally Lipschitz costs . . . . .	7
3.2	Upper bound for locally semi-concave costs . . . . .	10
<b>4</b>	<b>Lower bound for <math>\mathcal{C}^2</math> costs with a signature condition</b>	<b>14</b>
<b>5</b>	<b>Examples and matching bound</b>	<b>17</b>

## Notations

$X_i$	a subset of $\mathbb{R}^N$ for any index $i \in \{1, \dots, m\}$
$\mathbf{X}$	product space $X_1 \times \dots \times X_m$ ;
$x_i, x, \mathbf{x}$	a point in $X_i$ , in some $X_j, j \in \{1, \dots, m\}$ , and in $\mathbf{X}$ respectively;
$x_q$	$(x_i)_{i \in q}$ if $q \subseteq \{1, \dots, m\}$ and $\mathbf{x} \in \mathbf{X}$ ;
$ \cdot $	Euclidean norm on $\mathbb{R}^N$ ;
$\ \cdot\ $	norm on $\mathbb{R}^N \times \dots \times \mathbb{R}^N$ defined by $\ \mathbf{x}\  = \max_{1 \leq i \leq m}  x_i $ if $\mathbf{x} = (x_1, \dots, x_m)$ ;
$B_r(x), B_r(\mathbf{x})$	open ball of radius $r$ centered at $x \in \mathbb{R}^N$ or $\mathbf{x} \in (\mathbb{R}^N)^m$ for the above norms;
$\mathcal{E}_{\text{loc}}^{0,1}(X)$	space of real-valued locally Lipschitz functions on $X$ which is a sub-manifold of $\mathbb{R}^N$ or $(\mathbb{R}^N)^m$ ;
$[f]_{\mathcal{E}^{0,1}(X)}$	Lipschitz constant of $f : X \rightarrow \mathbb{R}$ where $X$ is a subset of $\mathbb{R}^N$ or $(\mathbb{R}^N)^m$ for the above norms;
$\mathcal{E}_{\text{loc}}^{1,1}(X)$	space of differentiable real-valued functions on $X$ , a sub-manifold of $\mathbb{R}^N$ or $(\mathbb{R}^N)^m$ , with locally Lipschitz differential;
$\mathcal{P}(X)$	space of probability measures on a metric space $X$ ;
$\text{spt } \mu$	support of the measure $\mu$ ;
$M_N(\mathbb{R})$	space of real matrices of size $N \times N$ , endowed with the Frobenius norm induced by the scalar product $A \cdot B := \text{Tr}(A^T B)$ , for $A, B \in M_N(\mathbb{R})$ ;
$S_N(\mathbb{R})$	subspace of real symmetric matrices of size $N \times N$ ;
$\mathcal{H}_X^s$	$s$ -dimensional Hausdorff measure on the metric space $X$ endowed with the Borel $\sigma$ -algebra (the subscript $X$ will often be dropped);
$\Delta_P$	simplex of P-uples $t = (t_p)_{p \in P}$ such that $t_p \geq 0$ for all $p \in P$ and $\sum_{p \in P} t_p = 1$ .

## 1 Introduction

We consider a  $m$ -uple of probability measures  $\mu_i$  compactly supported on sub-manifolds  $X_i \subseteq \mathbb{R}^N$  of dimension  $d_i$  and a cost function  $c : X_1 \times \dots \times X_m \rightarrow \mathbb{R}_+$ . The Entropic Multi-Marginal Optimal Transport problem is defined as :

$$\text{MOT}_\varepsilon := \inf \left\{ \int_{X_1 \times \dots \times X_m} c \, d\gamma + \varepsilon \text{Ent}(\gamma | \otimes_{i=1}^m \mu_i) \mid \gamma \in \Pi(\mu_1, \dots, \mu_m) \right\}, \quad (\text{MOT}_\varepsilon)$$

where  $\Pi(\mu_1, \dots, \mu_m)$  denotes the set of all probability measures  $\gamma$  having  $\mu_i$  for  $i \in \{1, \dots, m\}$  as marginals. The classical multi-marginal optimal transport problem cor-

responds to the case where  $\varepsilon = 0$ . In the last decade, these two classes of problems (entropic optimal transport and multi-marginal optimal transport) have witnessed a growing interest and they are now an active research topic.

Entropic optimal transport (EOT) has found applications and proved to be an efficient way to approximate Optimal Transport (OT) problems, especially from a computational viewpoint. Indeed, when it comes to solving EOT by alternating Kullback-Leibler projections on the two marginal constraints, by the algebraic properties of the entropy such iterative projections correspond to the celebrated Sinkhorn's algorithm [Sin64], applied in this framework in the pioneering works [Cut13; Ben+15]. The simplicity and the good convergence guarantees (see [FL89; MG20]) of this method compared to the algorithms used for the OT problems, then determined the success of EOT for applications in machine learning, statistics, image processing, language processing and other areas (see the monograph [PC19] and references therein).

As concerns multi-marginal optimal transport (MOT), it arises naturally in many different areas of applications, including economics [CE10], financial mathematics [BHP13; DS14a; DS14b; Enn+22], statistics [BK18; CCG16], image processing [Rab+11], tomography [Abr+17], machine learning [Haa+21; TJK22], fluid dynamics [Bre89] and quantum physics and chemistry, in the framework of density functional theory [BDG12; CFK13]. The structure of solutions to the multi-marginal optimal transport problem is a notoriously delicate issue, and is still not well understood, despite substantial efforts on the part of many researchers [GŚ98; Car03; CN08; Hei02; Pas11; Pas12; KP14; KP15; CDD15; CS16; PV21b; MP17; PV21a]; see also the surveys [Pas15] and [DGN15]. Since  $\text{MOT}_\varepsilon$  can be seen a perturbation of  $\text{MOT}_0$ , it is natural to study the behaviour as  $\varepsilon$  vanishes. In this paper we are mainly interested in investigating the rate of convergence of the entropic cost  $\text{MOT}_\varepsilon$  to  $\text{MOT}_0$  under some mild assumptions on the cost functions and marginals.

In particular we are going to extend the techniques introduced in [CPT23] for two marginals to the multi-marginal case which will also let us generalize the bounds in [CPT23] to the case of degenerate cost functions. For the two marginals and non-degenerate case we also refer the reader to a very recent (and elegant) paper [MS23] where the authors push a little further the analysis of the convergence rate by disentangling the roles of  $\int c d\gamma$  and the relative entropy in the total cost and deriving convergence rate for both these terms. Notice that concerning the convergence rate of the entropic multi-marginal optimal transport an upper bound has been already established in [EN22], which depends on the number of marginals and the quantization dimension of the optimal solutions to  $(\text{MOT}_\varepsilon)$  with  $\varepsilon = 0$ . Here we provide a improved, smaller, upper bound, which will depend only on the marginals, but not on the optimal transport plans for the un-regularized problem, and we also provide a lower bound depending on a signature condition on the mixed second derivatives of the cost function, that was introduced in [Pas12].

Our main findings can be summarized as follows: we establish two **upper bounds**, one valid for locally Lipschitz costs and a finer one valid for locally semi-concave costs. The proofs rely, as in [CPT23], on a multi-marginal variant of the block approximation introduced in [Car+17]. Notice that in this case the bound will depend only on the

dimension of the support of the marginals. Moreover, for locally semi-concave cost functions, by exploiting Alexandrov-type results as in [CPT23], we improve the upper bound by a 1/2 factor, obtaining the following inequality for some  $C^* \in \mathbb{R}_+$

$$\boxed{\text{MOT}_\varepsilon \leq \text{MOT}_0 + \frac{1}{2} \left( \sum_{i=1}^m d_i - \max_{1 \leq i \leq m} d_i \right) \varepsilon \log(1/\varepsilon) + C^* \varepsilon.} \quad (1.1)$$

We stress that this upper bound is smaller or equal than the one provided in [EN22, Theorem 3.8], which is of the form  $\frac{1}{2}(m-1)D\varepsilon \log(1/\varepsilon) + O(\varepsilon)$  where  $D$  is a quantization dimension of the support of an optimal transport plan. Thus  $D$  must be greater or equal than the maximum dimension of the support of the marginals, and of course  $\sum_{i=1}^m d_i - \max_{1 \leq i \leq m} d_i \leq (m-1) \max_{1 \leq i \leq m} d_i$ . The inequality may be strict for example in the two marginals case with unequal dimension, as shown in Section 5.

For the **lower bound**, from the dual formulation of (MOT $_\varepsilon$ ) we have

$$\text{MOT}_\varepsilon \geq \text{MOT}_0 - \varepsilon \log \int_{\prod_{i=1}^m X_i} e^{-\frac{E(x_1, \dots, x_m)}{\varepsilon}} d \otimes_{i=1}^m \mu_i(x_i),$$

where  $E(x_1, \dots, x_m) = c(x_1, \dots, x_m) - \oplus_{i=1}^m \phi_i(x_i)$  is the duality gap and  $(\phi_1, \dots, \phi_m)$  are Kantorovich potentials for the un-regularized problem (MOT $_\varepsilon$ ) with  $\varepsilon = 0$ . By using the singular values decomposition of the bilinear form obtained as an average of mixed second derivatives of the cost and a signature condition introduced in [Pas11], we are able to prove that  $E$  detaches quadratically from the set  $\{E = 0\}$  and this allows us to estimate the previous integral in the desired way as in [CPT23] and improve the results in [EN22] where only an upper bound depending on the quantization dimension of the solution to the un-regularized problem is provided. Moreover, this slightly more flexible use of Minty's trick compared to [CPT23] allows us to obtain a lower bound also for degenerate cost functions in the two marginals setting. Given a  $\kappa$  depending on a signature condition (see (PS( $\kappa$ ))) on the second mixed derivatives of the cost, the lower bound can be summarized as follows

$$\boxed{\text{MOT}_\varepsilon \geq \text{MOT}_0 + \frac{\kappa}{2} \varepsilon \log(1/\varepsilon) - C_* \varepsilon.} \quad (1.2)$$

for some  $C_* \in \mathbb{R}_+$ .

The paper is organized as follows: in Section 2 we recall the multi-marginal optimal transport problem, some results concerning the structure of the optimal solution, in particular the ones in [Pas11], and define its entropy regularization. Section 3 is devoted to the upper bounds stated in Proposition 3.2 and Proposition 3.7. In Section 4 we establish the lower bound stated in Proposition 4.2. Finally, in Section 5 we provide some examples for which we can get the matching bounds.

## 2 Preliminaries

Given  $m$  probability compactly supported measures  $\mu_i$  on sub-manifolds  $X_i$  of dimension  $d_i$  in  $\mathbb{R}^N$  for  $i = 1, 2, \dots, m$  and a continuous cost function  $c : X_1 \times X_2 \times \dots \times X_m \rightarrow \mathbb{R}_+$ ,

the multi-marginal optimal transport problem consists in solving the following optimization problem

$$\text{MOT}_0 := \inf_{\gamma \in \Pi(\mu_1, \dots, \mu_m)} \int_{\mathbf{X}} c(x_1, \dots, x_m) d\gamma \quad (\text{MOT})$$

where  $\mathbf{X} := X_1 \times X_2 \times \dots \times X_m$  and  $\Pi(\mu_1, \dots, \mu_m)$  denotes the set of probability measures on  $\mathbf{X}$  whose marginals are the  $\mu_i$ . The formulation above is also known as the *Kantorovich problem* and it amounts to a linear minimization problem over a convex, weakly compact set; it is then not difficult to prove the existence of a solution by the direct method of calculus of variations. Much of the attention in the optimal transport community is rather focused on uniqueness and the structure of the minimizers. In particular one is mainly interested in determining if the solution is concentrated on the graph of a function  $(T_2, \dots, T_m)$  over the first marginal, where  $(T_i)_\# \mu_1 = \mu_i$  for  $i = 1, \dots, m$ , in which case this function induces a solution *à la Monge*, that is  $\gamma = (\text{Id}, T_2, \dots, T_m)_\# \mu_1$ .

