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► **To cite this version:**

Fernando Chamizo, Bruno Martin. The approximate functional equation of some Diophantine series. Monatshefte für Mathematik, 2023, 10.1007/s00605-023-01859-6 . hal-04154260

HAL Id: hal-04154260

<https://hal.science/hal-04154260>

Submitted on 6 Jul 2023

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The approximate functional equation of some Diophantine series

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July 13, 2022

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Abstract

We prove that a family of Diophantine series satisfies an approximate functional equation. It generalizes a result by Rivoal and Roques and proves an extended version of a conjecture posed in their paper. We also characterize the convergence points.

1 Introduction

Consider $g : \mathbb{R} \rightarrow \mathbb{R}$ an odd 1-periodic C^1 function and $f : \mathbb{R} - \mathbb{Z} \rightarrow \mathbb{R}$ a 1-periodic continuous function such that

$$(1.1) \quad L = \lim_{x \rightarrow 0} \left(f(x) - \frac{\lambda}{x} \right)$$

exists (and it is finite) for some $\lambda \neq 0$. This means that the only singularities of f , as a real function, are simple pole like singularities at the integers. For $(w, \alpha) \in (\mathbb{R}^+)^2$, we introduce the Diophantine sum

$$(1.2) \quad \Phi_w(\alpha) = \sum_{m=1}^{\lfloor w \rfloor} \frac{g(m^2\alpha)}{m^2} f(m\alpha)$$

where $\Phi_w(\alpha)$ is defined by continuity for $\alpha = p/q \in \mathbb{Q}$ with $q \leq w$. Note that it makes sense because $g(x) \sim C(x-n)$ when $x \rightarrow n \in \mathbb{Z}$. Note also that with these assumptions, we have the uniform trivial bound $g(m^2\alpha)f(m\alpha) \ll m$.

Motivated by a Diophantine approximation problem raised in [7], Rivoal and Roques proved in [9] that when $g(x) = \sin(2\pi x)$ and $f(x) = \cot(\pi x)$,

*Partially supported by the PID2020-113350GB-I00 grant of the MICINN (Spain) and by “Severo Ochoa Programme for Centres of Excellence in R&D” (CEX2019-000904-S).

†Supported by the joint FWF-ANR project Arithrand: FWF: I 4945-N and ANR-20-CE91-0006.

$\Phi_N(\alpha)$ satisfies an approximate functional equation for $N \in \mathbb{Z}^+$. Namely, they show that

$$(1.3) \quad G_N(\alpha) = \Phi_N(\alpha) - \alpha\Phi_{N\alpha}(1/\alpha) - \frac{1}{\pi\alpha} \sum_{m=1}^N \frac{\sin(2\pi m^2\alpha)}{m^3}$$

has a limit when N goes to infinity for each $\alpha \in (0, 1]$, and that it is uniformly bounded in this interval. As a matter of fact, the last term could be replaced by $\log(1/\alpha)$. They conjecture that the limiting function is not only bounded but continuous. The existence of this limiting function contrasts with the fact, also proved for this choice of g and f in [9] (see also [7] for a weaker result), that $\lim_{N \rightarrow \infty} \Phi_N(\alpha)$ exists if and only if α is a Brjuno number, that is, an irrational number such that the convergents p_n/q_n in its continued fraction satisfy $\sum q_n^{-1} \log q_{n+1} < \infty$.

As an aside, if we choose formally g as a constant, even replacing n^2 by n^s , approximate functional equations of similar kind (very explicit ones in some cases) can be established, although the convergence conditions are tighter (see [8], [2], [5]). These variants are related to some formulas of Ramanujan for ζ at odd values (see [10]) and to previous works of Lerch (see [6]). For other references and a historical account on approximate functional equations, see [1, §3].

The proof of the existence of the limiting function given in [9] uses heavily the additive properties of the sine function and the partial fraction expansion of the cotangent. We show here that this approximate functional equation also holds with the general definition of Φ_N as above and that it can be deduced from a mainly combinatorial argument not depending on the choice of f and g . We also prove the continuity of the limiting function for $\alpha > 0$ and its continuous extension to $[0, \infty)$, settling in particular the conjecture posed in [9]. In the last section we apply a general result of [1] for certain approximate functional equations to characterize completely the convergence points of Φ_N when $N \rightarrow \infty$.

