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# Polyhedral Neighborhoods vs. Tubular Neighborhoods:

## New Insights for Fractal Zeta Functions

### and Complex Dimensions

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#### Abstract

We provide exact expressions for the volume of *polyhedral neighborhoods* for the sequence of pre-fractal graphs which converge to the Weierstrass Curve, called *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), associated with a suitable (and geometrically meaningful) sequence of values of a parameter  $\epsilon$  tending to zero, also known as *the cohomology infinitesimal*, due to its connections with fractal cohomology, as developed in an earlier work of the authors. We then introduce the associated *local* and *global polyhedral fractal zeta functions*, and prove that the poles of the global polyhedral fractal zeta function are exactly the same as the Complex Dimensions of the Weierstrass function itself. Contrary to the set of possible Complex Dimensions usually (and classically) obtained by means of tubular neighborhoods, as in an earlier work of ours about the Weierstrass IFD, not only is our result more precise (we obtain the set of *actual* or *exact Complex Dimensions*, instead of the set of *possible Complex Dimensions*), but it also enables us to build a bridge with fractal cohomology, where, for any nonnegative integer  $m$ , the  $m^{\text{th}}$  cohomology group is constituted of continuous functions which possess a generalized Taylor expansion, with fractional derivatives of orders the underlying – and actual – Complex Dimensions.

Then, the aforementioned exact expressions of the polyhedral neighborhoods also enables us to revisit the computation of the box-counting (or Minkowski) dimension of the Weierstrass Curve, in a fully rigorous manner, and thereby to provide a complete and geometric proof of part of Mandelbrot's conjecture about the fractal dimension of the Weierstrass Curve.

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**Keywords:** Weierstrass Curve, iterated fractal drum (IFD), fractal zeta functions, Complex Dimen-

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sions of an IFD, box-counting (or Minkowski) dimension, cohomology infinitesimal, intrinsic scale, Hodge Diamond Star relation, polyhedral measure, tubular neighborhoods, polyhedral (or polygonal) neighborhoods, effective local and global polyhedral zeta functions.

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## 1 Introduction

Fractal zeta functions are classically often obtained in terms of tube formulas (see [LvF13], [LRŽ17], [LRŽ18], [Lap19]). However, when it comes to fractals defined by means of a nonlinear and noncontractive iterated function system (i.f.s.), as is the case for the Weierstrass Curve (see [Dav18]), difficulties arise. Indeed, we cannot obtain an explicit expression for the volume of the general tubular neighborhood of the  $m^{\text{th}}$  prefractal approximation – and, hence, of the associated geometric tube zeta function. Note that, in addition, nonlinearity makes the geometry awfully complicated; in particular, one cannot obtain the exact values of the underlying elementary lengths and angles. As a consequence, we only dispose of an approximate expression for the tube zeta function. In practice, since the approximation only concerns the coefficients in factor of the residues, and since we dispose of sharp estimates for those coefficients, we can obtain the exact values of the poles – and, hence, of the Complex Dimensions associated to each prefractal approximation. However, a natural question is whether the Complex Dimensions of the fractal itself are the same as those of the prefractal approximations, at least for sufficiently large values of these approximations.

In [DL24a], we gave a partial answer to the latter question, by introducing the concept of *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* involved, for instance, in the case of the Cantor Staircase, in [LRŽ17], Section 5.5.4, and in [LRŽ18]. Weierstrass IFDs simply consist in a sequence of polygonal domains which contain the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , and are sufficiently close to  $\Gamma_{\mathcal{W}}$ . As is first explained in [DL22] and [DL24a], we could expect the Complex Dimensions of the IFD to be the same as the Complex Dimensions of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ . An interesting argument arises if we consider the fractal Complex Dimensions as dynamical quantities, which evolve with the scales: to each prefractal approximation  $\Gamma_{\mathcal{W}_m}$  of  $\Gamma_{\mathcal{W}}$ , we can associate specific values of the Complex Dimensions, as is shown in [DL24d]. In this light, it is natural to define the Complex Dimensions of the Weierstrass Curve as the set of the Complex Dimensions of the Weierstrass IFD.

In [DL24c], we have introduced a specific polyhedral measure, by means of polygonal neighborhoods of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ . It so happens that the sequence of polygonal neighborhoods and the sequence of tubular neighborhoods associated with the Weierstrass IFD are nested: given any

integer  $m \in \mathbb{N}$ , there exists  $k_1 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the polygonal neighborhood of the  $m^{\text{th}}$  prefractal approximation contains, but for a finite number of wedges, the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood. On the other hand, there also exists  $k_2 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the polygonal neighborhood of the  $m^{\text{th}}$  tubular neighborhood contains the  $(m+k, \varepsilon_{m+k}^{m+k})$  polygonal neighborhood. Here, the sequence  $(\varepsilon_m^m)_{m \in \mathbb{N}}$  is the so-called *cohomology infinitesimal* introduced in [DL22] and [DL24d] and expressed in terms of the geometry of the Weierstrass Curve – where  $\varepsilon_m^m = (\varepsilon_m)^m$ , with  $\varepsilon_m > 0$ ,  $\varepsilon_m^m \xrightarrow{m \rightarrow \infty} 0$  and  $\varepsilon_m \xrightarrow{m \rightarrow \infty} L$ , for some  $L > 0$ .

In light of this fact, it was natural to consider the fractal zeta functions associated with the sequence of polygonal neighborhoods, instead of the fractal zeta functions associated with the sequence of tubular neighborhoods. For this sequence, as will be shown, in the present paper, not only do we dispose of the exact expression for the volume (i.e., area) of the polygonal neighborhood, at a given step of the prefractal approximation, but we also have the recurrence relation between consecutive fractal zeta functions. This, in particular, enables us to obtain the poles of the limit fractal zeta function – the one associated with the limit fractal object – and, hence, the Complex Dimensions of the fractal.

More specifically, a key outcome of our study is that the (intrinsic) Complex Dimensions of the Weierstrass IFD are all simple, *exact* and given by the following infinite sequence:

$$\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) \pm i \ell_{j_{k,m},k} \mathbf{p} \quad , \quad \text{with } m \in \mathbb{N} \text{ arbitrary, } 0 \leq k \leq m,$$

where the nonnegative integers  $\ell_{j_{k,m},k} \in \mathbb{N}$  (which depend on  $k$  and  $m$ , called *the cohomological vertex integers* and are given in Theorem 5.4, on page 36), form a suitable infinite sequence having a natural geometric and dynamical interpretation (see the discussion below). Here,  $D_{\mathcal{W}}$  is the Minkowski (or, equivalently, box) dimension of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  and  $\mathbf{p}$  denotes the oscillatory period of the associated IFD. The codimension  $2 - D_{\mathcal{W}}$  is the sharp Hölder exponent of the Weierstrass function  $\mathcal{W}$ .

Note that, since  $\ell_{j_{k,m},k} = 0$  if and only if  $\omega_{j_{k,m},k}$  is real (i.e. if and only if  $\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ ). Furthermore, on each vertical line of Complex Dimensions (of abscissa  $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ , for some  $k = 0, 1, 2, \dots$ , say), there are countably infinitely many nonreal complex conjugate pairs of Complex Dimensions, as should be the case, geometrically.

We deduce from these results and from the construction of the cohomological vertex integers  $\ell_{j_{k,m},k}$  – provided in Definition 5.2, on page 35, in terms of the labeling of the vertices of the  $m^{\text{th}}$  prefractal graph approximation of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  – that the oscillations (or vibrations) associated with the Complex Dimensions (see [LvF00], [LvF06], [LvF13], [LRŽ17], [Lap19]) have a very natural geometric and dynamical interpretation in the context of our extended theory of Complex Dimensions (developed in [DL22], [DL24d], [DL24a], [DL24c], [DL24b]) as well as in the present paper. As a result, so do the (intrinsic) Complex Dimensions of  $\Gamma_{\mathcal{W}}$  themselves.

We point out that our result is stronger than the one obtained in [DL22], where the values of the possible Complex Dimensions of the Weierstrass IFD – obtained via fractal zeta functions first defined in terms of tubular neighborhoods – included, in addition to the aforementioned values, the values  $-2, 0$  and  $1 - 2k \pm i \ell_{j_{k,m},k}$ . While such a result can seem intriguing, it simply comes from the fact that tubular neighborhoods involve wedges, whose contribution leads to the occurrence of  $-2$  as a possible pole of the associated local fractal effective tube zeta function. Indeed, as we will see, polyhedral neighborhoods, which are the natural neighborhoods adapted to fractal curves, do not lead to this value as a possible pole of the associated local fractal effective polyhedral zeta function. More precisely, since the volume of the polyhedral neighborhood of each prefractal approximation can be expressed by means of values of the Weierstrass function, the possible Complex Dimensions are exactly the same as the Complex Dimensions of the Weierstrass function itself, as obtained in [DL24d].

Our main results in the present setting can be found in the following places:

- i.* In Definition 3.5, on page 11 and Property 4.4, on page 25, where we introduce and study the sequence of polyhedral neighborhoods and show that it converges to the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ .
- ii.* In Theorem 4.10, on page 28, and Corollary 4.11, on page 29, where we show that our polyhedral measure is a quasi self-similar measure. This result extends the classical notion of self-similar measure.
- iii.* In Property 5.1, on page 30, where we show that the polyhedral and tubular neighborhoods are nested.
- iv.* In Theorem 5.4, on page 36, where we obtain recurrence relations satisfied by the coefficients of the exact fractal (or Complex Dimensions) expansion for the complexified and the ordinary Weierstrass functions.
- v.* In Property 5.5, on page 40, where we show that the (complex) coefficients associated with the Complex Dimensions series expansion of the Weierstrass function satisfy a *Hodge Diamond Star* relation, reminiscent of Poincaré duality (see [DL24d]).
- vi.* In Theorem 5.6, on page 41, where we obtain an exact fractal power series expansion for the volume of suitable  $m^{\text{th}}$  polyhedral neighborhoods of the Weierstrass Curve.
- vii.* In Theorem 5.9, on page 61, where we introduce *the local and global effective polyhedral zeta function*, obtain fractal power series expansions and recurrence relations for the local zeta functions, and show that the global zeta function is well defined, meromorphic in all of  $\mathbb{C}$ , and is given by an explicit fractal power series, evaluated along the cohomology infinitesimal.
- viii.* In Theorem 5.12, on page 74, where we prove that the *global effective tube zeta function* of the Weierstrass IFD exists and coincides with the global effective polyhedral zeta function.
- ix.* In Theorem 6.1, on page 81, where we introduce a new and complete proof of the computation of the box-counting dimension (or, equivalently, of the Minkowski dimension) of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , by simply using the covers of  $\Gamma_{\mathcal{W}}$  by polyhedral neighborhoods – therefore fully establishing part of Mandelbrot’s conjecture [Man82] about the fractal dimension of  $\Gamma_{\mathcal{W}}$ .

## 2 Basic Notation

**Notation 1 (Set of all Natural Numbers, and Intervals).**

As in Bourbaki [Bou04] (Appendix E. 143), we denote by  $\mathbb{N} = \{0, 1, 2, \dots\}$  the set of all natural numbers, and set  $\mathbb{N}^{\star} = \mathbb{N} \setminus \{0\}$ .

Given  $a, b$  with  $-\infty \leq a \leq b \leq \infty$ ,  $]a, b[ = (a, b)$  denotes an open interval, while, for example,  $]a, b] = (a, b]$  denotes a half-open, half-closed interval.

**Notation 2 (Lebesgue Measure on  $\mathbb{R}^2$ ).**

In the sequel, we denote by  $\mu_{\mathcal{L}}$  the Lebesgue measure on  $\mathbb{R}^2$ .

**Notation 3 (Logarithm).**

Given  $y > 0$ ,  $\ln y$  denotes the natural logarithm of  $y$ .

**Notation 4 (Signum Function).**

The *signum function* of a real number  $x$  is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ +1, & \text{if } x > 0. \end{cases}$$

### 3 Geometry of the Weierstrass Curve

We begin by reviewing the main geometric properties of the Weierstrass Curve (and of the associated IFD), which will be needed in this paper.

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by  $(x, y)$ . The horizontal and vertical axes will be respectively referred to as  $(x'x)$  and  $(y'y)$ .

**Notation 5 (Weierstrass Parameters).**

In the sequel,  $\lambda$  and  $N_b$  are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^* \quad \text{and} \quad \lambda N_b > 1 \quad . \quad (\clubsuit) \quad (\mathcal{R}1)$$

Note that this implies that  $N_b > 1$ ; i.e.,  $N_b \geq 2$ , if  $N_b \in \mathbb{N}^*$ , as will be the case in this paper.

As explained in [Dav19], we deliberately made the choice to introduce the notation  $N_b$  which replaces the initial number  $b$ , in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass' original article [Wei75]),  $b$  is any positive real number satisfying  $\lambda b > 1$ , whereas we deal here with the specific case of a natural integer, which accounts for the natural notation  $N_b$ .

**Definition 3.1 (Weierstrass Function, Weierstrass Curve).**

We consider the *Weierstrass function*  $\mathcal{W}$  (also called, in short, the *W-function*) defined, for any real number  $x$ , by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n x). \quad (\mathcal{R}2)$$

We call the associated graph the *Weierstrass Curve*, and denote it by  $\Gamma_{\mathcal{W}}$ .

Due to the one-periodicity of the Weierstrass function (since  $N_b \in \mathbb{N}^*$ ), from now on, and without loss of generality, we restrict our study to the interval  $[0, 1[ = [0, 1)$ . Note that  $\mathcal{W}$  is continuous, and hence, bounded on all of  $\mathbb{R}$ . In particular,  $\Gamma_{\mathcal{W}}$  – which is also the graph of  $\mathcal{W}$  restricted to  $[0, 1[$  – is a (nonempty) compact subset of  $\mathbb{R}^2$ .

**Definition 3.2 (Complexified Weierstrass function).**

We introduce the *Complexified Weierstrass function*  $\mathcal{W}_{comp}$ , defined, for any real number  $x$ , by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n e^{2i\pi N_b^n x}.$$

Clearly,  $\mathcal{W}_{comp}$  is also a continuous and 1-periodic function on  $\mathbb{R}$  and  $\mathcal{W}$  is the real part of  $\mathcal{W}_{comp}$ .

**Notation 6 (Minkowski Dimension and Hölder Exponent).**

For the parameters  $\lambda$  and  $N_b$  satisfying condition  $(\clubsuit)$  (see Notation 5, on page 5), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in ]1, 2[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. We point out that the results in our previous paper [DL22], combined with those obtained in the present paper in Section 6 below, will provide a direct geometric and fully rigorous proof of the fact that  $D_{\mathcal{W}}$ , the Minkowski dimension (or box-counting dimension) of  $\Gamma_{\mathcal{W}}$ , exists and takes the above values, as well as of the fact that  $\mathcal{W}$  is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda}.$$

**Convention (The Weierstrass Curve as a Cyclic Curve).**

In the sequel, we identify the points  $(0, \mathcal{W}(0))$  and  $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$ . This is justified by the fact that the Weierstrass function  $\mathcal{W}$  is 1-periodic, since  $N_b$  is an integer.

*Remark 3.1.* The above convention makes sense, because, in addition to the periodicity property of the  $\mathcal{W}$ -function, the points  $(0, \mathcal{W}(0))$  and  $(1, \mathcal{W}(1))$  have the same vertical coordinate.

**Property 3.1 (Symmetry with Respect to the Vertical Line  $x = \frac{1}{2}$ ).**

Since, for any  $x \in [0, 1]$ ,

$$\mathcal{W}(1-x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n - 2\pi N_b^n x) = \mathcal{W}(x),$$

the Weierstrass Curve is symmetric with respect to the vertical straight line  $x = \frac{1}{2}$ .

**Proposition 3.2 (Nonlinear and Noncontractive Iterated Function System (IFS)).**

Following our previous work [Dav18], we approximate the restriction  $\Gamma_{\mathcal{W}}$  to  $[0, 1] \times \mathbb{R}$ , of the Weierstrass Curve, by a sequence of finite graphs, built via an iterative process. For this purpose, we use the nonlinear (and noncontractive) iterated function system (IFS) consisting of a finite family of  $C^\infty$  bijective maps from  $\mathbb{R}^2$  to  $\mathbb{R}^2$  and denoted by

$$\mathcal{T}_{\mathcal{W}} = \{T_0, \dots, T_{N_b-1}\},$$

where, for any integer  $i$  belonging to  $\{0, \dots, N_b - 1\}$  and any point  $(x, y)$  of  $\mathbb{R}^2$ ,

$$T_i(x, y) = \left( \frac{x+i}{N_b}, \lambda y + \cos\left(2\pi \left(\frac{x+i}{N_b}\right)\right) \right).$$

Note that unlike in the classical situation, these maps are not contractions. Nevertheless,  $\Gamma_{\mathcal{W}}$  can be recovered from this IFS in the usual way, as we next explain.

**Remark 3.2 (Further Comments About Nonlinear and Noncontractive Iterated Function Systems).**

In [Dav19], the first author has begun to lay the foundations for a theory of non-classical, non-affine iterated function systems, of the form

$$\{T_0, \dots, T_{N-1}\} \quad , \quad N \in \mathbb{N}, N \geq 2,$$

where, for any integer  $i \in \{0, \dots, N-1\}$ ,  $T_i$  is a map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  obtained by means of:

- i.* a set  $\{\phi_i\}_{0 \leq i \leq N-1}$  of affine contractive maps of ratio  $\frac{1}{N}$  and integer constants from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}^{n-1}$ ;
- ii.* a set  $\{\varphi_i\}_{0 \leq i \leq N-1}$  of linear contractive maps of ratio  $r > \frac{1}{N}$  from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}$ ;
- iii.* a  $T$ -periodic, bounded function  $\psi$  from  $\mathbb{R}$  to  $\mathbb{R}$  satisfying a Lipschitz condition ( $T > 0$ ).



By placing oneself in the Euclidean space of dimension  $n$ , referred to a direct orthonormal frame, with Cartesian coordinates  $(x_1, \dots, x_n)$ , the maps  $T_i$ , for  $0 \leq i \leq N - 1$ , are defined, for any  $(x_1, \dots, x_n) \in \mathbb{R}^n$ , by

$$T_i(x_1, \dots, x_n) = (\phi_i(x_1, \dots, x_{n-1}), \varphi_i(x_n) + \psi \circ \phi_i(Tx_1, \dots, Tx_{n-1})) .$$

This i.f.s. is related to a function of Weierstrass type

$$(x_1, \dots, x_{n-1}) \mapsto \mathcal{W}(x_1, \dots, x_{n-1}) = \sum_{n=0}^{+\infty} r^n \psi(TN^n(x_1, \dots, x_{n-1})) .$$

Contrary to self-affine fractals such as the Sierpiński gasket or the Koch Curve, the Gluing Lemma [BD85] does not apply. Yet, one has an equivalent property, in so far as, for any integer  $i \in \{0, \dots, N\}$ , the map  $T_i$  is a bijection of the set  $(x_1, \dots, x_{n-1}, \mathcal{W}(x_1, \dots, x_{n-1})) \in \mathbb{R}^n$ .

In addition, those maps possess what can be viewed as a property that is essentially equivalent to the contractive property of classical IFSs, since, at each step of the iterative process which enables one to obtain the prefractal graphs, they reduce the two-dimensional Lebesgue measures of a given sequence of polyhedra covering the curve. Such maps play a role in the first step of the horseshoe map process introduced by Stephen Smale; we refer the interested reader to the book of Robert L. Devaney [Dev03].

**Property 3.3 (Attractor of the IFS [Dav18], [Dav19]).**

The Weierstrass Curve  $\Gamma_{\mathcal{W}}$  is the attractor of the IFS  $\mathcal{T}_{\mathcal{W}}$ , and hence, is the unique nonempty compact subset  $\mathcal{K}$  of  $\mathbb{R}^2$  satisfying  $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$ ; in particular, we have that  $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$ .

**Lemma 3.4 (Fixed Points).**

For any integer  $i$  belonging to  $\{0, \dots, N_b - 1\}$ , we denote by

$$P_i = (x_i, y_i) = \left( \frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos\left(\frac{2\pi i}{N_b - 1}\right) \right)$$

the unique fixed point of the map  $T_i$ ; see [Dav19].

**Definition 3.3 (Sets of Vertices, Prefractals).**

We denote by  $V_0$  the ordered set (according to increasing abscissae) of the points

$$\{P_0, \dots, P_{N_b-1}\} .$$

The set of points  $V_0$  – where, for any integer  $i$  in  $\{0, \dots, N_b - 2\}$ , the point  $P_i$  is linked to the point  $P_{i+1}$  – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by  $\Gamma_{\mathcal{W}_0}$ . Then,  $V_0$  is called *the set of vertices* of the graph  $\Gamma_{\mathcal{W}_0}$ .

For any positive integer  $m$ , i.e., for  $m \in \mathbb{N}^*$ , we set  $V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$ .

The set of points  $V_m$ , where two consecutive points are linked, is an oriented finite graph, ordered according to increasing abscissae, called the  $m^{\text{th}}$  order  $\mathcal{W}$ -prefractal. Then,  $V_m$  is called *the set of vertices* of the prefractal  $\Gamma_{\mathcal{W}_m}$ ; see Figure 2, on page 13. We call *Weierstrass Iterated Fractal Drums (IFD)* the sequence of prefractal graphs which converge to the Weierstrass Curve.

**Property 3.5 (Density of the Set  $V^* = \bigcup_{n \in \mathbb{N}} V_n$  in the Weierstrass Curve [DL24d]).**

*The set  $V^* = \bigcup_{n \in \mathbb{N}} V_n$  is dense in the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ .*

**Definition 3.4 (Adjacent Vertices, Edge Relation).**

For any  $m \in \mathbb{N}$ , the prefractal graph  $\Gamma_{\mathcal{W}_m}$  is equipped with an edge relation  $\sim_m$ , as follows: two vertices  $X$  and  $Y$  of  $\Gamma_{\mathcal{W}_m}$  (i.e., two points belonging to  $V_m$ ) are said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment  $[X, Y]$  is an edge of  $\Gamma_{\mathcal{W}_m}$ ; we then write  $X \sim_m Y$ . Note that this edge relation depends on  $m$ , which means that points adjacent in  $V_m$  might not remain adjacent in  $V_{m+1}$ .

**Property 3.6 (Scaling Properties of the Weierstrass Function, and Consequences [DL22]).**

*Since, for any real number  $x$ ,  $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^n x)$ , one also has*

$$\mathcal{W}(N_b x) = \sum_{n=0}^{\infty} \lambda^n \cos(2\pi N_b^{n+1} x) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos(2\pi N_b^n x) = \frac{1}{\lambda} (\mathcal{W}(x) - \cos(2\pi x)),$$

*which yields, for any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m\}$ ,*

$$\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) = \lambda \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^{m-1}}\right) + \cos\left(\frac{2\pi j}{(N_b - 1) N_b^m}\right).$$

*By induction, one then obtains that*

$$\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) = \lambda^m \mathcal{W}\left(\frac{j}{(N_b - 1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\pi N_b^k j}{(N_b - 1) N_b^m}\right).$$

We refer to part *iv.* of Property 3.7, on page 9, along with Figure 1, on page 12, for the definition of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$  associated with the Weierstrass Curve.

**Property 3.7.** [Dav18] *For any  $m \in \mathbb{N}$ , the following statements hold:*

- i.  $V_m \subset V_{m+1}$ .*

ii.  $\#V_m = (N_b - 1) N_b^m + 1$ , where  $\#V_m$  denotes the number of elements in the finite set  $V_m$ .

iii. The prefractal graph  $\Gamma_{\mathcal{W}_m}$  has exactly  $(N_b - 1) N_b^m$  edges.

iv. The consecutive vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  are the vertices of  $N_b^m$  simple nonregular polygons  $\mathcal{P}_{m,k}$  with  $N_b$  sides, where  $1 \leq k \leq N_b^m - 1$  (note that contrary to the numbering of the vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$ , the numbering of the polygons starts with  $k = 1$  and not  $k = 1$ ). For  $1 \leq k \leq N_b^m - 1$ , the vertices of  $\mathcal{P}_{m,k}$  are the following points

$$\left( \frac{(N_b - 1)k + k'}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k + k'}{(N_b - 1)N_b^m} \right) \right), \quad 0 \leq k' \leq N_b - 1.$$

Hence, for any strictly positive integer  $m$ , the junction point between two consecutive polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{P}_{m,k+1}$  is the point

$$\left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m} \right) \right), \quad 1 \leq k \leq N_b^m - 1.$$

Hence, the total number of junction points is  $N_b^m - 1$ . For instance, in the case  $N_b = 3$ , the polygons are all triangles; see Figure 1, on page 12.

We call extreme vertices of the polygon  $\mathcal{P}_{m,k}$  the junction points

$$\mathcal{V}_{\text{initial}}(\mathcal{P}_{m,k}) = \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m} \right) \right), \quad 0 \leq k \leq N_b^m - 1,$$

and

$$\mathcal{V}_{\text{end}}(\mathcal{P}_{m,k}) = \left( \frac{(N_b - 1)(k + 1)}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)(k + 1)}{(N_b - 1)N_b^m} \right) \right), \quad 0 \leq k \leq N_b^m - 2.$$

In the sequel, we will denote by  $\mathcal{P}_0$  **the initial polygon**, whose vertices are the fixed points of the maps  $T_i$ ,  $0 \leq i \leq N_b - 1$ , introduced in Notation 3.4, on page 8 and Definition 3.3, on page 8, i.e.,  $\{P_0, \dots, P_{N_b-1}\}$ ; see, again, Figure 1, on page 12.

In the same way, the consecutive vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$ , distinct from the fixed points  $P_0$  and  $P_{N_b-1}$  (see Notation 3.4, on page 8), are also the vertices of  $N_b^m - 1$  simple nonregular polygons  $\mathcal{Q}_{m,j}$ , for  $1 \leq j \leq N_b^m - 2$ . For any integer  $j$  such that  $1 \leq j \leq N_b^m - 2$ , one obtains each polygon  $\mathcal{Q}_{m,j}$  by connecting the point number  $j$  (i.e., with the notation of Definition 3.6, on page 14, the vertex  $M_{j,m}$ ) to the point number  $j + 1$  (i.e., the vertex  $M_{j+1,m}$ ) if  $j \equiv i \pmod{N_b}$ , for  $1 \leq i \leq N_b - 1$ , and the point number  $j$  to the point number  $j - N_b + 1$  if  $j \equiv 0 \pmod{N_b}$ .

For  $1 \leq k \leq N_b^m - 1$ , the vertices of  $\mathcal{Q}_{m,k}$  are the following points:

a. When the integer  $N_b$  is odd:

$$\left( \frac{(N_b - 1)k + \frac{N_b - 1}{2} + k'}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k + \frac{N_b - 1}{2} + k'}{(N_b - 1)N_b^m} \right) \right), \quad 0 \leq k' \leq N_b - 1.$$

b. When the integer  $N_b$  is even:

$$\left( \frac{(N_b - 1)k + \frac{N_b}{2} + k'}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k + \frac{N_b}{2} + k'}{(N_b - 1)N_b^m} \right) \right), \quad 0 \leq k' \leq N_b - 2.$$

As previously, we call extreme vertices of the polygon  $\mathcal{Q}_{m,k}$  the points

$$\mathcal{V}_{initial}(\mathcal{Q}_{m,k}) = \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k}{(N_b - 1)N_b^m} \right) \right), \quad 1 \leq k \leq N_b^m - 1,$$

and

$$\mathcal{V}_{end}(\mathcal{Q}_{m,k}) = \left( \frac{(N_b - 1)(k + 1)}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)(k + 1)}{(N_b - 1)N_b^m} \right) \right), \quad 1 \leq k \leq N_b^m - 2.$$

**Definition 3.5 (Polygonal Sets).**

For any  $m \in \mathbb{N}$ , we introduce the following *polygonal sets*

$$\mathcal{P}_m = \{\mathcal{P}_{m,k}, 0 \leq k \leq N_b^m - 1\} \quad \text{and} \quad \mathcal{Q}_m = \{\mathcal{Q}_{m,k}, 0 \leq k \leq N_b^m - 2\}.$$

We then introduce *the sequence of domains delimited by the Weierstrass IFD*, or *polygonal neighborhood of the Weierstrass Curve* as the sequence  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \in \mathbb{N}}$  of open, connected polygonal sets  $(\mathcal{P}_m \cup \mathcal{Q}_m)_{m \in \mathbb{N}}$ , where, for each  $m \in \mathbb{N}$ ,  $\mathcal{P}_m$  and  $\mathcal{Q}_m$  respectively denote the polygonal sets introduced just above.

Given  $m \in \mathbb{N}$ , we call  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  the  $m^{\text{th}}$  *polyhedral neighborhood* (of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ ).

**Notation 7.** For any  $m \in \mathbb{N}$ , we denote by:

- ii.  $X \in \mathcal{P}_m$  (resp.,  $X \in \mathcal{Q}_m$ ) a vertex of a polygon  $\mathcal{P}_{m,k}$ , with  $0 \leq k \leq N_b^m - 1$  (resp., a vertex of a polygon  $\mathcal{Q}_{m,k}$ , with  $1 \leq k \leq N_b^m - 2$ ).
- ii.  $\mathcal{P}_m \cup \mathcal{Q}_m$  the reunion of the polygonal sets  $\mathcal{P}_m$  and  $\mathcal{Q}_m$ , which consists in the set of all the vertices of the polygons  $\mathcal{P}_{m,k}$ , with  $0 \leq k \leq N_b^m - 1$ , along with the vertices of the polygons  $\mathcal{Q}_{m,k}$ , with  $1 \leq k \leq N_b^m - 2$ . In particular,  $X \in \mathcal{P}_m \cup \mathcal{Q}_m$  simply denotes a vertex in  $\mathcal{P}_m$  or  $\mathcal{Q}_m$ .
- iii.  $\mathcal{P}_m \cap \mathcal{Q}_m$  the intersection of the polygonal sets  $\mathcal{P}_m$  and  $\mathcal{Q}_m$ , which consists in the set of all the vertices of both a polygon  $\mathcal{P}_{m,k}$ , with  $0 \leq k \leq N_b^m - 1$ , and a polygon  $\mathcal{Q}_{m,k'}$ , with  $1 \leq k' \leq N_b^m - 2$ .