In the two marginals setting, the theory is fairly well understood and it is well-known that under mild conditions on the cost function (e.g. twist condition) and marginals (e.g. being absolutely continuous with respect to Lebesgue), the solution to (MOT) is unique and is concentrated on the graph of a function; we refer the reader to [San15] to have a glimpse of it. The extension to the multi-marginal case is still not well understood, but it has attracted recently a lot of attention due to a diverse variety of applications. In particular in his seminal works [Pas11; Pas12] Pass established some conditions, more restrictive than in the two marginals case, to ensure the existence of a solution concentrated on a graph. In this work we rely on the following (local) result in [Pas12] giving an upper bound on the dimension of the support of the solution to (MOT). Let  $P$  be the set of partitions of  $\{1, \dots, m\}$  into two non-empty disjoint subsets:  $p = \{p_-, p_+\} \in P$  if  $p_- \cup p_+ = \{1, \dots, m\}$ ,  $p_- \cap p_+ = \emptyset$  and  $p_-, p_+ \neq \emptyset$ . Then for each  $p \in P$  we denote by  $g_p$  the bilinear form on  $\mathbf{X}$  as

$$g_p = D_{p_- p_+}^2 c + D_{p_+ p_-}^2 c \quad \text{where} \quad D_{pq}^2 c := \sum_{i \in p, j \in q} D_{x_i x_j}^2 c$$

for every  $p, q \subseteq \{1, \dots, m\}$ , and  $D_{x_i x_j}^2 c := \sum_{\alpha_i, \alpha_j} \frac{\partial^2 c}{x_i^{\alpha_i} x_j^{\alpha_j}} dx_i^{\alpha_i} \otimes dx_j^{\alpha_j}$ , defined for every  $i, j$  on the whole tangent bundle  $T\mathbf{X}$ .

Define  $G := \{\sum_{p \in P} t_p g_p \mid (t_p)_{p \in P} \in \Delta_P\}$  to be the convex hull generated by the  $g_p$ , then it is easy to verify that each  $g \in G$  is symmetric and therefore its signature is well defined. Then, the following result from [Pas12] gives a control on the dimension of the support of the optimizer(s) in terms of these signatures.

**Theorem 2.1** (Part of [Pas12, Theorem 2.3]). *Let  $\gamma$  a solution to (MOT) and suppose that at some point  $\mathbf{x} \in \mathbf{X}$ , the signature of some  $g \in G$  is  $(d^+, d^-, d^0)$ , that is the number of positive, negative and zero eigenvalues. Then, there exists a neighbourhood  $N_{\mathbf{x}}$  of  $\mathbf{x}$  such that  $N_{\mathbf{x}} \cap \text{spt } \gamma$  is contained in a Lipschitz sub-manifold of  $\mathbf{X}$  with dimension no greater than  $\sum_{i=1}^m d_i - d^+(g(\mathbf{x}))$ .*

*Remark 2.2.* For the following it is important to notice that by standard linear algebra arguments we have for each  $g \in G$  that  $d^+(g) \leq \sum_{i=1}^m d_i - \max_i d_i$ . This implies that the smallest bound on the dimension of  $\text{spt } \gamma$  which [Theorem 2.1](#) can provide is  $\max_i d_i$ .

*Remark 2.3* (Two marginals case). When  $m = 2$ , the only  $g \in G$  coincides precisely with the pseudo-metric introduced by Kim-McCann in [\[KM10\]](#). Assuming for simplicity that  $d_1 = d_2 = d$ , they noted that  $g$  has signature  $(d, d, 0)$  whenever  $c$  is non-degenerate so [Theorem 2.1](#) generalizes their result since it applies even when non-degeneracy fails providing new information in the two marginals case: the signature of  $g$  is  $(r, r, 2d - 2r)$  where  $r$  is the rank of  $D_{x_1 x_2}^2 c$ . Notice that this will help us to generalize the results established in [\[CPT23; EN22\]](#) to the case of a degenerate cost function.

It is well known that under some mild assumptions the Kantorovich problem [\(MOT\)](#) is dual to the following

$$\sup \left\{ \sum_{i=1}^m \int_{X_i} \phi_i(x_i) d\mu_i \mid \phi_i \in \mathcal{C}_b(X_i), \sum_{i=1}^m \phi_i(x^i) \leq c(x_1, \dots, x_m) \right\}. \quad (\text{MD})$$

We recall the entropic counterpart of [\(MOT\)](#): given  $m$  probability measures  $\mu_i$  on  $X_i$  as before, and a continuous cost function  $c : \mathbf{X} \rightarrow \mathbb{R}_+$  the  $\text{MOT}_\varepsilon$  problem is

$$\text{MOT}_\varepsilon = \inf_{\gamma \in \Pi(\mu_1, \dots, \mu_m)} \int_{\mathbf{X}} c d\gamma + \varepsilon \text{Ent}(\gamma \mid \otimes_{i=1}^m \mu_i), \quad (\text{MOT}_\varepsilon)$$

where  $\text{Ent}(\cdot \mid \otimes_{i=1}^m \mu_i)$  is the Boltzmann-Shannon relative entropy (or Kullback-Leibler divergence) w.r.t. the product measure  $\otimes_{i=1}^m \mu_i$ , defined for general probability measures  $p, q$  as

$$\text{Ent}(p \mid q) = \begin{cases} \int_{\mathbb{R}^d} \rho \log(\rho) dq & \text{if } p = \rho q, \\ +\infty & \text{otherwise.} \end{cases}$$

The fact that  $q$  is a probability measure ensures that  $\text{Ent}(p \mid q) \geq 0$ . The dual problem of [\(MOT\)\\_\varepsilon](#) reads as

$$\text{MOT}_\varepsilon = \varepsilon + \sup \left\{ \sum_{i=1}^m \int_{X_i} \phi_i(x_i) d\mu_i - \varepsilon \int_{\mathbf{X}} e^{\frac{\sum_{i=1}^m \phi_i(x_i) - c(\mathbf{x})}{\varepsilon}} d \otimes_{i=1}^m \mu_i \mid \phi_i \in \mathcal{C}_b(X_i) \right\}, \quad (\text{MD}_\varepsilon)$$

which is invariant by  $(\phi_1, \dots, \phi_m) \mapsto (\phi_1 + \lambda_1, \dots, \phi_m + \lambda_m)$  where  $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$  and  $\sum_{i=1}^m \lambda_i = 0$ , see [\[Léo14; NW22; MG20\]](#) for some recent presentations. It admits an equivalent “log-sum-exp” form:

$$\text{MOT}_\varepsilon = \sup \left\{ \sum_{i=1}^m \int_{X_i} \phi_i(x_i) d\mu_i - \varepsilon \log \left( \int_{\mathbf{X}} e^{\frac{\sum_{i=1}^m \phi_i(x_i) - c(\mathbf{x})}{\varepsilon}} d \otimes_{i=1}^m \mu_i \right) \mid \phi_i \in \mathcal{C}_b(X_i) \right\}, \quad (\text{MD}'_\varepsilon)$$

which is invariant by the same transformations without assuming  $\sum_{i=1}^m \lambda_i = 0$ .

From [\(MOT\)\\_\varepsilon](#) and [\(MD\)\\_\varepsilon](#) we recover, as  $\varepsilon \rightarrow 0$ , the unregularized multi-marginal optimal transport [\(MOT\)](#) and its dual [\(MD\)](#) we have introduced above. The link between

multi-marginal optimal transport and its entropic regularization is very strong and a consequence of the  $\Gamma$ -convergence of  $(\text{MOT}_\varepsilon)$  towards  $(\text{MOT})$  (one can adapt the proof in [Car+17] or see [BCN19; GKR20] for  $\Gamma$ -convergence in some specific cases) is that

$$\lim_{\varepsilon \rightarrow 0} \text{MOT}_\varepsilon = \text{MOT}_0.$$

By the direct method in the calculus of variations and strict convexity of the entropy, one can show that  $(\text{MOT}_\varepsilon)$  admits a unique solution  $\gamma_\varepsilon$ , called *optimal entropic plan*. Moreover, there exist  $m$  real-valued Borel functions  $\phi_i^\varepsilon$  such that

$$\gamma_\varepsilon = \exp\left(\frac{\bigoplus_{i=1}^m \phi_i^\varepsilon - c}{\varepsilon}\right) \otimes_{i=1}^m \mu_i, \quad (2.1)$$

where  $\bigoplus_{i=1}^m \phi_i^\varepsilon := (x_1, \dots, x_m) \mapsto \sum_{i=1}^m \phi_i^\varepsilon(x_i)$ , and in particular we have that that

$$\text{MOT}_\varepsilon = \sum_{i=1}^m \int_{X_i} \phi_i^\varepsilon \, d\mu_i \quad (2.2)$$

and these functions, which have continuous representatives, are a.s. uniquely determined up to additive constants. The reader is referred to the analysis of [MG20], to [Nen16] for the extension to the multi-marginal setting, and to [BL92; BLN94; Csi75; FG97; RT98] for earlier references on the two marginals framework.

The functions  $\phi_i^\varepsilon$  in (2.1) are called *Schrödinger potentials*, the terminology being motivated by the fact that they solve the dual problem  $(\text{MD}_\varepsilon)$ . Furthermore,  $\phi_i^\varepsilon$  are the (unique) solutions to the so-called *Schrödinger system*

$$\phi_i(x_i) = -\varepsilon \log \int_{\mathbf{X}_{-i}} e^{\frac{\bigoplus_{j=1, j \neq i}^m \phi_j^\varepsilon - c(x)}{\varepsilon}} \, d \otimes_{j=1, j \neq i}^m \mu_j \text{ for } \mu_i\text{-a.e. } x_i, \forall i = 1, \dots, m, \quad (2.3)$$

where  $\mathbf{X}_{-i} = \prod_{j=1, j \neq i}^m X_j$ . Note that (2.3) is a softmin version of the multi-marginal  $c$ -conjugacy relation for Kantorovich potentials.

### 3 Upper bounds

We start by establishing an upper bound, which will depend on the dimension of the marginals, for locally Lipschitz cost functions. We will then improve it for locally semi-concave (in particular  $\mathcal{C}^2$ ) cost functions.

#### 3.1 Upper bound for locally Lipschitz costs

The natural notion of dimension which arises in our context is the *entropy dimension*, also called *information dimension* or *Rényi dimension* [Rén59].