Theorem 1.1. *For $N \in \mathbb{Z}^+$ and $\alpha \in (0, 1]$ consider*

$$G_N(\alpha) = \Phi_N(\alpha) - \alpha\Phi_{N\alpha}(1/\alpha) - T_N(\alpha) \quad \text{with} \quad T_N(\alpha) = \frac{\lambda}{\alpha} \sum_{m=1}^N \frac{g(m^2\alpha)}{m^3}$$

and Φ_N as in (1.2). Then the sequence of functions $\{G_N\}_{N=1}^\infty$ is uniformly bounded on $(0, 1]$ and it converges uniformly on each compact interval $I \subset (0, 1]$ to a continuous function $G(\alpha) = O(\alpha \log(2\alpha^{-1}))$.

REMARK. The limit of $T_N(\alpha)$ when $N \rightarrow \infty$ can be taken separately because the series converges but, as pointed out before, the separate existence of the limits of $\Phi_N(\alpha)$ and $\Phi_{N\alpha}(1/\alpha)$ is not assured in general (see Theorem 4.3). In [9], in the case $g(x) = \sin(2\pi x)$ and $f(x) = \cot(\pi x)$, the

value $G(0) = 0$ is implicitly assigned. We prefer here to let it undefined and to link it to the continuous extension through $G(0^+) = 0$. Once we have stated the convergence in $(0, 1]$ the extension to $\alpha > 0$ is rather easy.

Corollary 1.2. *If we let $\alpha \in \mathbb{R}^+$, $G(\alpha) = \lim G_N(\alpha)$ defines a continuous function on \mathbb{R}^+ satisfying $G(\alpha) = O(\alpha \log(\alpha^{-1} + \alpha))$. In particular $G(0^+) = 0$ and G extends to a continuous function on $[0, \infty)$.*

Notation. Given a real number x , $\|x\|$ stands for the distance from x to its nearest integer. Vinogradov's notation $A \ll B$ is employed here with the same meaning as $A = O(B)$.

2 Some reductions

We begin with some trivial remarks that eventually lead to some reductions in the proofs of Theorem 1.1 and Corollary 1.2.

Lemma 2.1. *If λ is as in (1.1), the series*

$$\sum_{m=1}^{\infty} \frac{g(m^2\alpha)}{m^2} \left(f(m\alpha) - \lambda\pi \cot(\pi m\alpha) \right)$$

converges uniformly to a continuous bounded function on \mathbb{R} .

Proof. We know that $f(x) - \lambda/x = O(1)$ and $\pi \cot(\pi x) - 1/x = O(x)$ for $x \rightarrow 0$. Then $f(x) - \lambda\pi \cot(\pi x)$ is bounded in a neighborhood of 0 and, by the 1-periodicity, it is bounded on \mathbb{R} . \square

Lemma 2.2. *Assume that $\Omega(\alpha) = \lim_{N \rightarrow \infty} (\Phi_N(\alpha) - \alpha\Phi_{N\alpha}(1/\alpha))$ is well defined for a given $\alpha > 0$. Then $\Omega(1/\alpha)$ is also well defined and we have $\Omega(\alpha) + \alpha\Omega(1/\alpha) = 0$.*

Proof. It follows easily from the definition of $\Omega(\alpha)$. \square

Lemma 2.3. *For $f(x) = \cot(\pi x)$ we have*

$$\Phi_N(\alpha) = \sum_{m=1}^N \frac{g(m\|m\alpha\|)}{m^2} f(\|m\alpha\|).$$

Proof. If $\|m\alpha\| = \delta$, there exists $n \in \mathbb{Z}$ such that $m\alpha - n = \pm\delta$. Then $f(m\alpha) = f(\pm\delta)$ and $g(m^2\alpha) = g(mn \pm m\delta) = f(\pm m\delta)$. When taking the product, the signs cancel because both functions f and g are odd. \square

The first two reductions and some simple manipulations with T_N are enough to conclude Corollary 1.2. We abbreviate $\lim_{N \rightarrow \infty} T_N$ as T .