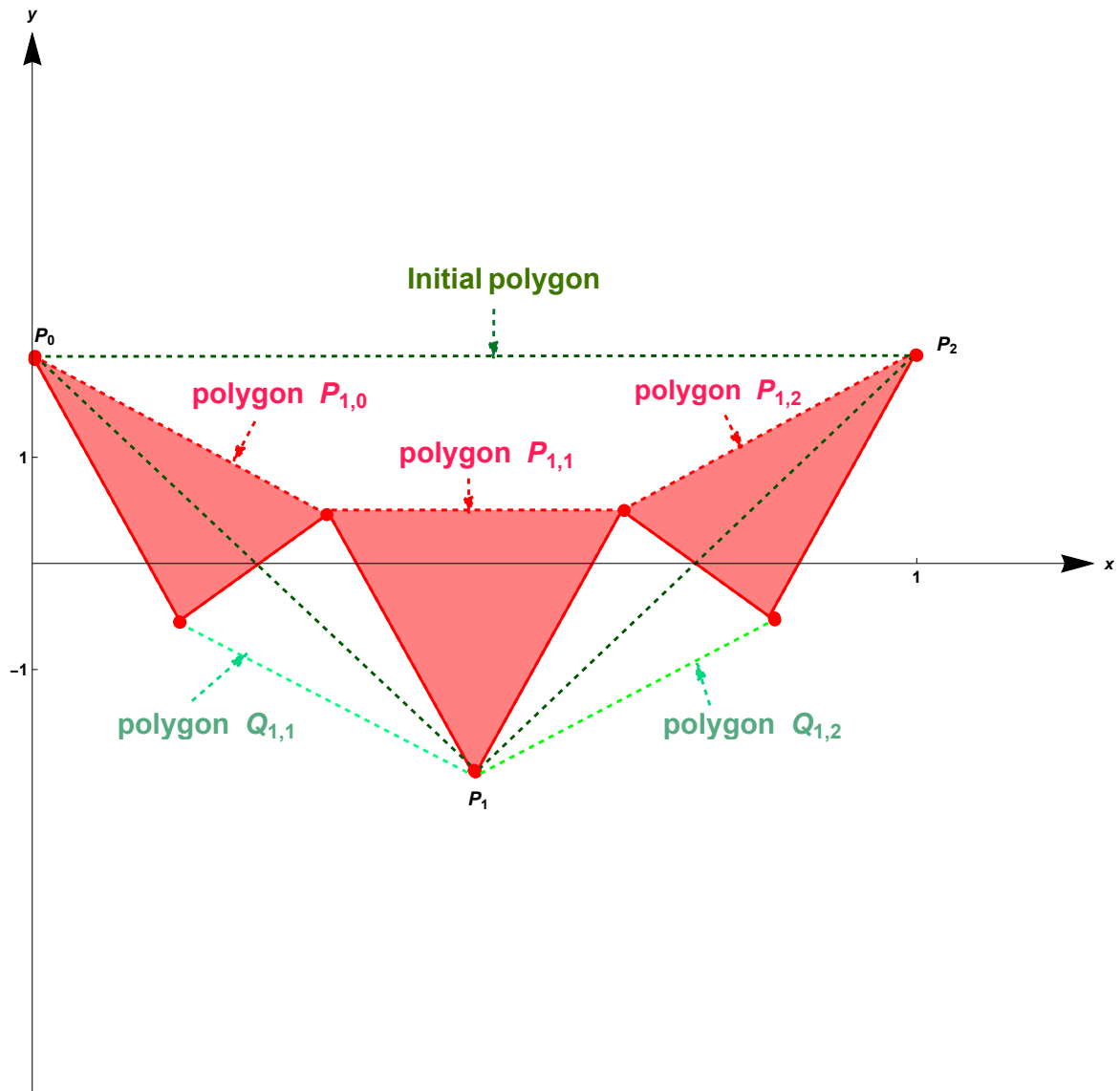
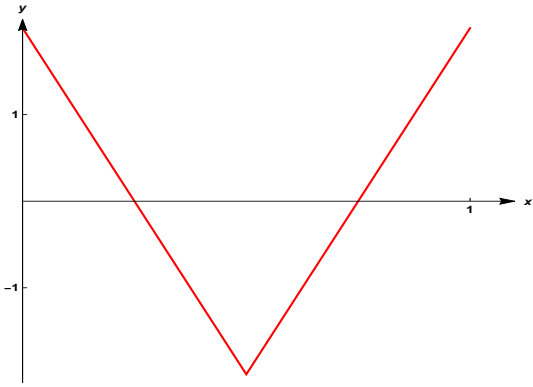
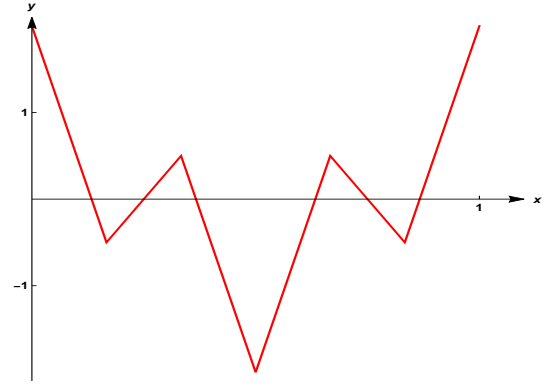


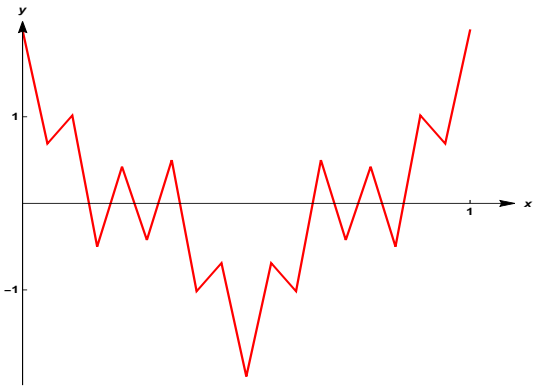
Figure 1: The initial polygon  $\mathcal{P}_0$ , and the respective polygons  $\mathcal{P}_{1,0}$ ,  $\mathcal{P}_{1,1}$ ,  $\mathcal{P}_{1,2}$ ,  $\mathcal{Q}_{1,1}$ ,  $\mathcal{Q}_{1,2}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ . (See also Figure 2, on page 13.)



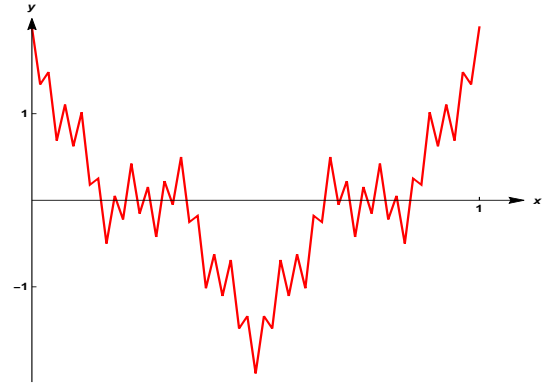
(a) The prefractal graph  $\Gamma_{W_0}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .



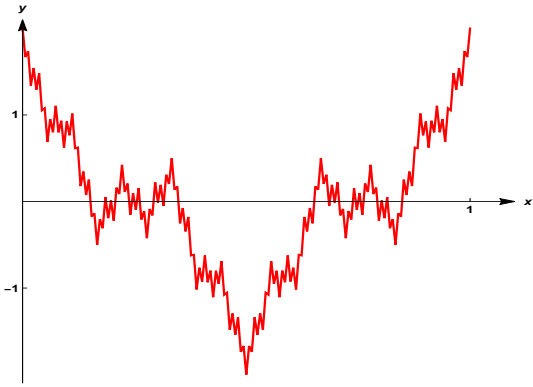
(b) The prefractal graph  $\Gamma_{W_1}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .



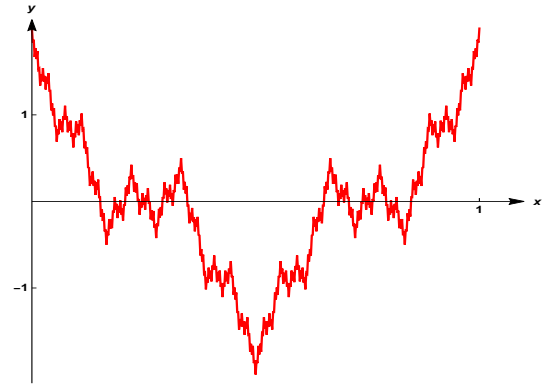
(c) The prefractal graph  $\Gamma_{W_2}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .



(d) The prefractal graph  $\Gamma_{W_3}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .



(e) The prefractal graph  $\Gamma_{W_4}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .



(f) The prefractal graph  $\Gamma_{W_5}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

Figure 2: The prefractal graphs  $\Gamma_{W_0}, \Gamma_{W_1}, \Gamma_{W_2}, \Gamma_{W_3}, \Gamma_{W_4}, \Gamma_{W_5}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

**Definition 3.6 (Vertices of the Prefractals, Elementary Lengths, Heights and Angles).**

Given a strictly positive integer  $m$ , we denote by  $(M_{j,m})_{0 \leq j \leq (N_b-1)N_b^m}$  the set of vertices of the prefractal graph  $\Gamma_{\mathcal{W}_m}$ . One thus has, for any integer  $j$  in  $\{0, \dots, (N_b-1)N_b^m\}$ :

$$M_{j,m} = \left( \frac{j}{(N_b-1)N_b^m}, \mathcal{W} \left( \frac{j}{(N_b-1)N_b^m} \right) \right).$$

We also introduce, for any integer  $j$  in  $\{0, \dots, (N_b-1)N_b^m - 1\}$ :

i. the elementary horizontal lengths:

$$L_m = \frac{1}{(N_b-1)N_b^m}; \quad (\mathcal{R}3)$$

ii. the elementary heights:

$$h_{j,j+1,m} = \left| \mathcal{W} \left( \frac{j+1}{(N_b-1)N_b^m} \right) - \mathcal{W} \left( \frac{j}{(N_b-1)N_b^m} \right) \right|;$$

iii. the elementary lengths:

$$l_{j,j+1,m} = d(M_{j,m}, M_{j+1,m}) = \sqrt{L_m^2 + h_{j,j+1,m}^2};$$

iv. the minimal height:

$$h_m^{inf} = \inf_{0 \leq j \leq (N_b-1)N_b^m - 1} h_{j,j+1,m}, \quad (\mathcal{R}4)$$

along with the maximal height:

$$h_m = \sup_{0 \leq j \leq (N_b-1)N_b^m - 1} h_{j,j+1,m}, \quad (\mathcal{R}5)$$

v. the geometric angles:

$$\theta_{j-1,j,m} = ((y'y), (\widehat{M_{j-1,m}M_{j,m}})) \quad , \quad \theta_{j,j+1,m} = ((y'y), (\widehat{M_{j,m}M_{j+1,m}})),$$

where  $(y'y)$  denotes the vertical axis, which yield **the following value of the geometric angle between consecutive edges**, namely,  $[M_{j-1,m}M_{j,m}, M_{j,m}M_{j+1,m}]$ , with  $\arctan = \tan^{-1}$ :

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{h_{j-1,j,m}} + \arctan \frac{L_m}{h_{j,j+1,m}}.$$

(Note that, of course,  $\theta_{j-1,j,m} = \arctan \frac{L_m}{h_{j-1,j,m}}$  and  $\theta_{j,j+1,m} = \arctan \frac{L_m}{h_{j,j+1,m}}$ .)

**Property 3.8.** For the geometric angle  $\theta_{j-1,j,m}$ ,  $0 \leq j \leq (N_b-1)N_b^m$ ,  $m \in \mathbb{N}$ , we have the following relation:

$$\tan \theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m}.$$

**Property 3.9 (A Consequence of the Symmetry with Respect to the Vertical Line  $x = \frac{1}{2}$ ).**

For any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m\}$ , we have that

$$\mathcal{W}\left(\frac{j}{(N_b - 1)N_b^m}\right) = \mathcal{W}\left(\frac{(N_b - 1)N_b^m - j}{(N_b - 1)N_b^m}\right),$$

which means that the points

$$\left(\frac{(N_b - 1)N_b^m - j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)N_b^m - j}{(N_b - 1)N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^m}\right)\right)$$

are symmetric with respect to the vertical line  $x = \frac{1}{2}$ ; see Figure 3, on page 15.

**Definition 3.7 (Left-Side and Right-Side Vertices).**

Given natural integers  $m, k$  such that  $0 \leq k \leq N_b^m - 1$ , and a polygon  $\mathcal{P}_{m,k}$ , we define:

- i. The set of its *left-side vertices* as the set of the first  $\left[\frac{N_b - 1}{2}\right]$  vertices, where  $[y]$  denotes the integer part of the real number  $y$ .
- ii. The set of its *right-side vertices* as the set of the last  $\left[\frac{N_b - 1}{2}\right]$  vertices.

When the integer  $N_b$  is odd, we define the bottom vertex as the  $\left(\frac{N_b - 1}{2}\right)^{th}$  one; see Figure 4, on page 16.

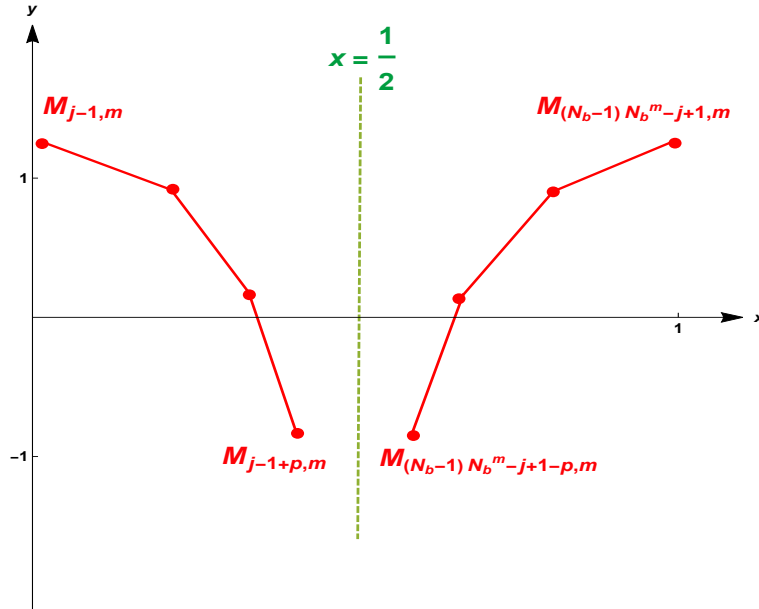


Figure 3: Symmetric points with respect to the vertical line  $x = \frac{1}{2}$ .



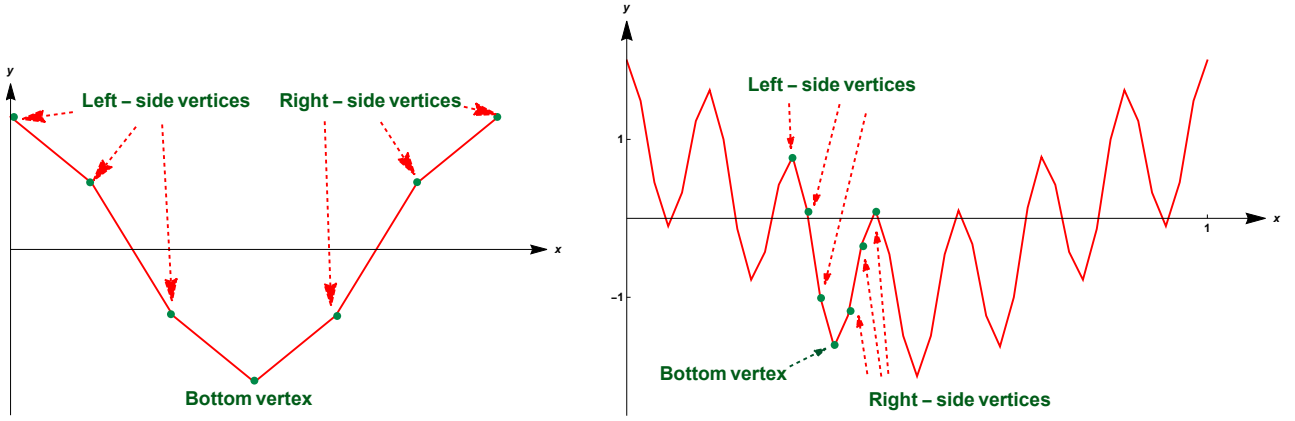


Figure 4: The Left and Right-Side Vertices, in the case when  $N_b = 7$ .

**Property 3.10** ([DL22]).

For any integer  $j$  in  $\{0, \dots, N_b - 1\}$ :

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n \frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\pi j}{N_b - 1}\right) = \frac{1}{1 - \lambda} \cos\left(\frac{2\pi j}{N_b - 1}\right).$$

**Property 3.11** ([DL22]).

For  $0 \leq j \leq \frac{(N_b - 1)}{2}$  (resp., for  $\frac{(N_b - 1)}{2} \leq j \leq N_b - 1$ ), we have that

$$\mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \leq 0 \quad \left(\text{resp., } \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \geq 0\right).$$

**Property 3.12** ([DL22]).

Given any strictly positive integer  $m$ , we have the following properties:

i. For any  $j$  in  $\{0, \dots, \#V_m\}$ , the point

$$\left(\frac{j}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right)\right)$$

is the image of the point

$$\left(\frac{j}{(N_b - 1) N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^{m-1}} - i\right)\right) = \left(\frac{j - i(N_b - 1) N_b^{m-1}}{(N_b - 1) N_b^{m-1}}, \mathcal{W}\left(\frac{j - i(N_b - 1) N_b^{m-1}}{(N_b - 1) N_b^{m-1}}\right)\right)$$

under the map  $T_i$ , where  $i \in \{0, \dots, N_b - 1\}$  is arbitrary.

Consequently, for  $0 \leq j \leq N_b - 1$ , **the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m,k}$** ,  $0 \leq k \leq N_b^m - 1$ , i.e., the point

$$\left( \frac{(N_b - 1)k + j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)k + j}{(N_b - 1)N_b^m}\right) \right),$$

is the image of the point

$$\left( \frac{(N_b - 1)(k - i(N_b - 1)N_b^{m-1}) + j}{(N_b - 1)N_b^{m-1}}, \mathcal{W}\left(\frac{(N_b - 1)(k - i(N_b - 1)N_b^{m-1}) + j}{(N_b - 1)N_b^{m-1}}\right) \right)$$

under the map  $T_i$ , where  $i \in \{0, \dots, N_b - 1\}$  is again arbitrary. This latter point is also **the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m-1, k-i(N_b-1)N_b^{m-1}}$** . Therefore, there is an exact correspondence between vertices of the polygons at consecutive steps  $m - 1, m$ .

ii. Given  $j$  in  $\{0, \dots, N_b - 2\}$  and  $k$  in  $\{0, \dots, N_b^m - 1\}$ , we have that

$$\text{sgn}\left(\mathcal{W}\left(\frac{k(N_b - 1) + j + 1}{(N_b - 1)N_b^m}\right) - \mathcal{W}\left(\frac{k(N_b - 1) + j}{(N_b - 1)N_b^m}\right)\right) = \text{sgn}\left(\mathcal{W}\left(\frac{j + 1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right)\right).$$

*Proof.*

i. Given  $m \in \mathbb{N}^*$ , let us consider  $i \in \{0, \dots, N_b - 1\}$ . The image of the point

$$\left( \frac{j}{(N_b - 1)N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^{m-1}} - i\right) \right)$$

under the map  $T_i$  is obtained by applying the analytic expression given in Property 3.2, on page 7, to the coordinates of this point, which, thanks to Property 3.6, on page 9 above, yields the expected result, namely,

$$\left( \frac{j}{(N_b - 1)N_b^m}, \lambda \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^{m-1}} - i\right) + \cos\frac{2\pi j}{(N_b - 1)N_b^m} \right) = \left( \frac{j}{(N_b - 1)N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^m}\right) \right),$$

since, by 1-periodicity,

$$\mathcal{W}\left(\frac{j}{(N_b - 1)N_b^{m-1}} - i\right) = \mathcal{W}\left(\frac{j}{(N_b - 1)N_b^{m-1}}\right).$$

ii. See [DL22]. □

**Property 3.13 (Lower Bound and Upper Bound for the Elementary Heights [DL22]).**

For any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, (N_b - 1)N_b^m\}$ , we have the following estimates, where  $L_m$  is the elementary horizontal length introduced in part i. of Definition 3.6, on page 14:

$$C_{inf} L_m^{2-D_{\mathcal{W}}} \leq \underbrace{|\mathcal{W}((j+1)L_m) - \mathcal{W}(jL_m)|}_{h_{j,j+1,m}} \leq C_{sup} L_m^{2-D_{\mathcal{W}}} \quad , \quad m \in \mathbb{N}, 0 \leq j \leq (N_b - 1) N_b^m, \quad (\star)$$

where the finite and positive constants  $C_{inf}$  and  $C_{sup}$  are given by

$$C_{inf} = (N_b - 1)^{2-D_{\mathcal{W}}} \min_{0 \leq j \leq N_b - 1, \mathcal{W}\left(\frac{j+1}{N_b-1}\right) \neq \mathcal{W}\left(\frac{j}{N_b-1}\right)} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathcal{W}}} \left( \max_{0 \leq j \leq N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right| + \frac{2\pi}{(N_b - 1)(\lambda N_b - 1)} \right).$$

One should note, in addition, that these constants  $C_{inf}$  and  $C_{sup}$  depend on the initial polygon  $\mathcal{P}_0$ , but are independent of  $m \in \mathbb{N}$  sufficiently large (the fact that  $m$  has to be sufficiently large is in relation with the fact that  $\varepsilon^m$  has to be close to 0).

As a consequence, we also have that

$$C_{inf} L_m^{2-D_{\mathcal{W}}} \leq h_m^{inf} \leq C_{sup} L_m^{2-D_{\mathcal{W}}} \quad \text{and} \quad C_{inf} L_m^{2-D_{\mathcal{W}}} \leq h_m \leq C_{sup} L_m^{2-D_{\mathcal{W}}},$$

where  $h_m^{inf}$  and  $h_m$  respectively denote the minimal and maximal heights introduced in part iv. of Definition 3.6, on page 14.

### Theorem 3.14 (Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22]).

For any natural integer  $m$  (i.e., for any  $m \in \mathbb{N}$ ), let us consider a pair of real numbers  $(x, x')$  such that

$$x = \frac{(N_b - 1)k + j}{(N_b - 1)N_b^m} = ((N_b - 1)k + j) L_m \quad , \quad x' = \frac{(N_b - 1)k + j + \ell}{(N_b - 1)N_b^m} = ((N_b - 1)k + j + \ell) L_m,$$

where  $0 \leq k \leq N_b^m - 1$ . We then have the following (discrete, local) reverse-Hölder inequality, with sharp Hölder exponent  $-\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$ :

$$C_{inf} |x' - x|^{2-D_{\mathcal{W}}} \leq |\mathcal{W}(x') - \mathcal{W}(x)|,$$

where  $(x, \mathcal{W}(x))$  and  $(x', \mathcal{W}(x'))$  are adjacent vertices of the same  $m^{\text{th}}$  prefractal approximation,  $\Gamma_{\mathcal{W}_m}$ , with  $m \in \mathbb{N}$  arbitrary. Here,  $C_{inf}$  is given as in Property 3.13, on page 17 just above.

**Corollary 3.15 (Optimal Hölder Exponent for the Weierstrass Function (see [DL22])).**

The local reverse Hölder property of Theorem 3.14, on page 18 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, page 47) – shows that the Codimension  $2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} \in ]0, 1[$  is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function  $\mathcal{W}$  is nowhere differentiable.

**Corollary 3.16 (of Property 3.13 (see [DL22])).**

Thanks to Property 3.13, on page 17, one may now write, for any strictly positive integer  $m$  and any integer  $j$  in  $\{0, \dots, (N_b - 1)N_b^m - 1\}$ , and with  $C_{inf}$  and  $C_{supf}$  defined as in Property 3.13, on page 17:

i. for the elementary heights:

$$h_{j-1,j,m} = L_m^{2-D_{\mathcal{W}}} \mathcal{O}(1) ; \quad (\mathcal{R}6)$$

ii. for the elementary quotients:

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathcal{W}}} \mathcal{O}(1) , \quad (\mathcal{R}7)$$

and where

$$0 < C_{inf} \leq \mathcal{O}(1) \leq C_{sup} < \infty .$$

Note that, thanks to Property 3.13, on page 17,  $\mathcal{O}(1)$  may depend on  $m$ , but is bounded away from 0 and  $\infty$ .

**Corollary 3.17 (Nonincreasing Sequence of Geometric Angles (Coming from Property 3.12; see [DL22])).**

For the **geometric angles**  $\theta_{j-1,j,m}$ ,  $0 \leq j \leq (N_b - 1)N_b^m$ ,  $m \in \mathbb{N}$ , introduced in part v. of Definition 3.6, on page 14, we have the following result:

$$\tan \theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}} (N_b - 1) > \tan \theta_{j-1,j,m+1} ,$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \text{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1} .$$

**Corollary 3.18 (Local Extrema (Coming from Property 3.12, on page 16; see [DL22])).**

i. The set of local maxima of the Weierstrass function on the interval  $[0, 1]$  is given by

$$\left\{ \left( \frac{(N_b - 1)k}{N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k}{N_b^m} \right) \right) : 0 \leq k \leq N_b^m - 1, m \in \mathbb{N} \right\},$$

and corresponds to the extreme vertices of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$  (i.e., they are the vertices connecting consecutive polygons; see part iv. of Property 3.7, on page 9).

ii. For odd values of  $N_b$ , the set of local minima of the Weierstrass function on the interval  $[0, 1]$  is given by

$$\left\{ \left( \frac{(N_b - 1)k + \frac{N_b - 1}{2}}{(N_b - 1)N_b^m}, \mathcal{W} \left( \frac{(N_b - 1)k + \frac{N_b - 1}{2}}{(N_b - 1)N_b^m} \right) \right) : 0 \leq k \leq N_b^m - 1, m \in \mathbb{N} \right\},$$

and corresponds to the bottom vertices of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$  at a given step  $m$ ; see also part iv. of Property 3.7, on page 9.

**Property 3.19 (Existence of Reentrant Angles [DL22]).**

i. The initial polygon  $\mathcal{P}_0$ , admits **reentrant interior angles**, at a vertex  $P_j$ , with  $0 < j \leq N_b - 1$ , in the sense that, with the right-hand rule (according to which angles are measured in a counterclockwise direction), we have that

$$((P_j P_{j+1}), \widehat{(P_j P_{j-1})}) > \pi,$$

for

$$0 < j \leq \frac{N_b - 3}{4} \quad \text{or} \quad \frac{3N_b - 1}{4} \leq j < N_b - 1$$

(see Figure 5, on page 21), which does not occur for values of  $N_b < 7$ .

The number of reentrant angles is then equal to  $2 \left\lceil \frac{N_b - 3}{4} \right\rceil$ .

ii. At a given step  $m \in \mathbb{N}^*$ , with the above convention, a polygon  $\mathcal{P}_{m,k}$  admits reentrant interior angles in the sole cases when  $N_b \geq 7$ , at vertices  $M_{k+j}$ ,  $1 \leq k \leq N_b^m$ ,  $0 < j \leq N_b - 1$ , as well as in the case when

$$0 < j \leq \frac{N_b - 3}{4} \quad \text{or} \quad \frac{3N_b - 1}{4} \leq j < N_b - 1.$$

The number of reentrant angles is then equal to  $2 N_b^m \left\lceil \frac{N_b - 3}{4} \right\rceil$ , where  $[y]$  denotes the integer part of the real number  $y$ .

*Remark 3.3.* Note that, still when  $N_b \geq 7$ , due to the respective definitions of the polygons  $\mathcal{P}_{m,k}$  and  $\mathcal{Q}_{m,k}$ , the existence of reentrant interior angles for  $\mathcal{P}_{m,k}$  at a vertex  $M_{k+j}$ ,  $1 \leq k \leq N_b^m$ ,  $0 < j \leq N_b - 1$ , also results in the existence of reentrant interior angles for  $\mathcal{Q}_{m,k}$  at the vertices  $M_{k+j-1}$ ,  $1 \leq k \leq N_b^m$ ,  $1 < j \leq N_b - 1$  and  $M_{k+j+1}$ ,  $1 \leq k \leq N_b^m$ ,  $0 < j \leq N_b - 2$ .

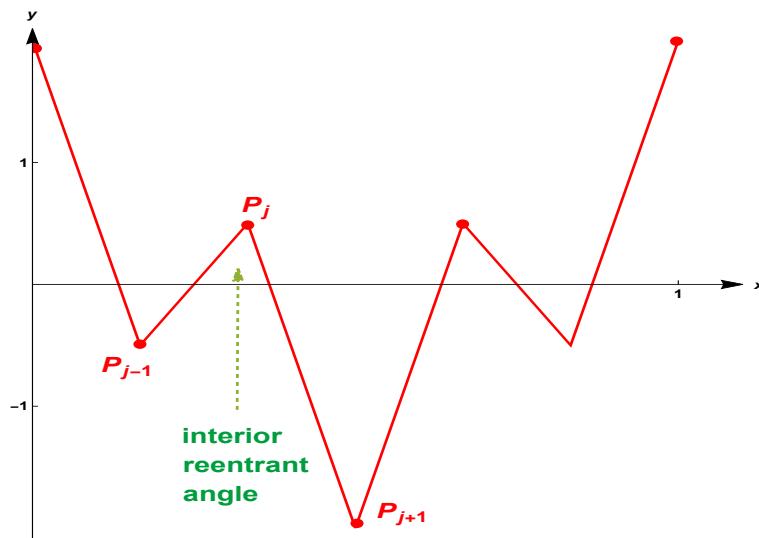


Figure 5: **An interior reentrant angle.** Here,  $N_b = 7$  and  $\lambda = \frac{1}{2}$ .

## 4 Polyhedral Measure

Our results on fractal cohomology obtained in [DL24d] (see also [DL22], [DL24a]) have highlighted the role played by specific threshold values for a sequence of geometric invariants at any step  $m \in \mathbb{N}$  of the prefractal graph approximation; namely, the sequence of the  $m^{\text{th}}$  *cohomology infinitesimals* introduced in Definition 4.1, on page 21 just below, which also plays a major part in studying the polyhedral neighborhoods associated with the sequence of prefractal approximations of the Weierstrass Curve (or, equivalently, of the Weierstrass IFD). Those polyhedral neighborhoods have enabled us in [DL24c] to construct a specific polyhedral measure. We hereafter recall the main corresponding results.

**Definition 4.1** ( $m^{\text{th}}$  *Cohomology Infinitesimal* [DL22], [DL24d] and  $m^{\text{th}}$  *Intrinsic Cohomology Infinitesimal*).