**Definition 3.1** (Rényi dimension [You82]). If  $\mu$  is a probability measure over a sub-manifold  $X$  of  $\mathbb{R}^d$ , we set for every  $\delta > 0$ ,

$$H_\delta(\mu) = \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) \log(1/\mu(A_n)) : \forall n, \text{diam}(A_n) \leq \delta, \text{ and } X = \bigsqcup_{n \in \mathbb{N}} A_n \right\},$$

where the infimum is taken over countable partitions  $(A_n)_{n \in \mathbb{N}}$  of  $X$  by Borel subsets of diameter less than  $\delta$ , and we define the *lower and upper entropy dimension* of  $\mu$  respectively by:

$$\underline{\dim}_R(\mu) := \liminf_{\delta \rightarrow 0^+} \frac{H_\delta(\mu)}{\log(1/\delta)}, \quad \overline{\dim}_R(\mu) := \limsup_{\delta \rightarrow 0^+} \frac{H_\delta(\mu)}{\log(1/\delta)}.$$

Notice that if  $\mu$  is compactly supported on a Lipschitz manifold of dimension  $d$ , then  $N_\delta(\text{spt } \mu) \leq d \log(1/\delta) + C$  for some constant  $C > 0$  and  $\delta \in (0, 1]$ , where  $N_\delta(\text{spt } \mu)$  is the box-counting number of  $\text{spt } \mu$ , i.e. the minimal number of sets of diameter  $\delta > 0$  which cover  $\text{spt } \mu$ . In particular, by concavity of  $t \mapsto t \log(1/t)$ , we have

$$H_\delta(\mu) \leq \log N_\delta(\text{spt } \mu). \quad (3.1)$$

We refer to the beginning of [CPT23, §3.1] for additional information and references on Rényi dimension.

The following proposition establishes an upper bound for locally Lipschitz costs.

**Proposition 3.2.** *Assume that  $\mu_i \in \mathcal{P}(X_i)$  for  $i \in \{1, \dots, m\}$  are compactly supported measures on Lipschitz sub-manifolds of dimension  $d_i$ , respectively, and  $c$  is of class  $\mathcal{C}_{\text{loc}}^{0,1}(\mathbf{X})$ , then*

$$\boxed{\text{MOT}_\varepsilon \leq \text{MOT}_0 + \left( \sum_{i=1}^m d_i - \max_{j=1, \dots, m} d_j \right) \varepsilon \log(1/\varepsilon) + O(\varepsilon).} \quad (3.2)$$

*Proof.* Given an optimal plan  $\gamma_0$  for  $\text{MOT}_0$ , we use the so-called “block approximation” introduced in [Car+17]. For every  $\delta > 0$  and  $i$ , consider a partition  $X_i = \bigsqcup_{n \in \mathbb{N}} A_i^n$  of Borel sets such that  $\text{diam}(A_i^n) \leq \delta$  for every  $n \in \mathbb{N}$ , and set

$$\mu_i^n := \begin{cases} \frac{\mu_i \llcorner A_i^n}{\mu_i(A_i^n)} & \text{if } \mu_i(A_i^n) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

then for every  $m$ -uple  $n = (n_1, \dots, n_m) \in \mathbb{N}^m$ ,

$$(\gamma_0)^n := \gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m}) \mu_1^{n_1} \otimes \dots \otimes \mu_m^{n_m},$$

and finally,

$$\gamma_\delta := \sum_{n \in \mathbb{N}^m} (\gamma_0)^n.$$

By definition,  $\gamma_\delta \ll \otimes_i \mu_i$  and we may check that its marginals are  $\mu_i$ . Besides, for  $\otimes_i \mu_i$ -almost every  $\mathbf{x} = (x_1, \dots, x_m) \in \prod_{i=1}^m A_i^{n_i}$ ,

$$\frac{d\gamma_\delta}{d \otimes_i \mu_i}(x_1, \dots, x_m) := \begin{cases} \frac{\gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m})}{\mu_1(A_1^{n_1}) \dots \mu_m(A_m^{n_m})} & \text{if } \mu_1(A_1^{n_1}) \dots \mu_m(A_m^{n_m}) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute its entropy and assume for simplicity that the measure  $\mu_m$  is the one such that  $\overline{\dim}_R(\mu_m) = \max_{i=1, \dots, m} \overline{\dim}(\mu_i)$ :

$$\begin{aligned} \text{Ent}(\gamma_\delta | \otimes_{i=1}^m \mu_i) &= \sum_{n \in \mathbb{N}^m} \int_{\prod_{i=1}^m A_i^{n_i}} \log \left( \frac{\gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m})}{\mu_1(A_1^{n_1}) \dots \mu_m(A_m^{n_m})} \right) d\gamma_\delta \\ &= \sum_{n \in \mathbb{N}^m} \gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m}) \log \left( \frac{\gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m})}{\mu_1(A_1^{n_1}) \dots \mu_m(A_m^{n_m})} \right) \\ &= \sum_{n \in \mathbb{N}^m} \gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m}) \log \left( \frac{\gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m})}{\mu_m(A_m^{n_m})} \right) \\ &\quad + \sum_{j=1}^{m-1} \sum_{n \in \mathbb{N}^m} \gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m}) \log(1/\mu_j(A_j^{n_j})) \\ &= \sum_{n \in \mathbb{N}^m} \gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m}) \log \left( \frac{\gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m})}{\mu_m(A_m^{n_m})} \right) \\ &\quad + \sum_{j=1}^{m-1} \sum_{n \in \mathbb{N}^{m-1}} \mu_j(A_j^{n_j}) \log(1/\mu_j(A_j^{n_j})) \\ &\leq \sum_{j=1}^{m-1} \sum_{n \in \mathbb{N}^{m-1}} \mu_j(A_j^{n_j}) \log(1/\mu_j(A_j^{n_j})). \end{aligned}$$

the last inequality coming from the inequality  $\gamma_0(A_1^{n_1} \times \dots \times A_m^{n_m}) \leq \mu_m(A_m^{n_m})$ . Taking 1-optimal partitions  $(A_j^n)_{n \in \mathbb{N}}$  of diameter smaller than  $\delta$  in the definition of  $H_\delta(\mu_j)$ , we get by definition of  $H_\delta$ ,

$$\text{Ent}(\gamma_\delta | \otimes_{i=1}^m \mu_i) \leq \sum_{j=1}^{m-1} H_\delta(\mu_j) + 1.$$

Notice that  $W_\infty(\gamma_\delta, \gamma_0) \leq \delta$  (taking the  $\ell^\infty$  norm), thus taking  $\gamma_\delta$  as competitor in

(MOT $_\varepsilon$ ) we obtain

$$\begin{aligned}
\text{MOT}_\varepsilon &\leq \int c \, d\gamma_\delta + \varepsilon \sum_{j \leq m-1} H_\delta(\mu_j) + \varepsilon \\
&\leq \text{MOT}_0 + \sum_{j_1, \dots, j_m} \int_{B_{j_1, \dots, j_m}} c \, d(\gamma_\delta - \gamma_0) + \varepsilon \sum_{j \leq m-1} H_\delta(\mu_j) + \varepsilon \\
&\leq \text{MOT}_0 + \sum_{j_1, \dots, j_m} [c]_{\mathcal{C}^{0,1}(B_{j_1, \dots, j_m})} W_1(\gamma_\delta \llcorner B_{j_1, \dots, j_m}, \gamma_0 \llcorner B_{j_1, \dots, j_m}) + \varepsilon \sum_{j \leq m-1} H_\delta(\mu_j) + \varepsilon \\
&\leq \text{MOT}_0 + L\delta + \varepsilon \sum_{j \leq m-1} \frac{H_\delta(\mu_j)}{\log(1/\delta)} \log(1/\delta) + \varepsilon,
\end{aligned} \tag{3.3}$$

where we have considered a covering of each sub-manifold  $X_i$ , that is  $X_i \subseteq \bigcup_{j_i} B_{\delta_0}(a_i^{j_i})$  with  $\delta \leq \delta_0$ ,  $B_{j_1, \dots, j_m} = B_{\delta_0}(a_1^{j_1}) \times \dots \times B_{\delta_0}(a_m^{j_m})$  for all  $j_1, \dots, j_m$  and

$$L = \sum_{j_1, \dots, j_m} [c]_{\mathcal{C}^{0,1}(B_{j_1, \dots, j_m})}.$$

Taking  $\delta = \varepsilon$  and considering that  $\mu_i$  are concentrated on sub-manifolds of dimension  $d_i$  then  $H_\delta(\mu_i) \leq d_i \log(1/\delta) + C^* - 1$  for some  $C^* \geq 1$  and we get

$$\text{MOT}_\varepsilon \leq \text{MOT}_0 + \left( \sum_{j \leq m-1} d_j \right) \varepsilon \log(1/\varepsilon) + C^* \varepsilon.$$

□

*Remark 3.3.* Notice that by taking  $m = 2$  and  $d_1 = d_2 = d$ , one easily retrieves the results in [CPT23]. Besides, if the  $\mu_i$ 's are merely assumed to have compact support (not necessarily supported on a sub-manifold), the same proof actually shows the slightly weaker estimate

$$\text{MOT}_\varepsilon \leq \text{MOT}_0 + \left( \sum_{i=1}^m \overline{\dim}_R(\mu_i) - \max_{j=1, \dots, m} \overline{\dim}_R(\mu_j) \right) \varepsilon \log(1/\varepsilon) + o(\varepsilon \log(1/\varepsilon)). \tag{3.4}$$

### 3.2 Upper bound for locally semi-concave costs

We provide now a finer upper bound under the additional assumptions that the  $X_i$ 's are  $\mathcal{C}^2$  sub-manifolds of  $\mathbb{R}^N$ ,  $c$  is locally semi-concave as in Definition 3.4 (in particular if  $c \in \mathcal{C}^2(\mathbf{X}, \mathbb{R}_+)$ ), and the  $\mu_i$ 's are measures in  $L^\infty(\mathcal{H}_{X_i}^{d_i})$  with compact support in  $X_i$ .

**Definition 3.4.** A function  $f : X \rightarrow \mathbb{R}$  defined on a  $\mathcal{C}^2$  sub-manifold  $X \subseteq \mathbb{R}^N$  of dimension  $d$  is *locally semiconcave* if for every  $x \in X$  there exists a local chart (i.e. a  $\mathcal{C}^2$  diffeomorphism)  $\psi : U \rightarrow \Omega$  where  $U \subseteq X$  is an open neighborhood of  $x$  and  $\Omega$  is an open convex subset of  $\mathbb{R}^d$ , such that  $f \circ \psi^{-1}$  is  $\lambda$ -concave for some  $\lambda \in \mathbb{R}$ , meaning  $f \circ \psi^{-1} - \lambda|\cdot|^2$  is concave on  $\Omega$ .