Proof of Corollary 1.2. By Lemma 2.1, it is enough to consider the case $f(x) = \cot(\pi x)$. Theorem 1.1 assures the existence and the continuity of G in $(0, 1]$ which are reflected into $[1, \infty)$ thanks to Lemma 2.2. The continuous extension to $[0, \infty)$ follows from the bound $G(\alpha) = O(\alpha \log(2\alpha)^{-1})$. The only missing point is the bound $G(\alpha) = O(\alpha \log \alpha)$ for large α . By Lemma 2.2, for $\alpha > 0$,

$$\alpha^{-1}G(\alpha) = -G(\alpha^{-1}) - \alpha^{-1}T(\alpha) - T(\alpha^{-1}).$$

By Theorem 1.1, $G(\alpha^{-1}) \rightarrow 0$ when $\alpha \rightarrow \infty$ and $\alpha^{-1}T(\alpha) \rightarrow 0$ is obvious. Finally, since $g(x)/x$ is bounded we have

$$\alpha \sum_{m=1}^{\infty} \frac{g(m^2/\alpha)}{m^3} \ll 1 + \sum_{m < \sqrt{\alpha}} \frac{g(m^2/\alpha)\alpha/m^2}{m} \ll \sum_{m < \sqrt{\alpha}} \frac{1}{m} \ll \log \alpha,$$

showing that $T(\alpha^{-1}) = O(\log \alpha)$ when $\alpha \rightarrow \infty$. \square

3 Proof of the main result

The key argument to prove Theorem 1.1 is that the terms in $\alpha\Phi_{N\alpha}(1/\alpha)$ are almost completely canceled by the terms in $\sum_{m=1}^N \frac{g(m\|m\alpha\|)}{m^2} f(\|m\alpha\|)$ such that $m\alpha$ is close to a positive integer. The following elementary result will be used to identify precisely these integers m . We remind the reader that if $y \in \mathbb{R}$ is not a half-integer, then $\lfloor y + 1/2 \rfloor$ gives the nearest integer to y . Note that for any $y \in \mathbb{R}$, $y = \lfloor y + 1/2 \rfloor \pm \|y\|$.

Lemma 3.1. *For $\alpha \in (0, 1]$ fixed, the map $r_\alpha : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ given by the formula $r_\alpha(n) = \lfloor n/\alpha + 1/2 \rfloor$ is one-to-one and for $m \in \text{Im } r_\alpha$ it can be inverted with $r_\alpha^{-1}(m) = \lfloor m\alpha + 1/2 \rfloor$.*

Proof. We can write $m = r_\alpha(n)$ as $m + \delta = n/\alpha + 1/2$ for some $0 \leq \delta < 1$. Eliminating n we have $n = m\alpha + (\delta - 1/2)\alpha$ and so n is the nearest integer to $m\alpha$ because $|(\delta - 1/2)\alpha| < 1/2$ (the case $(\delta, \alpha) = (0, 1)$ cannot occur since it would imply $m = n \pm 1/2$). Consequently, $n = \lfloor m\alpha + 1/2 \rfloor$. This proves that r_α is injective, hence the conclusion. \square

An important, and still elementary, remark is that although r_α may not map $[1, N\alpha] \cap \mathbb{Z}$ onto $[1, N] \cap \text{Im } r_\alpha$, the surjectivity only may fail for one element.

Lemma 3.2. *For $N \in \mathbb{Z}_{>1}$ and $\alpha \in (0, 1]$, let $N^* = N$ if $r_\alpha^{-1}([1, N]) \subset [1, N\alpha]$ and $N^* = N - 1$ otherwise. Then*

$$r_\alpha : [1, N\alpha] \cap \mathbb{Z} \rightarrow [1, N^*] \cap \text{Im } r_\alpha$$

is a bijection.

Proof. Lemma 3.1 assures the injectivity. If $N^* = N$ the surjectivity and the well-definition are obvious. If $N^* = N - 1$, they follow because $N^* \geq m \in \text{Im } r_\alpha$ implies $N - 1 \geq m = \lfloor n/\alpha + 1/2 \rfloor$ and $n > N\alpha$ would give a contradiction. \square

In the sequel, ψ is a fixed 1-periodic continuous even function such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ for $x \in [-1/4, 1/4]$ and $\psi(x) = 0$ for $x \in [3/8, 5/8]$.

In the following result, we quantify the cancellation phenomenon described above between $\alpha\Phi_{N\alpha}(1/\alpha)$ and $\Phi_N(\alpha)$. It is the main responsible for the very existence of an approximate functional equation.