From now on, given any  $m \in \mathbb{N}$ , we will call  $m^{\text{th}}$  *cohomology infinitesimal* the number  $\varepsilon_m^m > 0$  which, modulo a multiplicative constant equal to  $\frac{1}{N_b - 1}$ , i.e.,  $\varepsilon_m^m = \frac{1}{N_b - 1} \frac{1}{N_b^m}$  (recall that  $N_b > 1$ ), stands as the elementary horizontal length introduced in part  $i$ . of Definition 3.6, on page 14; i.e.,

$$\frac{1}{N_b^m}.$$

Observe that, clearly,  $\varepsilon_m$  itself – and not just  $\varepsilon_m^m$  – depends on  $m$ ; hence, we should really write  $\varepsilon_m^m = (\varepsilon_m)^m$ , for all  $m \in \mathbb{N}$ .

In addition, since  $N_b > 1$ ,  $\varepsilon_m^m$  satisfies the following asymptotic behavior,

$$\varepsilon_m^m \rightarrow 0, \text{ as } m \rightarrow \infty,$$

which, naturally, results in the fact that the larger  $m$ , the smaller  $\varepsilon_m^m$ . It is for this reason that we call  $\varepsilon_m^m$  – or rather, the *infinitesimal sequence*  $(\varepsilon_m^m)_{m=0}^\infty$  of positive numbers tending to zero as  $m \rightarrow \infty$ , with  $\varepsilon_m^m = (\varepsilon_m)^m$ , for each  $m \in \mathbb{N}$  – an *infinitesimal*. Note that this  $m^{\text{th}}$  cohomology infinitesimal is the one naturally associated to the scaling relation of Property 3.6, on page 9.

In the sequel, it is also useful to keep in mind that the sequence of positive numbers  $(\varepsilon_m)_{m=0}^\infty$  itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}, \text{ as } m \rightarrow \infty ;$$

i.e.,  $\varepsilon_m \rightarrow \frac{1}{N_b}$ , as  $m \rightarrow \infty$ . In particular,  $\varepsilon_m \not\rightarrow 0$ , as  $m \rightarrow \infty$ , but, instead,  $\varepsilon_m$  tends to a strictly positive and finite limit.

We also introduce, given any  $m \in \mathbb{N}$ , the  $m^{\text{th}}$  *intrinsic cohomology infinitesimal*, denoted by  $\varepsilon^m > 0$ , such that

$$\varepsilon^m = \frac{1}{N_b^m},$$

where

$$\varepsilon = \frac{1}{N_b}.$$

We call  $\varepsilon$  *the intrinsic scale*, or *intrinsic subdivision scale*.

*Remark 4.1.* Note that

$$\varepsilon_m^m = \frac{\varepsilon^m}{N_b - 1}.$$

and that the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$  is asymptotic (when  $m$  tends to  $\infty$ , and up to a positive multiplicative constant) to the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$ .

*Remark 4.2.* We point out that the choice of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal – instead of the  $m^{\text{th}}$  cohomology infinitesimal, as is done in [DL22], [DL24d] – will considerably simplify the computation of the polyhedral fractal volumes, the polyhedral global zeta functions, and hence also, of the (intrinsic) Complex Dimensions of the Weierstrass Curve.

*Remark 4.3 (Addressing Numerical Estimates).*

From a practical point of view (i.e., the amount of detail that becomes visible at certain levels), an important question is the choice of the integer  $m$ , in connection with the value of the ratio

$$\frac{\text{Cohomology infinitesimal}}{\text{Maximal height}} = \frac{\varepsilon_m^m}{h_m};$$

see relation ( $\mathcal{R}5$ ), on page 14.

In order to quantify  $m$ , we rely on the fact that the ration  $\frac{\varepsilon_m^m}{h_m}$  decreases to 0 when  $m$  increases. It is therefore useful to consider the number of digits  $q$  such that

$$\frac{1}{10^q} C_{inf} \leq \frac{\varepsilon_m^m}{h_m} \leq \frac{1}{10^q} C_{sup},$$

since, thanks to the estimates given in relation (R7), on page 19, we have that

$$\frac{\varepsilon_m^m}{h_m} = L_m^{1-D_W} \mathcal{O}(1) = \varepsilon_m^{m(1-D_W)} \mathcal{O}(1),$$

with

$$0 < C_{inf} \leq \mathcal{O}(1) \leq C_{sup} < \infty,$$

where  $C_{inf}$  and  $C_{sup}$  are given in Property 3.13, on page 17.

Given  $q \in \mathbb{N}^*$ , we then have

$$\frac{1}{10^q} C_{inf} \leq \frac{\varepsilon_m^m}{h_m} \leq \frac{1}{10^q} C_{sup}$$

when

$$\frac{C_{inf}}{10^q} \leq e^{(1-D_W) \ln L_m} \leq \frac{C_{sup}}{10^q},$$

or, equivalently, when

$$-\frac{1}{\ln N_b} \ln \left( (N_b - 1) \left( \frac{C_{sup}}{10^q} \right)^{\frac{1}{1-D_W}} \right) \leq m \leq -\frac{1}{\ln N_b} \ln \left( (N_b - 1) \left( \frac{C_{inf}}{10^q} \right)^{\frac{1}{1-D_W}} \right).$$

Numerical values for  $N_b = 3$  and  $\lambda = \frac{1}{2}$  yield:

- i. For  $q = 1$ :  $2 \leq m \leq 3$ .
- ii. For  $q = 2$ :  $7 \leq m \leq 9$ .
- iii. For  $q = 3$ :  $13 \leq m \leq 15$ .

Hence, when  $m$  increases, the ratio  $\frac{\varepsilon_m^m}{h_m}$  decreases, and tends to 0. This numerical – but very practical and explicit argument – also accounts for our forthcoming use of the sequence of polyhedral neighborhoods, of width equal to the cohomology infinitesimal.

We refer to part *iv.* of Property 3.7, on page 9 above, along with Figure 1, on page 12 for the definition of the polygons  $\mathcal{P}_{m,j}$  (resp.,  $\mathcal{Q}_{m,j}$ ) associated with the Weierstrass Curve in the next definition, as well as throughout the rest of this section. See also Definition 3.5, on page 11 where the polygonal families are introduced.

**Definition 4.2 (Power of a Vertex of the Prefractal Graph  $\Gamma_{W_m}$ ,  $m \in \mathbb{N}^*$ , with Respect to the Polygonal Families  $\mathcal{P}_m$  and  $\mathcal{Q}_m$ ).**

Given a strictly positive integer  $m$ , a vertex  $X$  of the prefractal graph  $\Gamma_{W_m}$  will be said to be:

- i. of power one relative to the polygonal family  $\mathcal{P}_m$  if  $X$  is a vertex of one and only one  $N_b$ -gon  $\mathcal{P}_{m,j}$ ,  $0 \leq j \leq N_b^m - 1$ ;
- ii. of power  $\frac{1}{2}$  relative to the polygonal family  $\mathcal{P}_m$  if  $X$  is a common vertex to two consecutive  $N_b$ -gons  $\mathcal{P}_{m,j}$  and  $\mathcal{P}_{m,j+1}$ ,  $0 \leq j \leq N_b^m - 2$ ;
- iii. of power zero relative to the polygonal family  $\mathcal{P}_m$  if  $X$  is not a vertex of any  $N_b$ -gon  $\mathcal{P}_{m,j}$ ,  $0 \leq j \leq N_b^m - 1$ .



Similarly, given  $m \in \mathbb{N}$ , a vertex  $X$  of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  is said to be:

- i. of power one relative to the polygonal family  $\mathcal{Q}_m$  if  $X$  is a vertex of one and only one  $N_b$ -gon  $\mathcal{Q}_{m,j}$ ,  $0 \leq j \leq N_b^m - 2$ ;
- ii. of power  $\frac{1}{2}$  relative to the polygonal family  $\mathcal{Q}_m$  if  $X$  is a common vertex to two consecutive  $N_b$ -gons  $\mathcal{Q}_{m,j}$  and  $\mathcal{Q}_{m,j+1}$ ,  $0 \leq j \leq N_b^m - 3$ ;
- iii. of power zero relative to the polygonal family  $\mathcal{Q}_m$  if  $X$  is not a vertex of any  $N_b$ -gon  $\mathcal{Q}_{m,j}$ ,  $0 \leq j \leq N_b^m - 2$ .

**Notation 8.** In the sequel, given a strictly positive integer  $m$ , the *power of a vertex  $X$  of the prefractal graph  $\Gamma_{\mathcal{W}_m}$  relative to the polygonal families  $\mathcal{P}_m$  and  $\mathcal{Q}_m$*  will be respectively denoted by

$$p(X, \mathcal{P}_m) \quad \text{and} \quad p(X, \mathcal{Q}_m).$$

**Notation 9.** For any  $m \in \mathbb{N}$ , and any vertex  $X$  of  $V_m$ , we set:

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \begin{cases} \frac{1}{N_b} p(X, \mathcal{P}_m) \sum_{0 \leq j \leq N_b^m - 1, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}), & \text{if } X \notin \mathcal{Q}_m, \\ \frac{1}{N_b} p(X, \mathcal{Q}_m) \sum_{1 \leq j \leq N_b^m - 2, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}), & \text{if } X \notin \mathcal{P}_m, \\ \frac{1}{2N_b} \left\{ p(X, \mathcal{P}_m) \sum_{0 \leq j \leq N_b^m - 1, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) + p(X, \mathcal{Q}_m) \sum_{1 \leq j \leq N_b^m - 2, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \right\}, & \\ \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m. & \end{cases}$$

**Property 4.1.** For any  $m \in \mathbb{N}$ , and any pair  $(X, Y)$  of adjacent vertices of  $V_m$  belonging to the same polygon  $\mathcal{P}_{m,j}$ , with  $0 \leq j \leq N_b^m - 1$  (resp.,  $\mathcal{Q}_{m,j}$ , with  $0 \leq j \leq N_b^m - 2$ ), we have that (with  $d_{Eucl}$  denoting the Euclidean distance on  $\mathbb{R}^2$ ),

$$d_{Eucl}(X, Y) = \sqrt{h_{j,j+1,m}^2 + L_m^2} > |h_{j,j+1,m}|,$$

which, due to the inequality given in Property 3.13, page 17, ensures that

$$\frac{1}{d_{Eucl}(X, Y)} < \frac{1}{|h_{j,j+1,m}|} = \mathcal{O}(L_m^{D_{\mathcal{W}}-2}) = \mathcal{O}(N_b^{(2-D_{\mathcal{W}})m}).$$

At the same time, we also have that

$$d_{Eucl}(X, Y) = \mathcal{O}(h_m) = \mathcal{O}(L_m^{2-D_{\mathcal{W}}}) = \mathcal{O}(N_b^{(D_{\mathcal{W}}-2)m}).$$

*Proof.* This follows at once from Property 3.13, on page 17. □

**Corollary 4.2.** For any  $m \in \mathbb{N}$ , any natural integer  $j$  of  $\{0, \dots, N_b^m - 1\}$ , and any pair of points  $(X, Y)$  of  $\mathcal{P}_{m,j}$  or of  $\mathcal{Q}_{m,j}$ , we have that

$$\frac{1}{d_{Eucl}(X, Y)} = \mathcal{O}\left(L_m^{D_{\mathcal{W}}-2}\right) = \mathcal{O}\left(N_b^{(2-D_{\mathcal{W}})m}\right)$$

and

$$d_{Eucl}(X, Y) = \mathcal{O}(h_m) = \mathcal{O}\left(L_m^{2-D_{\mathcal{W}}}\right) = \mathcal{O}\left(N_b^{(D_{\mathcal{W}}-2)m}\right).$$

**Property 4.3.** For any  $m \in \mathbb{N}$ , and any vertex  $X$  of  $V_m$ :

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \mathcal{O}(h_m L_m) = \mathcal{O}\left(L_m^{3-D_{\mathcal{W}}}\right) = \mathcal{O}\left(N_b^{(D_{\mathcal{W}}-3)m}\right)$$

and

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \mathcal{O}(h_m L_m) = \mathcal{O}\left(L_m^{3-D_{\mathcal{W}}}\right) = \mathcal{O}\left(N_b^{(D_{\mathcal{W}}-3)m}\right).$$

*Proof.* This also directly follows from Property 3.13, on page 17. □

**Definition 4.3 (Trace of a Polygon on the Weierstrass Curve [DL24c]).**

Given  $m \in \mathbb{N}$ , and  $0 \leq j \leq N_b^m - 1$  (resp.,  $0 \leq j \leq N_b^m - 2$ ), of extreme vertices  $\mathcal{V}_{initial}(\mathcal{P}_{m,j}) \in V_m$  and  $\mathcal{V}_{end}(\mathcal{P}_{m,j}) \in V_m$  (resp.,  $\mathcal{V}_{initial}(\mathcal{Q}_{m,j}) \in V_m$  and  $\mathcal{V}_{end}(\mathcal{Q}_{m,j}) \in V_m$ ; see Property 3.7, on page 9), we define *the trace* of the polygon  $\mathcal{P}_{m,j}$  (resp.,  $\mathcal{Q}_{m,j}$ ) on the Weierstrass Curve as the set  $tr_{\gamma_{\mathcal{W}}}(\mathcal{P}_{m,j})$  (resp.,  $tr_{\gamma_{\mathcal{W}}}(\mathcal{Q}_{m,j})$ ) of points  $\{\mathcal{V}_{initial}(\mathcal{P}_{m,j}), M_{\star}, \mathcal{V}_{end}(\mathcal{P}_{m,j})\}$  (resp.,  $\{\mathcal{V}_{initial}(\mathcal{Q}_{m,j}), M_{\star}, \mathcal{V}_{end}(\mathcal{Q}_{m,j})\}$ ), where we denote by  $M_{\star}$  any point of the Weierstrass Curve strictly located between  $\mathcal{V}_{initial}(\mathcal{P}_{m,j})$  and  $\mathcal{V}_{end}(\mathcal{P}_{m,j})$  (resp.,  $\mathcal{V}_{initial}(\mathcal{Q}_{m,j})$  and  $\mathcal{V}_{end}(\mathcal{Q}_{m,j})$ ).

**Property 4.4 (Domain Delimited by the Weierstrass IFD).**

We call domain, delimited by the Weierstrass IFD, the set which is equal to the following limit,

$$\mathcal{D}(\Gamma_{\mathcal{W}}) = \lim_{m \rightarrow \infty} \mathcal{D}(\Gamma_{\mathcal{W}_m}),$$

where the convergence is interpreted in the sense of the Hausdorff metric on  $\mathbb{R}^2$ ; see [DL24c]. In fact, we have that

$$\mathcal{D}(\Gamma_{\mathcal{W}}) = \Gamma_{\mathcal{W}}.$$

**Proposition 4.5 (Minimal and Maximal Values of the Weierstrass Function  $\mathcal{W}$  on  $[0, 1]$ ).**

We set

$$m_{\mathcal{W}} = \min_{t \in [0,1]} \mathcal{W}(t) = -\frac{1}{1-\lambda} \quad , \quad M_{\mathcal{W}} = \max_{t \in [0,1]} \mathcal{W}(t) = \frac{1}{1-\lambda}.$$

Then, when the integer  $N_b$  is odd, we have that

$$m_{\mathcal{W}} = -\frac{1}{1-\lambda} \quad , \quad M_{\mathcal{W}} = \frac{1}{1-\lambda} .$$

**Notation 10.** Henceforth, for a given  $m \in \mathbb{N}$ , the notation  $\sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m}$  means that the associated finite sum runs through all of the vertices of the polygons belonging to the sets  $\mathcal{P}_m$  or  $\mathcal{Q}_m$  introduced in Definition 3.5, on page 11; see also Notation 7, on page 11 following that definition.

**Lemma 4.6** ([DL24c]).

Given a continuous function  $u$  on  $[0, 1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}]$ , we have that, for any  $m \in \mathbb{N}$ , and any vertex  $X$  of  $V_m$ :

$$\left| \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X) \right| \leq \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \left( \max_{[0,1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}]} |u| \right) .$$

Consequently, with the notation of Definition 4.1, on page 21, we have that

$$\varepsilon^{m(D_{\mathcal{W}}-2)} \left| \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X) \right| = \mathcal{O}(\varepsilon_m^{-m}) .$$

Since the sequence  $\left( \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \varepsilon^{-m} \right)_{m \in \mathbb{N}}$  is a positive and increasing sequence (the number of vertices involved increases as  $m$  increases), this ensures the existence of the finite limit

$$\lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X) ,$$

where we have used Notation 10, on page 26.

**Property 4.7 (Polyhedral Measure on the Weierstrass IFD [DL24c]).**

We introduce the polyhedral measure on the Weierstrass IFD, denoted by  $\mu$ , such that for any continuous function  $u$  on the Weierstrass Curve, with the use of Notation 9, on page 24 and Notation 10, on page 26,

$$\int_{\Gamma_{\mathcal{W}}} u d\mu = \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) u(X) , \quad (\star)$$

which, thanks to Property 3.5, on page 11, can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u d\mu = \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u d\mu .$$

*Remark 4.4.* In a sense, our polyhedral measure can be seen as a measure which is an extension of the Riemann integral, where the step functions are replaced by upper and lower affine functions which approximate the Weierstrass Curve.

**Theorem 4.8** ([DL24c]).

The polyhedral measure  $\mu$  is well defined, positive, as well as a bounded, nonzero, Borel measure on  $\mathcal{D}(\Gamma_{\mathcal{W}})$ . The associated total mass is given by

$$\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) = \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m), \quad (\star\star)$$

and satisfies the following estimate:

$$0 < \mu(\mathcal{D}(\Gamma_{\mathcal{W}})) \leq \frac{2}{N_b} (N_b - 1)^2 C_{sup}. \quad (\star\star\star)$$

Furthermore, the support of  $\mu$  coincides with the entire curve:

$$\text{supp } \mu = \mathcal{D}(\Gamma_{\mathcal{W}}) = \Gamma_{\mathcal{W}}.$$

In addition,  $\mu$  is the weak limit as  $m \rightarrow \infty$  of the following discrete measures (or Dirac Combs), given, for each  $m \in \mathbb{N}$ , by

$$\mu_m = \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \delta_X,$$

where  $\varepsilon_m^m$  denotes the cohomology infinitesimal introduced in Definition 4.1, on page 21,  $\delta_X$  is the Dirac measure concentrated at  $X$ , and we have used Notation 9, on page 24 for  $\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)$ , along with Notation 10, on page 26.

We next establish a useful new result concerning the polyhedral measure, according to which it is a quasi self-similar measure; see Property 4.9, on page 27 and Theorem 4.10, on page 28.

**Property 4.9 (The Quasi Self-Similar Sequence of Discrete Polyhedral Measures).**

The sequence of discrete polyhedral measures  $(\mu_m)_{m \in \mathbb{N}}$  introduced in Theorem 4.8, on page 27 just above, satisfies the following recurrence relation, for all  $m \in \mathbb{N}^*$ ,

$$\mu_m = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \mu_{m+1} \circ T_j^{-1}, \quad (\spadesuit)$$

where for  $\mathcal{T}_{\mathcal{W}} = \{T_0, \dots, T_{N_b-1}\}$  is the nonlinear iterated function system (IFS) introduced in Proposition 3.2, on page 7.

Note that relation  $(\spadesuit)$  can be viewed as a generalization of classical self-similar measures, as considered in [Hut81], page 714.

*Proof.* First, we can note that, for  $m \in \mathbb{N}^*$ ,

$$\varepsilon_{m+1}^{m+1} = \frac{1}{N_b} \varepsilon_m^m,$$

which ensures that

$$\varepsilon_{m+1}^{(m+1)(D_{\mathcal{W}}-2)} = \frac{1}{N_b^{D_{\mathcal{W}}-2}} \varepsilon_m^{m(D_{\mathcal{W}}-2)} = N_b^{2-D_{\mathcal{W}}} \varepsilon_m^{m(D_{\mathcal{W}}-2)}.$$

We then simply use the result of Property 3.12, on page 16, according to which, for  $0 \leq j \leq N_b - 1$ , the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m+1,k}$ ,  $0 \leq k \leq N_b^m - 1$ , is the image of the  $j^{\text{th}}$  vertex of the polygon  $\mathcal{P}_{m,k-i(N_b-1)N_b^m}$  under the bijective map  $T_j$ , where  $0 \leq j \leq N_b - 1$  is arbitrary. Therefore, there is an exact correspondence between polygons at consecutive steps  $m$ ,  $m + 1$ : indeed, polygons at the  $(m + 1)^{\text{th}}$  step of the prefractal approximation process are obtained by applying each (bijective) map  $T_i$ , for  $0 \leq i \leq N_b - 1$ , to the polygons at the  $m^{\text{th}}$  step of the prefractal approximation process:

$$\bigcup_{j=0}^{N_b-1} \{T_j^{-1}(X), X \in V_{m+1}\} = V_m,$$

which can equivalently be written as

$$\bigcup_{j=0}^{N_b-1} \{T_j^{-1}(X), X \in \mathcal{P}_{m+1} \cup \mathcal{Q}_{m+1}\} = \{X \in \mathcal{P}_m \cup \mathcal{Q}_m\}.$$

We can then deduce that

$$\sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \delta_X = \sum_{T_j \in \mathcal{T}_W} \sum_{X \in \mathcal{P}_{m+1} \cup \mathcal{Q}_{m+1}} \mu^{\mathcal{L}}(X, T_j^{-1}(\mathcal{P}_{m+1}), T_j^{-1}(\mathcal{Q}_{m+1})) \delta_X,$$

which yields the desired result. □

#### Theorem 4.10 (The Quasi Self-Similar Polyhedral Measure).

The polyhedral measure  $\mu$  satisfies the following quasi self-similarity relation,

$$\mu = N_b^{D_W-2} \sum_{T_j \in \mathcal{T}_W} \mu \circ T_j^{-1}, \quad (\spadesuit\spadesuit)$$

where  $\mathcal{T}_W = \{T_0, \dots, T_{N_b-1}\}$  is the nonlinear iterated function system (IFS) introduced in Proposition 3.2, on page 7.

*Proof.* As in the proof of Property 4.9, on page 27, we use the fact that, for all  $m \in \mathbb{N}^*$ ,

$$\bigcup_{j=0}^{N_b-1} \{T_j^{-1}(X), X \in \mathcal{P}_m \cup \mathcal{Q}_m\} = \{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}\}.$$

We then have that, for any continuous function  $u$  on the Weierstrass Curve,

$$\begin{aligned}
\sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \int_{\Gamma_{\mathcal{W}}} u d(\mu \circ T_j^{-1}) &= \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \cup \mathcal{Q}_m} \mu^{\mathcal{L}}(T_j^{-1}(X), \mathcal{P}_m, \mathcal{Q}_m) u(T_j^{-1}(X)) \\
&= \lim_{m \rightarrow \infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}) u(X) \\
&= \lim_{m \rightarrow \infty} N_b^{D_{\mathcal{W}}-2} \varepsilon_{m-1}^{(m-1)(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}) u(X) \\
&= N_b^{2-D_{\mathcal{W}}} \lim_{m \rightarrow \infty} \varepsilon_{m-1}^{(m-1)(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_{m-1} \cup \mathcal{Q}_{m-1}} \mu^{\mathcal{L}}(X, \mathcal{P}_{m-1}, \mathcal{Q}_{m-1}) u(X) \\
&= N_b^{2-D_{\mathcal{W}}} \int_{\Gamma_{\mathcal{W}}} u d\mu,
\end{aligned}$$

from which we deduce, as desired, that

$$\mu = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \mu \circ T_j^{-1}.$$

□

**Corollary 4.11.** *The polyhedral measure is the unique fixed point, with respect to the Kantorovich distance, of the map  $\Phi$  which, to any Borel probability measure  $\nu$  associates*

$$\Phi(\nu) = N_b^{D_{\mathcal{W}}-2} \sum_{T_j \in \mathcal{T}_{\mathcal{W}}} \nu \circ T_j^{-1}.$$

*Remark 4.5.* The results of Theorem 4.10, on page 29 and of Corollary 4.11, on page 29 just above, foreshadow and announce the recurrence relation satisfied by the sequence of effective polyhedral volumes, and established in the proof of Theorem 5.9, on page 61.

## 5 Polyhedral Neighborhoods

**Definition 5.1** ( $(m, \varepsilon_m^m)$ -Neighborhood [DL22]).

Given  $m \in \mathbb{N}$  sufficiently large (so that  $\varepsilon_m^m$  be a sufficiently small positive number), we define the  $(m, \varepsilon_m^m)$ -neighborhood of the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  as follows:

$$\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m) = \left\{ M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_m}) \leq \varepsilon_m^m \right\},$$

where  $d(M, \Gamma_{\mathcal{W}_m}) = \min \{ d_{\text{Eucl}}(M, P), P \in \Gamma_{\mathcal{W}_m} \}$ .

**Property 5.1 (The Nested Neighborhoods).**

- i.* Given  $m \in \mathbb{N}$  sufficiently large, there exists  $k_1 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.5, on page 11 contains, but for a finite number of wedges, the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood; see Figures 6–9, on pages 32–33.
- ii.* Given  $m \in \mathbb{N}$  sufficiently large, there exists  $k_2 \in \mathbb{N}$  such that, for all  $k \geq k_2$ , the  $(m, \varepsilon_m^m)$ -neighborhood contains the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}})$  introduced in Definition 3.5, on page 11; see Figure 10, on page 34.
- iii.* Given  $m \in \mathbb{N}$  sufficiently large, there exists  $k_3 \in \mathbb{N}$  such that, for all  $k \geq k_3$ , the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.5, on page 11 contains the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}})$ .

*Proof.*

- i.* At a given step  $m \geq 0$ , for  $k \geq 0$ , between two adjacent vertices  $M_{i,m}$  and  $M_{i+1,m}$  of  $V_m$ , there are  $N_b^k - 1$  consecutive vertices of  $V_{m+k} \setminus V_m$ ,  $(M_{j+1,m+k}, \dots, M_{j+N_b^k-1,m+k}) \in V_{m+k}^{N_b^k-1}$  such that

$$M_{i,m} = M_{j,m+k} \quad \text{and} \quad M_{i+1,m} = M_{j+N_b^k,m+k}.$$

Thanks to Property 3.12, on page 16, we obtain an exact correspondence between vertices of the polygons at the step  $m+k$ , and at the initial step  $m=0$ . Since reentrant angles occur when  $N_b \geq 7$  (see Property 3.19, on page 20), we can restrict ourselves to the cases  $N_b \leq 6$  (in the case of reentrant angles, the following arguments can be suitably adjusted). We then simply have to consider the  $\left\lceil \frac{N_b-2}{2} \right\rceil$  vertices  $M_{j+k',m+k}$ , with  $1 \leq k' \leq \left\lceil \frac{N_b-2}{2} \right\rceil$  (the same arguments hold for the vertices  $M_{j+N_b-k',m+k}$ ). Also, again thanks to Property 3.12, on page 16, given  $j$  in  $\{0, \dots, N_b-2\}$  and  $k'$  in  $\{0, \dots, N_b^{m+k}-1\}$ , we have that

$$\text{sgn} \left( \mathcal{W} \left( \frac{k'(N_b-1) + j + 1}{(N_b-1)N_b^{m+k}} \right) - \mathcal{W} \left( \frac{k'(N_b-1) + j}{(N_b-1)N_b^{m+k}} \right) \right) = \text{sgn} \left( \mathcal{W} \left( \frac{j+1}{N_b-1} \right) - \mathcal{W} \left( \frac{j}{N_b-1} \right) \right);$$

i.e., equivalently,

$$\text{sgn} \left( \mathcal{W} \left( (k'(N_b-1) + j + 1) L_{m+k} \right) - \mathcal{W} \left( (k'(N_b-1) + j) L_{m+k} \right) \right) = \text{sgn} \left( \mathcal{W} \left( \frac{j+1}{N_b-1} \right) - \mathcal{W} \left( \frac{j}{N_b-1} \right) \right),$$

where  $\text{sgn}$  has been defined in Notation 4, on page 5.

Due to the symmetry of the initial polygon  $\mathcal{P}_0$  (or, equivalently, of the initial prefractal graph  $\Gamma_{\mathcal{W}_0}$ ) with respect to the vertical line  $x = \frac{1}{2}$  (see Property 3.1, on page 7), this means that we can restrict ourselves to the case when

$$\mathcal{W}(j L_{m+k}) \geq \mathcal{W}((j+1) L_{m+k}) \geq \dots \geq \mathcal{W} \left( \left( j + \left\lceil \frac{N_b-2}{2} \right\rceil \right) L_{m+k} \right)$$

and

$$\mathcal{W}(j L_{m+k}) \geq \mathcal{W}((j + N_b) L_{m+k}),$$

since

$$\mathcal{W}(j L_{m+k}) = \mathcal{W}(i L_m) \quad \text{and} \quad \mathcal{W}((j + N_b^k) L_{m+k}) = \mathcal{W}((i + 1) L_m).$$

Thanks to Property 3.13, on page 17, we deduce, by the triangle inequality, for  $1 \leq k' \leq \left\lceil \frac{N_b - 2}{2} \right\rceil$ , that

$$\left| \frac{\mathcal{W}((j + k') L_{m+k}) - \mathcal{W}(j L_{m+k})}{\mathcal{W}(i L_m)} \right| \leq \left\lceil \frac{N_b - 2}{2} \right\rceil C_{sup} L_{m+k}^{2-D_{\mathcal{W}}}.$$

Since

$$L_{m+k} = \frac{L_m}{N_b^k},$$

we then obtain that

$$\left| \frac{\mathcal{W}((j + k') L_{m+k}) - \mathcal{W}(j L_{m+k})}{\mathcal{W}(i L_m)} \right| \leq \left\lceil \frac{N_b - 2}{2} \right\rceil N_b^{k(D_{\mathcal{W}} - 2)} C_{sup} L_m^{2-D_{\mathcal{W}}}.$$

Recall now (see Property 3.13, on page 17) that

$$C_{inf} = (N_b - 1)^{2-D_{\mathcal{W}}} \min_{0 \leq j \leq N_b - 1, \mathcal{W}\left(\frac{j+1}{N_b-1}\right) \neq \mathcal{W}\left(\frac{j}{N_b-1}\right)} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right|$$

and

$$C_{sup} = (N_b - 1)^{2-D_{\mathcal{W}}} \left( \max_{0 \leq j \leq N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \right| + \frac{2\pi}{(N_b - 1)(\lambda N_b - 1)} \right).$$

Here, we have that

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \frac{1}{1 - \lambda} \cos \frac{2\pi j}{N_b - 1}.$$

This ensures that

$$\left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right| \leq \frac{2\pi}{N_b - 1} \frac{1}{1 - \lambda}.$$

Since  $N_b \geq 3$  and  $0 < D_{\mathcal{W}} - 2 < 1$ , we can check that there exist  $m_0 \geq 0$  such that, for all  $k \geq m_0$ ,

$$\left\lceil \frac{N_b - 2}{2} \right\rceil N_b^{k(D_{\mathcal{W}} - 2)} C_{sup} \leq C_{inf},$$

from which we immediately deduce that for,  $1 \leq k' \leq \left\lceil \frac{N_b - 2}{2} \right\rceil$ ,

$$\left| \frac{\mathcal{W}((j + k') L_{m+k}) - \mathcal{W}(j L_{m+k})}{\mathcal{W}(i L_m)} \right| \leq |\mathcal{W}((i + 1) L_m) - \mathcal{W}(i L_m)|.$$



For  $1 \leq k \leq \left\lceil \frac{N_b - 2}{2} \right\rceil$ , the vertices  $M_{j+k',m+k}$  are then strictly between the vertices  $M_{i,m}$  and  $M_{i+1,m}$ .