**Lemma 3.5** (Local semiconcavity and covering). *Let  $c : \mathbf{X} \rightarrow \mathbb{R}_+$  be a locally semiconcave cost function and  $(\phi_i)_{1 \leq i \leq m} \in \prod_{1 \leq i \leq m} \mathcal{C}(K_i)$  be a system of  $c$ -conjugate functions defined on compact subsets  $K_i \subseteq X_i$ . We can find for every  $i \in \{1, \dots, m\}$  a finite open covering  $(U_i^j)_{1 \leq j \leq J}$  of  $K_i$  and local charts  $\psi_i^j : U_i^j \rightarrow \Omega_i^j$  such that for some  $\lambda \in \mathbb{R}$ :*

(i) *for every  $\mathbf{j} = (j_1, \dots, j_m) \in \{1, \dots, J\}^m$ ,  $c \circ (\psi^{\mathbf{j}})^{-1}$  is  $\lambda$ -concave on  $\Omega^{\mathbf{j}} := \prod_{1 \leq i \leq m} \Omega_i^{j_i}$ ,*

(ii) *for every  $(i, j) \in \{1, \dots, m\} \times \{1, \dots, J\}$ ,  $\phi_i \circ (\psi_i^j)^{-1}$  is  $\lambda$ -concave on  $\Omega_i^j$ .*

In particular, all the  $\phi_i$ 's are locally semiconcave.

*Proof.* For every  $i$ , by compactness of the  $K_i$ 's we can find a finite open covering  $(U_i^j)_{1 \leq j \leq J}$  of  $K_i$  and local charts  $\psi_i^j : U_i^j \rightarrow \Omega_i^j$  such that for every  $\mathbf{j} = (j_1, \dots, j_m) \in \{1, \dots, J\}^m$ ,  $c \circ (\psi^{\mathbf{j}})^{-1} - \lambda |\cdot|^2$  is concave for some  $\lambda_j \in \mathbb{R}$ , where  $\psi^{\mathbf{j}} := (\psi_1^{j_1}, \dots, \psi_m^{j_m})$ . We may assume that  $\lambda_j = \lambda$  for every  $\mathbf{j}$ , by taking  $\lambda = \max\{\lambda_j : \mathbf{j} \in \{1, \dots, J\}^m\}$ . Fix  $i \in \{1, \dots, m\}$ ,  $j \in \{1, \dots, J\}$ , then for every  $\mathbf{k} = (k_\ell)_{\ell \neq i} \in \{1, \dots, J\}^{m-1}$  set  $\Omega^{\mathbf{k}} = \prod_{\ell \neq i} \Omega_\ell^{k_\ell}$  and  $\psi^{\mathbf{k}} = (\psi_1^{k_1}, \dots, \psi_{i-1}^{k_{i-1}}, \psi_i^j, \psi_{i+1}^{k_{i+1}}, \dots)$ . Notice that for every  $y \in \Omega_i^j$ ,

$$\begin{aligned} \phi_i \circ (\psi_i^j)^{-1}(y) &= \inf_{(x_\ell)_{\ell \neq i}} c(x_1, \dots, x_{i-1}, (\psi_i^j)^{-1}(y), x_{i+1}, \dots) - \sum_{\ell \neq i} \phi_\ell(x_\ell) \\ &= \min_{\mathbf{k}} \inf_{(y_\ell)_{\ell \neq i} \in \Omega^{\mathbf{k}}} c \circ (\psi^{\mathbf{k}})^{-1}(y_1, \dots, y_{i-1}, y, y_{i+1}, \dots) - \sum_{\ell \neq i} \phi_\ell \circ (\psi_\ell^{k_\ell})^{-1}(y_\ell), \end{aligned}$$

and we see that it is  $\lambda$ -concave as an infimum of  $\lambda$ -concave functions.  $\square$

We are going to use an integral variant of Alexandrov Theorem which is proved in [CPT23].

**Lemma 3.6** (Lemma 3.6 [CPT23]). *Let  $f : \Omega \rightarrow \mathbb{R}$  be a  $\lambda$ -concave function defined on a convex open set  $\Omega \subseteq \mathbb{R}^d$ , for some  $\lambda \geq 0$ . There exists a constant  $C \geq 0$  depending only on  $d$  such that:*

$$\int_{\Omega} \sup_{y \in B_r(x) \cap \Omega} |f(y) - (f(x) + \nabla f(x) \cdot (y - x))| dx \leq Cr^2 \mathcal{H}^{d-1}(\partial\Omega) ([f]_{\mathcal{C}^{0,1}(\Omega)} + \lambda \text{diam}(\Omega)). \quad (3.5)$$

**Proposition 3.7.** *Let  $c \in \mathcal{C}^2(\mathbf{X})$  and assume that for every  $i \in \{1, \dots, m\}$ ,  $X_i \subseteq \mathbb{R}^N$  is a  $\mathcal{C}^2$  sub-manifold of dimension  $d_i$  and  $\mu_i \in L^\infty(\mathcal{H}_{X_i}^{d_i})$  is a probability measure compactly supported in  $X_i$ . Then there exists a constant  $C^* \geq 0$  such that*

$$\boxed{\text{MOT}_\varepsilon \leq \text{MOT}_0 + \frac{1}{2} \left( \sum_{i=1}^m d_i - \max_{1 \leq i \leq m} d_i \right) \varepsilon \log(1/\varepsilon) + C^* \varepsilon.} \quad (3.6)$$

*Proof.* The measures  $\mu_i$  being compactly supported in  $X_i$ , take for every  $i \in \{1, \dots, m\}$  an open subset  $U_i$  such that  $\text{spt } \mu_i \subseteq U_i \Subset X_i$  and define the compact set  $K_i \subseteq \bar{U}_i$ .

Take  $(\phi_i)_{1 \leq i \leq m} \in \prod_{1 \leq i \leq m} \mathcal{C}(K_i)$  a  $m$ -uple of  $c$ -conjugate Kantorovich potentials and a transport plan  $\gamma_0 \in \bar{\Pi}(\mu_1, \dots, \mu_m)$  which are optimal for the unregularized problems. In particular,

$$\begin{aligned} E &:= c - \bigoplus_{i=1}^m \phi_i \geq 0 \quad \text{on} \quad \mathbf{U} := \prod_{1 \leq i \leq m} U_i, \\ E &= 0 \quad \text{on} \quad \text{spt } \gamma_0 \subseteq \mathbf{U}. \end{aligned} \tag{3.7}$$

For every  $i$  we consider the coverings  $(U_i^j)_{1 \leq j \leq J}$  provided by [Lemma 3.5](#) and we notice by compactness that there exists open subsets  $\tilde{U}_i^j \Subset U_i^j$  such that  $(\tilde{U}_i^j)_{1 \leq j \leq J}$  is still an open covering of  $K_i$  and such that for a small  $\delta_0 > 0$ , the  $\delta_0$ -neighbourhood of  $\tilde{\Omega}_i^j := \psi_i^j(\tilde{U}_i^j)$  is included in  $\Omega_i^j$ . For  $\delta \in (0, \delta_0)$  we consider the block approximation  $\gamma_\delta$  of  $\gamma_0$  built in the proof of [Proposition 3.2](#), as well as  $\kappa_\delta \in \Pi(\gamma_0, \gamma_\delta)$  such that  $\sup_{(\mathbf{x}_0, \mathbf{x}) \in \text{spt } \kappa_\delta} \|\mathbf{x}_0 - \mathbf{x}\| \leq \delta$ . For every  $\mathbf{j} = (j_1, \dots, j_m) \in \{1, \dots, m\}$ , we set  $E^{\mathbf{j}} := E \circ (\psi^{\mathbf{j}})^{-1}$  and  $U^{\mathbf{j}} := \prod_{i=1}^m U_i^{j_i}$ , and we write:

$$\begin{aligned} \int_{\mathbf{X}} c \, d\gamma_\delta - \int_{\mathbf{X}} c \, d\gamma_0 &= \int_{\mathbf{X}} E \, d\gamma_\delta \\ &= \int_{\mathbf{X} \times \mathbf{X}} E(\mathbf{x}) \, d\kappa_\delta(\mathbf{x}_0, \mathbf{x}) \\ &\leq \sum_{\mathbf{j}=(j_1, \dots, j_m)} \int_{(\mathbf{x}_0, \mathbf{x}) \in \tilde{U}^{\mathbf{j}} \times \mathbf{X}} E(\mathbf{x}) \, d\kappa_\delta(\mathbf{x}_0, \mathbf{x}) \\ &\leq \sum_{\mathbf{j}=(j_1, \dots, j_m)} \int_{(\mathbf{x}_0, \mathbf{x}) \in (U^{\mathbf{j}})^2} E^{\mathbf{j}}(\psi^{\mathbf{j}}(\mathbf{x})) \, d\kappa_\delta(\mathbf{x}_0, \mathbf{x}). \end{aligned}$$

Notice that for  $\gamma_0$ -a.e.  $\mathbf{x}_0 \in U^{\mathbf{j}}$ ,  $E^{\mathbf{j}}$  is differentiable at  $\psi^{\mathbf{j}}(\mathbf{x}_0)$ , or equivalently  $E$  is differentiable at  $\mathbf{x}_0$ . Indeed  $c$  is differentiable everywhere, and for every  $i, j$ ,  $\phi_i \circ (\psi_i^j)^{-1}$  is semi-concave thus differentiable  $\mathcal{L}^{d_i}$ -a.e. and hence  $\phi_i$  is differentiable  $\mu_i$ -a.e. because  $\mu_i \ll \mathcal{H}^{d_i}$  and  $\psi_i^j$  is bi-Lipschitz, which in turn implies that  $\bigoplus_{i=1}^m \phi$  is differentiable  $\gamma_0$ -a.e. because  $\gamma_0 \in \Pi(\mu_1, \dots, \mu_m)$ . Moreover, by [\(3.7\)](#) we have  $T_{\psi^{\mathbf{j}}(\mathbf{x}_0)} E^{\mathbf{j}} \equiv 0$  for  $\gamma_0$ -a.e.  $\mathbf{x}_0 \in U^{\mathbf{j}}$ . We may then compute:

$$\begin{aligned} &\int_{\mathbf{X}} c \, d\gamma_\delta - \int_{\mathbf{X}} c \, d\gamma_0 \\ &\leq \sum_{\mathbf{j}=(j_1, \dots, j_m)} \int_{(\mathbf{x}_0, \mathbf{x}) \in (U^{\mathbf{j}})^2} \left( E^{\mathbf{j}}(\psi^{\mathbf{j}}(\mathbf{x})) - T_{\psi^{\mathbf{j}}(\mathbf{x}_0)} E^{\mathbf{j}}(\psi^{\mathbf{j}}(\mathbf{x}) - \psi^{\mathbf{j}}(\mathbf{x}_0)) \right) \, d\kappa_\delta(\mathbf{x}_0, \mathbf{x}) \\ &= \sum_{\mathbf{j}=(j_1, \dots, j_m)} \left( \int_{(\mathbf{x}_0, \mathbf{x}) \in (U^{\mathbf{j}})^2} \left( c^{\mathbf{j}}(\psi^{\mathbf{j}}(\mathbf{x})) - T_{\psi^{\mathbf{j}}(\mathbf{x}_0)} c^{\mathbf{j}}(\psi^{\mathbf{j}}(\mathbf{x}) - \psi^{\mathbf{j}}(\mathbf{x}_0)) \right) \, d\kappa_\delta(\mathbf{x}_0, \mathbf{x}) + \right. \\ &\quad \left. \sum_{1 \leq i \leq m} \int_{(x_0, x) \in (U_i^{j_i})^2} \left( \phi_i^{j_i}(\psi_i^{j_i}(x)) - T_{\psi_i^{j_i}(x_0)} \phi_i^{j_i}(\psi_i^{j_i}(x) - \psi_i^{j_i}(x_0)) \right) \, d(e_i, e_i) \# \kappa_\delta(x_0, x) \right). \end{aligned} \tag{3.8}$$