Lemma 3.3. *Given $n \in \mathbb{Z}^+$, consider $f_n : (0, 1] \rightarrow \mathbb{R}$ given by*

$$f_n(\alpha) = \frac{n^2}{m^2} g(m\|m\alpha\|) \cot(\pi\|m\alpha\|) - \alpha g(n\|n/\alpha\|) \cot(\pi\|n/\alpha\|)$$

with $m = r_\alpha(n)$. Then $f_n(\alpha) = O(\alpha)$ with an O -constant only depending on g . Moreover, $\psi(n/\alpha)f_n(\alpha)$ is continuous and bounded on $(0, 1]$.

Proof. Let us abbreviate $\delta = \|m\alpha\|$. By Lemma 3.1, $n = \lfloor m\alpha + 1/2 \rfloor$, then $m - n/\alpha = \pm\delta/\alpha$ and, since $m = \lfloor n/\alpha + 1/2 \rfloor$, we have $\|n/\alpha\| = \delta/\alpha$. It follows that $f_n(\alpha) = n^2 g(m\delta)/m^2 \cot(\pi\delta) - \alpha g(n\delta/\alpha) \cot(\pi\delta/\alpha)$. Since $\alpha \cot(\pi\delta/\alpha) - \alpha^2 \cot(\pi\delta) = O(\delta)$,

$$f_n(\alpha) = \alpha^2 \cot(\pi\delta) \left(\frac{\alpha^{-2}n^2}{m^2} g(m\delta) - g(n\delta/\alpha) \right) + O(\delta).$$

The relation $m - n/\alpha = \pm\delta/\alpha$ implies $\alpha^{-2}n^2/m^2 = 1 + O(\alpha^{-1}m^{-1}\delta)$. Since g is differentiable, we also have $g(m\delta) - g(n\delta/\alpha) \ll |m - n/\alpha|\delta \ll \delta^2/\alpha$. These bounds give

$$f_n(\alpha) \ll \alpha^2 \delta^{-1} (\alpha^{-1}\delta^2 + \alpha^{-1}m^{-1}\delta) + \delta \ll \alpha$$

as expected.

For the continuity, note that f_n is a composition of continuous functions except for the function $\alpha \mapsto r_\alpha(n)$ giving m , which is discontinuous when n/α is a half-integer. These discontinuities are eliminated by the factor $\psi(n/\alpha)$ because ψ vanishes at the half-integers. \square

Lemma 3.4. *Given $m \in \mathbb{Z}^+$, the function*

$$h_m(\alpha) = \begin{cases} 1 & \text{if } m \notin \text{Im } r_\alpha, \\ 1 - \psi(\alpha^{-1}\|m\alpha\|) & \text{if } m \in \text{Im } r_\alpha \end{cases}$$

is continuous on $(0, 1]$.

Proof. By definition, $m \in \text{Im } r_\alpha$ means that $n/\alpha + 1/2 = m + \delta$ for some $0 \leq \delta < 1$ and some $n \in \mathbb{Z}^+$ and this is equivalent to

$$\alpha \in \bigcup_{n=1}^{\infty} \left(\frac{2n}{2m+1}, \frac{2n}{2m-1} \right] \cap (0, 1].$$

Note that $\alpha \mapsto \psi(\alpha^{-1}\|m\alpha\|)$ is continuous on $(0, 1]$. Hence, if we check that both cases in the definition of h_m coincide at the boundary points $\alpha = 2n/(2m+1)$ and $\alpha = 2n/(2m-1)$ for $1 \leq n < m$ then h_m is continuous by the pasting lemma. This is the same as checking that $\psi(\alpha^{-1}\|m\alpha\|)$ vanishes at $\alpha = 2n/(2m \pm 1)$. Writing $m\alpha = n \mp n/(2m \pm 1)$, we have $\alpha^{-1}\|m\alpha\| = \alpha^{-1}n/(2m \pm 1) = 1/2$ and $\psi(1/2) = 0$. \square

The proof of Theorem 1.1 is based on a rearrangement of the terms in $\Phi_N(\alpha) - \alpha\Phi_{N\alpha}(1/\alpha)$ involving the functions f_n and h_m introduced in the previous lemmas. In the rest of this section we assume $f(x) = \cot(\pi x)$ and consequently $\lambda = 1/\pi$ in the definition of T_N . We can do it without loss of generality by Lemma 2.1.