As was explained previously, we can show, in a similar way, that for  $1 \leq k \leq \left\lceil \frac{N_b - 2}{2} \right\rceil$ , the vertices  $M_{j+N_b-k',m+k}$  are also strictly between the vertices  $M_{i,m}$  and  $M_{i+1,m}$ .

By induction, we then obtain that, given  $N_b + 1$  consecutive adjacent vertices  $M_{i,m}, M_{i+1,m}, M_{i+3,m}$  and  $M_{i+N_b,m}$  of  $V_m$ , with  $0 \leq i \leq \#V_m - N_b$  and  $k \in \mathbb{N}$ , the vertices of  $V_{m+k} \setminus V_m$  located between  $M_{i,m}$  and  $M_{i+N_b,m}$  can be all contained in the simple and convex polygon  $M_{i,m}M_{i+1,m}M_{i+2,m} \dots M_{i+N_b,m}$ , which coincides with the union of two consecutive polygons  $\mathcal{P}_{m,j}$  and  $\mathcal{Q}_{m,j}$ . Thus, there exists  $m_0 \in \mathbb{N}$  such that, for all  $k \geq m_0$ , the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood

$$\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k}) = \{M = (x, y) \in \mathbb{R}^2, d(M, \Gamma_{\mathcal{W}_{m+k}}) \leq \varepsilon_{m+k}^{m+k}\},$$

from which we remove the wedges associated to the vertices  $M_{i,m}, M_{i+1,m}, M_{i+3,m}$  and  $M_{i+4,m}$  (see [DL22]), can be fully contained in the polygon  $M_{i,m}M_{i+1,m}M_{i+2,m} \dots M_{i+N_b,m}$ . Hence, there exists  $k_1 \in \mathbb{N}$  such that, for all  $k \geq k_1$ , the  $(m, \varepsilon_m^m)$ -neighborhood but for a finite number of wedges, the  $(m+k, \varepsilon_{m+k}^{m+k})$ -neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_{m+k}}, \varepsilon_{m+k}^{m+k})$ , can be fully contained in the polygonal domain  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$ .

*ii.* This latter result has been obtained in [DL24c]. It comes from the fact that, in the sense of the Hausdorff metric on  $\mathbb{R}^2$ ,

$$\lim_{m \rightarrow \infty} \mathcal{D}(\Gamma_{\mathcal{W}_m}) = \Gamma_{\mathcal{W}}.$$

*iii.* This is an immediate consequence of *i.* and *ii.*

□

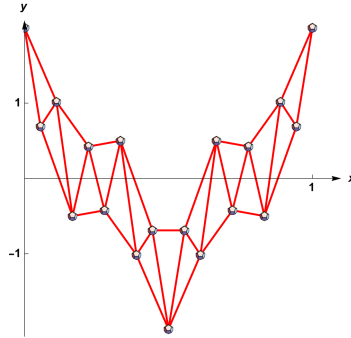


Figure 6: The polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_2})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

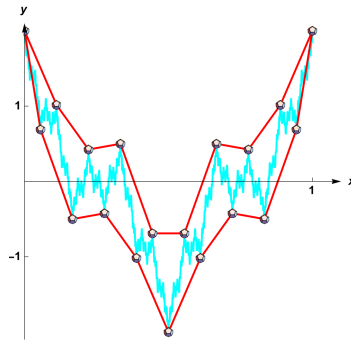


Figure 7: The exterior boundary of the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_2})$  (in red), and the tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7)$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

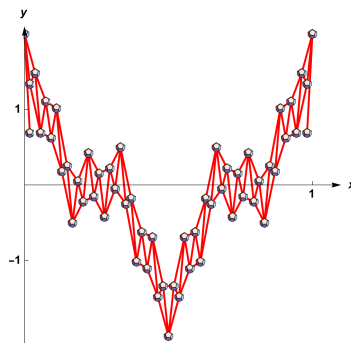


Figure 8: The polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_3})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

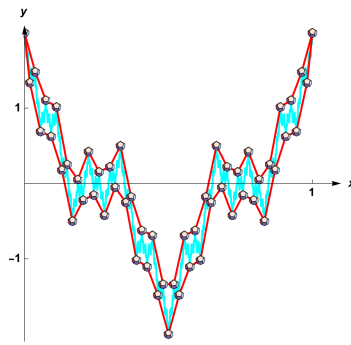


Figure 9: The exterior boundary of the polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_3})$  (in red), and the tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7)$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

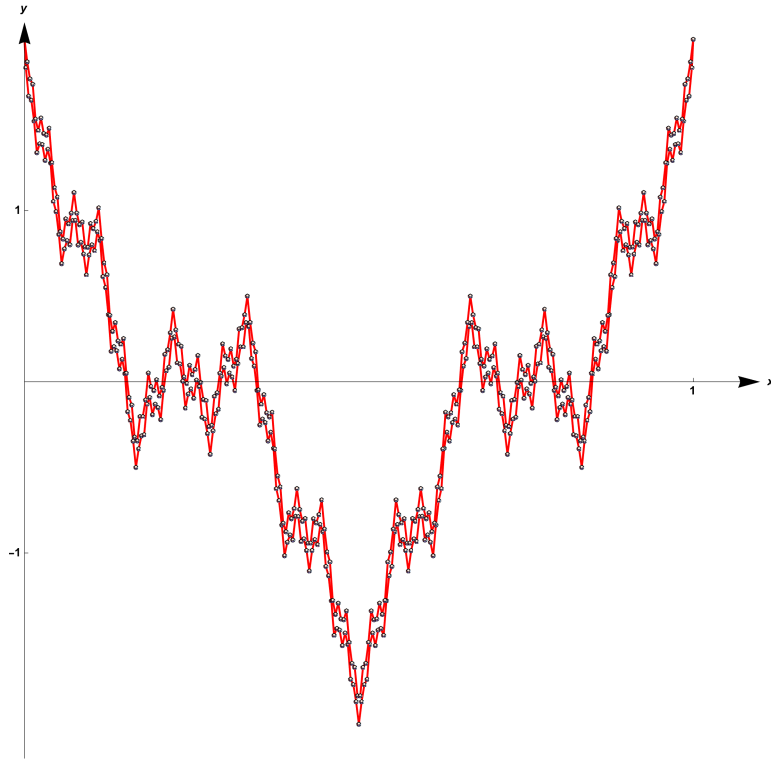


Figure 10: The polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_5})$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 3$ .

**Corollary 5.2** (of Property 5.1, given on page 30).

We immediately deduce from Property 5.1 that there exists  $m_0 \in \mathbb{N}$  such that, for all  $k \geq m_0$ , the polyhedral neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.5, on page 11 contains the vertices of the prefractal approximation  $\Gamma_{\mathcal{W}_{m+k}}$  to the the Weierstrass Curve; i.e., the set of vertices  $V_{m+k}$ . In particular, the density Property 3.5, on page 9, also ensures that the Weierstrass Curve is contained in  $\Gamma_{\mathcal{W}_m}$ :

$$\forall m \geq m_0 : \Gamma_{\mathcal{W}} \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}) .$$

Next, we give the Complex Dimensions series expansion of the Weierstrass function, required to obtain a key exact fractal formula for the volume (i.e., area) of the polyhedral neighborhoods of the prefractal approximations. Note that the classical theory of Complex Dimensions (see, for instance, [LRŽ17]) cannot be applied in the context of our fractal curve. Building on the fact that the imaginary part of the Complex Dimensions aims at characterizing the oscillations of the fractal under study, which are, also, connected to the evolution in scales – in real life (fractal-shaped living forms), the occurrence of new details keeps on appearing with characteristic spatial oscillations – we choose to use the existing cohomological vertex integers, which enable us to *label* each vertex at a given step of the prefractal approximation. By introducing *cross scale paths* (see Definition 5.3 below, on page 35), we can, in some sense, *track* the evolution of vertices starting from the initial step of the prefractal approximation. In this light, our approach differs from the classical one – we *a priori* introduce the imaginary parts of the expected Complex Dimensions, by means of a very useful term  $e^{2i\ell\pi}$ , where  $\ell$  is an integer (in  $\mathbb{Z}$ ). Of course, in the theory developed by the second author and his collaborators in [LvF13], [LRŽ17], [Lap19], a geometric object is said to be *fractal* if it admits at least one nonreal Complex Dimension (defined as a pole of the associated geometric or fractal zeta function, see below),

where the imaginary part of the complex dimension characterizes the local oscillations (changes of direction) of the object structure in the neighborhood of points as one zooms in the object, i.e. as a function of the scale of observation of the object. We are still working along these lines: in our present work, our cross scale paths and associated sequences of cohomological vertex integers characterize fractality, in the sense that they provide a faithful reflection of the change of shape and the occurrence of new details when switching from a prefractal graph approximation to the next one.

Another important point has to be taken into account: indeed, the sequence of prefractal graphs involves discrete graphs. In real life – for instance, in biology – the process which gives birth to fractal-shaped living forms is a continuous one. To name a few, ferns, cauliflowers (see the recent work of the biomathematician Christophe Godin, the biologist François Parcy and their collaborators in [ATM<sup>+</sup>21]), trees, clouds, mountains, coastlines, rivers, lungs, networks of blood vessels, etc. . . . Indeed, in nature, growth is a continuous process. In this light, the notion of *oscillatory period* introduced by the second author and his collaborators in [LvF00], [LvF06], [LRŽ17], [LP06], appears as a useful and powerful tool, by providing a way to bridge the gap between the consecutive discrete approximations. Intuitively, it relies on the fact that the *oscillations* intrinsic to the fractal involved come from the subdivision process. Hence, it naturally involves the number of divisions ( $N_b$  in the present context), and is obtained by considering the Fourier series expansion of a suitable 1-periodic map, where the 1-periodicity reflects, in a sense, the *conservation* of intrinsic characteristics at each step  $m \in \mathbb{N}$  of the prefractal approximation process.

Much of the remainder of this paper will rely on our new method.

**Definition 5.2 (Cohomological Vertex Integers [DL24c]).**

Given  $m \in \mathbb{N}$ , and a vertex  $M_{j,m} = M_{(N_b-1)k'+k'',m} \in V_m$ , of abscissa  $\left((N_b - 1)k' + k''\right) \varepsilon_m^m$ , where  $0 \leq k' \leq N_b^m - 1$  and  $0 \leq k'' \leq N_b - 1$ , we introduce the *cohomological vertex integer*  $\ell_{j,m}$  associated to the vertex  $M_{j,m}$  (which is also the  $(k'')$ <sup>th</sup> vertex of the polygon  $\mathcal{P}_{m,k'}$ ; see part *iv.* of Property 3.7, on page 9), as

$$\ell_{j,m} = \ell_{k',k'',m} = (N_b - 1)k' + k'' . \tag{R8}$$

Depending on the context; that is,

- i.* when the cohomological vertex integer enables one to locate the vertex  $M_{j,m}$ .
- ii.* When it is used in a more general framework, i.e., in order to describe the generators of cohomology groups (see [DL24b]);

we will use the best suited notation between  $\ell_{j,m}$ , in case *i.*, or  $\ell_{k',k'',m}$ , in case *ii.*

**Proposition 5.3 (Cross-Scales Paths, and Associated Sequence of Vertex Integers).**

Given  $m \in \mathbb{N}$ ,  $0 \leq j \leq \#V_m - 1$  and a vertex  $M_{j,m} = M_{(N_b-1)k'+k'',m}$  in  $V_m$ , with  $0 \leq k' \leq N_b^m - 1$  and  $0 \leq k'' \leq N_b - 1$ , we introduce the cross-scales path  $\text{Path}(P_{k''}, M_{j,m})$ , where  $P_{k'}$  is the  $(k')$ <sup>th</sup> fixed point of the map  $T_{k'}$  (see Proposition 3.2, on page 7, along with Notation 3.4, on page 8), as the ordered set  $\left(M_{j_{k,m},k}\right)_{0 \leq k \leq m}$  such that:

i. For  $0 \leq k \leq m$ , each vertex  $M_{j_{k,m},k}$  is in  $V_k \setminus V_k \cap V_m$  (which means that  $M_{j_{k,m},k}$  strictly belongs to  $V_k$ , i.e., it is in the  $k^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_k}$ , and not in  $\Gamma_{\mathcal{W}_{k+1}}$ ).

ii. For  $1 \leq k \leq m$ , each vertex  $M_{j_{k,m},k} = M_{(N_b-1)k'_{k,m}+k''}$ , with  $0 \leq k'_{k,m} \leq N_b^k - 1$ , is the image of the point  $M_{j_{k-1,m},k-1}$  under the map  $T_i$  (see again Proposition 3.2, on page 7), where  $i \in \{0, \dots, N_b - 1\}$  is the smallest admissible value. We thus also have that

$$M_{j_{k-1,m},k-1} = \left( \frac{(N_b - 1) (k'_{k,m} - i (N_b - 1) N_b^{k-1}) + k''}{(N_b - 1) N_b^{k-1}}, \mathcal{W} \left( \frac{(N_b - 1) (k'_{k,m} - i (N_b - 1) N_b^{k-1}) + k''}{(N_b - 1) N_b^{k-1}} \right) \right).$$

This latter point is also **the  $(k'')$ <sup>th</sup> vertex of the polygon  $k'_{k,m} - i (N_b - 1) N_b^{k-1}$**  (see part iv. of Property 3.7, on page 9).

The sequence of vertex integers associated with the cross-scales path  $\text{Path}(P_{k''}, M_{j,m})$  (or, in short, and equivalently, also referred to as the sequence of vertex integers associated with  $M_{j,m}$ ) is the sequence  $(\ell_{j_{k,m},k})_{0 \leq k \leq m}$ , where, for  $0 \leq k \leq m$ ,  $\ell_{j_{k,m},k}$  is the cohomological vertex integer associated with the vertex  $M_{j_{k,m},k}$  (see Definition 5.2, on page 35).

*Proof.* We simply use the results of Property 3.12, on page 16. □

**Theorem 5.4 (Complex Dimensions Series Expansion of the Complexified Weierstrass function  $\mathcal{W}_{\text{comp}}$  [DL24d], and of the Weierstrass function  $\mathcal{W}$ ).**

For any sufficiently large positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$ , we have the following exact expansion, indexed by the Complex Codimensions  $k(D_{\mathcal{W}} - 2) + i k \ell_{j_{k,m},k} \mathbf{p}$ , with  $0 \leq k \leq m$ ,

$$\begin{aligned} \mathcal{W}_{\text{comp}}(j \varepsilon_m^m) &= \mathcal{W}_{\text{comp}}\left(\frac{j \varepsilon^m}{N_b - 1}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{\text{comp}}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k,j,m} \mathbf{p}} \quad (\mathcal{R}9) \\ &= \sum_{k=0}^m c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{k,j,m} \mathbf{p}}, \end{aligned}$$

where, for  $0 \leq k \leq m$ ,  $\varepsilon^k$  is the  $k^{\text{th}}$  intrinsic cohomology infinitesimal, introduced in Definition 4.1, on page 21, with  $\mathbf{p} = \frac{2\pi}{\ln N_b}$  denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22] and where:

i.  $\ell_{j_{k,m},k} \in \mathbb{Z}$  is the cohomological vertex integer associated with the vertex  $M_{j_{k,m},k}$  (see Definition 5.2, on page 35);

ii.  $c_{m,j,m} = \mathcal{W}_{\text{comp}}\left(\frac{j}{N_b - 1}\right)$  and, for  $0 \leq k \leq m - 1$ ,  $c_{k,j,m} \in \mathbb{C}$  is given by

$$c_{k,j,m} = \exp\left(\frac{2i\pi}{N_b - 1} j \varepsilon^{m-k}\right). \quad (\diamond\diamond) \quad (\mathcal{R}10)$$

For  $0 \leq k \leq m$ , the coefficient  $c_{k,j,m}$  will also be referred to as the  $k^{\text{th}}$  Weierstrass coefficient associated with the vertex  $M_{j_k,m,k} \in V_k$ .

For any  $m \in \mathbb{N}$ , the complex numbers  $\{c_{0,j,m+1}, \dots, c_{m+1,j,m+1}\}$  satisfy the following recurrence relations:

$$c_{m+1,j,m+1} = \mathcal{W}\left(\frac{j}{N_b - 1}\right) = c_{m,j,m} \quad (\mathcal{R}11)$$

and

$$\forall k \in \{1, \dots, m\} : \quad c_{k,j,m+1} = c_{k-1,j,m}. \quad (\mathcal{R}12)$$

In addition, since relation (R9) is valid for any  $m \in \mathbb{N}^*$  (and since, clearly, relation (R10) implies that the coefficients  $c_{k,j,m}$  are nonzero for  $0 \leq k \leq m$ ), we deduce that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the Weierstrass function) are

$$D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i \ell_{j_k,m,k} \mathbf{P},$$

where  $m \in \mathbb{N}^*$  is arbitrary,  $0 \leq k \leq m$  and  $\ell_{j_k,m,k} \in \mathbb{Z}$  is the cohomological vertex integer associated with the vertex  $M_{j_k,m,k}$  (see Definition 5.2, on page 35). Those Complex Dimensions are all exact and simple.

This immediately ensures, for the Weierstrass function (i.e., the real part of the Complexified Weierstrass function  $\mathcal{W}_{\text{comp}}$ ), that, for any strictly positive integer  $m$  and for any  $j$  in  $\{0, \dots, \#V_m - 1\}$ ,

$$\begin{aligned} \mathcal{W}(j \varepsilon_m^m) &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{\text{comp}}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e\left(c_{k,j,m} \varepsilon_k^{i \ell_{j_k,m,k} \mathbf{P}}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}_{\text{comp}}\left(\frac{j}{N_b - 1}\right) + \frac{1}{2} \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \left(c_{k,j,m} \varepsilon^{i \ell_{j_k,m,k} \mathbf{P}} + \overline{c_{k,j,m}} \varepsilon^{-i \ell_{j_k,m,k} \mathbf{P}}\right) \\ &= \frac{1}{2} \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(c_{k,j,m} \varepsilon^{i \ell_{j_k,m,k} \mathbf{P}} + \overline{c_{k,j,m}} \varepsilon^{-i \ell_{j_k,m,k} \mathbf{P}}\right), \end{aligned} \quad (\mathcal{R}13)$$

where  $\bar{z}$  denotes the complex conjugate of  $z \in \mathbb{C}$ .

More generally, for any strictly positive integer  $m$  and for any integer  $j$ ,

$$\mathcal{W}_{\text{comp}}(j \varepsilon^m) = \sum_{k=0}^{\infty} \varepsilon^{k(2-D_{\mathcal{W}})} c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{j_k,m,k} \mathbf{P}}, \quad (\mathcal{R}14)$$

where, for all  $k \in \mathbb{N}$ ,

$$c_{k,j,m} = \varepsilon^{2i\pi N_b^k j \varepsilon^m}. \quad (\mathcal{R}15)$$

We also note that, if a vertex  $M_{j,m} = M_{j',m+m'}$  is in  $V_m \cap V_{m+m'}$ , for  $m' \in \mathbb{N}$ , we of course have that, for  $0 \leq k \leq m$ ,

$$c_{k,j,m} = c_{k,j',m+m'}, \quad (\mathcal{R} 16)$$

along with

$$\varepsilon^{i \ell_{j_k,m,k}} = \varepsilon^{i \ell_{j_k,m+m',k}}. \quad (\mathcal{R} 17)$$

For  $m+1 \leq k \leq m+m'$ , we have that

$$c_{k,j,m} = c_{k,j',m+m'} = 0. \quad (\mathcal{R} 18)$$

In addition, we have that, for  $m' \in \mathbb{N}$ ,

$$c_{k,j,m+m'} = \lambda^{m'} c_{k,j,m} = \varepsilon^{m'(2-D_{\mathcal{W}})} c_{k,j,m}, \quad (\mathcal{R} 19)$$

*Proof.* Thanks to Property 3.6, on page 9 (the Scaling Property of the Weierstrass function  $\mathcal{W}$ ), for any strictly positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$ , we then have that

$$\mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) = \lambda^m \mathcal{W}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\pi N_b^k j}{(N_b - 1) N_b^m}\right).$$

By using the Minkowski Dimension  $D_{\mathcal{W}}$  introduced in Notation 6, on page 6, this latter expression can be rewritten as

$$\begin{aligned} \mathcal{W}(j \varepsilon_m^m) &= N_b^{m(D_{\mathcal{W}}-2)} \mathcal{W}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} N_b^{k(D_{\mathcal{W}}-2)} \cos\left(\frac{2\pi}{N_b - 1} j \varepsilon^{m-k}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \cos\left(\frac{2\pi}{N_b - 1} j \varepsilon^{m-k}\right) \\ &= \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b - 1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e\left(\exp\left(\frac{2i\pi}{N_b - 1} j \varepsilon^{m-k}\right)\right). \end{aligned}$$

We must then show that there exist complex numbers  $\{c_{0,j,m}, \dots, c_{m,j,m}\}$  such that, for any integer  $k$  in  $\{0, \dots, m\}$ ,

$$\exp\left(\frac{2i\pi}{N_b - 1} j \varepsilon^{m-k}\right) = c_{k,j,m} \varepsilon^{i \ell_{j_k,m,k} \mathbf{P}} = c_{k,j,m} \exp(i \ell_{j_k,m,k} \mathbf{P} \ln \varepsilon) = c_{k,j,m} \exp(-2i\pi \ell_{j_k,m,k}),$$

where, for  $0 \leq k \leq m$ ,  $\ell_{j_k,m,k} \in \mathbb{Z}$  is the cohomological vertex integer associated with the vertex  $M_{j_k,m,k}$  (see Definition 5.2, on page 35).

We obviously obtain that  $c_{m,j,m} = \mathcal{W}_{comp}\left(\frac{j}{N_b - 1}\right)$  and that, for  $0 \leq k \leq m-1$ ,

$$c_{k,j,m} = \exp\left(\frac{2i\pi}{N_b - 1} j \varepsilon^{m-k}\right).$$

We note that, since, for any positive integer  $m$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$  (see Property 3.6, on page 9),

$$\mathcal{W}_{comp}\left(\frac{j}{(N_b - 1) N_b^{m+1}}\right) = \lambda \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) + \exp\left(\frac{2\pi j}{(N_b - 1) N_b^{m+1}}\right),$$

this ensures that

$$\begin{aligned}
\mathcal{W}_{comp}\left(\frac{j}{(N_b-1)N_b^{m+1}}\right) &= \lambda \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \lambda \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right) \\
&\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\
&= \varepsilon^{2-D_{\mathcal{W}}} \varepsilon^{m(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \varepsilon^{2-D_{\mathcal{W}}} \sum_{k=0}^{m-1} \varepsilon^{k(2-D_{\mathcal{W}})} \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m-k}\right) \\
&\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\
&= \varepsilon^{(m+1)(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{(k+1)(2-D_{\mathcal{W}})} \exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m+1-k-1}\right) \\
&\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\
&= \varepsilon^{(m+1)(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=1}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(\exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m+1-k}\right)\right) \\
&\quad + \exp\left(\frac{2\pi j}{(N_b-1)N_b^{m+1}}\right) \\
&= \varepsilon^{(m+1)(2-D_{\mathcal{W}})} \mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \left(\exp\left(\frac{2i\pi}{N_b-1} j \varepsilon^{m+1-k}\right)\right).
\end{aligned}$$

At the same time, we have that

$$\mathcal{W}_{comp}\left(\frac{j}{(N_b-1)N_b^{m+1}}\right) = \sum_{k=0}^{m+1} c_{k,j,m+1} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i\ell_{k,j,m+1}\mathbf{P}},$$

We thus deduce that the stated recurrence relations ( $\mathcal{R}11$ ), on page 37, hold:

$$c_{m+1,j,m+1} = \mathcal{W}\left(\frac{j}{N_b-1}\right) = c_{m,j,m}$$

and that, for  $1 \leq k \leq m+1$ ,

$$c_{k,j,m+1} = c_{k-1,j,m} \quad , \quad \varepsilon^{i\ell_{j_k,m+1,k}\mathbf{P}} = \varepsilon^{i\ell_{j_k,m,k}\mathbf{P}}.$$

Note that, by using the series expansion of the Weierstrass function, we also obtain that, since, for any positive integer  $m$  and any integer  $j$ ,

$$\mathcal{W}_{comp}(j \varepsilon^m) = \sum_{k=0}^{\infty} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{2i\pi N_b^k j \varepsilon^m},$$

which can be written in the following form,



$$\mathcal{W}_{comp}(j \varepsilon^m) = \sum_{k=0}^{\infty} \varepsilon^{k(2-D_{\mathcal{W}})} c_{k,j,m} \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{j_k, m, k} \mathbf{P}},$$

where, for all  $k \in \mathbb{N}$ ,

$$c_{k,j,m} = \varepsilon^{2i\pi N_b^k j \varepsilon^m}.$$

In addition, by applying the results in Property 3.6, on page 9, we have that, for all  $m' \in \mathbb{N}$ ,

$$\mathcal{W}_{comp}\left(\frac{j}{(N_b - 1) N_b^{m_0+m}}\right) = \lambda^m \mathcal{W}_{comp}\left(\frac{j}{(N_b - 1) N_b^{m_0}}\right) + \sum_{k=0}^{m_0-1} \lambda^k \exp\left(\frac{2i\pi N_b^k j}{(N_b - 1) N_b^{m_0+m}}\right),$$

or, equivalently,

$$\begin{aligned} \sum_{k=0}^{m+m'} \varepsilon^{k(2-D_{\mathcal{W}})} c_{k,j,m+m'} \varepsilon^{i \ell_{j_k, m+m', k} \mathbf{P}} &= \lambda^{m'} \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right) + \sum_{k=0}^{m-1} \lambda^k c_{k,j,m+m'} \varepsilon^{i \ell_{j_k, m+m', k} \mathbf{P}} \\ &= \lambda^{m'} \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \varepsilon^{i \ell_{j_k, m, k} \mathbf{P}} c_{k,j,m} + \sum_{k=0}^{m-1} \lambda^k c_{k,j,m+m'} \varepsilon^{i \ell_{j_k, m+m', k} \mathbf{P}} \end{aligned}$$

This ensures that, for all  $m' \in \mathbb{N}$ , and all  $k \in \{m, \dots, m+m'\}$ ,

$$c_{k,j,m+m'} \varepsilon^{i \ell_{j_k, m+m', k} \mathbf{P}} = \lambda^{m'} c_{k,j,m} = \varepsilon^{m'(2-D_{\mathcal{W}})} c_{k,j,m} \varepsilon^{i \ell_{j_k, m, k} \mathbf{P}},$$

or, in an equivalent form,

$$c_{k,j,m+m'} = \lambda^{m'} c_{k,j,m} = \varepsilon^{m'(2-D_{\mathcal{W}})} c_{k,j,m}. \quad (\mathcal{R}20)$$

□

*Remark 5.1.* We note that the results obtained in Theorem 5.4 will play a key role in the remainder of this paper. In particular, relations (R11) and (R12), on page 37 above, will be the key to obtain the recurrence relations satisfied by the local polyhedral effective fractal zeta functions and thereby, to establish the existence of the global polyhedral effective fractal zeta function.

### Property 5.5 (A Hodge Diamond Star Relation).

For any  $m \in \mathbb{N}^*$ , any  $k$  in  $\{1, \dots, m\}$  and any  $j$  in  $\{0, \dots, \#V_m - 1\}$ , we have the following Hodge Diamond Star relation

$$c_{k, (N_b-1) N_b^m - j, m} = \overline{c_{k,j,m}}. \quad (\mathcal{R}21)$$

*Remark 5.2.* The Hodge Diamond Star relation (R21) just above is directly connected to the symmetry with respect to the vertical line  $x = \frac{1}{2}$ , stated in Property 3.9, on page 15, since the points

$$\left(\frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{(N_b - 1) N_b^m - j}{(N_b - 1) N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right)\right)$$

are symmetric with respect to the vertical line  $x = \frac{1}{2}$ ; see Figure 3, on page 15. It is also reminiscent of Poincaré duality (see our previous work [DL24d]).

*Proof.* (of Property 5.5, on page 40)

We have that, for  $1 \leq k \leq m - 1$ ,

$$\begin{aligned}
c_{k,(N_b-1)N_b^{m-1}-j,m} &= \exp\left(\frac{2i\pi}{N_b-1} \left((N_b-1)N_b^{m-1} - j\right) \varepsilon^{m-k}\right) \\
&= \exp\left(2i\pi N_b^{m-1} \varepsilon^{m-k}\right) \exp\left(-\frac{2i\pi j}{N_b-1} \varepsilon^{m-k}\right) \\
&= \underbrace{\exp\left(2i\pi N_b^{m-1} N_b^{k-m}\right)}_1 \exp\left(-\frac{2i\pi j}{N_b-1} \varepsilon^{m-k}\right) \\
&= \overline{c_{k,j,m}}.
\end{aligned}$$

□

### Definition 5.3 (The Weierstrass Complex Curve).

We place ourselves in the Riemann sphere (or complex projective line)  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ . We define the *Weierstrass Curve* as the graph in  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$  associated with the Weierstrass complexified function  $\mathcal{W}_{comp}$ ; i.e., the set denoted by  $\Gamma_{\mathcal{W},comp}$  and such that

$$\Gamma_{\mathcal{W},comp} = \{(x, \mathcal{W}_{comp}(x)) \mid x \in [0, 1]\} = \Gamma_{\mathcal{W}} + i\Gamma_{\mathcal{W},\mathcal{I}},$$

where  $\Gamma_{\mathcal{W},\mathcal{I}}$  is the graph (in  $\mathbb{R}^2$ ) associated with the imaginary part  $\mathcal{W}_{\mathcal{I}}$  of the Weierstrass complexified function  $\mathcal{W}_{comp}$  and defined, for any real number  $x$ , by

$$\mathcal{W}_{\mathcal{I}}(x) = \sum_{n=0}^{\infty} \lambda^n \sin(2\pi N_b^n x). \quad (\mathcal{R} 22)$$

### Theorem 5.6 (Exact Expression for the Volume of the $m^{\text{th}}$ Polyhedral Neighborhood (or $m^{\text{th}}$ Natural Polyhedral Volume)).