Now, since  $c^j$  is  $\lambda$ -concave on each  $\Omega^j := \psi^j(U^j)$ , whenever  $\|\mathbf{x}_0 - \mathbf{x}\| \leq \delta$  we have

$$c^j(\psi^j(\mathbf{x})) - T_{\psi^j(\mathbf{x}_0)} c^j(\psi^j(\mathbf{x}) - \psi^j(\mathbf{x}_0)) \leq \lambda \frac{\|\psi^j(\mathbf{x}) - \psi^j(\mathbf{x}_0)\|^2}{2} \leq \frac{\lambda L^j}{2} \delta^2, \quad (3.9)$$

where  $L^j := \max\{\text{Lip}(\psi^j), \text{Lip}((\psi^j)^{-1})\}$ . Besides, we may apply [Lemma 3.6](#) to each  $\phi_i^{j_i}$  over  $\Omega_i^{j_i}$  to get

$$\begin{aligned} & \int_{(x_0, x) \in (U_i^{j_i})^2} \left( \phi_i^{j_i}(\psi_i^{j_i}(x)) - T_{\psi_i^{j_i}(x_0)} \phi_i^{j_i}(\psi_i^{j_i}(x) - \psi_i^{j_i}(x_0)) \right) \\ & \leq \int_{x_0 \in U_i^{j_i}} \sup_{y \in B_{L^j \delta}(\psi_i^{j_i}(x_0)) \cap \Omega_i^{j_i}} \left| \phi_i^{j_i}(y) - T_{\psi_i^{j_i}(x_0)} \phi_i^{j_i}(y - \psi_i^{j_i}(x_0)) \right| d(e_i, e_i)_{\#} \kappa_{\delta}(x_0, x) \\ & = \int_{U_i^{j_i}} \sup_{y \in B_{L^j \delta}(\psi_i^{j_i}(x_0)) \cap \Omega_i^{j_i}} \left| \phi_i^{j_i}(y) - T_{\psi_i^{j_i}(x_0)} \phi_i^{j_i}(y - \psi_i^{j_i}(x_0)) \right| d\mu_i(x_0) \\ & \leq \int_{\Omega_i^{j_i}} \sup_{y \in B_{L^j \delta}(y_0) \cap \Omega_i^{j_i}} \left| \phi_i^{j_i}(y) - T_{y_0} \phi_i^{j_i}(y - y_0) \right| d(\psi_i^{j_i})_{\#} \mu_i(y_0) \\ & \leq \|\mu_i\|_{L^\infty(\mathcal{H}^{d_i})} L^j C (L^j \delta)^2 \mathcal{H}^{d_i-1}(\partial \Omega_i^{j_i}) \left( [\phi_i^{j_i}]_{\mathcal{C}^{0,1}(\Omega_i^{j_i})} + \lambda \text{diam}(\Omega_i^{j_i}) \right) \\ & \leq C^j \delta^2, \end{aligned} \quad (3.10)$$

for some constant  $C^j \in (0, +\infty)$ . Reporting [\(3.9\)](#) and [\(3.10\)](#) in [\(3.8\)](#) yields

$$\int_{\mathbf{X}} c d\gamma_{\delta} - \int_{\mathbf{X}} c d\gamma_0 \leq \sum_{j=1}^m \left( \frac{\lambda L^j}{2} + C^j \right) \delta^2 =: C' \delta^2.$$

Finally, we proceed as in the end of the proof of [Proposition 3.2](#), taking  $\gamma_{\delta}$  as competitor in the primal formulation [\(MOT \$\_{\varepsilon}\$ \)](#), so as to obtain

$$\begin{aligned} \text{MOT}_{\varepsilon} - \text{MOT}_0 & \leq \int_{\mathbf{X}} c d\gamma_{\delta} - \int_{\mathbf{X}} c d\gamma_0 + \varepsilon \sum_{i \leq m-1} H_{\delta}(\mu_j) + \varepsilon \\ & \leq C' \delta^2 + \varepsilon \sum_{i \leq m-1} (d_i \log(1/\delta) + C''), \end{aligned}$$

where  $C'' \in (0, +\infty)$  is a constant such that  $H_{\delta}(\mu_i) \leq d_i \log(1/\delta) + C'' - 1$ . Taking  $\delta = \sqrt{\varepsilon}$  provides the desired estimate

$$\text{MOT}_{\varepsilon} - \text{MOT}_0 \leq \frac{1}{2} \left( \sum_{i=1}^{m-1} d_i \right) \varepsilon \log(1/\varepsilon) + (C' + C'') \varepsilon,$$

recalling that the index  $i = m$  was chosen merely to simplify notations.  $\square$

## 4 Lower bound for $\mathcal{C}^2$ costs with a signature condition

In this section we consider a cost  $c \in \mathcal{C}^2(\mathbf{X}, \mathbb{R}_+)$  where  $\mathbf{X} = X_1 \times \dots \times X_m$  and  $X_i \subseteq \mathbb{R}^N$  is a  $\mathcal{C}^2$  sub-manifold of dimension  $d_i$  and  $\text{spt } \mu_i$  is a compact subset of  $X_i$  for every  $i \in \{1, \dots, m\}$ . If  $g$  is a bilinear form we denote by  $(d^+(g), d^-(g), d^0(g))$  its signature.

**Lemma 4.1.** *Let  $c \in \mathcal{C}^2(\mathbf{X}, \mathbb{R}_+)$  and  $(\phi_1, \dots, \phi_m) \in \mathcal{C}(K_1) \times \dots \times \mathcal{C}(K_m)$  be a system of  $c$ -conjugate functions on subsets  $K_i \subseteq X_i$ . We set  $E := c - \phi_1 \oplus \dots \oplus \phi_m$  on  $\mathbf{K} := K_1 \times \dots \times K_m$ . Take  $\bar{\mathbf{x}} \in \mathbf{K}$  and  $g = \{\sum_{p \in P} t_p g_p(\bar{\mathbf{x}}) \text{ for } (t_p)_{p \in P} \in \Delta_P\}$ . Then there exists local coordinates around  $\bar{\mathbf{x}}$ , i.e.  $\mathcal{C}^2$  diffeomorphisms*

$$u = (u^0, u^-, u^+) : U_{\bar{\mathbf{x}}} \subseteq \mathbf{X} \rightarrow \mathbb{R}^{d^0(g)} \times \mathbb{R}^{d^-(g)} \times \mathbb{R}^{d^+(g)}$$

defined on a neighborhood of  $\bar{\mathbf{x}}$  such that if  $\mathbf{x}, \mathbf{x}' \in B_r(\bar{\mathbf{x}}) \subseteq U_{\bar{\mathbf{x}}}$ ,

$$\frac{E(\mathbf{x}') + E(\mathbf{x})}{2} \geq |u^+(\mathbf{x}') - u^+(\mathbf{x})|^2 - |u^-(\mathbf{x}') - u^-(\mathbf{x})|^2 - \varepsilon(r)|u(\mathbf{x}') - u(\mathbf{x})|^2 \quad (4.1)$$

where  $\varepsilon(r) \geq 0$  tends to 0 as  $r \rightarrow 0$ .

*Proof.* Let  $p = \{p_-, p_+\} \in P$ . For  $y \in \prod_{i \in p_{\pm}} K_i$ , we set

$$\phi_{p_{\pm}}(y) = \sum_{i \in p_{\pm}} \phi_i(y_i).$$

We identify any  $\mathbf{x} \in \mathbf{K}$  with  $(x_{p_-}, x_{p_+})$ . Since the  $(\phi_i)$  are  $c$ -conjugate, if  $\mathbf{x}, \mathbf{x}' \in \mathbf{K}$ , we have:

$$\begin{aligned} E(\mathbf{x}') &= c(x'_{p_-}, x'_{p_+}) - \phi_{p_-}(x'_{p_-}) - \phi_{p_+}(x'_{p_+}) \\ &\geq c(x'_{p_-}, x'_{p_+}) - (c(x'_{p_-}, x_{p_+}) - \phi_{p_+}(x_{p_+})) - (c(x_{p_-}, x'_{p_+}) - \phi_{p_-}(x_{p_-})) \\ &= c(x'_{p_-}, x'_{p_+}) - c(x'_{p_-}, x_{p_+}) - c(x_{p_-}, x'_{p_+}) + c(x_{p_-}, x_{p_+}) - E(\mathbf{x}). \end{aligned}$$

Now we do computations in local charts  $\psi_i : U_i \subseteq X_i \rightarrow \psi_i(U_i) \subseteq \mathbb{R}^{d_i}$  which are  $\mathcal{C}^2$  diffeomorphisms such that  $B_R(\bar{x}_i) \subseteq U_i$  and  $\psi_i(U_i)$  are convex. We keep the same notation with a slight abuse<sup>1</sup>, and use Taylor's integral formula:

$$E(\mathbf{x}') + E(\mathbf{x}) \geq \int_0^1 \int_0^1 D_{p_- p_+}^2 c(\mathbf{x}_{s,t})(x'_{p_-} - x_{p_-}, x'_{p_+} - x_{p_+})$$

where  $\mathbf{x}_{s,t} := (x_{p_-} + (1-s)x'_{p_-}, x_{p_+} + (1-t)x'_{p_+})$  for  $s, t \in [0, 1]$ . Since  $|D_{p_- p_+}^2 c(\mathbf{x}_{s,t}) - D_{p_- p_+}^2 c(\bar{\mathbf{x}})| \leq \varepsilon(\bar{\mathbf{x}}, \mathbf{r})$  where  $\varepsilon(\bar{\mathbf{x}}, \mathbf{r})$  is independent from  $p$  and tends to 0 as  $r \rightarrow 0$ , and since by definition  $D^2 c(\bar{\mathbf{x}})(x'_{p_-} - x_{p_-}, x'_{p_+} - x_{p_+}) = \frac{1}{2} g_p(\bar{\mathbf{x}})(\mathbf{x}' - \mathbf{x})$ , it holds:

$$E(\mathbf{x}) + E(\mathbf{x}') \geq \frac{1}{2} g_p(\bar{\mathbf{x}})(\mathbf{x}' - \mathbf{x}) - \varepsilon(\bar{\mathbf{x}}, \mathbf{r}) \|\mathbf{x}' - \mathbf{x}\|^2.$$

<sup>1</sup>Any linear combination  $az_i + by_i$  will designate  $\psi_i^{-1}(a\psi_i(z_i) + b\psi_i(y_i))$ .