For $w \in \mathbb{R}_+$ and $\alpha \in (0, 1]$, we set

$$\begin{aligned} S_w^{(1)}(\alpha) &= \sum_{n=1}^{\lfloor w \rfloor} \psi(n/\alpha) \frac{f_n(\alpha)}{n^2}, \\ S_w^{(2)}(\alpha) &= \alpha \sum_{n=1}^{\lfloor w \rfloor} (1 - \psi(n/\alpha)) \frac{g(n\|n/\alpha\|)}{n^2} \cot(\pi\|n/\alpha\|), \\ S_w^{(3)}(\alpha) &= \sum_{m=1}^{\lfloor w \rfloor} h_m(\alpha) \frac{g(m\|m\alpha\|)}{m^2} \cot(\pi\|m\alpha\|) \end{aligned}$$

and we denote by $S^{(j)}(\alpha)$ the corresponding infinite series.

Lemma 3.5. *For N, N^* and α as in Lemma 3.2, we have the decomposition*

$$(3.1) \quad \Phi_{N^*}(\alpha) - \alpha\Phi_{N\alpha}(1/\alpha) = S_{N\alpha}^{(1)}(\alpha) - S_{N\alpha}^{(2)}(\alpha) + S_{N^*}^{(3)}(\alpha).$$

Proof. Let us set $a_m = g(m\|m\alpha\|) \cot(\pi\|m\alpha\|)/m^2$ and write $a_m = b_m + c_m$ with $b_m = a_m$ if $m \in \text{Im } r_\alpha$ and $b_m = 0$ otherwise. With this notation, according to Lemma 2.3, we have $\Phi_{N^*}(\alpha) = \sum_{m=1}^{N^*} (b_m + c_m)$. Lemma 3.3 and Lemma 3.2 show that

$$\begin{aligned} \sum_{m=1}^{N^*} b_m - \alpha\Phi_{N\alpha}(1/\alpha) &= \sum_{n=1}^{\lfloor N\alpha \rfloor} \frac{f_n(\alpha)}{n^2} = S_{N\alpha}^{(1)}(\alpha) + \sum_{n=1}^{\lfloor N\alpha \rfloor} \left(1 - \psi\left(\frac{n}{\alpha}\right)\right) \frac{f_n(\alpha)}{n^2} \\ &= S_{N\alpha}^{(1)}(\alpha) - S_{N\alpha}^{(2)}(\alpha) + \sum_{m=1}^{N^*} \left(1 - \psi\left(\frac{r_\alpha^{-1}(m)}{\alpha}\right)\right) b_m \end{aligned}$$

where in the last sum, abusing of the notation, we assume that the undefined values of $r_\alpha^{-1}(m)$ when $m \notin \text{Im } r_\alpha$ are ignored because $b_m = 0$. Using the identity $m\alpha = r_\alpha^{-1}(m) \pm \|m\alpha\|$ for $m \in \text{Im } r_\alpha$ and recalling that ψ is even and 1-periodic, the last sum is $\sum_{m=1}^{N^*} h_m(\alpha) b_m = S_{N^*}^{(3)}(\alpha) - \sum_{m=1}^{N^*} c_m$. \square

Lemma 3.6. *For each $j \in \{1, 2, 3\}$, the series $S^{(j)}(\alpha)$ converges normally to a continuous function on each compact interval $I \subset (0, 1]$.*

Proof. It is enough to check that the three functions $\psi(n/\alpha)f_n(\alpha)$, $(1 - \psi(n/\alpha)) \cot(\pi\|n/\alpha\|)$ and $h_m(\alpha) \cot(\pi\|m\alpha\|)$ are continuous and bounded on each compact interval $I \subset (0, 1]$. For the first function, this is the content of Lemma 3.3. For the second, it follows from the fact that $\psi(x) = 1$ when $\|x\| \leq 1/4$. For the third, it follows from Lemma 3.4, noting that when $\|m\alpha\| < \alpha/4$, we have $m \in \text{Im } r_\alpha$ and $\psi(\alpha^{-1}\|m\alpha\|) = 1$ by the defining properties of ψ . \square

Lemma 3.7. *For $N \geq 1$ and $\alpha \in (0, 1]$,*

$$S_{N\alpha}^{(j)}(\alpha) \ll \alpha \quad \text{for } j = 1, 2 \quad \text{and} \quad S_N^{(3)}(\alpha) - T_N(\alpha) \ll \alpha \log(2\alpha^{-1}).$$