Given  $m \in \mathbb{N}^*$  sufficiently large (i.e., for all  $m \geq m_0$ , where  $m_0 \in \mathbb{N}^*$  is optimal, i.e., smallest), the volume (or two-dimensional Lebesgue measure)  $\mathcal{V}_m(\varepsilon)$  of the  $m^{\text{th}}$  polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$ , or  $m^{\text{th}}$  natural polyhedral volume, is given by

$$\begin{aligned}
\mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right) \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e\left(c_{k,(N_b-1)j+q,m} \varepsilon^{i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}\right),
\end{aligned} \tag{R 23}$$

where  $\varepsilon^m = (N_b - 1) \varepsilon_m^m$  is the  $m^{\text{th}}$  intrinsic cohomology infinitesimal introduced in Definition 4.1, on page 21, while the real numbers  $\alpha_q(N_b)$ , with  $0 \leq q \leq N_b$ , are coefficients which depend on  $N_b$  and where the complex numbers  $c_{k,(N_b-1)j+q,m} \in \mathbb{C}$  have been introduced in part ii. of Theorem 5.4, on page 36, while  $(\ell_{k,(N_b-1)j+q,m})_{0 \leq k \leq m}$  denotes the sequence of cohomological vertex integers associated with the vertex  $M_{(N_b-1)j+q,m} \in V_m$  (see Definition 5.3, on page 35, along with Theorem 5.4, on page 36), with

a. When the integer  $N_b$  is odd:

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b-1}{2}}(N_b) = -\alpha_{\frac{N_b-1}{2}+N_b-1}(N_b) = \frac{N_b-2}{2(N_b-1)}$$

and for  $1 \leq q \leq N_b - 2$ ,

$$\alpha_{\frac{N_b-1}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m,(N_b-1)N_b^m-(N_b-1)+q}(N_b) = \frac{1}{N_b-1},$$

along with

$$\alpha_{N_b^m,(N_b-1)N_b^m-(N_b-1)}(N_b) = \alpha_{N_b^m,1}(N_b) = \frac{N_b-2}{2(N_b-1)}.$$

b. When the integer  $N_b$  is even:

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b}{2}}(N_b) = -\alpha_{\frac{N_b}{2}+N_b-1}(N_b) = \frac{N_b-2}{2(N_b-1)}$$

and for  $1 \leq q \leq N_b - 2$ ,

$$\alpha_{\frac{N_b}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m,(N_b-1)N_b^m-(N_b-1)+q}(N_b) = -\frac{1}{N_b-1},$$

along with

$$\alpha_{N_b^m,(N_b-1)N_b^m-(N_b-1)}(N_b) = \alpha_{N_b^m,1}(N_b) = \frac{N_b-2}{2(N_b-1)}.$$

Given the form of the expression in relation (R23) just above, it is natural to introduce, for any  $m \in \mathbb{N}^*$  sufficiently large, the associated  $m^{\text{th}}$  complex natural polyhedral volume  $\mathcal{V}_{m,\text{comp}}(\varepsilon)$ , such that

$$\mathcal{V}_{m,\text{comp}}(\varepsilon) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}. \tag{R 24}$$

By using relation (R14), on page 37, we also have that

$$\mathcal{V}_{m,comp}(\varepsilon) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+ik} \ell_{k,(N_b-1)j+q,m} \mathbf{P}. \quad (\mathcal{R}25)$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation (R15), on page 37, with  $\ell_{k,(N_b-1)j+q,m} \in \mathbb{Z}$ .

By considering, in the same manner, given  $m \in \mathbb{N}^*$  sufficiently large, the volume (or two-dimensional Lebesgue measure)  $\mathcal{V}_{\mathcal{I},m}(\varepsilon_m^m)$  of the  $m^{\text{th}}$  polygonal neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W},\mathcal{I}_m})$ , associated with the  $m^{\text{th}}$  prefractal approximation to the graph  $\Gamma_{\mathcal{W},\mathcal{I}}$  of the imaginary part  $\mathcal{W}_{\mathcal{I}}$  of the Weierstrass complexified function  $\mathcal{W}_{comp}$  (see Definition 5.3, on page 41 above), we note that the complex volume  $\mathcal{V}_{m,comp}(\varepsilon_m^m)$  can be envisioned as the volume associated with the Weierstrass complex curve  $\Gamma_{\mathcal{W},comp}$  (see again Definition 5.3, on page 41) in the Riemann sphere (or complex projective line)  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ .

*Remark 5.3.* Observe that much as for its Euclidean counterpart obtained in our earlier work, [DL22], the  $m^{\text{th}}$  polyhedral fractal formula obtained in relation (R23) of Theorem 5.6, on page 41, is expressed at the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$ , for any  $m \in \mathbb{N}^*$  sufficiently large (instead of being expressed at any  $\varepsilon > 0$ ).

Also, the complex powers of  $\varepsilon$  are required, for the sake of the determination of the Complex Dimensions; see the comments just before Definition 5.2, on page 35.

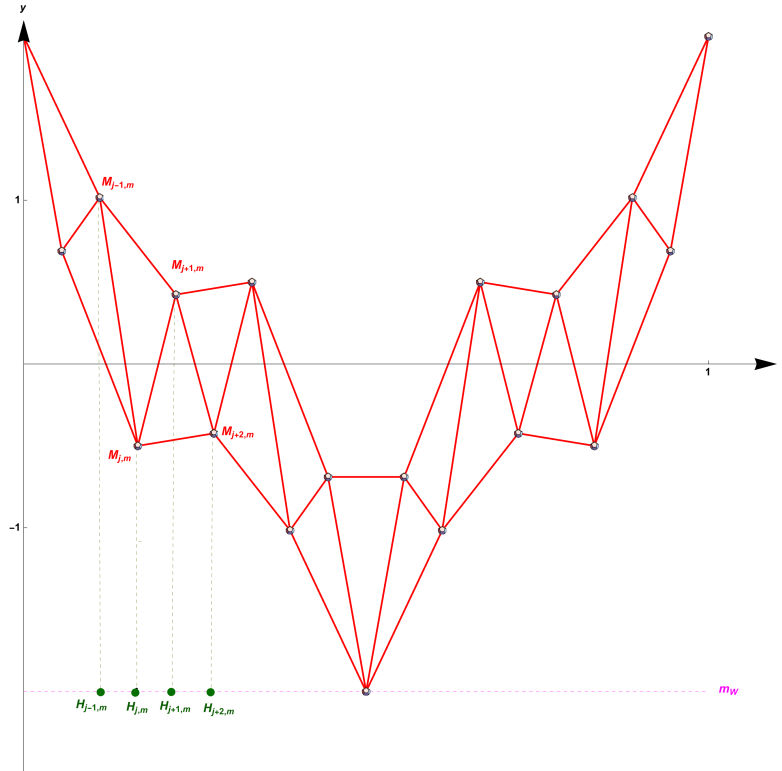


Figure 11: The orthogonal projections  $H_{j,m}$  of the vertices  $M_{j,m}$ , for  $0 \leq j \leq \#V_m - 1$ , onto the horizontal line  $y = m_{\mathcal{W}}$ .

*Proof.*

i. The special case  $N_b = 3$

First, note that since  $N_b = 3$ , the polygons involved are triangles.

Given  $m \in \mathbb{N}^*$  sufficiently large, and  $0 \leq j \leq \#V_m$ , we introduce the (vertical) orthogonal projection  $H_{j,m}$  of the vertex  $M_{j,m}$  onto the horizontal line  $y = m_{\mathcal{W}}$ , where  $m_{\mathcal{W}}$  is introduced in Notation 4.5, on page 25; see Figure 11, on page 43.

We note that, for  $0 \leq j \leq 3^m - 1$ , with the notation of Definition 3.6, on page 14, the two-dimensional Lebesgue measure (i.e., area) of the polygon  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{2j,m} \quad , \quad M_{2j+1,m} \quad \text{and} \quad M_{2j+2,m} \quad ,$$

is obtained by subtracting from the area of the trapezoid

$$H_{2j,m}M_{2j,m}M_{2j+2,m}H_{2j+2,m}$$

the area of the trapezoid

$$H_{2j,m}M_{2j,m}M_{2j+1,m}H_{2j+1,m}$$

and the area of the trapezoid

$$H_{2j+1,m}M_{2j+1,m}M_{2j+2,m}H_{2j+2,m} \quad ;$$

i.e.,

$$\begin{aligned} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) &= \frac{2\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) \right) \\ &= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) \right) \end{aligned}$$

Similarly, for  $0 \leq j \leq 3^m - 1$ , still with the notation of Definition 3.6, on page 14, the area of the polygon (triangle)  $\mathcal{Q}_{m,j+1}$ , with (consecutive) vertices

$$M_{2j+1,m} \quad , \quad M_{2j+2,m} \quad \text{and} \quad M_{2j+3,m} \quad ,$$

is obtained by adding the area of the trapezoid

$$H_{2j+1,m}M_{2j+1,m}M_{2j+2,m}H_{2j+2,m}$$

to the area of the trapezoid

$$H_{2j+2,m}M_{2j+2,m}M_{2j+3,m}H_{2j+3,m}$$

and then, by subtracting from this result the area of the trapezoid

$$H_{2j+1,m}M_{2j+1,m}M_{2j+3,m}H_{2j+3,m};$$

i.e.,

$$\begin{aligned}\mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+3}{2 \times 3^m}\right) \right) \\ &\quad - \frac{2\varepsilon_m^m}{2} \left( m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) + m_{\mathcal{W}} + \mathcal{W}\left(\frac{2j+3}{2 \times 3^m}\right) \right) \\ &= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) - \mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2j+3}{2 \times 3^m}\right) \right).\end{aligned}$$

We thus have that

$$\begin{aligned}\mathcal{V}_m(\varepsilon) &= \sum_{j=0}^{3^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,3^m}) \\ &= \sum_{j=0}^{3^m-2} \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) \right) \\ &\quad + \sum_{j=0}^{3^m-2} \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + \mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) - \mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2j+3}{2 \times 3^m}\right) \right) \\ &\quad + 2\frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2 \times 3^m - 2}{2 \times 3^m}\right) + \mathcal{W}\left(\frac{2 \times 3^m}{2 \times 3^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2 \times 3^m - 2}{2 \times 3^m}\right) + 2\mathcal{W}\left(\frac{2 \times 3^m - 1}{2 \times 3^m}\right) + \mathcal{W}\left(\frac{2 \times 3^m}{2 \times 3^m}\right) \right) \\ &= \sum_{j=0}^{3^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + 2\mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) - 3\mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2j+3}{2 \times 3^m}\right) \right) \\ &\quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{2 \times 3^m - 2}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2 \times 3^m - 1}{2 \times 3^m}\right) + 3\mathcal{W}(1) \right) \\ &= \sum_{j=0}^{3^m-2} \frac{\varepsilon_m^m}{4} \left( 2\mathcal{W}\left(\frac{2j}{2 \times 3^m}\right) + 2\mathcal{W}\left(\frac{2j+2}{2 \times 3^m}\right) - 3\mathcal{W}\left(\frac{2j+1}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2j+3}{2 \times 3^m}\right) \right) \\ &\quad + \frac{\varepsilon_m^m}{4} \left( \mathcal{W}\left(\frac{2 \times 3^m - 2}{2 \times 3^m}\right) - 2\mathcal{W}\left(\frac{2 \times 3^m - 1}{2 \times 3^m}\right) + 3\mathcal{W}(1) \right),\end{aligned}$$

since

$$\varepsilon_m^m = \frac{1}{2} \varepsilon^m.$$

For the sake of concision, we will write it in the following form

$$\mathcal{V}_m(\varepsilon) = \varepsilon^m \sum_{j=0}^{3^m-1} \sum_{q=0}^3 \alpha_q(3) \mathcal{W}\left(\frac{2j+q}{2 \times 3^m}\right),$$

where, for  $0 \leq j \leq 3^m - 2$ ,

$$\alpha_0(3) = \frac{1}{2} \quad , \quad \alpha_1(3) = \frac{1}{2} \quad , \quad \alpha_2(3) = -\frac{3}{4} \quad , \quad \alpha_3(3) = -\frac{1}{2}$$

and where

$$\alpha_{3^m-1,0}(3) = \frac{1}{4} \quad , \quad \alpha_{3^m-1,1}(3) = -\frac{1}{2} \quad , \quad \alpha_{3^m-1,2}(3) = \frac{3}{4} \quad , \quad \alpha_{3^m-1,4}(3) = 0.$$

Hence, by using the respective expressions for the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and for the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = 2\varepsilon_m^m$  given in Definition 4.1, on page 21, we have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{3^m-1} \sum_{q=0}^3 \alpha_q(3) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{k,2j+q,m} \varepsilon^{ik \ell_{k,2j+q,m} \mathbf{P}} \right), \end{aligned}$$

where the complex coefficients  $c_{k,2j+q,m}$  are given in part *ii.* of Theorem 5.4, on page 36, with  $\ell_{m,k,2j+q} \in \mathbb{Z}$  given in part *i.* of Theorem 5.4, on page 36.

An important comment: by using relation (R14), on page 37, we also have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{3^m-1} \sum_{q=0}^3 \alpha_q(3) \sum_{k=0}^{\infty} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{k,2j+q,m} \varepsilon^{ik \ell_{k,2j+q,m} \mathbf{P}} \right), \end{aligned}$$

where the complex coefficients  $c_{k,2j+q,m}$  are given by relation (R15), on page 37, with  $\ell_{k,2j+q,m} \in \mathbb{Z}$ .

*ii.* The general case  $N_b \geq 4$

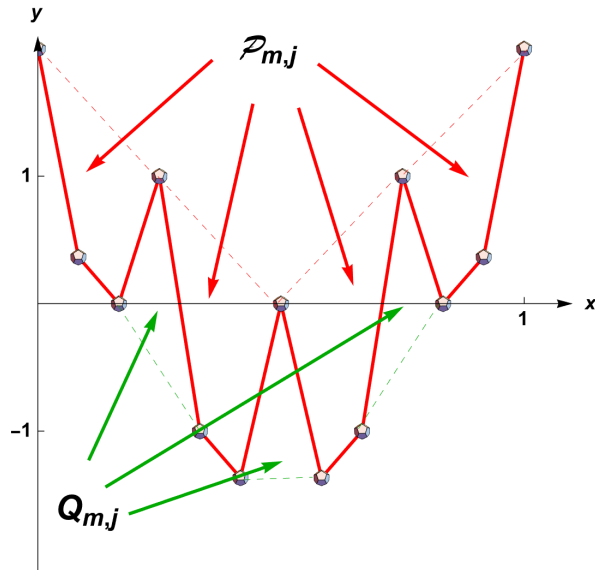


Figure 12: The polygons  $\mathcal{P}_{m,j}$  and  $\mathcal{P}_{m,j}$ , in the case when  $\lambda = \frac{1}{2}$  and  $N_b = 4$ .

As for the general case, when  $N_b \geq 4$ , it can be obtained in a similar manner.

$\rightsquigarrow$  More generally, given  $N_b \geq 6$ , the polygons  $\mathcal{P}_{m,j}$  involved are  $N_b$ -gons, while the polygons  $\mathcal{Q}_{m,j}$  are  $N_{b'}$ -gons, with  $N_{b'} \leq N_b$ . The cases  $N_b$  odd and  $N_b$  even have to be studied separately.

a. When  $N_b$  is an odd number:

For  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the  $N_b$ -gon  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j,m}, \dots, M_{(N_b-1)j+N_b-1,m}$$

is obtained by subtracting from the area of the trapezoid

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+N_b-1,m} H_{(N_b-1)j+N_b-1,m}$$

the respective areas of the following  $N_b - 1$  trapezoids

$$H_{(N_b-1)j,m} M_{(N_b-1)j,m} M_{(N_b-1)j+1,m} H_{(N_b-1)j+1,m}$$

⋮

$$H_{(N_b-1)j+N_b-2,m} M_{(N_b-1)j+N_b-2,m} M_{(N_b-1)j+N_b-1,m} H_{(N_b-1)j+N_b-1,m} ;$$

i.e.,

$$\begin{aligned} \mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) &= \frac{(N_b - 1) \varepsilon_m^m}{2} \left( \mathcal{W} \left( \frac{(N_b - 1)j}{(N_b - 1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b - 1)j + N_b - 1}{(N_b - 1)N_b^m} \right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W} \left( \frac{(N_b - 1)j + q}{(N_b - 1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b - 1)j + q + 1}{(N_b - 1)N_b^m} \right) \right) \\ &= \frac{(N_b - 1) \varepsilon_m^m}{2} \left( \mathcal{W} \left( \frac{(N_b - 1)j}{(N_b - 1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b - 1)j + N_b - 1}{(N_b - 1)N_b^m} \right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-2} 2 \mathcal{W} \left( \frac{(N_b - 1)j + q}{(N_b - 1)N_b^m} \right) - \frac{\varepsilon_m^m}{2} \mathcal{W} \left( \frac{(N_b - 1)j}{(N_b - 1)N_b^m} \right). \end{aligned}$$

We note that, for  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the non necessarily convex  $N_b$ -gon  $\mathcal{Q}_{m,j+1}$ , with (consecutive) vertices

$$M_{(N_b-1)j+\frac{N_b-1}{2},m}, \dots, M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m}$$

is obtained by subtracting the area of the trapezoid

$$H_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m},$$

from the sum of the respective areas of the trapezoids

$$H_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2},m} M_{(N_b-1)j+\frac{N_b-1}{2}+1,m} H_{(N_b-1)j+\frac{N_b-1}{2}+1,m}$$



⋮

$$H_{(N_b-1)j+\frac{N_b-1}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b-1}{2}+N_b-1,m};$$

i.e.,

$$\begin{aligned} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q+1}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right) \\ &= \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q}{(N_b-1)N_b^m}\right) \\ &\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right). \end{aligned}$$

We thus have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon) &= \sum_{j=0}^{N_b^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}) \\ &= \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+N_b-1}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-2} 2 \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right) - \frac{\varepsilon_m^m}{2} \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) \\ &\quad + \sum_{j=0}^{N_b^m-2} \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+q}{(N_b-1)N_b^m}\right) \\ &\quad - \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b-1}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right) \\ &\quad + \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-(N_b-1)}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)N_b^m}{(N_b-1)N_b^m}\right) \right) \\ &\quad - \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W}\left(\frac{(N_b-1)N_b^m-(N_b-1)+q}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)N_b^m-(N_b-1)+q+1}{(N_b-1)N_b^m}\right) \right). \end{aligned}$$

As previously, taking into account the fact that

$$\varepsilon_m^m = \frac{1}{N_b - 1} \varepsilon^m,$$

we write  $\mathcal{V}_m(\varepsilon)$  in the following form

$$\mathcal{V}_m(\varepsilon) = \varepsilon^m \sum_{j=0}^{N_b^m - 1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W} \left( \frac{(N_b - 1)j + q}{(N_b - 1)N_b^m} \right),$$

where

$$\alpha_0(N_b) = \frac{N_b - 1}{2(N_b - 1)} - \frac{1}{2(N_b - 1)} = \frac{N_b - 2}{2(N_b - 1)}$$

and for  $1 \leq q \leq N_b - 2$ ,

$$\alpha_{\frac{N_b - 1}{2} + q}(N_b) = \frac{2}{2(N_b - 1)} = \frac{1}{N_b - 1}, \quad \alpha_q(N_b) = -\frac{2}{2(N_b - 1)} = -\frac{1}{N_b - 1},$$

$$\alpha_{N_b^m, (N_b - 1)N_b^m - (N_b - 1) + q}(N_b) = -\frac{2}{2(N_b - 1)} = -\frac{1}{N_b - 1},$$

along with

$$\alpha_{\frac{N_b - 1}{2}}(N_b) = \frac{1}{2(N_b - 1)} - \frac{(N_b - 1)}{2(N_b - 1)} = -\frac{(N_b - 2)}{2(N_b - 1)},$$

$$\alpha_{\frac{N_b - 1}{2} + N_b - 1}(N_b) = \frac{1}{2(N_b - 1)} - \frac{(N_b - 1)}{2(N_b - 1)} = -\frac{(N_b - 2)}{2(N_b - 1)},$$

$$\alpha_{N_b - 1}(N_b) = -\frac{1}{2(N_b - 1)} + \frac{N_b - 1}{2(N_b - 1)} = \frac{N_b - 2}{2(N_b - 1)} = \alpha_0(N_b),$$

$$\alpha_{N_b^m, (N_b - 1)N_b^m - (N_b - 1)}(N_b) = \alpha_{N_b^m, 1}(N_b) = \frac{N_b - 1}{2(N_b - 1)} = \frac{1}{N_b - 1}.$$

Hence, by using the respective expressions of the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = (N_b - 1)\varepsilon_m^m$  given in Definition 4.1, on page 21, we have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{N_b^m - 1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{k, (N_b - 1)j + q, m} \varepsilon^{ik \ell_{k, (N_b - 1)j + q, m} \mathbf{p}} \right), \end{aligned}$$

where the complex coefficients  $c_{k, (N_b - 1)j + q, m}$  are given in part *ii.* of Theorem 5.4, on page 36, with  $\ell_{m, k, (N_b - 1)j + q} \in \mathbb{Z}$  given in part *i.* of Theorem 5.4, on page 36.

*b.* When  $N_b$  is an even number:

For  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the  $N_b$ -gon  $\mathcal{P}_{m, j+1}$ , with (consecutive) vertices

$$M_{(N_b - 1)j, m}, \dots, M_{(N_b - 1)j + N_b - 1, m}$$

is obtained exactly as in part *a*. It is the computation, for  $0 \leq j \leq N_b^m - 1$ , of the two-dimensional Lebesgue measure (i.e., area) of the nonconvex  $N_b$ -gon  $\mathcal{Q}_{m,j+1}$  with slight changes. More precisely, for  $0 \leq j \leq N_b^m - 1$ , the area of the non necessarily convex  $N_b$ -gon  $\mathcal{Q}_{m,j+1}$  with (consecutive) vertices

$$M_{(N_b-1)j+\frac{N_b}{2},m}, \dots, M_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m}$$

is obtained by subtracting the area of the trapezoid

$$H_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m},$$

from the sum of the respective areas of the trapezoids

$$H_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2},m} M_{(N_b-1)j+\frac{N_b}{2}+1,m} H_{(N_b-1)j+\frac{N_b}{2}+1,m}$$

⋮

$$H_{(N_b-1)j+\frac{N_b}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b}{2}+N_b-2,m} M_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m} H_{(N_b-1)j+\frac{N_b}{2}+N_b-1,m};$$

i.e.,

$$\begin{aligned} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \sum_{q=0}^{N_b-2} \left( \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2} + q}{(N_b-1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2} + q + 1}{(N_b-1)N_b^m} \right) \right) \\ &\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2}}{(N_b-1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2} + N_b - 1}{(N_b-1)N_b^m} \right) \right) \\ &= \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2} + 1}{(N_b-1)N_b^m} \right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2} + q}{(N_b-1)N_b^m} \right) \\ &\quad - \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2}}{(N_b-1)N_b^m} \right) + \mathcal{W} \left( \frac{(N_b-1)j + \frac{N_b}{2} + N_b - 1}{(N_b-1)N_b^m} \right) \right). \end{aligned}$$

We thus have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon) &= \sum_{j=0}^{N_b^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}) \\
&= \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-2} 2 \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right) - \frac{\varepsilon_m^m}{2} \mathcal{W}\left(\frac{(N_b-1)j}{(N_b-1)N_b^m}\right) \\
&\quad + \sum_{j=0}^{N_b^m-2} \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b}{2}+1}{(N_b-1)N_b^m}\right) + \frac{\varepsilon_m^m}{2} \sum_{q=1}^{N_b-1} 2 \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b}{2}+q}{(N_b-1)N_b^m}\right) \\
&\quad - \sum_{j=0}^{N_b^m-2} \frac{(N_b-1)\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b}{2}}{(N_b-1)N_b^m}\right) + \mathcal{W}\left(\frac{(N_b-1)j+\frac{N_b}{2}+N_b-1}{(N_b-1)N_b^m}\right) \right) \\
&\quad + \mu_{\mathcal{L}}(\mathcal{P}_{m,N_b^m}).
\end{aligned}$$

As previously, taking into account the fact that

$$\varepsilon_m^m = \frac{1}{N_b-1} \varepsilon^m,$$

we write  $\mathcal{V}_m(\varepsilon)$  in the following form,

$$\mathcal{V}_m(\varepsilon) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \mathcal{W}\left(\frac{(N_b-1)j+q}{(N_b-1)N_b^m}\right),$$

where

$$\alpha_0(N_b) = \frac{N_b-1}{2(N_b-1)} - \frac{1}{2(N_b-1)} = \frac{N_b-2}{2(N_b-1)}$$

and for  $1 \leq q \leq N_b-2$ ,

$$\alpha_{\frac{N_b}{2}+q}(N_b) = \frac{2}{2(N_b-1)} = \frac{1}{N_b-1}, \quad \alpha_q(N_b) = -\frac{2}{2(N_b-1)} = -\frac{1}{N_b-1},$$

along with

$$\begin{aligned}
\alpha_{\frac{N_b}{2}}(N_b) &= \frac{1}{2(N_b-1)} - \frac{(N_b-1)}{2(N_b-1)} = -\frac{(N_b-2)}{2(N_b-1)}, \\
\alpha_{\frac{N_b}{2}+N_b-1}(N_b) &= \frac{1}{2(N_b-1)} - \frac{(N_b-1)}{2(N_b-1)} = -\frac{(N_b-2)}{2(N_b-1)}, \\
\alpha_{N_b-1}(N_b) &= -\frac{1}{2(N_b-1)} + \frac{N_b-1}{2(N_b-1)} = \frac{N_b-2}{2(N_b-1)} = \alpha_0(N_b).
\end{aligned}$$

Hence, by using the respective expressions of the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = (N_b-1)\varepsilon_m^m$  given in Definition 4.1, on page 21, we have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{k,(N_b-1)j+q,m} \varepsilon^{i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right),
\end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given in part *ii.* of Theorem 5.4, on page 36, with  $\ell_{m,k,(N_b-1)j+q} \in \mathbb{Z}$  given in part *i.* of Theorem 5.4, on page 36.

As previously, by using relation (R14), on page 37, we also have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{\infty} \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e \left( c_{k,(N_b-1)j+q,m} \varepsilon^{i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right),
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\mathcal{V}_m(\varepsilon) &= \\
&= \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{\infty} \varepsilon^{k(2-D_{\mathcal{W}})} \left( c_{k,(N_b-1)j+q,m} \varepsilon^{i k \ell_{k,(N_b-1)j+q,m} \mathbf{P}} + \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{-i k \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right),
\end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation (R15), on page 37, with  $\ell_{k,(N_b-1)j+q,m} \in \mathbb{Z}$ .

### iii. Examples

$\rightsquigarrow$  The case  $N_b = 4$ .

Still with the notation of Definition 3.6, on page 14, the polygons  $\mathcal{P}_{m,j}$  involved are quadrilaterals, while the polygons  $\mathcal{Q}_{m,j}$  are triangles (see Figure 12 above). We note that, for  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of the quadrilateral  $\mathcal{P}_{m,j+1}$ , with (consecutive) vertices

$$M_{3j,m} \quad , \quad M_{3j+1,m} \quad , \quad M_{3j+2,m} \quad \text{and} \quad M_{3j+3,m} \quad ,$$

is obtained by subtracting from the area of the trapezoid

$$H_{3j,m} M_{3j,m} M_{3j+3,m} H_{3j+3,m} \quad ,$$

the respective areas of the following three trapezoids

$$H_{3j,m} M_{3j,m} M_{3j+1,m} H_{3j+1,m} \quad ,$$

$$H_{3j+1,m} M_{3j+1,m} M_{3j+2,m} H_{3j+2,m}$$

and

$$H_{3j+2,m} M_{3j+2,m} M_{3j+3,m} H_{3j+3,m} \quad ;$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) &= \frac{3\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{3j}{3 \times 4^m}\right) + 2\mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) + 2\mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) \right).
\end{aligned}$$

For  $0 \leq j \leq 4^m - 2$ , the area of the triangles  $\mathcal{Q}_{m,j+1}$  are obtained in the same manner as in the case when  $4 = 3$ ; i.e., in the context of  $4 = 4$ ; i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1}) &= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+4}{3 \times 4^m}\right) \right) \\
&\quad - \frac{2\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+4}{3 \times 4^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) - \mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) - \mathcal{W}\left(\frac{3j+4}{3 \times 4^m}\right) \right).
\end{aligned}$$

We thus have that

$$\begin{aligned}
\mathcal{V}_m(\varepsilon) &= \sum_{j=0}^{4^m-2} (\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) + \mu_{\mathcal{L}}(\mathcal{Q}_{m,j+1})) + \mu_{\mathcal{L}}(\mathcal{P}_{m,4^m}) \\
&= \sum_{j=0}^{4^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{3j}{3 \times 4^m}\right) - 2\mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) - 2\mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) + 2\mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) \right) \\
&\quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) - \mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) - \mathcal{W}\left(\frac{3j+4}{3 \times 4^m}\right) \right) \\
&\quad + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3(4^m-1)}{3 \times 4^m}\right) - 2\mathcal{W}\left(\frac{3(4^m-1)+1}{3 \times 4^m}\right) - 2\mathcal{W}\left(\frac{3(4^m-1)+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3(4^m-1)+3}{3 \times 4^m}\right) \right) \\
&= \sum_{j=0}^{4^m-2} \frac{\varepsilon_m^m}{2} \left( 2\mathcal{W}\left(\frac{3j}{3 \times 4^m}\right) - \mathcal{W}\left(\frac{3j+1}{3 \times 4^m}\right) - 3\mathcal{W}\left(\frac{3j+2}{3 \times 4^m}\right) \right) \\
&\quad + \sum_{j=0}^{4^m-2} \frac{\varepsilon_m^m}{2} \left( 3\mathcal{W}\left(\frac{3j+3}{3 \times 4^m}\right) - \mathcal{W}\left(\frac{3j+4}{3 \times 4^m}\right) \right) + \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{3(4^m-1)}{3 \times 4^m}\right) - 2\mathcal{W}\left(\frac{3(4^m-1)+1}{3 \times 4^m}\right) \right. \\
&\quad \left. - 2\mathcal{W}\left(\frac{3(4^m-1)+2}{3 \times 4^m}\right) + \mathcal{W}\left(\frac{3(4^m-1)+3}{3 \times 4^m}\right) \right).
\end{aligned}$$

As previously, taking into account the fact that

$$\varepsilon_m^m = \frac{1}{3} \varepsilon^m,$$

we write  $\mathcal{V}_m(\varepsilon)$  in the following form,

$$\mathcal{V}_m(\varepsilon) = \varepsilon^m \sum_{j=0}^{4^m-1} \sum_{q=0}^4 \alpha_q(4) \mathcal{W}\left(\frac{3j+q}{3 \times 4^m}\right),$$

where, for  $0 \leq j \leq 4^m - 2$ ,

$$\alpha_0(4) = \frac{1}{3} \quad , \quad \alpha_1(4) = -\frac{1}{6} \quad , \quad \alpha_2(4) = -\frac{3}{6},$$

$$\alpha_3(4) = \frac{3}{6} \quad , \quad \alpha_4(4) = -\frac{1}{6}$$

and

$$\alpha_{4^m-1,0}(4) = \frac{1}{6} \quad , \quad \alpha_{4^m-1,1}(4) = -\frac{1}{3} \quad , \quad \alpha_{4^m-1,2}(4) = -\frac{1}{3},$$

$$\alpha_{4^m-1,3}(4) = \frac{1}{6} \quad , \quad \alpha_{4^m-1,4}(4) = 0.$$

Hence, by using the respective expressions of the  $m^{\text{th}}$  cohomology infinitesimal  $\varepsilon_m^m$  and of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m = 3\varepsilon_m^m$  given in Definition 4.1, on page 21, we have that

$$\begin{aligned} \mathcal{V}_m(\varepsilon) &= \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})) \\ &= \varepsilon^m \sum_{j=0}^{4^m-1} \sum_{q=0}^4 \alpha_q(4) \sum_{k=0}^m \varepsilon^{k(2-D_{\mathcal{W}})} \mathcal{R}e\left(c_{k,3j+q,m} \varepsilon^{ik\ell_{k,3j+q,m}} \mathbf{P}\right), \end{aligned}$$

where the complex coefficients  $c_{k,3j+q,m}$  are given in part *ii.* of Theorem 5.4, on page 36, with  $\ell_{m,k,3j+q} \in \mathbb{Z}$  given in part *i.* of Theorem 5.4, on page 36.