Taking  $g = \sum_{p \in P} t_p g_p(\bar{\mathbf{x}})$  and averaging the previous inequality yields:

$$E(\mathbf{x}) + E(\mathbf{x}') \geq \frac{1}{2} g(\mathbf{x}' - \mathbf{x}) - \varepsilon(\bar{\mathbf{x}}, r) \|\mathbf{x}' - \mathbf{x}\|^2. \quad (4.2)$$

Finally, we can find a linear isomorphism  $Q \in GL(\sum_i d_i, \mathbb{R})$  which trivializes  $g$ , i.e. such that after setting  $u := Q \circ (\psi_1, \dots, \psi_m)$  and denoting  $u = (u^+, u^-, u^0) : \prod_i U_i \rightarrow \mathbb{R}^{d^+(g)} \times \mathbb{R}^{d^-(g)} \times \mathbb{R}^{d^0(g)}$  then:

$$\frac{1}{4} g(\mathbf{x}' - \mathbf{x}) = |u^+(\mathbf{x}') - u^+(\mathbf{x})|^2 - |u^-(\mathbf{x}') - u^-(\mathbf{x})|^2.$$

We get the result by replacing  $\varepsilon$  with  $\|Q\|^{-1}\varepsilon$ .  $\square$

We will use the following positive signature condition:

$$\begin{aligned} & \text{for every } \mathbf{x} \in X_1 \times \dots \times X_m, \quad d_c^+(\mathbf{x}) \geq \kappa \quad \text{where} \\ & d_c^\pm(\mathbf{x}) := \max \{ d^\pm(g) : g \in \text{conv}_{p \in P} \{ g_p(\mathbf{x}) \} \}. \end{aligned} \quad (\text{PS}(\kappa))$$

**Proposition 4.2.** *We assume that for every  $i$ ,  $\mu_i \in L^\infty(\mathcal{H}^{d_i} \llcorner X_i)$  and has compact support in  $X_i$ . If  $(\text{PS}(\kappa))$  is satisfied, then there exists a constant  $C_* \in [0, \infty)$  such that for every  $\varepsilon > 0$ ,*

$$\boxed{\text{MOT}_\varepsilon \geq \text{MOT}_0 + \frac{\kappa}{2} \varepsilon \log(1/\varepsilon) - C_* \varepsilon.} \quad (4.3)$$

*Proof.* The measures  $\mu_i$  being supported on some compact subsets  $K_i \subseteq X_i$ , consider a family  $(\phi_i)_{1 \leq i \leq m} \in \mathcal{C}(K_1) \times \dots \times \mathcal{C}(K_m)$  of  $c$ -conjugate Kantorovich potentials. Taking  $(\phi_i)_{1 \leq i \leq m}$  as competitor in  $(\text{MD}'_\varepsilon)$ , we get the lower bound:

$$\begin{aligned} \text{MOT}_\varepsilon & \geq \sum_{i=1}^m \int_{X_i} \phi_i d\mu_i - \varepsilon \log \left( \int_{\prod_{1 \leq i \leq m} X_i} e^{-\frac{E}{\varepsilon}} d \otimes_{1 \leq i \leq m} \mu_i \right) \\ & = \text{MOT}_0 - \varepsilon \log \left( \int_{\prod_{1 \leq i \leq m} X_i} e^{-\frac{E}{\varepsilon}} d \otimes_{1 \leq i \leq m} \mu_i \right), \end{aligned}$$

where  $E := c - \bigoplus_{1 \leq i \leq m} \phi_i$  on  $\mathbf{X} := \prod_{1 \leq i \leq m} X_i$  as in [Lemma 4.1](#). We are going to show that for some constant  $C > 0$  and for every  $\varepsilon > 0$ ,

$$\int_{\mathbf{X}} e^{-E/\varepsilon} d \otimes_{1 \leq i \leq m} \mu_i \leq C \varepsilon^{\kappa/2},$$

which yields [\(4.3\)](#) with  $C_* = \log(C)$ .

For every  $\bar{\mathbf{x}} \in \mathbf{K} := \prod_{1 \leq i \leq m} K_i$ , we consider a quadratic form  $g_{\bar{\mathbf{x}}} = \sum_p t_p g_p(\bar{\mathbf{x}})$  of signature  $(\kappa, d_{\bar{\mathbf{x}}}^-, d_{\bar{\mathbf{x}}}^0)$ , which is possible thanks to  $(\text{PS}(\kappa))$ , and take a local chart

$$u_{\bar{\mathbf{x}}} : U_{\bar{\mathbf{x}}} \subseteq \mathbf{X} \rightarrow B_{R_{\bar{\mathbf{x}}}}^\kappa \times B_{R_{\bar{\mathbf{x}}}}^{d_{\bar{\mathbf{x}}}^-} \times B_{R_{\bar{\mathbf{x}}}}^{d_{\bar{\mathbf{x}}}^0}$$

as given by [Lemma 4.1](#), such that [\(4.1\)](#) holds with  $\varepsilon(R_{\bar{\mathbf{x}}}) \leq 1/2$ . Notice that  $u_{\bar{\mathbf{x}}}$  is bi-Lipschitz with some constant  $L_{\bar{\mathbf{x}}}$  on  $V_{\bar{\mathbf{x}}} := u_{\bar{\mathbf{x}}}^{-1}(B_{R_{\bar{\mathbf{x}}}/2}^\kappa \times B_{R_{\bar{\mathbf{x}}}/2}^{d_{\bar{\mathbf{x}}}^-} \times B_{R_{\bar{\mathbf{x}}}/2}^{d_{\bar{\mathbf{x}}}^0})$ .



For every  $i$  we may write  $\mu_i = f_i \mathcal{H}_{X_i}^{d_i}$ . By applying several times the co-area formula [Fed96, Theorem 3.2.22] to the projection maps onto  $X_i$ , we may justify that

$$\mathcal{H}_X^d = \otimes_{1 \leq i \leq m} \mathcal{H}_{X_i}^{d_i} \quad \text{where} \quad d = \sum_i d_i.$$

We set  $E_{\bar{x}} := E \circ u_{\bar{x}}^{-1} : B_{R_{\bar{x}}}^\kappa \times B_{R_{\bar{x}}}^{d_{\bar{x}}^-} \times B_{R_{\bar{x}}}^{d_{\bar{x}}^0} \rightarrow [0, +\infty]$  and we apply the area formula:

$$\begin{aligned} \int_{V_{\bar{x}}} e^{-E/\varepsilon} d \otimes_{1 \leq i \leq m} \mu_i &= \int_{V_{\bar{x}}} e^{-E/\varepsilon} (\otimes_{1 \leq i \leq m} f_i) d \mathcal{H}_X^d \\ &= \int_{B_{R_{\bar{x}/2}^\kappa} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^-}} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^0}} e^{-E_{\bar{x}}/\varepsilon} (\otimes_{1 \leq i \leq m} f_i) J u_{\bar{x}}^{-1} d \mathcal{L}^\kappa \otimes \mathcal{L}^{d_{\bar{x}}^-} \otimes \mathcal{L}^{d_{\bar{x}}^0} \\ &\leq L_{\bar{x}} \prod_{1 \leq i \leq m} \|\mu_i\|_{L^\infty(\mathcal{H}^{d_i} \llcorner X_i)} \int_{B_{R_{\bar{x}/2}^\kappa} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^-}} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^0}} e^{-E_{\bar{x}}(u^+, u^-, u^0)/\varepsilon} d(u^+, u^-, u^0). \end{aligned}$$

Now, for every  $(u^-, u^0) \in B_{R_{\bar{x}/2}^{d_{\bar{x}}^-}} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^0}$ , consider  $f^+(u^-, u^0)$  a minimizer of  $E_{\bar{x}}(\cdot, u^-, u^0)$  over  $\bar{B}_{R_{\bar{x}/2}^\kappa}$ . By (4.1) of Lemma 4.1, for every  $(u^+, u^-, u^0) \in B_{R_{\bar{x}/2}^\kappa} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^-}} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^0}$ ,

$$\begin{aligned} E_{\bar{x}}(u^+, u^-, u^0) &\geq \frac{1}{2} (E_{\bar{x}}(f^+(u^-, u^0), u^-, u^0) + E_{\bar{x}}(u^+, u^-, u^0)) \\ &\geq (1 - 1/2) |u^+ - f^+(u^-, u^0)|^2 = \frac{1}{2} |u^+ - f^+(u^-, u^0)|^2. \end{aligned}$$

As a consequence we obtain:

$$\begin{aligned} &\int_{B_{R_{\bar{x}/2}^\kappa} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^-}} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^0}} e^{-E_{\bar{x}}(u^+, u^-, u^0)/\varepsilon} d(u^+, u^-, u^0) \\ &\leq \int_{B_{R_{\bar{x}/2}^{d_{\bar{x}}^-}} \times B_{R_{\bar{x}/2}^{d_{\bar{x}}^0}} \int_{B_{R_{\bar{x}/2}^\kappa} e^{-\frac{|u^+ - f^+(u^-, u^0)|^2}{2\varepsilon}} du^+ d(u^-, u^0) \\ &\leq \varepsilon^{\kappa/2} \omega_{d_{\bar{x}}^-} \omega_{d_{\bar{x}}^0} R_{\bar{x}}^{d_{\bar{x}}^- + d_{\bar{x}}^0} \int_{\mathbb{R}^\kappa} e^{-|u|^2/2} du = C_{\bar{x}} \varepsilon^{\kappa/2} \end{aligned}$$

for some constant  $C_{\bar{x}} > 0$ . The sets  $\{V_{\bar{x}}\}_{\bar{x} \in \Sigma}$  form an open covering of the compact set  $\Sigma := \{\mathbf{x} \in \mathbf{K} : E(\mathbf{x}) = 0\}$ , hence we may extract a finite covering  $V_{\bar{x}_1}, \dots, V_{\bar{x}_I}$  and:

$$\int_{\bigcup_{i=1}^I V_{\bar{x}_i}} e^{-E/\varepsilon} d \otimes_{1 \leq i \leq m} \mu_i \leq \varepsilon^{\kappa/2} \left( \sum_{1 \leq i \leq I} L_{\bar{x}_i} C_{\bar{x}_i} \right) \left( \prod_{1 \leq i \leq m} \|\mu_i\|_{L^\infty(\mathcal{H}^{d_i} \llcorner X_i)} \right) = c_1 \varepsilon^{\kappa/2},$$

for some constant  $c_1 \in (0, +\infty)$ . Finally, since  $E$  is continuous and does not vanish on the compact set  $\mathbf{K}' = \mathbf{K} \setminus \bigcup_{i=1}^I V_{\bar{x}_i}$ , it is bounded from below on  $\mathbf{K}'$  by some constant  $c_2 > 0$ . Therefore, for every  $\varepsilon > 0$ ,

$$\int_{\mathbf{X}} e^{-E/\varepsilon} d \otimes_{1 \leq i \leq m} \mu_i \leq c_1 \varepsilon^{\kappa/2} + e^{-c_2/\varepsilon} \leq C \varepsilon^{\kappa/2},$$

for some constant  $C > 0$ . This concludes the proof.  $\square$

## 5 Examples and matching bound

We devote this section to applying the results we have stated above to several cost functions. For simplicity we can assume that the dimensions of the  $X_i$  are all equal to some common  $d$  and the cost function  $c$  is  $\mathcal{C}^2$ . As in [Pas12] we consider, for the lower bound, the metric  $\bar{g}$  such that  $t_p = \frac{1}{2^{m-1}-1}$  for all  $p \in P$ , we remind that  $P$  is the set of partition of  $\{1, \dots, m\}$  into two non empty disjoint subsets.