Proof. By Lemma 3.3, we have $S_{N\alpha}^{(1)}(\alpha) = O(\alpha)$ and the same bound applies to $S_{N\alpha}^{(2)}(\alpha)$ because $(1 - \psi(n/\alpha)) \cot(\pi\|n/\alpha\|)$ is uniformly bounded. It remains to prove that

$$\sum_{m=1}^N \left(h_m(\alpha) \frac{g(m^2\alpha)}{m^2} \cot(\pi m\alpha) - \frac{g(m^2\alpha)}{\pi\alpha m^3} \right) = O(\alpha \log(2\alpha^{-1})).$$

For $m \leq \alpha^{-1}/2$, clearly $m \notin \text{Im } r_\alpha$ and hence $h_m(\alpha) = 1$. Using $\cot x - 1/x = O(x)$, we obtain that the part of the sum corresponding to this range is $O(\alpha \log(2\alpha^{-1}))$. On the other hand, in the complementary range $m > \alpha^{-1}/2$, we have $\sum g(m^2\alpha)/m^3 \ll \alpha^2$ and it only remains to show that the part of the sum involving h_m is $O(\alpha \log(2\alpha^{-1}))$ in this complementary range. We deduce it from the bounds

$$\sum_{m > \frac{1}{2}\alpha^{-1}} h_m(\alpha) \frac{|\cot(\pi m\alpha)|}{m^2} \ll \sum_{\substack{m > \frac{1}{2}\alpha^{-1} \\ \|m\alpha\| > \alpha/4}} \frac{1}{m^2 \|m\alpha\|} \ll \sum_{q=1}^{\infty} \frac{1}{(q\alpha^{-1})^2} \sum_{k < \frac{1}{2}\alpha^{-1}} \frac{1}{\alpha k}.$$

The last inequality follows by writing $m = q\lfloor \alpha^{-1}/2 \rfloor + r$ with $0 \leq r < \lfloor \alpha^{-1}/2 \rfloor$ and by noting that $\|m\alpha\|$ presents gaps of size α when r varies. The last double sum amounts $O(\alpha \log(2\alpha^{-1}))$. \square

Proof of Theorem 1.1. Recall that we have defined $T(\alpha) = \lim T_N(\alpha)$. The convergence is uniform, hence T is continuous on $(0, 1]$. On the other hand, the trivial bound gives $\Phi_N(\alpha) - \Phi_{N^*}(\alpha) \ll N^{-1}$. Hence, considering the

decomposition (3.1), the uniform convergence of $G_N(\alpha)$ on every interval $[a, b] \subset (0, 1]$ to

$$(3.2) \quad G(\alpha) = S^{(1)}(\alpha) - S^{(2)}(\alpha) + S^{(3)}(\alpha) - T(\alpha)$$

follows from Lemma 3.6. The uniform boundedness of $G_N(\alpha)$ on $(0, 1]$ and the bound $G(\alpha) = O(\alpha \log(2\alpha^{-1}))$ are consequences of Lemma 3.7. \square

We can also state a result à la Hardy-Littlewood (cf. [3, Th. 2.128]).

Proposition 3.8. *For $N \geq 1$ and $\alpha \in (0, 1]$, we have*

$$\Phi_N(\alpha) - \alpha \Phi_{N\alpha}(1/\alpha) = T_N(\alpha) + G(\alpha) + O(\min((N\alpha)^{-1}, \alpha \log(2\alpha^{-1})) + N^{-1}).$$

Proof. The trivial bound proves that $\Phi_N(\alpha) - T_N(\alpha)$ and $\Phi_{N^*}(\alpha) - T_{N^*}(\alpha)$ differ in $O(N^{-1})$ which is absorbed by the error term, then it is enough to prove that $\Phi_{N^*}(\alpha) - \alpha \Phi_{N\alpha}(1/\alpha) - T_{N^*}(\alpha) - G(\alpha)$ is $\ll \alpha \log(2\alpha^{-1})$ and $\ll (N\alpha)^{-1}$. Thanks to (3.1) and (3.2), it equals

$$(S_{N\alpha}^{(1)}(\alpha) - S^{(1)}(\alpha)) - (S_{N\alpha}^{(2)}(\alpha) - S^{(2)}(\alpha)) + (S_{N^*}^{(3)}(\alpha) - T_{N^*}(\alpha)) - (S^{(3)}(\alpha) - T(\alpha))$$

which is $\ll \alpha \log(2\alpha^{-1})$ according to Lemma 3.7. On the other hand, arranging the last two terms as $(S_{N^*}^{(3)}(\alpha) - S^{(3)}(\alpha)) + (T(\alpha) - T_{N^*}(\alpha))$ and following the proof of Lemma 3.6, we have $S_{N\alpha}^{(j)}(\alpha) - S^{(j)}(\alpha) \ll (N\alpha)^{-1}$ for $j = 1, 2$, $S_{N^*}^{(3)}(\alpha) - S^{(3)}(\alpha) \ll N^{-1}$ and trivially $T(\alpha) - T_{N^*}(\alpha) \ll N^{-2}\alpha^{-1}$. \square