$\rightsquigarrow$  The case  $N_b = 5$ .

The polygons involved are pentagons and quadrilaterals. We note that, for  $0 \leq j \leq N_b^m - 1$ , the two-dimensional Lebesgue measure (i.e., area) of a pentagon  $\mathcal{P}_{m,j+1}$ , (consecutive) vertices

$$M_{4j,m} \quad , \quad M_{4j+1,m} \quad , \quad M_{4j+2,m} \quad , \quad M_{4j+3,m} \quad \text{and} \quad M_{4j+4,m},$$

is obtained by subtracting from the area of the trapezoid

$$H_{4j,m} M_{4j,m} M_{4j+3,m} H_{4j+4,m}$$

the respective areas of the following four trapezoids

$$H_{4j,m} M_{4j,m} M_{4j+1,m} H_{4j+1,m} \quad , \quad H_{4j+1,m} M_{4j+1,m} M_{4j+2,m} H_{4j+2,m} \quad ,$$

$$H_{4j+2,m} M_{4j+2,m} M_{4j+3,m} H_{4j+3,m} \quad , \quad H_{4j+3,m} M_{4j+3,m} M_{4j+4,m} H_{4j+4,m} \quad ;$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}(\mathcal{P}_{m,j+1}) &= \frac{4\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{4j}{4 \times 5^m}\right) + \mathcal{W}\left(\frac{4j+3}{4 \times 5^m}\right) \right) - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{4j}{4 \times 5^m}\right) + \mathcal{W}\left(\frac{4j+1}{4 \times 5^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{4j+1}{4 \times 5^m}\right) + \mathcal{W}\left(\frac{4j+2}{4 \times 5^m}\right) \right) - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{4j+2}{4 \times 5^m}\right) + \mathcal{W}\left(\frac{4j+3}{4 \times 5^m}\right) \right) \\
&\quad - \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{4j+3}{4 \times 5^m}\right) + \mathcal{W}\left(\frac{4j+4}{4 \times 5^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( \mathcal{W}\left(\frac{4j}{4 \times 5^m}\right) - 2\mathcal{W}\left(\frac{4j+1}{4 \times 5^m}\right) - 2\mathcal{W}\left(\frac{4j+2}{4 \times 5^m}\right) \right) \\
&= \frac{\varepsilon_m^m}{2} \left( 3\mathcal{W}\left(\frac{4j+3}{4 \times 5^m}\right) - 2\mathcal{W}\left(\frac{4j+4}{4 \times 5^m}\right) \right).
\end{aligned}$$

As previously, for  $0 \leq j \leq 5^m - 2$ , the area of the quadrilaterals are obtained similarly as in the case when  $N_b = 4$ .

□

**Definition 5.4** ( $m^{\text{th}}$  Tubular Volume [DL22]).

Given  $m \in \mathbb{N}$  sufficiently large (so that  $\varepsilon_m^m$  be a sufficiently small positive number), we call  $m^{\text{th}}$  tubular volume the two-dimensional Lebesgue measure  $\mathcal{V}_m^{\text{tube}}(\varepsilon_m^m)$  of the  $(m, \varepsilon_m^m)$ -neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)$  of the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  introduced in Definition 5.1, on page 29.

There now remains to overcome the following difficulty: contrary to our previous work involving ordinary tubular neighborhoods (see [DL22]), the formulae respectively obtained in relation (R23), on page 42 above and in relation (R27), on page 56 below, which give, for all  $m \in \mathbb{N}$ , the respective expressions for the  $m^{\text{th}}$  natural polyhedral volume  $\mathcal{V}_m(\varepsilon_m^m)$  and the  $m^{\text{th}}$  natural complex polyhedral volume  $\mathcal{V}_{m,\text{comp}}(\varepsilon)$ , do not have the same form as the expression computed in [DL22] for the  $m^{\text{th}}$  tubular volume  $\mathcal{V}_m^{\text{tube}}$ .

In our present context, the  $m^{\text{th}}$  natural polyhedral volume  $\mathcal{V}_m(\varepsilon_m^m)$  and the  $m^{\text{th}}$  natural complex polyhedral volume  $\mathcal{V}_{m,\text{comp}}(\varepsilon)$  are expressed in function of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$ , but associated with a sum of  $N_b^m = \varepsilon^{-m}$  terms, themselves expressed in terms of the  $k^{\text{th}}$  intrinsic cohomology infinitesimals  $\varepsilon^k$ , for  $0 \leq k \leq m$ . Therefore,  $\mathcal{V}_m(\varepsilon_m^m)$  and  $\mathcal{V}_{m,\text{comp}}(\varepsilon)$  cannot thus be expressed in an explicit way at an arbitrary value  $\varepsilon > 0$ .

For those reasons, when it comes to obtaining the associated fractal zeta function, we cannot, as in the case of an arbitrary bounded subset of  $\mathbb{R}^2$  (see [LRŽ17], Definition 2.2.8, page 118), directly use an integral formula of the form

$$\tilde{\zeta}_m(s) = \int_0^{\varepsilon_m} t^{s-3} \mathcal{V}_m(t) dt = \int_0^{\varepsilon_m} t^{s-2} \mathcal{V}_m(t) \frac{dt}{t}. \quad (\mathcal{R} 26)$$

We can bypass these difficulties by noting that, given  $m \in \mathbb{N}$  sufficiently large,



i. the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$  associated with the aforementioned sum of  $N_b^m = \varepsilon^{-m}$  terms has to be considered as a constant value in any computation of a fractal zeta function associated with the  $m^{\text{th}}$  polyhedral volume. This means, in particular, that we cannot substitute an integration variable for this specific  $\varepsilon^m$ .

ii. As for the terms of the sum of  $N_b^m = \varepsilon^{-m}$  terms, themselves expressed in terms of the  $k^{\text{th}}$  intrinsic cohomology infinitesimals  $\varepsilon^k$ , for  $0 \leq k \leq m$ , we note that due to Property 3.6, on page 9 above, they can be interpreted as the expansion of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$  with respect to a specific polynomial basis.

**Lemma 5.7 (Natural Polyhedral Volume Extension Formula).**

Given  $m \geq 1$  sufficiently large, we set

$$\mathcal{V}_{m,comp}(\varepsilon) = \varepsilon^m \mathcal{V}_{partial,m,comp}(\varepsilon), \quad (\mathcal{R} 27)$$

where

$$\begin{aligned} \mathcal{V}_{partial,m,comp}(\varepsilon) &= \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \mathcal{V}_{partial,m,comp}(\varepsilon) &= \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation (R15), on page 37, with  $\ell_{k,(N_b-1)j+q,m} \in \mathbb{Z}$ .

Here,  $\varepsilon^m = (N_b - 1) \varepsilon_m^m$  is the  $m^{\text{th}}$  intrinsic cohomology infinitesimal introduced in Definition 4.1, on page 21, while the real numbers  $\alpha_q(N_b)$ , with  $0 \leq q \leq N_b$ , have been introduced in Theorem 5.6, on page 41, and the complex numbers  $c_{k,(N_b-1)j+q,m} \in \mathbb{C}$  have been introduced in part ii. of Theorem 5.4, on page 36, while  $(\ell_{k,(N_b-1)j+q,m})_{0 \leq k \leq m}$  denotes the sequence of cohomological vertex integers associated with the vertex  $M_{(N_b-1)j+q,m} \in V_m$  (see also Definition 5.2, on page 35). We have used the notation  $\mathcal{V}_{partial,m,comp}(\varepsilon^m)$ , since, as was mentioned previously, the sum

$$\begin{aligned} &\frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_W)+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ & + \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{k(2-D_W)+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \end{aligned}$$

(where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation (R15), on page 37, with  $\ell_{k,(N_b-1)j+q,m} \in \mathbb{Z}$ ), can be considered as a function of the  $m^{\text{th}}$  intrinsic cohomology infinitesimal  $\varepsilon^m$ .

We then introduce, for all sufficiently large  $m \in \mathbb{N}^*$ ,  $\tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  as the continuous function defined for all  $t \in [0, \varepsilon]$  by substituting  $t$  for  $\varepsilon$  on the right-hand side of the expression for  $\mathcal{V}_{\text{partial},m,\text{comp}}(\varepsilon)$ , in relation (R27) just above; i.e.,

$$\begin{aligned} \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t) &= \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) c_{k,(N_b-1)j+q,m} t^{k(2-D_W)+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^m \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} t^{k(2-D_W)-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned} \tag{R28}$$

or, equivalently,

$$\begin{aligned} \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t) &= \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m} t^{k(2-D_W)+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} t^{k(2-D_W)-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned} \tag{R29}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation (R15), on page 37, with  $\ell_{k,(N_b-1)j+q,m} \in \mathbb{Z}$ .

As is explained in [DL22] (in the case of the ordinary Euclidean volume), one can think of  $\varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t)$  as being the effective polyhedral volume of the  $m^{\text{th}}$  prefractal approximation to the Weierstrass Curve.

Also,  $\tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(\varepsilon)$  and  $\tilde{\mathcal{V}}_{\text{partial},m+1,\text{comp}}(\varepsilon)$  satisfy the following recurrence relation,

$$\begin{aligned} & \varepsilon^{m+1} \mathcal{V}_{\text{partial},m+1,\text{comp}}(\varepsilon) = \\ & = \varepsilon \times \varepsilon^m \mathcal{V}_{\text{partial},m,\text{comp}}(\varepsilon) + \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_W)+i((N_b-1)j+q) \mathbf{P}} \\ & + \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \varepsilon^{k(2-D_W)-i((N_b-1)j+q) \mathbf{P}}. \end{aligned} \tag{R30}$$

*Proof.* Given  $m \in \mathbb{N}$ , we aim at obtaining a recurrence relation satisfied by  $\tilde{\mathcal{V}}_{partial,m,comp}(\varepsilon)$  and  $\tilde{\mathcal{V}}_{partial,m+1,comp}(\varepsilon)$ .

We note that, given  $m \in \mathbb{N}$ , the expression of  $\tilde{\mathcal{V}}_{partial,m,comp}(\varepsilon)$  involves vertices in  $V_m \subset V_{m+1}$ , which correspond to the value  $q = 0$ . By using relation (R16), on page 38, we thus have that

$$\begin{aligned} & \sum_{j=0}^{N_b^{m+1}-1} \sum_{k=0}^{\infty} \alpha_0(N_b) c_{k,(N_b-1)j,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j)\mathbf{p}} = \\ & = \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \end{aligned}$$

and, similarly,

$$\begin{aligned} & \sum_{j=0}^{N_b^{m+1}-1} \sum_{k=0}^{\infty} \alpha_0(N_b) \overline{c_{k,(N_b-1)j,m+1}} \varepsilon^{k(2-D_{\mathcal{W}})-i((N_b-1)j)\mathbf{p}} = \\ & = \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{k(2-D_{\mathcal{W}})+-((N_b-1)j+q)\mathbf{p}} \end{aligned}$$

(Note that the indices  $j$  in the first and second parts in the equality are dummy indices; in the first expression, they refer to the vertices  $M_{j,m+1} \in V_m \cap V_{m+1}$ , while, in the second expression, they refer to the vertices  $M_{j,m} \in V_m$ .)

We then obtain, for the first sum involved in the expression for  $\mathcal{V}_{partial,m+1,comp}(\varepsilon)$  (i.e., the expression without the complex conjugates) that

$$\sum_{j=0}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} =$$

$$\begin{aligned}
&= \sum_{j=0}^{N_b^{m+1}-1} \sum_{k=0}^{\infty} \alpha_0(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&+ \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&= \sum_{j=0}^{N_b^{m+1}-1} \sum_{k=0}^{\infty} \alpha_0(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j)\mathbf{p}} \\
&+ \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&= \sum_{j=0}^{N_b^{m+1}-1} \sum_{k=0}^{\infty} \alpha_0(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j)\mathbf{p}} \\
&= \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&+ \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}
\end{aligned}$$

(and similarly, for the second sum involved in the expression for  $\mathcal{V}_{partial,m+1,comp}(\varepsilon)$  (i.e., the expression with the complex conjugates)), from which we deduce that

$$\begin{aligned}
\varepsilon^{m+1} \mathcal{V}_{partial,m+1,comp}(\varepsilon) &= \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \varepsilon^{k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}} \\
&= \varepsilon \times \varepsilon^m \mathcal{V}_{partial,m,comp}(\varepsilon) \\
&+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \varepsilon^{k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}.
\end{aligned}$$

as desired. □

**Notation 11 (Natural Polyhedral Complex Volume Extension).**

For the sake of simplicity, given  $m \in \mathbb{N}^*$  sufficiently large, we will from now on call *the  $m^{\text{th}}$  natural polyhedral complex volume extension*, the volume extension function  $\tilde{\mathcal{V}}_{m,\text{comp}} = \varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  associated in Lemma 5.7, on page 56, with the  $m^{\text{th}}$  natural polyhedral complex volume  $\mathcal{V}_{m,\text{comp}}$  introduced in Theorem 5.6, on page 41. Alternatively,  $\tilde{\mathcal{V}}_{m,\text{comp}}$  will be called *the  $m^{\text{th}}$  effective polyhedral complex volume*.

In the same way, as is done in [DL22], and given  $m \in \mathbb{N}^*$  sufficiently large, we call *the  $m^{\text{th}}$  natural volume extension*, the volume extension function  $\tilde{\mathcal{V}}_m^{\text{tube}}$  associated with the *the  $m^{\text{th}}$  tubular volume*  $\mathcal{V}_m^{\text{tube}}$ . Alternatively,  $\tilde{\mathcal{V}}_m^{\text{tube}}$  will be called *the  $m^{\text{th}}$  effective tubular volume*.

**Proposition 5.8 (Zeta Function Associated With the Weierstrass Function at a Vertex  $X$ ).**

We introduce the zeta function associated with the Weierstrass function at a vertex  $X \in V^* = \bigcup_{n \in \mathbb{N}} V_n$ , of associated sequence of vertex integers  $(\ell_{X,k})_{0 \leq k \leq m}$  defined, for all  $s \in \mathbb{C}$ , by the convergent series

$$\zeta_{\mathcal{W},X}(s) = \sum_{k=1}^{\infty} c_{X,k} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{X,k}\mathbf{P}}}{s-2+k(2-D_{\mathcal{W}})+i\ell_{X,k}\mathbf{P}}, \quad (\mathcal{R}31)$$

where, for each integer  $k \geq 1$ ,  $c_{X,k} \in \mathbb{C}$  is the  $k^{\text{th}}$  Weierstrass coefficient associated with the vertex  $X$  (see Theorem 5.4, on page 36).

We note that the (global) zeta function associated with the Weierstrass function can be defined, for all  $s \in \mathbb{C}$ , by the following convergent Riemann sums:

$$\begin{aligned} \zeta_{\mathcal{W}}(s) &= \lim_{m \rightarrow \infty} \frac{1}{N_b^m} \sum_{j=0}^{N_b^m-1} \zeta_{\mathcal{W},M_{j,m}}(s) \\ &= \lim_{m \rightarrow \infty} \varepsilon^m \sum_{j=0}^{N_b^m-1} \zeta_{\mathcal{W},M_{j,m}}(s). \\ &= \lim_{m \rightarrow \infty} \varepsilon^m \sum_{q=1}^{N_b-1} \sum_{j=0}^{N_b^m-1} \zeta_{\mathcal{W},M_{(N_b-1)j+q}}(s) \\ &= \int_0^\varepsilon \sum_{X \in V^*} \sum_{k=1}^{\infty} c_{X,k} t^{s-3} t^{k(2-D_{\mathcal{W}})+i\ell_{k,X}\mathbf{P}} dt. \end{aligned} \quad (\mathcal{R}32)$$

For the sake of concision, we note that the convergence of the Riemann sum in relation (R32), on page 60, can be established exactly in the same way as the convergence of the Riemann sum in relation (R46) in the proof of Theorem 5.9 below (the same proof holds by letting, for  $1 \leq q \leq N_b - 1$ ,  $\alpha_q(N_b) = 1$ ). Furthermore, one shows similarly that the convergence of the Riemann sum in relation (R32), on

page 60, is locally uniform on  $\mathbb{C}$ ; see Remark 5.5 below, on page 63.

The following key theorem completes the main results of [DL22].

**Theorem 5.9 (Local and Global Polyhedral Effective Zeta Functions).**

Given any  $m \in \mathbb{N}^*$  sufficiently large (i.e., for all  $m \geq m_0$ , where  $m_0 \in \mathbb{N}^*$  is optimal, as in Theorem 5.6, on page 41), we introduce the  $m^{\text{th}}$  local polyhedral effective zeta function  $\tilde{\zeta}_m^e$ , such that, for  $\text{Re}(s) > D_{\mathcal{W}}$ ,

$$\tilde{\zeta}_m^e(s) = \int_0^\varepsilon t^{s-3} \varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t) dt \quad (\mathcal{R} 33)$$

where  $\varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  is the  $m^{\text{th}}$  effective polyhedral complex volume, introduced in Notation 11, on page 60 above, with  $\varepsilon$  denoting the  $m^{\text{th}}$  intrinsic cohomology infinitesimal introduced in Definition 4.1, on page 21. We caution the reader that  $\tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}$  depends on  $m$ , and is such that, for all  $t \in [0, \varepsilon]$ ,  $\varepsilon^m \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t^m)$  is bounded, independently of  $m$ .

Then, still for all  $m \in \mathbb{N}^*$  sufficiently large (i.e., for all  $m \geq m_0$ ), we have that, for all  $s \in \mathbb{C}$ ,

$$\begin{aligned} \tilde{\zeta}_m^e(s) &= \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-2+k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned} \quad (\mathcal{R} 34)$$

or, equivalently,

$$\begin{aligned} \tilde{\zeta}_m^e(s) &= \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\ &+ \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}, \end{aligned} \quad (\mathcal{R} 35)$$

where, for  $0 \leq j \leq N_b^m - 1$  and  $0 \leq q \leq N_b$ , the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation (R10), on page 37,  $(\ell_{k,(N_b-1)j+q,m})_{0 \leq k \leq m}$  denotes the sequence of cohomological vertex integers associated with the vertex  $M_{(N_b-1)j+q,m} \in V_m$  (see Definition 5.2, on page 35, along with Definition 5.3, on page 35), and where  $\varepsilon$  is the intrinsic scale introduced in Definition 4.1, on page 21). In addition, the coefficients  $\alpha_q(N_b) \in \mathbb{R}$  are introduced in Theorem 5.6, on page 41, while the coefficients  $c_{k,(N_b-1)j+q,m}$  belong to  $\mathbb{C}$ . (See also Remark 5.6 below, on page 63.)

We note that, still for all  $m \in \mathbb{N}^*$  sufficiently large, by letting, for all  $s \in \mathbb{C}$ ,

$$Z_m(s) =$$

$$\begin{aligned}
&= \ln \left( \prod_{j=0}^{N_b^m-1} \prod_{q=0}^{N_b} \prod_{k=1}^m (s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i \ell_{k,(N_b-1)j+q,m} \mathbf{P})^{\alpha_q(N_b) c_{k,(N_b-1)j+q,m}} \varepsilon^{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right) \\
&+ \ln \left( \prod_{j=0}^{N_b^m-1} \prod_{q=0}^{N_b} \prod_{k=1}^m (s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i \ell_{k,(N_b-1)j+q,m} \mathbf{P})^{\alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}}} \varepsilon^{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \right),
\end{aligned} \tag{R 36}$$

we then obtain, for the  $m^{\text{th}}$  local polyhedral effective zeta function  $\tilde{\zeta}_m^e$ , the following expression, valid for all  $s \in \mathbb{C}$ :

$$\tilde{\zeta}_m^e(s) = \frac{1}{2} \varepsilon^m \left( \frac{dZ_m(s)}{ds} - \frac{1}{\ln \varepsilon} Z_m(s) \right) = \frac{1}{2} \varepsilon^m \left( \frac{dZ_m(s)}{ds} + \frac{\mathbf{P}}{2\pi} Z_m(s) \right). \tag{R 37}$$

Furthermore,  $Z_m$  is meromorphic on all of  $\mathbb{C}$ .

In addition, still for all  $m \in \mathbb{N}^*$  sufficiently large, the function  $\tilde{\zeta}_m^e = \zeta_m^e(s)$  – initially given by relation (R33) for  $\text{Re}(s) > D_{\mathcal{W}}$  – admits a (necessarily unique) meromorphic extension to  $\mathbb{C}$  (still denoted by  $\tilde{\zeta}_m^e$ ) and is given, for all  $s \in \mathbb{C}$ , by the well-defined expressions in relation (R34) (or, equivalently, (R35)) above.

Moreover, the associated sequence  $(\tilde{\zeta}_m^e)_{m \in \mathbb{N}}$  satisfies the following recurrence relation, for all values of the positive integer  $m$  sufficiently large, and for all  $s \in \mathbb{C}$ :

$$\begin{aligned}
\tilde{\zeta}_{m+1}^e(s) &= \varepsilon \tilde{\zeta}_m^e(s) \\
&+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i((N_b-1)j+q)\mathbf{P}}}{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i((N_b-1)j+q)\mathbf{P}} \\
&+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \frac{\varepsilon^{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i((N_b-1)j+q)\mathbf{P}}}{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i((N_b-1)j+q)\mathbf{P}},
\end{aligned} \tag{R 38}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m+1}$  are given by relation (R10), on page 37.

This ensures the existence of the limit fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , i.e., the fractal zeta function associated with the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (or, rather, with the Weierstrass IFD), called the global polyhedral effective zeta function and given, for all  $s \in \mathbb{C}$ , by

$$\tilde{\zeta}_{\mathcal{W}}^e(s) = \lim_{m \rightarrow \infty} \tilde{\zeta}_m^e(s), \tag{R 39}$$

where the convergence in relation (R39) is locally uniform on  $\mathbb{C}$  (see Remark 5.5 below, on page 63), along with the existence of an integer  $m_0 \in \mathbb{N}$  such that, for any integer  $m \geq m_0$ , the set of poles of  $\tilde{\zeta}_m^e$  is a subset of the set of poles of  $\tilde{\zeta}_{m+1}^e$  – and hence also, of that of  $\tilde{\zeta}_{\mathcal{W}}^e$ . More specifically,  $\tilde{\zeta}_{\mathcal{W}}^e$  is meromorphic in all of  $\mathbb{C}$  and its (necessarily unique) meromorphic extension to  $\mathbb{C}$  (still denoted  $\tilde{\zeta}_{\mathcal{W}}^e$ ) is given, for all  $s \in \mathbb{C}$ , by the following limit of Riemann sums:

$$\begin{aligned}
\tilde{\zeta}_{\mathcal{W}}^e(s) &= \frac{1}{2} \frac{1}{1 - \varepsilon} \lim_{m \rightarrow \infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i((N_b-1)j+q)\mathbf{P}}}{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i((N_b-1)j+q)\mathbf{P}} \\
&+ \frac{1}{2} \frac{1}{1 - \varepsilon} \lim_{m \rightarrow \infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \frac{\varepsilon^{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i((N_b-1)j+q)\mathbf{P}}}{s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i((N_b-1)j+q)\mathbf{P}},
\end{aligned} \tag{R 40}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m+1}$  are given by relation (R10), on page 37.

In addition,  $\mathcal{P}(\tilde{\zeta}_{\mathcal{W}}^e)$ , the set of poles, in  $\mathbb{C}$ , of the meromorphic function  $\tilde{\zeta}_{\mathcal{W}}^e$  on  $\mathbb{C}$ , is given by the following discrete (and countably infinite) subset of  $\mathbb{C}$ :

$$\mathcal{P}(\tilde{\zeta}_{\mathcal{W}}^e) = \mathcal{P}^* = \bigcup_{m \in \mathbb{N}, m \geq m_0} \mathcal{P}(\tilde{\zeta}_m^e);$$

see Lemma 5.10 below, on page 64, along with Theorem 5.13 above, on page 77.

In other words,  $\tilde{\zeta}_{\mathcal{W}}^e$  admits a (necessarily unique) meromorphic continuation, given, for all  $s \in \mathbb{C}$ , by relation (R40) just above.

Here,  $m_0 \in \mathbb{N}^*$  is the optimal (i.e., the smallest) positive integer such that  $\tilde{\zeta}_m^e$  is meromorphic, for all  $m \geq m_0$  (i.e., such that Theorem 5.6 above, on page 41 and Lemma 5.10 below, on page 64, hold).

*Remark 5.4.* As was mentioned in the introduction, we note that our result is stronger than the one previously obtained in [DL22], where, in particular, the values of the possible Complex Dimensions of the Weierstrass IFD included  $-2$ ,  $0$  and  $1 - 2k$ , with  $k \in \mathbb{N}$  arbitrary. As we can see in relation (40) just above, the poles of the limit effective fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$  are exactly the same as the Complex Dimensions of the Weierstrass function itself; see Theorem 5.4, on page 36. Note that, in [DL22], the Complex Dimensions are defined in terms of the volume of the tubular (rather than polyhedral) neighborhood of the Weierstrass IFD.

*Remark 5.5.* Both the local uniform convergence on  $\mathbb{C}$  of the Riemann sums in relation (R32), on page 60 (in Proposition 5.8) and of the Riemann sums in relation (R39) in Theorem 5.9, hold when the meromorphic functions involved are viewed as continuous functions on  $\mathbb{C}$  with values in the Riemann sphere  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ , equipped with the chordal metric (as, for example, in [LvF00], [LvF13], [LRŽ17]); see the discussion surrounding relations (R50) and (R51), on page 72, towards the end of the proof of Theorem 5.9 given below.

*Remark 5.6.* As we will see in the proof of Theorem 5.9 below, the zeta functions  $\tilde{\zeta}_m^e$  used in the statement of Theorem 5.9, on page 61, are the restricted ones (still denoted  $\tilde{\zeta}_m^e$ , for simplicity), obtained from the original ones defined by relation (R33), on page 61, by removing the pole part corresponding to the simple pole at  $s = 2$ ; see the discussion immediately following relation (R41) below, on page 66.

We will implicitly make the same assumption throughout the remainder of this paper – including in Definition 5.5, on page 76, in Theorem 5.12, on page 74, and Theorem 5.13, on page 77, along with its corollaries.

*Remark 5.7.* The zeta functions introduced in Theorem 5.9, on page 61, have a tilde, in order to distinguish them from the zeta functions of the classical theory [LRŽ17].

The following lemma will be used in part *iii.* of the proof of Theorem 5.9.



**Lemma 5.10 (A Preliminary Result About the Sets of Poles of the Local Polyhedral Zeta Functions).**

Let us denote by  $\mathcal{P}^\star = \bigcup_{m \geq m_0} \mathcal{P}(\tilde{\zeta}_m^e)$  the union of the discrete sets of poles of the local polyhedral zeta functions  $\tilde{\zeta}_m^e$ , for all  $m \in \mathbb{N}$  such that  $m \geq m_0$ . Then, the sequence  $(\mathcal{P}(\tilde{\zeta}_m^e))_{m \geq m_0}$  of subsets of  $\mathbb{C}$  is increasing.

Furthermore, the set  $\mathcal{P}^\star$  is discrete and countably infinite. It is, also, the set of poles of the global polyhedral zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ .

In addition, its complement  $\mathcal{U}$  in  $\mathbb{C}$  is an open, connected subset of  $\mathbb{C}$  (since  $\mathcal{P}^\star$  is necessarily closed).

*Proof.* (of Lemma 5.10)

The proof relies on the fact that, for all  $m \in \mathbb{N}$  sufficiently large,

$$\mathcal{P}(\tilde{\zeta}_m^e) = \{D_{\mathcal{W}} - (k-1)(2 - D_{\mathcal{W}}) \pm i \ell_{k, (N_b-1)j+q, m} \mathbf{p}, 1 \leq k \leq m\} \not\subseteq \mathcal{P}(\tilde{\zeta}_{m+1}^e).$$

This ensures that, as a countable union of discrete sets, and without any accumulating points, the set  $\mathcal{P}^\star$  is itself discrete and, hence, countably infinite. This ensures that its complement  $\mathcal{U} \subset \mathbb{C}$  is an open, connected subset of  $\mathbb{C}$ . This implies that  $\mathcal{P}^\star$  intersects each compact set  $\mathcal{K} \subset \mathbb{C}$  at a finite number of points.

The fact that  $\mathcal{P}^\star$  is the set of poles of the global polyhedral zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$  comes from the expression given in relation (R40), on page 62.

Also, the fact that, for all  $m \geq m_0$ ,  $\mathcal{P}(\tilde{\zeta}_m^e) \subseteq \mathcal{P}(\tilde{\zeta}_{m+1}^e)$ , is established in the proof of Theorem 5.9 below. □

*Proof.* (of Theorem 5.9)

*i.* We first give the explicit expression for the  $m^{\text{th}}$  local effective polyhedral zeta function  $\tilde{\zeta}_m^e$ .

In light of relation (R33), on page 61, along with relation (R28), on page 57, or relation (R29), on page 57, we restrict ourselves to sufficiently large values of  $m \in \mathbb{N}$ , i.e.,  $m \geq m_0$ , for some suitable integer  $m_0 \in \mathbb{N}$ .