*Example 5.1 (Two marginals case).* In previous works [CPT23; EN22] concerning the rate of convergence for the two marginals problem, it was assumed that the cost function must satisfy a non degeneracy condition, that is  $D_{x_1 x_2}^2 c$  must be of full rank. A direct consequence of our analysis is that we can provide a lower bound (the upper bound does not depend on such a condition) for costs for which the non-degeneracy condition fails. Let  $r$  be the rank of  $D_{x_1 x_2}^2 c$  at the point where the non-degeneracy condition fails, then the signature of  $\bar{g}$  at this point is given by  $(r, r, 2d - 2r)$  meaning that locally the support of the optimal  $\gamma_0$  is at most  $2d - r$  dimensional. Thus, the bounds become

$$\boxed{\frac{r}{2}\varepsilon \log(1/\varepsilon) - C_*\varepsilon \leq \text{OT}_\varepsilon - \text{OT}_0 \leq \frac{d}{2}\varepsilon \log(1/\varepsilon) + C^*\varepsilon,}$$

for some constants  $C_*, C^* > 0$ . Notice that if  $D_{x_1, x_2}^2 c$  has full rank then  $r = d$  and we retrieve the matching bound results of [CPT23; EN22].

*Example 5.2 (Two marginals case and unequal dimension).* Consider now the two marginals case but unequal dimensional, that is for example  $d_1 > d_2$ . Then, if  $D_{x_1, x_2}^2 c$  has full rank, that is  $r = d_2$ , we obtain a matching bound depending only on the lower dimensional marginal

$$\boxed{\frac{d_2}{2}\varepsilon \log(1/\varepsilon) - C_*\varepsilon \leq \text{OT}_\varepsilon - \text{OT}_0 \leq \frac{d_2}{2}\varepsilon \log(1/\varepsilon) + C^*\varepsilon,}$$

for some constants  $C_*, C^* > 0$ . If  $\mu_1$  is absolutely continuous with respect to  $\mathcal{H}^{d_1}$  on some smooth sub-manifold of dimension  $d_1$ , then any optimal transport plan would be concentrated on a set of Hausdorff dimension no less than  $d_1$ , and thus the upper bound given in [EN22, Theorem 3.8] would be  $\frac{d_1}{2}\varepsilon \log(1/\varepsilon) + O(\varepsilon)$ , which is strictly worse than our estimate.

*Example 5.3 (Negative harmonic cost).* Consider the cost  $c(x_1, \dots, x_m) = h(\sum_{i=1}^m x_i)$  where  $h$  is  $\mathcal{C}^c$  and  $D^2 h > 0$ . Assuming that the marginals have finite second moments, when  $h(x) = |x|^2$  this kind of cost is equivalent to the harmonic negative cost that is  $c(x_1, \dots, x_m) = -\sum_{i < j} |x_i - x_j|^2$  (here  $|\cdot|$  denotes the standard euclidean norm), see [DGN15] for more details. It follows now that the signature of the metric  $\bar{g}$  is  $(d, (m-1)d, 0)$  thus the bounds between  $\text{MOT}_\varepsilon$  and  $\text{MOT}_0$  that we obtain are

$$\boxed{\frac{d}{2}\varepsilon \log(1/\varepsilon) - C_*\varepsilon \leq \text{MOT}_\varepsilon - \text{MOT}_0 \leq \frac{1}{2}\left((m-1)d\right)\varepsilon \log(1/\varepsilon) + C^*\varepsilon,}$$

for some constants  $C_*, C^* > 0$ . We remark that it is known from [Pas12; DGN15] that a transport plan  $\gamma_0$  is optimal if and only if it is supported on the set  $\{(x_1, \dots, x_m) \mid \sum_{i=1}^m x_i = l\}$ , where  $l \in \mathbb{R}^d$  is any constant and there exists solutions whose support has dimension exactly  $(m-1)d$ .

*Example 5.4 (Gangbo-Świąćh cost and Wasserstein barycenter).* Suppose that  $c(x_1, \dots, x_m) = \sum_{i < j} |x_i - x_j|^2$ , known as the Gangbo-Świąćh cost [GŚ98]. Notice that the cost is equivalent to  $c(x_1, \dots, x_m) = h(\sum_{i < j} x_i)$  where  $h$  is  $\mathcal{C}^2$  and  $D^2h < 0$ , then the signature of  $\bar{g}$  is  $((m-1)d, d, 0)$  and we have a matching bound

$$\boxed{\frac{1}{2} \left( (m-1)d \right) \varepsilon \log(1/\varepsilon) - C_* \varepsilon \leq \text{MOT}_\varepsilon - \text{MOT}_0 \leq \frac{1}{2} \left( (m-1)d \right) \varepsilon \log(1/\varepsilon) + C^* \varepsilon.}$$

Notice now that considering the  $\text{MOT}_0$  problem with a cost  $c(x_1, \dots, x_m) = \sum_i |x_i - T(x_1, \dots, x_m)|^2$ , where  $T(x_1, \dots, x_m) = \sum_{i=1}^m \lambda_i x_i$  is the Euclidean barycenter, is equivalent to the  $\text{MOT}_0$  with the Gangbo-Świąćh cost and the matching bound above still holds. Moreover, the multi-marginal problem with this particular cost has been shown [AC11] to be equivalent to the Wasserstein barycenter, that is  $T_{\#} \gamma_0 = \nu$  is the barycenter of  $\mu_1, \dots, \mu_m$ .

**Acknowledgments.** L.N. is partially on academic leave at Inria (team Matherials) for the year 2022-2023 and acknowledges the hospitality of this institution during this period. His work was supported by a public grant as part of the Investissement d’avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH and from H-Code, Université Paris-Saclay. P.P. acknowledges the academic leave provided by Inria Paris (team Mokaplan) for the year 2022-2023.

## References

- [Abr+17] I. Abraham, R. Abraham, M. Bergounioux, and G. Carlier. “Tomographic Reconstruction from a Few Views: A Multi-Marginal Optimal Transport Approach”. In: *Applied Mathematics & Optimization* 75.1 (2017), pp. 55–73.
- [AC11] M. Agueh and G. Carlier. “Barycenters in the Wasserstein Space”. In: *SIAM Journal on Mathematical Analysis* 43.2 (2011), pp. 904–924. ISSN: 0036-1410, 1095-7154. DOI: [10.1137/100805741](https://doi.org/10.1137/100805741).
- [BCN19] J.-D. Benamou, G. Carlier, and L. Nenna. “Generalized Incompressible Flows, Multi-Marginal Transport and Sinkhorn Algorithm”. In: *Numerische Mathematik* 142.1 (2019), pp. 33–54. ISSN: 0945-3245. DOI: [10.1007/s00211-018-0995-x](https://doi.org/10.1007/s00211-018-0995-x).
- [BDG12] G. Buttazzo, L. De Pascale, and P. Gori-Giorgi. “Optimal-Transport Formulation of Electronic Density-Functional Theory”. In: *Physical Review A* 85.6 (2012), p. 062502.

- [Ben+15] J.-D. Benamou, G. Carlier, M. Cuturi, L. Nenna, and G. Peyré. “Iterative Bregman Projections for Regularized Transportation Problems”. In: *SIAM Journal on Scientific Computing* 37.2 (2015), A1111–A1138. ISSN: 1064-8275. DOI: [10.1137/141000439](https://doi.org/10.1137/141000439).
- [BHP13] M. Beiglböck, P. Henry-Labordere, and F. Penkner. “Model-Independent Bounds for Option Prices—a Mass Transport Approach”. In: *Finance and Stochastics* 17.3 (2013), pp. 477–501.
- [BK18] J. Bigot and T. Klein. “Characterization of Barycenters in the Wasserstein Space by Averaging Optimal Transport Maps”. In: *ESAIM: Probability and Statistics* 22 (2018), pp. 35–57.
- [BL92] J. M. Borwein and A. S. Lewis. “Decomposition of Multivariate Functions”. In: *Canadian Journal of Mathematics* 44.3 (1992), pp. 463–482. ISSN: 0008-414X, 1496-4279. DOI: [10.4153/CJM-1992-030-9](https://doi.org/10.4153/CJM-1992-030-9).
- [BLN94] J. M. Borwein, A. S. Lewis, and R. D. Nussbaum. “Entropy Minimization, DAD Problems, and Doubly Stochastic Kernels”. In: *Journal of Functional Analysis* 123.2 (1994), pp. 264–307. ISSN: 0022-1236. DOI: [10.1006/jfan.1994.1089](https://doi.org/10.1006/jfan.1994.1089).
- [Bre89] Y. Brenier. “The Least Action Principle and the Related Concept of Generalized Flows for Incompressible Perfect Fluids”. In: *Journal of the American Mathematical Society* 2.2 (1989), pp. 225–255.
- [Car+17] G. Carlier, V. Duval, G. Peyré, and B. Schmitzer. “Convergence of Entropic Schemes for Optimal Transport and Gradient Flows”. In: *SIAM Journal on Mathematical Analysis* 49.2 (2017), pp. 1385–1418. ISSN: 0036-1410. DOI: [10.1137/15M1050264](https://doi.org/10.1137/15M1050264).
- [Car03] G. Carlier. “On a Class of Multidimensional Optimal Transportation Problems”. In: *Journal of Convex Analysis* 10.2 (2003), pp. 517–529. ISSN: 0944-6532.
- [CCG16] G. Carlier, V. Chernozhukov, and A. Galichon. “Vector Quantile Regression: An Optimal Transport Approach”. In: *The Annals of Statistics* 44.3 (2016), pp. 1165–1192.
- [CDD15] M. Colombo, L. De Pascale, and S. Di Marino. “Multimarginal Optimal Transport Maps for One-Dimensional Repulsive Costs”. In: *Canadian Journal of Mathematics* 67.2 (2015), pp. 350–368. ISSN: 0008-414X. DOI: [10.4153/CJM-2014-011-x](https://doi.org/10.4153/CJM-2014-011-x).
- [CE10] G. Carlier and I. Ekeland. “Matching for Teams”. In: *Economic theory* 42.2 (2010), pp. 397–418.
- [CFK13] C. Cotar, G. Friesecke, and C. Klüppelberg. “Density Functional Theory and Optimal Transportation with Coulomb Cost”. In: *Communications on Pure and Applied Mathematics* 66.4 (2013), pp. 548–599. ISSN: 1097-0312. DOI: [10.1002/cpa.21437](https://doi.org/10.1002/cpa.21437).