4 Convergence

If we slightly strengthen the regularity requirements for g , it is possible to characterize the convergence points of Φ_N when $N \rightarrow \infty$, extending [9, Th. 2]. Namely, we impose in this section $g \in C^{1,\gamma}$ for some $0 < \gamma < 1$. This means that g' satisfies locally the Hölder condition $g'(y) - g'(x) = O(|y-x|^\gamma)$. In fact it is enough to require it for $x = 0$.

The characterization of the convergence points is a quite direct application of a general result in [1] on approximate functional equations (see [1, Prop. 2] and [1, §6.2]). The point to be checked is a simple analytic fact.

Lemma 4.1. *Let g be a bounded function in $C^{1,\gamma}$ for some $0 < \gamma < 1$ with $g(0) = 0$. Then*

$$\frac{1}{x} \sum_{m=1}^{\infty} \frac{g(m^2 x)}{m^3} - \frac{1}{2} g'(0) \log \frac{1}{x}$$

is a bounded function on $(0, 1]$.

Proof. We can restrict the sum to $m < x^{-1/2}$ because the rest of the terms give a bounded contribution to the function by the trivial estimate. On the other hand, we know that $g'(\xi) - g'(0) = O(|\xi|^\gamma)$ and the mean value theorem shows that $g(t) = g(0) + g'(0)t + O(t^{\gamma+1})$ for $0 \leq t < 1$. Then

$$\sum_{m < x^{-1/2}} \frac{g(m^2x)/(m^2x)}{m} = \sum_{m < x^{-1/2}} \frac{g'(0)}{m} + O\left(\sum_{m < x^{-1/2}} x^\gamma m^{2\gamma-1}\right).$$

The first sum is $g'(0) \log x^{-1/2} + O(1)$ and the second is $O(1)$. \square

The previous result shows that the case $g'(0) = 0$ is special and that in fact, the sum in Theorem 1.1 could be omitted. The series appearing in the next result is a model of what happens in this case under $g \in C^{1,\gamma}$. The purpose of introducing it here is to give a common treatment to $g'(0) \neq 0$ and $g'(0) = 0$ in the proof of Theorem 4.3 (in fact, the absolute convergence in the latter case can be obtained as a quite direct consequence of it). A secondary reason is to show that an extra factor $\|n^2x\|^\gamma$ forces the convergence everywhere of the series in [7, §4] (recall that $\|n\|nx\| = \|n^2x\|$).

Lemma 4.2. *Let $\gamma > 0$. The series*

$$\sum_{n=1}^{\infty} \frac{\|n\|nx\|^{\gamma+1}}{n^2\|nx\|}$$

converges for every $x \in \mathbb{R}$ and determines a bounded function, where the terms with $\|nx\| = 0$ are defined (by continuity) as 0.

Proof. Of course, $\|nx\| = 0$ only occurs if $x = a/q \in \mathbb{Q}$ with $q \mid n$. On the other hand, $\sum_{n=1}^{q-1} \|na/q\|^{-1} \ll q \log q$ if $q \nmid n$. Then subdividing the range of n into blocks of length q the series for $x = a/q$ is majorized by $\sum_{k=1}^{\infty} (qk)^{-2} q \log q$, which converges and is uniformly bounded.