We have that, for  $\text{Re}(s) > D_{\mathcal{W}}$ ,

$$\begin{aligned}
\tilde{\zeta}_m^e(s) &= \int_0^\varepsilon t^{s-3} \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t) dt \quad (\text{for } \mathcal{R}e(s) > D_{\mathcal{W}}) \\
&= \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&\quad + \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^m \frac{\overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{s-2+k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}},
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
\tilde{\zeta}_m^e(s) &= \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^{\infty} \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&\quad + \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^{\infty} \frac{\overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{s-2+k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}},
\end{aligned}$$

where the complex coefficients  $c_{k,(N_b-1)j+q,m}$  are given by relation  $(\mathcal{R}10)$ , on page 37.

We note that this expression involves the pole  $s = 2$ , which comes from the Euclidean polygons themselves involved in the sequence of polyhedral neighborhoods (see Theorem 5.6, on page 41). For this reason, we hereafter exclude the terms  $\frac{c_{0,(N_b-1)j+q,m} \varepsilon^{s-2+i\ell_{0,(N_b-1)j+q,m} \mathbf{P}}}{s-2+i\ell_{0,(N_b-1)j+q,m} \mathbf{P}}$  from the expression for the  $m^{\text{th}}$  local effective polyhedral zeta function  $\tilde{\zeta}_m^e$ , so that

$$\begin{aligned}
\tilde{\zeta}_m^e(s) &= \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^m \frac{c_{k,(N_b-1)j+q,m} \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}+i\ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&\quad + \frac{1}{2} \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^m \frac{\overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{s-2+k(2-D_{\mathcal{W}})-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s+2(k-1)-kD_{\mathcal{W}}-i\ell_{k,(N_b-1)j+q,m} \mathbf{P}}.
\end{aligned}$$

For the sake of simplicity, we still use the same notation  $\tilde{\zeta}_m^e$  for this  $m^{\text{th}}$  restricted local effective polyhedral zeta function.

By using the recurrence relation  $(\mathcal{R}30)$ , on page 57, we then compute

$$\begin{aligned}
\tilde{\zeta}_{m+1}^e(s) &= \int_0^\varepsilon t^{s-3} \tilde{\mathcal{V}}_{\text{partial},m+1,\text{comp}}(t) dt \quad (\text{for } \mathcal{R}e(s) > D_{\mathcal{W}}) \\
&= \varepsilon \int_0^\varepsilon t^{s-3} \tilde{\mathcal{V}}_{\text{partial},m,\text{comp}}(t) dt \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \int_0^\varepsilon t^{s-3} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \varepsilon^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} dt \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \int_0^\varepsilon t^{s-3} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \varepsilon^{k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}} dt \\
&= \varepsilon \tilde{\zeta}_m^e(s) \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \int_0^\varepsilon t^{s-3} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} t^{k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} dt \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \int_0^\varepsilon t^{s-3} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} t^{k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}} dt \\
&= \varepsilon \tilde{\zeta}_m^e(s) \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}.
\end{aligned} \tag{R 41}$$

As previously, we note that this expression involves the pole  $s = 2$ , which comes from the Euclidean line segments themselves involved in the sequence of polyhedral neighborhoods (see Theorem 5.6, on page 41). For this reason, we also exclude the terms  $\frac{c_{0,(N_b-1)j+q,m+1} \varepsilon^{s-2+i\ell_{k,(N_b-1)j+q,m}\mathbf{p}}}{s-2+i\ell_{0,(N_b-1)j+q,m}\mathbf{p}}$  from the last expression, so that the recurrence relation we will use is

$$\begin{aligned}
\tilde{\zeta}_{m+1}^e(s) &= \varepsilon \tilde{\zeta}_m^e(s) \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
&\quad + \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}.
\end{aligned} \tag{R 42}$$

Thanks to relation  $(\mathcal{R}16)$ , on page 38, given  $m \in \mathbb{N}$ , we have that, for all  $k \geq m + 2$

$$c_{k,(N_b-1)j+q,m+1} = 0.$$

We then obtain that

$$\begin{aligned} \tilde{\zeta}_{m+1}^e(s) &= \varepsilon \tilde{\zeta}_m^e(s) \\ &+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\ &+ \frac{1}{2} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m+1}} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}. \end{aligned} \tag{\mathcal{R}43}$$

ii. For all  $m \in \mathbb{N}^*$  sufficiently large, by letting, for all  $s \in \mathbb{C}$ ,

$$\begin{aligned} Z_m(s) &= \\ &= \ln \left( \prod_{j=0}^{N_b^m-1} \prod_{q=0}^{N_b} \prod_{k=1}^m (s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i \ell_{k,(N_b-1)j+q,m} \mathbf{p})^{\alpha_q(N_b) c_{k,(N_b-1)j+q,m}} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{p}} \right) \\ &+ \ln \left( \prod_{j=0}^{N_b^m-1} \prod_{q=0}^{N_b} \prod_{k=1}^m (s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i \ell_{k,(N_b-1)j+q,m} \mathbf{p})^{\alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}}} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i \ell_{k,(N_b-1)j+q,m} \mathbf{p}} \right) \\ &= \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m \ln \left( (s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) + i \ell_{k,(N_b-1)j+q,m} \mathbf{p})^{\alpha_q(N_b) c_{k,(N_b-1)j+q,m}} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{p}} \right) \\ &+ \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m \ln \left( (s - D_{\mathcal{W}} + (k-1)(2 - D_{\mathcal{W}}) - i \ell_{k,(N_b-1)j+q,m} \mathbf{p})^{\alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}}} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i \ell_{k,(N_b-1)j+q,m} \mathbf{p}} \right), \end{aligned} \tag{\mathcal{R}44}$$

we then obtain that, for all  $s \in \mathbb{C}$ ,

$$\frac{dZ_m(s)}{ds} =$$

$$\begin{aligned}
&= \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m \frac{\alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&+ \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m (\ln \varepsilon) \alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \times \\
&\quad \times \ln (s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{P}) \\
&+ \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m \frac{\alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i \ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \\
&+ \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m (\ln \varepsilon) \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m}} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i \ell_{k,(N_b-1)j+q,m} \mathbf{P}} \times \\
&\quad \times \ln (s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})-i \ell_{k,(N_b-1)j+q,m} \mathbf{P}) \\
&= \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=1}^m \frac{\alpha_q(N_b) c_{k,(N_b-1)j+q,m} \varepsilon^{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{P}}}{s-D_{\mathcal{W}}+(k-1)(2-D_{\mathcal{W}})+i \ell_{k,(N_b-1)j+q,m} \mathbf{P}}
\end{aligned}$$

i.e., equivalently,

$$\frac{dZ_m(s)}{ds} = \frac{2}{\varepsilon^m} \tilde{\zeta}_m^e(s) + (\ln \varepsilon) Z_m(s),$$

since, for all  $s \in \mathbb{C}$ ,

$$\frac{d\varepsilon^s}{ds} = \frac{de^{s \ln \varepsilon}}{ds} = (\ln \varepsilon) e^{s \ln \varepsilon} = (\ln \varepsilon) \varepsilon^s.$$

We then obtain, for the  $m^{\text{th}}$  local polyhedral effective zeta function  $\tilde{\zeta}_m^e$ , the following expression, valid for all  $s \in \mathbb{C}$ :

$$\tilde{\zeta}_m^e(s) = \frac{1}{2} \varepsilon^m \left( \frac{dZ_m(s)}{ds} - \frac{1}{\ln \varepsilon} Z_m(s) \right) = \frac{1}{2} \varepsilon^m \left( \frac{dZ_m(s)}{ds} + \frac{\mathbf{p}}{2\pi} Z_m(s) \right). \quad (\mathcal{R} 45)$$

Note that  $Z_m$  is meromorphic on all of  $\mathbb{C}$ .

iii. We first note that, by using relation (R32), on page 60, we have the following Riemann sum

$$\begin{aligned}
&\lim_{m \rightarrow \infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} = \\
&= \lim_{m \rightarrow \infty} \frac{1}{N_b^{m+1}} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \alpha_q(N_b) \zeta_{\mathcal{W},M_{(N_b-1)j+q,m+1}}(s).
\end{aligned}$$

(R 46)

We then aim at proving that the convergence of the Riemann sum in relation (R46) is locally uniform.

We note that, for all  $m_0 \in \mathbb{N}$ , and all  $m \geq m_0$ ,

$$\begin{aligned}
& \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} = \\
& \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=1}^{m_0-1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
& + \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \quad (\mathcal{R}47) \\
& = \sum_{k=1}^{m_0-1} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\
& + \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}.
\end{aligned}$$

By placing ourselves in a compact set  $\mathcal{K} \subset \mathbb{C}$ , located away from the potential poles of  $\tilde{\zeta}_m^e$ , for all  $m \geq m_0$ , we have that

$$\begin{aligned}
& \left| \sum_{k=1}^{m_0-1} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \right| \leq \\
& \leq \sum_{k=1}^{m_0-1} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} |\alpha_q(N_b)| \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})}}{\inf_{0 \leq j \leq N_b^{m+1}+1, s \in \mathcal{K}} |s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}|} \\
& \leq \sum_{k=1}^{m_0-1} \varepsilon^{m+1} N_b^{m+1} \left( \max_{1 \leq q \leq N_b-1} |\alpha_q(N_b)| \right) \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})}}{\inf_{0 \leq j \leq N_b^{m+1}+1, s \in \mathcal{K}} |s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}|} \\
& \leq \sum_{k=1}^{m_0-1} \left( \max_{1 \leq q \leq N_b-1} |\alpha_q(N_b)| \right) \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})}}{\inf_{0 \leq j \leq N_b^{m+1}+1, s \in \mathcal{K}} |s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}|},
\end{aligned}$$

which ensures the (local) normal convergence – and hence, the (local) uniform convergence – of the first part of the sum involved in relation (R47), on page 69.

We then aim at proving the (local) uniform convergence of the second part of the sum involved in relation (R47), on page 69; i.e., the (local) uniform convergence of

$$\varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}.$$

For this purpose, we will rely on the Cauchy criterion (Weierstrass  $M$  test). We thus consider, for all  $m \geq m_0$ , the difference

$$\begin{aligned} & \varepsilon^{m+1} \sum_{j=0}^{N_b^{m_0+m}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0+m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \\ & - \varepsilon^{m+1} \sum_{j=0}^{N_b^{m_0}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}. \end{aligned} \quad (\mathcal{R}48)$$

The first sum in relation ( $\mathcal{R}47$ ) just above, on page 69, involves terms associated with the vertices of  $V_{m_0+m}$ , while the second sum (still in relation ( $\mathcal{R}47$ )) comprises terms associated with the vertices of  $V_{m_0} \subset V_{m_0+m}$ , so that, for all  $k \geq m_0$  (see relation ( $\mathcal{R}18$ ), on page 38), we have that, for all  $k \geq m_0$ ,

$$c_{k,(N_b-1)j+q,m_0+1} = 0.$$

This ensures that

$$\sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} = 0.$$

We then simply have to prove that

$$\lim_{m \rightarrow \infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^{m_0+m}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0+m+1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} = 0.$$

Thanks to relation ( $\mathcal{R}19$ ), on page 38, we have that, for all  $k \geq m_0$ ,

$$c_{k,j,m_0+m+1} = \lambda^{m+1} c_{k,j,m_0} = \varepsilon^{(m+1)(2-D_{\mathcal{W}})} c_{k,j,m_0}.$$

Therefore, this amounts to proving that

$$\lim_{m \rightarrow \infty} \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \sum_{j=0}^{N_b^{m_0+m}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,j,m_0} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} = 0.$$

Note that, for all  $0 \leq j \leq N_b^{m_0+m}$ , the sum

$$\sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \quad (\mathcal{R}49)$$

is convergent, uniformly for all  $s \in \mathbb{C}$  in a given bounded set not containing a pole of  $\tilde{\zeta}_m^e$ . Indeed,  $0 < \varepsilon < 1$  and all the coefficients of the fractal power series are uniformly bounded.

We then place ourselves in a compact set  $\mathcal{K} \subset \mathbb{C}$  contained in the connected open set  $\mathcal{U} \subset \mathbb{C}$  which is the complement in  $\mathbb{C}$  of the discrete set  $\mathcal{P}^* = \bigcup_{n \in \mathbb{N}, n \geq m_0} \mathcal{P}(\tilde{\zeta}_n^e)$ , the countable (and hence, closed) union of the sets of poles of the local polyhedral zeta functions  $\tilde{\zeta}_n^e$ , for all  $n \geq m_0$  (see Lemma 5.10, on page 64).

We then have that, for all  $m \geq m_0$ ,

$$\begin{aligned}
& \left| \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \sum_{j=0}^{N_b^{m_0+m}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \right| \leq \\
& \leq \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \left( \max_{1 \leq q \leq N_b-1} |\alpha_q(N_b)| \right) \sum_{j=0}^{N_b^{m_0+m}-1} \frac{|c_{k,(N_b-1)j+q,m_0}| \varepsilon^{s-2+k(2-D_{\mathcal{W}})}}{\inf_{0 \leq j \leq N_b^{m_0+m}-1, s \in \mathcal{K}} |s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}|} \\
& \leq \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \left( \max_{1 \leq q \leq N_b-1} |\alpha_q(N_b)| \right) \sum_{j=0}^{N_b^{m_0+m}-1} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})}}{\inf_{0 \leq j \leq N_b^{m_0+m}-1, s \in \mathcal{K}} |s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}|} \\
& \leq \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \left( \max_{1 \leq q \leq N_b-1} |\alpha_q(N_b)| \right) N_b^{m_0+m} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})}}{\inf_{0 \leq j \leq N_b^{m_0+m}-1, s \in \mathcal{K}} |s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}|}.
\end{aligned}$$

Note that the above infimum is really a minimum and is strictly positive because  $\mathcal{K} \subseteq \mathbb{C}$  is a compact set and, by construction and, by construction, does not intersect  $\mathcal{P}(\tilde{\zeta}_m^e)$ .

Since

$$\varepsilon^{m+1+m(2-D_{\mathcal{W}})} N_b^{m_0+m} = \varepsilon^{m(2-D_{\mathcal{W}})-m_0},$$

where  $m(2-D_{\mathcal{W}})-m_0 > 0$  and  $0 < \varepsilon < 1$ , we obviously have that

$$\lim_{m \rightarrow \infty} \varepsilon^{m+1+m(2-D_{\mathcal{W}})} N_b^{m_0+m} = \varepsilon^{m(2-D_{\mathcal{W}})-m_0} = 0,$$

from which we deduce the desired result; i.e., uniformly for all  $s \in \mathcal{K}$ ,

$$\lim_{m \rightarrow \infty} \left| \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \sum_{j=0}^{N_b^{m_0+m}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) c_{k,(N_b-1)j+q,m_0} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i((N_b-1)j+q)\mathbf{p}} \right| = 0.$$

The same result obviously also holds for the conjugate sum (recall that all of the coefficients  $\alpha_q(N_b)$  are real); namely,

$$\lim_{m \rightarrow \infty} \left| \varepsilon^{m+1+m(2-D_{\mathcal{W}})} \sum_{j=0}^{N_b^{m_0+m}-1} \sum_{q=1}^{N_b-1} \sum_{k=m_0}^{\infty} \alpha_q(N_b) \overline{c_{k,(N_b-1)j+q,m_0}} \frac{\varepsilon^{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})-i((N_b-1)j+q)\mathbf{p}} \right| = 0.$$

For any integer  $m \geq m_0$ , we now denote by  $\mathcal{P}(\tilde{\zeta}_m^e) \subset \mathbb{C}$  the set of poles of the zeta function  $\tilde{\zeta}_m^e$ .

We set

$$\mathcal{U} = \{s \in \mathbb{C}, \operatorname{Re}(s) < 3\}.$$

We note that, for all  $m \geq m_0$ , we have that

$$\mathcal{P}(\tilde{\zeta}_{m_0}^e) \subset \mathcal{P}(\tilde{\zeta}_m^e) \subset \mathcal{U}.$$



The remainder of our proof – in the case when we place ourselves in a compact set  $\mathcal{K} \subset \mathbb{C}$  containing some of the potential poles (necessarily, finitely many of them, since  $\mathcal{K}$  is compact and the poles are isolated) of  $\tilde{\zeta}_m^e$  – can be obtained by using the chordal metric.

For this purpose, a (complex-valued) meromorphic function  $f$  is viewed as a continuous function with values in  $\mathbb{P}^1(\mathbb{C})$ , equipped with the chordal metric, and such that, for any pole  $\omega$  of  $f$ ,  $f(\omega)$  takes the value  $\infty$  (for instance, as in [LvF13], Section 3.4 and Appendix C).

More precisely, if  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$  denotes the Riemann sphere (or complex projective line), we can show that, for *the chordal metric*, defined, for all  $(z_1, z_2) \in (\mathbb{P}^1(\mathbb{C}))^2$  by

$$\|z_1, z_2\| = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1^2|} \sqrt{1 + |z_2^2|}}, \text{ if } z_1 \neq \infty \text{ and } z_2 \neq \infty, \quad (\mathcal{R}50)$$

and

$$\|z_1, \infty\| = \frac{1}{\sqrt{1 + |z_1^2|}}, \text{ if } z_1 \neq \infty. \quad (\mathcal{R}51)$$

Since the discrete set  $\mathcal{P}^\star = \bigcup_{m \in \mathbb{N}} \mathcal{P}(\tilde{\zeta}_m^e)$  intersects each compact set  $\mathcal{K} \subset \mathbb{C}$  at a countably finite number of points (see Lemma 5.10 above, on page 64), we can restrict ourselves to the case when  $\mathcal{K}$  contains one single pole of a given  $\tilde{\zeta}_n^e$ , for some  $n \geq m_0$ .

The only difference with the first case, when the compact set  $\mathcal{K} \subset \mathbb{C}$  does not intersect  $\mathcal{P}^\star$ , is about the (previously) involved infimum  $\inf_{N_b^{m_1} \leq j \leq N_b^{m_2} + 1, s \in \mathcal{K}} |s - 2 + k(2 - D_{\mathcal{W}}) + i((N_b - 1)j + q) \mathbf{p}|$ , which has to be replaced by

$$\sup_{N_b^{m_1} \leq j \leq N_b^{m_2} + 1, s \in \mathcal{K}} \frac{1}{\sqrt{1 + |s - 2 + k(2 - D_{\mathcal{W}}) + i((N_b - 1)j + q) \mathbf{p}|^2}}.$$

The limit of the Riemann sums in relation (R46), on page 68, i.e., the (uniform) limit fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , is holomorphic on  $\mathcal{U}$ , as the (locally) uniform limit of a sequence of holomorphic functions on  $\mathcal{U}$ . By applying Weierstrass' Theorem, we then deduce that, for all  $m \geq m_0$ , the zeta function  $\tilde{\zeta}_m^e$  is meromorphic on  $\mathbb{C}$ .

Moreover, the counterpart of the results obtained in [DL22] for the sequence of tube zeta functions associated with the Weierstrass IFD, which admit a meromorphic continuation to all of  $\mathbb{C}$ , obviously holds for the sequence of polyhedral tube zeta functions: hence,  $\tilde{\zeta}_m^e$  is meromorphic on  $\mathbb{C}$ , with only simple poles, as specified in Theorem 5.13, on page 77 below.

More specifically, our earlier discussion (ending just after relation (R49), on page 70, shows the uniform convergence of  $(\tilde{\zeta}_m^e)_{m \geq m_0}$  on  $\mathcal{K}$  – and, hence also, the existence of the global polyhedral effective zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , whenever the compact set  $\mathcal{K} \subset \mathbb{C}$  does not intersect  $\mathcal{P}^\star$  (i.e.,  $\mathcal{K} \subseteq \mathcal{U} = \mathbb{C} \setminus \mathcal{P}^\star$ ). The fact that  $\tilde{\zeta}_{\mathcal{W}}^e$  is holomorphic on the (connected) open set  $\mathcal{U} \subseteq \mathbb{C}$  then follows at once from Weierstrass' Theorem about the holomorphicity of a (locally) uniformly convergent sequence of holomorphic functions (here, the sequence of holomorphic functions  $(\tilde{\zeta}_m^e)_{m \geq m_0}$  on  $\mathcal{U}$ ).

In the above case when  $\mathcal{K} \cap \mathcal{P}^\star$  is nonempty – and, without loss of generality, when  $\mathcal{K} \cap \mathcal{P}^\star = \{\omega\}$ , where  $\omega$  is a pole of  $\tilde{\zeta}_{m_1}^e$ , for some integer  $m_1 \geq m_0$  (and, thus, for any  $m \geq m_1$ ), our above argument shows that  $\tilde{\zeta}_{\mathcal{W}}^e$  is well defined and is continuous on  $\mathcal{K}$  (as the uniform limit on the compact

set  $\mathcal{K}$  of the sequence  $(\tilde{\zeta}_m^e)_{m \geq m_0}$  of continuous functions on  $\mathcal{K}$ , with all the functions involved being viewed as taking their values in the compact – and thus, complete – metric space  $(\mathbb{P}^1(\mathbb{C}), \|\cdot\|)$  (i.e.,  $\lim_{m \rightarrow \infty} \sup_{s \in \mathcal{K}} \|\tilde{\zeta}_m^e(s), \tilde{\zeta}_{\mathcal{W}}^e(s)\| = 0$ ) and such that  $\tilde{\zeta}_{\mathcal{W}}^e(\omega) = \infty$  since  $\tilde{\zeta}_m^e(\omega) = \infty$ , for all  $m \geq m_1$ . Consequently,  $\tilde{\zeta}_{\mathcal{W}}^e$  is meromorphic on  $\mathcal{U}$  and  $\omega$  is also a pole of  $\tilde{\zeta}_{\mathcal{W}}^e$  (i.e.,  $\omega \in \mathcal{P}(\tilde{\zeta}_{\mathcal{W}}^e)$ ).

In summary, we deduce that  $\tilde{\zeta}_{\mathcal{W}}^e$  is well defined (by the locally uniform limit in relation (R39), on page 62), is meromorphic on all of  $\mathbb{C}$  and  $\mathcal{P}(\tilde{\zeta}_{\mathcal{W}}^e) = \mathcal{P}^*$ .

This concludes the proof of Theorem 5.9, on page 61. □

**Property 5.11 (From the  $m^{\text{th}}$  Local Effective Polyhedral Zeta Function to the  $m^{\text{th}}$  Local Effective Tube Zeta Function).**

Given  $m \in \mathbb{N}^*$  sufficiently large (i.e.,  $m \geq m_0$ ), the Lebesgue measure  $\mathcal{V}_m^{\text{tube}}(\varepsilon_m^m)$  of the tubular neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m)$  can be connected to the Lebesgue measure  $\mathcal{V}_m(\varepsilon_m^m)$  of the  $m^{\text{th}}$  polyhedral neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  introduced in Definition 3.5, on page 11, by means of the following relation:

$$\mathcal{V}_m(\varepsilon) = \mathcal{V}_m^{\text{tube}}(\varepsilon) + \mathcal{R}_m, \quad (\mathcal{R}52)$$

where  $\mathcal{R}_m$  is a suitable error term and

$$\mathcal{V}_m^{\text{tube}}(\varepsilon) = \mu_{\mathcal{L}}(\mathcal{D}(\Gamma_{\mathcal{W}_m})).$$

As is done in Lemma 5.7, on page 56, we extend relation (R55) to the corresponding effective volumes; i.e., for  $0 \leq t \leq \varepsilon_m^m$ ,

$$\tilde{\mathcal{V}}_m(t) = \tilde{\mathcal{V}}_m^{\text{tube}}(t) + \tilde{\mathcal{R}}_m(t), \quad (\mathcal{R}53)$$

where, for all sufficiently large  $n \in \mathbb{N}^*$ ,  $\tilde{\mathcal{V}}_n$  (resp.,  $\tilde{\mathcal{V}}_n^{\text{tube}}$  or  $\tilde{\mathcal{R}}_n$ ) is the continuous function defined for all  $t \in [0, \varepsilon]$  by substituting  $t$  for  $\varepsilon$  in the expression for  $\mathcal{V}_n(\varepsilon)$  (resp.,  $\mathcal{V}_n^{\text{tube}}$  or  $\mathcal{R}_n$ ), obtained in Theorem 5.6, on page 41.

This ensures, for the associated zeta function  $\mathcal{Z}_{m, \mathcal{R}}$ , given by

$$s \mapsto \int_0^\varepsilon t^{s-3} \tilde{\mathcal{R}}_m(t) dt,$$

that (locally, uniformly on  $\mathbb{C}$ )

$$\lim_{m \rightarrow \infty} \int_0^\varepsilon t^{s-3} \tilde{\mathcal{R}}_m(t) dt = 0.$$

*Proof.* This follows directly from Property 3.5, on page 11, in conjunction with Property 5.1, on page 30. Given  $\eta > 0$ , there exists  $m_0 \in \mathbb{N}$  such that, for all  $m \geq m_0$ ,

$$|\mathcal{V}_m(\varepsilon) - \mathcal{V}_m^{\text{tube}}(\varepsilon)| \leq \eta.$$

We then proceed as in the proof of Theorem 5.9, on page 61. Since, for  $\text{Re}(s) > D_{\mathcal{W}}$ ,

$$\int_0^\varepsilon t^{s-3} \widetilde{\mathcal{R}}_m(t) dt = \int_0^\varepsilon t^{s-3} (\widetilde{\mathcal{V}}_m(t) - \widetilde{\mathcal{V}}_m^{tube}(t)) dt,$$

we obtain the expected (local) uniform convergence. □

**Theorem 5.12 (Local and Global Tube Zeta Function for the Weierstrass Iterated Fractal Drums [DL22], [DL24a]).**

Given  $m \in \mathbb{N}^*$  sufficiently large, we denote by  $\zeta_{m, \Gamma_{\mathcal{W}_m}}^{e, exact}$  the exact expression for the  $m^{\text{th}}$  local effective tube zeta function by exact, we refer to the expression which could be obtained if we had the knowledge of the exact expression for the volume of the  $m^{\text{th}}$  tubular neighborhood, which is not the case and might never be obtained). We then set, for any  $s \in \mathbb{C}$ :

$$\zeta_{m, \Gamma_{\mathcal{W}_m}}^{e, strict}(s) = \zeta_{m, \Gamma_{\mathcal{W}_m}}^{e, exact}(s) - \frac{\pi \varepsilon^s}{s} + \frac{\pi \varepsilon^{s+2}}{4(s+2)},$$

since the contribution of the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  to  $\zeta_{\mathcal{W}}^e$ , the global effective tube zeta function of the Weierstrass IFD, is obtained by excluding the (artificial) terms  $\frac{\pi \varepsilon^s}{s}$  and  $-\frac{\pi \varepsilon^{s+2}}{4(s+2)}$  coming from the extreme wedges (see Remark 5.8 just below, on page 74). Here,  $\zeta_{m, \Gamma_{\mathcal{W}_m}}^{e, exact} = \zeta_m^{e, tube}$ , the  $m^{\text{th}}$  local tubular effective zeta function of  $\Gamma_{\mathcal{W}}$  studied in [DL22]. Note that we thereby consider the exact expression for this local tubular effective zeta function, which, thus far, cannot be determined explicitly (in [DL22], we only give an approximate expression for this local tubular effective zeta function; we cannot go to the limit in this approximate expression).

The global effective tube zeta function of the Weierstrass IFD exists and coincides with the global effective polyhedral zeta function  $\zeta_{\mathcal{W}}^e$ . Hence, it is given not only by the (locally) uniform limit of the strict prefractal zeta functions but also by the limit of the Riemann sums defining  $\zeta_{\mathcal{W}}^e$ , given in relation (R40) of Theorem 5.9, on page 61. It admits a (necessarily unique) meromorphic continuation to all of  $\mathbb{C}$ , and is given, for any  $s \in \mathbb{C}$ , by the following expression:

$$\zeta_{\mathcal{W}}^e(s) = \lim_{m \rightarrow \infty} \zeta_{m, \Gamma_{\mathcal{W}_m}}^{e, strict}(s). \tag{R54}$$

Note that  $\zeta_{m, \Gamma_{\mathcal{W}_m}}^{e, strict}$  is a (tamed) Dirichlet-type integral (in the sense of [LRŽ17], Appendix A) and hence, admits, for all sufficiently large positive integers  $m$ , an abscissa of (absolute) convergence.

Furthermore, the abscissa of convergence of  $\zeta_{\mathcal{W}}^e$  is equal to

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln b} = 2 - \ln_b \frac{1}{\lambda}.$$

The proof of Theorem 5.12 is given on page 75, right after Remark 5.9.

**Remark 5.8 (About the excluded poles 0 and  $-2$  [DL24a]).**

We do not include the (artificial) terms coming from the extreme wedges. Indeed, due to the periodicity of the Weierstrass function, we have restricted our study to the values of the abscissa  $x \in [0, 1]$ . In a sense, this amounts to *cut* the Curve, so that the poles arising because of this *cut* do not have to be taken into account. More precisely, given  $m \in \mathbb{N}^*$  sufficiently large, the involved terms are  $-\frac{\pi (\varepsilon_m^m)^4}{2}$

and  $\pi (\varepsilon_m^m)^2$ , which would have given rise to the terms  $\frac{\pi \varepsilon^s}{s}$  and  $-\frac{\pi \varepsilon^{s+2}}{2(s+2)}$  in the expression for the associated  $m^{\text{th}}$  local effective tube zeta function).

*Remark 5.9 (About the possible pole 1 obtained in [DL22]).*

In our previous work [DL22], we obtained the value 1 as a possible pole of the  $m^{\text{th}}$  local effective tube zeta function. Note that this possible pole was associated to the approximate expression for the  $m^{\text{th}}$  local effective tube zeta function (coming from the contributions of triangles and parallelograms to the tubular neighborhood, for which the associated residues may cancel each other), and not to the exact expression, which explains why this value 1 is not *an actual pole* of the global effective tube zeta function of the Weierstrass IFD.