- [CN08] G. Carlier and B. Nazaret. “Optimal Transportation for the Determinant”. In: *ESAIM. Control, Optimisation and Calculus of Variations* 14.4 (2008), pp. 678–698. ISSN: 1292-8119. DOI: [10.1051/cocv:2008006](https://doi.org/10.1051/cocv:2008006).
- [CPT23] G. Carlier, P. Pegon, and L. Tamanini. “Convergence Rate of General Entropic Optimal Transport Costs”. In: *Calculus of Variations and Partial Differential Equations* 62.4 (2023). Preprint (submitted), p. 116. ISSN: 1432-0835. DOI: [10.1007/s00526-023-02455-0](https://doi.org/10.1007/s00526-023-02455-0).
- [CS16] M. Colombo and F. Stra. “Counterexamples in Multimarginal Optimal Transport with Coulomb Cost and Spherically Symmetric Data”. In: *Mathematical Models & Methods in Applied Sciences* 26.6 (2016), pp. 1025–1049. ISSN: 0218-2025. DOI: [10.1142/S021820251650024X](https://doi.org/10.1142/S021820251650024X).
- [Csi75] I. Csiszar. “ $I$ -Divergence Geometry of Probability Distributions and Minimization Problems”. In: *The Annals of Probability* 3.1 (1975), pp. 146–158. ISSN: 0091-1798, 2168-894X. DOI: [10.1214/aop/1176996454](https://doi.org/10.1214/aop/1176996454).
- [Cut13] M. Cuturi. “Sinkhorn Distances: Lightspeed Computation of Optimal Transport”. In: *Advances in Neural Information Processing Systems*. Vol. 26. Curran Associates, Inc., 2013.
- [DGN15] S. Di Marino, A. Gerolin, and L. Nenna. *Optimal Transportation Theory with Repulsive Costs*. Comment: Survey for the special volume for RICAM (Special Semester on New Trends in Calculus of Variations). 2015. URL: <http://arxiv.org/abs/1506.04565>. preprint.
- [DS14a] Y. Dolinsky and H. M. Soner. “Martingale Optimal Transport and Robust Hedging in Continuous Time”. In: *Probability Theory and Related Fields* 160.1 (2014), pp. 391–427.
- [DS14b] Y. Dolinsky and H. M. Soner. “Robust Hedging with Proportional Transaction Costs”. In: *Finance and Stochastics* 18.2 (2014), pp. 327–347.
- [EN22] S. Eckstein and M. Nutz. *Convergence Rates for Regularized Optimal Transport via Quantization*. Comment: Fixed a typo in Theorem 3.8. 2022. URL: <http://arxiv.org/abs/2208.14391>. preprint.
- [Enn+22] H. Ennaji, Q. Mérigot, L. Nenna, and B. Pass. *Robust Risk Management via Multi-Marginal Optimal Transport*. 2022. DOI: [10.48550/ARXIV.2211.07694](https://doi.org/10.48550/ARXIV.2211.07694). URL: <https://arxiv.org/abs/2211.07694>.
- [Fed96] H. Federer. *Geometric Measure Theory*. Classics in Mathematics. Berlin, Heidelberg: Springer, 1996. ISBN: 978-3-642-62010-2. DOI: [10.1007/978-3-642-62010-2](https://doi.org/10.1007/978-3-642-62010-2).
- [FG97] H. Föllmer and N. Gantert. “Entropy Minimization and Schrödinger Processes in Infinite Dimensions”. In: *The Annals of Probability* 25.2 (1997), pp. 901–926. ISSN: 0091-1798, 2168-894X. DOI: [10.1214/aop/1024404423](https://doi.org/10.1214/aop/1024404423).

- [FL89] J. Franklin and J. Lorenz. “On the Scaling of Multidimensional Matrices”. In: *Linear Algebra and its Applications*. Special Issue Dedicated to Alan J. Hoffman 114–115 (1989), pp. 717–735. ISSN: 0024-3795. DOI: [10.1016/0024-3795\(89\)90490-4](https://doi.org/10.1016/0024-3795(89)90490-4).
- [GKR20] A. Gerolin, A. Kausamo, and T. Rajala. “Multi-Marginal Entropy-Transport with Repulsive Cost”. In: *Calculus of Variations and Partial Differential Equations* 59.3 (2020), p. 90. ISSN: 1432-0835. DOI: [10.1007/s00526-020-01735-3](https://doi.org/10.1007/s00526-020-01735-3).
- [GŚ98] W. Gangbo and A. Świąch. “Optimal Maps for the Multidimensional Monge-Kantorovich Problem”. In: *Communications on Pure and Applied Mathematics* 51.1 (1998), pp. 23–45. ISSN: 1097-0312. DOI: [10.1002/\(SICI\)1097-0312\(199801\)51:1<23::AID-CPA2>3.0.CO;2-H](https://doi.org/10.1002/(SICI)1097-0312(199801)51:1<23::AID-CPA2>3.0.CO;2-H).
- [Haa+21] I. Haasler, R. Singh, Q. Zhang, J. Karlsson, and Y. Chen. “Multi-Marginal Optimal Transport and Probabilistic Graphical Models”. In: *IEEE Transactions on Information Theory* 67.7 (2021), pp. 4647–4668. DOI: [10.1109/TIT.2021.3077465](https://doi.org/10.1109/TIT.2021.3077465).
- [Hei02] H. Heinich. “Problème de Monge Pour n Probabilités”. In: *Comptes Rendus Mathématique. Academie des Sciences. Paris* 334.9 (2002), pp. 793–795. ISSN: 1631-073X. DOI: [10.1016/S1631-073X\(02\)02341-5](https://doi.org/10.1016/S1631-073X(02)02341-5).
- [KM10] Y.-H. Kim and R. J. McCann. “Continuity, Curvature, and the General Covariance of Optimal Transportation”. In: *Journal of the European Mathematical Society* 12.4 (2010), pp. 1009–1040. ISSN: 1435-9855. DOI: [10.4171/jems/221](https://doi.org/10.4171/jems/221).
- [KP14] Y.-H. Kim and B. Pass. “A General Condition for Monge Solutions in the Multi-Marginal Optimal Transport Problem”. In: *Siam Journal On Mathematical Analysis* 46.2 (2014), pp. 1538–1550. ISSN: 0036-1410. DOI: [10.1137/130930443](https://doi.org/10.1137/130930443).
- [KP15] Y.-H. Kim and B. Pass. “Multi-Marginal Optimal Transport on Riemannian Manifolds”. In: *American Journal of Mathematics* 137.4 (2015), pp. 1045–1060. ISSN: 0002-9327. DOI: [10.1353/ajm.2015.0024](https://doi.org/10.1353/ajm.2015.0024).
- [Léo14] C. Léonard. “A Survey of the Schrödinger Problem and Some of Its Connections with Optimal Transport”. In: *Discrete and Continuous Dynamical Systems* 34.4 (2014), p. 1533. DOI: [10.3934/dcds.2014.34.1533](https://doi.org/10.3934/dcds.2014.34.1533).
- [MG20] S. D. Marino and A. Gerolin. “An Optimal Transport Approach for the Schrödinger Bridge Problem and Convergence of Sinkhorn Algorithm”. In: *Journal of Scientific Computing* 85.2 (2020), p. 27. ISSN: 1573-7691. DOI: [10.1007/s10915-020-01325-7](https://doi.org/10.1007/s10915-020-01325-7).
- [MP17] A. Moameni and B. Pass. “Solutions to Multi-Marginal Optimal Transport Problems Concentrated on Several Graphs”. In: *ESAIM. Control, Optimisation and Calculus of Variations* 23.2 (2017), pp. 551–567. ISSN: 1292-8119. DOI: [10.1051/cocv/2016003](https://doi.org/10.1051/cocv/2016003).

- [MS23] H. Malamut and M. Sylvestre. *Convergence Rates of the Regularized Optimal Transport : Disentangling Suboptimality and Entropy*. 2023. URL: <http://arxiv.org/abs/2306.06940>. preprint.
- [Nen16] L. Nenna. “Numerical Methods for Multi-Marginal Optimal Transportation”. PhD thesis. Université Paris sciences et lettres, 2016.
- [NW22] M. Nutz and J. Wiesel. “Stability of Schrödinger Potentials and Convergence of Sinkhorn’s Algorithm”. 2022.
- [Pas11] B. Pass. “Uniqueness and Monge Solutions in the Multimarginal Optimal Transportation Problem”. In: *SIAM Journal on Mathematical Analysis* 43.6 (2011), pp. 2758–2775. ISSN: 0036-1410. DOI: [10.1137/100804917](https://doi.org/10.1137/100804917).
- [Pas12] B. Pass. “On the Local Structure of Optimal Measures in the Multi-Marginal Optimal Transportation Problem”. In: *Calculus of Variations and Partial Differential Equations* 43.3 (2012), pp. 529–536. ISSN: 1432-0835. DOI: [10.1007/s00526-011-0421-z](https://doi.org/10.1007/s00526-011-0421-z).
- [Pas15] B. Pass. “Multi-Marginal Optimal Transport: Theory and Applications”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 49.6 (6 2015), pp. 1771–1790. ISSN: 0764-583X, 1290-3841. DOI: [10.1051/m2an/2015020](https://doi.org/10.1051/m2an/2015020).
- [PC19] G. Peyré and M. Cuturi. “Computational Optimal Transport: With Applications to Data Science”. In: *Foundations and Trends in Machine Learning* 11.5-6 (2019), pp. 355–607. ISSN: 1935-8237, 1935-8245. DOI: [10.1561/22000000073](https://doi.org/10.1561/22000000073).
- [PV21a] B. Pass and A. Vargas-Jiménez. “Monge Solutions and Uniqueness in Multi-Marginal Optimal Transport via Graph Theory”. 2021.
- [PV21b] B. W. Pass and A. Vargas-Jiménez. “Multi-Marginal Optimal Transportation Problem for Cyclic Costs”. In: *Siam Journal On Mathematical Analysis* 53.4 (2021), pp. 4386–4400. ISSN: 0036-1410. DOI: [10.1137/19M130889X](https://doi.org/10.1137/19M130889X).
- [Rab+11] J. Rabin, G. Peyré, J. Delon, and M. Bernot. “Wasserstein Barycenter and Its Application to Texture Mixing”. In: *International Conference on Scale Space and Variational Methods in Computer Vision*. Springer. 2011, pp. 435–446.
- [Rén59] A. Rényi. “On the Dimension and Entropy of Probability Distributions”. In: *Acta Mathematica Academiae Scientiarum Hungarica* 10.1 (1959), pp. 193–215. ISSN: 1588-2632. DOI: [10.1007/BF02063299](https://doi.org/10.1007/BF02063299).
- [RT98] L. Rüschendorf and W. Thomsen. “Closedness of Sum Spaces and the Generalized Schrödinger Problem”. In: *Theory of Probability & Its Applications* 42.3 (1998), pp. 483–494. ISSN: 0040-585X. DOI: [10.1137/S0040585X97976301](https://doi.org/10.1137/S0040585X97976301).
- [San15] F. Santambrogio. *Optimal Transport for Applied Mathematicians*. Vol. 87. Progress in Nonlinear Differential Equations and Their Applications. Calculus of variations, PDEs, and modeling. Cham: Springer International Publishing, 2015. ISBN: 978-3-319-20828-2. DOI: [10.1007/978-3-319-20828-2](https://doi.org/10.1007/978-3-319-20828-2).

- [Sin64] R. Sinkhorn. “A Relationship Between Arbitrary Positive Matrices and Doubly Stochastic Matrices”. In: *The Annals of Mathematical Statistics* 35.2 (1964), pp. 876–879. ISSN: 0003-4851, 2168-8990. DOI: [10 . 1214 / aoms / 1177703591](https://doi.org/10.1214/aoms/1177703591).
- [TJK22] N. G. Trillos, M. Jacobs, and J. Kim. *The Multimarginal Optimal Transport Formulation of Adversarial Multiclass Classification*. 2022. DOI: [10.48550/ARXIV.2204.12676](https://doi.org/10.48550/ARXIV.2204.12676). URL: <https://arxiv.org/abs/2204.12676>.
- [You82] L.-S. Young. “Dimension, Entropy and Lyapunov Exponents”. In: *Ergodic Theory and Dynamical Systems* 2.1 (1982), pp. 109–124. ISSN: 1469-4417, 0143-3857. DOI: [10.1017/S0143385700009615](https://doi.org/10.1017/S0143385700009615).