For $x \notin \mathbb{Q}$, let S be the part of the series corresponding to $q \leq n < Q$ with p/q and P/Q consecutive convergents in the continued fraction of x . It is enough to prove $S = O(q^{-\sigma})$ for some $\sigma > 0$ because the denominators of the convergents grow exponentially [4, Th.12]. Replacing $\|n\|nx\|$ by 1, the terms in S with $q \nmid n$ contribute $O(q^{-1} \log(2q))$ as shown in the proof of [2, Prop.3.1] (where they are labeled S_2 and S_3) taking $k = 1$, $s = 2$. If $q \mid n$, [2, Lem. 2.3] proves that $n\|nx\| \leq 1$ and $n\|nx\| \geq 1$ imply, respectively, $n \ll \sqrt{qQ}$ and $n \gg \sqrt{qQ}$ because $qQ\|nx\| \asymp n$. Summing up,

$$S \ll \sum_{\substack{q \leq n \ll \sqrt{qQ} \\ q \mid n}} \frac{(n^2/qQ)^{1+\gamma}}{n^2(n/qQ)} + \sum_{\substack{\sqrt{qQ} \ll n < Q \\ q \mid n}} \frac{1}{n^2(n/qQ)} + \frac{\log(2q)}{q}.$$

Writing $n = kq$, we obtain that each sum is $O(q^{-1})$. \square

Now, we are ready to decide about the convergence of Φ_N when $N \rightarrow \infty$. To keep the coherence with the previous parts, we state the result for positive numbers, but the periodicity allows to extend it to \mathbb{R} .

Theorem 4.3. *Let f and g as in the introduction but imposing also $g \in C^{1,\gamma}$ for some $0 < \gamma < 1$. For $x \in \mathbb{R}^+$ and $g'(0) \neq 0$, the series*

$$\sum_{m=1}^{\infty} \frac{g(m^2x)}{m^2} f(mx)$$

converges if and only if x is a Brjuno number. On the other hand, it converges for every $x \in \mathbb{R}^+$ if $g'(0) = 0$.

Proof. By the 1-periodicity of the series, we can restrict ourselves to $x \in (0, 1]$. And we can also suppose that $f(x) = \cot(\pi x)$ (see Lemma 2.1). In this interval, the result for x irrational is an application of Proposition 2 in [1] with (we rename g there as g^* to avoid the clash of notation) $a = \theta = s = 1$, $f(x, v) = \Phi_v(x) - \mathcal{S}_{G+H}(x)$, $g^*(x) = \frac{1}{2}\lambda g'(0) \log x^{-1}$ and $H(x)$ the function in Lemma 4.1 multiplied by λ . We have to check that

$$\varepsilon(x, v) = f(x, v) - xf(1/x, xv) - g^*(x)$$

satisfies the four conditions **i)**, **ii)**, **iii)** and **iv)** in [1, §4.1]. This is done in [1, §6.2] for the case $g(x) = \sin(2\pi x)$ and $f(x) = \cot(\pi x)$ and the arguments can be repeated almost word by word here. We refer the reader to look up [1] for more details. Set $\eta(x, v) = x\Phi_{xv}(1/x) - x\Phi_{x[v]}(1/x)$. We have¹

$$\varepsilon(x, v) = G_v(x) - G(x) - \frac{\lambda}{x} \sum_{m>v} \frac{g(m^2x)}{m^3} - \eta(x, v),$$

and since $\eta(x, v) \ll x \sum_{x[v] < n \leq xv} n^{-1} \ll v^{-1}$,

$$\varepsilon(x, y) = G_v(x) - G(x) + O(x^{-1}v^{-2} + v^{-1}).$$

Conditions **i)** and **ii)** follow from Theorem 1.1. The condition **iii)** is obvious in our case. Finally, the condition **iv)** requires $\Phi_v(x) \ll \log(v+2)$ if $g'(0) \neq 0$ and $\Phi_v(x) \ll 1$ if $g'(0) = 0$. The first case follows since $g(m^2x)f(mx) \ll g(m\|mx\|)/\|mx\| \ll m$ and the second case, using $g(t) = O(|t|^{\gamma+1})$, follows from $g(m^2x)f(mx) \ll \|m\|mx\|^{\gamma+1}/\|mx\|$ and Lemma 4.2.

It remains to consider $x = a/q \in \mathbb{Q}^+$. Lemma 4.2 covers the case $g'(0) = 0$ as before. If $g'(0) \neq 0$ it is easy to check that it does not converge. Note that $g(m^2x)f(mx)$ tends to $\lambda g'(0)m$ when $q \mid m$ and remains bounded otherwise giving a divergence as in the harmonic series (see [9, p.98]). \square

¹The term $\eta(x, v)$ was forgotten in [1, §4.1].

Acknowledgements. This work is the result of the visit of the first author to the Université du Littoral Côte d’Opale. He appreciates the invitation, the support and the hospitality during this period.

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