*Proof. (of Theorem 5.12, on page 74)*

Thanks to Property 5.11, on page 73, (and since  $\lim_{m \rightarrow \infty} \varepsilon_m^m = 0$ ), and given  $m \in \mathbb{N}^*$  sufficiently large, the  $m^{\text{th}}$  strict effective tubular volume

$$\tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}}(\varepsilon) = \tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{exact}}(\varepsilon) + \frac{\pi (\varepsilon_m^m)^4}{2} - \pi (\varepsilon_m^m)^2$$

can be connected to the  $m^{\text{th}}$  effective polyhedral volume  $\tilde{\mathcal{V}}_m^{\mathcal{P}}(\varepsilon_m^m)$  by means of the following relation:

$$\tilde{\mathcal{V}}_m^{\mathcal{P}}(\varepsilon) = \tilde{\mathcal{V}}_{m, \Gamma_{\mathcal{W}_m}}^{\text{strict}}(\varepsilon) + \tilde{\mathcal{R}}_m, \quad (\mathcal{R} 55)$$

where  $\tilde{\mathcal{R}}_m$  is a suitable error term such that the associated zeta function  $\mathcal{Z}_{m, \mathcal{R}}$ , given by

$$s \mapsto \int_0^\varepsilon t^{s-3} \tilde{\mathcal{R}}_m(t) dt$$

(where  $\varepsilon = \frac{1}{N_b}$  is the intrinsic scale, as introduced in Definition 4.1, on page 21) locally, uniformly on  $\mathbb{C}$ , converges to 0:

$$\lim_{m \rightarrow \infty} \int_0^\varepsilon t^{s-3} \tilde{\mathcal{R}}_m(t) dt = 0.$$

We then obtain that (locally, uniformly on  $\mathbb{C}$ )

$$\lim_{m \rightarrow \infty} \tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}}(s) = \lim_{m \rightarrow \infty} \tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{\mathcal{P}, e}(s) = \tilde{\zeta}_{\Gamma_{\mathcal{W}}}^e(s),$$

where, for all  $m \in \mathbb{N}^*$ ,  $\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{\mathcal{P}, e}$  is the effective fractal zeta function associated with the sequence of polyhedral neighborhoods of the Weierstrass IFD (and called *the  $m^{\text{th}}$  local polyhedral effective zeta function*).

We note that both sequences of local effective zeta functions  $(\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{\mathcal{P}, e})_{m \in \mathbb{N}}$  and  $(\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}})_{m \in \mathbb{N}}$  thus have the same limit, the global fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$  of the Weierstrass IFD. As was mentioned above, an interesting comment to be made is that we do not pass to the limit in the approximate expression for the strict tubular zeta function  $\tilde{\zeta}_{m, \Gamma_{\mathcal{W}_m}}^{e, \text{strict}}$  obtained in [DL22]). Indeed, the exact expression for this local tubular effective zeta function cannot, thus far, be determined explicitly. (It is the reason why, in [DL22], we only give an approximate expression for this local tubular effective zeta function:

we cannot go to the limit in this approximate expression.)

Hereafter (as was the case in the proof of Theorem 5.9), a (complex-valued) meromorphic function  $f$  is viewed as a continuous function with values in the Riemann sphere (or complex projective line)  $\mathbb{P}^1(\mathbb{C})$ , equipped with the chordal metric, and such that, for any pole  $\omega$  of  $f$ ,  $f(\omega)$  takes the value  $\infty$  (for instance, as in [LvF13], Section 3.4 and Appendix C).

We can show that, for *the chordal metric*, defined, for all  $(z_1, z_2) \in (\mathbb{P}^1(\mathbb{C}))^2$  by

$$\|z_1, z_2\| = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1^2|} \sqrt{1 + |z_2^2|}}, \text{ if } z_1 \neq \infty \text{ and } z_2 \neq \infty,$$

and

$$\|z_1, \infty\| = \frac{1}{\sqrt{1 + |z_1^2|}}, \text{ if } z_1 \neq \infty,$$

we have, thanks to the local uniform convergence on  $\mathbb{C}$ ,

$$\lim_{m \rightarrow \infty} \|\tilde{\zeta}_m^{e,strict}, \tilde{\zeta}_W^e\| = 0.$$

Indeed, for any  $\eta > 0$  and any compact set  $\mathcal{K} \subset \mathbb{C}$ , we can choose  $m_1 \in \mathbb{N}^*$  such that, for all integers  $m \geq m_1$  and all  $s \in \mathcal{K}$ , we have that

$$|\tilde{\zeta}_m^{e,strict}(s) - \tilde{\zeta}_W^e(s)| \leq \eta,$$

and hence, still for  $m \geq m_1$  and all  $s \in \mathcal{K}$ ,

$$\|\tilde{\zeta}_m^{e,strict}(s), \tilde{\zeta}_W^e(s)\| \leq |\tilde{\zeta}_m^{e,strict}(s) - \tilde{\zeta}_W^e(s)| \leq \eta.$$

□

**Definition 5.5 (Complex Dimensions of the Weierstrass Curve and of its Prefractal Approximations).**

The *Complex Dimensions* of the Weierstrass Curve  $\Gamma_W$  (or, rather, of the Weierstrass IFD) are the poles of the global polyhedral effective zeta function  $\tilde{\zeta}_W^e$ . To distinguish these Complex Dimensions from those defined in our earlier paper [DL22] (via local effective tube zeta functions, we may wish to call them *intrinsic Complex Dimensions* of  $\Gamma_W$  (or, rather, of the associated IFD).

More specifically, the phrase “*possible* (or *potential*) Complex Dimensions” refers to those  $\omega \in \mathbb{C}$  which appear as an  $\omega$ -exponent in the fractal power series expansion (R40), on page 62; accordingly, its associated residue may vanish (i.e.,  $\text{res}(\tilde{\zeta}_W^e, \omega) = 0$ ) or else, it may be nonzero (i.e.,  $\text{res}(\tilde{\zeta}_W^e, \omega) \neq 0$ ). In the latter case,  $\omega$  is called an *exact Complex Dimension* of  $\Gamma_W$  (or of the associated IFD).

We use an entirely similar definition for the Complex Dimensions of the prefractal approximations  $\Gamma_{W_m}$ , for any  $m \in \mathbb{N}^*$  sufficiently large, except for  $\tilde{\zeta}_m^e$  instead of  $\tilde{\zeta}_W^e$  and for the fractal powers series in relation (R34), on page 61, instead of in relation (R40), on page 62.

**Theorem 5.13 (Exact (Intrinsic) Complex Dimensions of the Weierstrass Curve and of its Prefractal Approximations).**

*i. Intrinsic Complex Dimensions of the  $m^{\text{th}}$  Prefractal Approximation.*

For all  $m \in \mathbb{N}^*$  sufficiently large (i.e., for  $m \geq m_0$ , for some optimal  $m_0 \in \mathbb{N}^*$ ), the Complex Dimensions of the  $m^{\text{th}}$  Prefractal Approximation  $\Gamma_{\mathcal{W}_m}$  to the Weierstrass Curve (i.e., the poles of  $\tilde{\zeta}_m^e$ ) are given as follows:

$$\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) \pm i \ell_{j_{k,m},k} \mathbf{P} \quad , \quad \text{with } m \in \mathbb{N}, 0 \leq k \leq m, \quad (\mathcal{R}56)$$

where the integers  $\ell_{j_{k,m},k} \in \mathbb{N}$  (which depend on  $k$  and  $m$ ) are given in Theorem 5.4, on page 36). Note that the  $\pm$  sign comes from the conjugate sums in the expression of  $\tilde{\zeta}_m^e$  given in relation (R35), on page 61.

We note that they only partly coincide with the possible poles of the tube fractal zeta function  $\tilde{\zeta}_m^{e,\text{tube}} = \tilde{\zeta}_{m,\Gamma_{\mathcal{W}_m}}^{e,\text{exact}}$ . In particular, we then deduce that the following possible Complex Dimensions previously obtained in [DL22], i.e.,  $1 - 2k \pm i \ell \mathbf{P}$ , with  $k \in \mathbb{N}$  and  $\ell \in \mathbb{Z}$ , along with  $-2$  and  $0$ , are not actual Complex Dimensions, in the sense introduced in the present paper.

Furthermore, without loss of generality, we may assume that for  $0 \leq k \leq m$ ,  $\ell_{j_{k,m},k} \geq 0$  (i.e.,  $\ell_{j_{k,m},k} \in \mathbb{N}$ ). In addition,  $\ell_{j_{k,m},k} = 0$  if and only if  $\omega_{j_{k,m},k}$  is real (i.e., if and only if  $\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ ).

So that, as should be the case, the nonreal prefractal Complex Dimensions come in complex conjugate pairs.

*ii. Intrinsic Complex Dimensions of the Weierstrass Curve.*

The intrinsic Complex Dimensions of the Weierstrass Curve (i.e., the poles of  $\tilde{\zeta}_{\mathcal{W}}^e$ ) are given as follows:

$$\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) \pm i \ell_{j_{k,m},k} \mathbf{P} \quad , \quad \text{with } m \in \mathbb{N} \text{ arbitrary and } 0 \leq k \leq m, \quad (\mathcal{R}57)$$

where the integers  $\ell_{j_{k,m},k} \in \mathbb{N}$  (which depend on  $k$  and  $m$ ) are given in Theorem 5.4, on page 36). Also,  $\ell_{j_{k,m},k} = 0$  if and only if  $\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ .

Moreover, they are simple and actual Complex Dimensions of  $\Gamma_{\mathcal{W}}$ ; i.e., they are simple and exact Complex Dimensions of  $\Gamma_{\mathcal{W}}$ , in the sense of Definition 5.5, on page 76.

So that, as should be the case, the nonreal Complex Dimensions come in conjugate pairs. In short, all of the possible (intrinsic) Complex Dimensions of  $\Gamma_{\mathcal{W}}$  (or of the associated IFD) are given by relation (R57) above, are exact (still in the sense of Definition 5.5). There are infinitely many (countably) of them. Furthermore, there are no other Complex Dimensions.

For the exceptional cases (depending on the values of the parameters  $\lambda$  and  $N_b$ ), we refer to [DL22] for a closely related discussion.

*Proof.*

Let  $\omega$  be a pole (i.e., a possible Complex Dimension) of the global effective fractal zeta function  $\tilde{\zeta}_{\mathcal{W}}^e$ , given by relation (R40) of Theorem 5.9, on page 61.  $\omega$  is of the following form

$$\omega = \omega_{k,\ell} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell_{k,(N_b-1)j+q,m+1} \mathbf{P},$$

with  $(k, j, m) \in \mathbb{N}^3$ ,  $0 \leq q \leq N_b$  and  $\ell_{k,(N_b-1)j+q,m+1} \in \mathbb{Z}$  arbitrary.

The residue of  $\tilde{\zeta}_{\mathcal{W}}^e$  at  $\omega_{k,\ell}$  is given by

$$\begin{aligned} \operatorname{res} \left( \tilde{\zeta}_{\mathcal{W}}^e, \omega_{k,\ell} \right) &= \lim_{s \rightarrow \omega_{k,\ell}} (s - \omega_{k,\ell}) \tilde{\zeta}_{\mathcal{W}}^e(s) \\ &= \varepsilon^{m+1} \alpha_q(N_b) c_{k,(N_b-1)j+q,m+1}, \end{aligned} \tag{R58}$$

where the nonzero coefficients  $\alpha_q(N_b)$  and  $c_{k,(N_b-1)j+q,m+1}$  are respectively given in Theorem 5.6, on page 41 and in part *ii.* of Theorem 5.4, on page 36. This ensures that the residue of  $\tilde{\zeta}_{\mathcal{W}}^e$  at  $\omega_{k,\ell}$  is nonzero:  $\omega_{k,\ell}$  is an actual Complex Dimension of the Weierstrass Curve. □

Recall that in the standard mathematical theory of Complex Dimensions (see, e.g., [LvF13], [LRŽ17], [Lap19]), a geometric object is said to be *fractal* if it admits at least one *nonreal* Complex Dimension (defined as a pole of the associated geometric or fractal zeta function). [In the present context, the fractal zeta function is the global polyhedral effective zeta function in the case of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (resp., the  $m^{\text{th}}$  local polyhedral effective zeta function of its  $m^{\text{th}}$  prefractal approximation, with  $m \geq m_0$ ; see Definition 5.5, on page 76 above).]

In addition, given  $d \in \mathbb{R}$ , it is *fractal in dimension  $d$*  if it has at least one *nonreal* Complex Dimension with real part  $d$ . (Note that nonreal Complex Dimensions come in complex conjugate pairs.) In particular, it is *principally fractal* if it is *fractal in dimension  $d$*  (here,  $D_{\mathcal{W}}$ ), the abscissa of convergence of  $\tilde{\zeta}_{\mathcal{W}}^e$  (resp., of  $\tilde{\zeta}_m^e$ , with  $m \geq m_0$ ), which is the largest possible value of  $d$ .

Accordingly, we have the following two corollaries (Corollaries 5.14 and 5.15) of Theorems 5.9, on page 61 and 5.13, on page 77.

**Corollary 5.14 (Fractality of the Weierstrass Curve).**

*The Weierstrass Curve  $\Gamma_{\mathcal{W}}$  (or the Weierstrass IFD) is fractal; in fact, it has infinitely (and countably) many intrinsic Complex Dimensions given by Theorem 5.13, on page 77. In particular, it is both principally fractal (since it has infinitely many intrinsic Complex Dimensions with real part  $D_{\mathcal{W}}$ , the Minkowski dimension of  $\Gamma_{\mathcal{W}}$ ) and is fractal in infinitely (and countably) many values of  $d$  (namely,  $d = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ , with  $k \in \mathbb{N}$  arbitrary and unrestricted).*

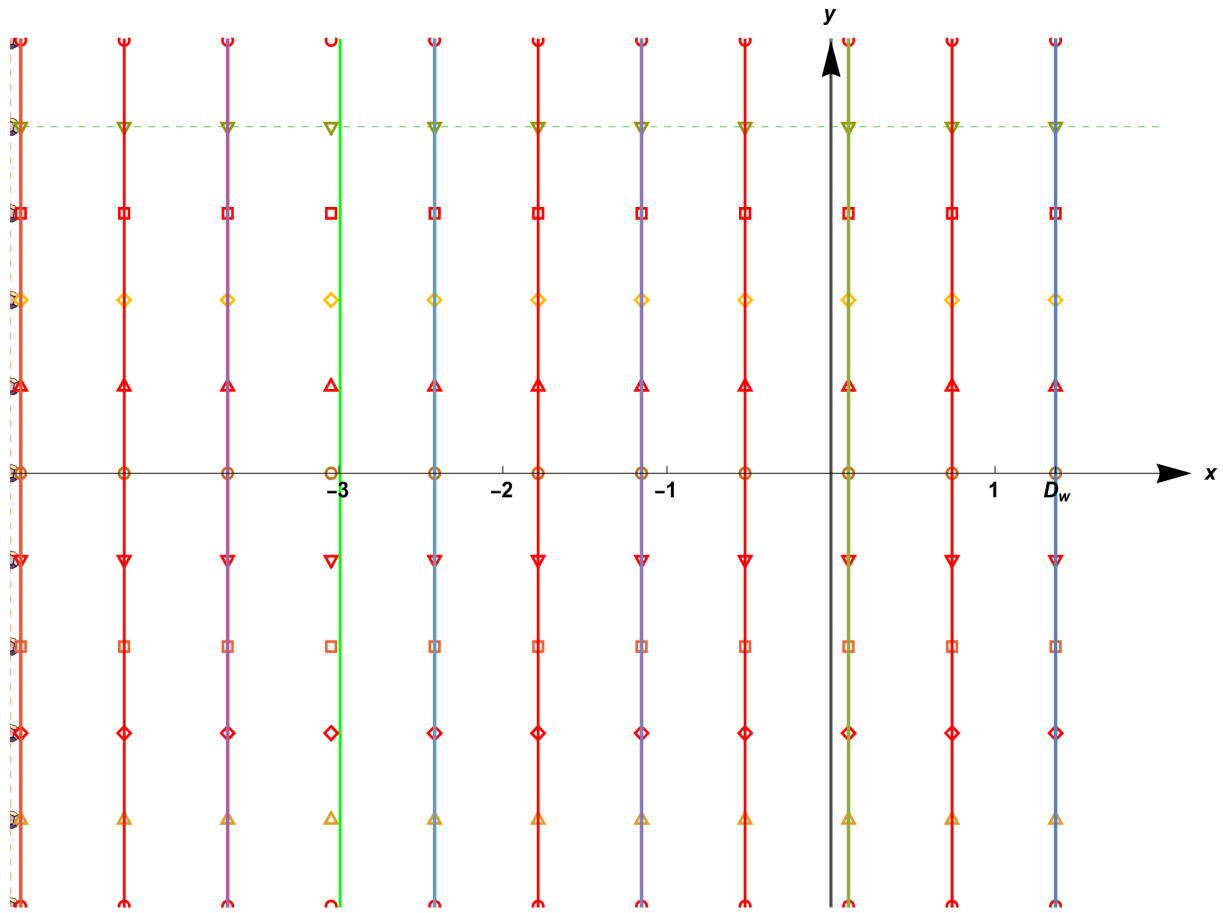


Figure 13: **The Complex Dimensions of the Weierstrass IFD.** The nonzero Complex Dimensions are periodically distributed (with the same period  $p = \frac{2\pi}{\ln N_b}$ , the oscillatory period of the Weierstrass IFD) along countably many vertical lines, with abscissae  $D_W - (k - 1)(2 - D_W)$ , where  $k \in \mathbb{N}$  is arbitrary.

For the sake of representation, there is a different color for each vertical line, and a specific symbol is used to plot the imaginary parts of the Complex Dimensions associated with a given vertical line.

For the sake of simplicity, we have omitted the exact Complex Dimensions 2 and 0, as well as the possible Complex Dimensions  $1 + inp$ , with  $n \in \mathbb{Z}$  arbitrary. Note that in light of Theorem 5.13, on page 77, the *intrinsic* Complex Dimensions form an appropriate subset of the set of Complex Dimensions depicted in the figure. They appear on each vertical line (which they necessarily intersect at countably many points), both on the real axis and in (nonreal) complex conjugate pairs.

**Corollary 5.15 (Fractality of the Prefractal Approximations).**

For all sufficiently large positive integers  $m$  (i.e., for all  $m \geq m_0$ ; see the proof on page 64 (of Theorem 5.9, on page 61), the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{W_m}$  to the Weierstrass Curve  $\Gamma_W$  is fractal and even, principally fractal. More specifically, it is fractal in dimension  $d$ , for precisely the following values of  $d$ :

$$d = D_W - k(2 - D_W),$$

with  $k$  being an arbitrary integer satisfying  $0 \leq k \leq m$ .



Indeed, the intrinsic Complex Dimensions of  $\Gamma_{\mathcal{W}_m}$  (for any  $m \geq m_0$ ) are all simple and exact; furthermore, they are given by

$$\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) \pm i \ell_{j_{k,m},k} \mathbf{p} \quad , \quad \text{with } m \in \mathbb{N}, 0 \leq k \leq m,$$

where the integers  $k \in \mathbb{N}$  and the integers  $\ell_{j_{k,m},k} \in \mathbb{N}$  (which depend on  $k$  and  $m$ ) are given in Theorem 5.4, on page 36, with  $\ell_{j_{k,m},k} = 0$  if and only if  $\omega_{j_{k,m},k}$  is real, i.e.,  $\omega_{j_{k,m},k} = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ . Note that the  $\pm$  sign comes from the complex conjugate sums in the expression of  $\tilde{\zeta}_m^e$  given in relation (R35), on page 61.

*Proof.* Corollary 5.15 follows at once from Theorem 5.13, on page 77.

The statement regarding the exactness of the Complex Dimensions in Corollary 5.14, on page 78, is proved in the same way as its counterpart in the proof of Theorem 5.13. □

**Remark 5.10 (Difference Between the Fractal and Prefractal Cases).**

Observe that there is an important and natural difference between  $\Gamma_{\mathcal{W}}$  and of the  $m^{\text{th}}$  prefractal approximation  $\Gamma_{\mathcal{W}_m}$  itself, for any  $m \geq m_0$ . Namely,  $\Gamma_{\mathcal{W}}$  is fractal for infinitely many values of  $d$  (namely,  $d = d_k = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ , for any  $k \in \mathbb{N}$ ), whereas, for all  $m \geq m_0$ ,  $\Gamma_{\mathcal{W}_m}$  is only fractal for finitely many values of  $d \in \mathbb{R}$  (namely,  $d = d_k = D_{\mathcal{W}} - k(2 - D_{\mathcal{W}})$ , for any  $k \in \mathbb{N}$  such that  $0 \leq k \leq m$ ).

Here, by analogy with [LRŽ17], [Lap19], given  $d \in \mathbb{R}$ , we say that  $\Gamma_{\mathcal{W}}$  is *fractal in dimension  $d$*  if  $\Gamma_{\mathcal{W}}$  has *nonreal* (intrinsic) Complex Dimensions with real part  $d$ ; and similarly for  $\Gamma_{\mathcal{W}_m}$ .

**Remark 5.11 (Connections with Fractal Cohomology).**

In our previous work [DL24d], we proved that the cohomology groups associated with the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  consist of continuous functions (on  $\Gamma_{\mathcal{W}}$ ) which are expressed in terms of (finite, or countably infinite) sums indexed by the Complex Dimensions  $2 - k(2 - D_{\mathcal{W}}) - i \ell_k \mathbf{p}$ , where  $\mathbf{p}$  denotes the oscillatory period of the Weierstrass Curve, for  $k \in \mathbb{N}$  arbitrary, and where  $\ell_k \in \mathbb{Z}$  satisfies a specific condition governed by the underlying geometry of the Curve. Except for this latter and specific condition, our results in Theorem 5.13, on page 77 just above, are precisely of the same form as in [DL24d], with an additional condition expressed in terms of the integers involved in the imaginary part of the Complex Dimensions associated with the fractal cohomology group, which appears as a natural feature in the fractal cohomological context of [DL24d].

## 6 Revisiting the Computation of the Minkowski Dimension

In this section, we provide a fully rigorous proof of the fact that the Minkowski dimension – which, as is well known, coincides with the box-counting (or box) dimension  $D_{\mathcal{W}}$  of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  – is given by the expected formula, originally conjectured by Benoît Mandelbrot in [Man82]

(in Chapter XI, on top of page 390). Note that B. Mandelbrot only mentioned he was talking about the *fractal dimension* and did not specify whether it was the Hausdorff or the Minkowski dimension.

In our present context (see Theorem 6.1, on page 81 below), we have that  $D_{\mathcal{W}} = 2 - \frac{\ln \frac{1}{\lambda}}{\ln N_b}$ .

It appears that earlier proofs of this fact were either not full rigorous or complete. The methods developed in the present paper and in our earlier work, especially, [DL22] and [DL24c], building in part upon the geometric approach of [Dav18], enable us to establish, in a new manner, Mandelbrot's conjecture.

When it comes to computing the box-counting – or Minkowski dimension – of a fractal, one usually proceeds by using a (classical) cover of the fractal under study, constituted of balls; namely, in dimension 2, disks or squares; see Definition 6.1, on page 81 below.

However, when a non self-affine, complicated fractal curve, as the Weierstrass Curve is involved, such methods are not optimal. In fact, covering such a curve with disks or squares is not natural. We hereafter propose to consider, instead, the polyhedral neighborhoods introduced in Definition 3.5, on page 11.

**Definition 6.1 (Box-Counting Dimension).**

As can be found, for instance, in [Fal86], we recall that the *box-counting dimension* (or *box dimension*, in short), of  $\Gamma_{\mathcal{W}}$ , is given by

$$D_{\mathcal{W}} = - \lim_{\delta \rightarrow 0^+} \frac{\ln N_{\delta}(\Gamma_{\mathcal{W}})}{\ln \delta}, \quad (\diamond)$$

where  $N_{\delta}(\Gamma_{\mathcal{W}})$  stands for any of the following quantities:

- i.* the smallest number of sets (here, subsets of  $\mathbb{R}^2$ ) of diameter at most  $\delta$  that cover  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$  ;
- ii.* the smallest number of closed balls (disks, here) of radius  $\delta$  that cover  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$  ;
- iii.* the smallest number of cubes (squares, here) of side  $\delta$  that cover  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$ ;
- iv.* the number of  $\delta$ -mesh cubes (squares, here) that intersect  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$ ;
- v.* the largest number of disjoint balls (disks, here) of radius  $\delta$  with centers in  $\Gamma_{\mathcal{W}}$  on  $[0, 1[$ .

Furthermore, for the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , as, more generally, for any bounded subset of Euclidean space – the box-counting dimension coincides with the Minkowski dimension.

**Theorem 6.1.** *The box-counting dimension – or, equivalently, the Minkowski dimension – of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  exists and is equal to  $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$ .*

*Proof.* We simply apply the result given in Corollary 5.2, on page 34. We then have the existence of an integer  $m_0 \in \mathbb{N}$  such that

$$\forall m \geq m_0 : \Gamma_{\mathcal{W}} \subset \mathcal{D}(\Gamma_{\mathcal{W}_m}) .$$

In other words, for all integers  $m \geq m_0$ , the polyhedral neighborhood  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  covers the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ .

Note that we thus dispose, with the sequence of polyhedral domains  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \geq m_0}$ , of a nonusual (but admissible, in the sense of Definition 6.1, on page 81 above) sequence of covers. However,  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \geq m_0}$  is the most natural and optimal sequence of covers of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , insofar that for any  $m \geq m_0$ , each domain  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  contains the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , while, at the same time, the sequence  $(\mathcal{D}(\Gamma_{\mathcal{W}_m}))_{m \geq m_0}$  converges to  $\Gamma_{\mathcal{W}}$ . This, in particular, means that when  $m \rightarrow \infty$ , the two-dimensional Lebesgue measure (i.e., area) of each polygon belonging to  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  tends to 0.

For all  $m \in \mathbb{N}^*$ , each polygon belonging to  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  has a vertical height at most equal to  $(N_b - 1) h_m$  (this directly comes from the definition of  $h_m$ ; see relation (R5), on page 14, and an horizontal width at most equal to  $(N_b - 1) \varepsilon_m^m$ . Note now that, thanks to Remark 4.3, on page 22, we have that  $h_m \gg \varepsilon_m^m$ .

Hence, it is the cohomology infinitesimal  $\varepsilon_m^m$  which plays the role of the elementary diameter  $\delta$  in Definition 6.1, on page 81. Moreover, the area of each polygon belonging to  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  also corresponds to the area of the reunion of  $\frac{(N_b - 1) h_m}{\varepsilon_m^m}$  elementary and smaller polygons, each of diameter  $\varepsilon_m^m$ , where *the diameter* of a polygon is to be understood in the sense of the largest distance between any pair of vertices of the considered polygon.

Since  $\mathcal{D}(\Gamma_{\mathcal{W}_m})$  consists itself of the reunion of  $(2 N_b^m - 1) = \frac{2 \varepsilon_m^{-m}}{N_b - 1} - 1$  polygons, we then obtain the box-counting dimension as

$$- \lim_{m \rightarrow \infty} \frac{\ln \left( \left( \frac{2 \varepsilon_m^{-m}}{N_b - 1} - 1 \right) \frac{(N_b - 1) h_m}{\varepsilon_m^m} \right)}{\ln \varepsilon_m^m} = - \lim_{m \rightarrow \infty} \frac{\ln (\varepsilon_m^{-2m} h_m)}{\ln \varepsilon_m^m} .$$

Thanks to the estimate given in Property 3.13, on page 17, we have that

$$C_{inf} (\varepsilon_m^m)^{2-D_{\mathcal{W}}} \leq h_m \leq C_{sup} (\varepsilon_m^m)^{2-D_{\mathcal{W}}} ,$$

where the constants  $C_{inf}$  and  $C_{sup}$  do not depend on  $m$ , and are such that

$$0 < C_{inf} < C_{sup} < \infty ,$$

which ensures that

$$C_{inf} (\varepsilon_m^m)^{-D_{\mathcal{W}}} \leq \varepsilon_m^{-2m} h_m \leq C_{sup} (\varepsilon_m^m)^{-D_{\mathcal{W}}} ,$$

and, consequently,

$$\ln C_{inf} - D_{\mathcal{W}} \ln (\varepsilon_m^m) \leq \ln (\varepsilon_m^{-2m} h_m) \leq \ln C_{sup} - D_{\mathcal{W}} \ln (\varepsilon_m^m) .$$

Since  $\lim_{m \rightarrow \infty} \varepsilon_m^m = 0$ , we obviously have that

$$\lim_{m \rightarrow \infty} \frac{\ln C_{inf}}{\ln (\varepsilon_m^m)} = \lim_{m \rightarrow \infty} \frac{\ln C_{sup}}{\ln (\varepsilon_m^m)} = 0 ,$$

from which we immediately deduce that the value of the box-counting dimension of the Weierstrass Curve is exactly

$$-\lim_{m \rightarrow \infty} \frac{\ln(\varepsilon_m^{-2m} h_m)}{\ln \varepsilon_m^m} = D_{\mathcal{W}}.$$

This concludes the proof of the existence of the box-counting (i.e., the Minkowski) dimension and of the fact that it is given by  $D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$ . □

## 7 A Bridge to Fractal Cohomology

In [DL24d], we established the connections between the Complex Dimensions of a fractal object, and the associated fractal cohomology. In the test case of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ , we obtained a complete characterization of all the elements of the cohomology groups associated to the Curve, which consist of continuous functions (on the Weierstrass Curve  $\Gamma_{\mathcal{W}}$ ), expressed in terms of Taylor-like expansions, with fractional derivatives of orders the underlying – and actual – Complex Dimensions of the Weierstrass function  $\mathcal{W}$ . As powerful as this result stands, there remained an unanswered question, insofar as the Complex Dimensions involved did not comprise all the possible values obtained in [DL22] by means of the (classical) tube zeta functions, defined via ordinary (Euclidean) neighborhoods; namely,  $-2$ ,  $0$  and  $1$ .

Our effective local and global polyhedral fractal zeta functions enables us to finally answer this remaining question: their poles, which exactly correspond to the Complex Dimensions both of the Weierstrass Curve  $\Gamma_{\mathcal{W}}$  and of the Weierstrass function  $\mathcal{W}$ , do not include the values  $-2$ ,  $0$  and  $-1 + i\ell \mathbf{p}$  where, contrary to our present results,  $\ell \in \mathbb{Z}$  is arbitrary.

More specifically, the possible Complex Dimensions are all exact and precisely of the following form:  $D_{\mathcal{W}} - k(2 - D_{\mathcal{W}}) + i\ell \mathbf{p}$ , where  $k \in \mathbb{N}$  and for a specific choice of  $\ell \in \mathbb{Z}$ . Note that the aforementioned classical approach gives a less precise result, without any condition on  $\ell \in \mathbb{Z}$ .

For this reason, we claim that the polyhedral approach, by means of polyhedral neighborhoods and polyhedral fractal zeta functions, is intrinsic, as well as appears to be more natural and better suited than the tubular (or Euclidean) approach introduced in [DL22] to the study of a more general class of fractal curves, including the Takagi function, the Knopp functions, along with the Koch function and the Koch Curve. The connections with fractal cohomology – and the fact that we fall back on the same values of the Complex Dimensions of the fractal curve involved – open new perspectives in terms of the understanding of those everywhere singular objects. In particular, the polygons which constitute the polyhedral neighborhoods certainly play an important role in the simplicial (and natural) homology, dual to the fractal cohomology introduced in [DL23] and [DL24d], and to be developed in a later book of the authors.

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